ransX framework

Theory Guide

Miroslav Mocák, Casey Meakin, Maxime Viallet & David Arnett

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1 Introduction

We present a statistical analysis of turbulent convection in stars within our Reynolds-Averaged Navier Stokes (RANS) framework in spherical geometry which we derived from first principles (see Sect.17 and further sections for details). The analysed data include **core convection during oxygen burning in a 15 $M_\odot$ supernova progenitor, envelope convection in a 5 $M_\odot$ red giant and core convection during core helium flash and hydrogen injection flash of a 1.25 $M_\odot$ star**. These simulations have been partially already described by Meakin and Arnett [2007], Arnett et al. [2010], Viallet et al. [2013] and Mocák et al. [2009, 2011].

We obtain our 1D RANS equations by introducing two types of averaging: statistical averaging and horizontal averaging [Besnard et al., 1992, Viallet et al., 2013]. In practice, statistical averages are computed by performing a time average (the ergodic hypothesis). Therefore, the combined average of a quantity q is defined as

$$\overline{q}(r,t) = \frac{1}{T\Delta\Omega} \int_{t-T/2}^{t+T/2} q(r,\theta,\phi,t')\ d\Omega\ dt' \tag{1}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle in spherical coordinates, T is the averaging time period, and $\Delta\Omega$ is total solid angle being averaged over.

The flow variables are then decomposed into mean and fluctuation $q = \overline{q} + q'$, noting that $\overline{q'} = 0$ by construction. Similarly, we introduce Favre (or density weighted) averaged quantities by

$$\widetilde{q} = \frac{\overline{\rho q}}{\overline{\rho}} \tag{2}$$

which defines a complimentary decomposition of the flow into mean and fluctuations according to $q = \widetilde{q} + q''$. Here, q'' is the Favrian fluctuation and its mean is zero when Favre averaged $\widetilde{q''} = 0$. For a more complete elaboration on the algebra of these averaging procedures we refer the reader to Chassaing et al. [2010]. All calculated mean fields shown for a given equation in next sections were additionally multiplied by $4\pi r^2$. It gives us advantage to reflect changing volume of our computational domain as a function of radius and visualize better volume integral budgets of the individual mean fields, which are then equivalent to the area below corresponding mean field profile. We explore not only general properties of mean fields but also their resolution dependency, wedge-size dependency, convection zone depth dependency, position of driving source dependency and validity of few turbulence models.

2 Summary of Reynolds-averaged Navier Stokes equations in spherical geometry

2.1 Various mean fields equations (first-order moments)

$$
\begin{aligned}
\widetilde{D}_t \overline{\rho} &= -\overline{\rho}\widetilde{d} + \mathcal{N}_\rho & (3)\\
\overline{\rho}\widetilde{D}_t \widetilde{u}_r &= -\nabla_r \widetilde{R}_{rr} - \overline{G_r^M} - \partial_r \overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} & (4)\\
\overline{\rho}\widetilde{D}_t \widetilde{u}_\theta &= -\nabla_r \widetilde{R}_{\theta r} - \overline{G_\theta^M} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} & (5)\\
\overline{\rho}\widetilde{D}_t \widetilde{u}_\phi &= -\nabla_r \widetilde{R}_{\phi r} - \overline{G_\phi^M} + \mathcal{N}_{u\phi} & (6)\\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_I &= -\nabla_r (f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} & (7)\\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_k &= -\nabla_r (f_k + f_P) - \widetilde{R}_{ir}\partial_r \widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} & (8)\\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_t &= -\nabla_r (f_I + f_T + f_k + f_P) - \widetilde{R}_{ir}\partial_r \widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} & (9)\\
\overline{\rho}\widetilde{D}_t \widetilde{h} &= -\nabla_r f_h - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T + \mathcal{N}_h & (10)\\
\overline{\rho}\widetilde{D}_t \widetilde{s} &= -\nabla_r f_s + \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s & (11)\\
\overline{D}_t \overline{P} &= -\nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1-\Gamma_1) W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P & (12)\\
\overline{D}_t \overline{T} &= -\nabla_r f_T + (1-\Gamma_3)\overline{T}\ \overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot F_T)/\rho c_v} + \overline{(\tau_{ij}\partial_i u_j)/\rho c_v} + \overline{\epsilon_{\mathrm{nuc}}/c_v} + \mathcal{N}_T & (13)\\
\overline{\rho}\widetilde{D}_t \widetilde{X}_\alpha &= -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\mathrm{nuc}} + \mathcal{N}_\alpha & (14)\\
\overline{\rho}\widetilde{D}_t \widetilde{A} &= -\nabla_r f_A - \overline{\rho A^2 \Sigma_\alpha (\dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha)} + \mathcal{N}_A & (15)\\
\overline{\rho}\widetilde{D}_t \widetilde{Z} &= -\nabla_r f_Z - \overline{\rho Z A \Sigma_\alpha (\dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha)} + \overline{\rho A \Sigma_\alpha (Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha)} + \mathcal{N}_Z & (16)\\
\overline{\rho}\widetilde{D}_t \widetilde{j}_z &= -\nabla_r f_{jz} + \mathcal{N}_{jz} & (17)
\end{aligned}
$$

2.2 Mean turbulent mass flux and mean density-specific volume covariance equation (second-order moments)

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t \overline{u''_r} &= -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r \overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r \overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r \overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r \overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r \overline{P} + \overline{\rho' v \partial_r P'} + \mathcal{G}_a + \mathcal{N}_a & (18)\\
\overline{D}_t b &= +\overline{v}\nabla_r \overline{\rho}\overline{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\overline{v'd'} + \mathcal{N}_b & (19)
\end{aligned}
$$

2.3 Mean Reynolds stress equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{rrr} + f^r_p + f^r_p + f^{rr}_\tau + f^{rr}_\tau) + \overline{u''_r}\partial_r\overline{P} + \overline{u''_r}\partial_r\overline{P} - \widetilde{R}_{rr}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + \overline{P'\nabla_r u''_r} + \overline{P'\nabla_r u''_r} - \overline{u''_r G^M_r} - \overline{u''_r G^M_r} - \overline{G^R_{rr}} - \varepsilon^{rr}_\tau - \varepsilon^{rr}_\tau \tag{20}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{r\theta}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{r\theta r} + f^\theta_p + f^{r\theta}_\tau + f^{\theta r}_\tau) - \overline{u''_\theta}\partial_r\overline{P} - \widetilde{R}_{rr}\partial_r\widetilde{u}_\theta - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_r + \overline{P'\nabla_r u''_\theta} + \overline{P'\nabla_\theta u''_r} - \overline{u''_r G^M_\theta} - \overline{u''_\theta G^M_r} - \overline{G^R_{r\theta}} - \varepsilon^{r\theta}_\tau - \varepsilon^{\theta r}_\tau \tag{21}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{r\phi}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{r\phi r} + f^\phi_p + f^{r\phi}_\tau + f^{\phi r}_\tau) - \overline{u''_\phi}\partial_r\overline{P} - \widetilde{R}_{rr}\partial_r\widetilde{u}_\phi - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_r + \overline{P'\nabla_r u''_\phi} + \overline{P'\nabla_\phi u''_r} - \overline{u''_r G^M_\phi} - \overline{u''_\phi G^M_r} - \overline{G^R_{r\phi}} - \varepsilon^{r\phi}_\tau - \varepsilon^{\phi r}_\tau \tag{22}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta r}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\theta rr} + f^\theta_p + f^{\theta r}_\tau + f^{r\theta}_\tau) - \overline{u''_\theta}\partial_r\overline{P} - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{u}_\theta + \overline{P'\nabla_\theta u''_r} + \overline{P'\nabla_r u''_\theta} - \overline{u''_\theta G^M_r} - \overline{u''_r G^M_\theta} - \overline{G^R_{\theta r}} - \varepsilon^{\theta r}_\tau - \varepsilon^{r\theta}_\tau \tag{23}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\theta\theta r} + f^{\theta\theta}_\tau + f^{\theta\theta}_\tau) - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\theta u''_\theta} - \overline{u''_\theta G^M_\theta} - \overline{u''_\theta G^M_\theta} - \overline{G^R_{\theta\theta}} - \varepsilon^{\theta\theta}_\tau - \varepsilon^{\theta\theta}_\tau \tag{24}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\phi}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\theta\phi r} + f^{\theta\phi}_\tau + f^{\phi\theta}_\tau) - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_\phi - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\theta + \overline{P'\nabla_\theta u''_\phi} + \overline{P'\nabla_\phi u''_\theta} - \overline{u''_\theta G^M_\phi} - \overline{u''_\phi G^M_\theta} - \overline{G^R_{\theta\phi}} - \varepsilon^{\theta\phi}_\tau - \varepsilon^{\phi\theta}_\tau \tag{25}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi r}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\phi rr} + f^\phi_p + f^{\phi r}_\tau + f^{r\phi}_\tau) - \overline{u''_\phi}\partial_r\overline{P} - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{u}_\phi + \overline{P'\nabla_\phi u''_r} + \overline{P'\nabla_r u''_\phi} - \overline{u''_\phi G^M_r} - \overline{u''_r G^M_\phi} - \overline{G^R_{\phi r}} - \varepsilon^{\phi r}_\tau - \varepsilon^{r\phi}_\tau \tag{26}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi\theta}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\phi\theta r} + f^{\phi\theta}_\tau + f^{\theta\phi}_\tau) - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\theta - \widetilde{R}_{\theta r}\partial_r\widetilde{u}_\phi + \overline{P'\nabla_\phi u''_\theta} + \overline{P'\nabla_\theta u''_\phi} - \overline{u''_\phi G^M_\theta} - \overline{u''_\theta G^M_\phi} - \overline{G^R_{\phi\theta}} - \varepsilon^{\phi\theta}_\tau - \varepsilon^{\theta\phi}_\tau \tag{27}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi\phi}/\overline{\rho}\right) = -\nabla_r(\widetilde{F}^R_{\phi\phi r} + f^{\phi\phi}_\tau + f^{\phi\phi}_\tau) - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi - \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi + \overline{P'\nabla_\phi u''_\phi} + \overline{P'\nabla_\phi u''_\phi} - \overline{u''_\phi G^M_\phi} - \overline{u''_\phi G^M_\phi} - \overline{G^R_{\phi\phi}} - \varepsilon^{\phi\phi}_\tau - \varepsilon^{\phi\phi}_\tau \tag{28}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -\nabla_r(2f^r_k + 2f_P) + 2W_b - 2\widetilde{R}_{rr}\partial_r\widetilde{u}_r + 2\overline{P'\nabla_r u''_r} + 2\mathcal{G}^r_k + \mathcal{N}_{Rrr} \tag{29}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\nabla_r(2f^\theta_k) - 2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + 2\overline{P'\nabla_\theta u''_\theta} + 2\mathcal{G}^\theta_k + \mathcal{N}_{R\theta\theta} \tag{30}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi\phi}/\overline{\rho}\right) = -\nabla_r(2f^\phi_k) - 2\widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi + 2\overline{P'\nabla_\phi u''_\phi} + 2\mathcal{G}^\phi_k + \mathcal{N}_{R\phi\phi} \tag{31}$$

2.4 Mean turbulent kinetic energy equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{32}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f^r_k + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}^r_k + \mathcal{N}_{kr} \tag{33}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f^h_k - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}^h_k + \mathcal{N}_{kh} \tag{34}$$

$$\tag{35}$$

2.5 Mean flux equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) = -\nabla_r f_I^r - f_I \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{\epsilon}_I - \overline{\epsilon_I''}\partial_r \overline{P} - \overline{\epsilon_I''\partial_r P'} - \overline{u_r''(Pd)} + \overline{u_r''(\mathcal{S} + \nabla \cdot F_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \quad (36)$$

$$\overline{\rho}\widetilde{D}_t(f_h/\overline{\rho}) = -\nabla_r f_h^r - f_h \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{h} - \overline{h''}\partial_r \overline{P} - \overline{h''\partial_r P'} - \Gamma_1 \overline{u_r''(Pd)} + \Gamma_3 \overline{u_r''(\mathcal{S} + \nabla \cdot F_T)} + \mathcal{G}_h + \mathcal{N}_h \quad (37)$$

$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{s} - \overline{s''}\partial_r \overline{P} - \overline{s''\partial_r P'} + \overline{u_r''(\mathcal{S} + \nabla \cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \quad (38)$$

$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{X}_\alpha - \overline{X_\alpha''}\partial_r \overline{P} - \overline{X_\alpha''\partial_r P'} + \overline{u_r''\rho \dot{X}_\alpha^{\rm nuc}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \quad (39)$$

$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = -\nabla_r f_A^r - f_A \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{A} - \overline{A''}\partial_r \overline{P} - \overline{A''\partial_r P'} - \overline{u_r''\rho A^2 \Sigma_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha} + \mathcal{G}_A + \mathcal{N}_{fA} \quad (40)$$

$$\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = -\nabla_r f_Z^r - f_Z \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{Z} - \overline{Z''}\partial_r \overline{P} - \overline{Z''\partial_r P'} - \overline{u_r''\rho Z A \Sigma_\alpha (\dot{X}_\alpha^{\rm nuc}/A_\alpha)} - \overline{u_r''\rho A \Sigma_\alpha (Z_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z + \mathcal{N}_{fZ} \quad (41)$$

$$\overline{\rho}\widetilde{D}_t(f_{jz}/\rho) = -\nabla_r f_{jz}^r - f_{jz}\partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{j}_z - \overline{j_z''}\partial_r \overline{P} - \overline{j_z''\partial_r P'} + \mathcal{G}_{jz} + \mathcal{N}_{jz} \quad (42)$$

$$\widetilde{D}_t f_T = -\nabla_r f_T^r - f_T \partial_r \overline{u}_r - \overline{u_r' u_r''}\partial_r \overline{T} - \overline{T'\partial_r P/\rho} - (\Gamma_3 - 1)(\overline{T}\ \overline{u_r' d''} + \widetilde{d}\ \overline{u_r' T'} + \overline{u_r' T' d''}) + \overline{T' u_r' d''} + \overline{u_r' \epsilon_{\rm nuc}/c_v} + \overline{u_r' \nabla \cdot F_T/\rho c_v} + \mathcal{G}_T + \mathcal{N}_{fT} \quad (43)$$

$$\overline{\rho}\widetilde{D}_t(f_k/\overline{\rho}) = -\nabla_r(2f_k^r + f_P^r) - (1/2)(\nabla_r \overline{u_i'' u_i'' P'} + \nabla_r 2k\partial_r \widetilde{u}_r - \widetilde{u_i'' u_r''}\nabla_r \widetilde{R}_{rr} - 2k\partial_r \overline{P} + \overline{u_i'' u_i''}\partial_r \overline{P} - \overline{P'\nabla_r u_i'' u_i''}) - \quad (44)$$

$$-(k_i^r \partial_r \widetilde{u}_i - \widetilde{u_i'' u_r''}\nabla_r \widetilde{u_i'' u_r''} - 2k^r \partial_r \overline{P} + \overline{u_r'' u_r''}\partial_r \overline{P} - \overline{P'\nabla_i u_i'' u_r''}) + \mathcal{G}_{fk} + \mathcal{N}_{fk} \quad (45)$$

$$\widetilde{D}_t f_p = -\nabla_r f_p^r - f_p \partial_r \overline{u}_r + \overline{u_r'' u_r'}\partial_r \overline{P} + \Gamma_1 \overline{u_r' P d} + (\Gamma_3 - 1)\overline{u_r' \rho \varepsilon_{nuc}} + \overline{P' u_r'' d''} - \overline{P' G_r^M/\rho} - \overline{P'\partial_r P/\rho} + \mathcal{N}_{fp} \quad (46)$$

2.6 Mean variance equations (second-order moments)

$$\widetilde{D}_t \sigma_\rho = -\nabla_r \overline{(\rho'\rho' u_r'')} - 2\overline{\rho}\ \overline{\rho' d''} - 2\overline{\rho' u_r''}\partial_r \overline{\rho} - 2\widetilde{d}\ \sigma_\rho - \overline{\rho'\rho' d''} + \mathcal{N}_{\sigma_\rho} \quad (47)$$

$$\widetilde{D}_t \sigma_P = -\nabla_r \overline{(P'P'u_r'')} - 2\Gamma_1 \overline{P}\ W_P - 2f_P \partial_r \overline{P} - 2\Gamma_1 \widetilde{d}\ \sigma_P - (2\Gamma_1 - 1)\overline{P'P'd''} + 2(\Gamma_3 - 1)\overline{P'\mathcal{S}} + \mathcal{N}_{\sigma_P} \quad (48)$$

$$\widetilde{D}_t \sigma_T = -\nabla_r \overline{(T'T'u_r'')} - 2(\Gamma_3 - 1)\overline{T}\ \overline{T'd''} - 2\overline{T'u_r''}\partial_r \overline{T} - 2(\Gamma_3 - 1)\widetilde{d}\sigma_T + (3 - 2\Gamma_3)\overline{T'T'd''} + \overline{2T'\nabla \cdot F_T/\rho c_v} + \overline{2T'\epsilon_{\rm nuc}/c_v} + \mathcal{N}_{\sigma T} \quad (49)$$

$$\overline{\rho}\widetilde{D}_t \sigma_{ur} = -\nabla_r (\overline{\rho u_r'' u_r'' u_r''}) + 2\nabla_r f_P + 2W_b - 2\widetilde{R}_{rr}\partial_r \widetilde{u}_r + 2\overline{P'\nabla_r u_r''} + \mathcal{G}_{\sigma_{ur}} + \mathcal{N}_{\sigma_{ur}} \quad (50)$$

$$\overline{\rho}\widetilde{D}_t \sigma_{\epsilon I} = -\nabla_r (\overline{\rho \epsilon_I'' \epsilon_I'' u_r''}) - 2f_I \partial_r \widetilde{\epsilon}_I - 2\overline{\epsilon_I''}\ \overline{P}\ \widetilde{d} - 2\overline{P}\ \overline{\epsilon_I'' d''} - 2\widetilde{d}\ \overline{\epsilon_I'' P'} - 2\overline{\epsilon_I'' P' d''} + 2\overline{\epsilon_I''\mathcal{S}} + \mathcal{N}_{\sigma_{\epsilon I}} \quad (51)$$

$$\overline{\rho}\widetilde{D}_t \sigma_h = -\nabla_r (\overline{\rho h'' h'' u_r''}) - 2f_h \partial \widetilde{h} - 2\Gamma_1 \overline{h'' P d} + 2\Gamma_3 \overline{h''\rho \mathcal{S}} + \mathcal{N}_{\sigma_h} \quad (52)$$

$$\overline{\rho}\widetilde{D}_t \sigma_s = -\nabla_r (\overline{\rho s'' s'' u_r''}) - 2f_s \partial_r \widetilde{s} - 2\overline{s''\nabla \cdot F_T/T} + 2\overline{s''\mathcal{S}/T} + \mathcal{N}_{\sigma_s} \quad (53)$$

$$\overline{\rho}\widetilde{D}_t \sigma_\alpha = -\nabla_r (\overline{\rho X_\alpha'' X_\alpha'' u_r''}) - 2f_\alpha \partial_r \widetilde{X}_\alpha + 2\overline{X_\alpha''\rho \dot{X}_\alpha^{\rm nuc}} + \mathcal{N}_{\sigma_\alpha} \quad (54)$$

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
T temperature	$\mathcal{S} = \rho\epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r \sin\theta \ u_\phi$ specific angular momentum	$F_T = \chi\partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d \ ln \ P/d \ ln \ \rho)\vert_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d \ ln \ P/d \ ln \ T)\vert_s$
h specific enthalpy	$\Gamma_3 - 1 = (d \ ln \ T/d \ ln \ \rho)\vert_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho}\widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho}\widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho}\widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho}\widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P'd''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy
$f_P = \overline{P'u'_r}$ acoustic flux	$f_T = -\overline{T'u'_r}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon''_I u''_r}$ internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X''_\alpha u''_r}$ X_α flux
$f_s = \overline{\rho}\widetilde{s'' u''_r}$ entropy flux	$f_{jz} = \overline{\rho}\widetilde{j''_z u''_r}$ angular momentum flux
$f_T = \overline{u'_r T'}$ turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u''_r}$ A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u''_r}$ enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u''_r}$ Z (mean charge per isotope) flux
$b = \overline{v'\rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f_\tau^r = -\overline{\tau'_{rr} u'_r}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f_\tau^\theta = -\overline{\tau'_{\theta r} u'_\theta}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f_\tau^\phi = -\overline{\tau'_{\phi r} u'_\phi}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_I^r = \overline{\rho}\widetilde{\epsilon''_I u''_r u''_r}$ radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_s^r = \overline{\rho}\widetilde{s'' u''_r u''_r}$ radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$ numerical effect
$f_h^r = \overline{\rho}\widetilde{h'' u''_r u''_r}$ radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$f_T^r = \overline{T' u'_r u'_r}$ radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect
$f_{jz}^r = \overline{\rho}\widetilde{j''_z u''_r u''_r}$ radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect
$f_\alpha^r = \overline{\rho}\widetilde{X''_\alpha u''_r u''_r}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f_A^r = \overline{\rho}\widetilde{A'' u''_r u''_r}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect
$f_Z^r = \overline{\rho}\widetilde{Z'' u''_r u''_r}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u''_r G_r^M}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3 - 1)\tau_{ij}\partial_i u_j} - \overline{u''_r \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z \tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i \tau_{ri}/\rho} + \overline{u'_r \tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s''s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I \epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I \tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$\varepsilon_k^r = \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r}$$
$$\varepsilon_k^\theta = \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta}$$
$$\varepsilon_k^\phi = \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi}$$
$$\varepsilon_k = \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi$$
$$\varepsilon_k^h = \varepsilon_k^\theta + \varepsilon_k^\phi$$
$$\varepsilon_a = \overline{\rho' v \nabla_r \tau'_{rr}}$$
$$\varepsilon_I = \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I}$$
$$\varepsilon_s = \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''}$$
$$\varepsilon_\alpha = \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha}$$
$$\varepsilon_A = \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''}$$
$$\varepsilon_Z = \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''}$$
$$\varepsilon_h = \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''}$$
$$\varepsilon_{jz} = \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}$$

$$\overline{G_r^M} = -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r}$$
$$\overline{G_\theta^M} = +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)}$$
$$\overline{G_\phi^M} = +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)}$$
$$\overline{G_{rr}^R} = -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r}$$
$$\overline{G_{\theta\theta}^R} = +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)}$$
$$\overline{G_{\phi\phi}^R} = +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)}$$
$$\overline{G_r^I} = -\overline{\rho\epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho\epsilon''_I u''_\phi u''_\phi/r}$$
$$\overline{G_r^s} = -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^\alpha} = -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r}$$
$$\overline{G_r^A} = -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^Z} = -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^h} = -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^T} = -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r}$$
$$\overline{G_r^{jz}} = -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}$$

Table 5: Definitions (continued):

$\widetilde{F}^R_{ijk} = \overline{\rho} \widetilde{u''_i u''_j u''_k}$ Reynolds stress flux
$f^{ij}_\tau = \overline{u''_i \tau_{jr}}$ viscous flux
$\varepsilon^{ij}_\tau = \overline{\tau'_{ir}\partial_r u''_i} + \overline{\tau'_{i\theta}(1/r)\partial_\theta u''_j} + \overline{\tau'_{i\phi}(1/r\sin\theta)\partial_\phi u''_j}$ viscous dissipation
$G^M_r = -(\rho u^2_\theta - \tau_{\theta\theta})/r - (\rho u^2_\phi - \tau_{\phi\phi})/r$
$G^M_\theta = +(\rho u_\theta u_r - \tau_{\theta r})/r - (\rho u^2_\phi - \tau_{\phi\phi})\cos\theta/(r\sin\theta)$
$G^M_\phi = +(\rho u_\phi u_r - \tau_{\phi r})/r + (\rho u_\phi u_\theta - \tau_{\phi\theta})\cos\theta/(r\sin\theta)$

Below are non-vanishing terms of space-time average over $\nabla \cdot F_{ijk}$ (F_{ijk} is 3^{rd} order tensor)

$$
\begin{aligned}
\overline{G^R_{rr}} &= -\overline{F^R_{\theta\theta r}/r} - \overline{F^R_{\theta r\theta}/r} - \overline{F^R_{\phi\phi r}/r} - \overline{F^R_{\phi r\phi}/r} \\
\overline{G^R_{r\theta}} &= -\overline{F^R_{\theta\theta\theta}/r} + \overline{F^R_{\theta rr}/r} - \overline{F^R_{\phi\phi\theta}/r} - \overline{F^R_{\phi r\phi}\cos\theta/(r\sin\theta)} \\
\overline{G^R_{r\phi}} &= +\overline{F^R_{\theta\theta\phi}/r} - \overline{F^R_{\phi\phi\phi}} + \overline{F^R_{\phi r\phi}\cos\theta/(r\sin\theta)} \\
\overline{G^R_{\theta r}} &= +\overline{F^R_{\theta rr}/r} - \overline{F^R_{\theta\theta\theta}/r} - \overline{F^R_{\phi\phi r}\cos\theta/(r\sin\theta)} - \overline{F^R_{\phi\theta\phi}/r} \\
\overline{G^R_{\theta\theta}} &= +\overline{F^R_{\theta r\theta}/r} + \overline{F^R_{\theta\theta r}/r} - \overline{F^R_{\phi\phi\theta}\cos\theta/(r\sin\theta)} - \overline{F^R_{\phi\theta\phi}\cos\theta/(r\sin\theta)} \\
\overline{G^R_{\theta\phi}} &= +\overline{F^R_{\theta r\phi}/r} + \overline{F^R_{\phi\theta r}/r} + \overline{F^R_{\phi\theta\theta}\cos\theta/(r\sin\theta)} \\
\overline{G^R_{\phi r}} &= -\overline{F^R_{\theta\phi\theta}/r} + \overline{F^R_{\phi rr}/r} + \overline{F^R_{\phi\theta r}\cos\theta/(r\sin\theta)} - \overline{F^R_{\phi\phi\phi}/r} \\
\overline{G^R_{\phi\theta}} &= +\overline{F^R_{\theta\phi r}/r} + \overline{F^R_{\phi r\theta}/r} + \overline{F^R_{\phi\theta\theta}\cos\theta/(r\sin\theta)} - \overline{F^R_{\phi\theta\phi}\cos\theta/(r\sin\theta)} \\
\overline{G^R_{\phi\phi}} &= +\overline{F^R_{\phi r\phi}/r} + \overline{F^R_{\phi\theta\phi}\cos\theta/(r\sin\theta)} + \overline{F^R_{\phi\phi r}/r} + \overline{F^R_{\phi\phi\theta}\cos\theta/(r\sin\theta)}
\end{aligned}
\tag{55}
$$

$$\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2.) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta.) + \frac{1}{r\sin\theta}\partial_\phi(.)$$

3 Properties of our oxygen shell burning and red giant data

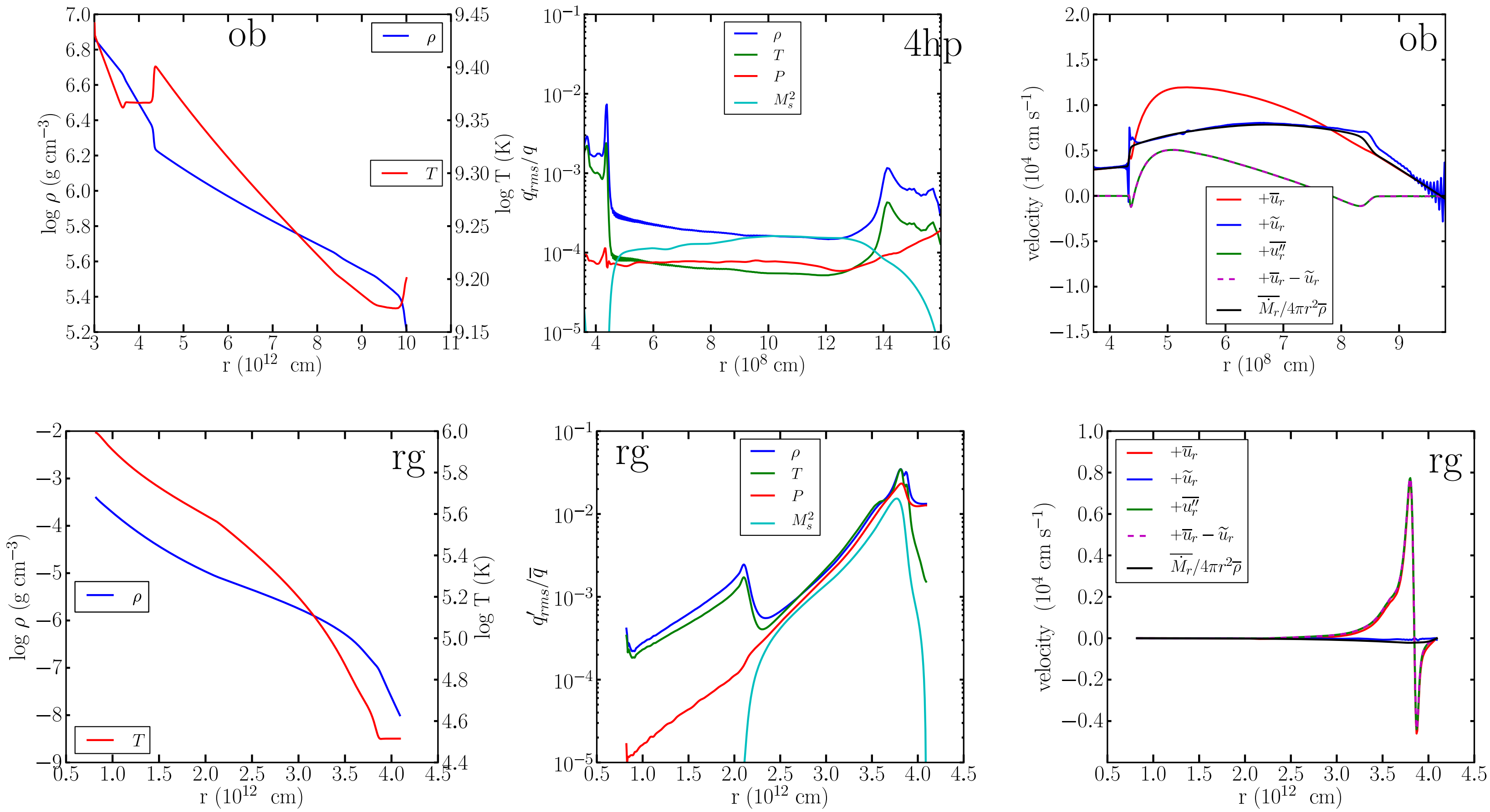

Figure 1: Properties of our data. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

3.1 Snapshots of turbulent kinetic energy in a meridional plane

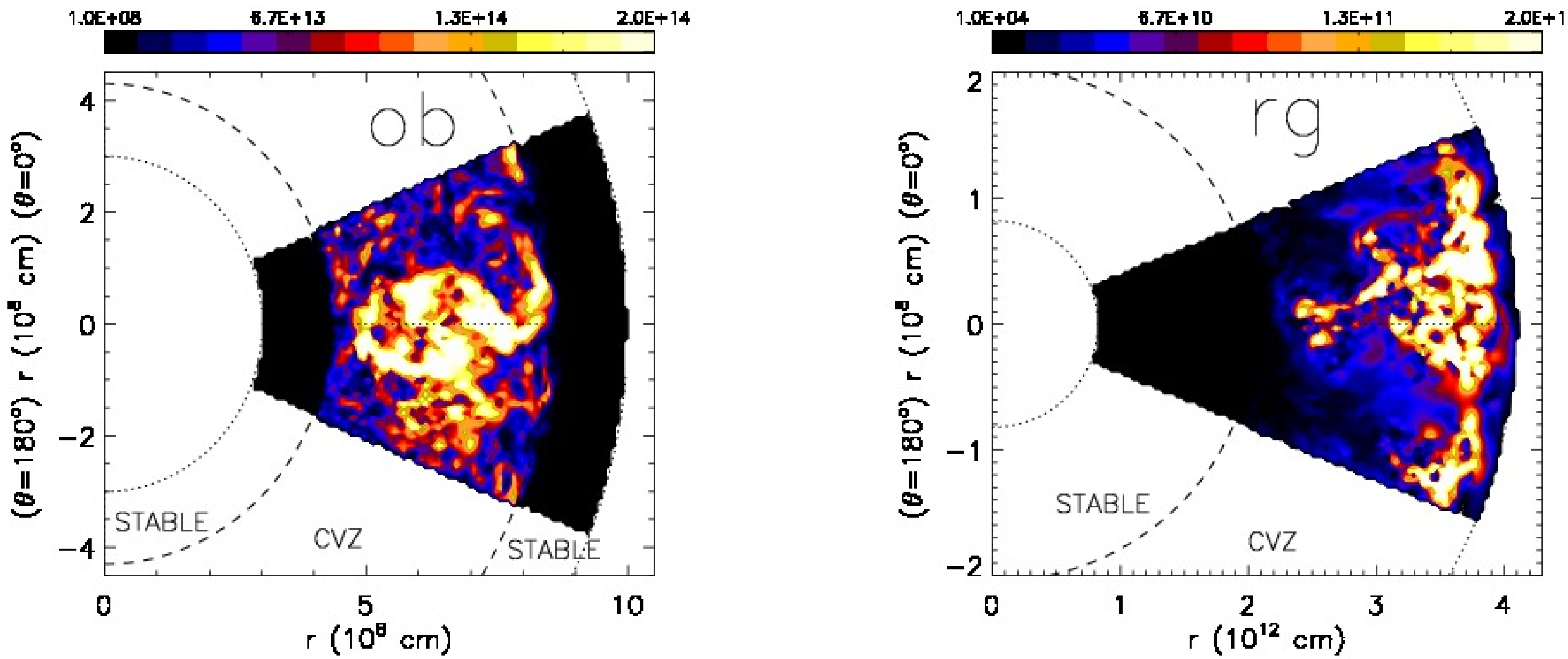

Figure 2: Snapshots of turbulent kinetic energy (in erg g^{-1}) in a meridional plane of 3D oxygen burning shell model ob.3D.mr (left) and red giant envelope convection model rg.3D.mr (right). Convectively unstable (CVZ) and stable layers (STABLE) are separated by dashed lines.

3.2 Summary of the Oxygen Burning Simulations and their properties

Parameter	ob.3D.lr	ob.3D.mr	ob.3D.hr	ob.3D.1hp	ob.3D.2hp	ob.3D.4hp	ob.3d.1hp.vc	ob.3d.1hp.vh
Grid zoning	$192{\times}128^2$	$384{\times}256^2$	$786{\times}512^2$	$200{\times}50^2$	$400{\times}100^2$	$320{\times}50^2$	$200{\times}50^2$	$400{\times}100^2$
$r_{\rm in}$, $r_{\rm out}$ (10^9 cm)	0.3, 1.0	0.3, 1.0	0.3, 1.0	0.3, 0.9	0.3, 1.0	0.3, 1.6	0.3, 0.9	0.3, 0.9
$r^c_{\rm b}$, $r^c_{\rm t}$ (10^9 cm)	0.43, 0.85	0.43, 0.85	0.43, 0.84	0.43, 0.68	0.43, 0.84	0.42, 1.4	0.43, 0.65	0.43, 0.68
$\Delta\theta$, $\Delta\phi$	45°	45°	45°	27.5°	27.5°	27.5°	27.5°	27.5°
CZ stratification (H_P)	1.9	1.9	1.9	1.2	1.9	4.1	1.1	1.2
$\Delta t_{\rm av}$ (s)	230	230	165	900	230	500	300	300
$v_{\rm rms}$ (10^6 cm/s)	10.7	10.9	10.9	5.28	9.15	4.88	4.66	4.97
$\tau_{\rm conv}$ (s)	78.2	77.1	75.6	94.7	89.2	403.	94.6	95.6
$P_{turb}/P_{gas}(10^{-4})$	3.88	4.05	4.03	0.96	3.01	1.73	0.79	0.96
L (10^{46} erg/s)	2.74	2.63	2.58	0.44	2.86	0.26	0.40	-0.42
$L_{\rm d}$ (10^{46} erg/s)	0.29	0.28	0.26	0.04	0.31	0.08	0.03	0.03
$l_{\rm d}$ (10^8 cm)	7.39	7.92	8.73	3.87	4.15	5.1	2.85	4.1
$\tau_{\rm d}$ (s)	34.48	36.47	39.98	36.72	22.64	52.04	30.56	41.1
$\tau_{\rm dr}$ (s)	38.06	39.27	-	75.65	46.59	130.73	90.20	90.8
$\tau_{\rm dh}$ (s)	30.77	32.14	-	22.91	14.21	28.42	18.4	25.24

Table 6: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r^c_{\rm b}$, $r^c_{\rm t}$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

3.3 Summary of Red Giant Simulations and their properties

Parameter	rg.3D.lr	rg.3D.mr	rg.3D.4hp
Grid zoning	216×128^2	432×256^2	176×128^2
$r_{\rm in}$, $r_{\rm out}$ (10^{12} cm)	0.82, 4.09	0.82, 4.09	0.82, 0.34
$r^c_{\rm in}$, $r^c_{\rm out}$ (10^{12} cm)	2.05, 3.86	2.07, 3.88	2.16, 3.33
$\Delta\theta$, $\Delta\phi$	45°	45°	45°
CZ stratification (H_p)	7.0	7.2	3.5
$\Delta t_{\rm av}$ (days)	800	800	800
$v_{\rm rms}$ (10^5 cm/s)	2.59	2.66	2.01
$\tau_{\rm conv}$ (days)	161.	158.	134.
$P_{turb}/P_{gas} (10^{-3})$	4.68	4.98	0.98
$L_{\rm cool}$ (10^{36} erg/s)	-8.57	-7.13	-9.2
$L_{\rm d}$ (10^{36} erg/s)	7.26	7.24	2.27
$l_{\rm d}$ (10^{11} cm)	9.95	10.4	11.6
$\tau_{\rm d}$ (days)	22.2	22.7	33.3
$\tau_{\rm dr}$ (days)	36.7	44.7	53.0
$\tau_{\rm dh}$ (days)	18.3	17.9	28.2

Table 7: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r^c_{\rm b}$, $r^c_{\rm t}$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

4 Profiles and intergral budgets of mean fields equations

4.1 Mean continuity equation

$$\widetilde{D}_t \overline{\rho} = -\overline{\rho}\widetilde{d} + \mathcal{N}_\rho \tag{56}$$

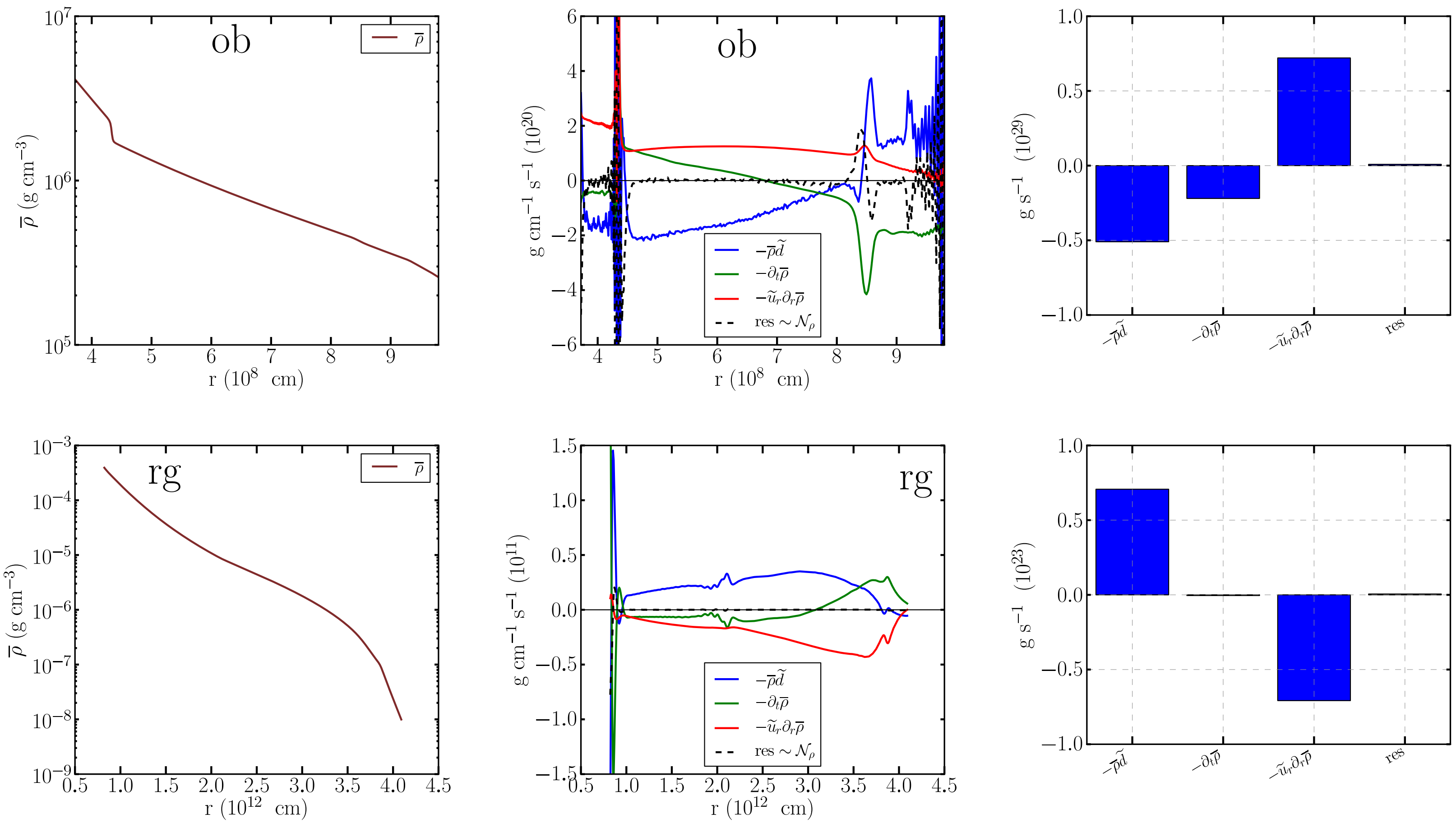

Figure 3: Mean continuity equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.2 Mean radial momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} \quad (57)$$

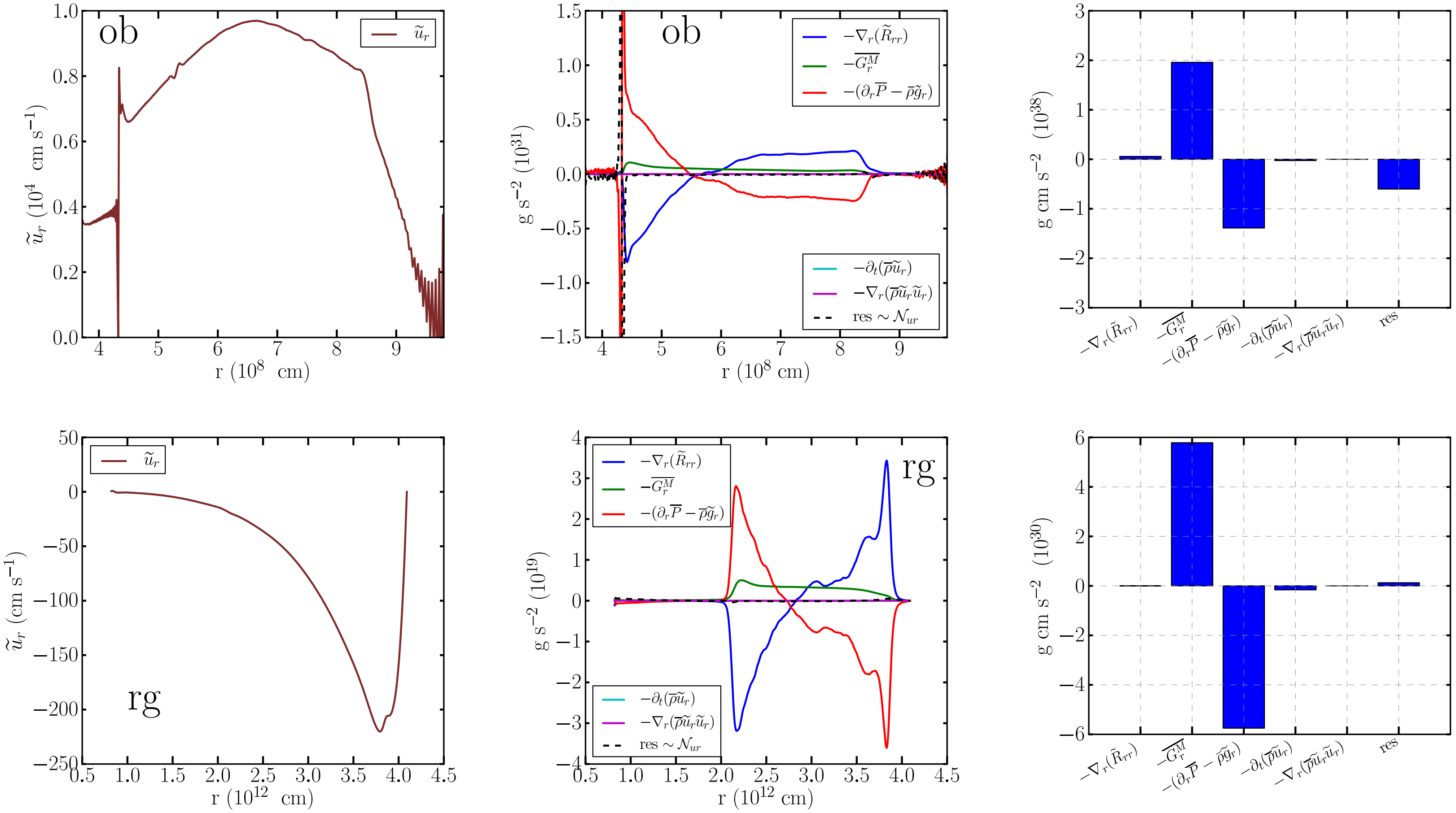

Figure 4: Mean radial momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.3 Mean azimutal momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G^M_\theta} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} \tag{58}$$

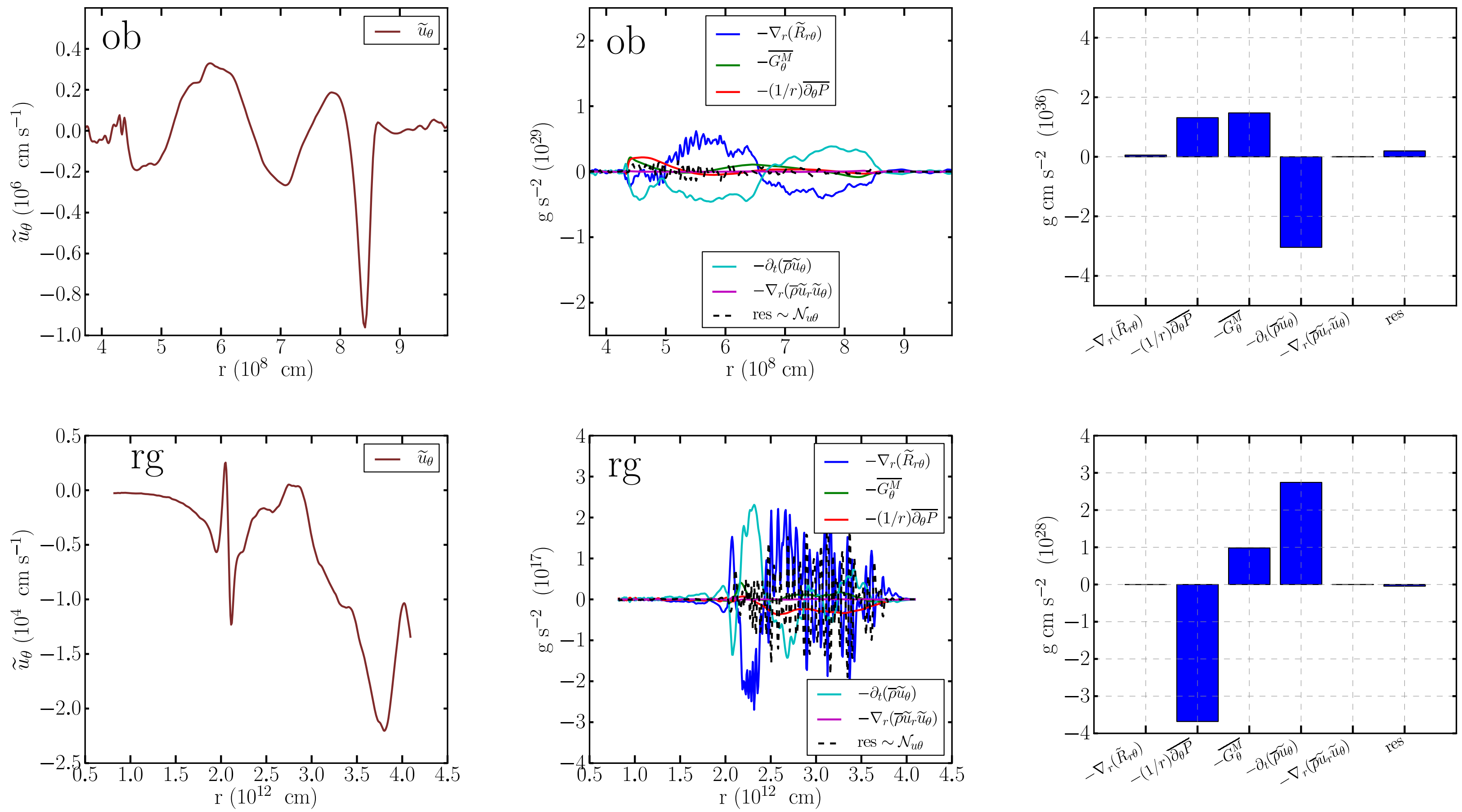

Figure 5: Mean azimuthal momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.4 Mean polar momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G_\phi^M} + \mathcal{N}_{u\phi} \tag{59}$$

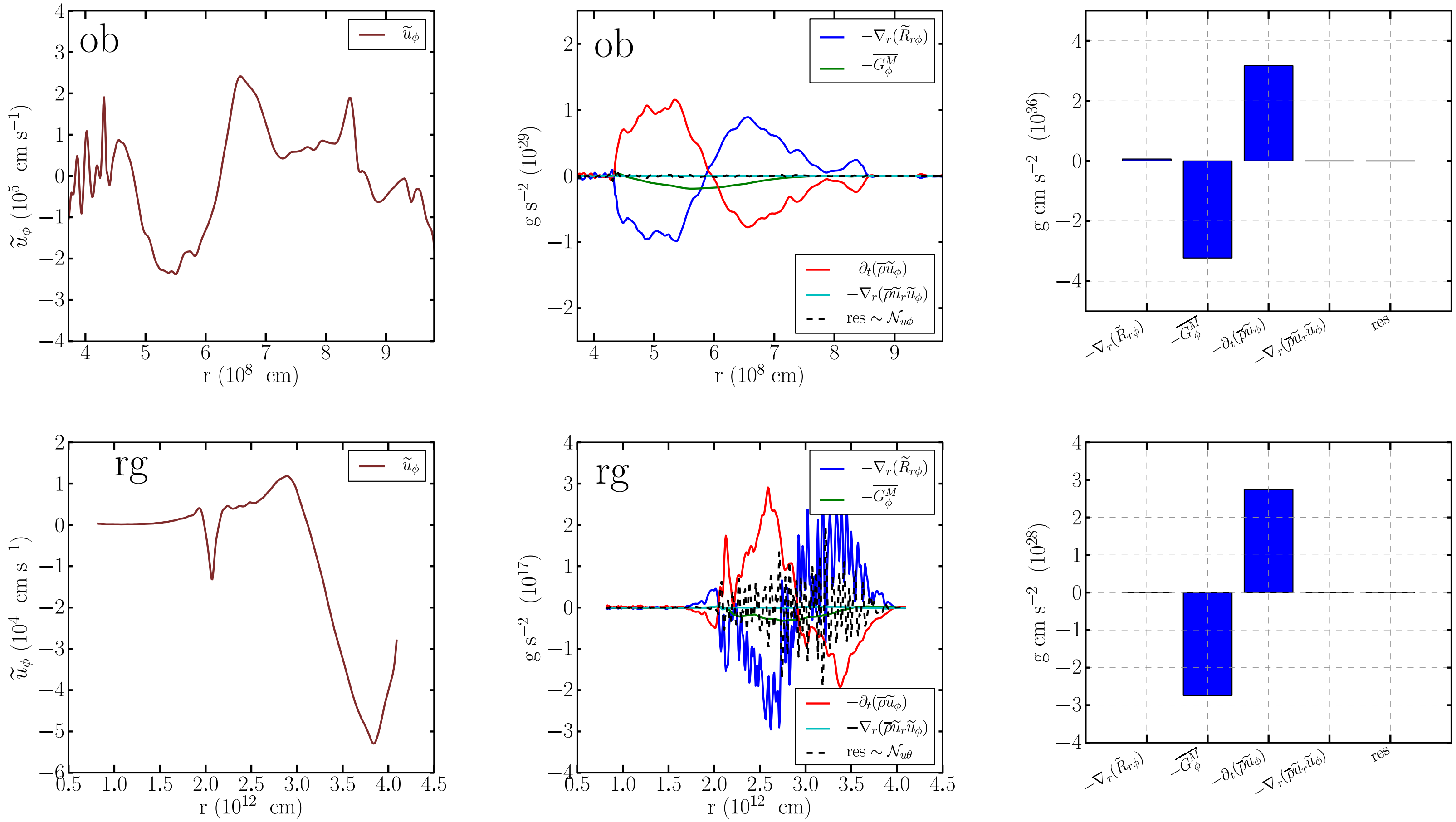

Figure 6: Mean polar omentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.5 Mean internal energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \tag{60}$$

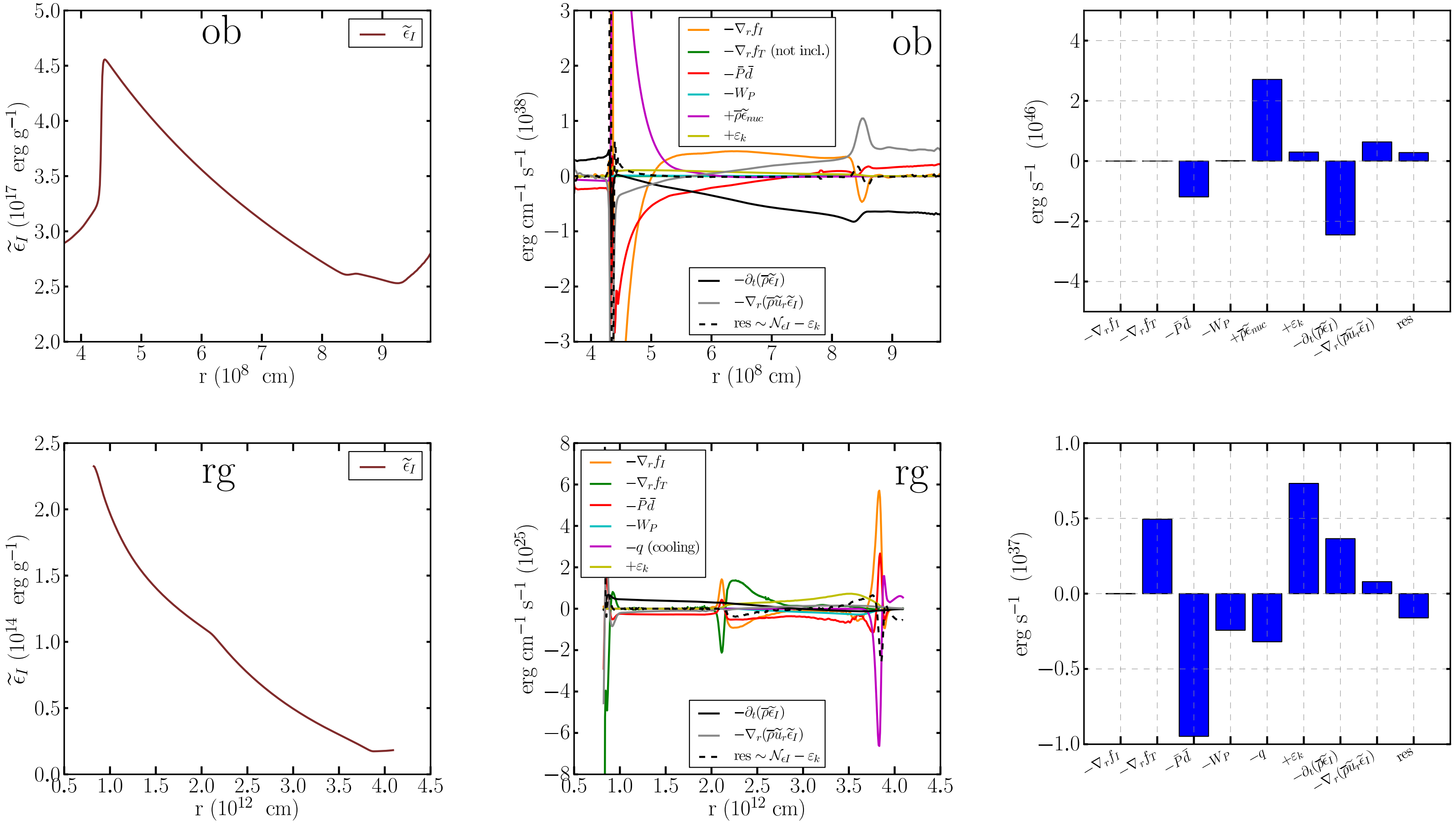

Figure 7: Mean internal energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.6 Mean kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_k = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} \quad (61)$$

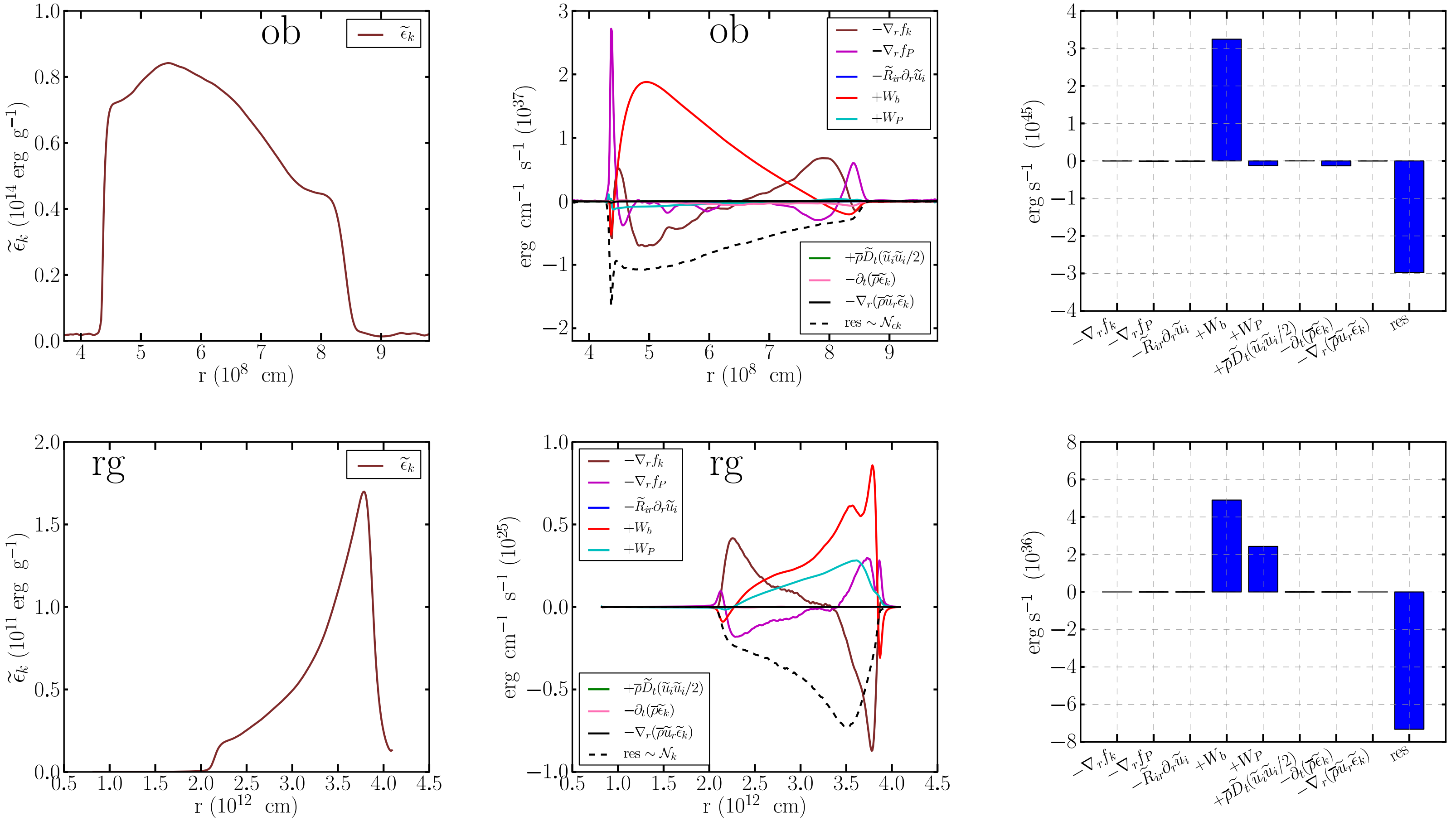

Figure 8: Mean kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.7 Mean total energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} \tag{62}$$

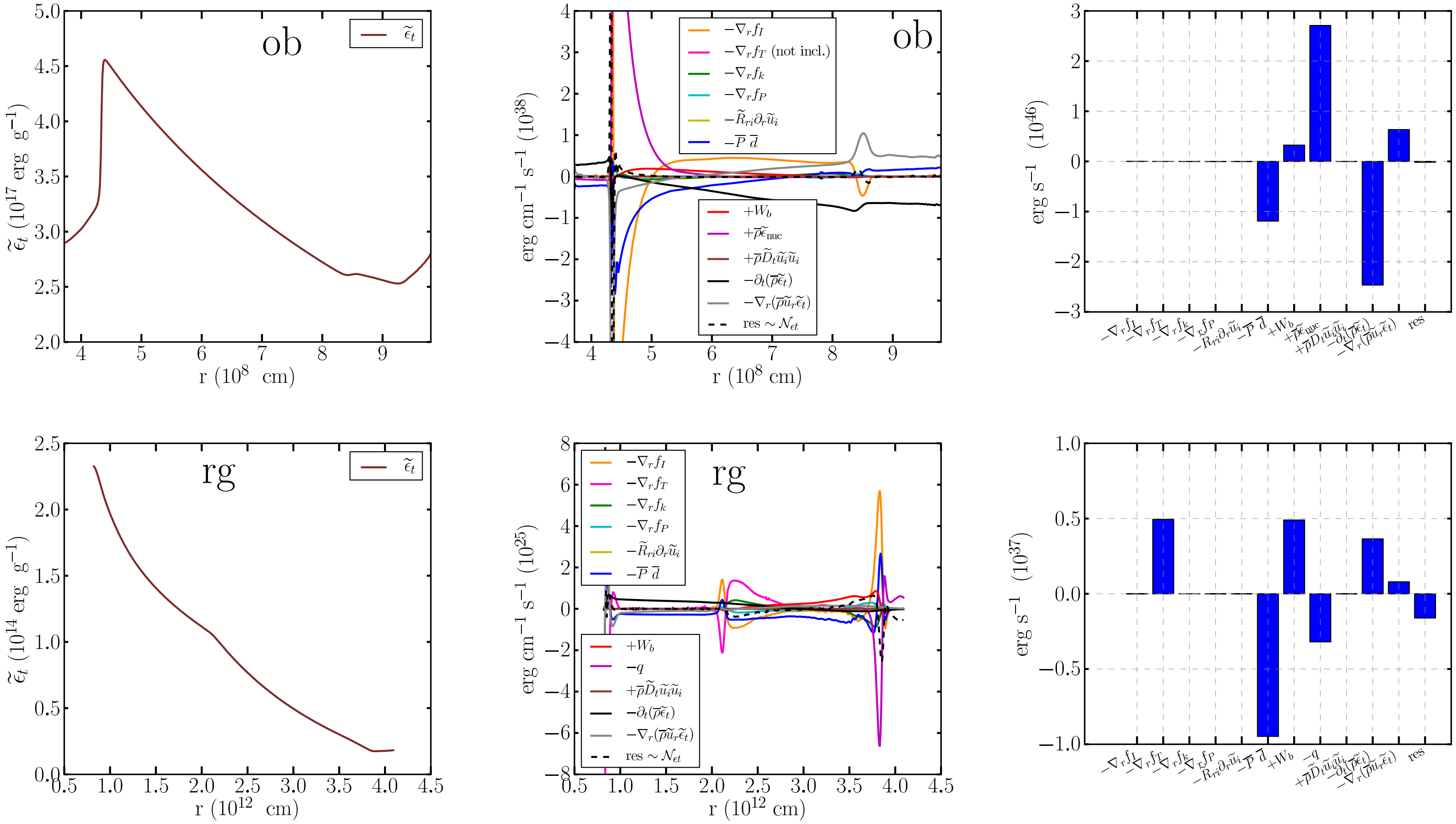

Figure 9: Mean total energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.8 Mean entropy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s - \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \tag{63}$$

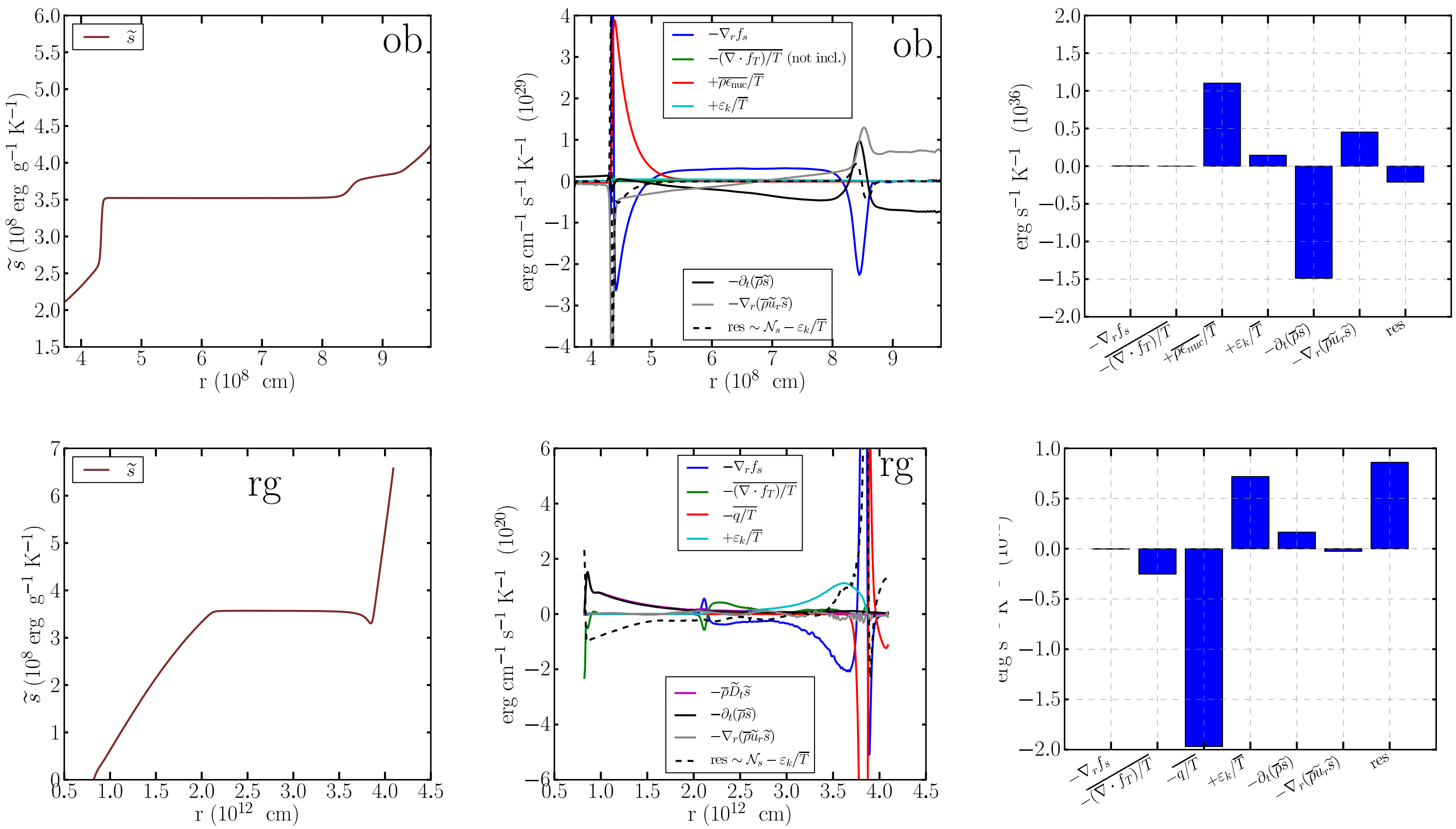

Figure 10: Mean entropy equation. Model ob.3D.2hp (upper panels) and model rg.3D.mr (lower panels).

4.9 Mean pressure equation

$$\overline{D}_t \overline{P} = -\nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1-\Gamma_1) W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P \tag{64}$$

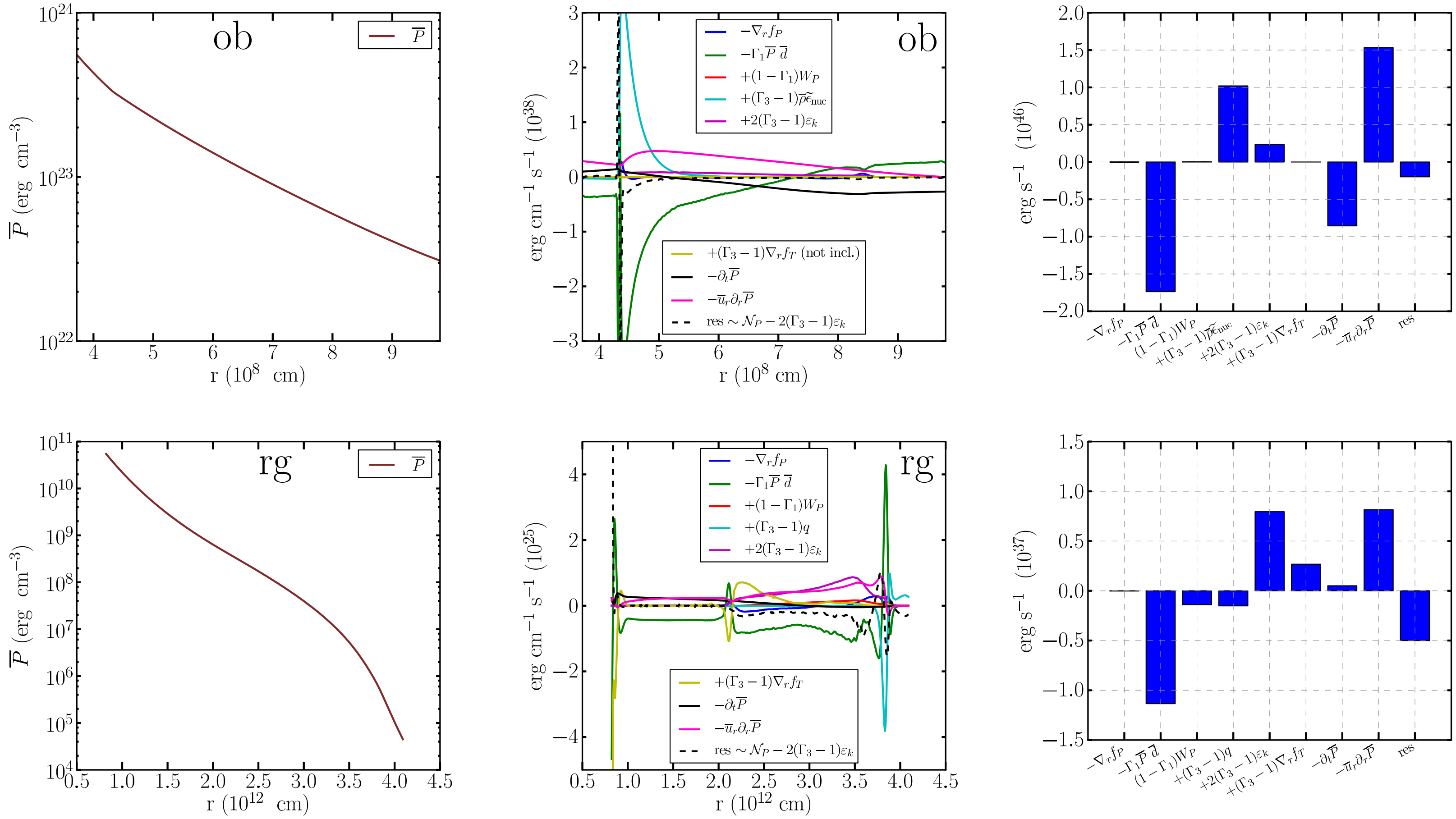

Figure 11: Mean pressure equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.10 Mean enthalpy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r f_h - \Gamma_1\overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3\mathcal{S} + \Gamma_3\nabla_r f_T + \mathcal{N}_h \tag{65}$$

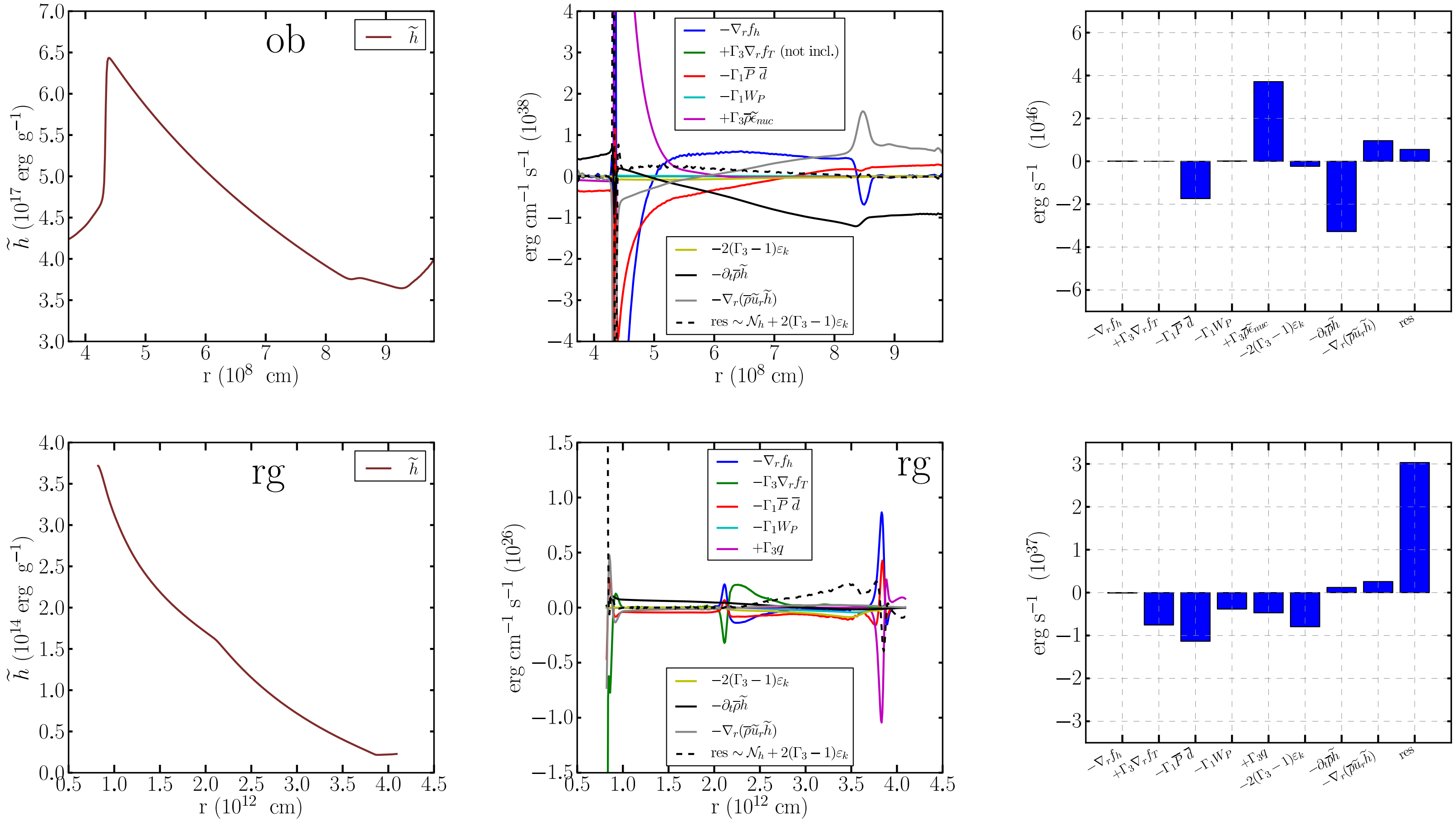

Figure 12: Mean enthalpy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.11 Mean angular momentum equation (z-component)

$$\overline{\rho}\widetilde{D}_t\widetilde{j}_z = -\nabla_r f_{jz} + \mathcal{N}_{jz} \tag{66}$$

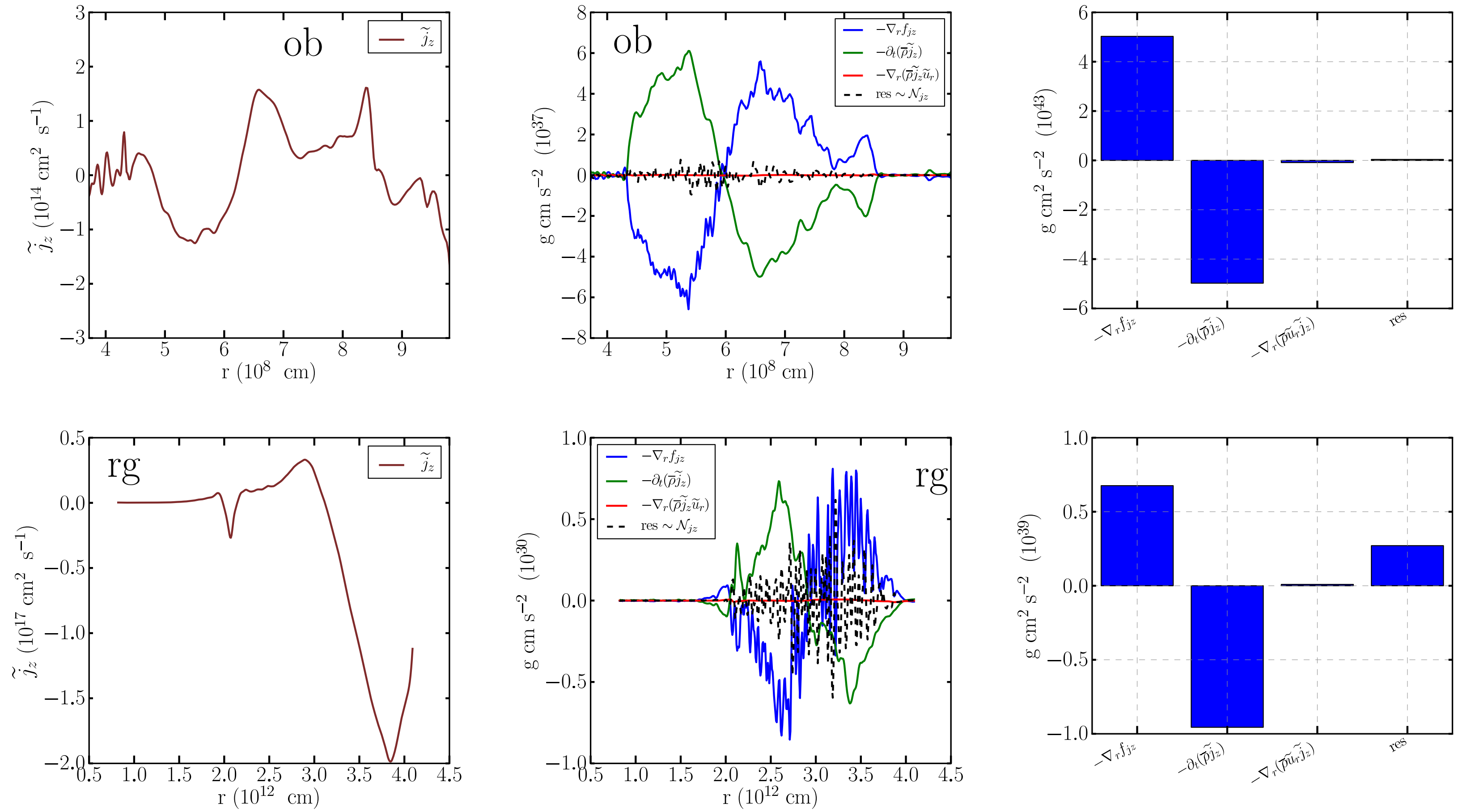

Figure 13: Mean angular momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.12 Mean composition equation

$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} + \mathcal{N}_\alpha \qquad\qquad \overline{\rho}\widetilde{D}_t\widetilde{A} = -\nabla_r f_A - \overline{\rho A^2 \Sigma_\alpha (\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{N}_A \tag{67}$$

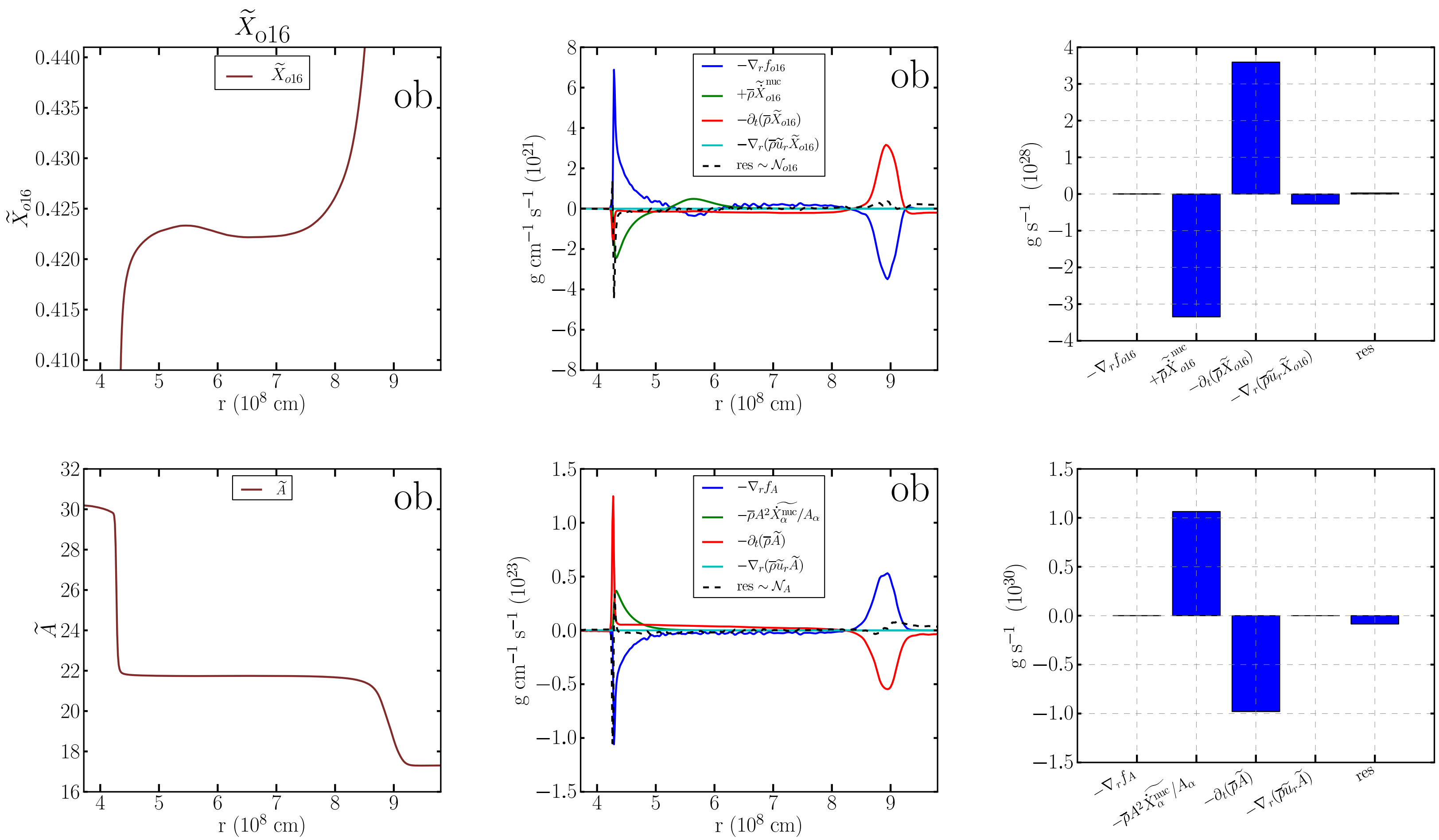

Figure 14: Mean composition equation for O^{16} and equation for mean number of nucleons per isotope. Model ob.3D.2hp.

4.13 Mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{68}$$

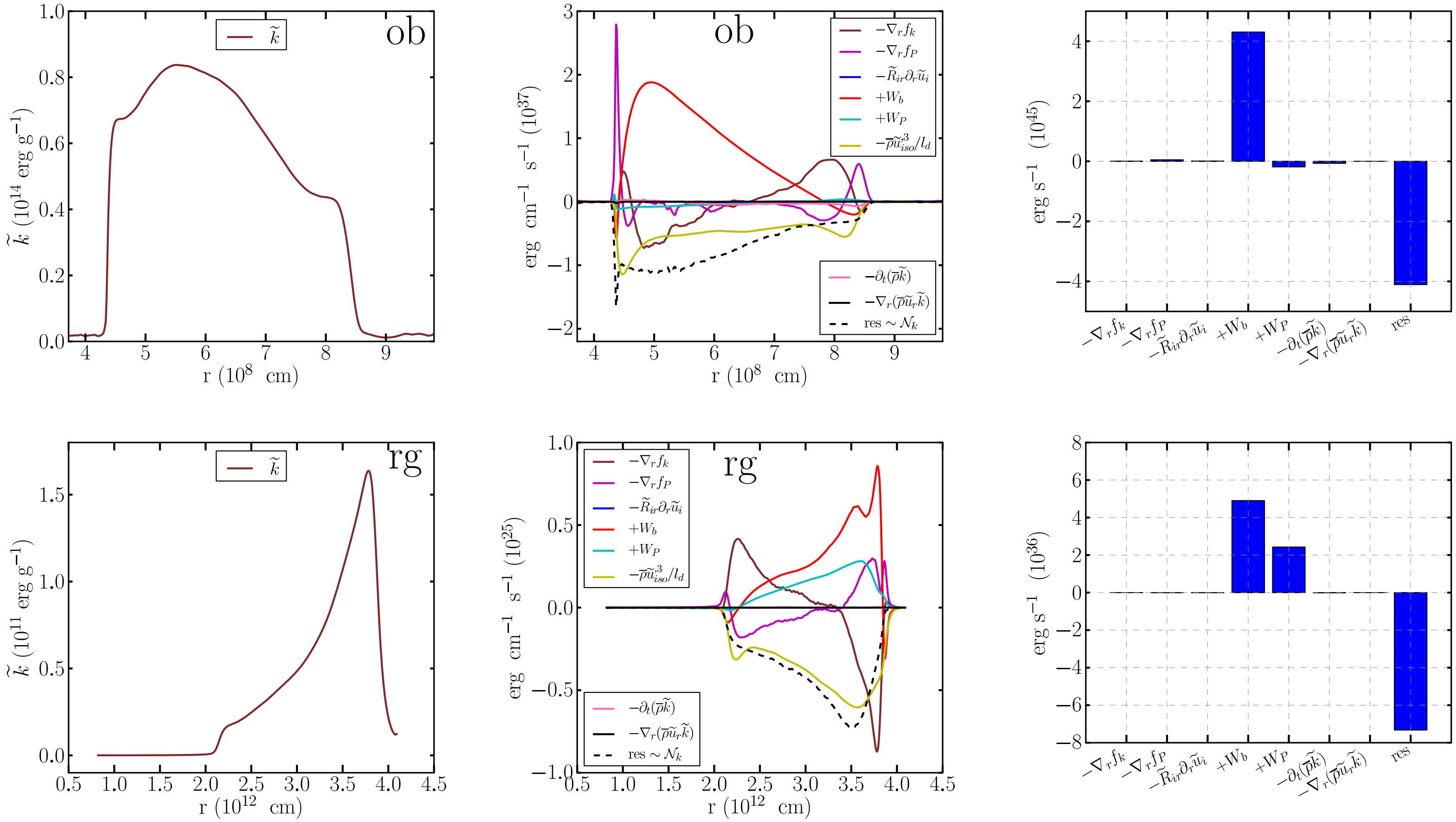

Figure 15: Turbulent kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.14 Mean turbulent kinetic energy equation (radial part)

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} \quad (69)$$

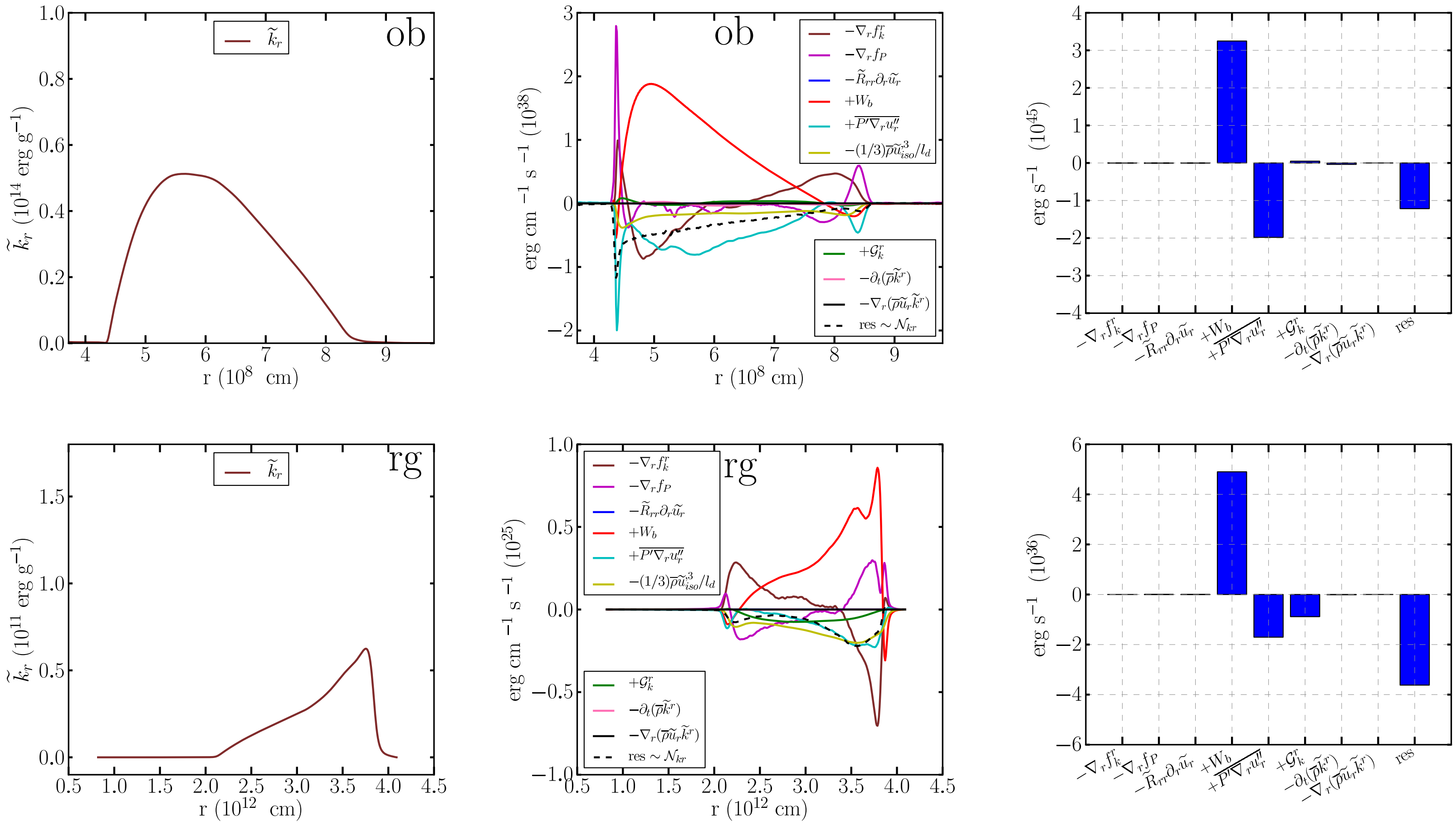

Figure 16: Turbulent radial kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.15 Mean turbulent kinetic energy equation (horizontal part)

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \quad (70)$$

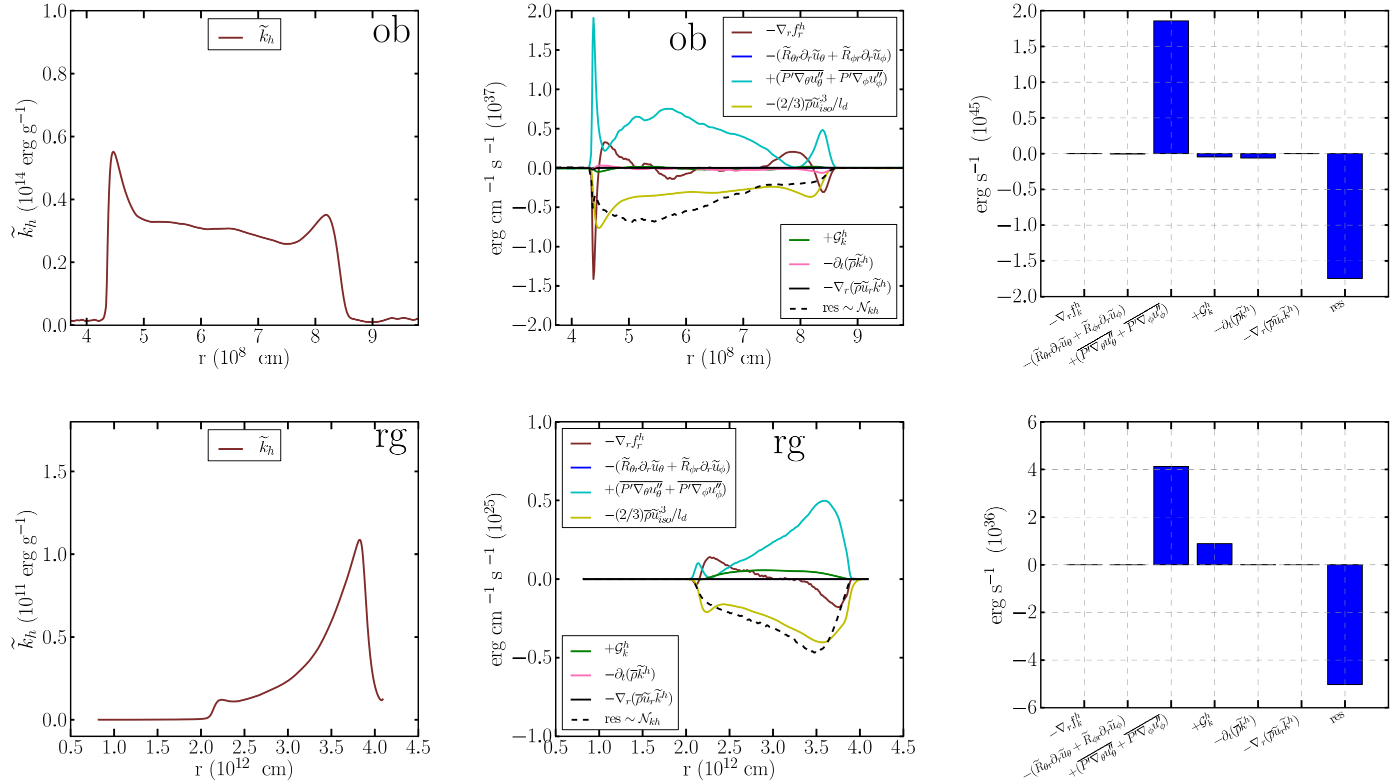

Figure 17: Turbulent horizontal kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.16 Mean turbulent mass flux equation

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = -\,(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} + \mathcal{G}_a + \mathcal{N}_a \tag{71}$$

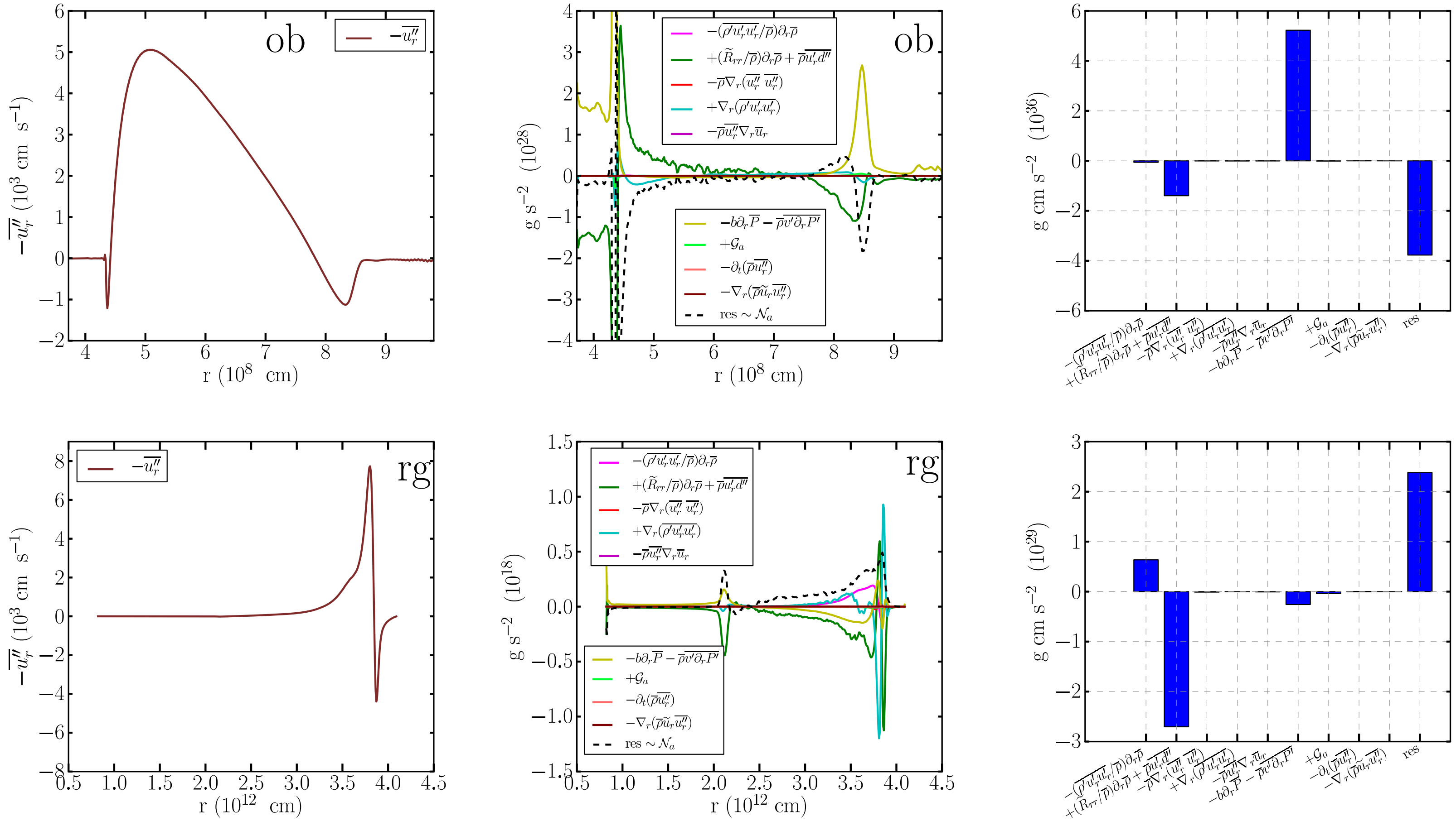

Figure 18: Turbulent mass flux equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.17 Mean density-specific volume covariance equation

$$\overline{D}_t b = +\,\overline{v}\nabla_r \overline{\rho} \widetilde{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\overline{v' d'} + \mathcal{N}_b \tag{72}$$

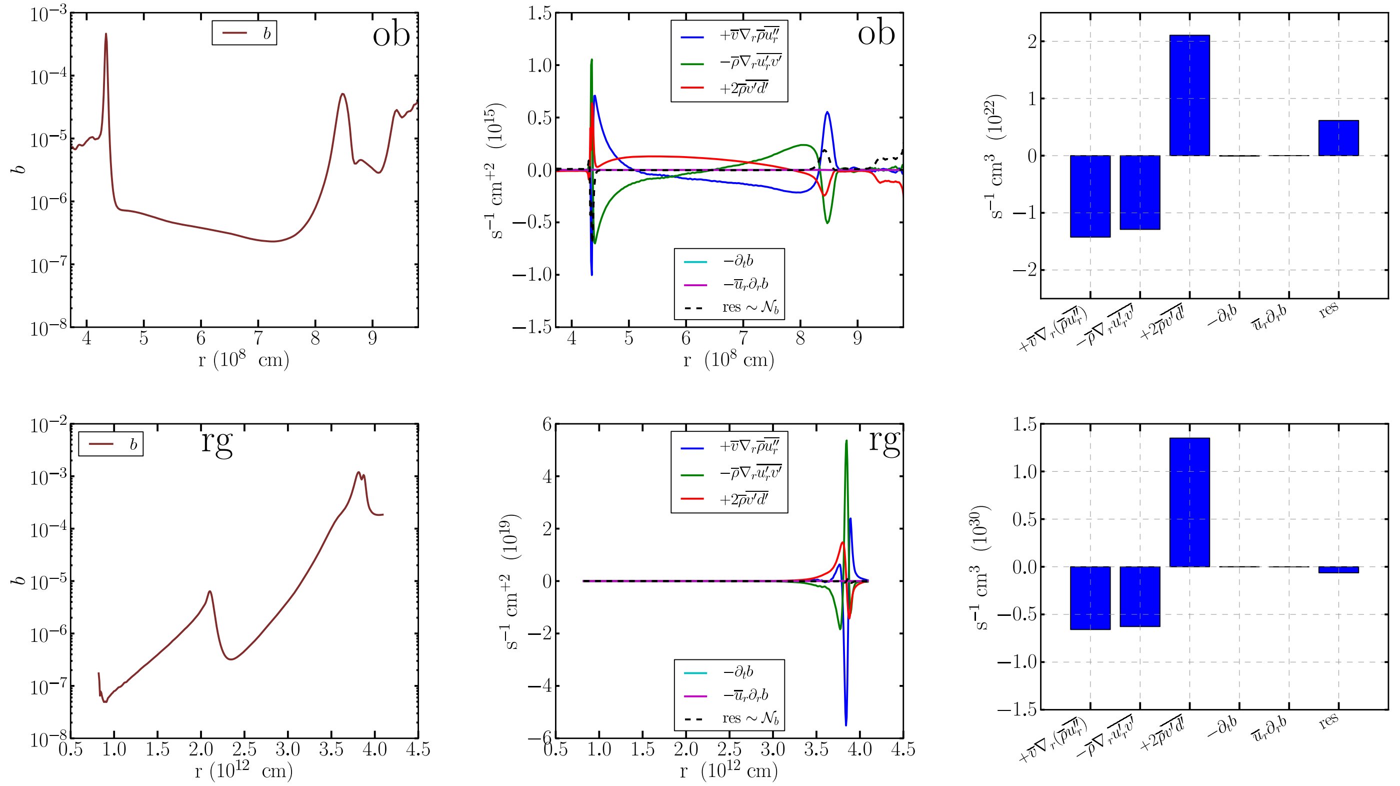

Figure 19: Density-specific volume covariance equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.18 Mean internal energy flux equation

$$\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) = \mathcal{N}_{fI} - \nabla_r f_I^r - f_I \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{\epsilon_I} - \overline{\epsilon_I''}\partial_r \overline{P} - \overline{\epsilon_I''\partial_r P'} - \overline{u_r''\,(Pd)} + \overline{u_r''(\mathcal{S} + \nabla\cdot f_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \quad (73)$$

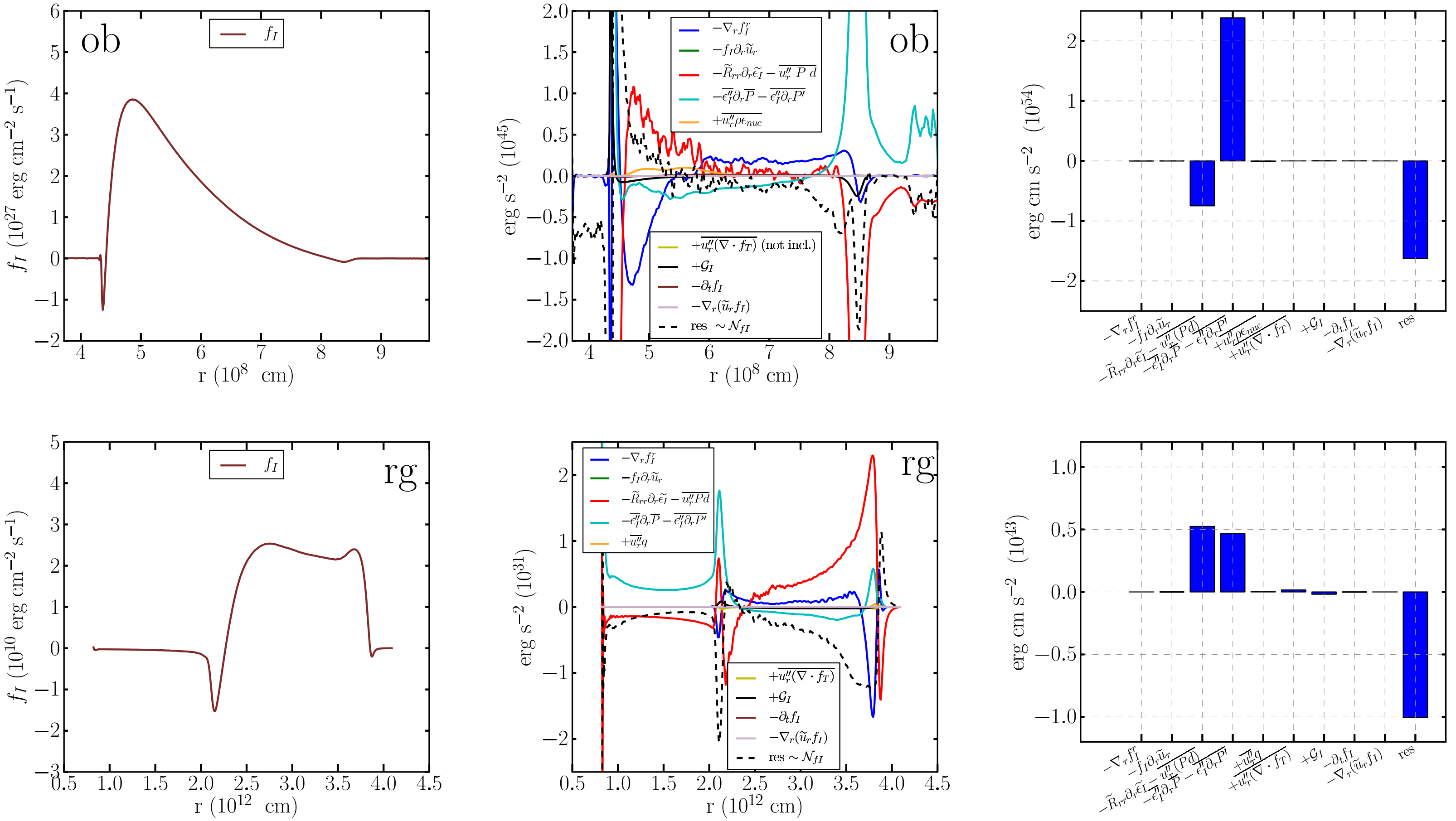

Figure 20: Mean internal energy flux equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.19 Mean entropy flux equation

$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{s} - \overline{s''}\partial_r \overline{P} - \overline{s''\partial_r P'} + \overline{u_r''(\mathcal{S} + \nabla \cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \tag{74}$$

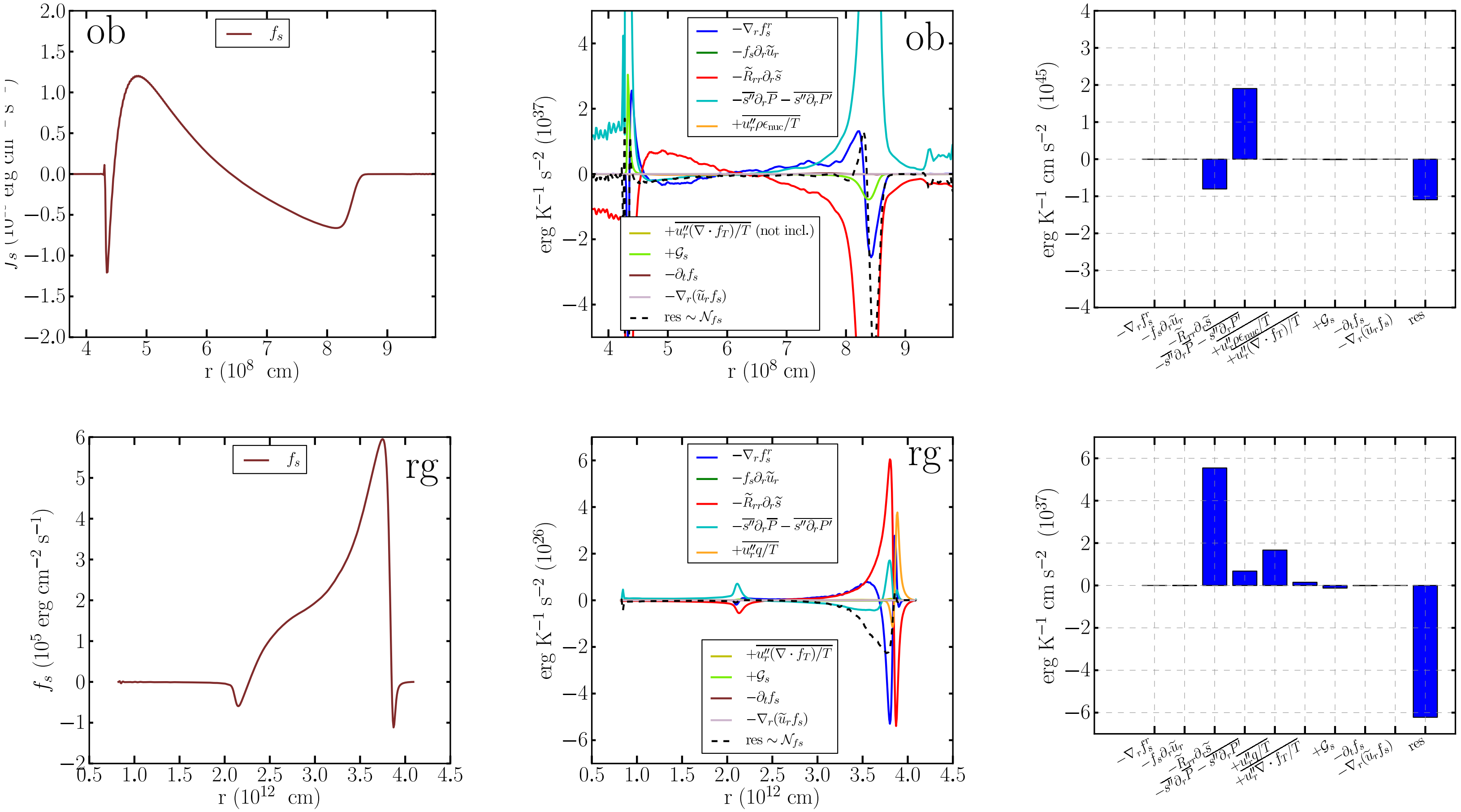

Figure 21: Mean entropy flux equation. Model ob.3D.2hp (upper panels) and model rg.3D.mr (lower panels).

4.20 Mean composition flux equation and mean A and Z flux equations

$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{X}_\alpha - \overline{X''_\alpha}\partial_r \overline{P} - \overline{X''_\alpha \partial_r P'} + \overline{u''_r \rho \dot{X}_\alpha^{\rm nuc}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \tag{75}$$

$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = \mathcal{N}_{fA} - \nabla_r f_A^r - f_A \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{A} - \overline{A''}\partial_r \overline{P} - \overline{A''\partial_r P'} - \overline{u''_r \rho A^2 \Sigma_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha} + \mathcal{G}_A \tag{76}$$

$$\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = \mathcal{N}_{fZ} - \nabla_r f_Z^r - f_Z \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{Z} - \overline{Z''}\partial_r \overline{P} - \overline{Z''\partial_r P'} - \overline{u''_r \rho Z A \Sigma_\alpha (\dot{X}_\alpha^{\rm nuc}/A_\alpha)} -$$
$$- \overline{u''_r \rho A \Sigma_\alpha (Z_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z \tag{77}$$

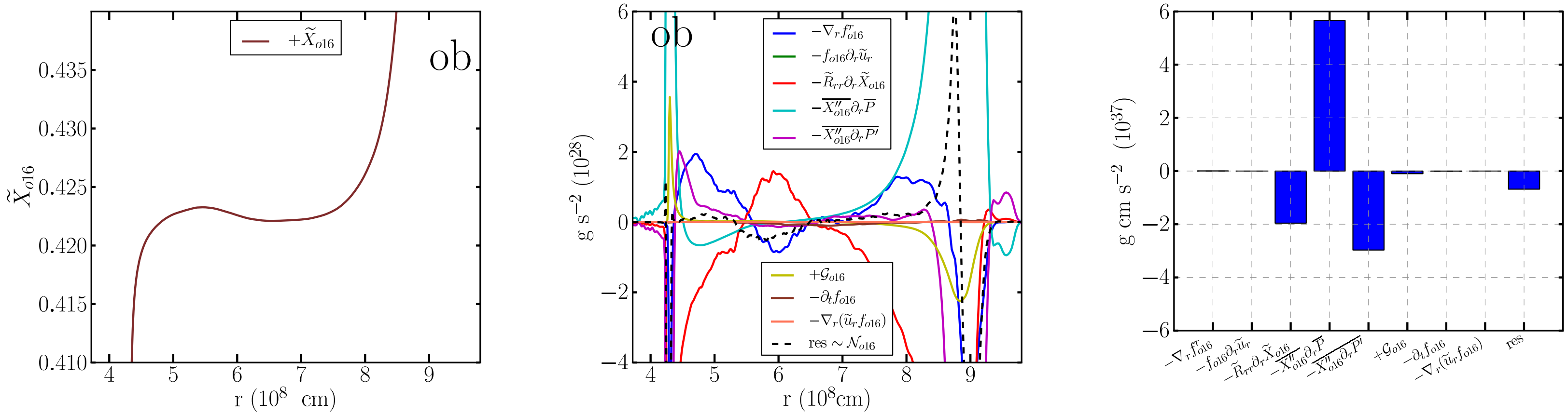

Figure 22: Mean composition flux equation. Model ob.3D.2hp.

4.21 Mean A and Z flux equations

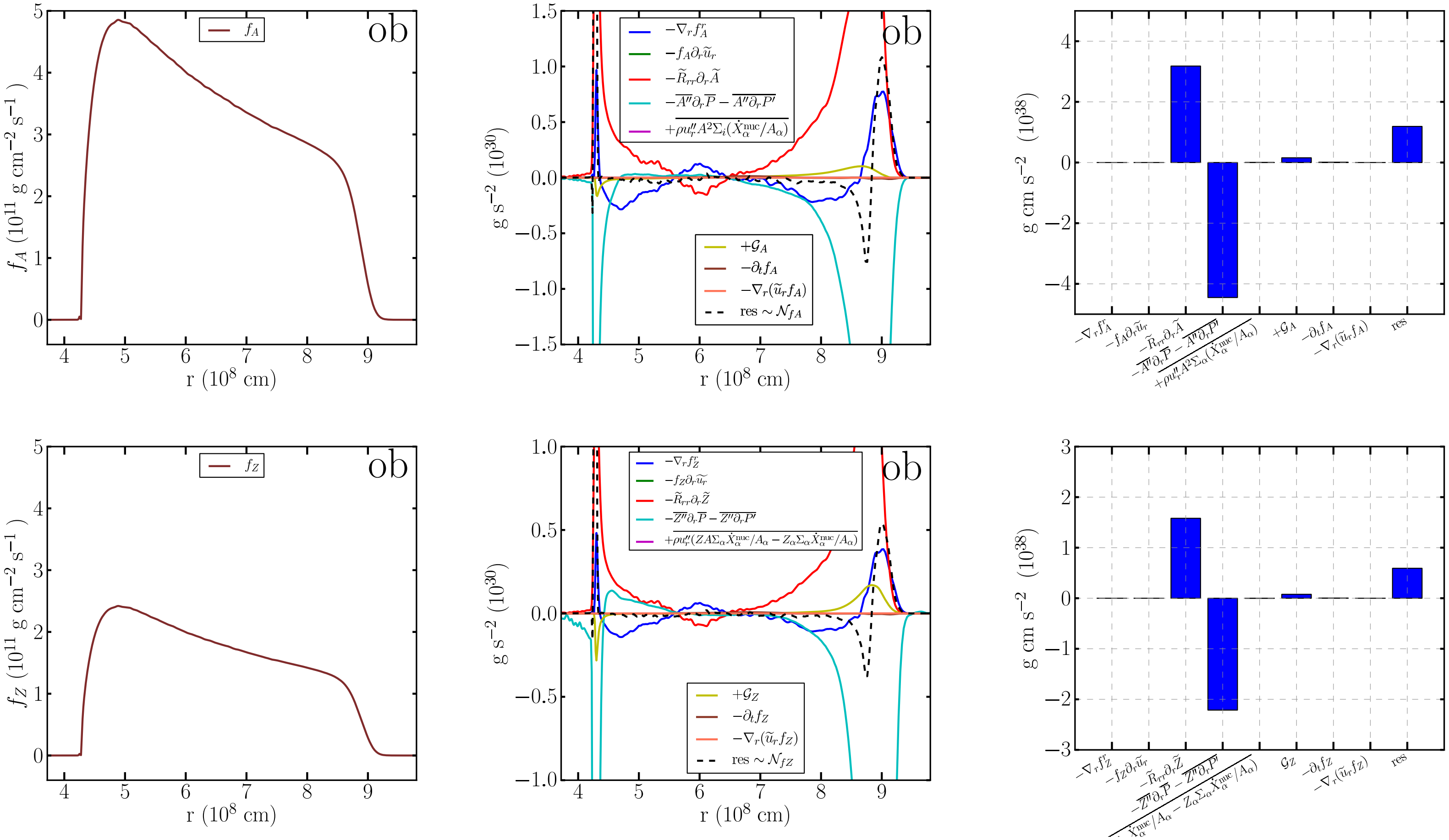

Figure 23: Mean A and Z flux equation. Model ob.3D.2hp.

5 Mean composition equations (for all elements in oxygen burning model ob.3D.2hp)

5.1 Mean C^{12} and O^{16} equation

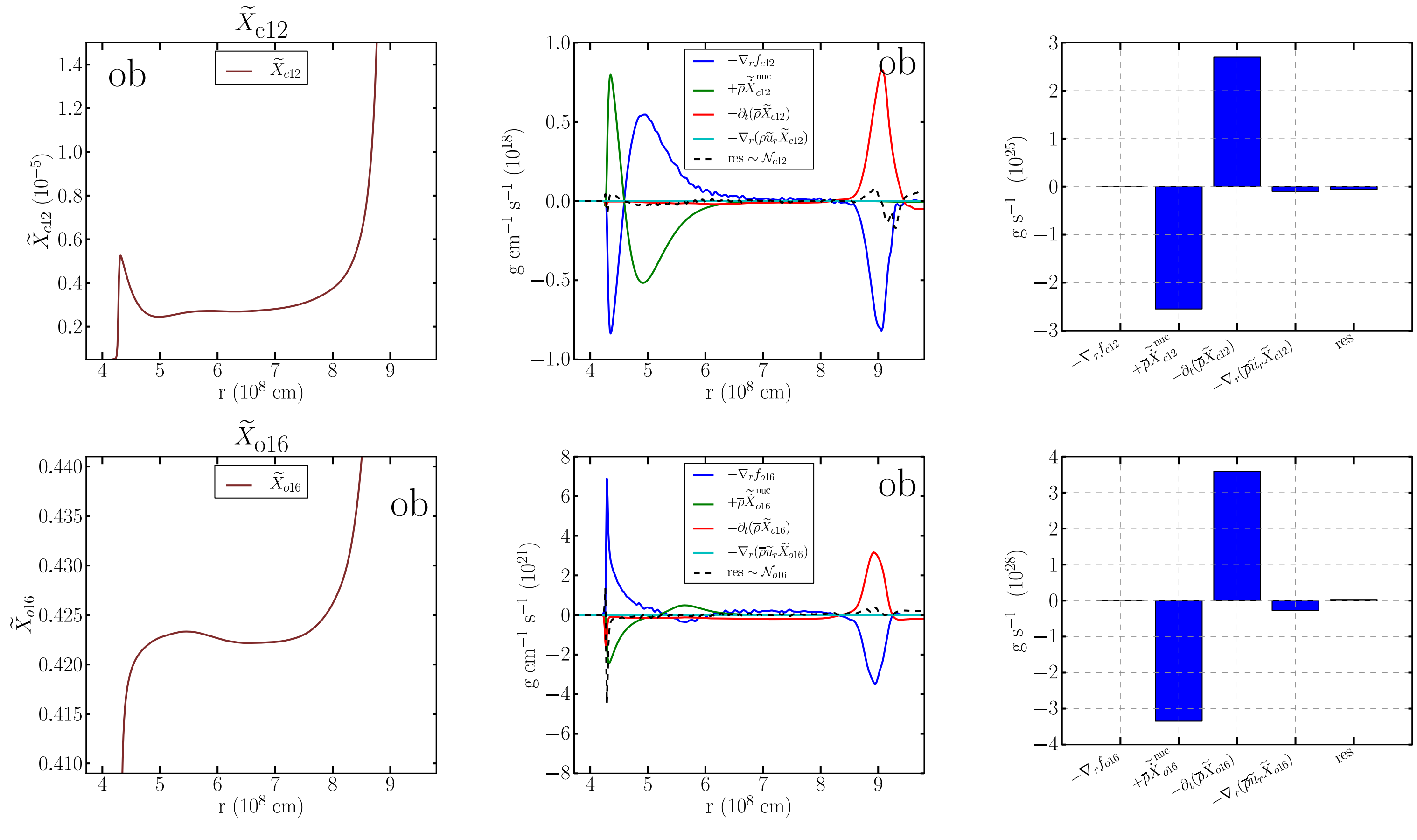

Figure 24: Mean composition equations. Model ob.3D.2hp.

5.2 Mean Ne^{20} and Na^{23} equation

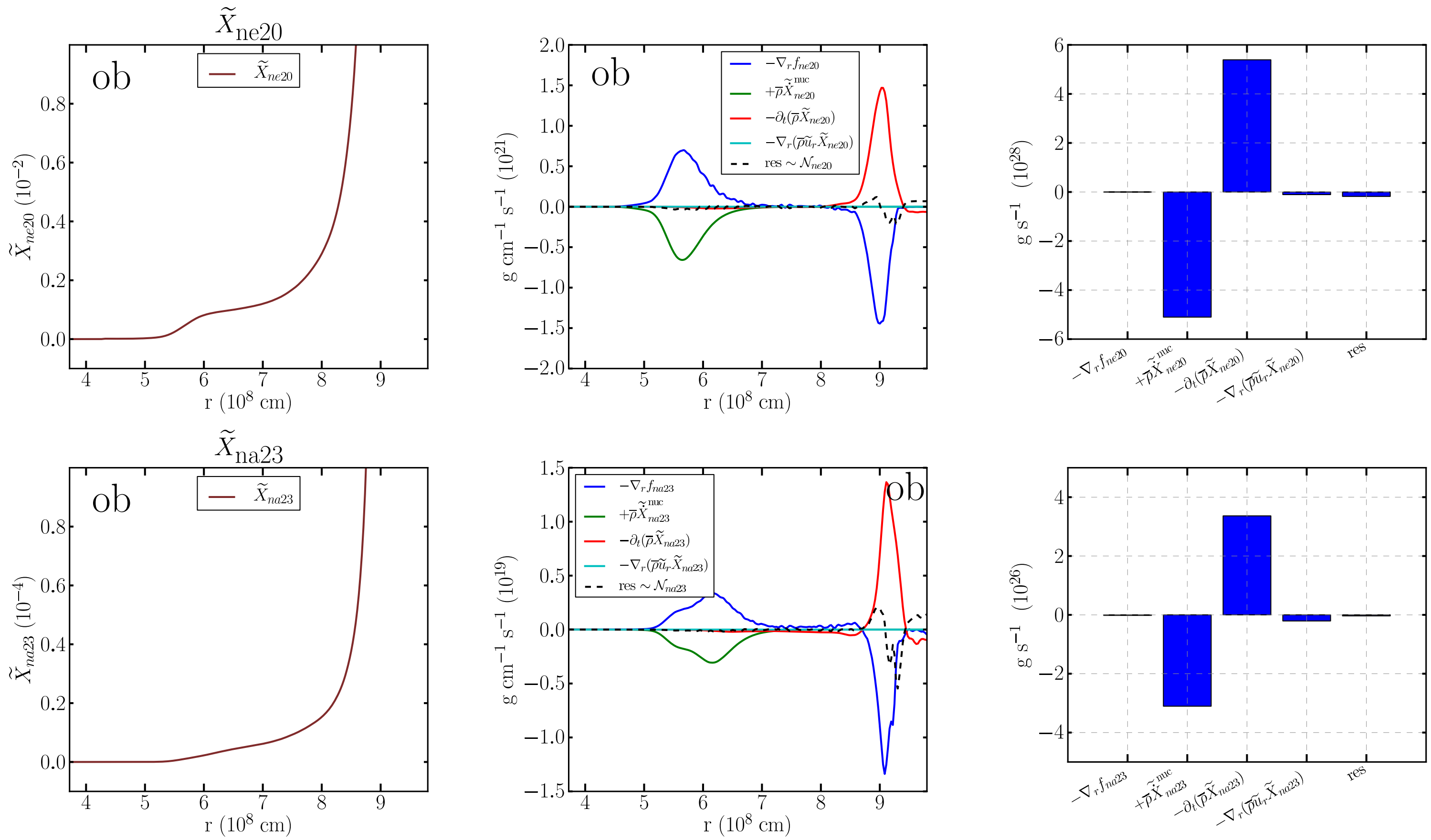

Figure 25: Mean composition equations. Model ob.3D.2hp.

5.3 Mean Mg^{24} and Si^{28} equation

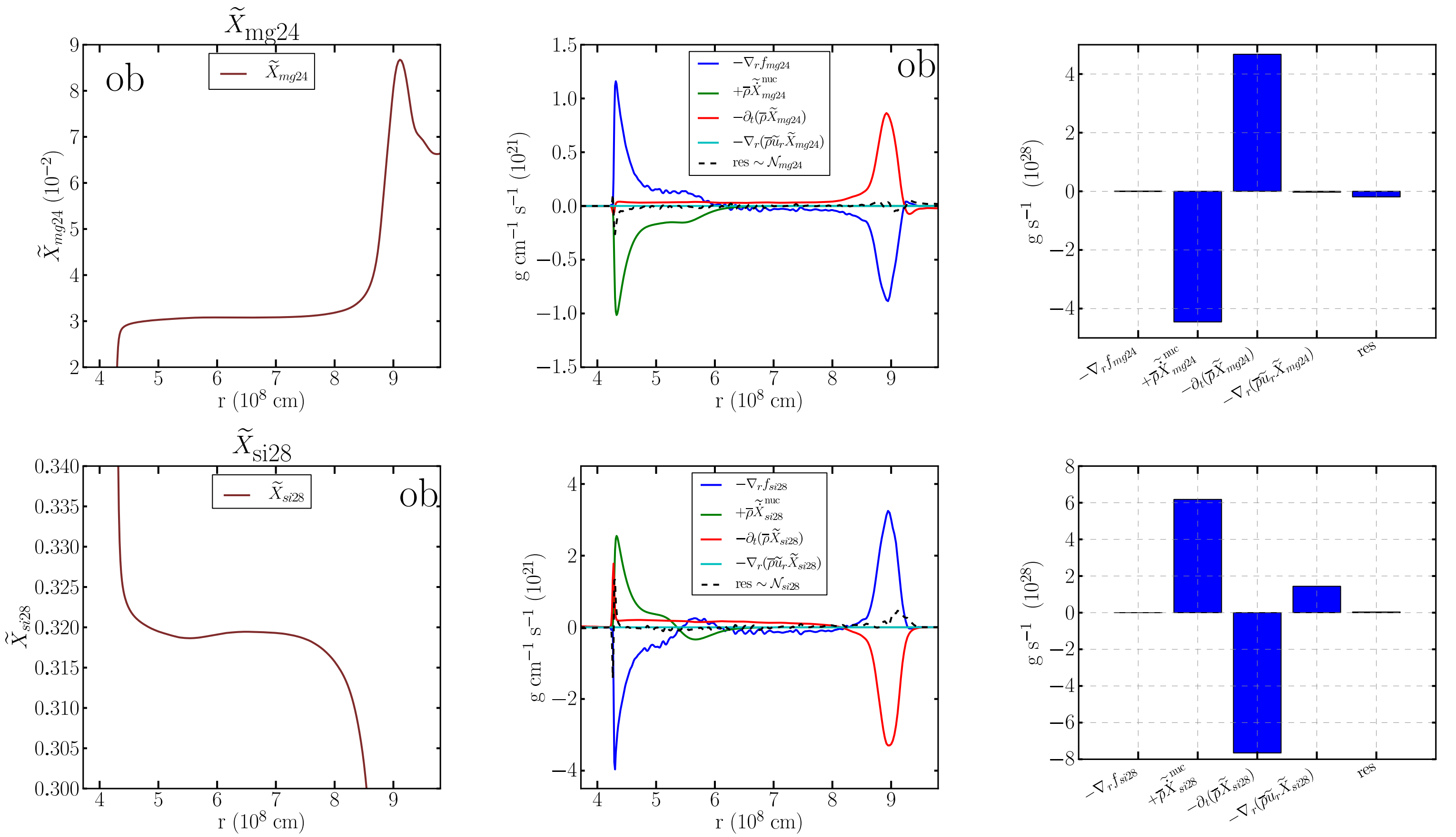

Figure 26: Mean composition equations. Model ob.3D.2hp.

5.4 Mean P^{31} and S^{32} equation

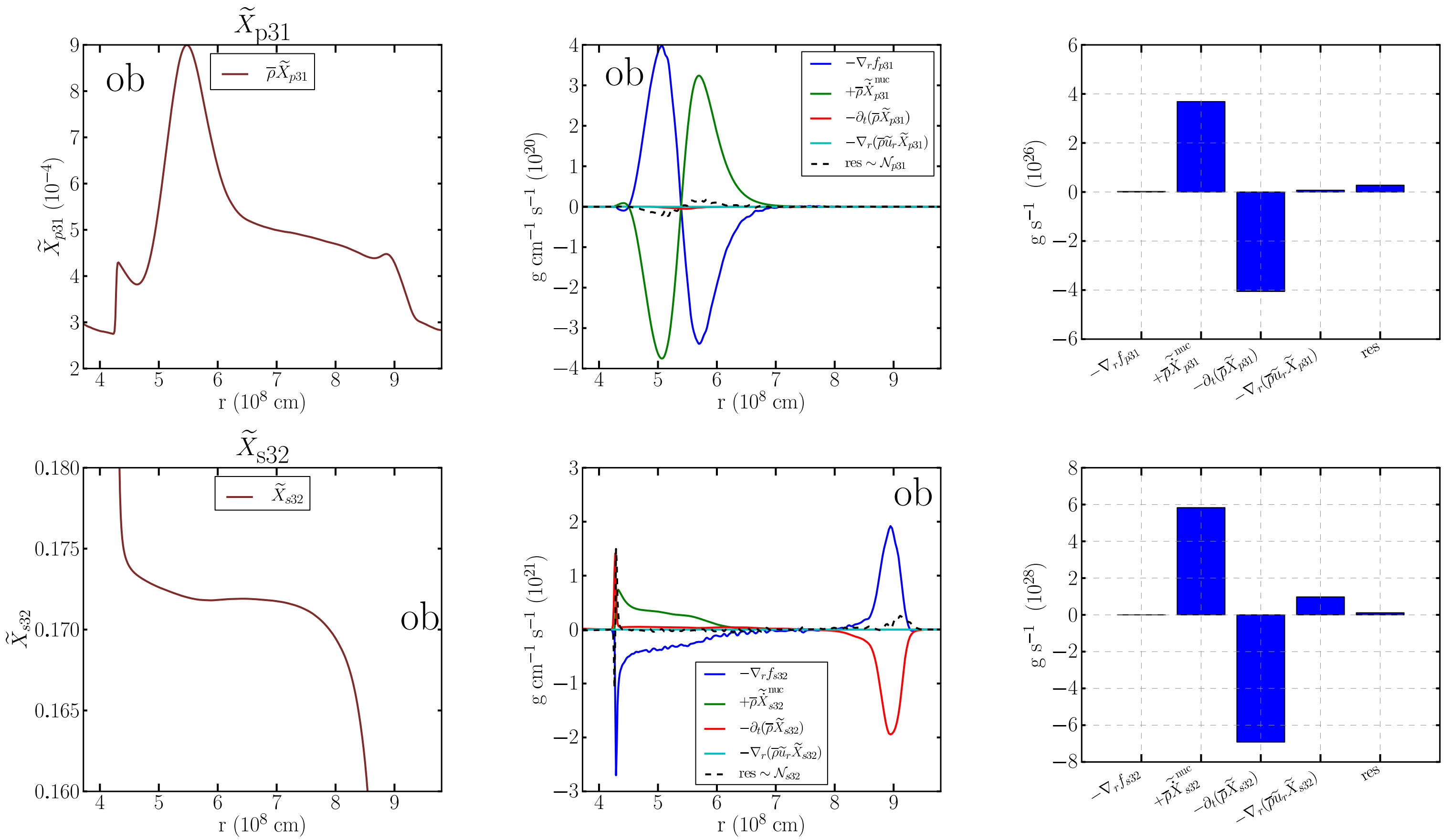

Figure 27: Mean composition equations. Model ob.3D.2hp.

5.5 Mean S^{34} and Cl^{35} equation

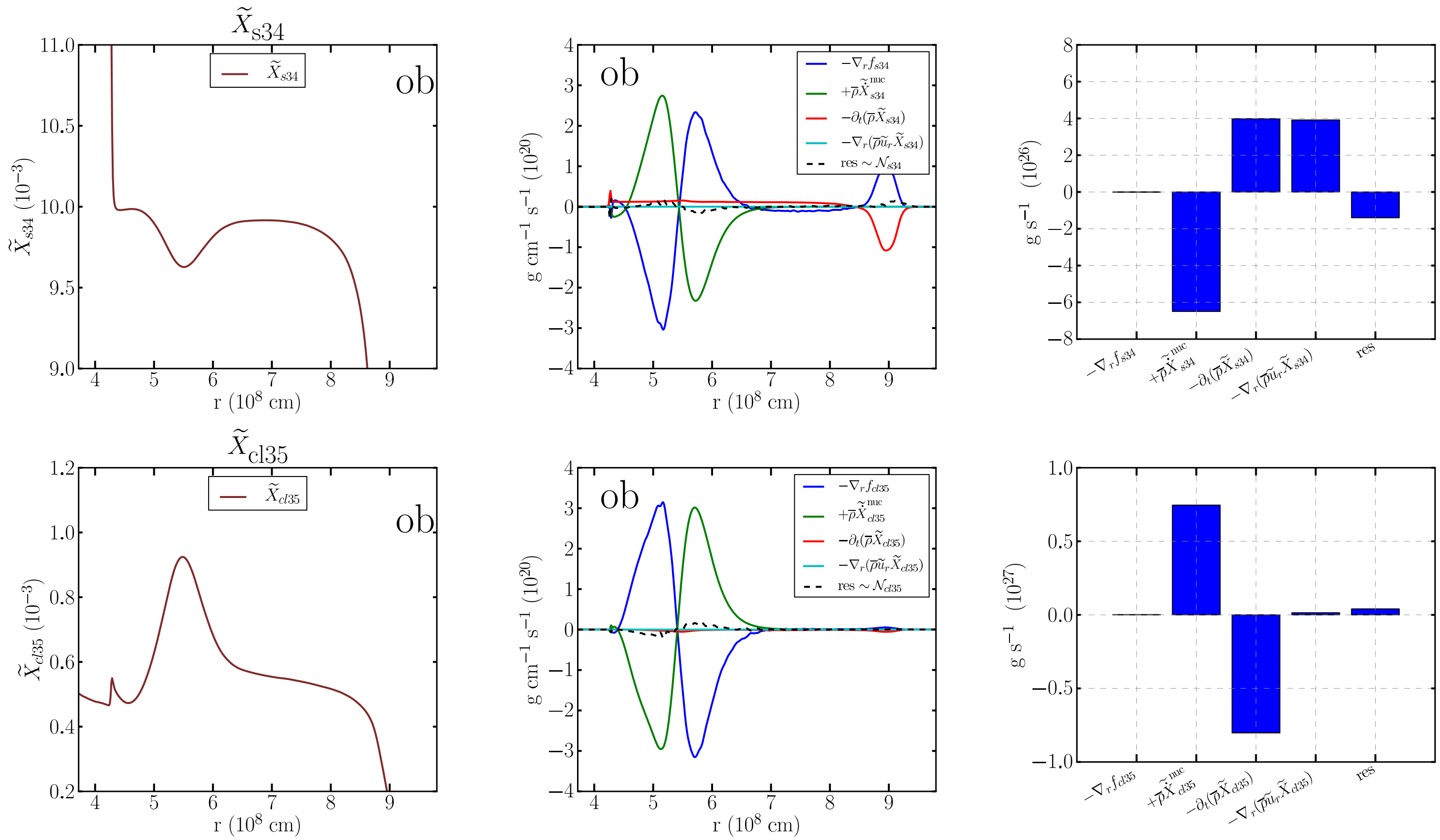

Figure 28: Mean composition equations. Model ob.3D.2hp.

5.6 Mean Ar^{36} and Ar^{38} equation

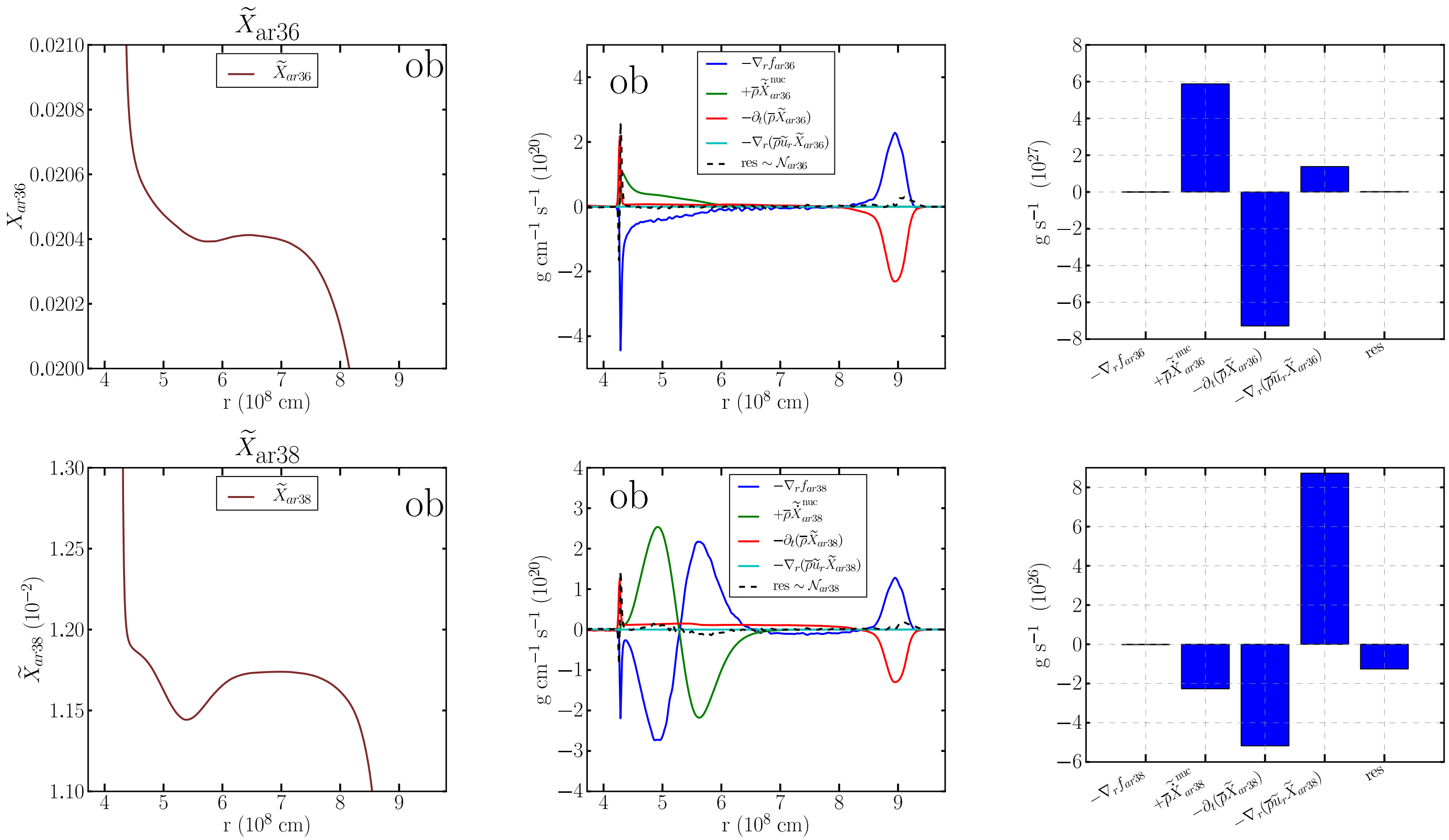

Figure 29: Mean composition equations. Model ob.3D.2hp.

5.7 Mean K^{39} and Ca^{40} equation

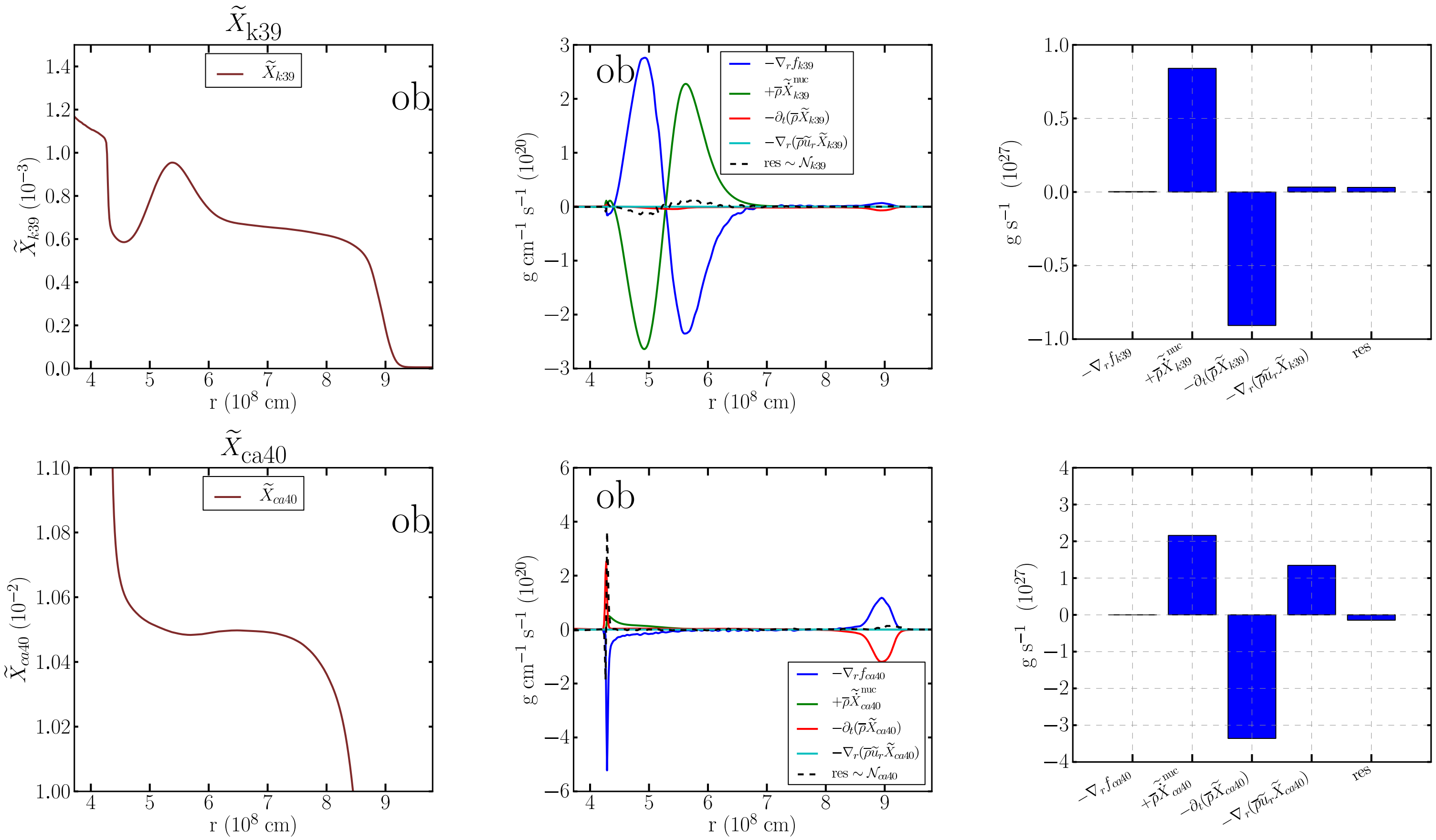

Figure 30: Mean composition equations. Model ob.3D.2hp.

5.8 Mean Ca^{42} and Ti^{44} equation

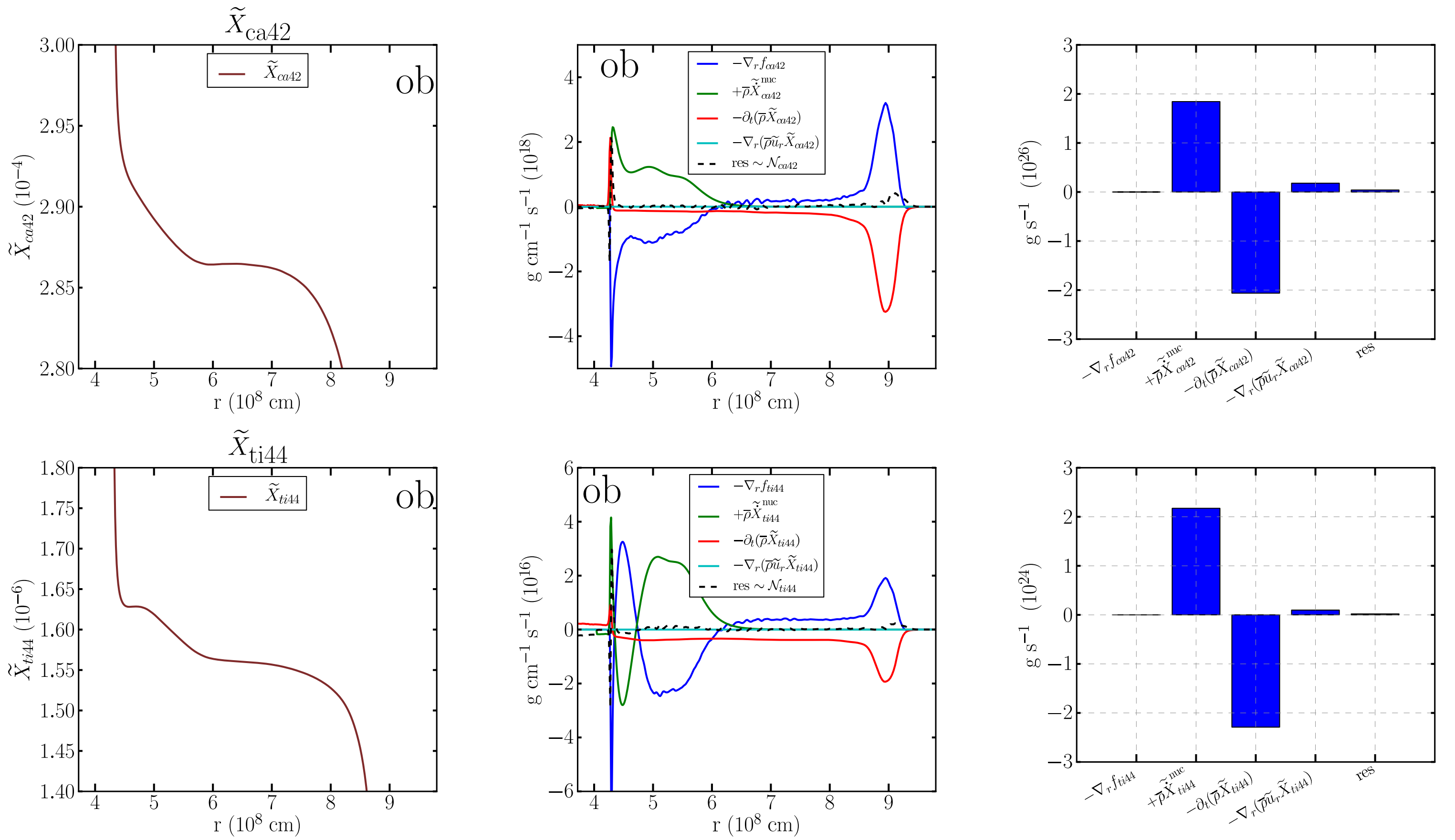

Figure 31: Mean composition equations. Model ob.3D.2hp.

5.9 Mean Ti^{46} and Cr^{48} equation

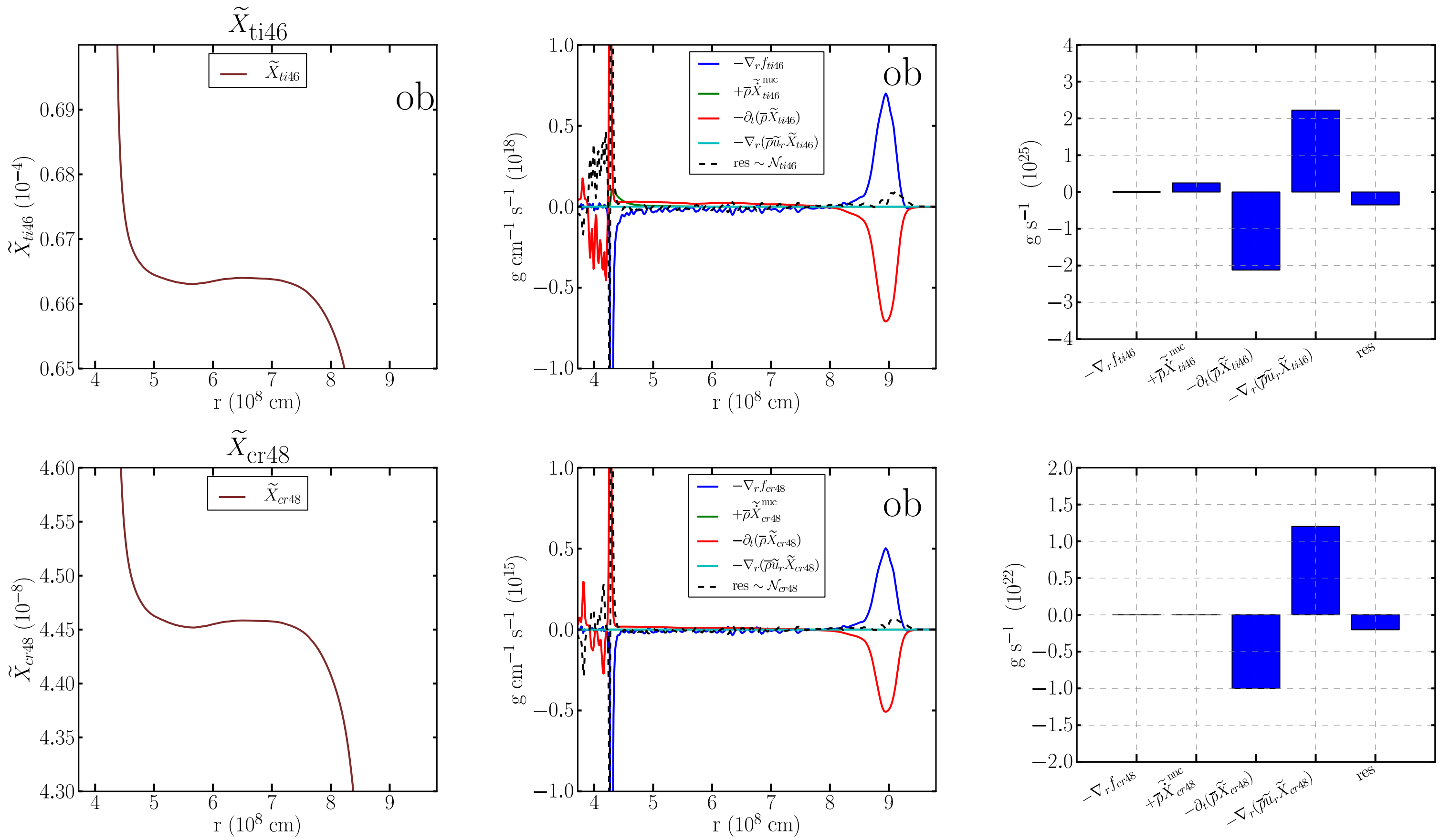

Figure 32: Mean composition equations. Model ob.3D.2hp.

5.10 Mean Cr^{50} and Fe^{52} equation

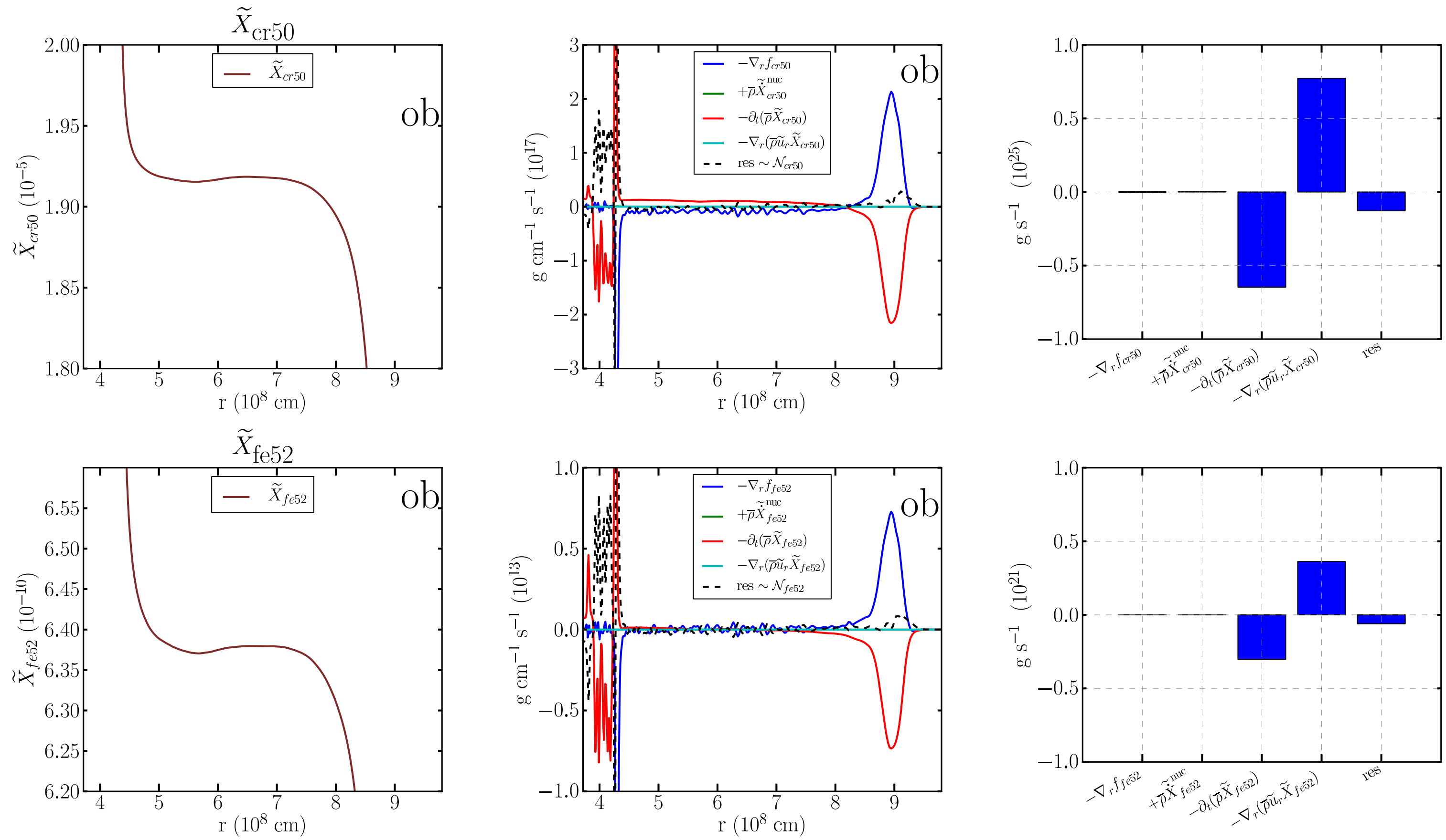

Figure 33: Mean composition equations. Model ob.3D.2hp.

5.11 Mean Fe^{54} and Ni^{56} equation

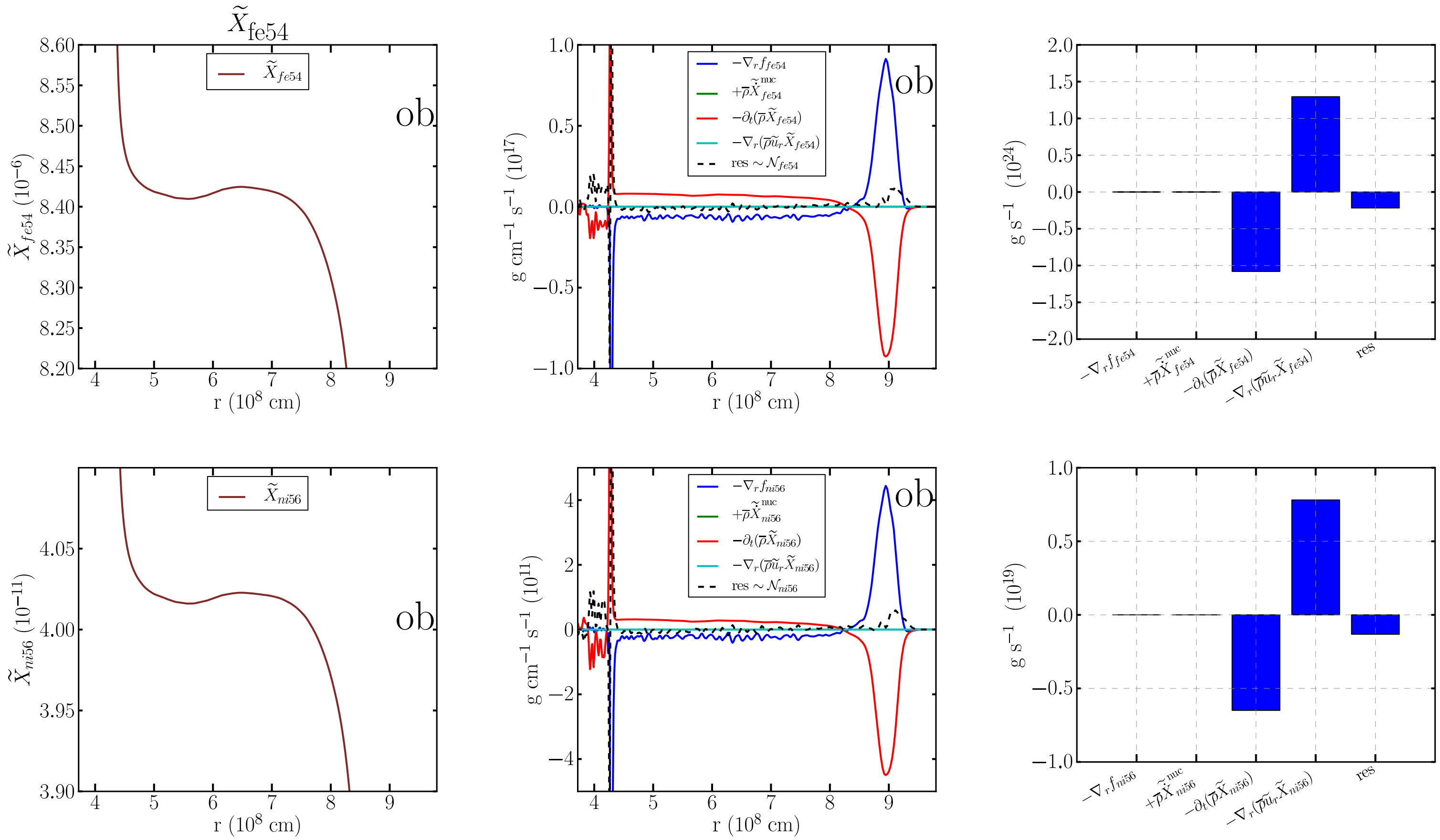

Figure 34: Mean composition equations. Model ob.3D.2hp.

6 Resolution effects

6.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

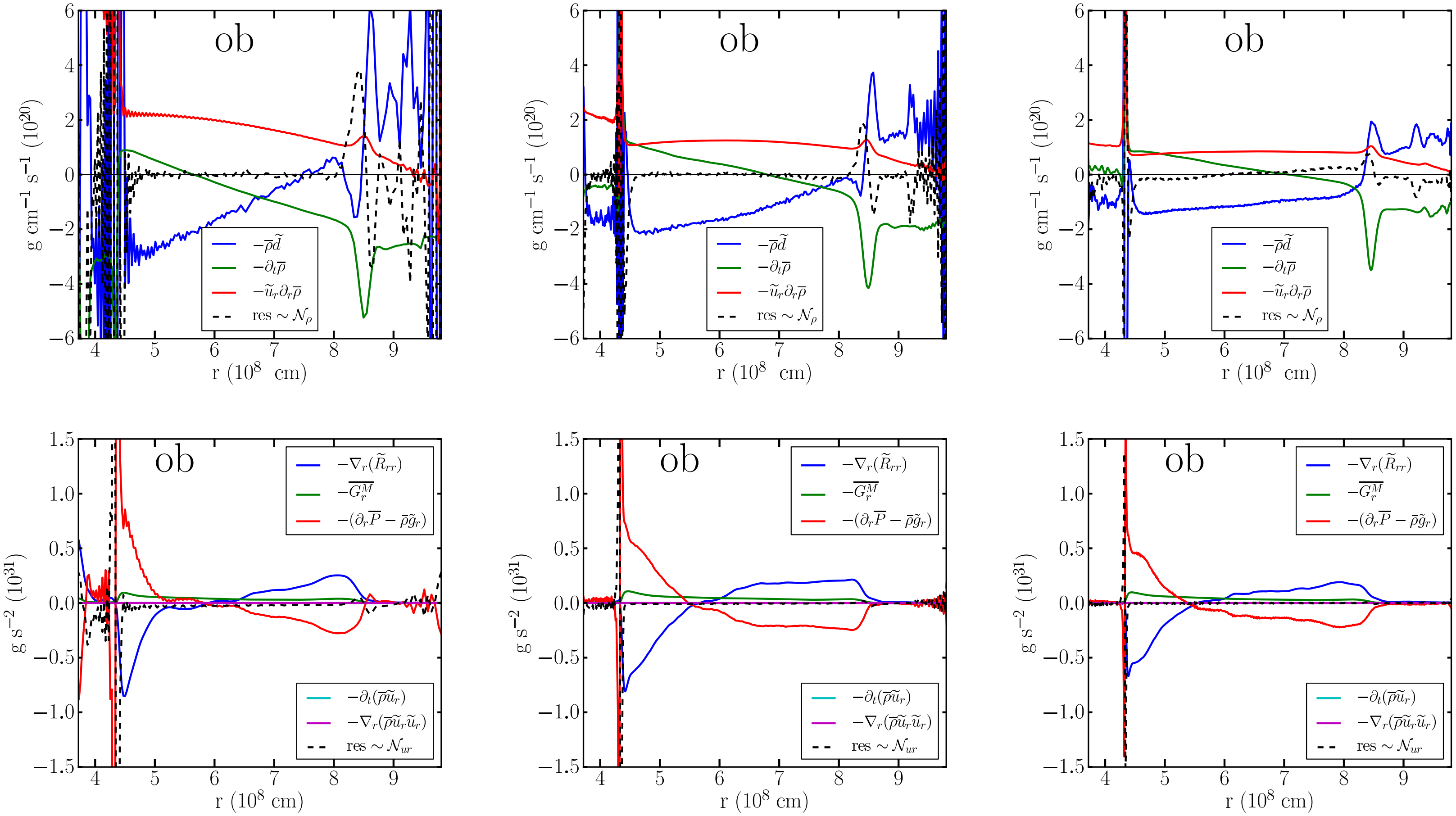

Figure 35: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean azimuthal and polar momentum equations

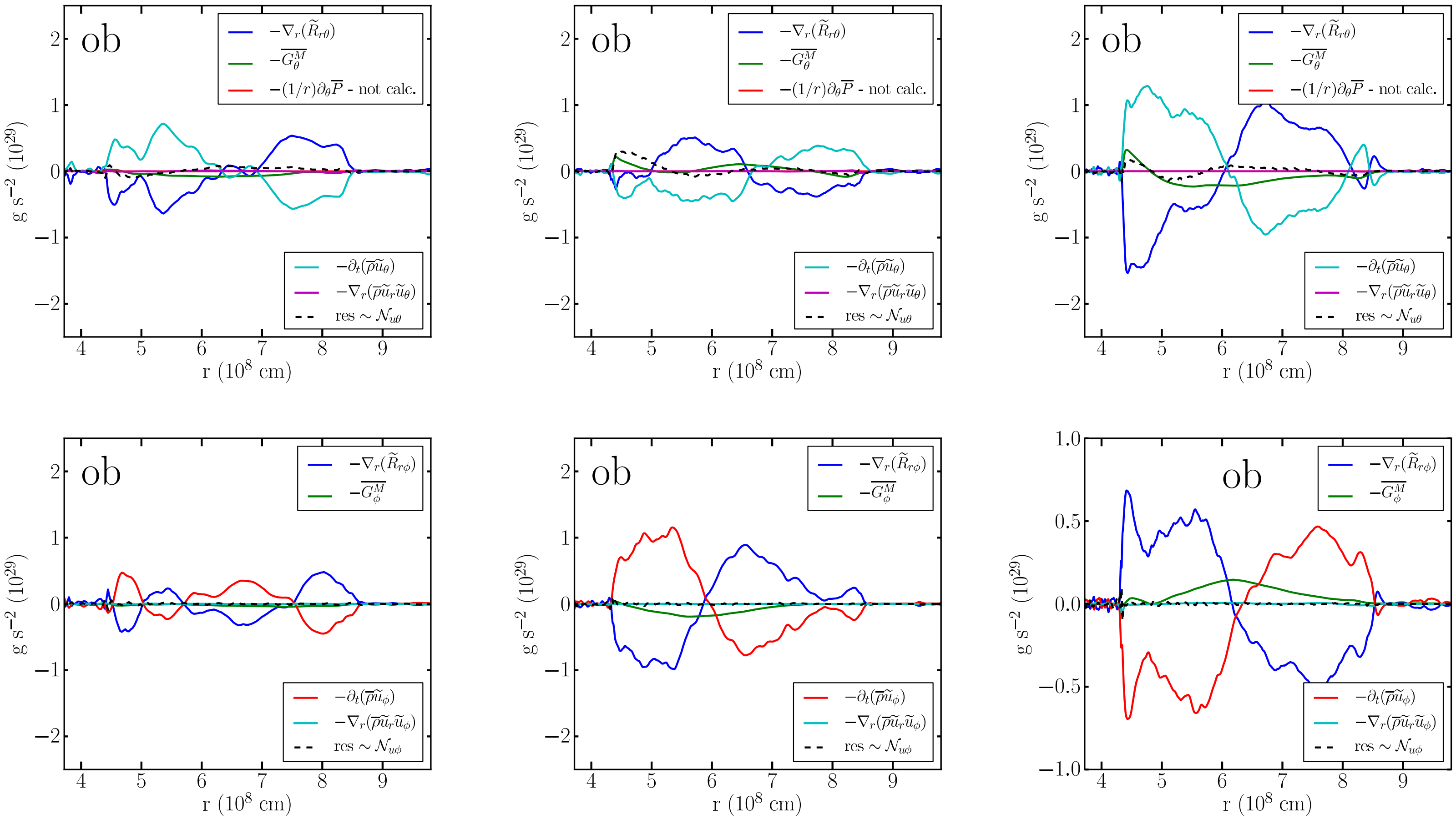

Figure 36: Mean azimuthal momentum equation (upper panels) and polar momentum equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean internal and kinetic energy equation

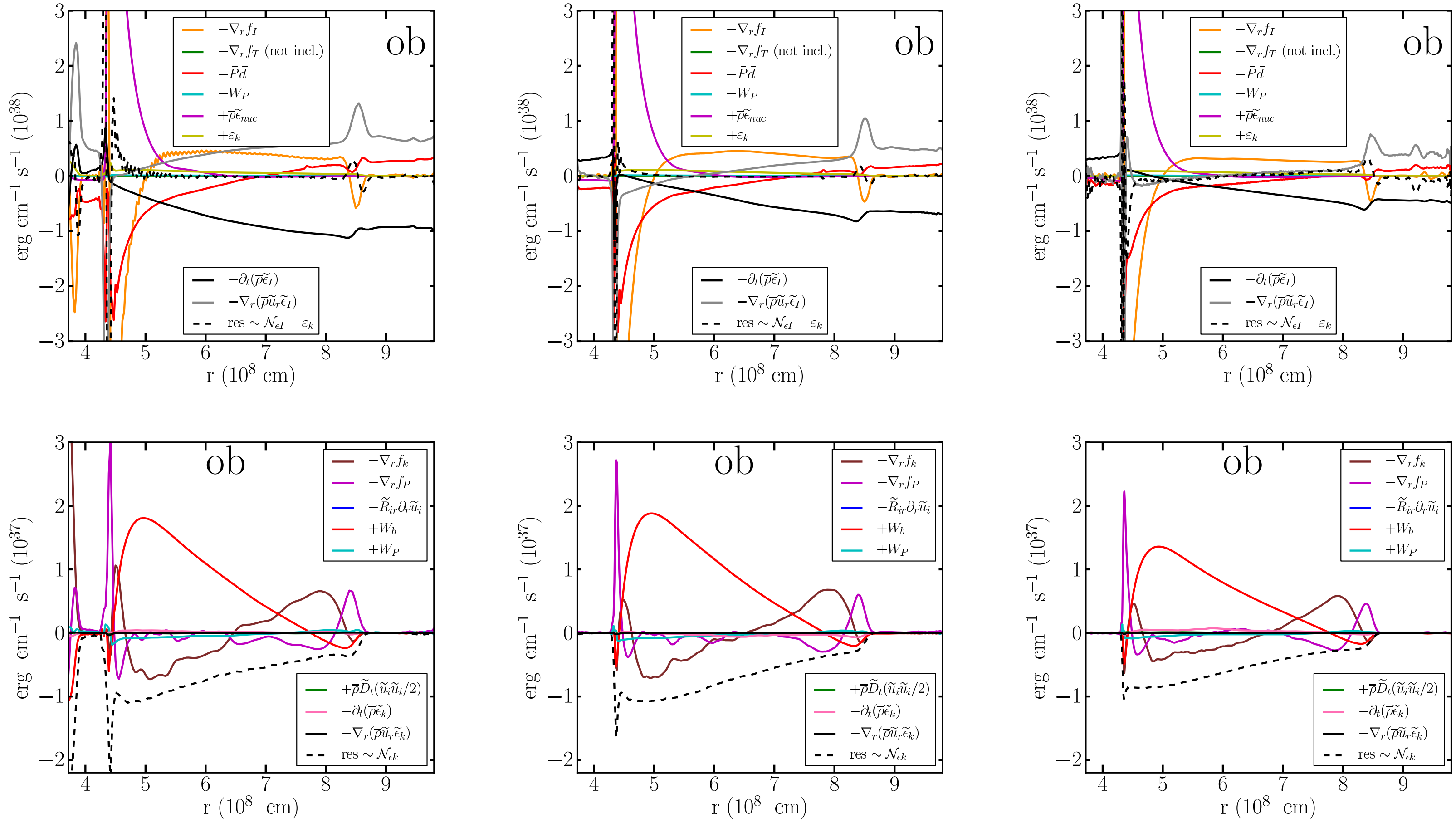

Figure 37: Mean internal energy equation (upper panels) and kinetic energy equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean total energy and entropy equation

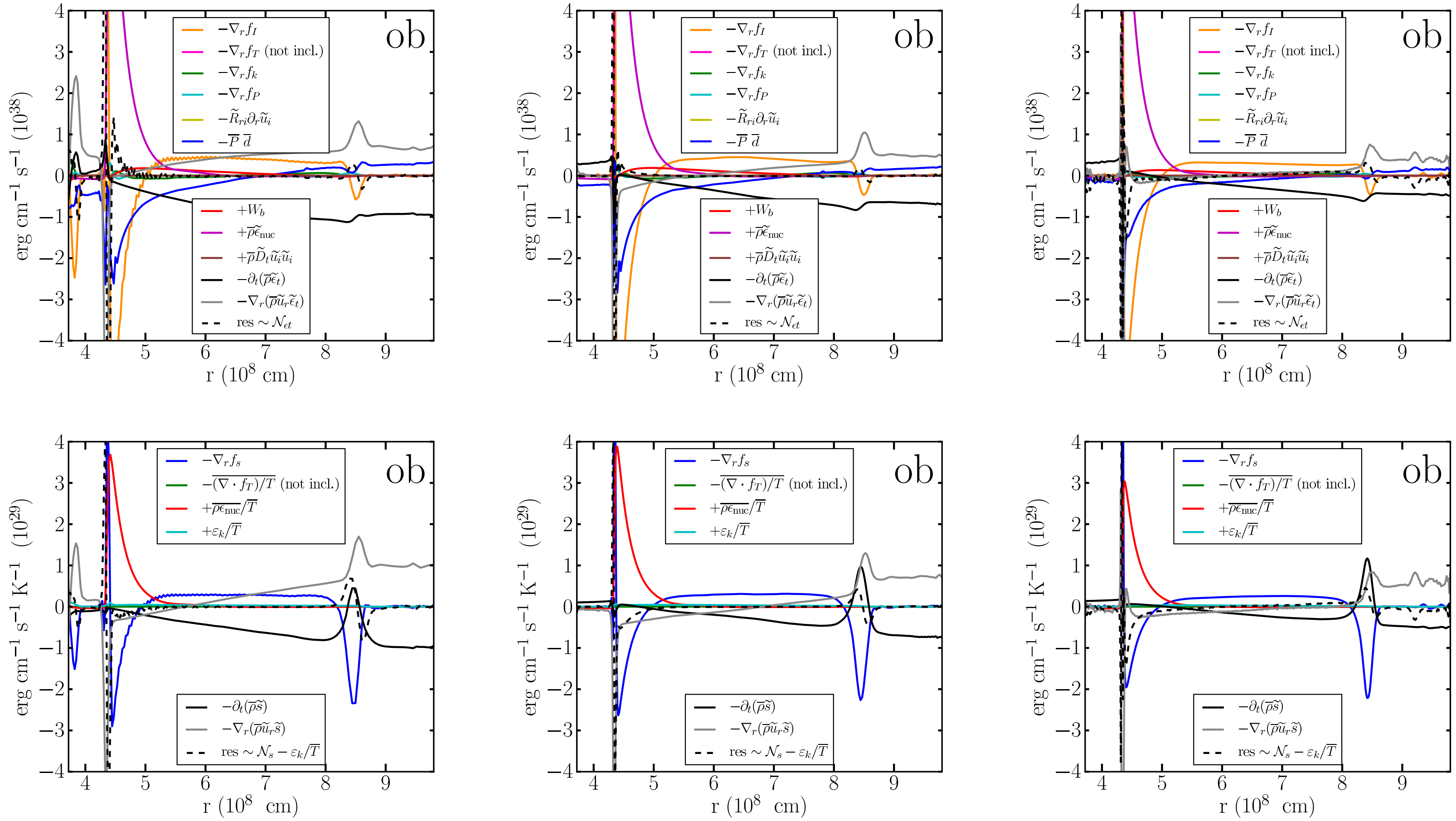

Figure 38: Mean total energy equation (upper panels) and mean entropy equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean density-specific volume covariance equation and mean number of nucleons per isotope equation

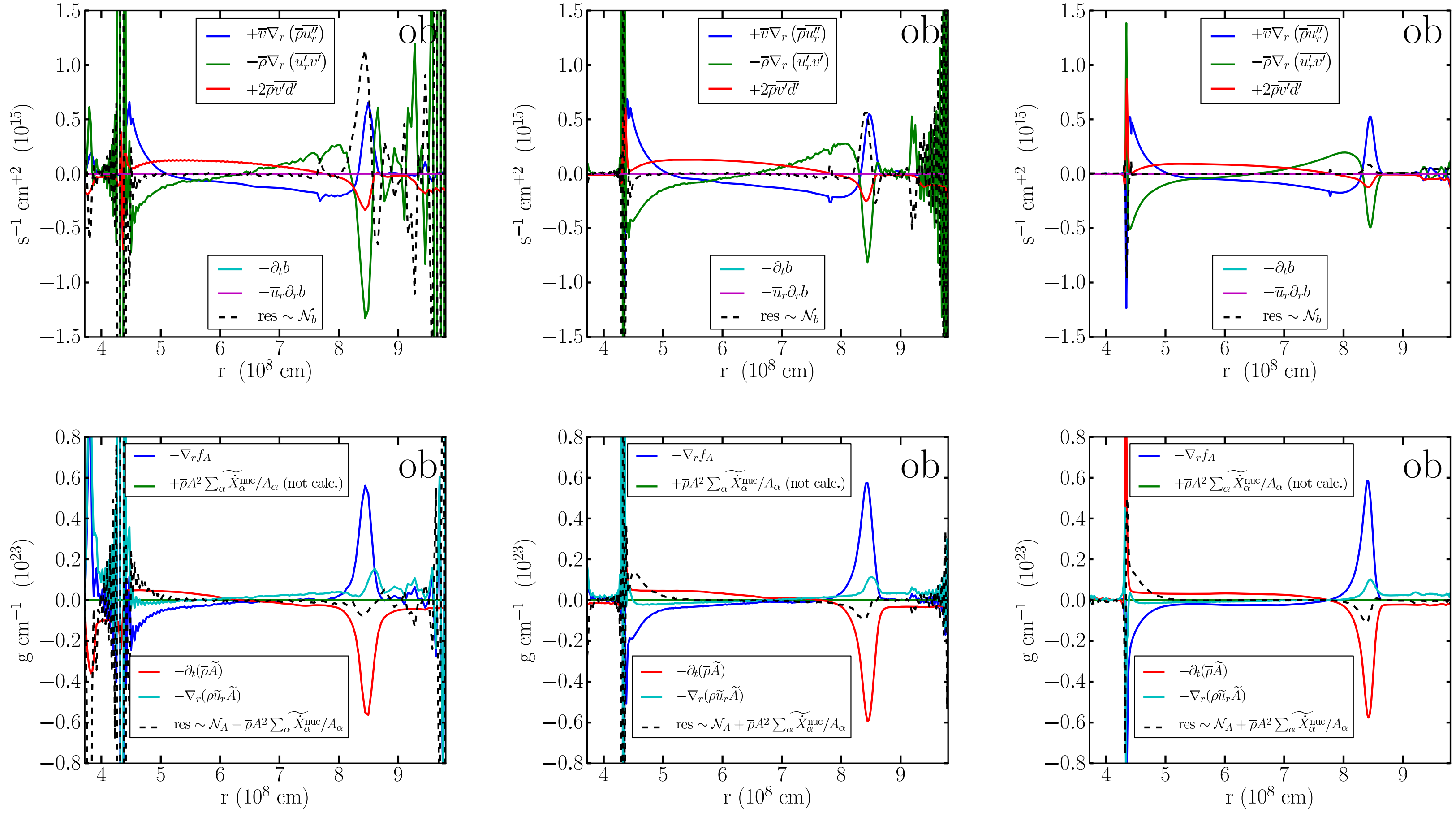

Figure 39: Mean density-specific volume covariance equation (upper panels) and mean number of nucleons per isotope equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean turbulent kinetic energy equation and mean velocities

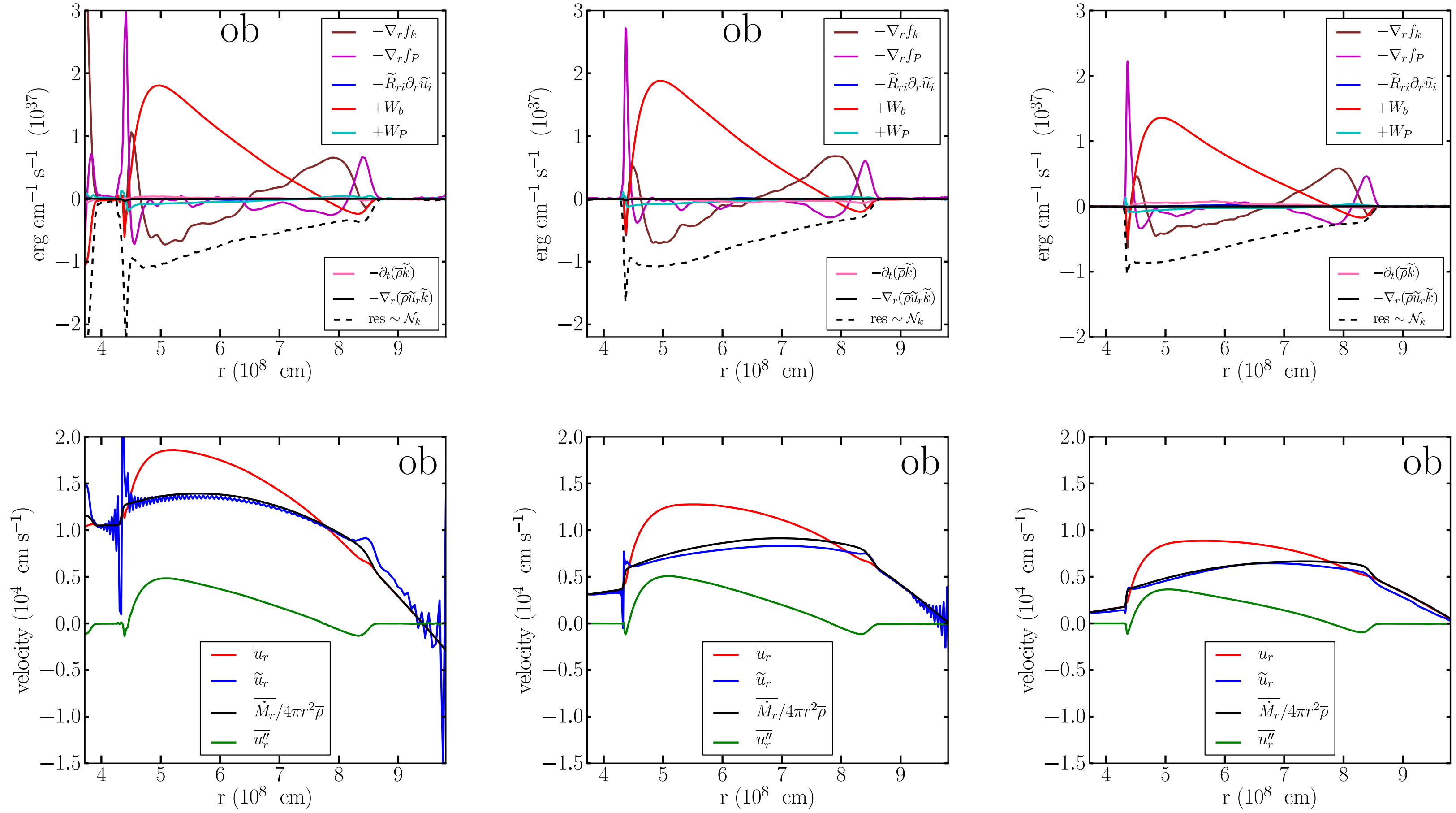

Figure 40: Mean turbulent kinetic energy equation (upper panels) and mean velocities (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

6.2 Red giant envelope convection

Mean continuity equation and mean radial momentum equation

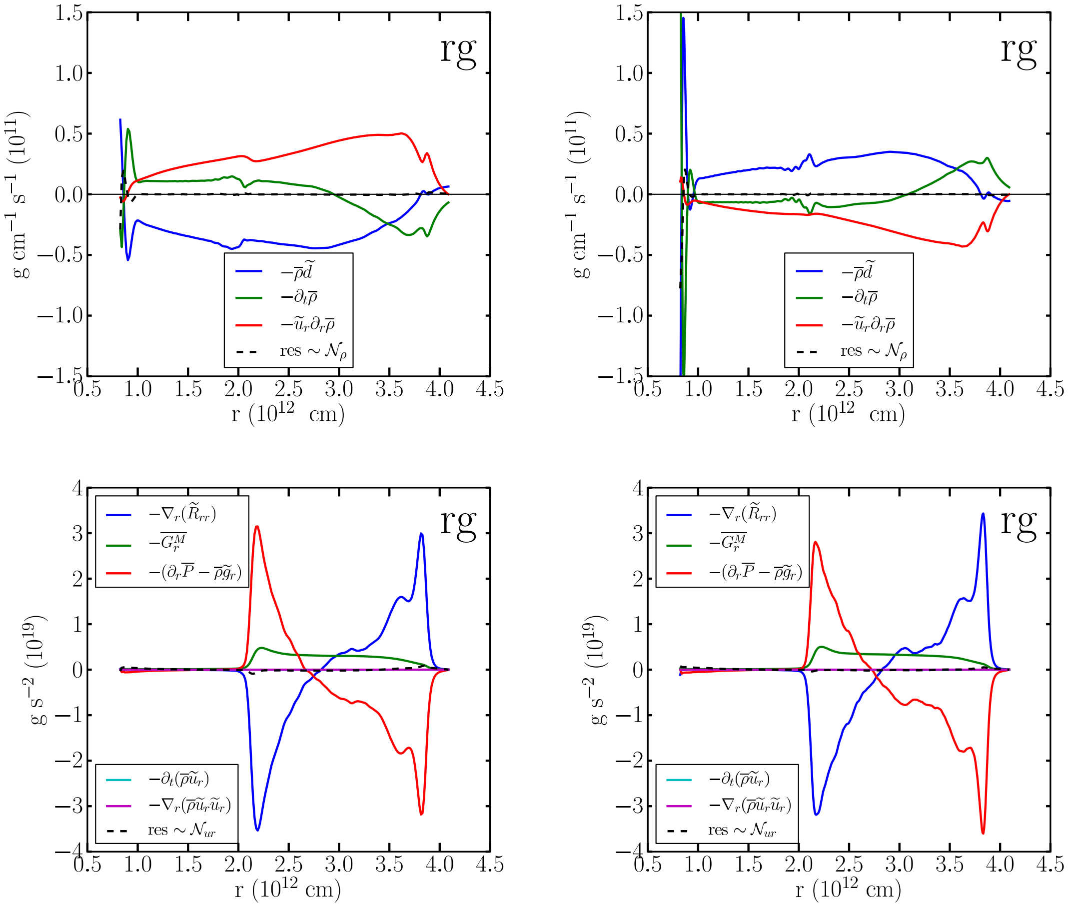

Figure 41: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean azimuthal and polar momentum equation

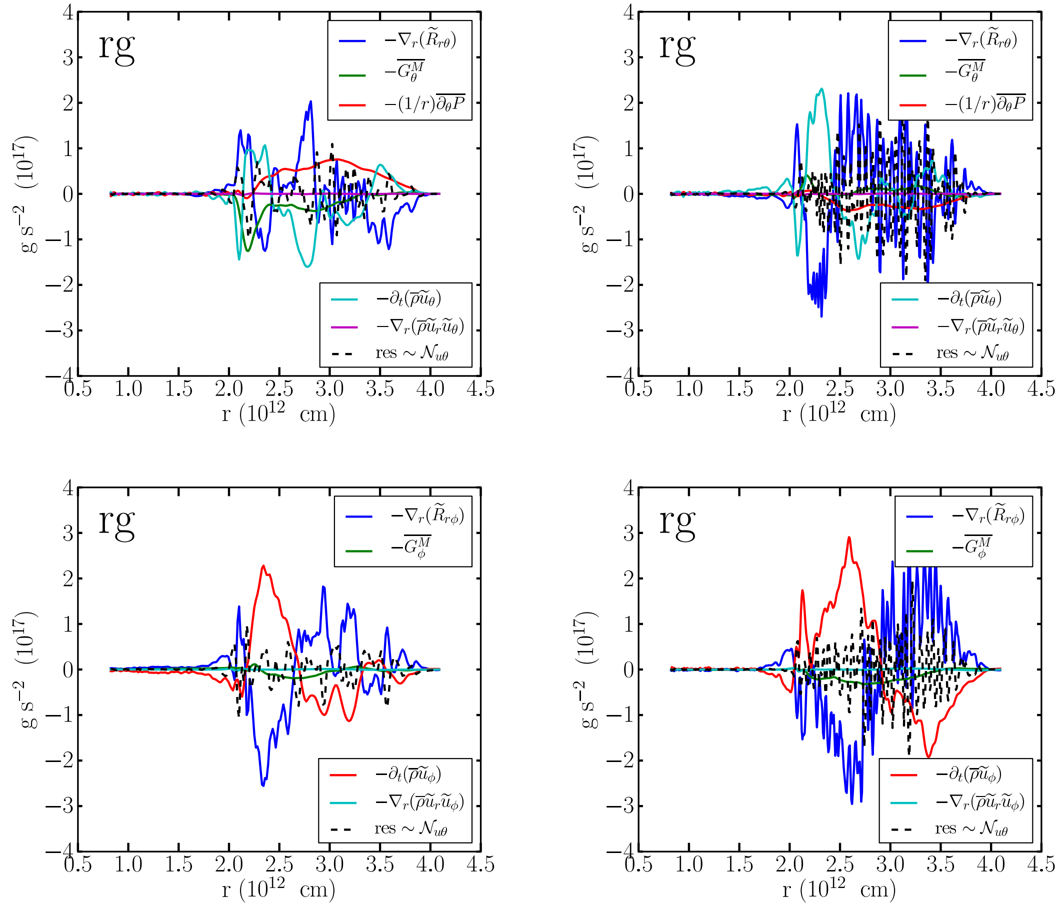

Figure 42: Mean azimuthal momentum (upper panels) and polar momentum equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean internal and kinetic energy equation

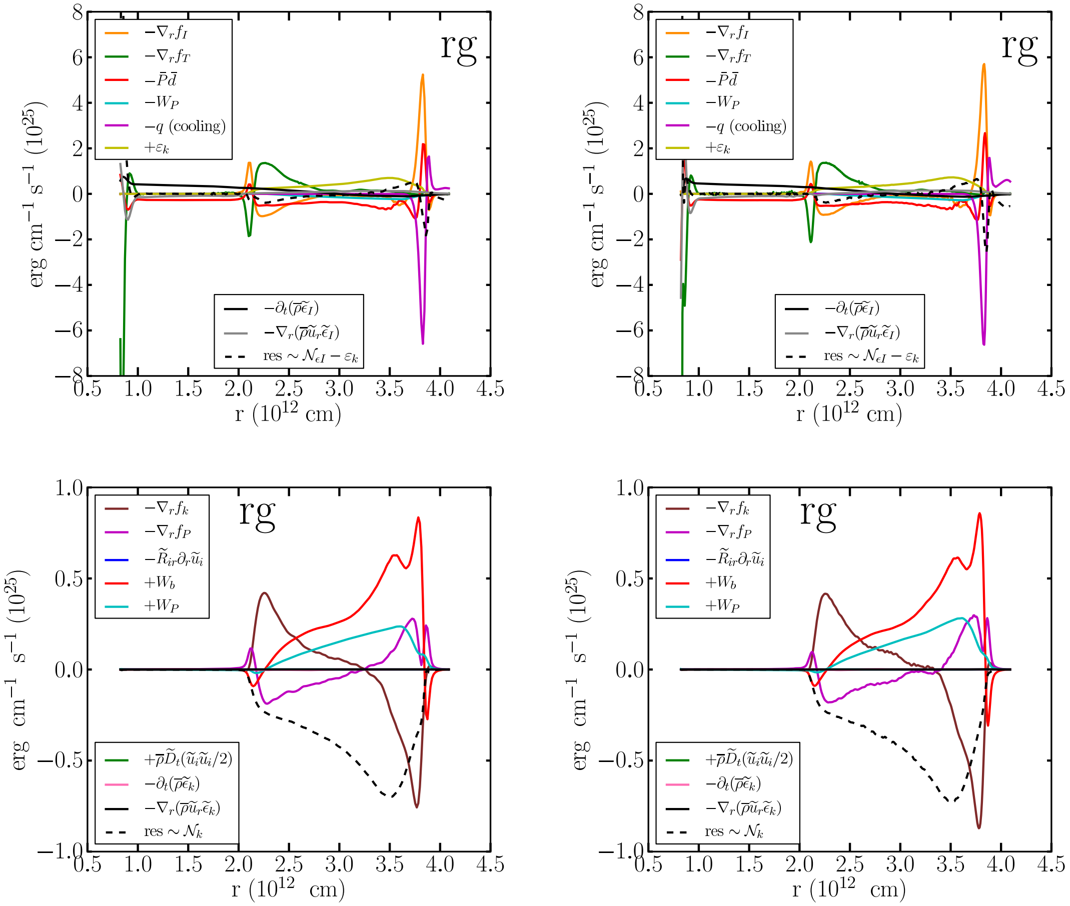

Figure 43: Mean internal energy equation (upper panels) and kinetic energy equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean total energy equation and mean entropy equation

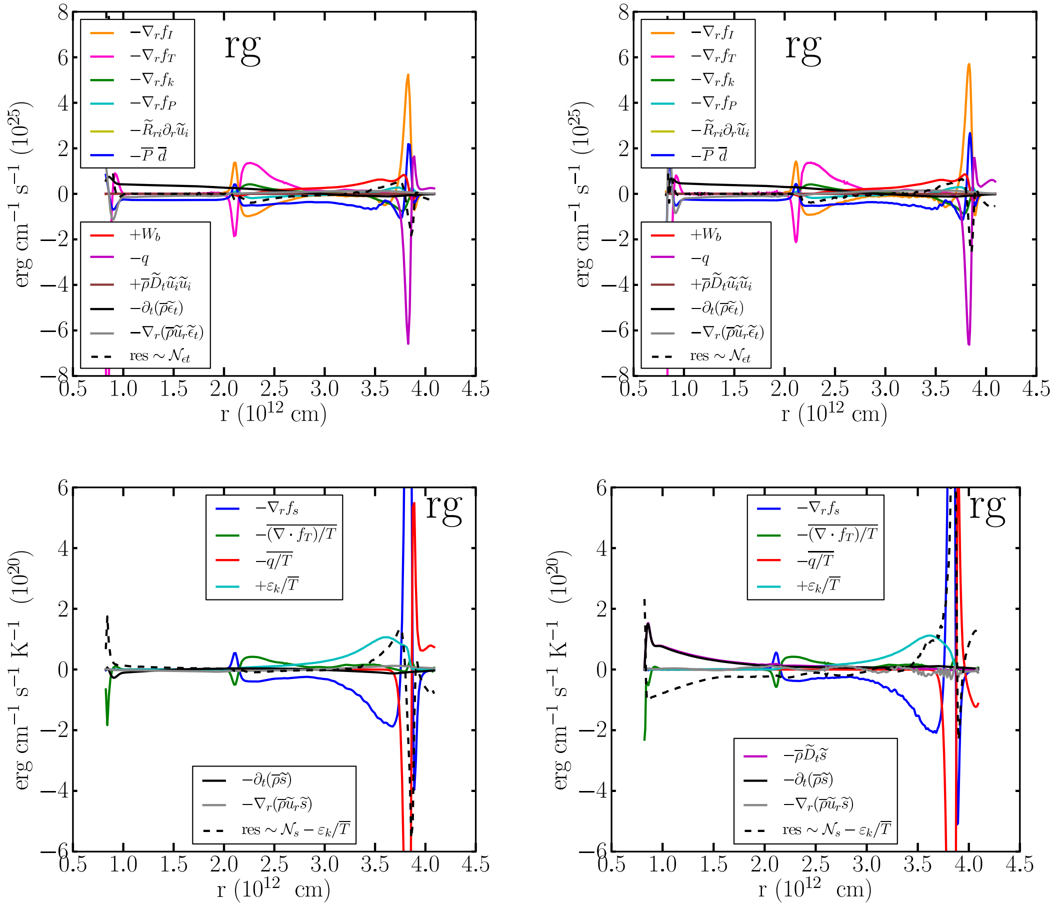

Figure 44: Mean total energy equation (upper panels) and mean entropy equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean density-specific volume covariance and entropy flux equation

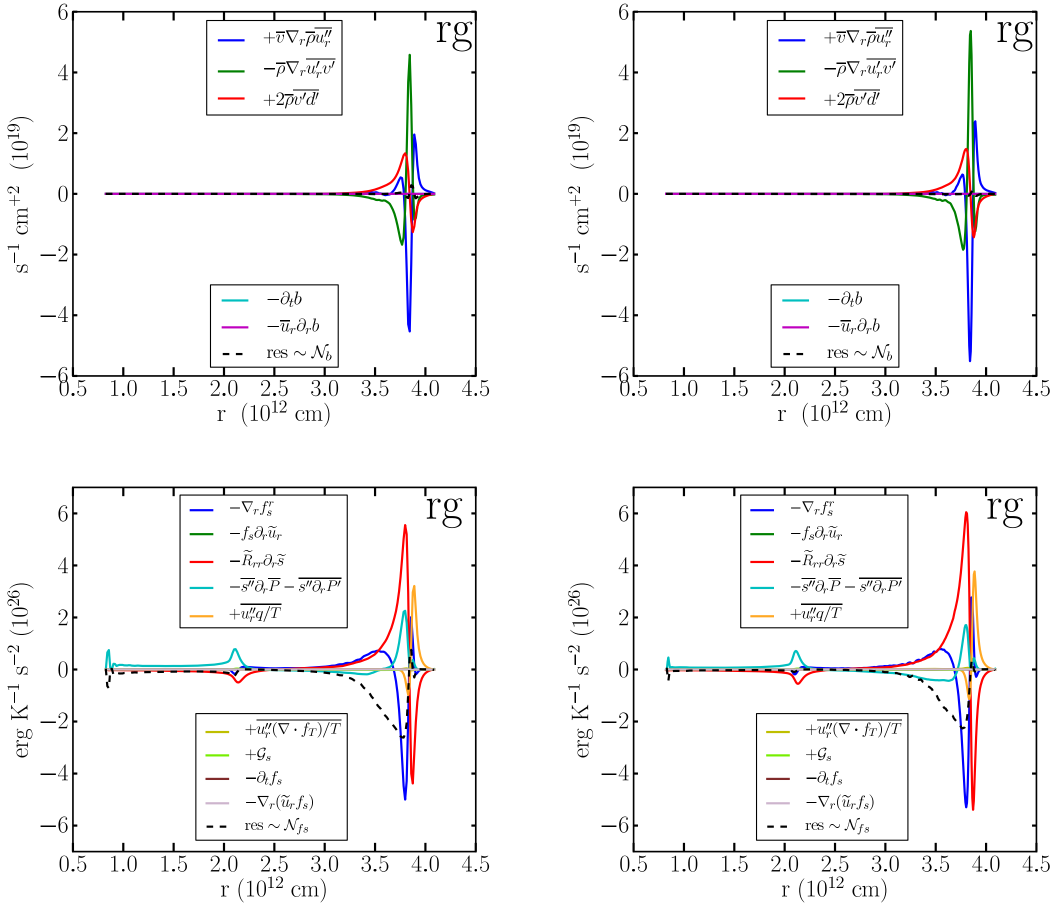

Figure 45: Mean density-specific volume covariance equation (upper panels) and entropy flux equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean turbulent kinetic energy equation and mean turbulent mass flux equation

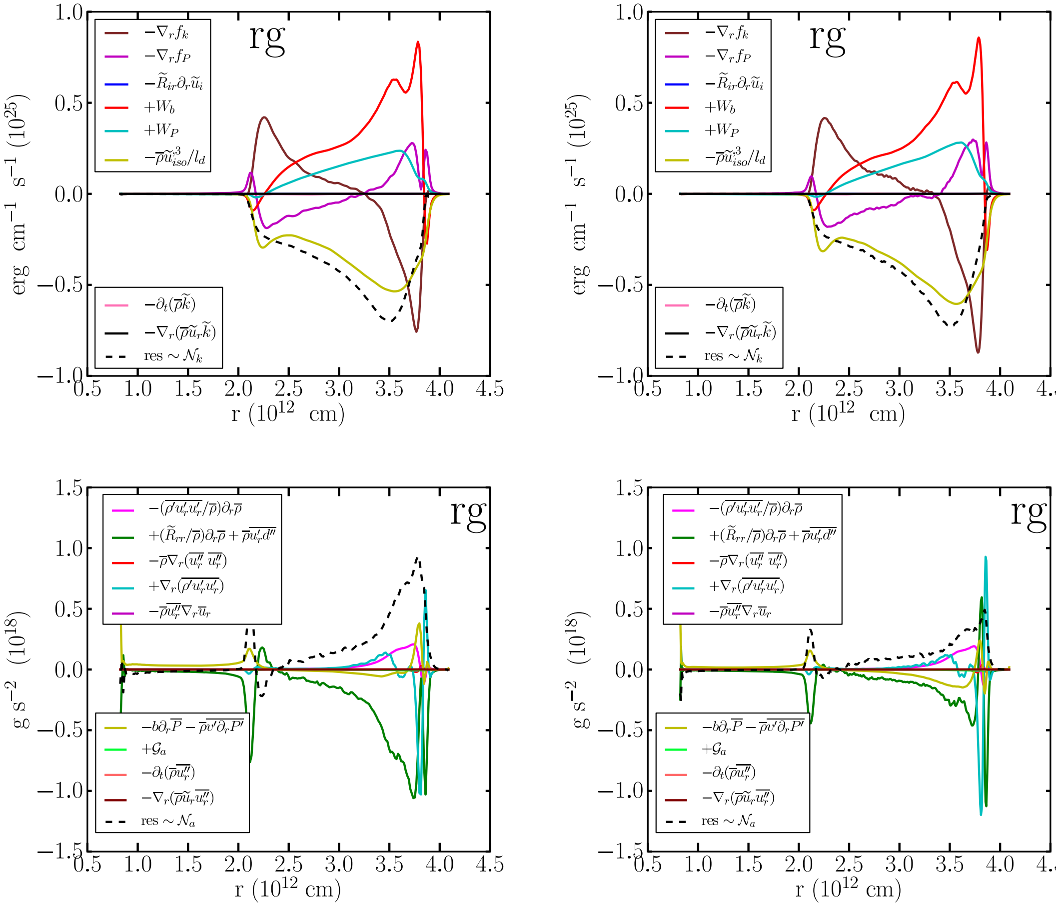

Figure 46: Mean turbulent kinetic energy equation (upper panels) and mean turbulent mass flux equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

7 Wedge-size effects

7.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

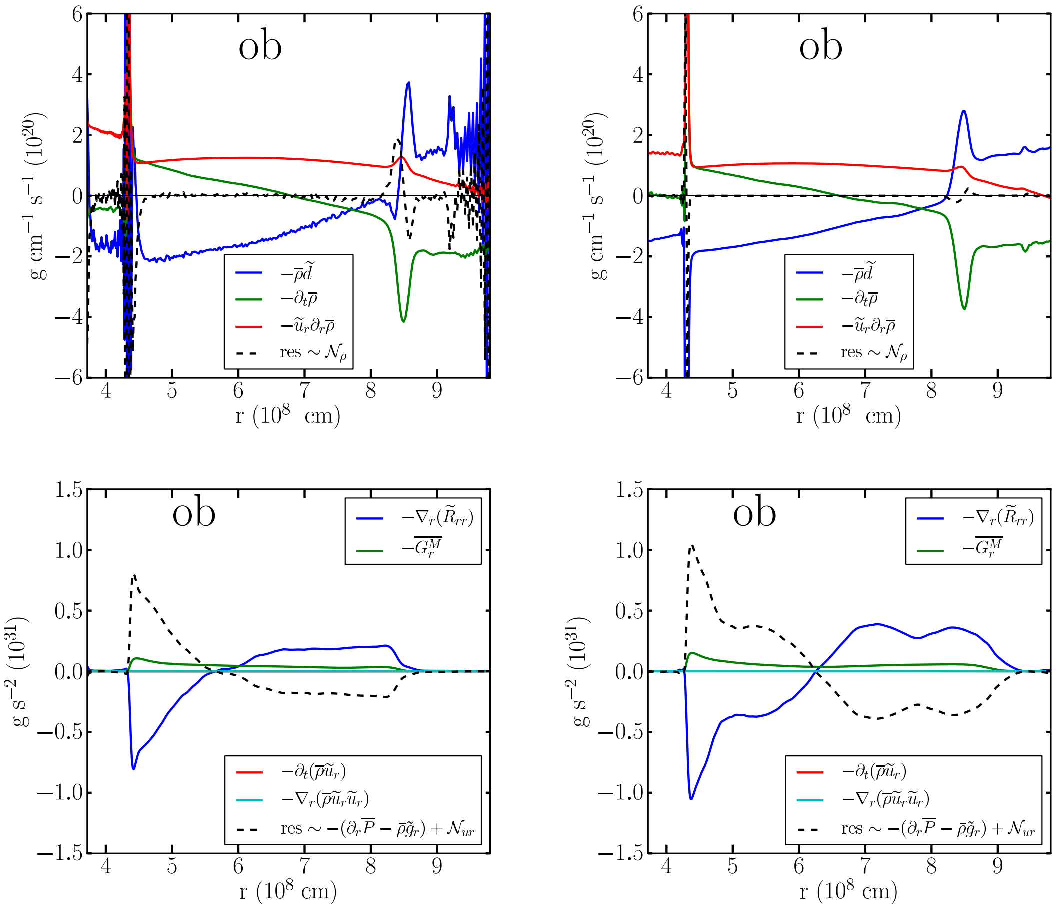

Figure 47: Continuity equation (upper panels) and radial momentum equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean azimuthal momentum and polar momentum equation

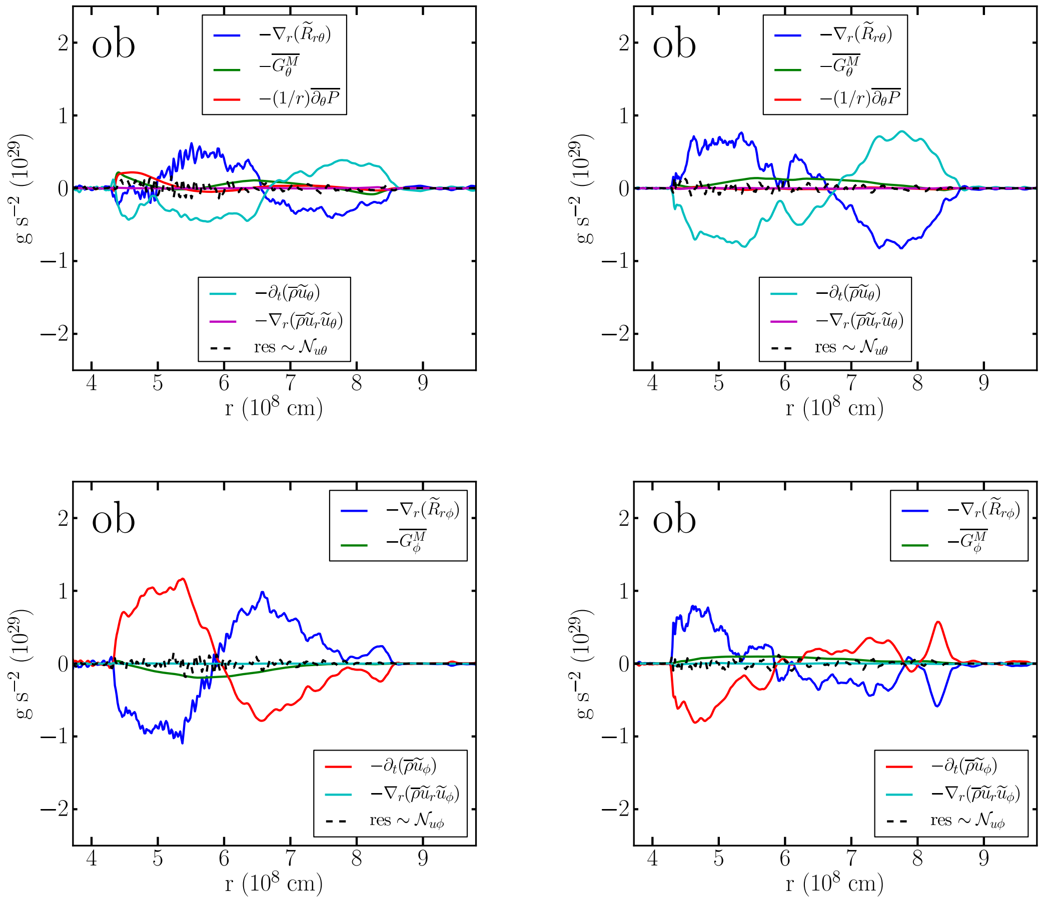

Figure 48: Mean azimuthal momentum (upper panels) and polar momentum equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean internal energy equation and total energy equation

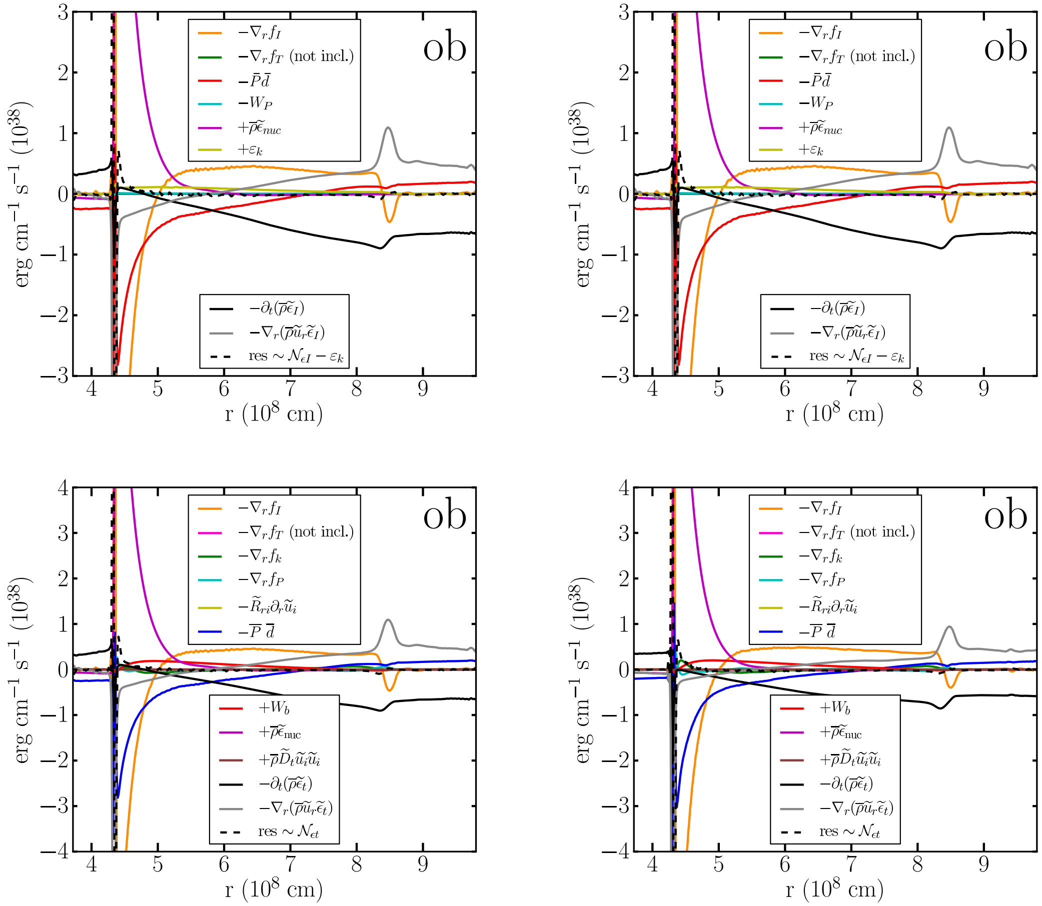

Figure 49: Mean internal energy (upper panels) equations and mean total energy equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean turbulent kinetic energy equation and turbulent mass flux equation

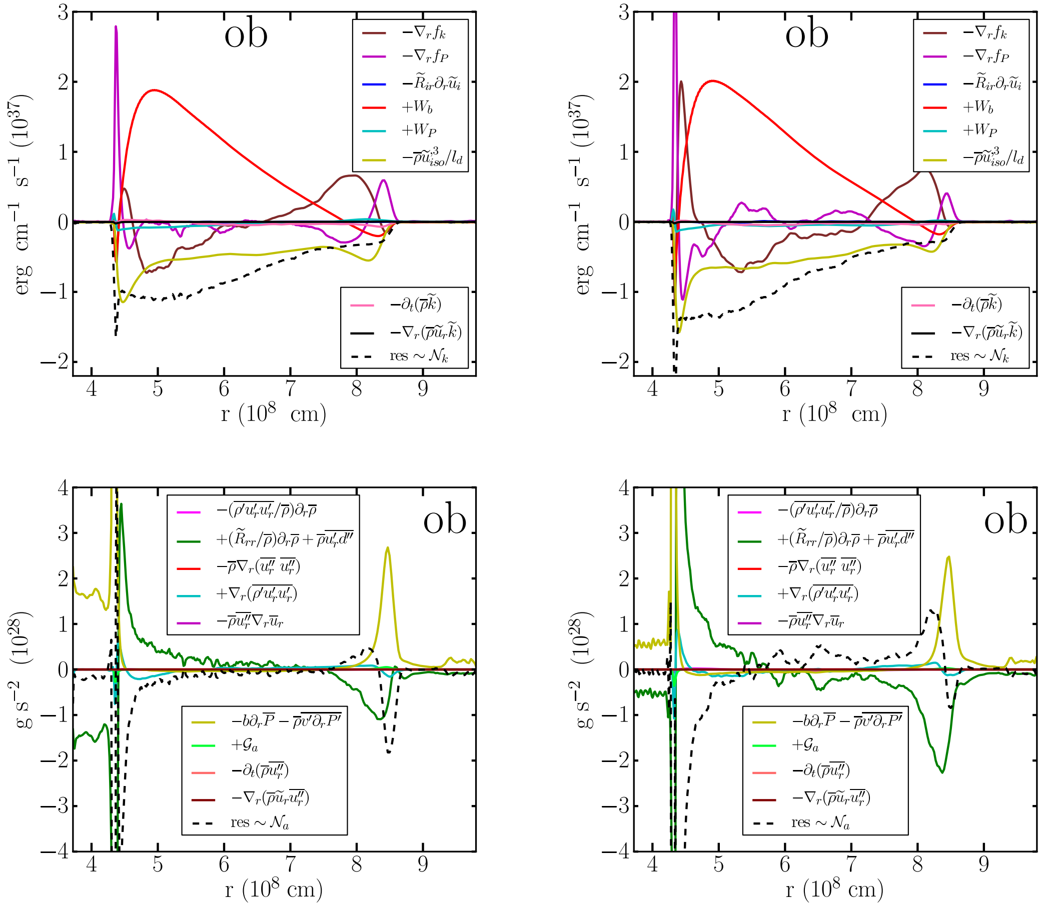

Figure 50: Mean turbulent kinetic energy equation and turbulent mass flux equation. Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean density-specific volume covariance and internal energy flux equation

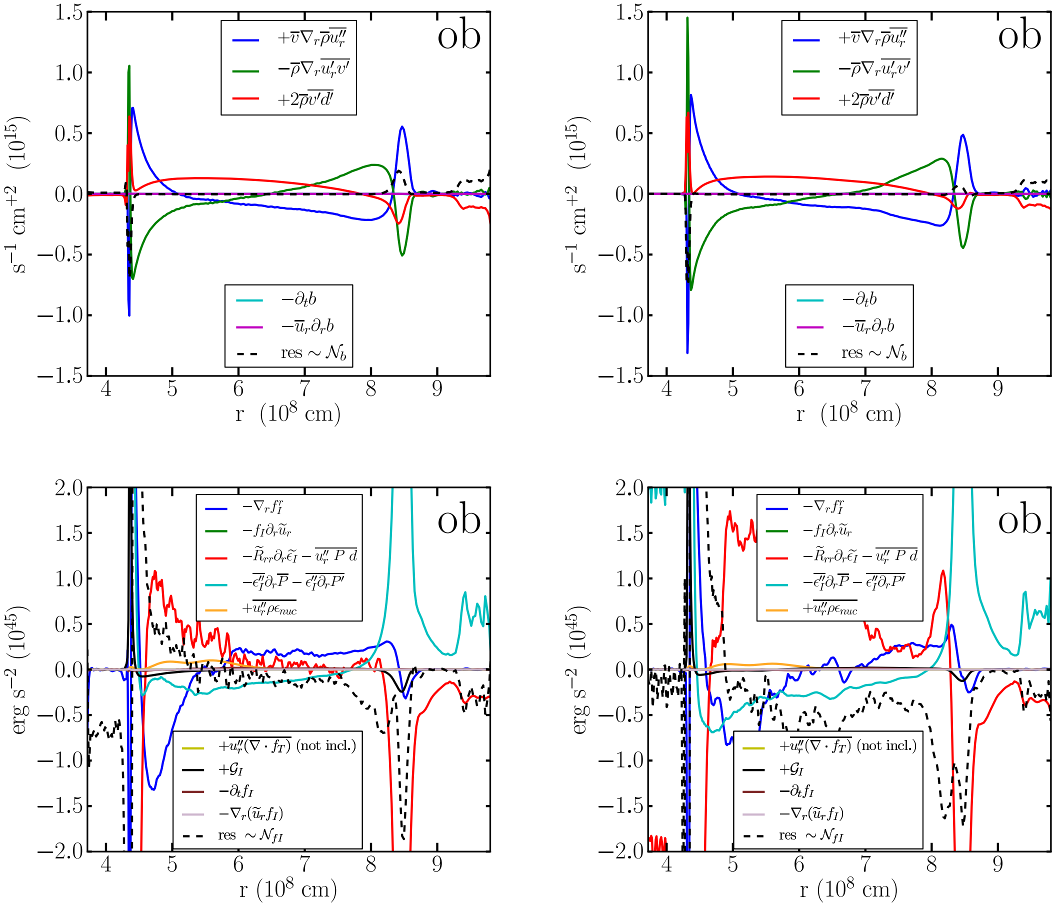

Figure 51: Mean density-specific volume covariance equation (upper panels) and mean internal energy flux equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean kinetic energy equation and mean velocities

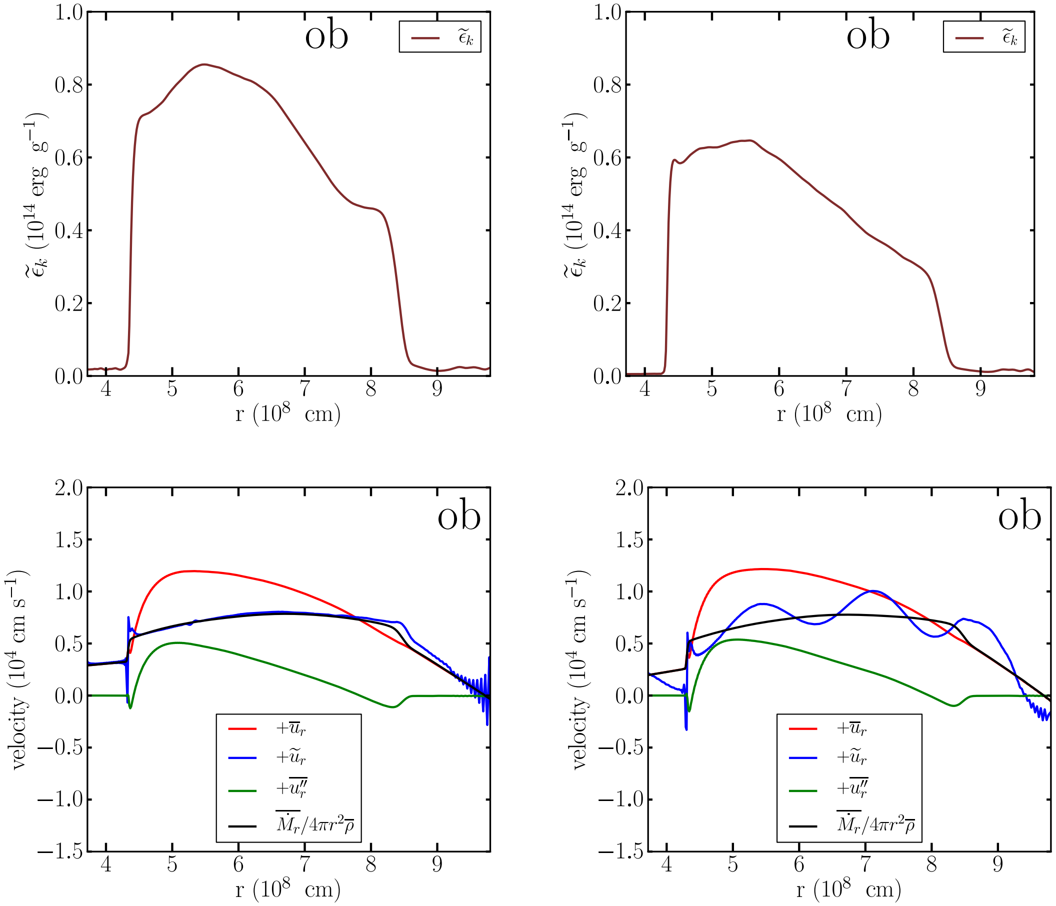

Figure 52: Mean kinetic energy equation (upper panels) mean velocities (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

8 Sensitivity to averaging window

8.1 Oxygen burning shell model

Mean continuity equation and mean radial momentum equation

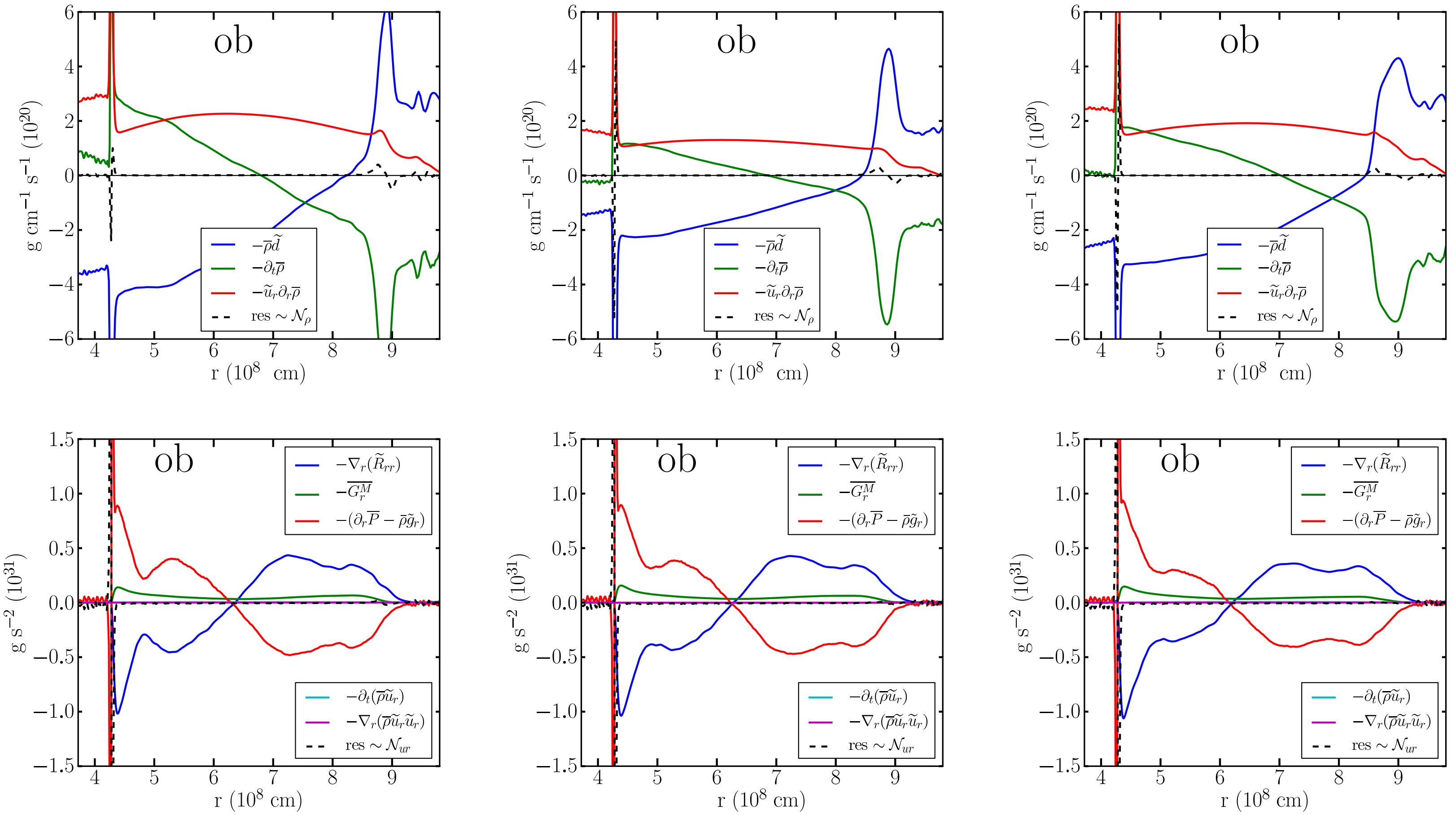

Figure 53: Mean continuity equation (upper panels) and radial momentum equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean azimuthal and polar momentum equations

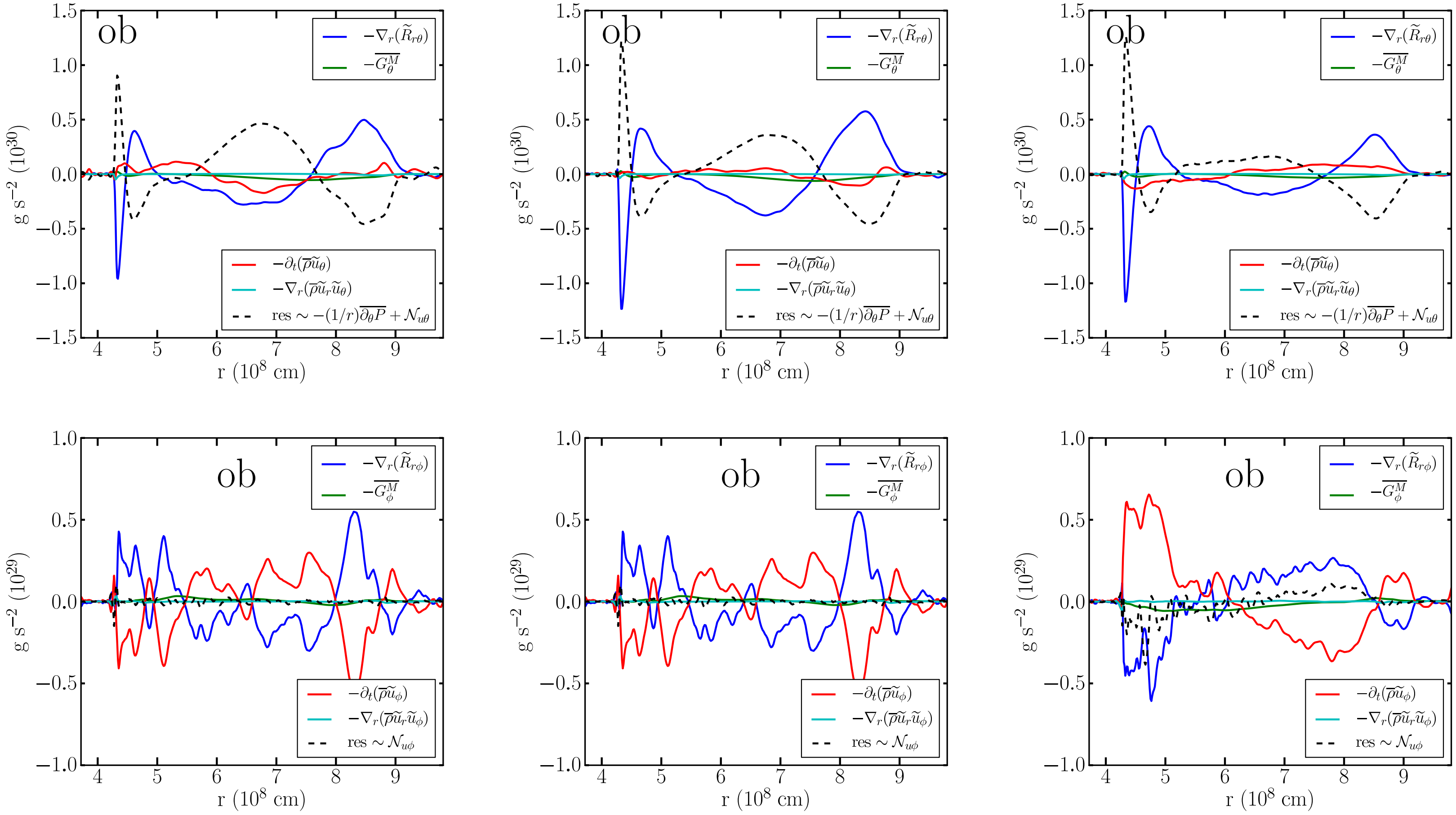

Figure 54: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean total energy equation and mean turbulent kinetic energy equation

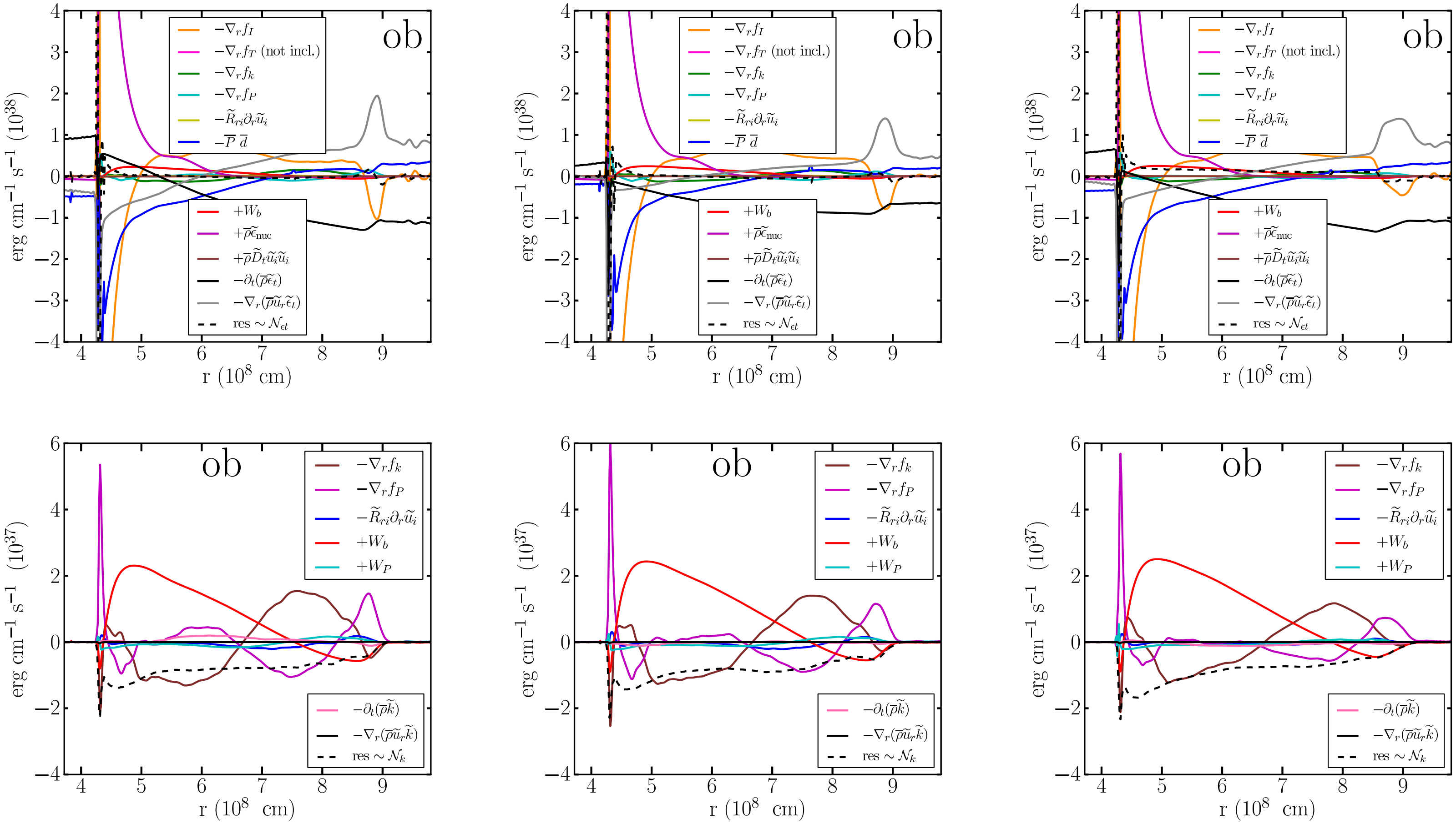

Figure 55: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean entropy equation and mean number of nucleons per isotope equation

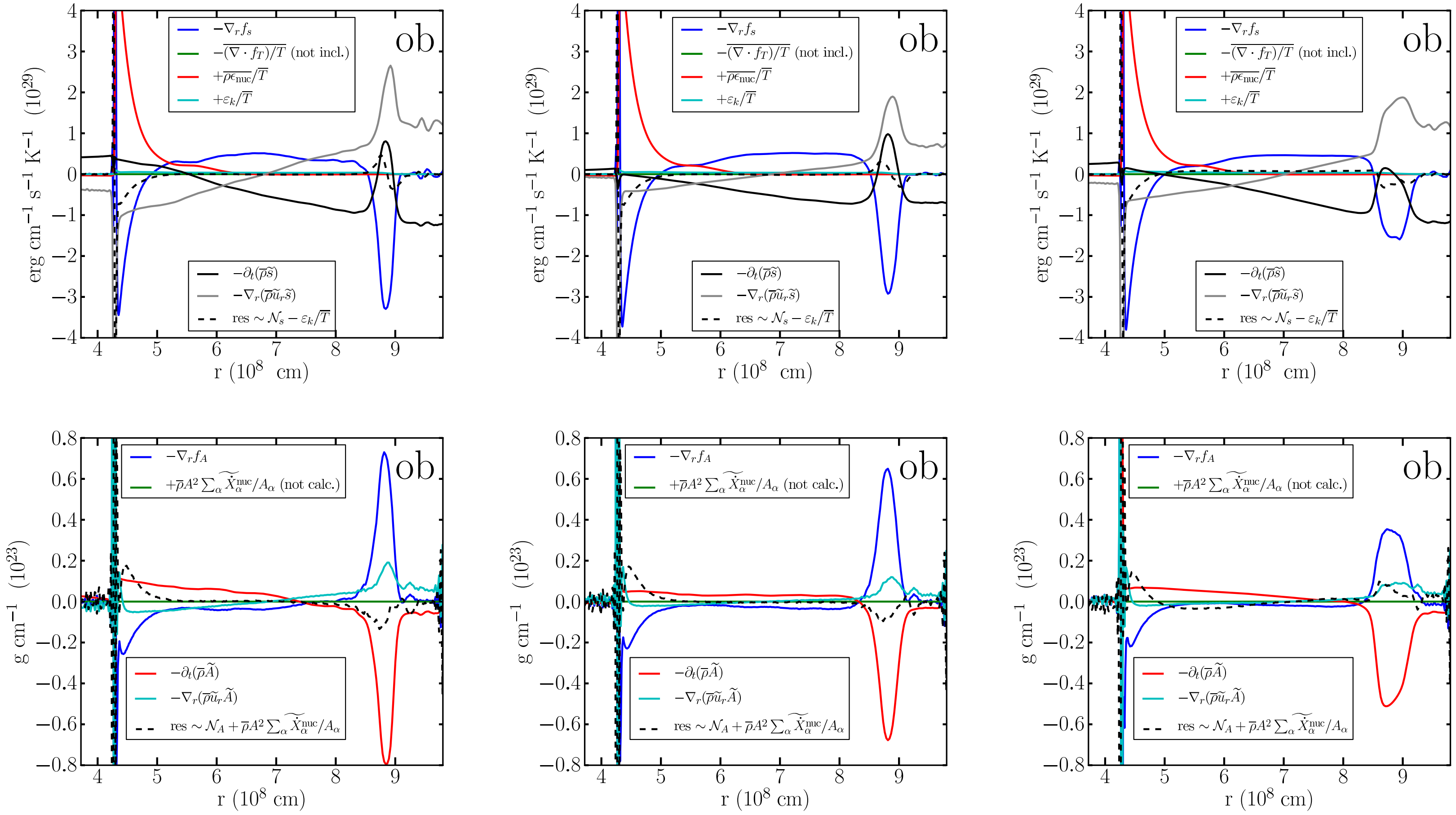

Figure 56: Mean entropy equtaion (upper panels) and mean number of nucleons per isotope (upper panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean turbulent kinetic energy and mean velocities

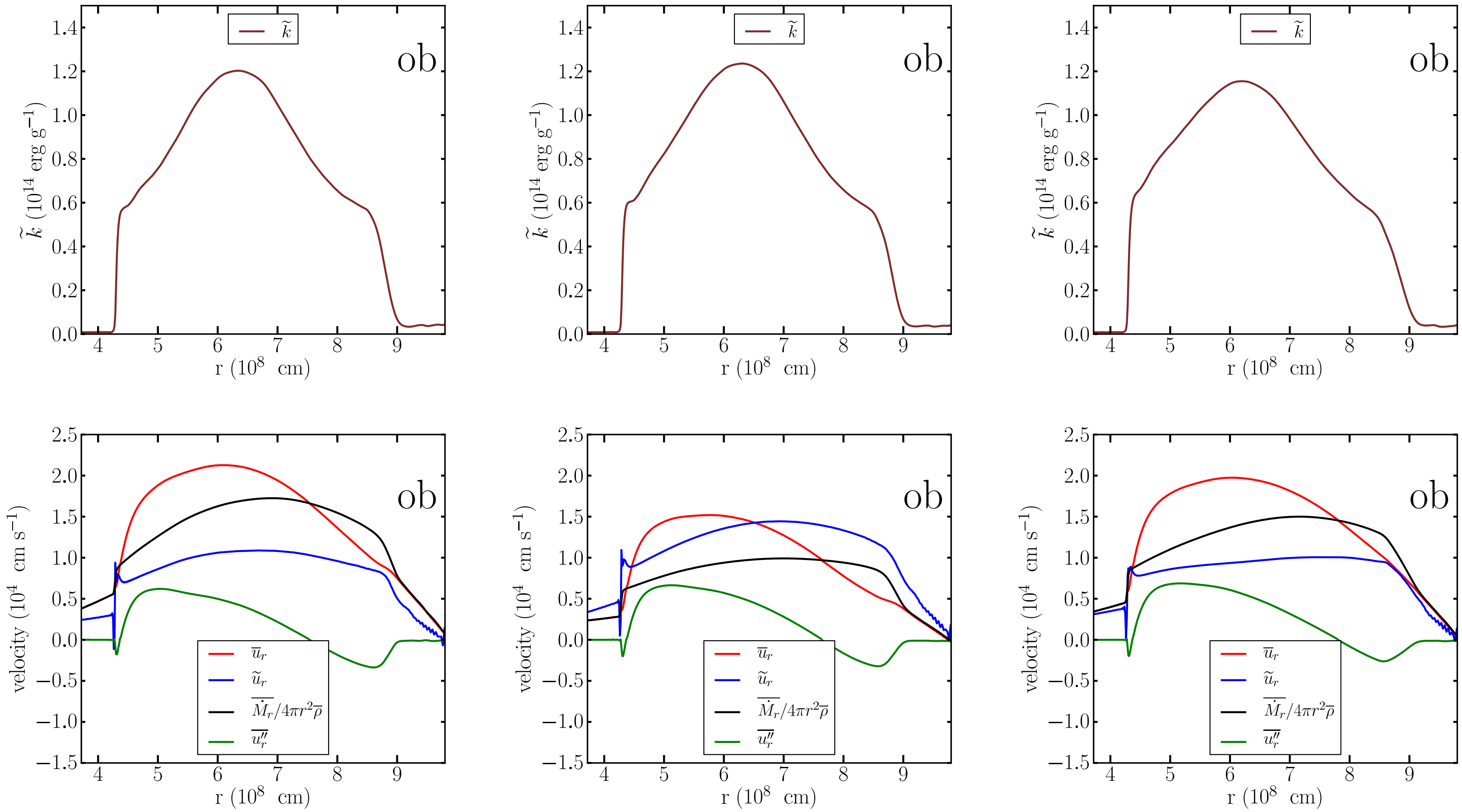

Figure 57: Mean turbulent kinetic energy equation (upper panels) and mean velocities from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

9 Depth dependence

9.1 Oxygen burning shell model

Mean continuity equation and mean radial momentum equation

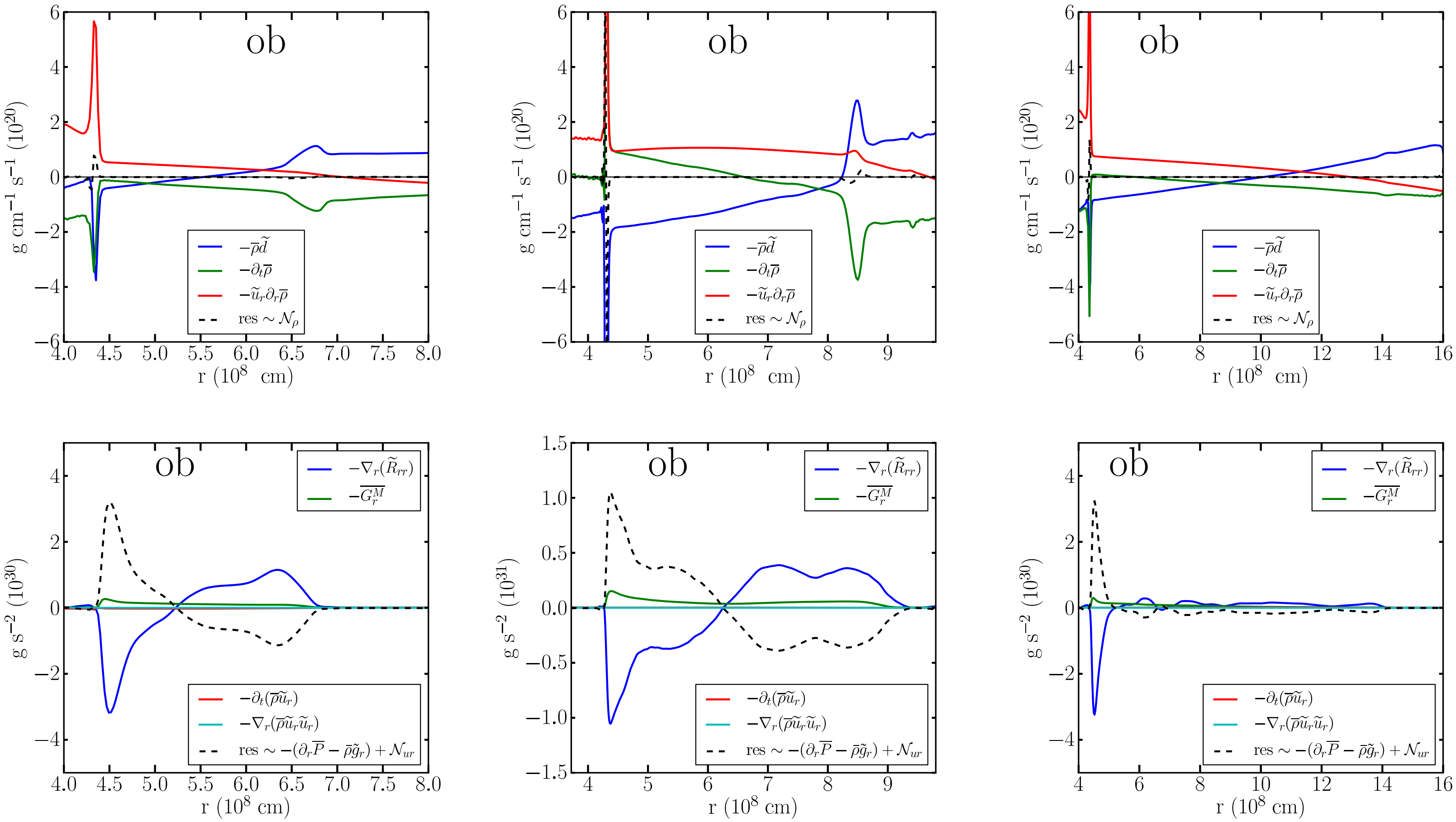

Figure 58: Mean continuity equation (upper panels) and radial momentum equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right)

Mean azimuthal and polar momentum equation

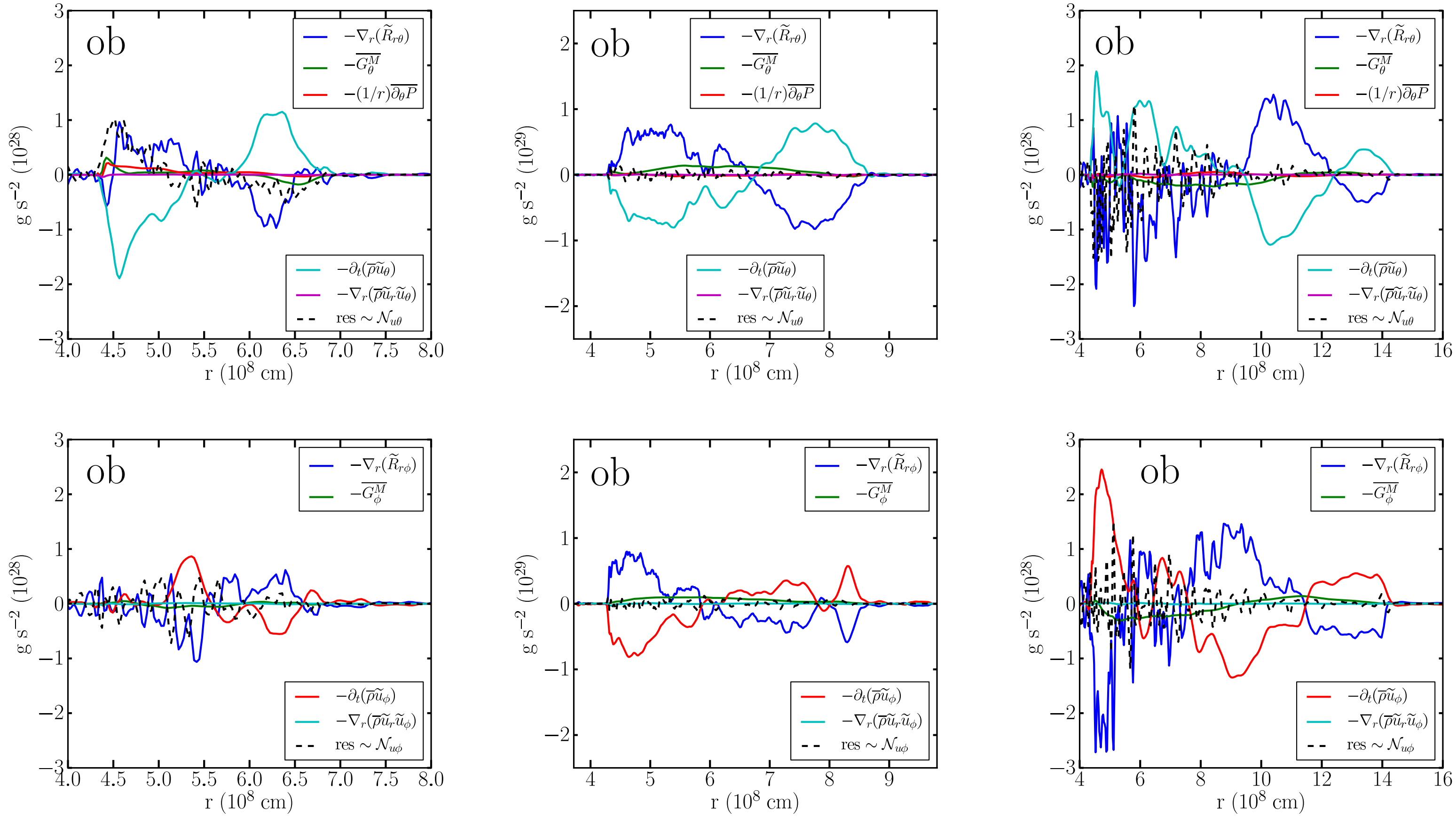

Figure 59: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean total energy equation and turbulent kinetic energy equation

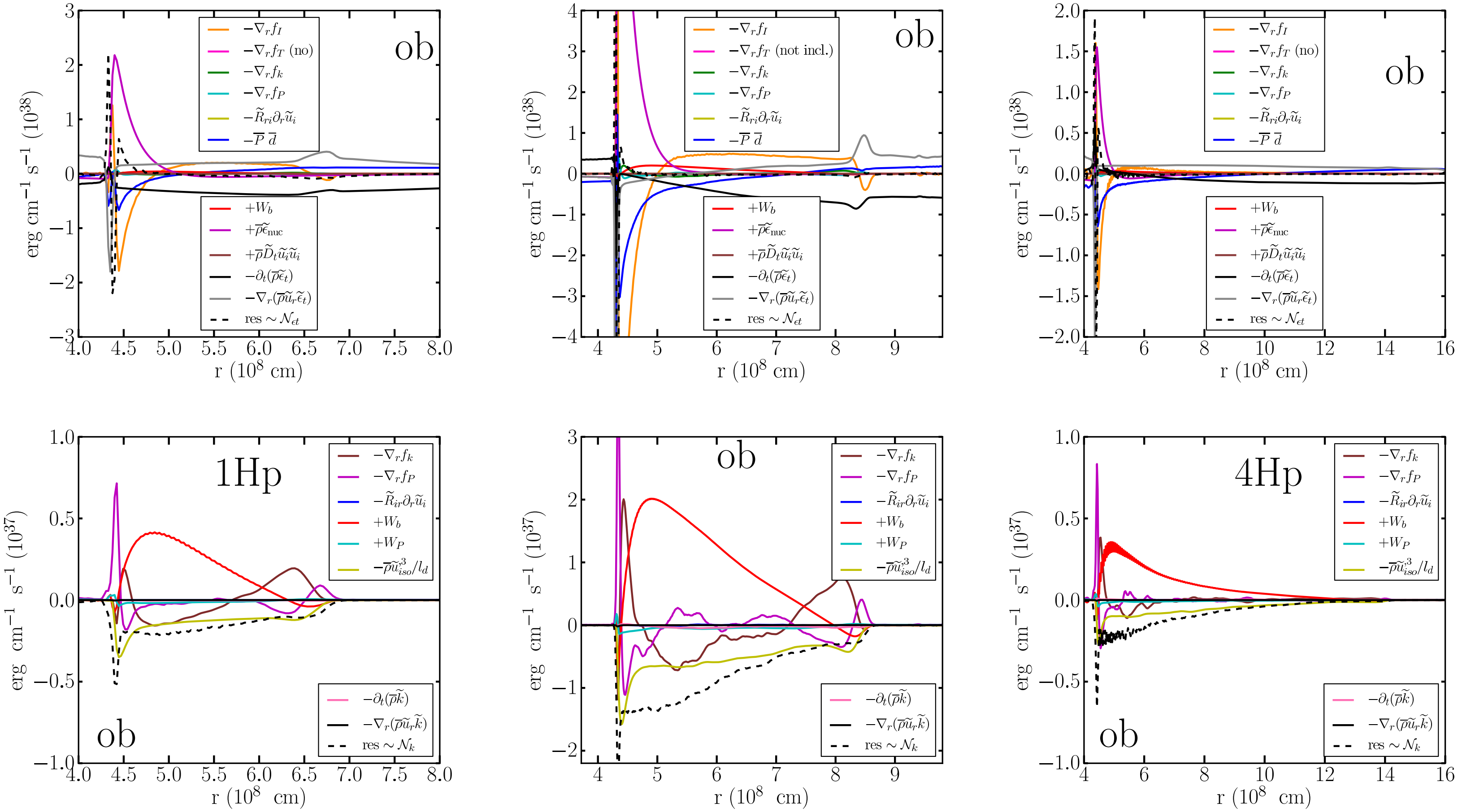

Figure 60: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent kinetic energy equation (radial + horizontal part)

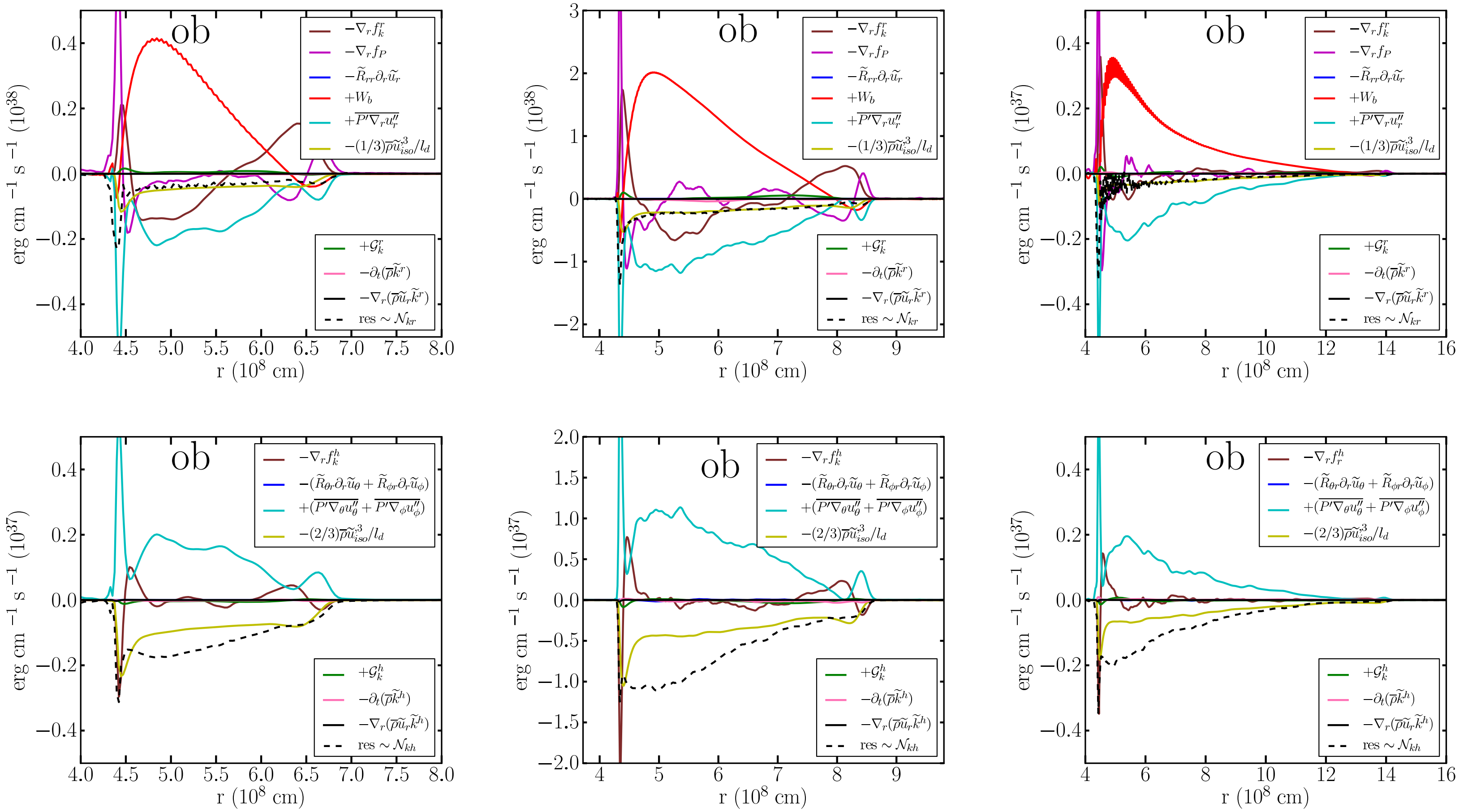

Figure 61: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent mass flux equation and mean density-specific volume covariance equation

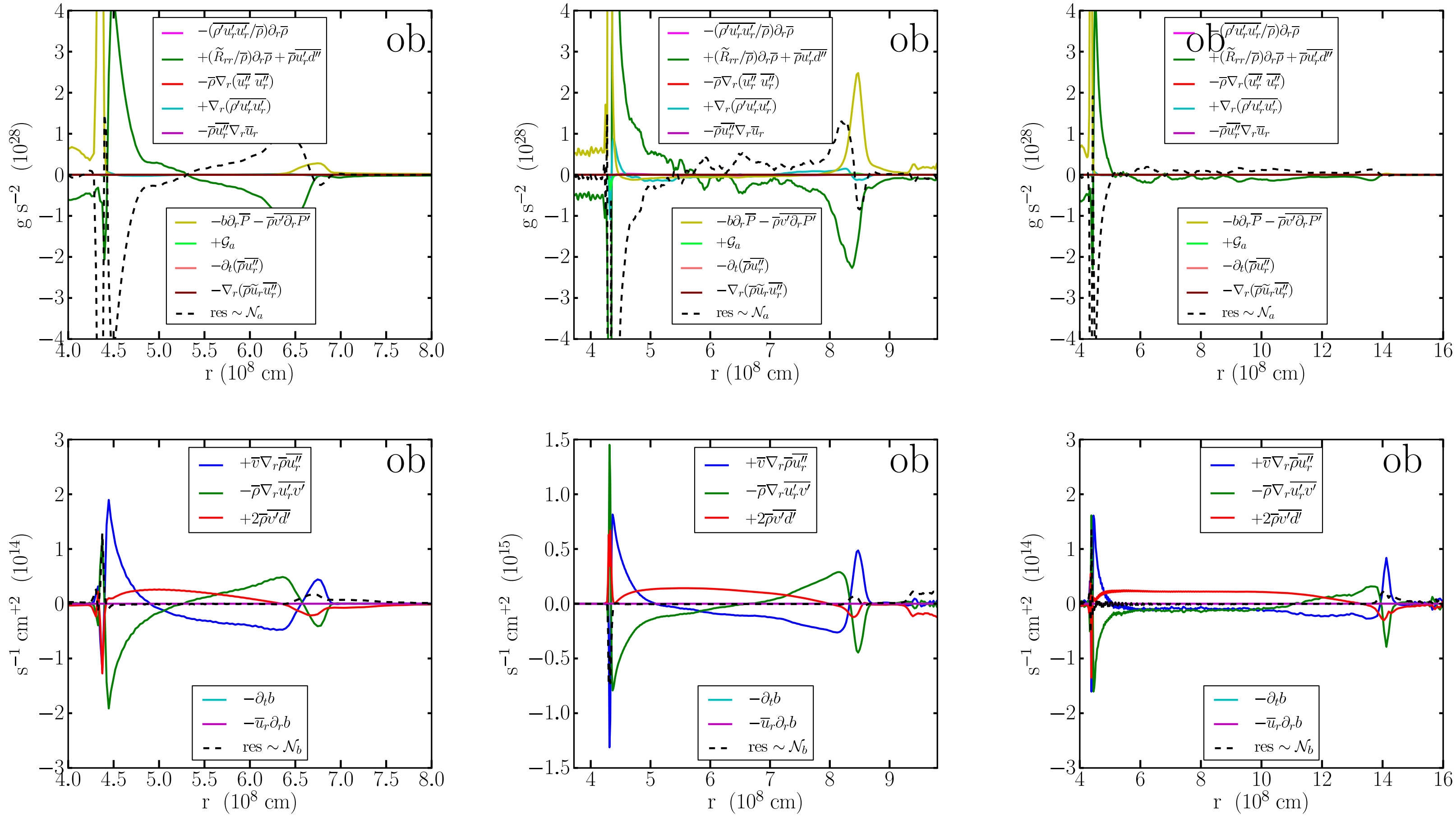

Figure 62: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean specific angular momentum equation and internal energy flux equation

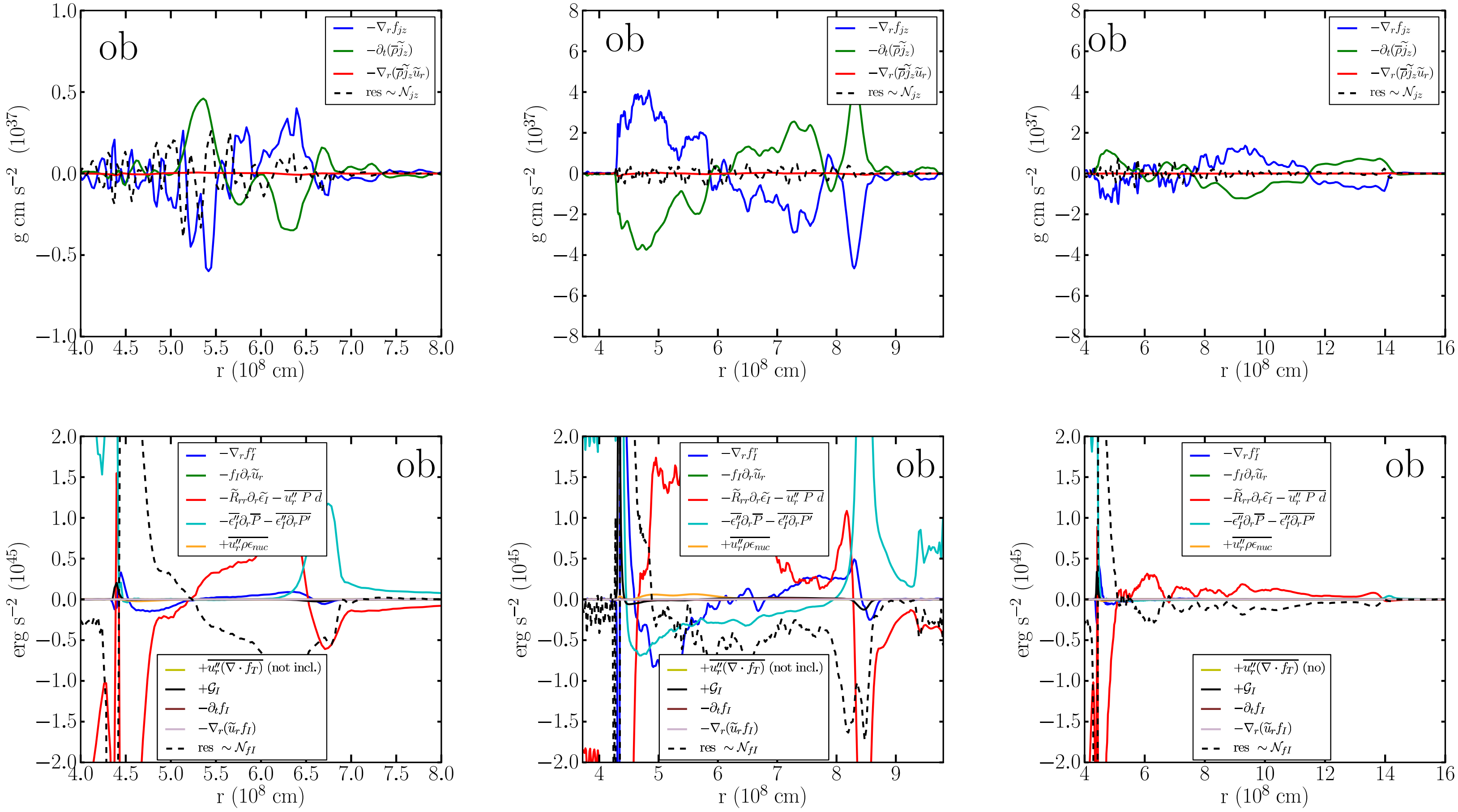

Figure 63: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean entropy equation and mean entropy flux equation

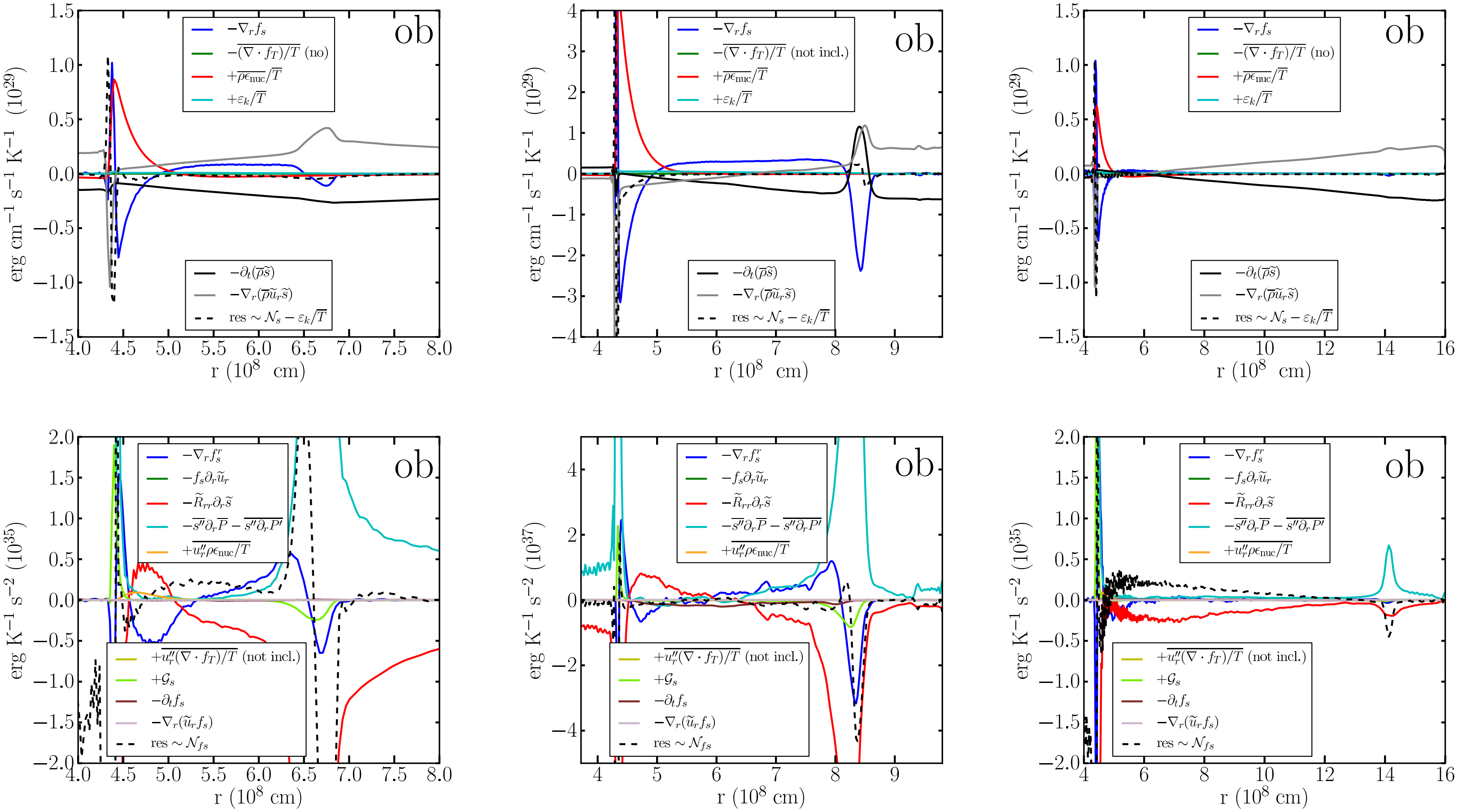

Figure 64: Mean entropy equation (upper panels) and mean entropy flux equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent kinetic energy and mean velocities

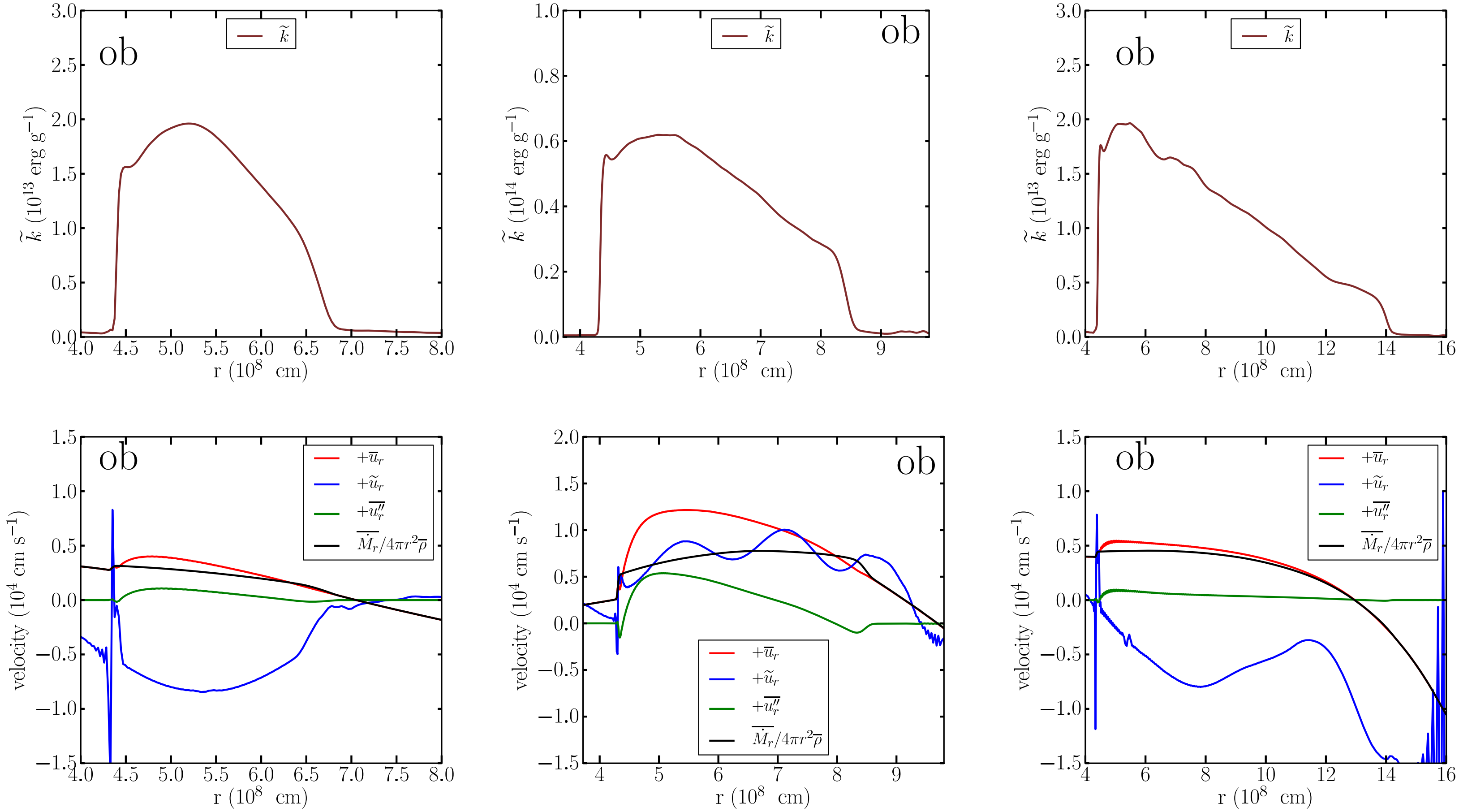

Figure 65: Mean turbulent kinetic energy (upper panels) and mean velocities (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

9.2 Red giant convection envelope model

Mean continuity equation and mean radial momentum equation

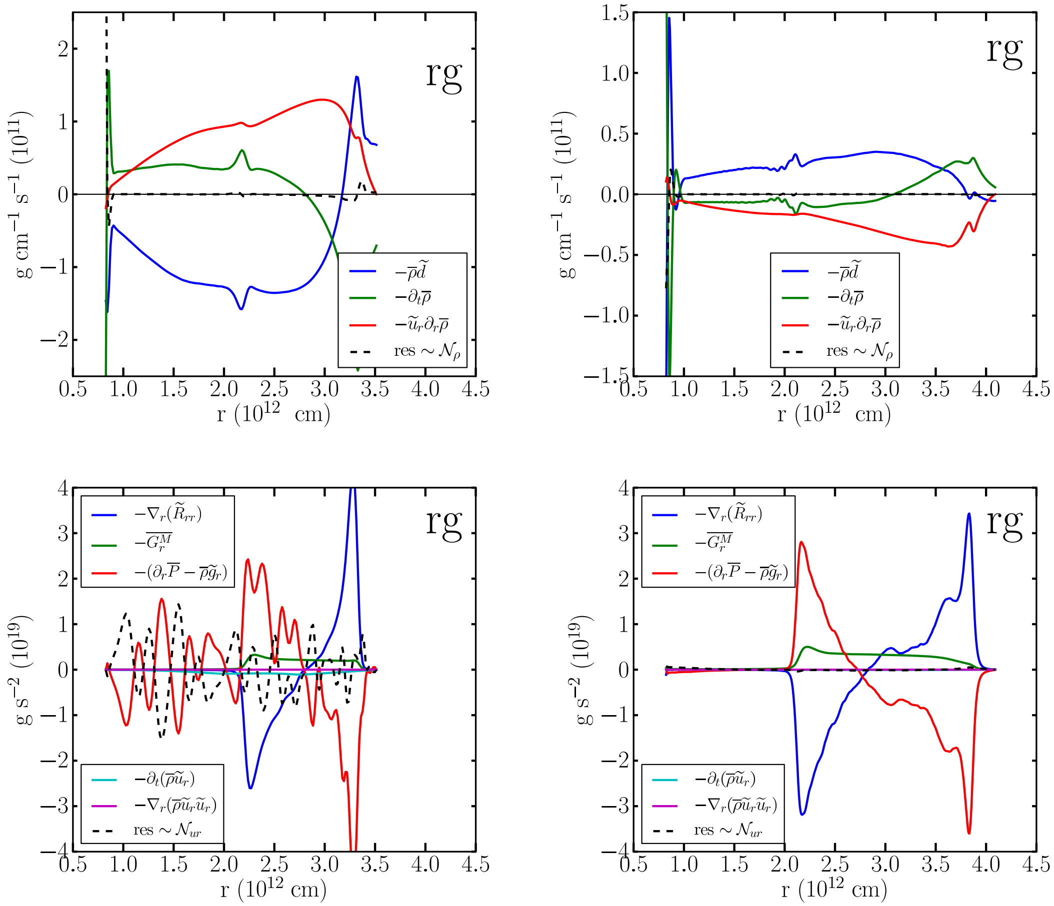

Figure 66: Mean continuity equation (upper panels) and radial momentum equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean azimuthal and polar momentum equation

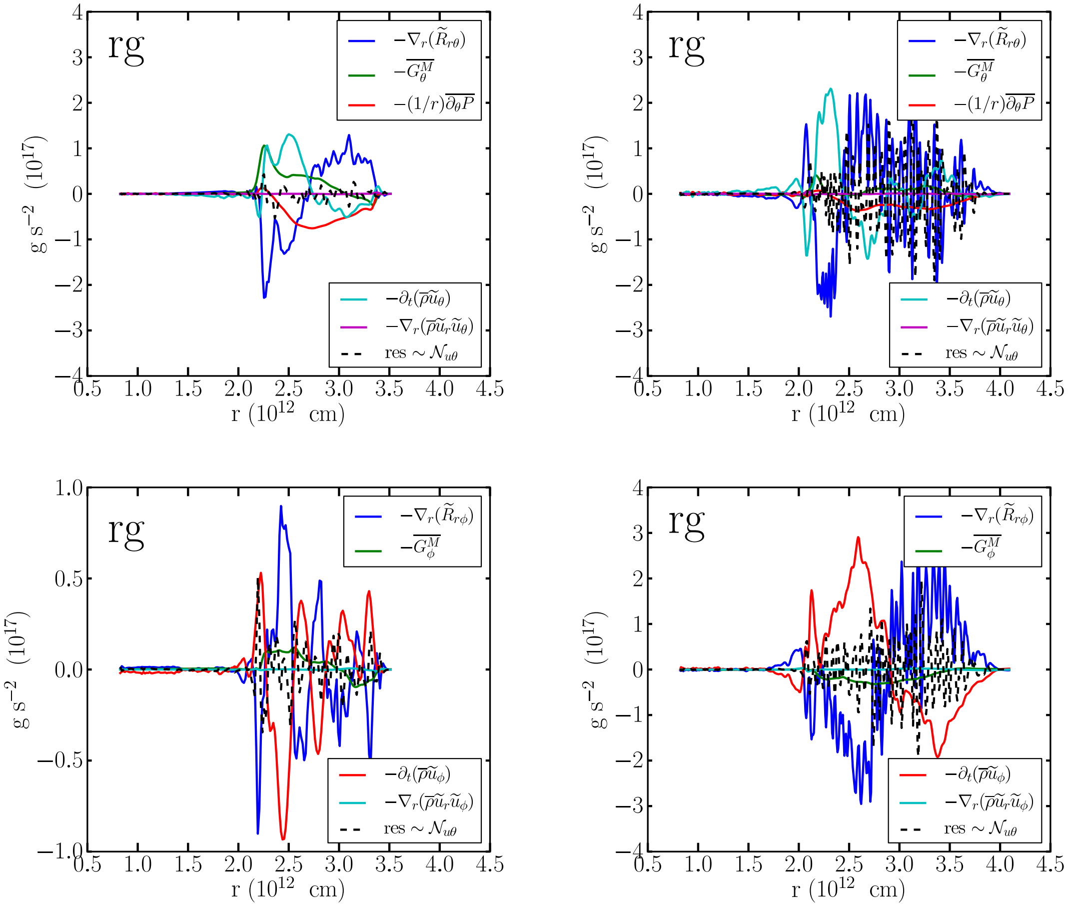

Figure 67: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean total energy equation and mean turbulent kinetic energy equation

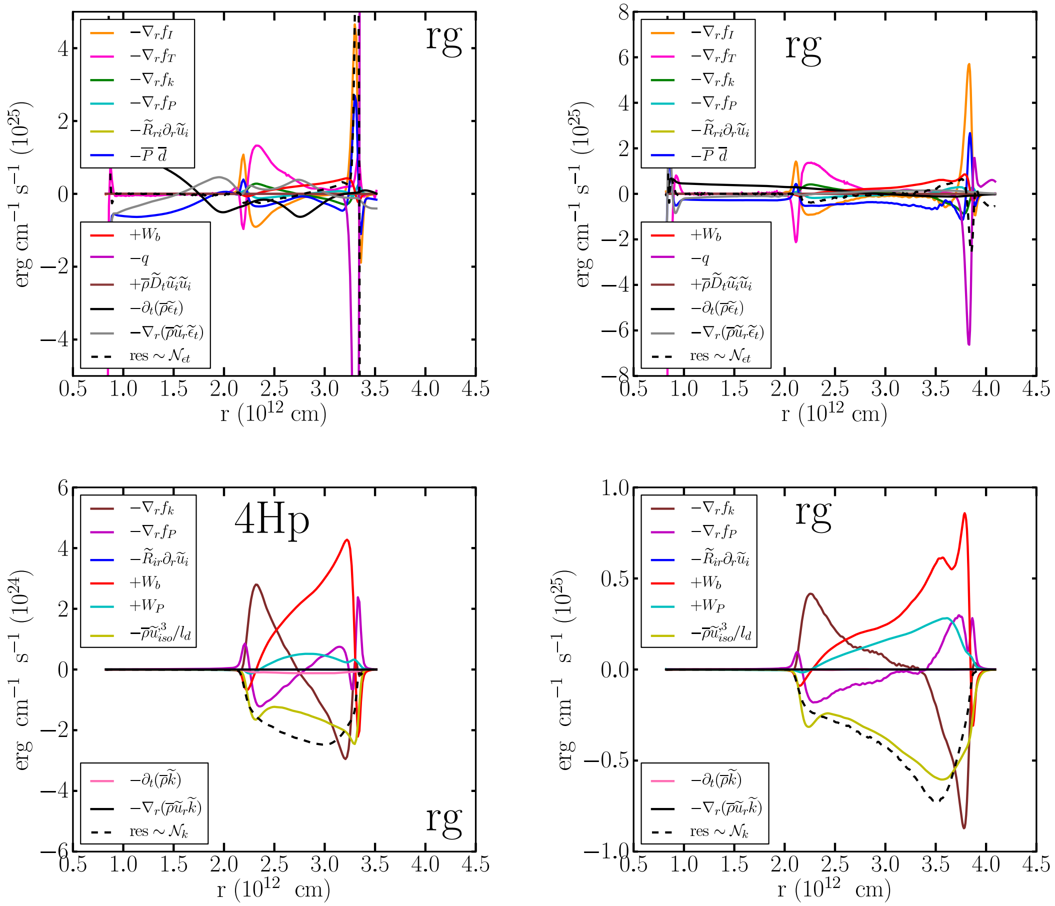

Figure 68: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent kinetic energy equation (radial + horizontal part)

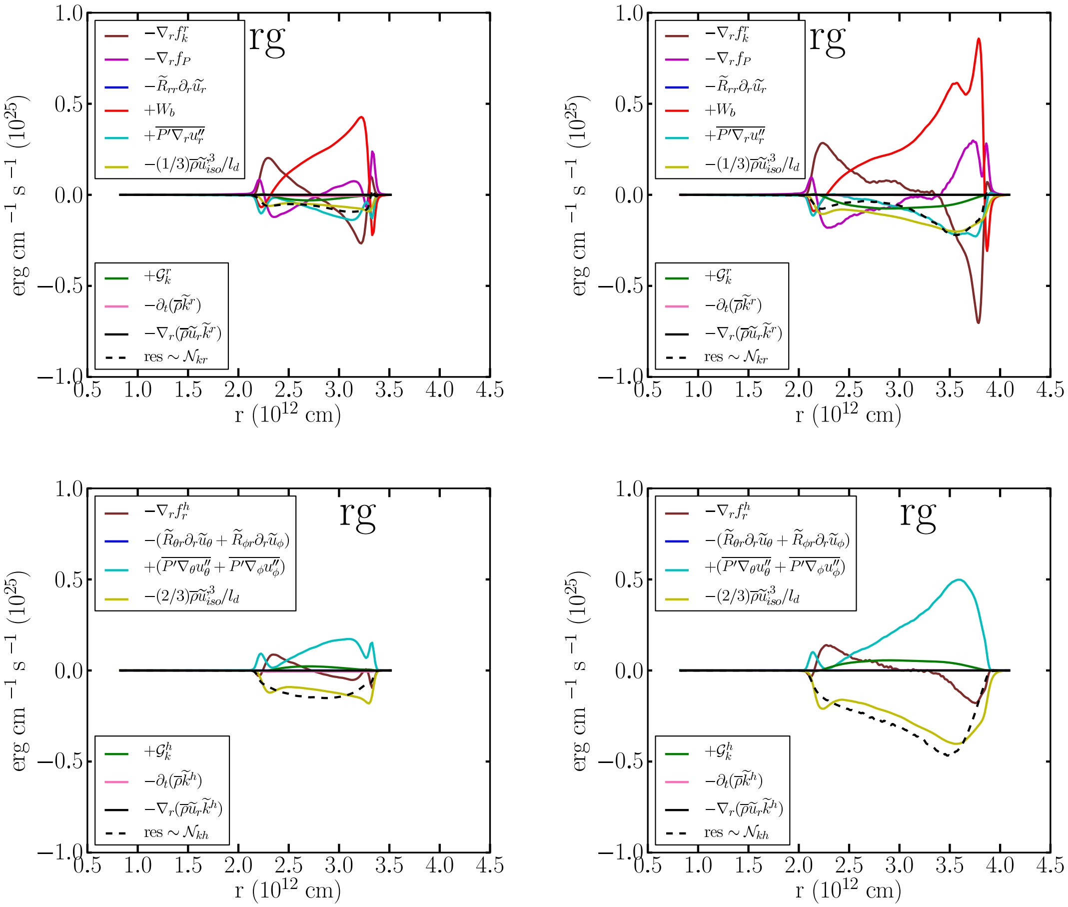

Figure 69: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent mass flux and mean density-specific volume covariance equation

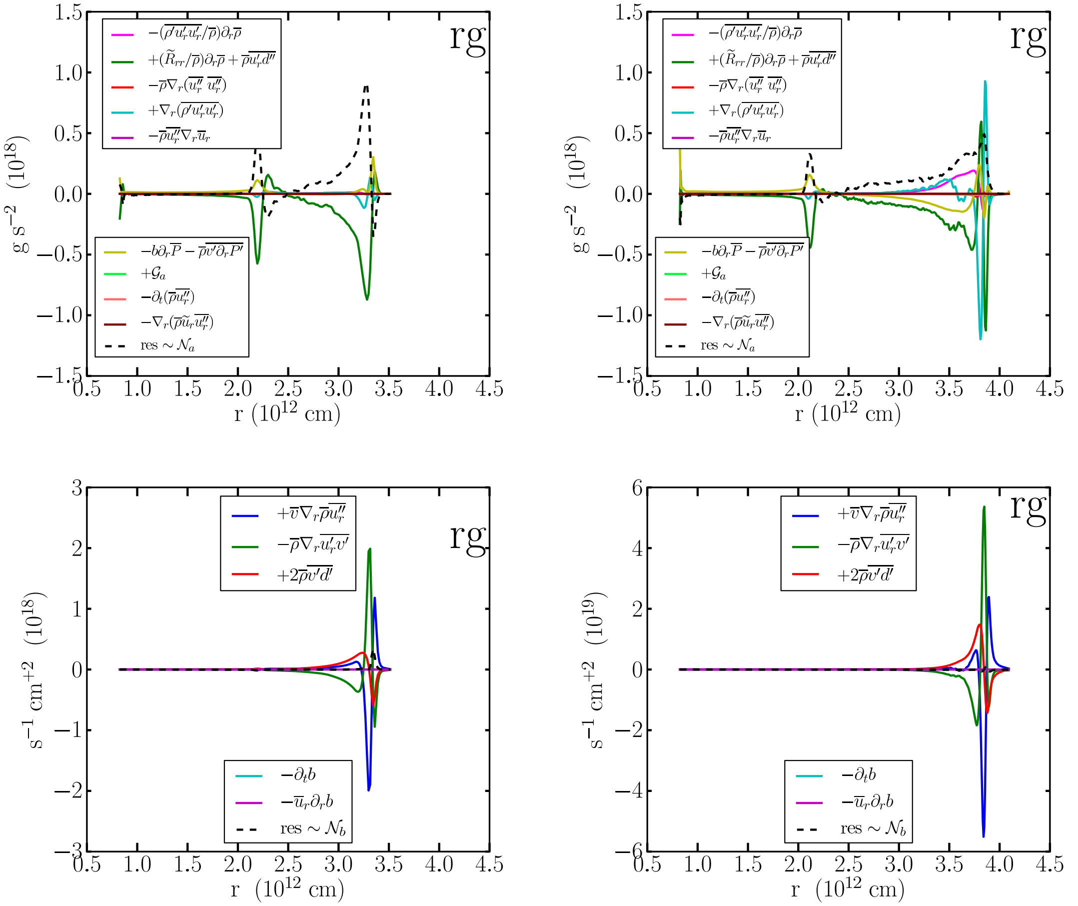

Figure 70: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean specific angular momentum equation and internal energy flux equation

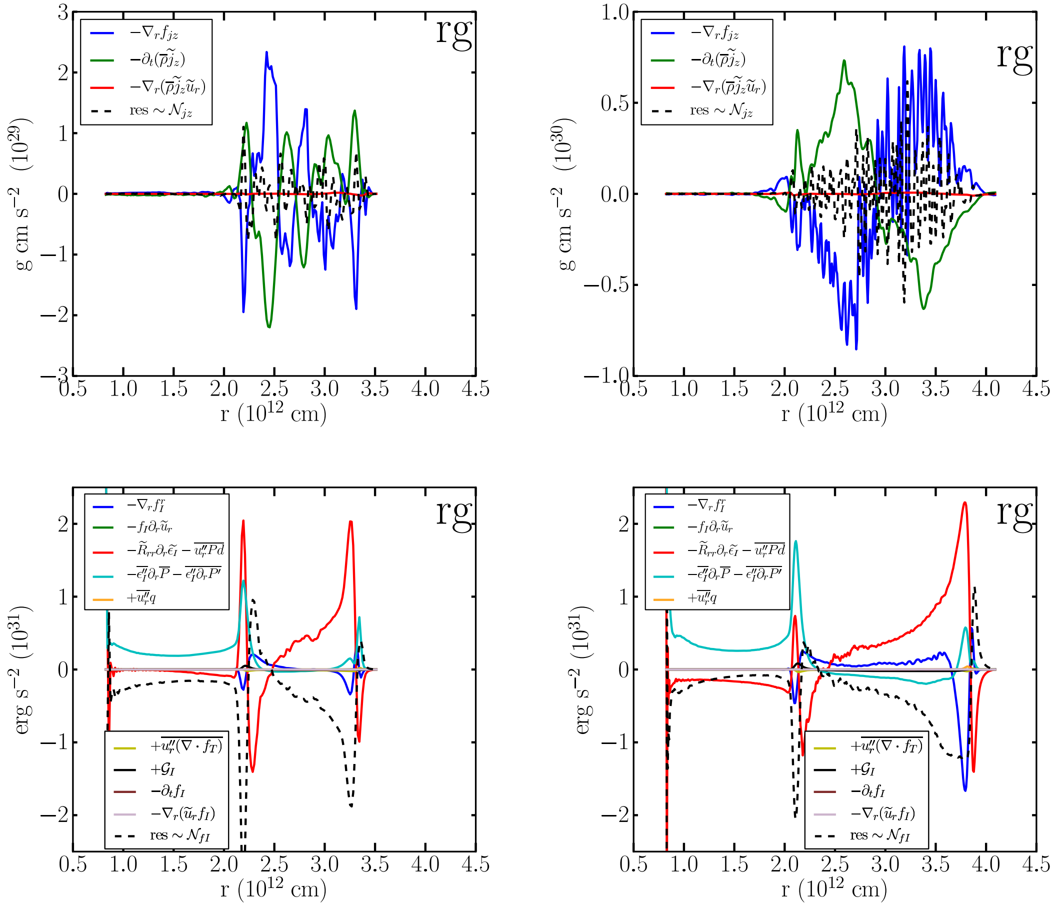

Figure 71: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean entropy equation and mean entropy flux equation

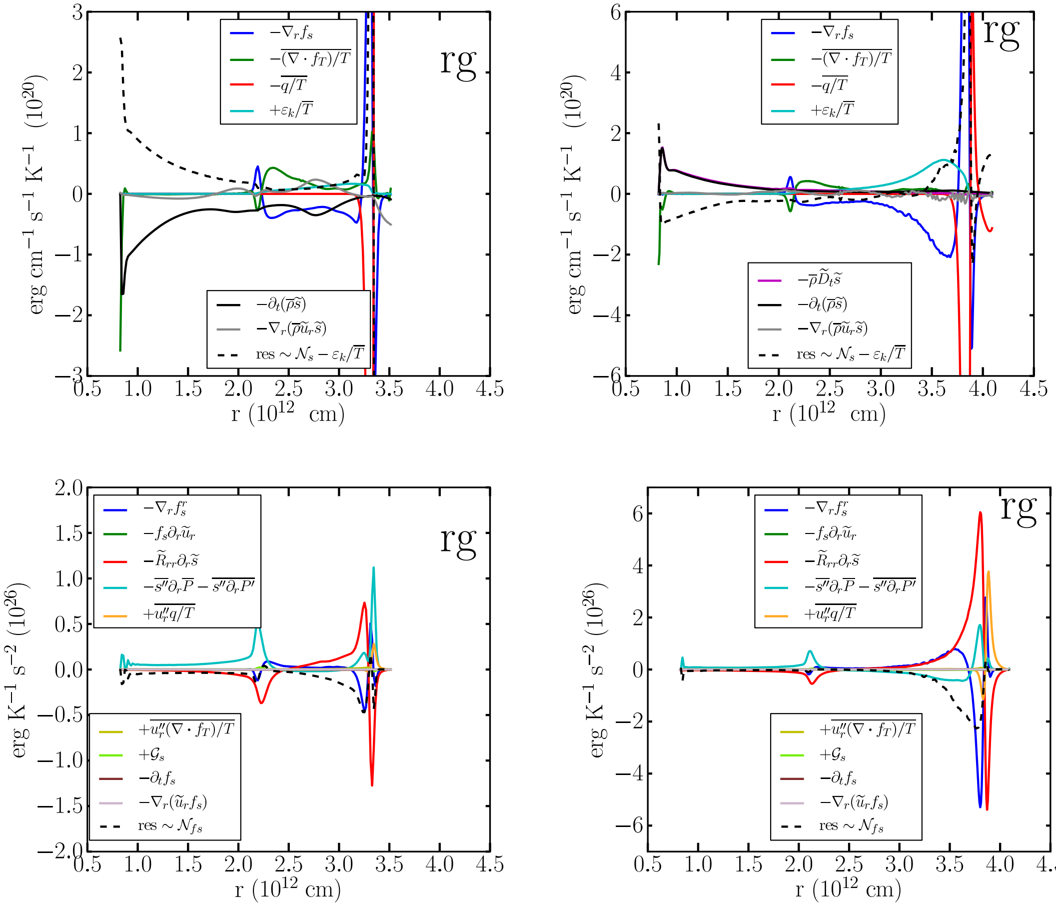

Figure 72: Mean entropy equation (upper panels) and mean entropy flux equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent kinetic energy and mean velocities

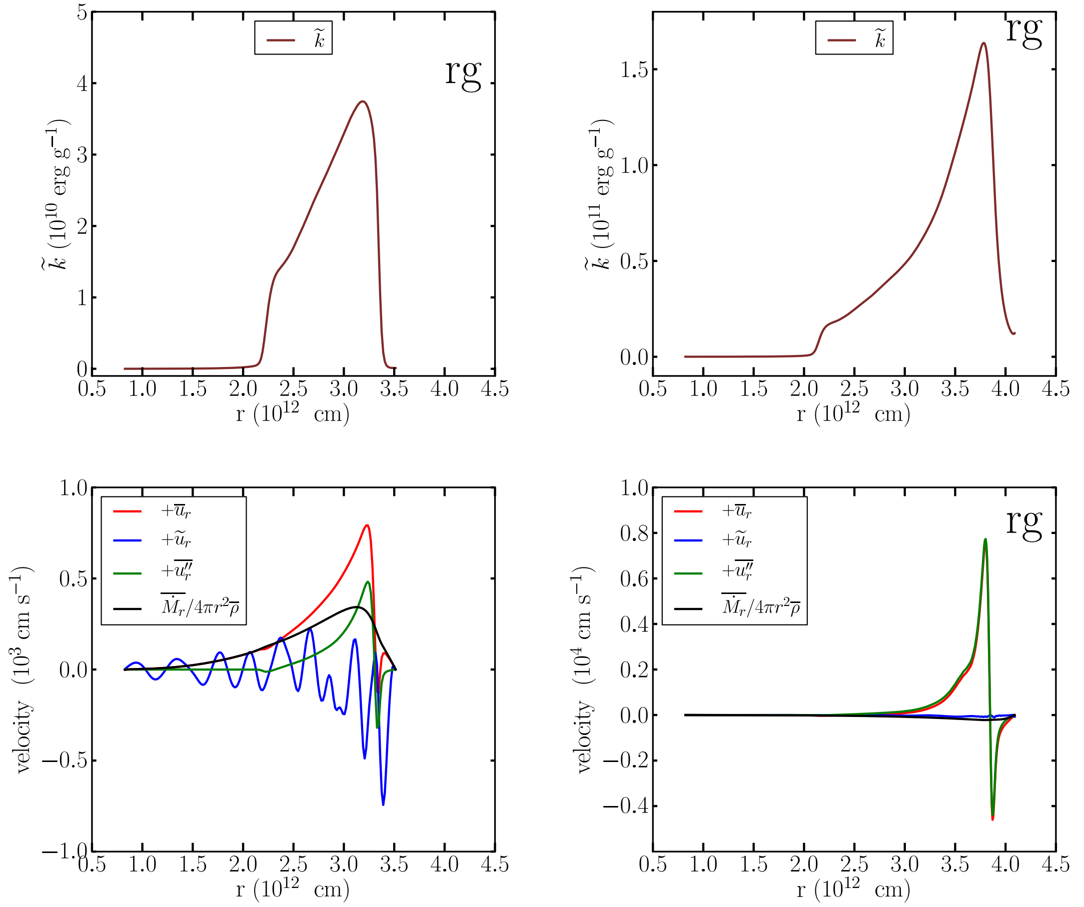

Figure 73: Mean turbulent kinetic energy (upper panels) and mean velocities (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

10 Position of convection driving source

10.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

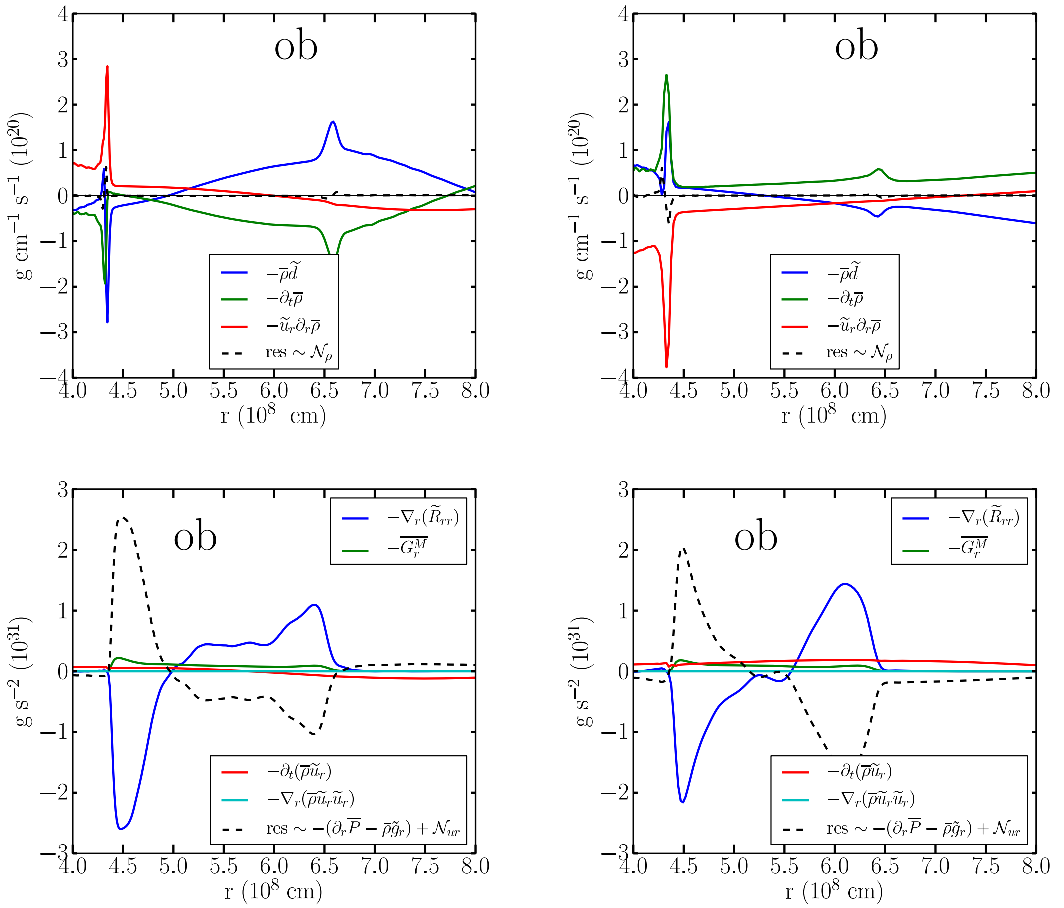

Figure 74: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean azimuthal and polar momentum equation

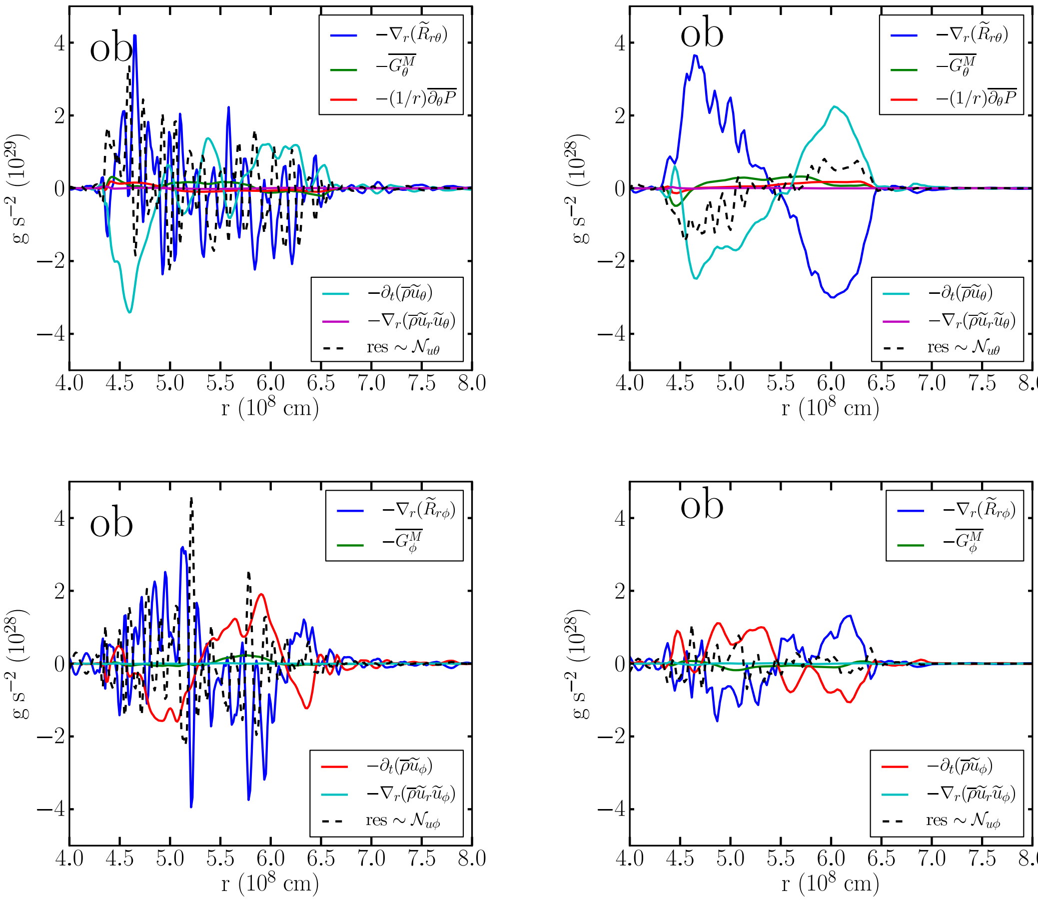

Figure 75: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean total energy equation and mean turbulent kinetic energy equation

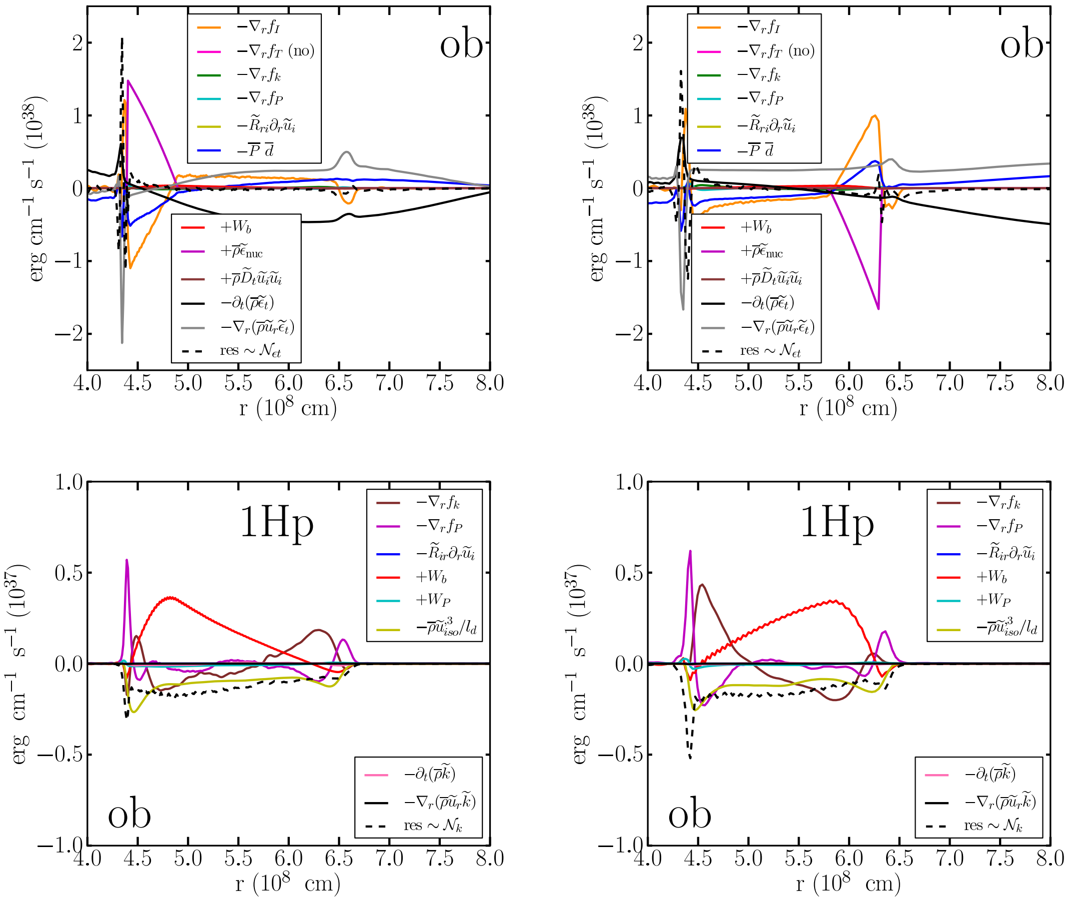

Figure 76: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent kinetic energy equations (radial + horizontal part)

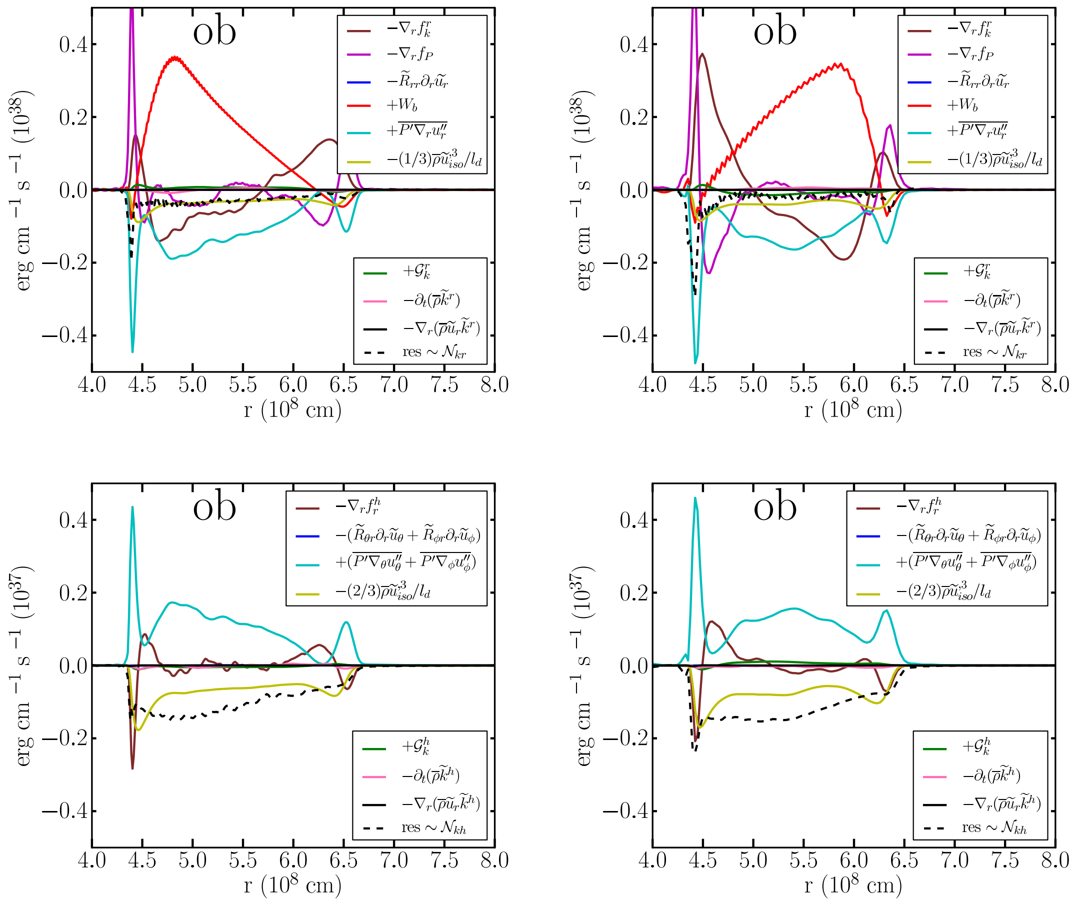

Figure 77: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent mass flux and mean density-specific volume covariance equations

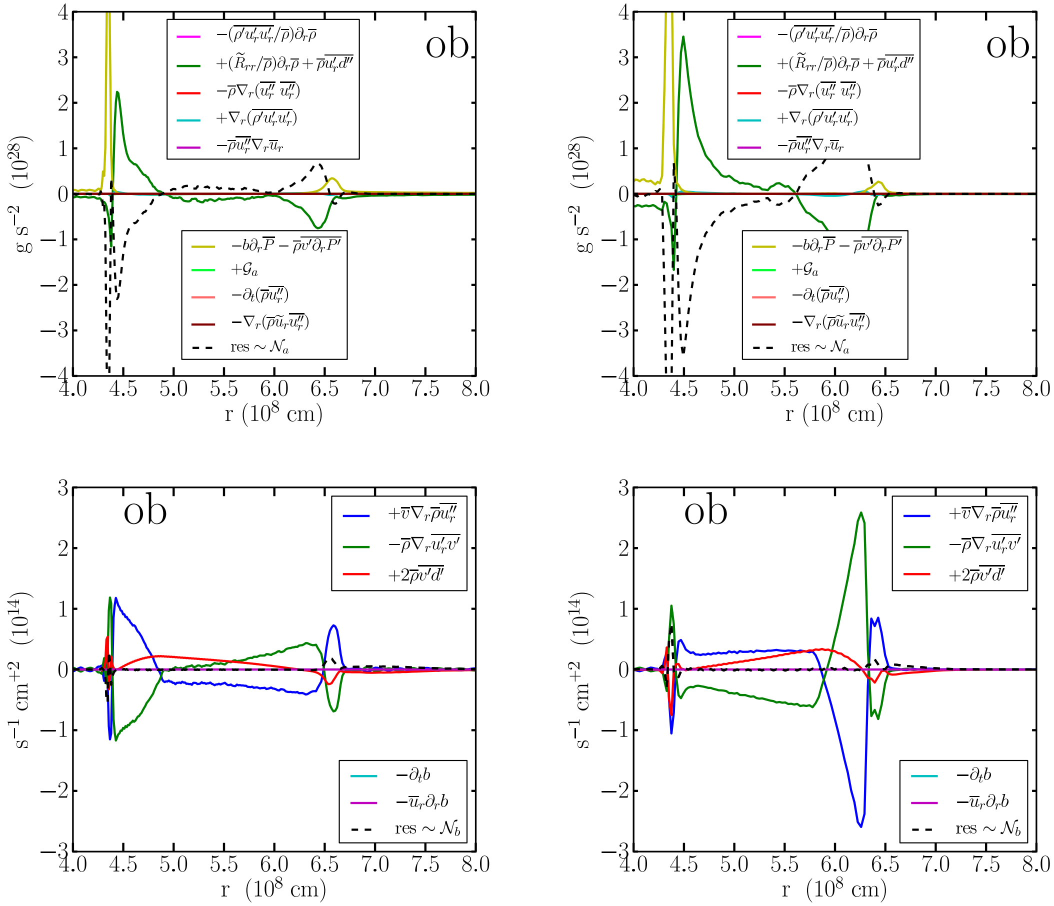

Figure 78: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean specific angular momentum equation and mean internal energy flux equation

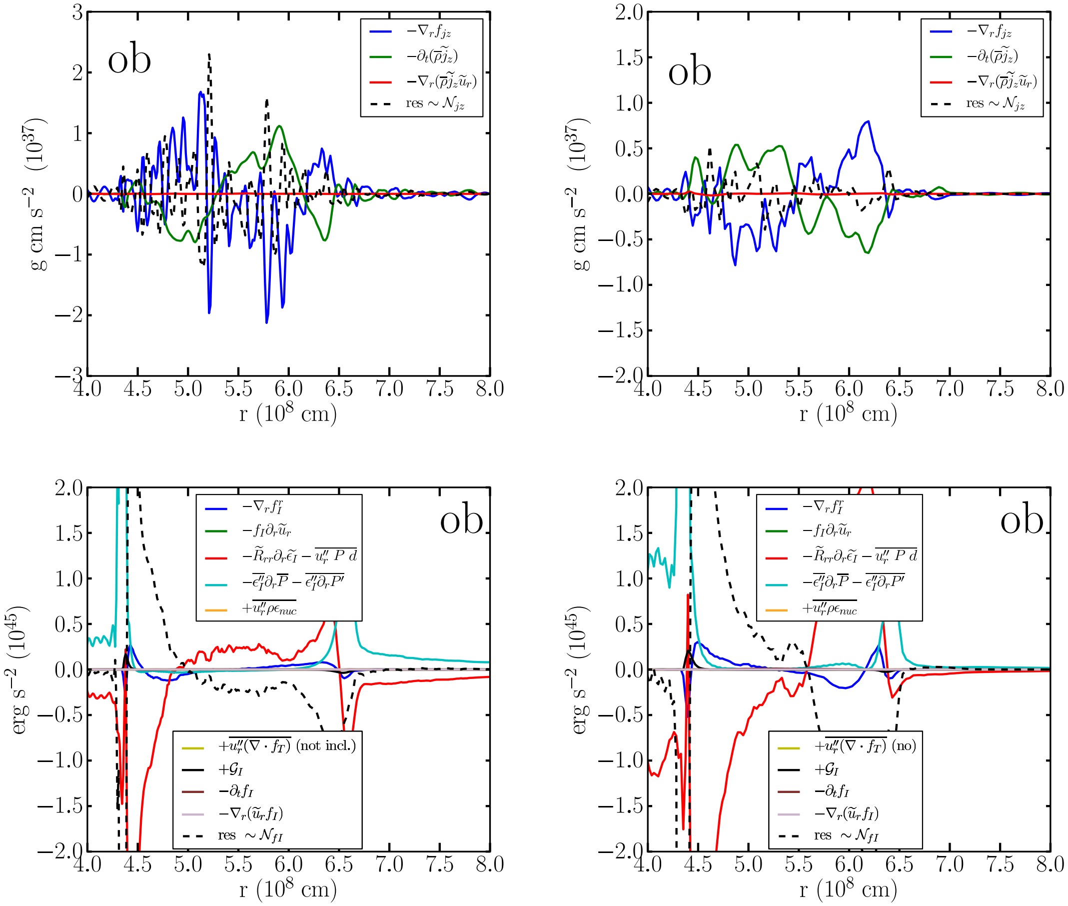

Figure 79: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent kinetic energy and mean velocities

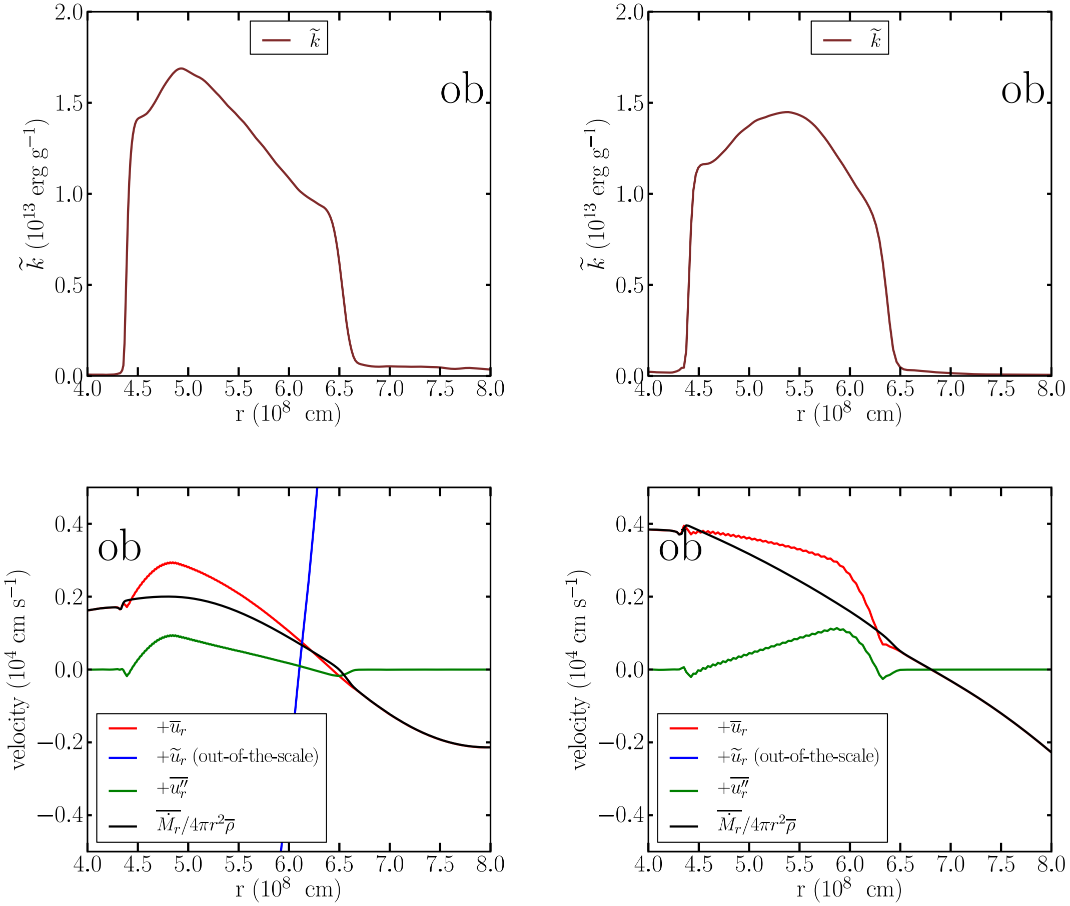

Figure 80: Mean turbulent kinetic energy (upper panels) and mean background velocities (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

11 Analysis of some turbulence one-point closure models

11.1 Downgradient approximation theory background

$$\widetilde{F}_i^q \sim -\Gamma_t \frac{\partial \widetilde{q}}{\partial x_i} \quad (\Gamma_t \text{ is turbulence diffusivity and } \widetilde{F}_i^q = \overline{\rho q'' u_i''} \text{ is a flux of } q)$$

- can be derived from a transport equation of a diffusive passive scalar (Harlow & Hirt, 1969; Daly & Harlow, 1970):

$$\partial_t \widetilde{F}_i^q - \overline{u_i'' q'' \partial_t \rho} - \widetilde{R}_{in} \partial_n \widetilde{q} + \widetilde{u}_n \overline{\rho \partial_n u_i'' q''} + \widetilde{F}_n^q \partial_n \widetilde{u}_i + \partial_n \overline{\rho u_n'' u_i'' q''} - \overline{u_i'' q'' \partial_n \rho u_n''} = -\overline{q''} \partial_i \overline{P} - \overline{q'' \partial_i P'} + \partial_n (\overline{\lambda \rho u_i'' \partial_n q''}) + f \widetilde{F}_i^q$$

where q is the passive scalar governed by a diffusion equation $D_t q = \lambda \nabla^2 q$.

It implies, that the downgradient approximation holds only for:

- a transport of a diffusive passive scalar
- a flow in steady state ($\partial_t \widetilde{F}_i^q = 0$)
- an incompressible flow ($\partial_t \rho = 0$)
- a flow with no background velocities ($\widetilde{u}_i = 0$)
- a flow with no pressure-scalar correlations ($\overline{q''} \partial_i \overline{P} = \overline{q'' \partial_i P'} = 0$)
- a homogeneous flow ($\partial_n \overline{\rho u_n'' u_i'' q''} = 0$)
- an isotropic flow (decay-rate assumption: $\overline{\partial_n q'' \partial_n \rho u_i''} \sim f \widetilde{F}_i^q$)

But, stellar turbulent convection is:

- stratified (not homogeneous)
- anisotropic
- compressible on expanding/contracting background
- **downgradient approximation is not suitable for modelling stellar processes**

11.2 Downgradient approximations

$$f_k = (C\ \rho\ \sqrt{\widetilde{k}}\ l_d)\ \partial_r \widetilde{k} \qquad f_I = (C\ \rho\ \sqrt{\widetilde{k}}\ l_d)\ \partial_r \widetilde{\epsilon}_I \qquad f_I = (C\ \rho\ \widetilde{k}^2/\varepsilon_k)\ \partial_r \widetilde{\epsilon}_I \tag{78}$$

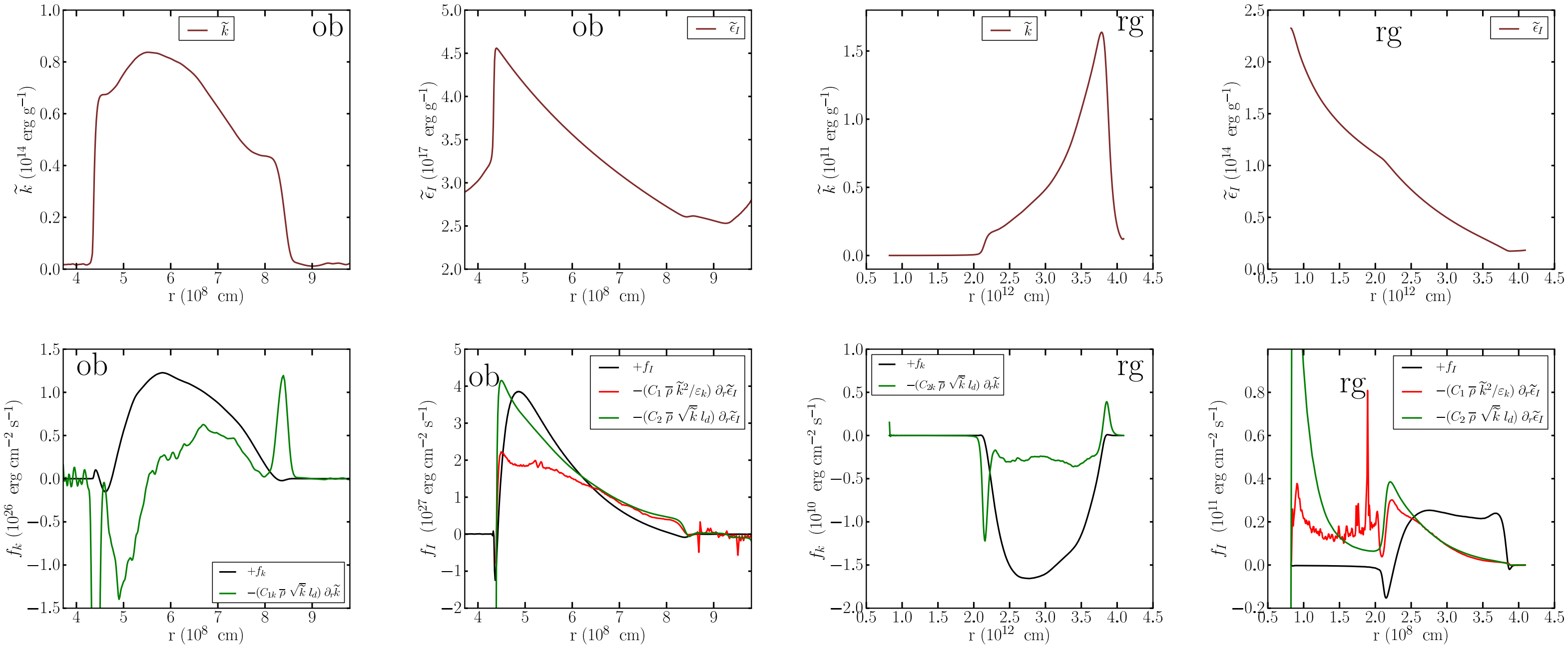

Figure 81: Profiles of mean turbulent kinetic energy and mean internal energy (upper panels) and various downgradient approximations to turbulent kinetic energy flux and internal energy flux (lower panels) derived from oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg). l_d is dissipation length scale and C is model constant.

11.3 Various approximations taken from Besnard-Harlow-Rauenzahn (BHR) model

$$\overline{\rho' u'_r u'_r} = C_{1a} \frac{l_d}{\sqrt{\widetilde{k}}} \widetilde{R}_{rr} \partial_r \overline{u''_r} \qquad \overline{v' \partial_r P'} = -C_{2a} \frac{\sqrt{k}}{l_d} \overline{u''_r} \qquad \overline{v' u'_r} = -C_{1b} \frac{l_d}{\sqrt{k}} \frac{\widetilde{R}_{rr}}{\overline{\rho}} \frac{\partial}{\partial r} \left(\frac{1+b}{\overline{\rho}} \right) \qquad \overline{v' d'} = -C_{2b} \frac{\sqrt{k}}{l_d} \frac{b}{\overline{\rho}}$$

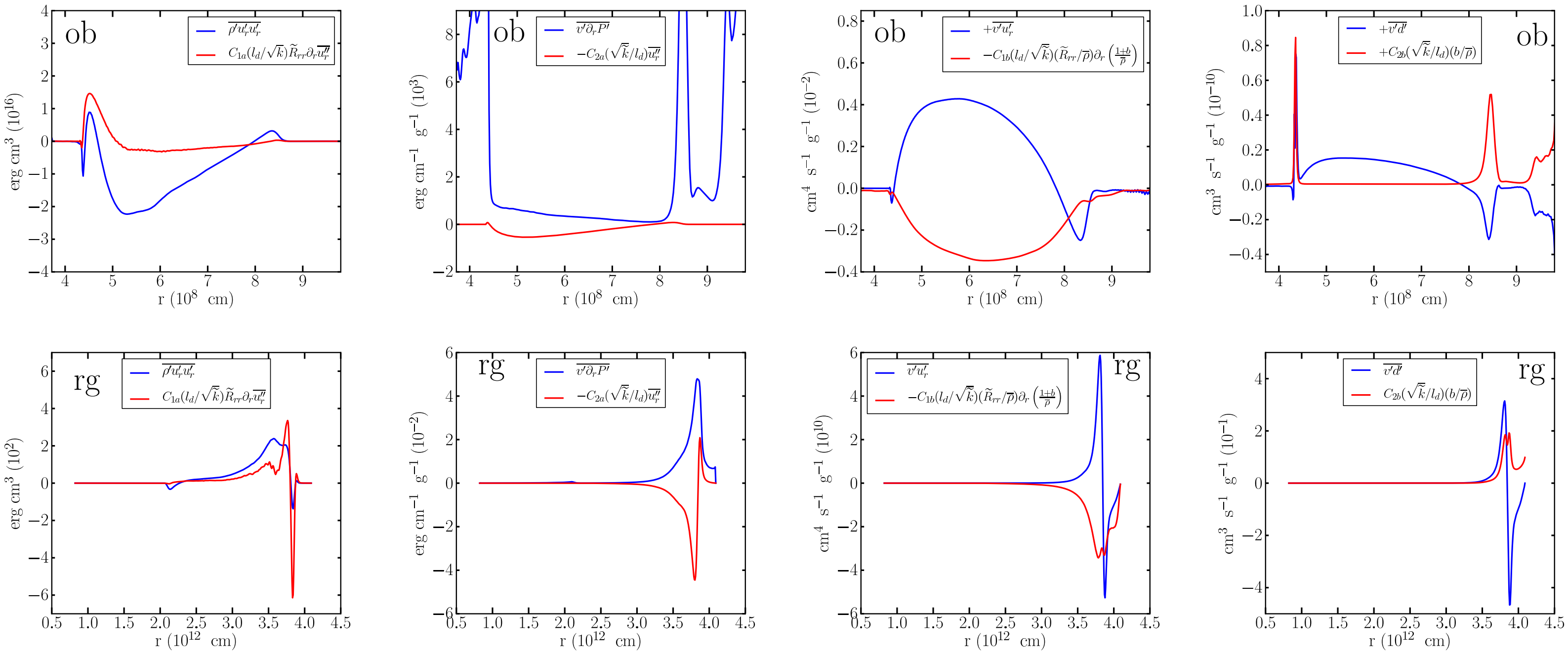

Figure 82: Various approximations taken from Besnard-Harlow-Rauenzahn (BHR) model derived from oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg). l_d is dissipation length scale and C is model constant.

11.4 Quasi-normal approximation and decay-rate assumption model

$$\overline{a'b'c'd'} = \overline{a'b'}\ \overline{c'd'} + \overline{a'c'}\ \overline{b'd'} + \overline{a'd'}\ \overline{b'c'} \qquad \overline{a'b'} = (l_d^2/\Delta)\overline{\partial_k a' \partial_k b'} \qquad \text{(original formulations)} \tag{79}$$

$$a''\widetilde{b''c''}d'' = \widetilde{a''b''}\ \widetilde{c''d''} + \widetilde{a''c''}\ \widetilde{b''d''} + \widetilde{a''d''}\ \widetilde{b''c''} \qquad \widetilde{a''b''} = (l_d^2/\Delta)\overline{\partial_k a'' \partial_k b''} \qquad \text{(assumed Favre eqvivalents)} \tag{80}$$

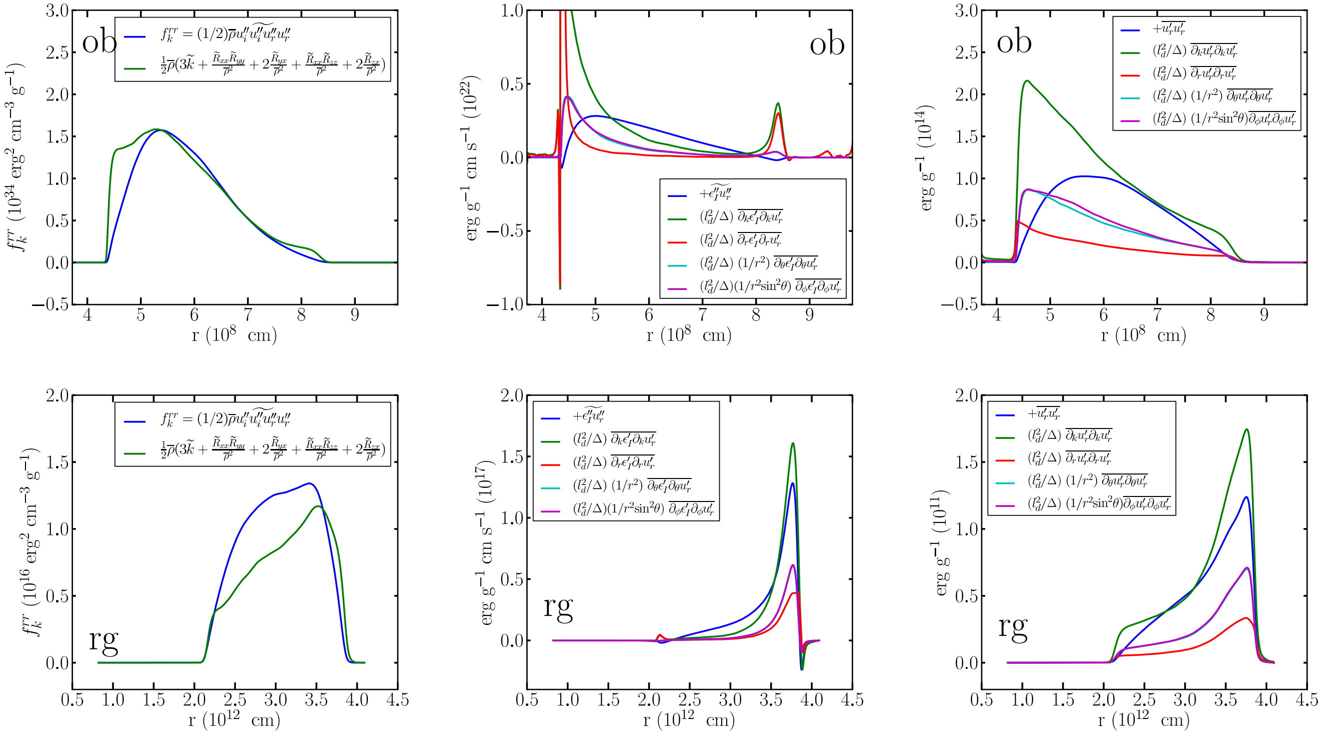

Figure 83: Quasi-normal approximations (left panels) and decay-rate assumption models (middle, right panels) derived from oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg).

11.5 Integral models

$$f_I = \int (\epsilon_{\rm nuc} + \varepsilon_k/\overline{\rho} - \overline{P}\ \overline{d}/\overline{\rho} - W_P/\overline{\rho} - T\dot{\widetilde{s}})\ dr \qquad\qquad f_k = \int (-(f_I(\Gamma_3 - 1) - f_P))/((\Gamma_3 - \Gamma_1 - 1)H_P) - \varepsilon_k\ - \nabla_r f_P + W_P\ )\ dr$$

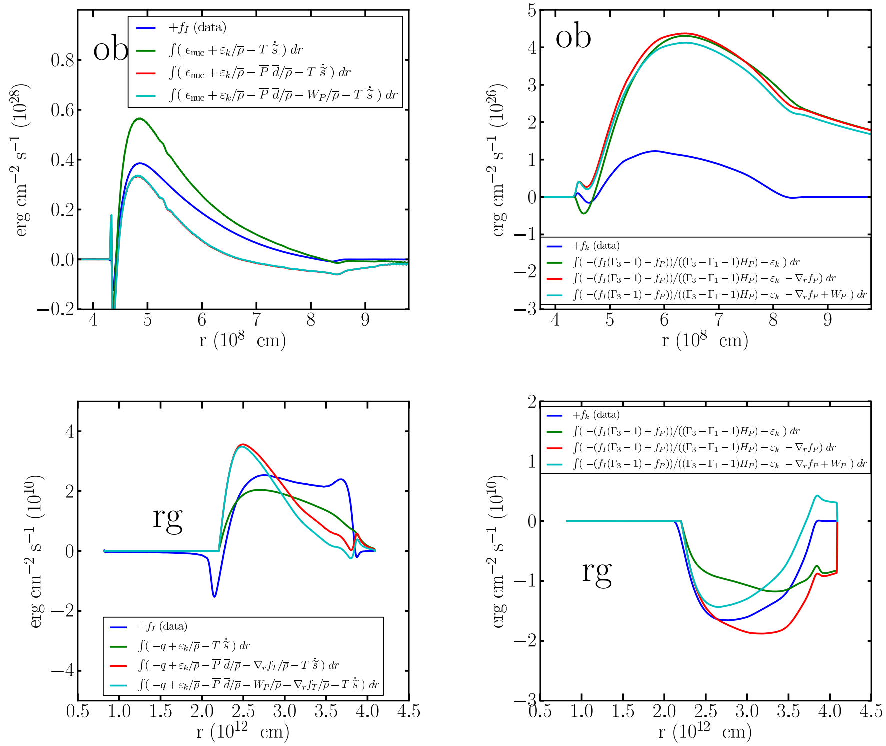

Figure 84: Integral models for internal energy flux f_I (left) and turbulent kinetic energy flux (right) derived from oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg).

12 Fourier scale analysis

We perform scale analysis on fluctuation correlations containing two constituents by using Fourier analysis and taking advantage of Parceval's theorem. If $a''(r,\theta,\phi)\delta(r')$ and $b''(r,\theta,\phi)\delta(r')$ are real-valued functions (in our case fluctuations in a two dimensional plane at radius r') and their Fourier transforms are $\widehat{a''}$ and $\widehat{b''}$ then "energy" contribution from large and small scales to horizontally averaged correlation $a''b''$ can be calculated as:

$$E(k) = \frac{1}{\Delta\Omega}\int_\Omega a''b''d\Omega = \int_k \widehat{a''}(k)\widehat{b''}^\star(k)dk = \\ = \underbrace{\int_{k\leq k_T}\widehat{a''}(k)\widehat{b''}^\star(k)dk}_{\text{large scales}} + \underbrace{\int_{k>k_T}\widehat{a''}(k)\widehat{b''}^\star(k)dk}_{\text{small scales}} \tag{81}$$

where this integral is evaluated by summing the integrand over spherical shells of constant radius $k = \sqrt{(k_\theta^2 + k_\phi^2)}$ and unit wavenumber. k_T is a threshold wave number which separates large and small scales. Ω is our two-dimensional cut domain.

The angular size of a structure in a horizontal plane can be at a given radius and wave number k expressed as $l \sim N/k$, where N is the angular size of simulated wedge. It means that in a 45°wedge simulation will structures at wave number $k = 10$ have approximately size 4.5°.

For a separation of contribution from large and small scales to a given correlation field, we choose a length scale equivalent to 1 H_P. This number is close to average vertical correlation length scales for u'_r, ρ', P' in middle of convection zones calculated by Viallet et al. [2013]. It is about 1.5×10^8 cm in the 2 H_P oxygen burning models and 3×10^{11} cm in 7 H_P red giant models. Corresponding separation wave numbers k_T are 4 for the oxygen case and 11 for the red giant case. These separation scales should be considered as a lower limit for a size of actual large scales, as we know that they are elongated in radial direction [Meakin and Arnett, 2007]. Here, we Fourier transform our data in two dimensional horizontal planes and essentially looking only at horizontal scales.

We present our scale analysis in a form of the cumulative Fourier spectra, which are calculated according the formula:

$$\overline{E}_c(k) = \sum_{i\leq k}\overline{E(i)/E_t} \tag{82}$$

where i is a wave number and $E(i)$ defined by Eq. 81. E_t is total energy of a correlation field or in other words a horizontally averaged value of the correlation field. The over bar represent time averaging of instantaneous values. Such a spectrum shows how big energy contribution to a field comes from first k Fourier modes.

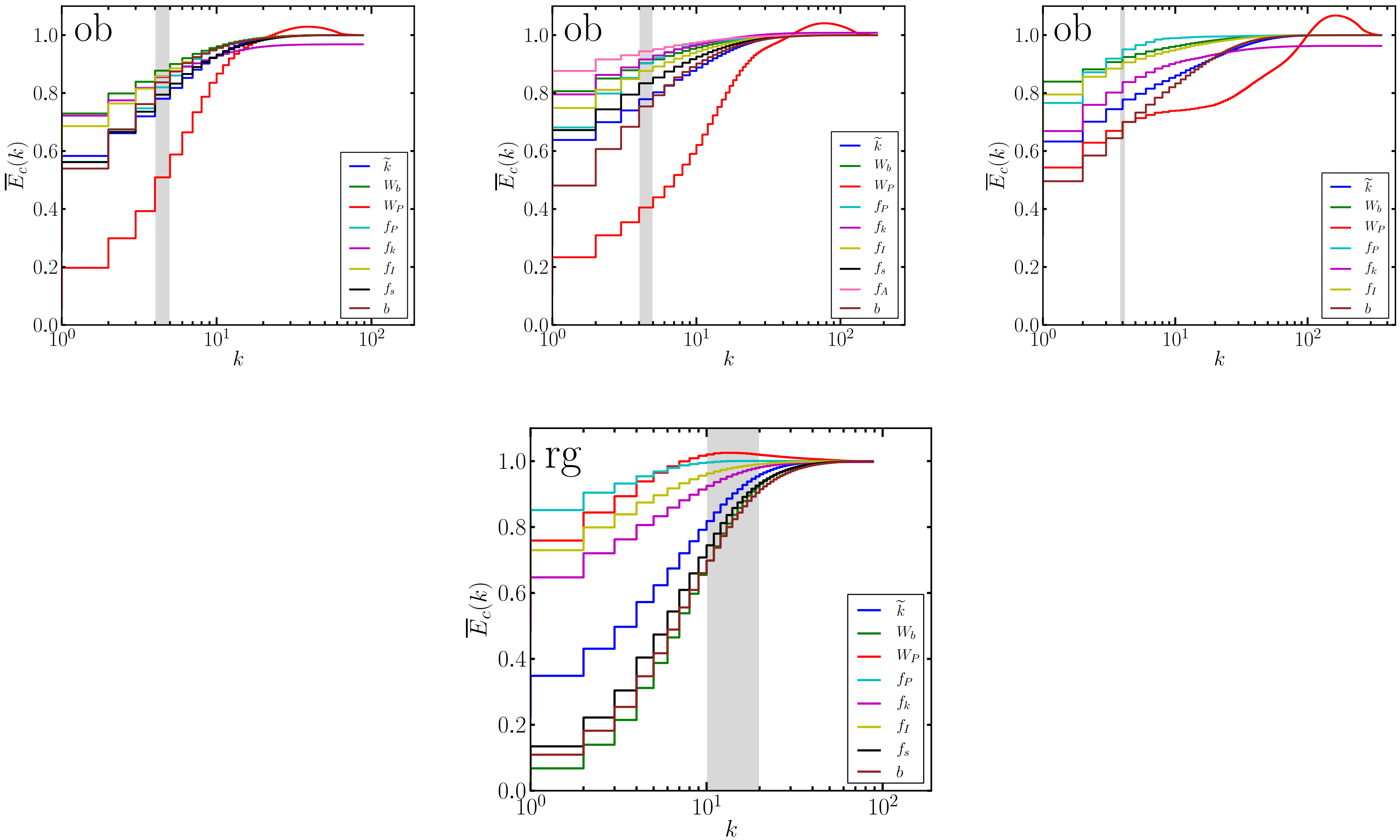

Figure 85: Cumulative Fourier spectra of relevant mean fields derived from oxygen burning shell models (upper panels) ob.3D.lr (left), ob.3D.mr (middle) and ob.3D.hr (right) derived at radius where corresponding mean field has a maximum value. The same is done for red giant envelope convection model (lower panel) rg.3D.lr. The shaded vertical lines separates cumulative contributions from large and small scales.

13 Properties of our hydrogen injection flash and core helium flash data

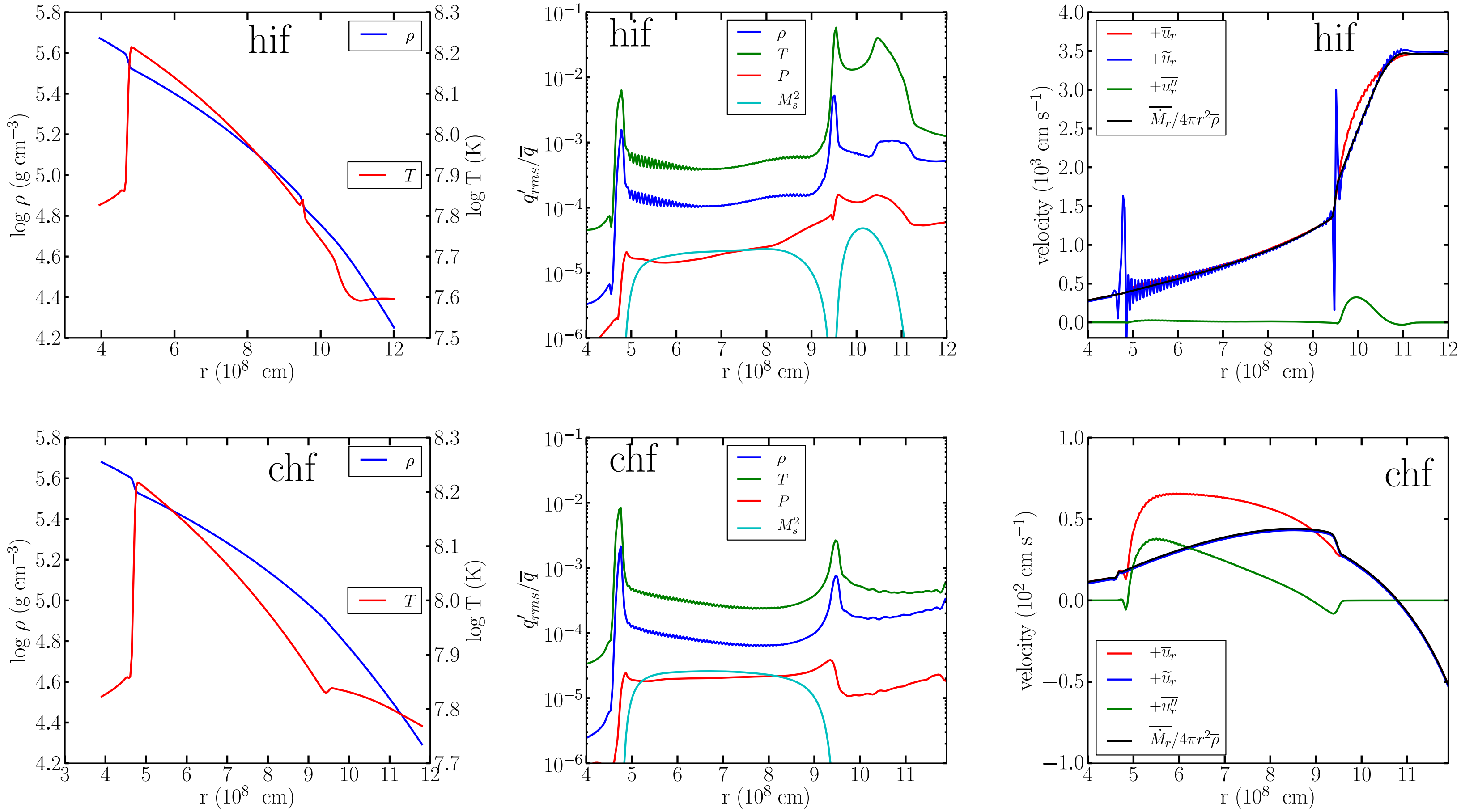

Figure 86: Properties of our data. Model hif.3D (upper panels) and model chf.3D (lower panels).

13.1 Snapshots of turbulent kinetic energy in a meridional plane

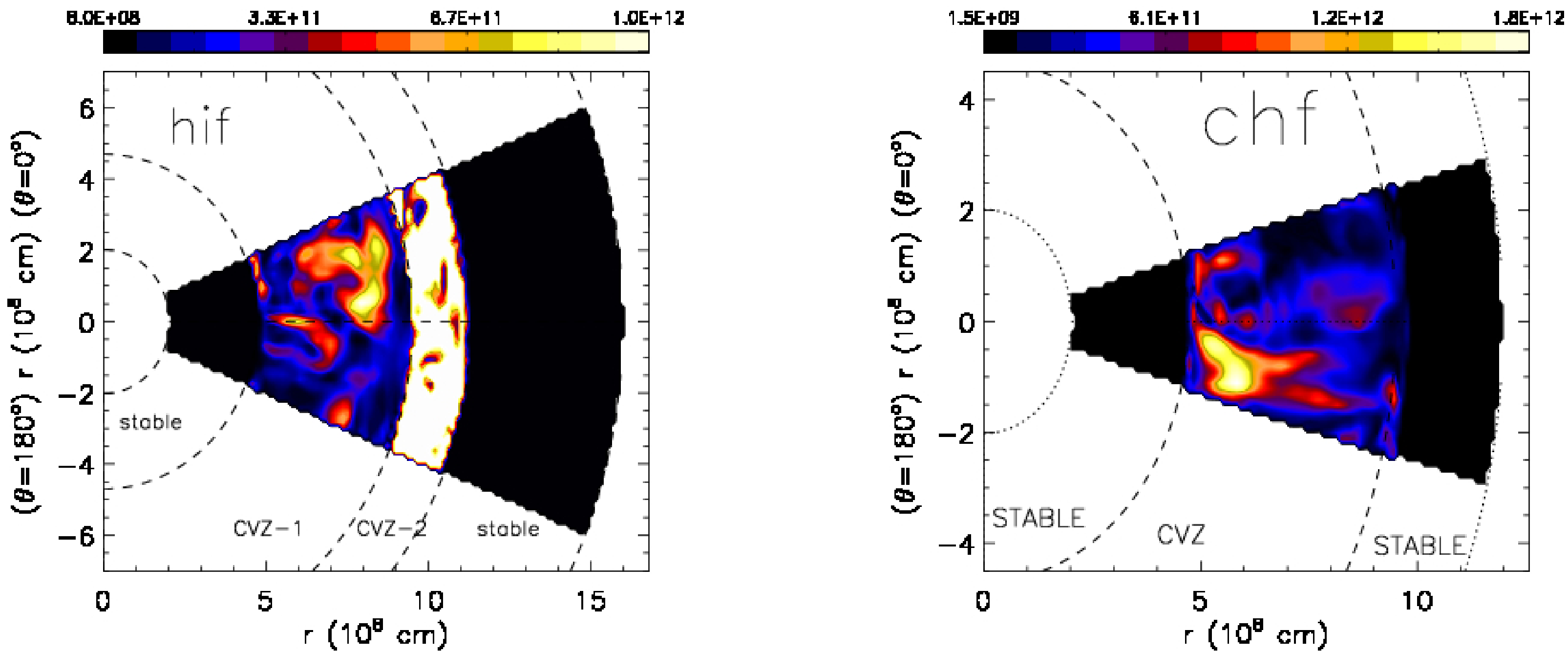

Figure 87: Snapshots of turbulent kinetic energy (in erg g^{-1}) in a meridional plane.

13.2 Summary of the hydrogen injection flash and core helium flash simulations and their properties

Parameter	hif.3D	hif.3D	chf.3D
Grid zoning	375×45^2	375×45^2	270×30^2
$r_{\rm in}$, $r_{\rm out}$ (10^8 cm)	2., 16.	2., 16.	2., 12.
$\Delta\theta$, $\Delta\phi$	45°		30°
Convection zone	He-burning (CVZ-1)	CNO-burning (CVZ-2)	He-burning (CVZ)
$r^c_{\rm in}$, $r^c_{\rm out}$ (10^8 cm)	4.7, 9.55	9.55, 11.3	4.7, 9.4
CZ stratification (H_p)	2.4	1.3	2.3
$\Delta t_{\rm av}$ (s)	4000	4000	18000
$v_{\rm rms}$ (10^5 cm/s)	8.2	11.8	8.5
$\tau_{\rm conv}$ (s)	1180	290	1100
$P_{turb}/P_{gas}(10^{-5})$	2.6	3.7	3.3
$L_{\rm heat}$ (10^{43} erg/s)	1.2	1.8	1.2
$L_{\rm d}$ (10^{41} erg/s)	5.3	6.1	6.
$l_{\rm d}$ (10^8 cm)	5.6	2.9	4.8
$\tau_{\rm d}$ (s)	337.4	123.6	281.9
$\tau_{\rm dr}$ (s)	719.7	175.8	659.7
$\tau_{\rm dh}$ (s)	211.2	116.1	142.5

Table 8: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r^c_{\rm b}$, $r^c_{\rm t}$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

14 Profiles and integral budgets of mean fields equations

14.1 Mean continuity equation

$$\widetilde{D}_t \overline{\rho} = - \overline{\rho} \widetilde{d} + \mathcal{N}_\rho \tag{83}$$

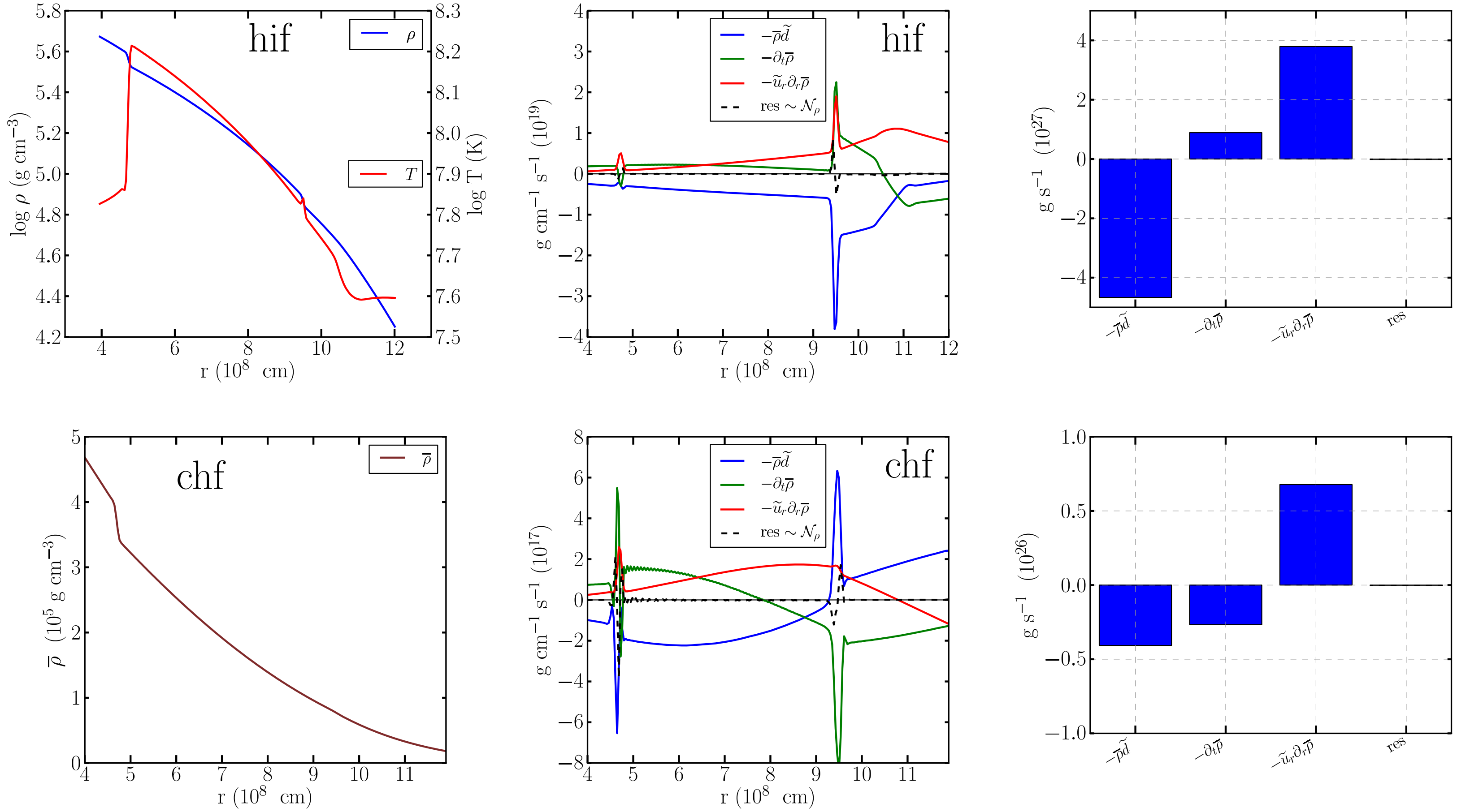

Figure 88: Mean continuity equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.2 Mean radial momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} \quad (84)$$

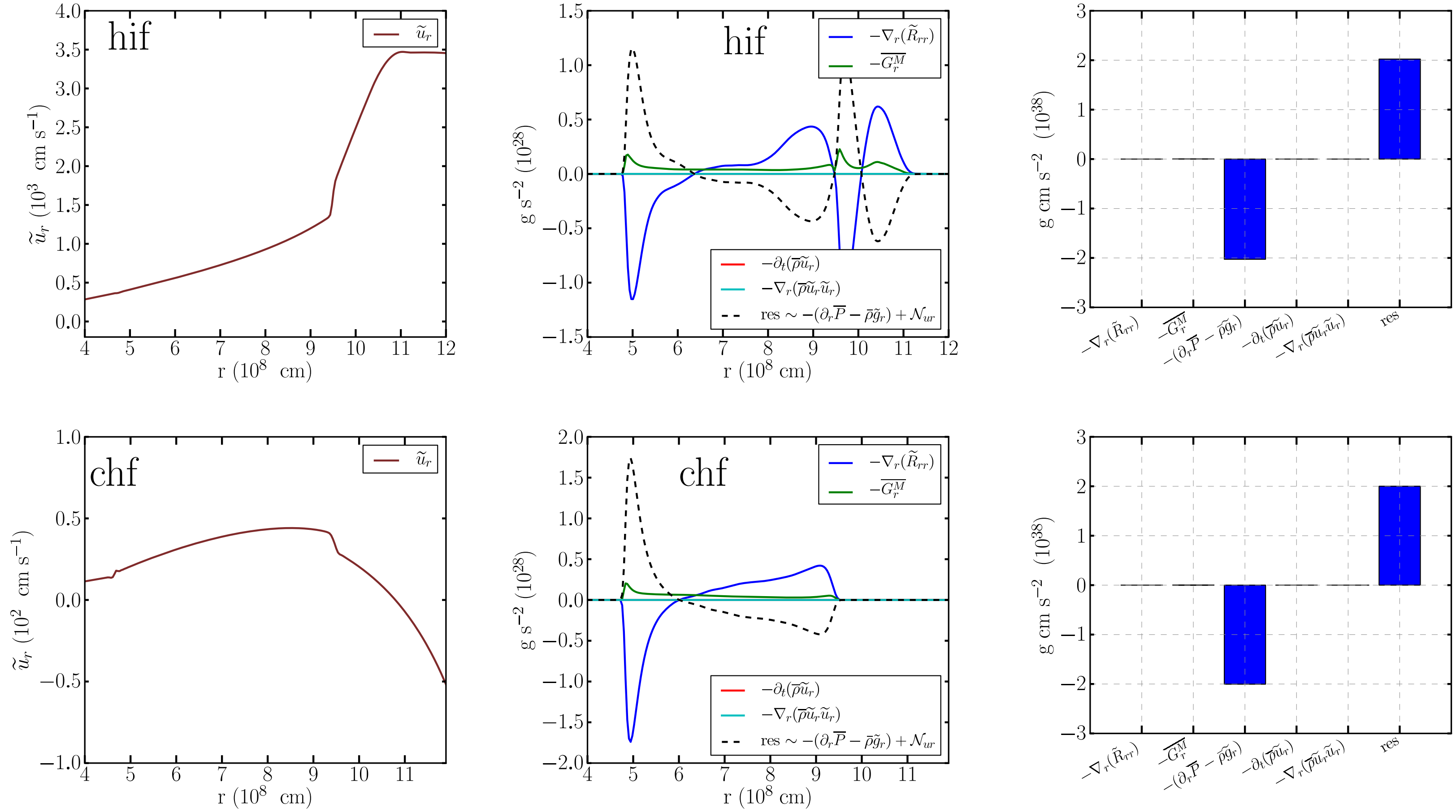

Figure 89: Mean radial momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.3 Mean azimuthal momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G^M_\theta} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} \tag{85}$$

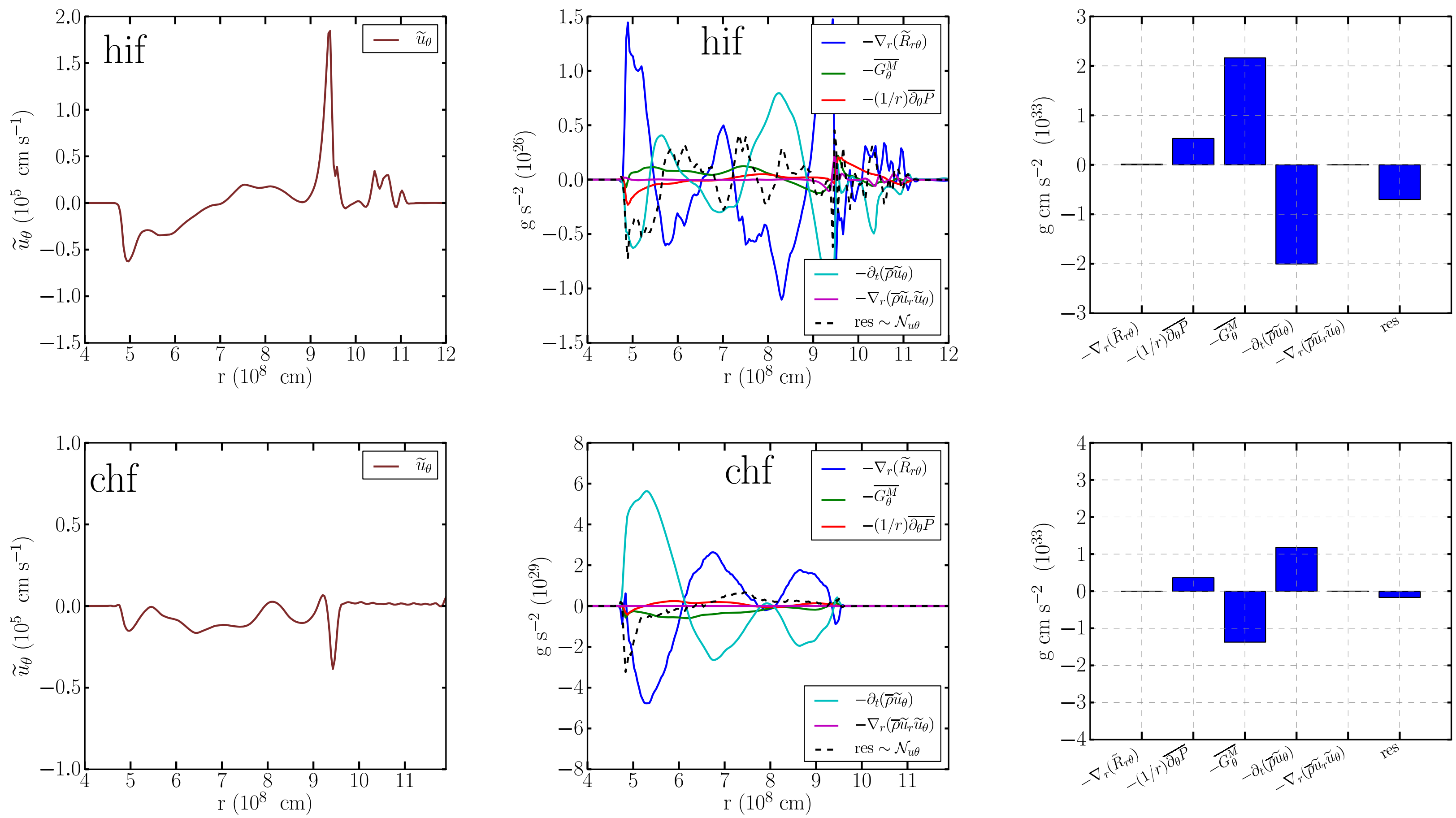

Figure 90: Mean azimuthal momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.4 Mean polar momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G^M_\phi} + \mathcal{N}_{u\phi} \qquad (86)$$

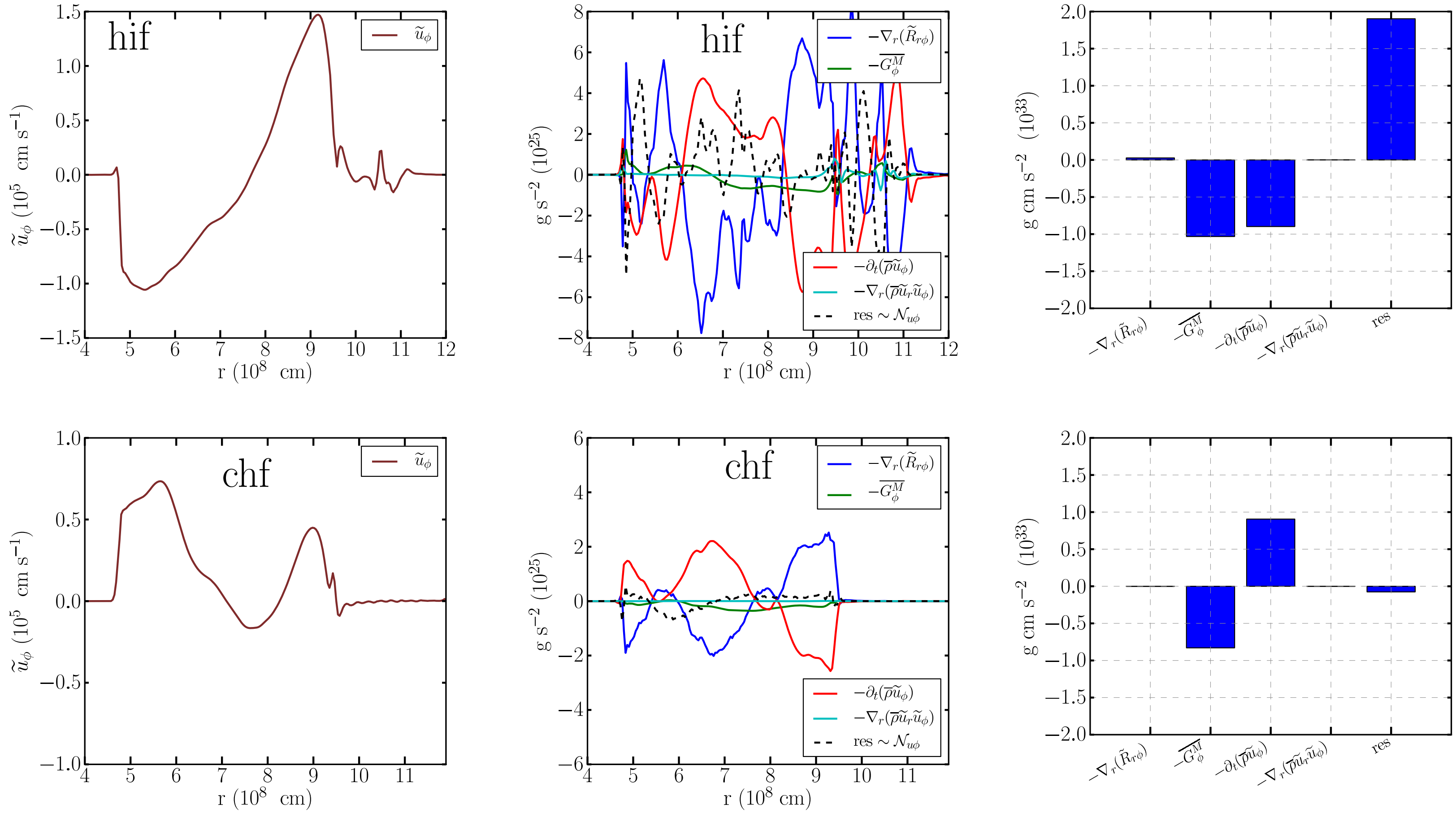

Figure 91: Mean polar momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.5 Mean internal energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \tag{87}$$

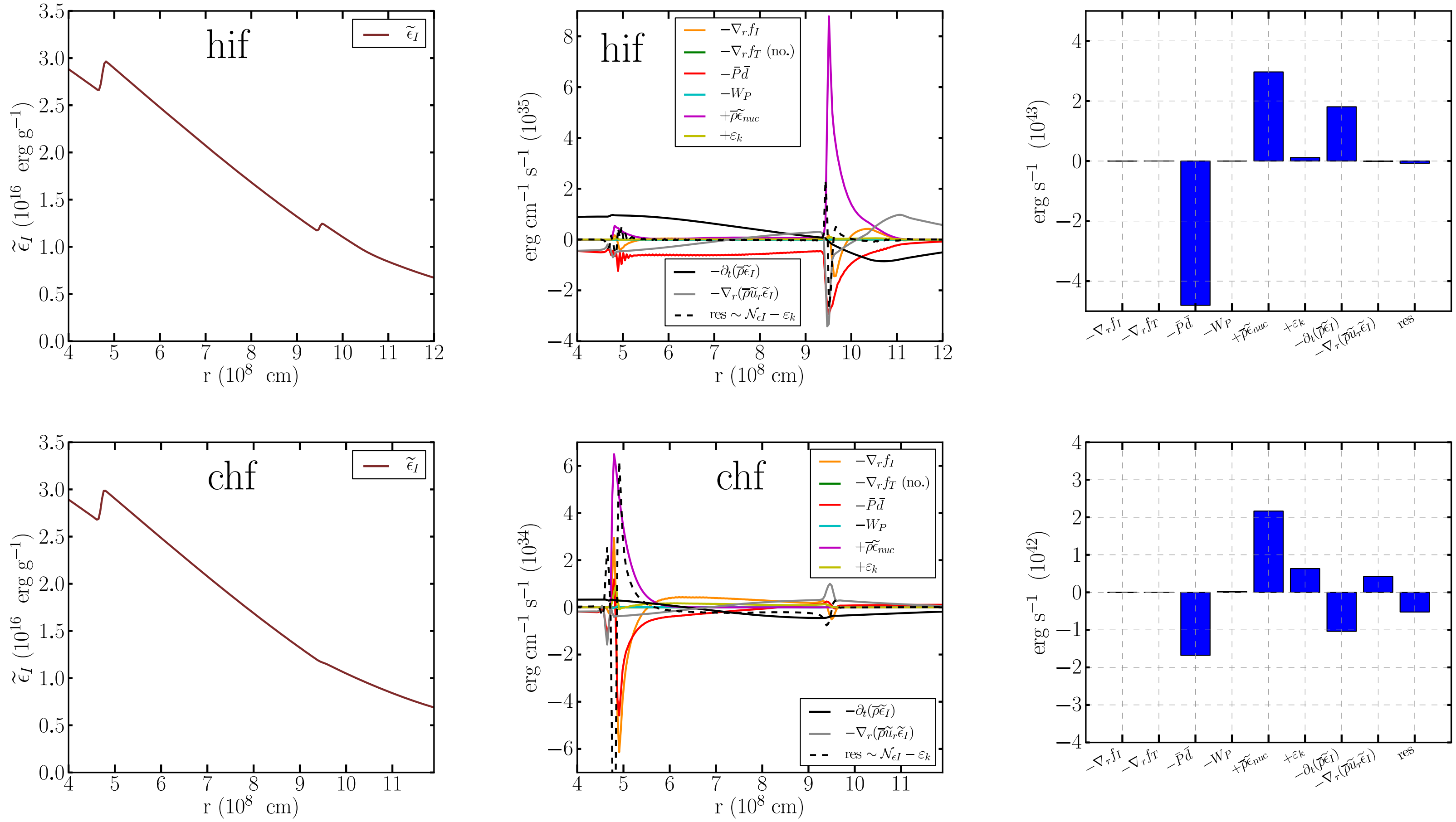

Figure 92: Mean internal energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.6 Mean kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_k = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} \tag{88}$$

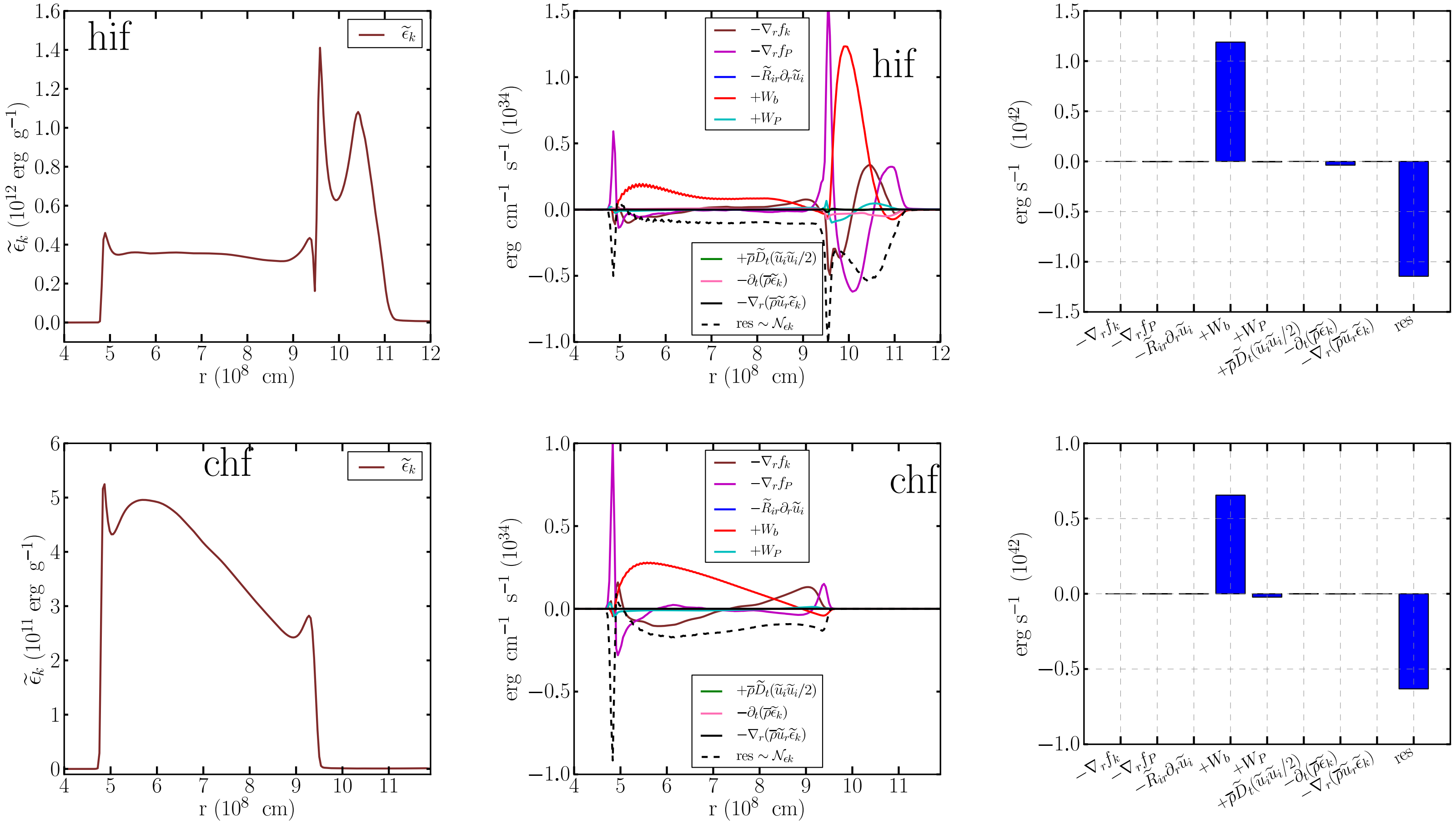

Figure 93: Mean kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.7 Mean total energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} \tag{89}$$

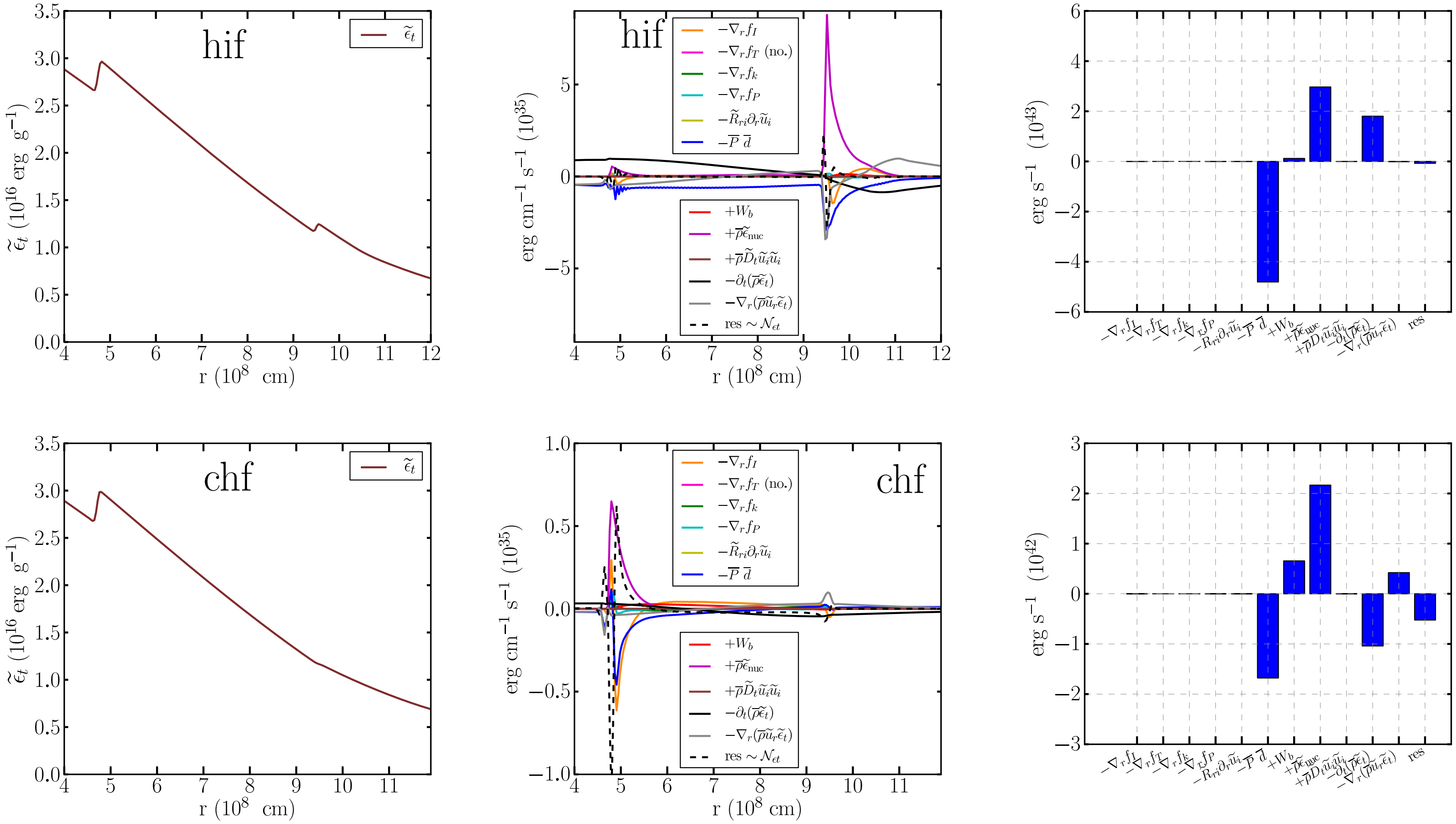

Figure 94: Mean total energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.8 Mean entropy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s - \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \tag{90}$$

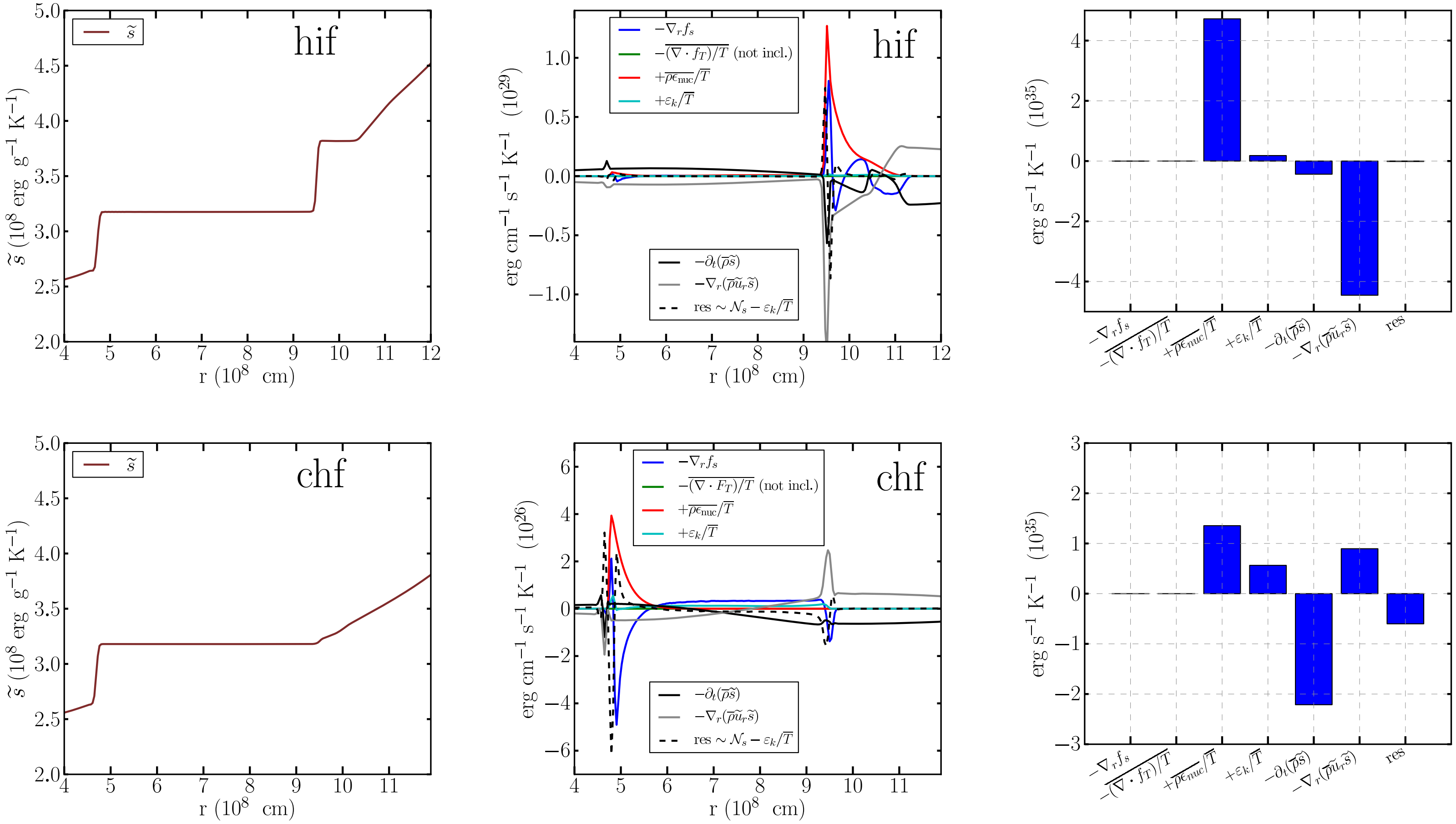

Figure 95: Mean entropy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.9 Mean pressure equation

$$\overline{D}_t \overline{P} = -\nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1-\Gamma_1) W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P \tag{91}$$

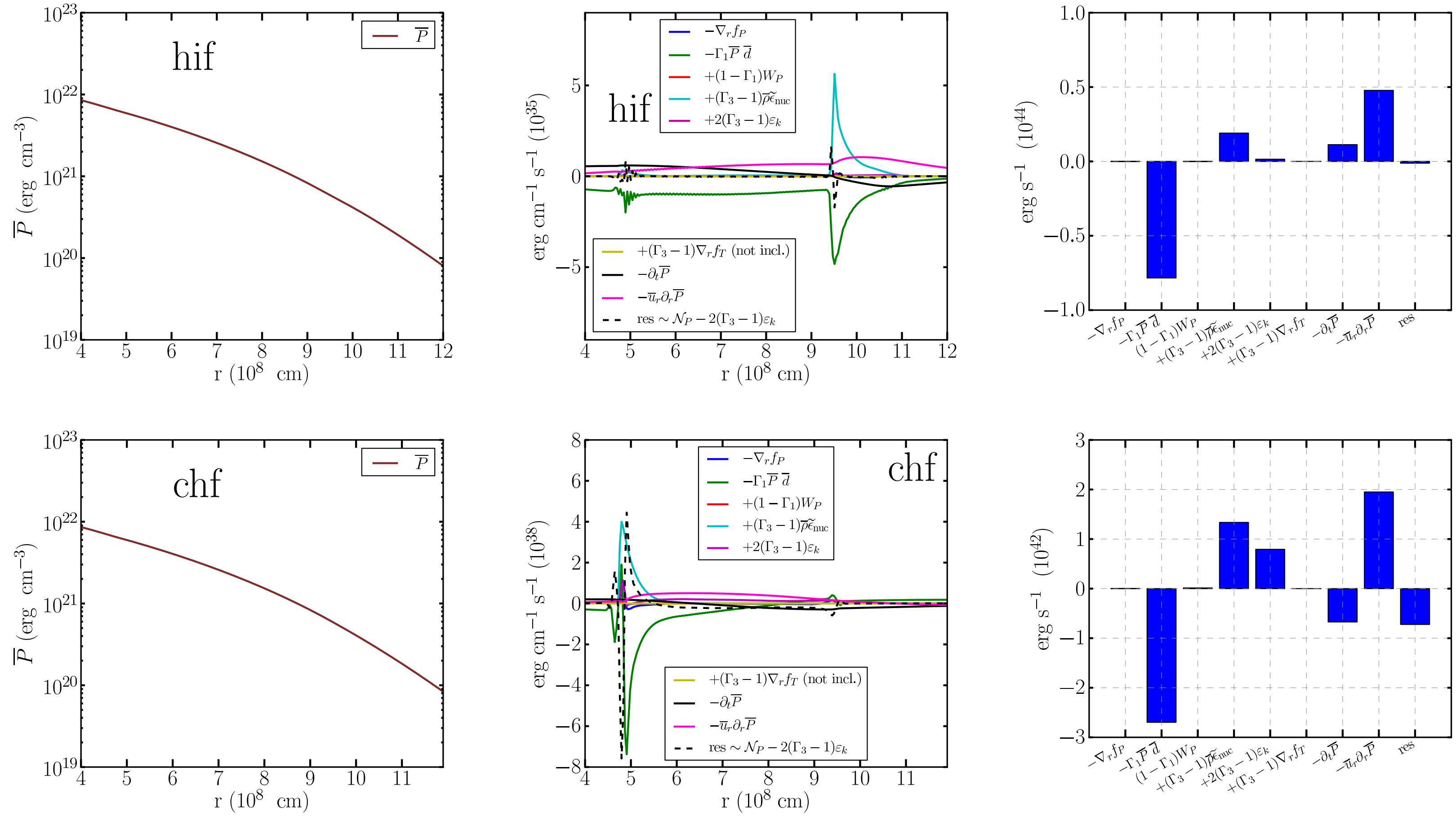

Figure 96: Mean pressure equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.10 Mean enthalpy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r f_h - \Gamma_1\overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3\mathcal{S} + \Gamma_3\nabla_r f_T + \mathcal{N}_h \quad (92)$$

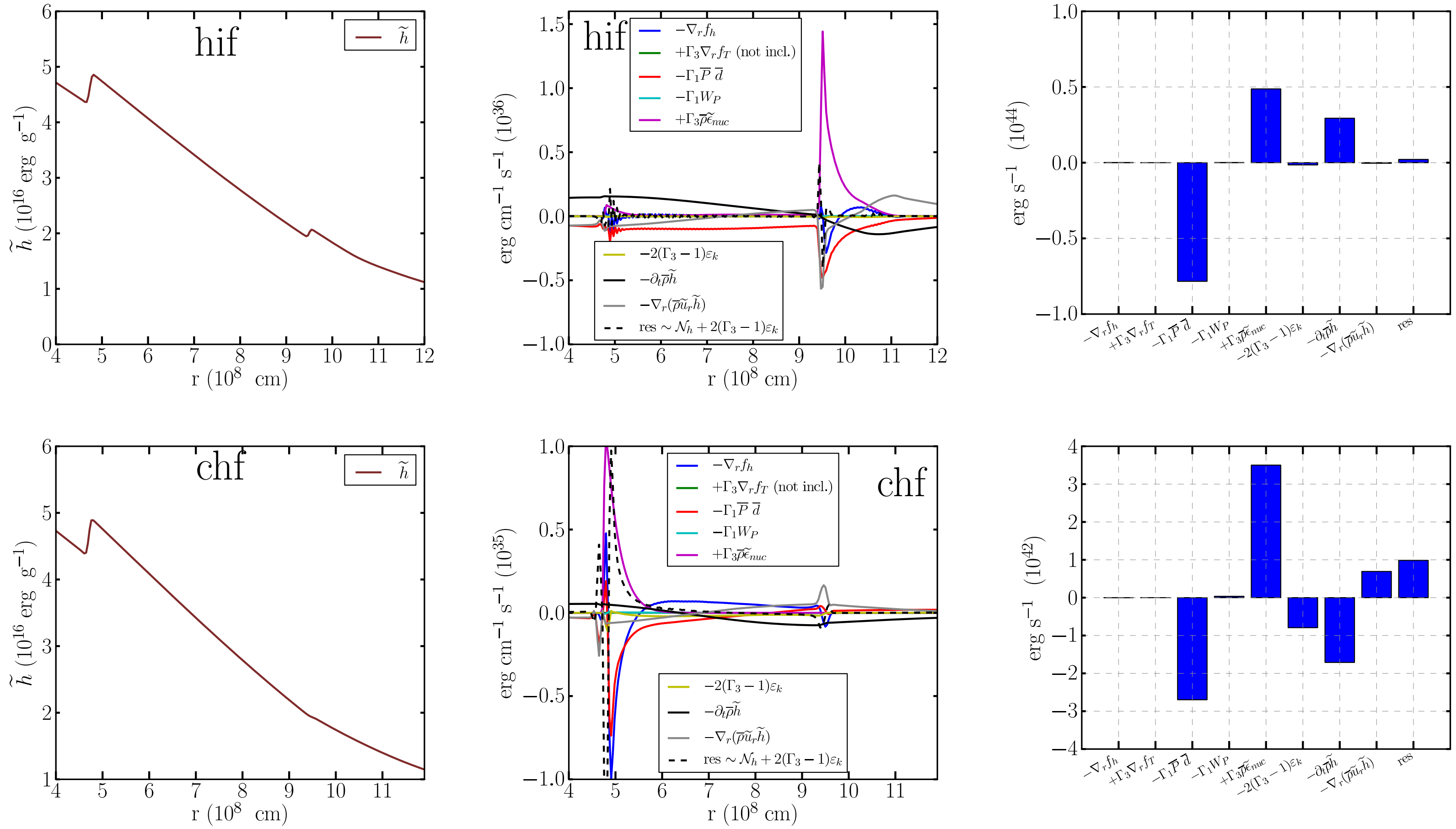

Figure 97: Mean enthalpy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.11 Mean angular momentum equation (z-component)

$$\overline{\rho}\widetilde{D}_t\widetilde{j}_z = -\nabla_r f_{jz} + \mathcal{N}_{jz} \tag{93}$$

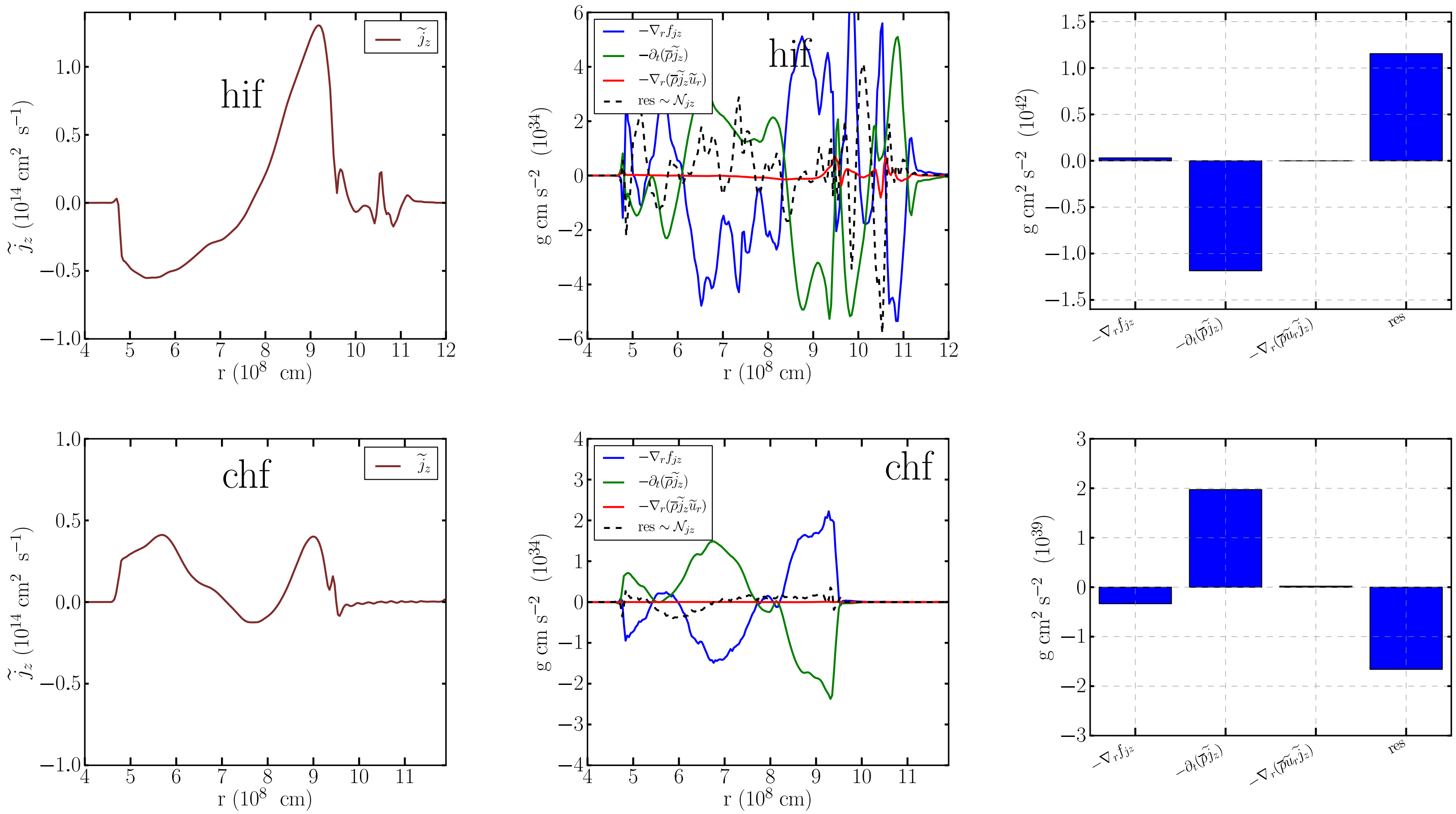

Figure 98: Mean angular momentum equation (z-component). Model hif.3D (upper panels) and model chf.3D (lower panels)

14.12 Mean composition equations

$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}^{\rm nuc}_\alpha + \mathcal{N}_\alpha \tag{94}$$

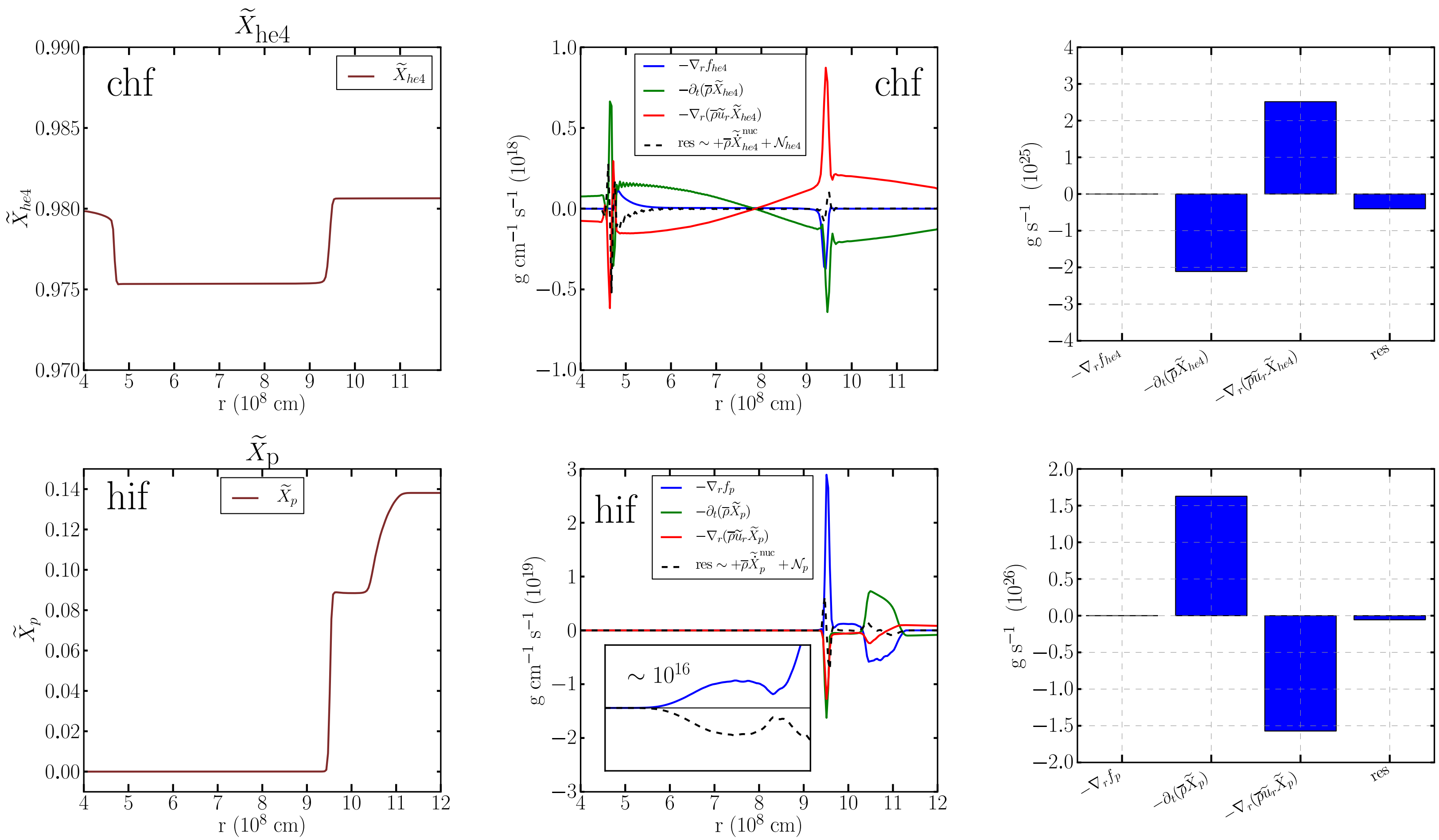

Figure 99: Mean composition equations. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.13 Mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \qquad (95)$$

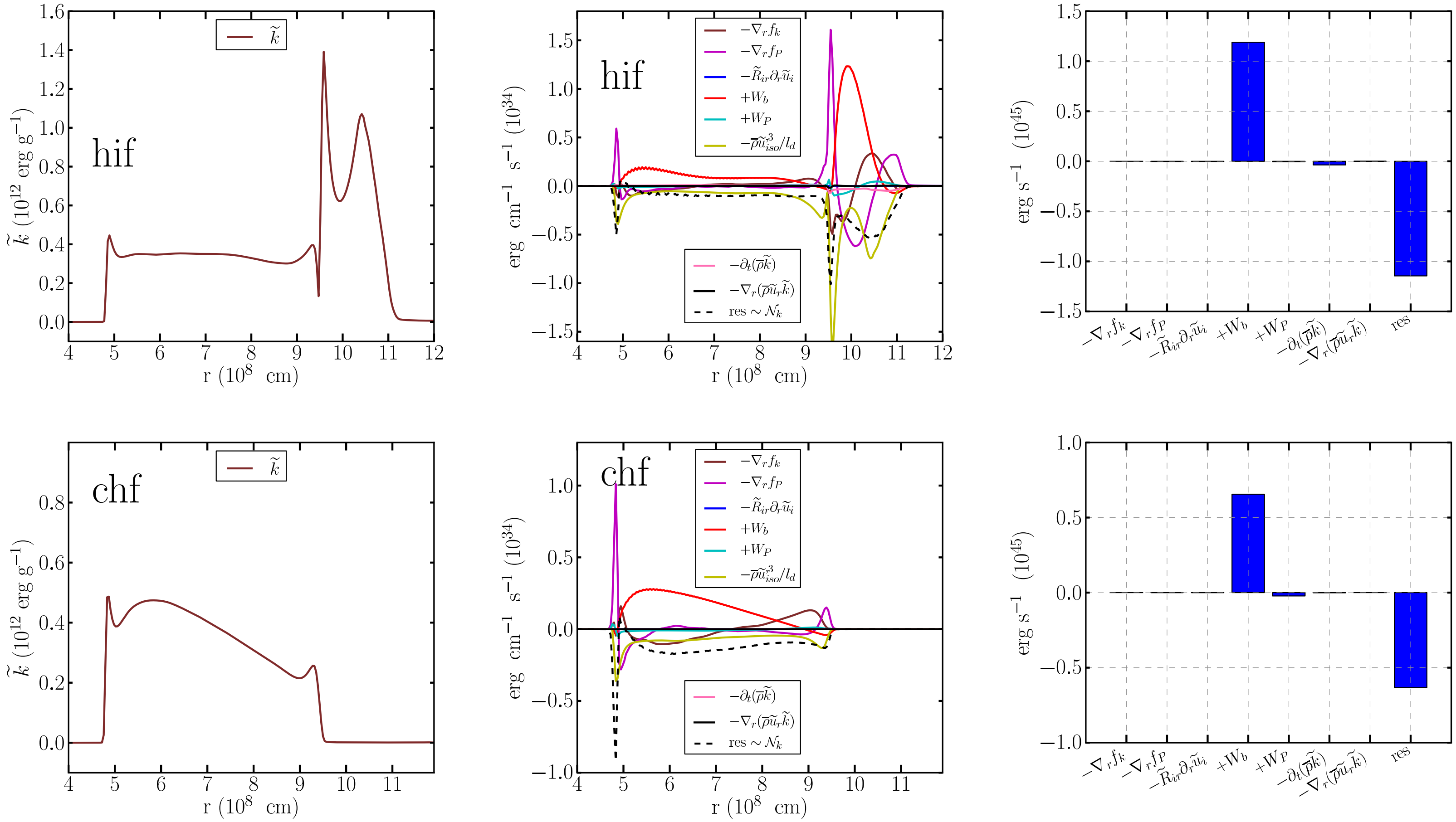

Figure 100: Mean turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.14 Radial part of mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} \quad (96)$$

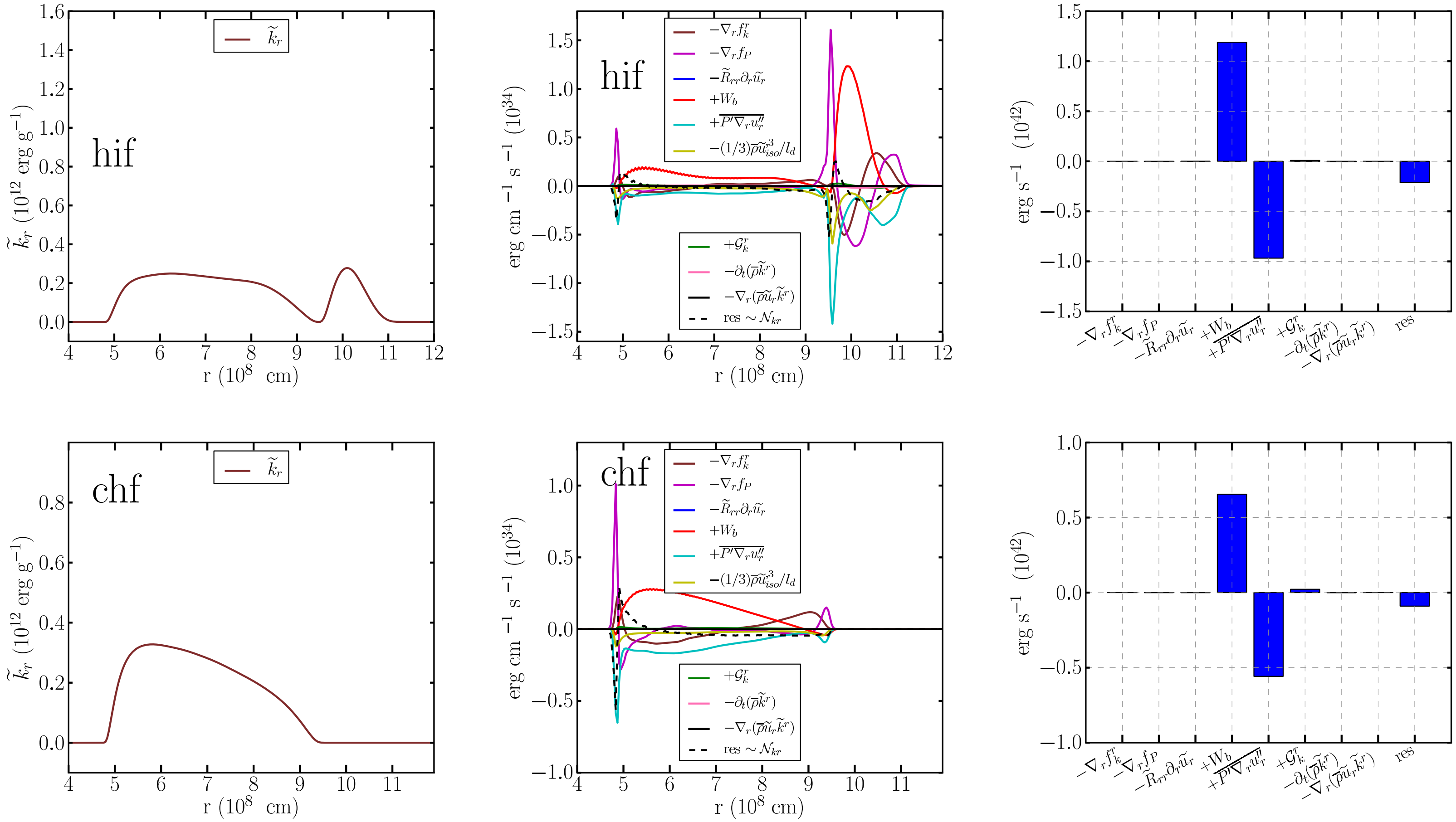

Figure 101: Radial turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.15 Horizontal part of mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \tag{97}$$

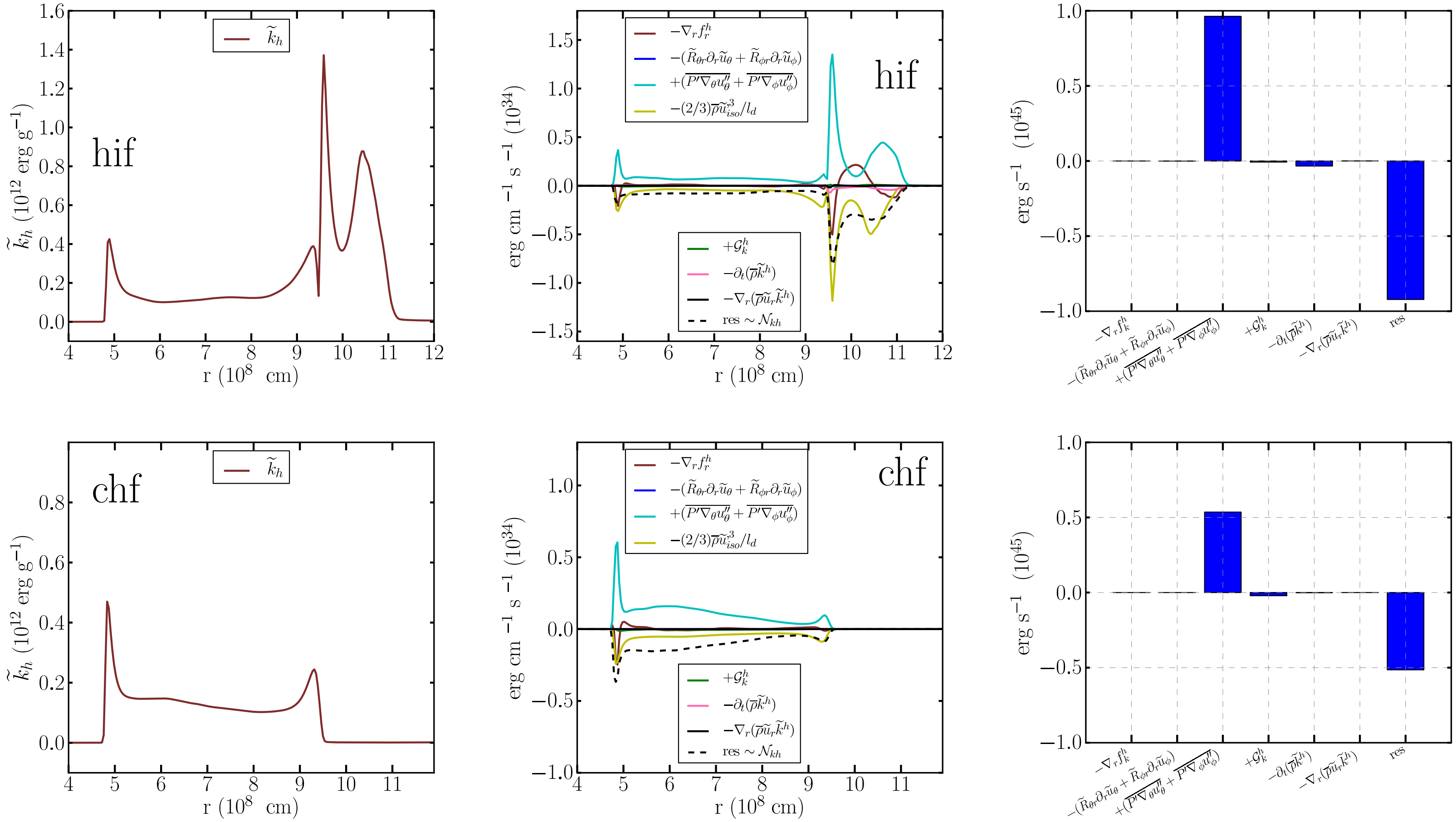

Figure 102: Horizontal turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

15 Compositon equations derived from hydrogen injection flash data

15.1 Mean H^1 and He^3 equation

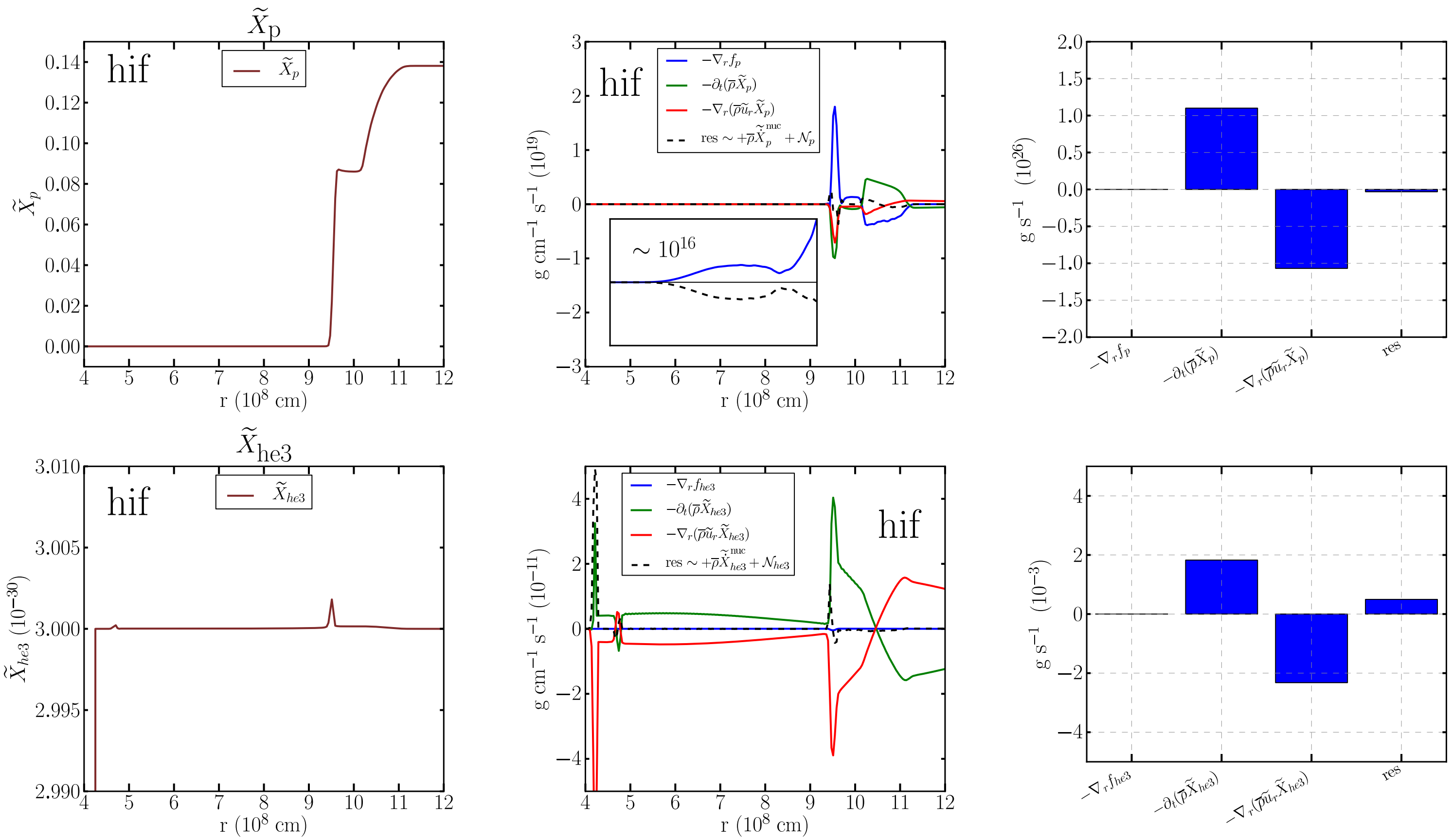

Figure 103: Mean composition equations derived from hif.3D.

15.2 Mean He^4 and C^{12} equation

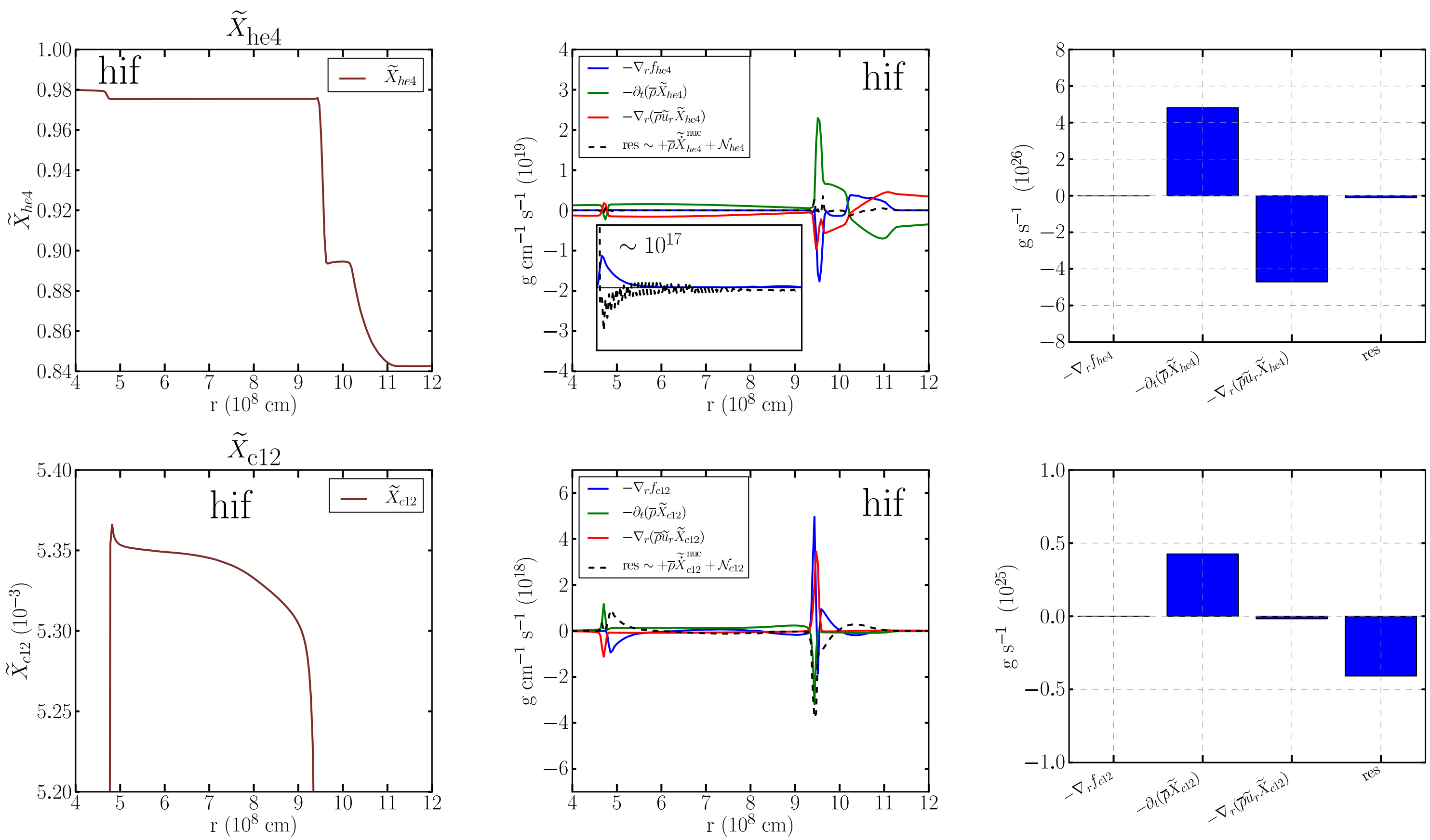

Figure 104: Mean composition equations derived from hif.3D.

15.3 Mean C^{13} and N^{13} equation

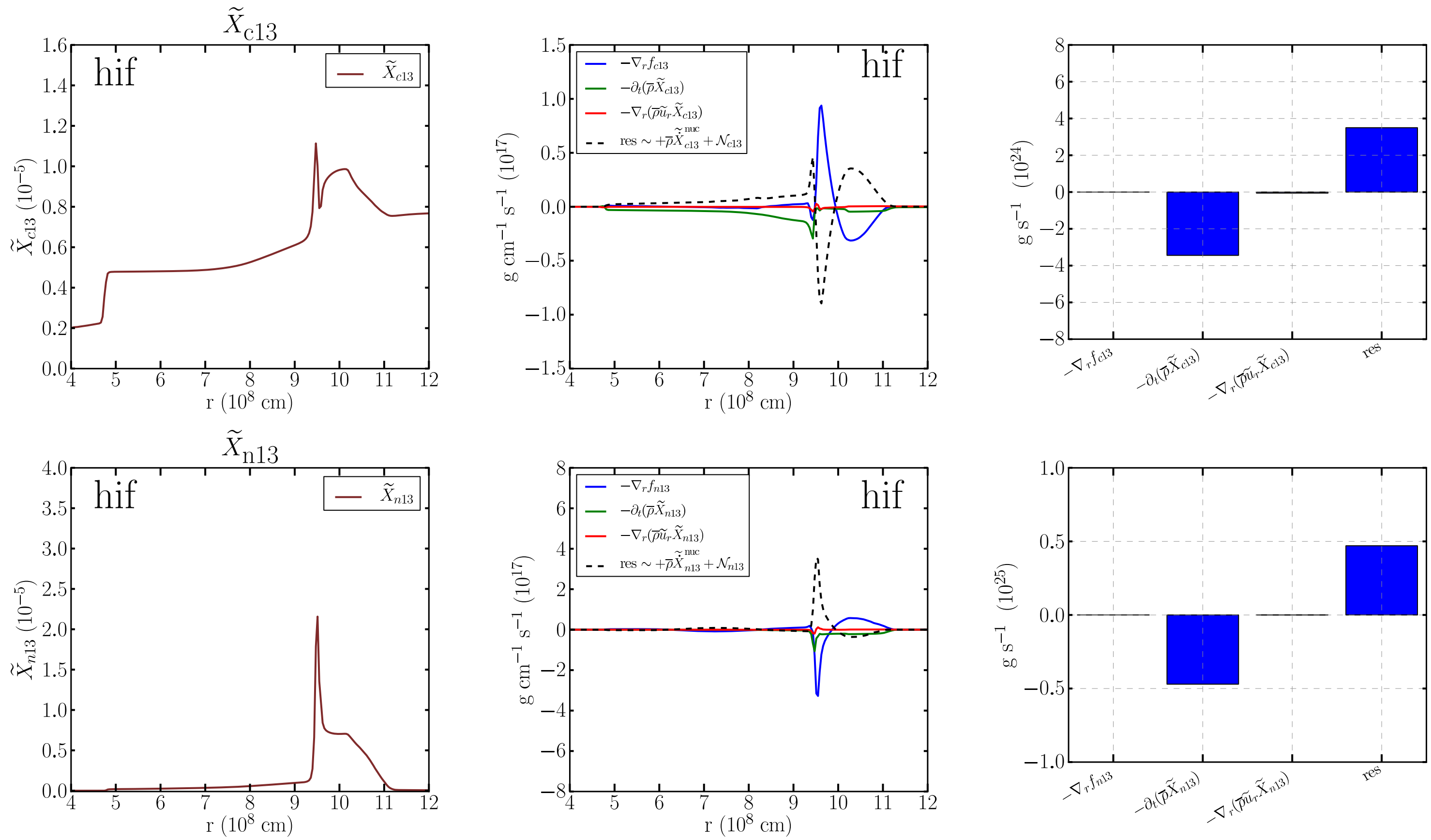

Figure 105: Mean composition equations derived from hif.3D.

15.4 Mean N^{14} and N^{15} equation

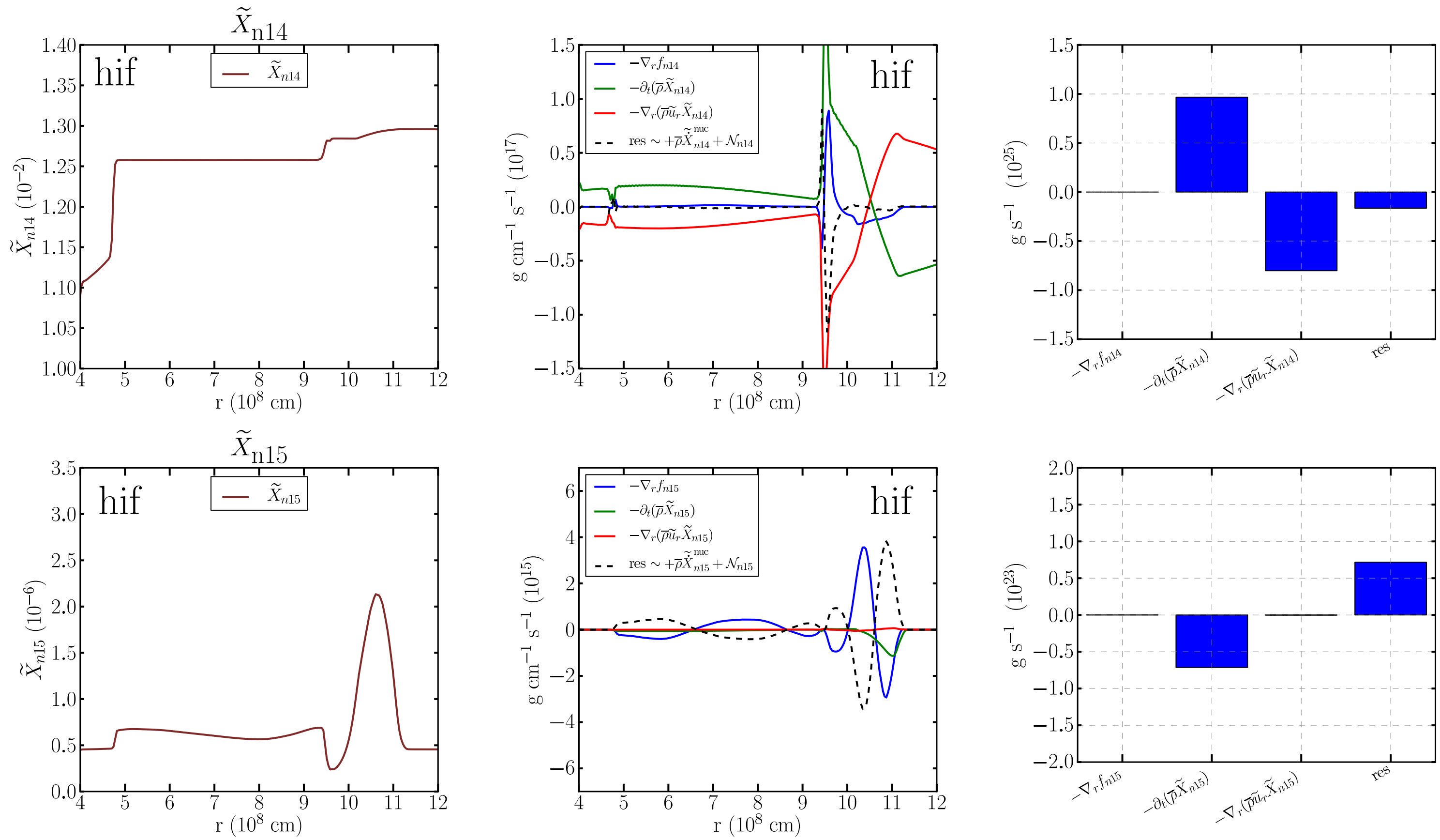

Figure 106: Mean composition equations derived from hif.3D.

15.5 Mean O^{15} and O^{16} equation

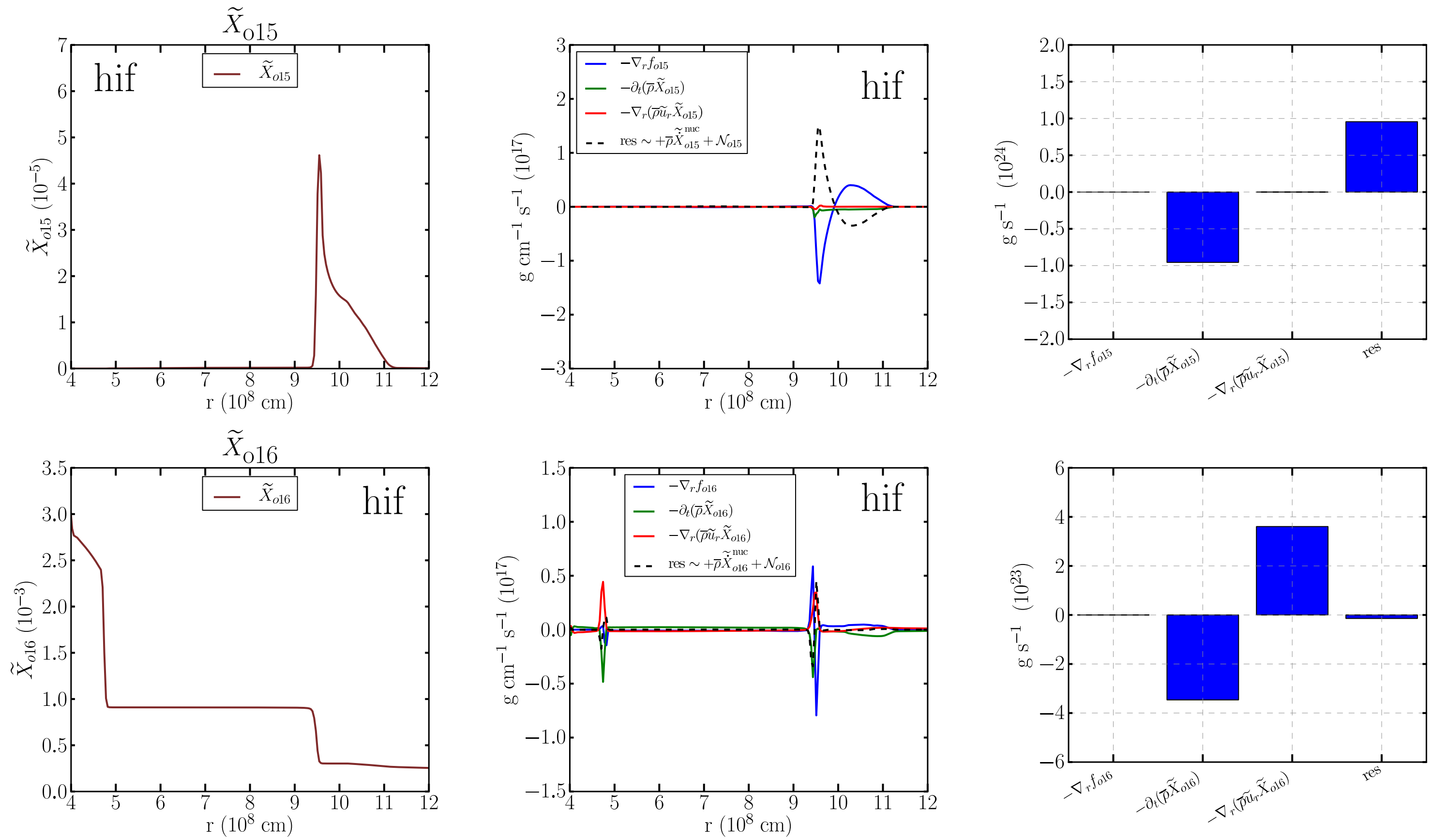

Figure 107: Mean composition equations derived from hif.3D.

15.6 Mean O^{17} and Ne^{20} equation

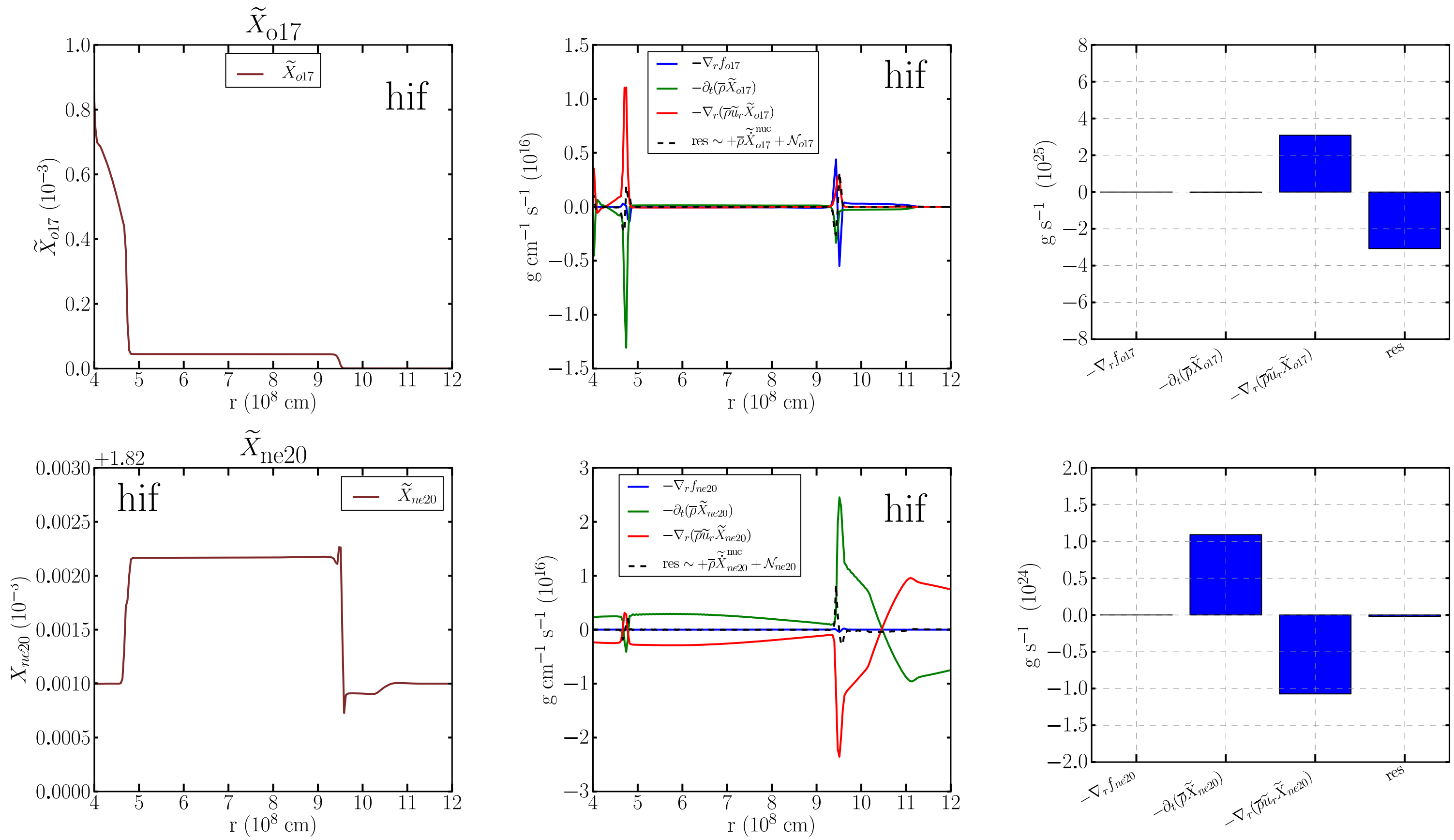

Figure 108: Mean composition equations derived from hif.3D.

15.7 Mean Mg^{24} and Si^{28} equation

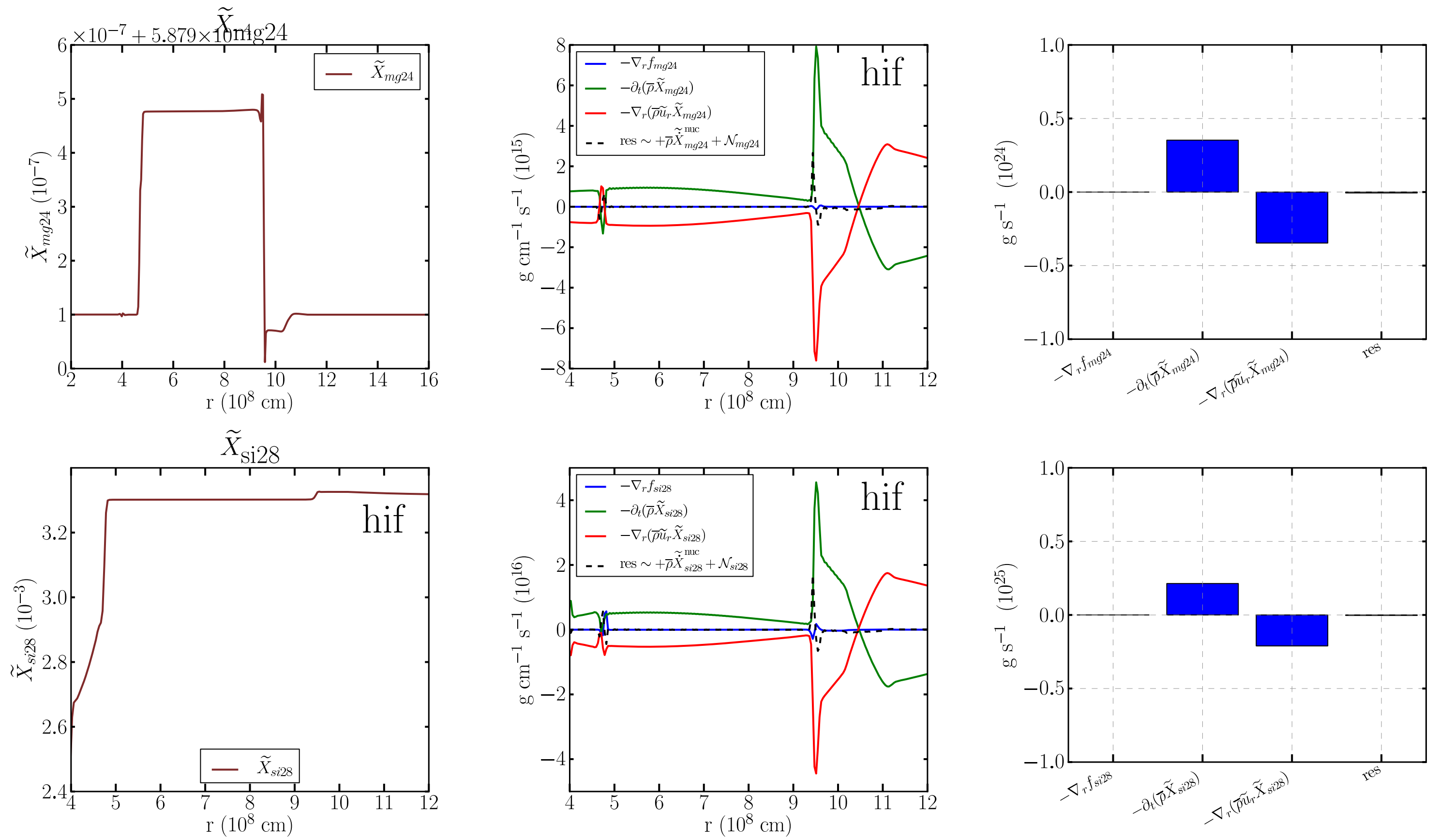

Figure 109: Mean composition equations derived from hif.3D

16 Compositon equations derived from core helium flash data

16.1 Mean He^4 and C^{12} equation

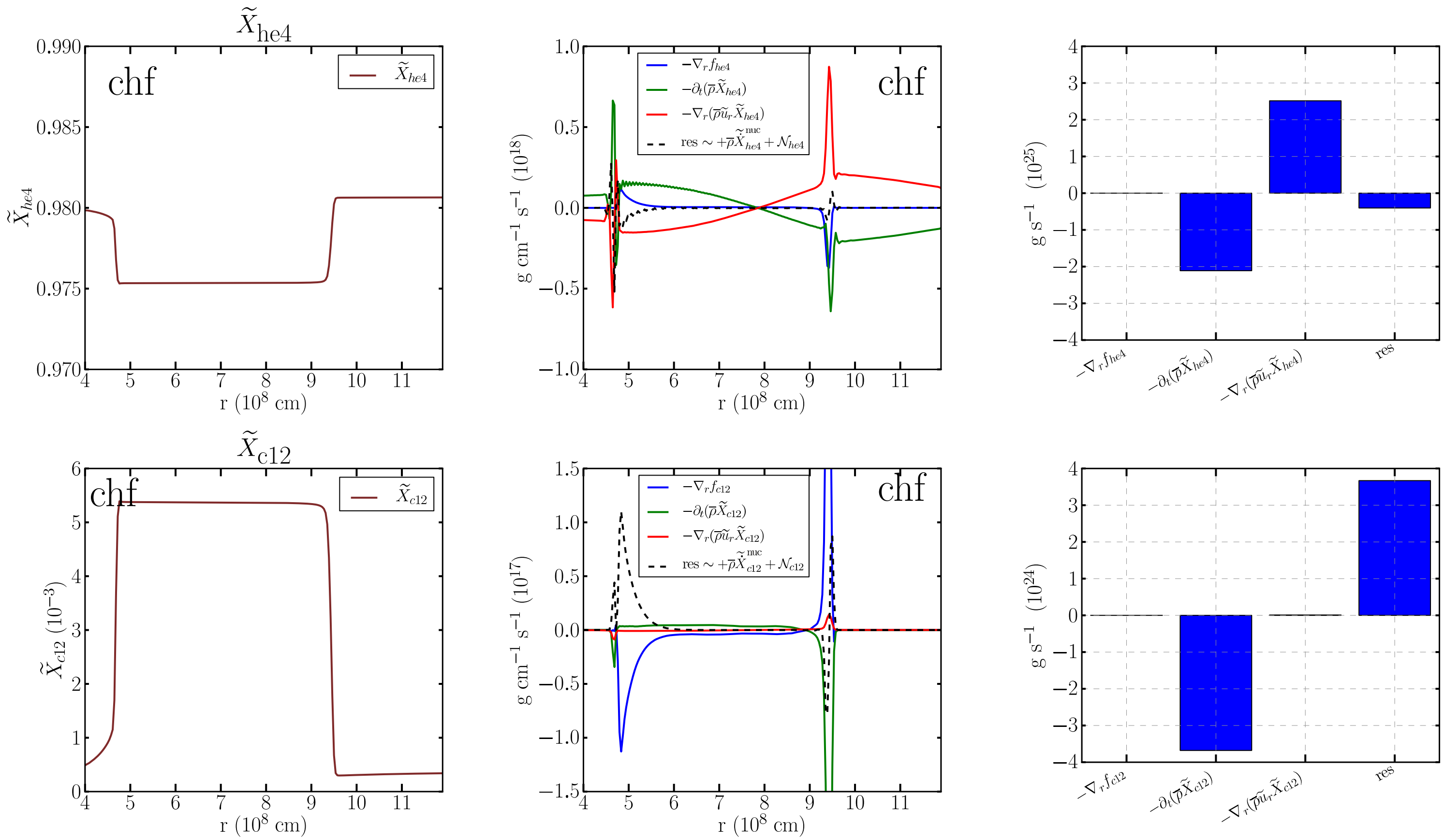

Figure 110: Mean composition equations derived from chf.3D.

16.2 Mean O^{16} and Ne^{20} equation

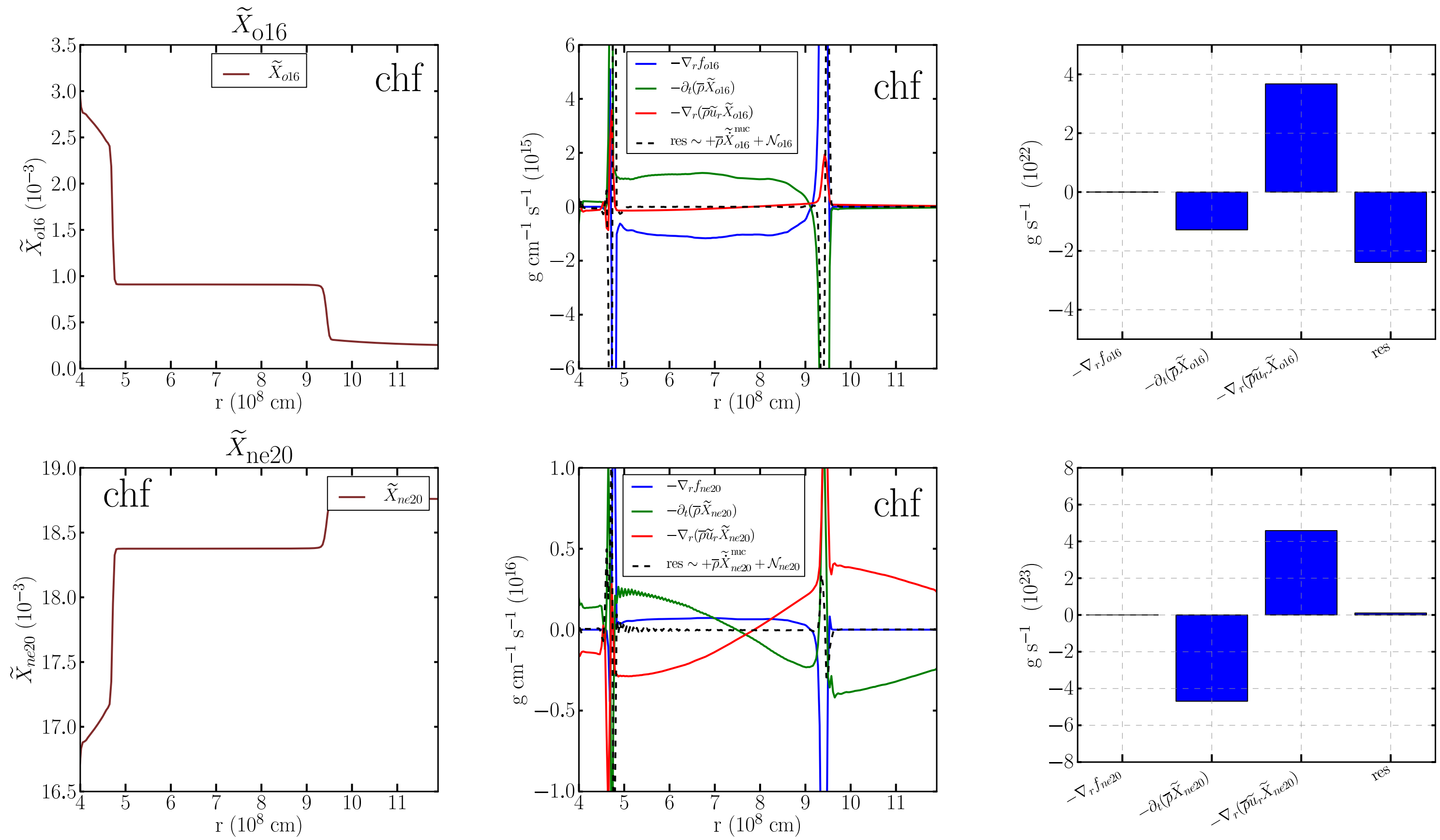

Figure 111: Mean composition equations derived from chf.3D.

17 Instantaneous hydrodynamic equations in spherical coordinates (Eulerian form)

The hydrodynamic equations of a viscous multi-component reactive gas subject to gravity and thermal transport in spherical coordinates (r, θ, ϕ):

$$\partial_t\big(\rho\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi]\right) \tag{98}$$

$$\partial_t\big(\rho u_r\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r^2 - \tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_r u_\phi - \tau_{r\phi}]\big) + G_r^M + \partial_r P\right) - \rho\partial_r\Phi \tag{99}$$

$$\partial_t\big(\rho u_\theta\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\theta u_r - \tau_{\theta r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big) + G_\theta^M + \frac{1}{r}\partial_\theta P\right) - \rho\frac{1}{r}\partial_\theta\Phi \tag{100}$$

$$\partial_t\big(\rho u_\phi\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \tau_{\phi r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \tau_{\phi\phi}]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) - \rho\frac{1}{r\sin\theta}\partial_\phi\Phi \tag{101}$$

$$\begin{aligned}\partial_t\big(\rho\epsilon_T\big) = &-\left(\frac{1}{r^2}\partial_r(r^2[u_r\big(\rho\epsilon_T + P\big) - K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta\big(\rho\epsilon_T + P\big) - K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi\big(\rho\epsilon_T + P\big) - K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \\ &+\left(\frac{1}{r^2}\partial_r\big(r^2[u_j\tau_{jr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_j\tau_{j\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}]\right) - \rho\big(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi) + \rho\epsilon_{nuc}\end{aligned} \tag{102}$$

$$\begin{aligned}\partial_t\big(\rho\epsilon_I\big) = &-\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_I]\right) - P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) + \\ &+\left(\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho\epsilon_{nuc}\end{aligned} \tag{103}$$

$$\begin{aligned}\partial_t\big(\rho\epsilon_K\big) = &-\left(\frac{1}{r^2}\partial_r[r^2\big(\rho u_r\epsilon_K - u_j\tau_{jr}\big)] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta\big(\rho u_\theta\epsilon_K - u_j\tau_{j\theta}\big)] + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_K - u_j\tau_{j\phi}]\right) - \\ &-\left(\frac{1}{r^2}\partial_r\big(r^2[Pu_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Pu_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) - \\ &-\left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) - \rho\big(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi)\end{aligned} \tag{104}$$

$$\partial_t\big(\rho X_k\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r X_k]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta X_k]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_k]\right) + \rho\dot{X}_k^n \qquad k = 1...N_{nuc} \tag{105}$$

$$G_r^M = -\frac{(\rho u_\theta^2 - \tau_{\theta\theta})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})}{r} \quad G_\theta^M = +\frac{(\rho u_\theta u_r - \tau_{\theta r})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})\cos\theta}{r\sin\theta} \quad G_\phi^M = +\frac{(\rho u_\phi u_r - \tau_{\phi r})}{r} + \frac{(\rho u_\phi u_\theta - \tau_{\phi\theta})\cos\theta}{r\sin\theta} \tag{106}$$

18 Instantaneous hydrodynamic equations in spherical coordinates (Lagrangian form)

The hydrodynamic equations of a viscous multi-component reactive gas subject to gravity and thermal transport in spherical coordinates (r,θ,ϕ) are ($D_t(.) = \partial_t(.) + u_n\partial_n(.)$ is advective derivative):

$$
\begin{aligned}
D_t(\rho) &= -\rho\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\;]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) && (107)\\
\rho D_t(u_r) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big) - G_r^M - \partial_r P\right) + \rho g_r && (108)\\
\rho D_t(u_\theta) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\theta r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{\theta\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi[\tau_{\theta\phi}]\big) - G_\theta^M - \frac{1}{r}\partial_\theta P\right) + \rho g_\theta && (109)\\
\rho D_t(u_\phi) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\phi r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{\phi\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{\phi\phi}]\big) - G_\phi^M - \frac{1}{r\sin\theta}\partial_\phi P\right) + \rho g_\phi && (110)\\
\rho D_t(\epsilon_T) &= -\left(\frac{1}{r^2}\partial_r\big(r^2[u_r P - K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta P - K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi P - K\frac{1}{r\sin\theta}\partial_\phi T]\right) - \\
&\quad + \left(\frac{1}{r^2}\partial_r[r^2(u_j\tau_{jr})] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta(u_j\tau_{j\theta})] + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}]\right) + \rho(u_r g_r + u_\theta g_\theta + u_\phi g_\phi) + \rho\epsilon_{nuc} && (111)\\
\rho D_t(\epsilon_I) &= -P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) + \left(\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \\
&\quad + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho\epsilon_{nuc} && (112)\\
\rho D_t(\epsilon_K) &= +\left(\frac{1}{r^2}\partial_r[r^2(u_j\tau_{jr})] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta(u_j\tau_{j\theta})] + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}]\right) - \left(\frac{1}{r^2}\partial_r\big(r^2[Pu_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Pu_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + \\
&\quad + P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\;]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) - \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho(u_r g_r + u_\theta g_\theta + u_\phi g_\phi) && (113)\\
\rho D_t(X_k) &= +\rho\dot{X}_k^n \qquad\qquad k = 1...N_{nuc} && (114)
\end{aligned}
$$

$$
G_r^M = -\frac{(\rho u_\theta^2 - \tau_{\theta\theta})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})}{r} \quad G_\theta^M = +\frac{(\rho u_\theta u_r - \tau_{\theta r})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})\cos\theta}{r\sin\theta} \quad G_\phi^M = +\frac{(\rho u_\phi u_r - \tau_{\phi r})}{r} + \frac{(\rho u_\phi u_\theta - \tau_{\phi\theta})\cos\theta}{r\sin\theta} \tag{115}
$$

where ρ, u_r, u_θ, u_ϕ, P, ϵ_T, ϵ_I, ϵ_K, T, ϵ_{nuc}, X_k, and $\dot{X}_k^n$ are the density, the radial velocity, the θ-velocity, the rotation velocity, the pressure, the total specific energy, the specific internal energy, the specific kinetic energy, the temperature, the energy generation rate per mass due to reactions, the mass fraction of species k, and the change of this mass fraction due to reactions, respectively. N_{nuc} is the number of species the gas is composed. $\tau_{ij} = 2\mu S_{ij} - 2/3\mu\nabla\cdot\mathbf{u}\delta_{ij}$ is the viscous stress, where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is strain-rate, $\mu = \rho\nu$ is dynamic viscosity and ν is the kinematic viscosity. K is thermal conductivity. g_i is gravitational acceleration in r,θ,ϕ and Φ is gravitational potential. G are geometric terms.

19 Reynolds decomposition

Reynolds decomposition:

$$A(r,\theta,\phi) = \overline{A}(r) + A'(r,\theta,\phi) \tag{116}$$

Definition of the averaging space-time operator:

$$\overline{A(r)} = \frac{1}{\Delta T \Delta\Omega} \int_{\Delta T} \int_{\Delta\Omega} A(r,\theta,\phi)\; dt\; d\Omega \tag{117}$$

Some properties of the operator:

$$\overline{A'} = 0 \tag{118}$$

$$\overline{\overline{A}} = \overline{A} \tag{119}$$

$$(A')' = A' \tag{120}$$

$$\overline{\overline{A}B} = \overline{A}\;\overline{B} \tag{121}$$

$$\overline{AB} = \overline{A}\;\overline{B} + \overline{A'B'} \tag{122}$$

$$\overline{A'B'} = \overline{A'B} \tag{123}$$

20 Favre decomposition

Favre decomposition:

$$F = \widetilde{F}(r) + F''(r,\theta,\phi) \tag{124}$$

Definition of the averaging operator:

$$\widetilde{F} = \frac{\overline{\rho F}}{\overline{\rho}} \tag{125}$$

Some properties of the operator:

$$\overline{\rho F''} = \widetilde{F''} = 0 \tag{126}$$

$$\widetilde{F} = \overline{F} + \frac{\overline{\rho F'}}{\overline{\rho}} \tag{127}$$

$$F'' = F' - \frac{\overline{\rho F'}}{\overline{\rho}} \rightarrow \overline{F''} = -\frac{\overline{\rho F'}}{\overline{\rho}} \tag{128}$$

$$\overline{\rho F'' G''} = \overline{\rho}\widetilde{F'' G''} \tag{129}$$

21 Derivation of first order moments

21.1 Mean continuity equation

We begin by instantaneous 3D continuity equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t \rho = -\left(\frac{1}{r^2}\partial_r\left(r^2[\rho u_r\ ]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi] \right) \tag{130}$$

$$\partial_t \overline{\rho} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[\rho u_r\ ]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi]}} \right) \quad \text{“ensemble” (space-time) averaging} \tag{131}$$

$$\partial_t \overline{\rho} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_r\ ]\right) \tag{132}$$

$$\partial_t \overline{\rho} = -\widetilde{u}_r\partial_r\overline{\rho} - \overline{\rho}\frac{1}{r^2}\partial_r(r^2\widetilde{u}_r) \tag{133}$$

$$\partial_t \overline{\rho} + \widetilde{u}_r\partial_r\overline{\rho} = -\overline{\rho}\frac{1}{r^2}\partial_r(r^2\widetilde{u}_r) \tag{134}$$

$$\widetilde{D}_t\overline{\rho} = -\overline{\rho}\widetilde{d} \tag{135}$$

21.2 Mean radial momentum equation

We begin by instantaneous 3D radial momentum equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t \rho u_r = -\left(\frac{1}{r^2}\partial_r\left(r^2[\rho u_r^2 - \tau_{rr}]\right) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\left([\rho u_r u_\phi - \tau_{r\phi}]\right) + G_r^M + \partial_r P \right) + \rho g_r \tag{136}$$

$$\partial_t \overline{\rho u_r} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[\rho u_r^2 - \tau_{rr}]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}])}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi\left([\rho u_r u_\phi - \tau_{r\phi}]\right)}} + \overline{G_r^M} + \partial_r \overline{P} \right) + \overline{\rho g_r} \tag{137}$$

$$\partial_t \overline{\rho u_r} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho u_r u_r}]\right) - \cancelto{0}{\frac{1}{r^2}\partial_r\left(r^2\overline{\tau}_{rr}\right)} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho g_r} \quad \text{we neglect mean viscosity } \overline{\tau} \tag{138}$$

$$\partial_t \overline{\rho}\widetilde{u_r} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r u_r}]\right) - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{139}$$

$$\partial_t \overline{\rho}\widetilde{u_r} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r}\widetilde{u_r} + \overline{\rho}\widetilde{u''_r u''_r}]\right) - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{140}$$

$$\partial_t \overline{\rho}\widetilde{u_r} + \frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r}\widetilde{u_r}]\right) = -\frac{1}{r^2}\partial_r\left(\overline{\rho}\widetilde{u''_r u''_r}\right) - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{141}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{142}$$

21.3 Mean polar momentum equation

We begin by instantaneous 3D polar momentum equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t \rho u_\theta = -\left(\frac{1}{r^2}\partial_r\left(r^2[\rho u_\theta u_r - \tau_{\theta r}]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]\right) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]\right) + G_\theta^M + \frac{1}{r}\partial_\theta P\right) - \rho\frac{1}{r}\partial_\theta\Phi \tag{143}$$

$$\partial_t \overline{\rho u_\theta} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[\rho u_\theta u_r - \tau_{\theta r}]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]\right)}} + \overline{G_\theta^M} + \frac{1}{r}\overline{\partial_\theta P} + \overline{\rho g_\theta}\right) \tag{144}$$

$$\partial_t \overline{\rho u_\theta} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho u_\theta u_r}]\right) + \cancelto{0}{\frac{1}{r^2}\partial_r\left(r^2\overline{\tau_{\theta r}}\right)} - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} + \cancelto{0}{\overline{\rho g_\theta}} \text{ we neglect mean viscosity } \overline{\tau} \text{ and gravity in } \theta \tag{145}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_\theta u_r}]\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{146}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_\theta\widetilde{u}_r + \overline{\rho}\widetilde{u''_\theta u''_r}\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{147}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta + \frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_\theta\widetilde{u}_r]\right) = -\frac{1}{r^2}\partial_r\left(r^2\overline{\rho}\widetilde{u''_\theta u''_r}\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{148}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G_\theta^M} - (1/r)\overline{\partial_\theta P} \tag{149}$$

21.4 Mean azimutal momentum equation

We begin by instantaneous 3D azimutal momentum equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\begin{aligned}
\partial_t\big(\rho u_\phi\big) = & -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \tau_{\phi r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \tau_{\phi\phi}]\big) + G^M_\phi + \frac{1}{r\sin\theta}\partial_\phi P\right) - \\
& - \rho\frac{1}{r\sin\theta}\partial_\phi\Phi
\end{aligned} \tag{150}$$

$$\begin{aligned}
\partial_t\overline{\rho u_\phi} = & -\left(\overline{\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \tau_{\phi r}]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \tau_{\phi\phi}]\big)}} + \overline{G^M_\phi} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi P}}\right) + \\
& + \overline{\rho g_\phi}
\end{aligned} \tag{151}$$

$$\partial_t\overline{\rho u_\phi} = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho u_\phi u_r}]\big) - \cancelto{0}{\frac{1}{r^2}\partial_r\big(r^2\overline{\tau_{\phi r}}\big)} + \overline{G^M_\phi} - \cancelto{0}{\overline{\rho g_\phi}} \text{ we neglect mean viscosity } \tau \text{ and gravity in } \phi \tag{152}$$

$$\partial_t\overline{\rho}\widetilde{u}_\phi = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u_\phi u_r}]\big) + \overline{G^M_\phi} \tag{153}$$

$$\partial_t\overline{\rho}\widetilde{u}_\phi = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_\phi\widetilde{u}_r - \overline{\rho}\widetilde{u''_\phi u''_r}]\big) - \overline{G^M_\phi} \tag{154}$$

$$\partial_t\overline{\rho}\widetilde{u}_\phi + \frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_\phi\widetilde{u}_r]\big) = -\frac{1}{r^2}\partial_r\big(r^2\overline{\rho}\widetilde{u''_\phi u''_r}\big) - \overline{G^M_\phi} \tag{155}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G^M_\phi} \tag{156}$$

21.5 Mean internal energy equation

We begin by instantaneous 3D internal energy equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\begin{aligned}
\partial_t\big(\rho\epsilon_I\big) = & -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_I]\right) - P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) + \\
& + \left(\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho\epsilon_{\rm nuc}
\end{aligned} \tag{157}$$

$$\partial_t \overline{\rho\epsilon_I} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[\rho u_r\epsilon_I]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta\epsilon_I]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_I]}}\right) - \overline{P\left(\frac{1}{r^2}\partial_r\left(r^2[u_r\ ]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[u_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right)} +$$
$$+\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[K\partial_r T]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[K\frac{1}{r}\partial_\theta T]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]}}\right) + \left(\overline{\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j}\right) + \overline{\rho\epsilon_{\mathrm{nuc}}} \tag{158}$$

$$\partial_t \overline{\rho\epsilon_I} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho u_r\epsilon_I}]\right) - \overline{Pd} + \frac{1}{r^2}\partial_r\left(r^2[\overline{\chi\partial_r T}]\right) + \overline{\tau_{ji}\partial_i u_j} + \overline{\rho\epsilon_{\mathrm{nuc}}} \tag{159}$$

$$\partial_t \overline{\rho}\widetilde{\epsilon_I} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r\epsilon_I}]\right) - \overline{Pd} + \frac{1}{r^2}\partial_r\left(r^2[\overline{\chi\partial_r T}]\right) + \overline{\tau_{ji}\partial_i u_j} + \overline{\rho\epsilon_{\mathrm{nuc}}} \tag{160}$$

$$\partial_t \overline{\rho}\widetilde{\epsilon_I} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_r\widetilde{\epsilon}_I]\right) - \frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u''_r\epsilon''_I}]\right) - \overline{Pd} + \frac{1}{r^2}\partial_r\left(r^2[\overline{\chi\partial_r T}]\right) + \cancelto{0}{\overline{\tau_{ji}}\partial_i\overline{u_j}} + \overline{\tau'_{ji}\partial_i u'_j} + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \quad \text{we neglect } \overline{\tau} \tag{161}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r f_I - \overline{Pd} + \nabla_r f_T + \overline{\tau'_{ji}\partial_i u'_j} + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \tag{162}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - \overline{P'd'} + \varepsilon_k + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \tag{163}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \varepsilon_k + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \tag{164}$$

21.6 Mean kinetic energy equation

We begin by instantaneous 3D kinetic energy equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t\left(\rho\epsilon_K\right) = -\left(\frac{1}{r^2}\partial_r[r^2\left(\rho u_r\epsilon_K - u_j\tau_{jr}\right)] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta\left(\rho u_\theta\epsilon_K - u_j\tau_{j\theta}\right)] + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_K - u_j\tau_{j\phi}]\right) -$$
$$-\left(\frac{1}{r^2}\partial_r\left(r^2[Pu_r]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[Pu_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + P\left(\frac{1}{r^2}\partial_r\left(r^2[u_r]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[u_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) -$$
$$-\left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) - \rho\left(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi\right) \tag{165}$$

$$
\begin{align}
\partial_t \overline{\rho \epsilon_K} =& -\left(\overline{\frac{1}{r^2}\partial_r[r^2(\rho u_r \epsilon_K - u_j \tau_{jr})]} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta[\sin\theta(\rho u_\theta \epsilon_K - u_j \tau_{j\theta})]}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi \epsilon_K - u_j \tau_{j\phi}]}} \right) - \nonumber \\
& - \left(\overline{\frac{1}{r^2}\partial_r(r^2[P u_r])} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta[P u_\theta])}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[P u_\phi]}} \right) + \overline{P\left(\frac{1}{r^2}\partial_r(r^2[u_r]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[u_\theta]) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right)} - \nonumber \\
& - \left(\overline{\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j}\right) + \overline{\rho(u_r g_r + \cancelto{0}{u_\theta g_\theta} + \cancelto{0}{u_\phi g_\phi})} \tag{166} \\
\partial_t \overline{\rho \epsilon_K} =& -\frac{1}{r^2}\partial_r[r^2(\overline{\rho u_r \epsilon_K})] - \frac{1}{r^2}\partial_r[r^2\overline{(u_j \tau_{jr})}] - \frac{1}{r^2}\partial_r(r^2[\overline{P u_r}]) + \overline{Pd} - \overline{\tau_{ji}\partial_i u_j} + \overline{\rho u_r g_r} \tag{167} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\frac{1}{r^2}\partial_r[r^2(\overline{\rho u''_r \epsilon''_K})] - \cancelto{0}{\frac{1}{r^2}\partial_r[r^2(\overline{u}_j\ \overline{\tau}_{jr})]} - \frac{1}{r^2}\partial_r[r^2\overline{(u'_j \tau'_{jr})}] - \frac{1}{r^2}\partial_r(r^2[\overline{P}\,\overline{u}_r]) - \frac{1}{r^2}\partial_r(r^2[\overline{P' u'_r}]) + \overline{P}\ \overline{d} + \overline{P'd'} - \varepsilon_k - \overline{\rho u''_r}\widetilde{g}_r + \overline{\rho}\ \overline{u}_r \widetilde{g}_r \tag{168} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\frac{1}{r^2}\partial_r[r^2(\overline{\rho u''_r \epsilon''_K})] - \frac{1}{r^2}\partial_r[r^2\overline{(u'_j \tau'_{jr})}] - \frac{1}{r^2}\partial_r(r^2[\overline{P}\,\overline{u}_r]) - \frac{1}{r^2}\partial_r(r^2[\overline{P' u'_r}]) + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r \widetilde{g}_r \tag{169} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\nabla_r \overline{\rho u''_r \epsilon''_K} - \nabla_r f_\tau - (\overline{P}\nabla_r \overline{u}_r + \overline{u}_r \partial_r \overline{P}) - \nabla_r f_P + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r \widetilde{g}_r \tag{170} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\nabla_r \overline{\rho u''_r \epsilon''_K} - \nabla_r f_\tau - \overline{P}\ \overline{d} - \overline{\rho}\ \overline{u}_r \widetilde{g}_r - \nabla_r f_P + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r \widetilde{g}_r \tag{171} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\nabla_r \overline{\rho u''_r \epsilon''_K} - \nabla_r (f_\tau + f_P) + W_P - \varepsilon_k + W_b \tag{172}
\end{align}
$$

Second way:

$$
\begin{align}
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& + \overline{\rho}\widetilde{D}_t \widetilde{u}_i \widetilde{u}_i + \overline{\rho}\widetilde{D}_t \widetilde{u''_i u''_i} \tag{173} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& + \overline{\rho}\widetilde{D}_t \widetilde{u}_i \widetilde{u}_i + \overline{\rho}\widetilde{D}_t \widetilde{k} \tag{174} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K =& -\nabla_r (f_k + f_P + f_\tau) - \widetilde{R}_{ir}\partial_r \widetilde{u}_i + W_b + W_P - \varepsilon_k + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i \widetilde{u}_i/2) \tag{175}
\end{align}
$$

where equation for the $\widetilde{k}$ is derived later.

21.7 Mean total energy equation

$$
\begin{align}
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_t =& + \overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_I + \overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_K \tag{176} \\
\overline{\rho}\widetilde{D}_t \widetilde{\epsilon}_t =& -\nabla_r (f_I + f_T + f_k + f_P + f_\tau) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r \widetilde{u}_i + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i \widetilde{u}_i/2) \tag{177}
\end{align}
$$

21.8 Mean pressure equation

We begin by deriving 3D instantaneous pressure equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$dP = \frac{\partial P}{\partial \rho}\Big|_{\epsilon_I} d\rho + \frac{\partial P}{\partial \epsilon_I}\Big|_{\rho} d\epsilon_I = \frac{P}{\rho}(1-\Gamma_3+\Gamma_1)d\rho + \rho(\Gamma_3-1)d\epsilon_I \tag{178}$$

$$D_t P = + \frac{P}{\rho}(1-\Gamma_3+\Gamma_1)D_t\rho + (\Gamma_3-1)\rho D_t\epsilon_I \tag{179}$$

$$D_t P = -(1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd+\mathcal{S}+\nabla\cdot F_T+\tau_{ij}\partial_i u_j) \tag{180}$$

$$\partial_t P = -u_n\partial_n P - (1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd+\mathcal{S}+\nabla\cdot F_T+\tau_{ij}\partial_i u_j) \tag{181}$$

$$\partial_t P = -\left(\frac{1}{r^2}\partial_r\left(r^2[Pu_r\ ]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[Pu_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + (1-\Gamma_1)Pd + (\Gamma_3-1)(\mathcal{S}+\nabla\cdot F_T+\tau_{ij}\partial_i u_j) \tag{182}$$

$$\partial_t\overline{P} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[Pu_r\ ]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[Pu_\theta]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]}}\right) + (1-\Gamma_1)\overline{Pd} + (\Gamma_3-1)(\mathcal{S}+\overline{\nabla\cdot F_T}+\overline{\tau_{ij}\partial_i u_j}) \tag{183}$$

$$\partial_t\overline{P} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{P}\,\overline{u}_r]\right) - \frac{1}{r^2}\partial_r\left(r^2[\overline{P'u'_r}]\right) + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)\overline{P'd'} + (\Gamma_3-1)(\mathcal{S}+\frac{1}{r^2}\partial_r\left(r^2\overline{\chi\partial_r T}\right) + \cancelto{0}{\overline{\tau_{ij}}\partial_i\overline{u'_j}} + \overline{\tau'_{ij}\partial_i u'_j}) \tag{184}$$

$$\partial_t\overline{P} = -\nabla_r\overline{P}\,\overline{u}_r - \nabla_r f_P + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S}+\nabla_r f_T+\varepsilon_k) \tag{185}$$

$$\partial_t\overline{P} = -\overline{u}_r\partial_r\overline{P} - \overline{P}\ \overline{d} - \nabla_r f_P + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\overline{\mathcal{S}}+\nabla_r f_T+\varepsilon_k) \tag{186}$$

$$\partial_t\overline{P} + \overline{u}_r\partial_r\overline{P} = -\nabla_r f_P - \Gamma_1\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S}+\nabla_r f_T+\varepsilon_k) \tag{187}$$

$$\overline{D}_t\overline{P} = -\nabla_r f_P - \Gamma_1\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S}+\nabla_r f_T+\varepsilon_k) \tag{188}$$

21.9 Mean enthalpy equation

We start from the total energy equation, where we can subsitute $\rho\epsilon_t = \rho h + \rho\epsilon_k - P$ (for clarity reasons we use here more compact vector notation):

$$\partial_t\epsilon_t + \vec{\nabla}\cdot\left((\rho\epsilon_t+P)\vec{u}\right) = \rho\vec{u}\cdot\vec{g} + \vec{\nabla}\cdot F_T + \mathcal{S} \tag{189}$$

$$\partial_t(\rho h + \rho\epsilon_k - P) + \vec{\nabla}\cdot(\rho h\vec{u} + \rho\epsilon_k\vec{u}) = \rho\vec{u}\cdot\vec{g} + \vec{\nabla}\cdot F_T + \mathcal{S} \tag{190}$$

$$\partial_t\rho h + \partial_t\rho\epsilon_k - \partial_t P = -\vec{\nabla}\cdot(\rho h\vec{u} + \rho\epsilon_k\vec{u}) + \rho\vec{u}\cdot\vec{g} + \vec{\nabla}\cdot F_T + \mathcal{S} \tag{191}$$

$$\partial_t\rho h + \left[-\vec{\nabla}\cdot(\rho\epsilon_k\vec{u}) - \vec{\nabla}\cdot(P\vec{u}) + P(\vec{\nabla}\cdot\vec{u}) + \rho\vec{u}\cdot\vec{g} + \nabla_i u_j\tau_{ji}\right] - \left[-\vec{\nabla}\cdot(P\vec{u}) + (1-\Gamma_1)P\vec{\nabla}\cdot\vec{u} + (\Gamma_3-1)(\mathcal{S}+\vec{\nabla}\cdot F_T+\tau_{ij}\partial_j u_i)\right] = \tag{192}$$

$$= -\vec{\nabla}\cdot(\rho h\vec{u} + \rho\epsilon_k\vec{u}) + \rho\vec{u}\cdot\vec{g} + \vec{\nabla}\cdot F_T + \mathcal{S} \tag{193}$$

$$\partial_t\rho h + \left[-\cancel{\vec{\nabla}\cdot(\rho\epsilon_k\vec{u})} - \cancel{\vec{\nabla}\cdot(P\vec{u})} + \cancel{P(\vec{\nabla}\cdot\vec{u})} + \cancel{\rho\vec{u}\cdot\vec{g}} + \nabla_i u_j\tau_{ji}\right] + \cancel{\vec{\nabla}\cdot(P\vec{u})} - \cancel{P\vec{\nabla}\cdot\vec{u}} + \Gamma_1 P\vec{\nabla}\cdot\vec{u} - \Gamma_3(\mathcal{S}+\vec{\nabla}\cdot F_T) + (\cancel{\mathcal{S}} + \cancel{\vec{\nabla}\cdot F_T}) - = \tag{194}$$

$$-(\Gamma_3-1)\tau_{ij}\partial_j u_i = -\vec{\nabla}\cdot(\rho h\vec{u} + \cancel{\rho\epsilon_k\vec{u}}) + \cancel{\rho\vec{u}\cdot\vec{g}} + \cancel{\vec{\nabla}\cdot F_T} + \cancel{\mathcal{S}} \tag{195}$$

So, from the above we have:

$$\partial_t \rho h + \nabla_i u_j \tau_{ji} + \Gamma_1 P \vec{\nabla} \cdot \vec{u} - \Gamma_3 \mathcal{S} - \Gamma_3 \vec{\nabla} \cdot F_T - (\Gamma_3 - 1) \tau_{ij} \partial_j u_i = -\vec{\nabla} \cdot (\rho h \vec{u}) \tag{196}$$

$$\rho D_t h = -\Gamma_1 P \vec{\nabla} \cdot \vec{u} + \Gamma_3 \mathcal{S} + \Gamma_3 \vec{\nabla} \cdot F_T - \nabla_i u_j \tau_{ji} + (\Gamma_3 - 1) \tau_{ij} \partial_j u_i \tag{197}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{h} = -\nabla_r f_h - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T - \nabla_r \overline{u_j \tau_{jr}} + (\Gamma_3 - 1) \overline{\tau_{ij} \partial_j u_i} \tag{198}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{h} = -\nabla_r (f_h + f_\tau) - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T + (\Gamma_3 - 1) \varepsilon_k \tag{199}$$

21.10 Mean temperature equation

We begin by deriving 3D instantaneous temperature equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$dT = \frac{\partial T}{\partial \rho}\Big|_{\epsilon_I} d\rho + \frac{\partial T}{\partial \epsilon_I}\Big|_{\rho} d\epsilon_I = \left(\frac{T}{\rho}(\Gamma_3 - 1) - \frac{P}{\rho^2} \frac{1}{c_v} \right) d\rho + \frac{1}{c_v} d\epsilon_I \tag{200}$$

$$D_t T = + \frac{T}{\rho}(\Gamma_3 - 1) D_t \rho - \frac{P}{\rho^2} \frac{1}{c_v} D_t \rho + \frac{1}{c_v} D_t \epsilon_I \tag{201}$$

$$D_t T = -\frac{T}{\rho}(\Gamma_3 - 1)(\rho d) + \frac{P}{\rho^2} \frac{1}{c_v}(\rho d) + \frac{1}{c_v} \left(-\frac{Pd}{\rho} + \frac{\nabla \cdot F_T}{\rho} + \frac{\tau_{ij} \partial_j u_i}{\rho} + \frac{\mathcal{S}}{\rho} \right) \tag{202}$$

$$D_t T = -(\Gamma_3 - 1) T d + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij} \partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} \tag{203}$$

$$\partial_t T = -u_n \partial_n T - (\Gamma_3 - 1) T d + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij} \partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} \tag{204}$$

$$\partial_t T = -\left(\frac{1}{r^2} \partial_r \left(r^2 [T u_r\ ] \right) + \frac{1}{r \sin \theta} \partial_\theta \left(\sin \theta [T u_\theta] \right) + \frac{1}{r \sin \theta} \partial_\phi [T u_\phi] \right) + T d - (\Gamma_3 - 1) T d + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij} \partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} \tag{205}$$

$$\partial_t \overline{T} = -\left(\overline{\frac{1}{r^2} \partial_r \left(r^2 [T u_r\ ] \right)} + \cancelto{0}{\overline{\frac{1}{r \sin \theta} \partial_\theta \left(\sin \theta [T u_\theta] \right)}} + \cancelto{0}{\overline{\frac{1}{r \sin \theta} \partial_\phi [T u_\phi]}} \right) + \overline{Td} - (\Gamma_3 - 1) \overline{Td} + \overline{\frac{\nabla \cdot F_T}{c_v \rho}} + \overline{\frac{\tau_{ij} \partial_j u_i}{c_v \rho}} + \overline{\frac{\mathcal{S}}{c_v \rho}} \tag{206}$$

$$\partial_t \overline{T} = -\nabla_r \overline{T} \overline{u}_r - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{207}$$

$$\partial_t \overline{T} + \overline{u}_r \partial_r \overline{T} = -\overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{208}$$

$$\overline{D}_t \overline{T} = -\overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{209}$$

$$\overline{D}_t \overline{T} = -\overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{T}\ \overline{d} + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{210}$$

$$\overline{D}_t \overline{T} = -\nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{211}$$

$$\overline{D}_t \overline{T} = -\nabla_r f_T + \overline{T}\ \overline{d} + \overline{T'd'} - \Gamma_3 (\overline{T}\ \overline{d} + \overline{T'd'}) + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{212}$$

$$\overline{D}_t \overline{T} = -\nabla_r f_T + (1 - \Gamma_3) \overline{T}\ \overline{d} + (2 - \Gamma_3) \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij} \partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} \tag{213}$$

21.11 Mean angular momentum equation (z-component)

Z component of the specific angular momentum is defined as $j_z = r\sin\theta u_\phi$. We begin by multiplying the instantaneous 3D azimutal momentum equation by $r\sin\theta$, neglect viscosity and ϕ component of gravity and obtain (Sect.1):

$$r\sin\theta\partial_t\big(\rho u_\phi\big) = -r\sin\theta\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \cancelto{0}{\tau_{\phi r}}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \cancelto{0}{\tau_{\theta\phi}}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \cancelto{0}{\tau_{\phi\phi}}]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) - \cancelto{0}{-r\sin\theta\rho\frac{1}{r\sin\theta}\partial_\phi\Phi} \quad (214)$$

$$r\sin\theta\partial_t\big(\rho u_\phi\big) = -r\sin\theta\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) \quad (215)$$

$$\partial_t\rho j_z = -\frac{1}{r^2}\partial_r(r^2\rho j_z u_r) - \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\rho j_z u_\theta) - \frac{1}{r\sin\theta}\partial_\phi(\rho j_z u_\phi) - \partial_\phi P \quad (216)$$

$$\partial_t\overline{\rho j_z} = -\overline{\frac{1}{r^2}\partial_r(r^2\rho j_z u_r)} - \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta\rho j_z u_\theta)}} - \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi(\rho j_z u_\phi)}} - \cancelto{0}{\overline{\partial_\phi P}} \quad (217)$$

$$\partial_t\overline{\rho j_z} = -\frac{1}{r^2}\partial_r(r^2\overline{\rho j_z u_r}) \quad (218)$$

$$\partial_t\overline{\rho}\widetilde{j_z} = -\frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z}\widetilde{u_r}) - \frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z''u_r''}) \quad (219)$$

$$\partial_t\overline{\rho}\widetilde{j_z} + \frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z}\widetilde{u_r}) = -\frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z''u_r''}) \quad (220)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{j_z} = -\nabla_r f_{jz} \quad (221)$$

21.12 Mean α equation

We begin by 3D instantaneous composition equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t\big(\rho X_\alpha\big) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r X_\alpha]) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta X_\alpha]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_\alpha]\right) + \rho\dot{X}_\alpha^{\rm nuc} \qquad \alpha = 1...N_{nuc} \quad (222)$$

$$\partial_t\overline{\rho X_\alpha} = -\left(\overline{\frac{1}{r^2}\partial_r(r^2[\rho u_r X_\alpha])} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta X_\alpha]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_\alpha]}}\right) + \overline{\rho\dot{X}_\alpha^{\rm nuc}} \quad (223)$$

$$\partial_t\overline{\rho}\widetilde{X}_\alpha = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{X}_\alpha + \overline{\rho}\widetilde{u_r''X_\alpha''}]) + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \quad (224)$$

$$\partial_t\overline{\rho}\widetilde{X}_\alpha + \frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{X}_\alpha]) = -\frac{1}{r^2}\partial_r(\overline{\rho}\widetilde{u_r''X_\alpha''}) + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \quad (225)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \quad (226)$$

21.13 Mean number of nucleons per isotope (A) equation

We begin by deriving 3D instantaneous A equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$A = + \left(\sum_\alpha \frac{X_\alpha}{A_\alpha} \right)^{-1} \tag{227}$$

$$D_t A = + D_t \left(\sum_\alpha \frac{X_\alpha}{A_\alpha} \right)^{-1} = +D_t \frac{1}{\sum_\alpha (X_\alpha / A_\alpha)} = - \frac{D_t \sum_\alpha (X_\alpha / A_\alpha)}{[\sum_\alpha (X_\alpha / A_\alpha)]^2} = -A^2 D_t \sum_\alpha \frac{X_\alpha}{A_\alpha} \tag{228}$$

$$D_t A = - A^2 D_t \sum_\alpha \frac{X_\alpha}{A_\alpha} = -A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha - X_\alpha D_t A_\alpha}{A_\alpha^2} = -A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha}{A_\alpha^2} \tag{229}$$

$$D_t A = - A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha}{A_\alpha^2} = -A^2 \sum_\alpha \frac{A_\alpha \dot{X}_\alpha^{\rm nuc}}{A_\alpha^2} = -A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha} \tag{230}$$

$$\rho D_t A = - \rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha} \tag{231}$$

$$\partial_t \rho A = - \left(\frac{1}{r^2} \partial_r (r^2 [\rho u_r A]) + \frac{1}{r \sin\theta} \partial_\theta \big(\sin\theta [\rho u_\theta A] \big) + \frac{1}{r \sin\theta} \partial_\phi [\rho u_\phi A] \right) - \rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha} \tag{232}$$

$$\partial_t \overline{\rho A} = - \left(\overline{\frac{1}{r^2} \partial_r (r^2 [\rho u_r A])} + \cancelto{0}{\overline{\frac{1}{r \sin\theta} \partial_\theta \big(\sin\theta [\rho u_\theta A] \big)}} + \cancelto{0}{\overline{\frac{1}{r \sin\theta} \partial_\phi [\rho u_\phi A]}} \right) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{233}$$

$$\partial_t \overline{\rho A} = - \frac{1}{r^2} \partial_r (r^2 [\overline{\rho u_r A}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{234}$$

$$\partial_t \overline{\rho} \widetilde{A} = - \frac{1}{r^2} \partial_r (r^2 [\overline{\rho} \widetilde{u_r A}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{235}$$

$$\partial_t \overline{\rho} \widetilde{A} = - \frac{1}{r^2} \partial_r (r^2 [\overline{\rho} \widetilde{u}_r \widetilde{A}]) - \frac{1}{r^2} \partial_r (r^2 [\overline{\rho} \widetilde{u''_r A''}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{236}$$

$$\partial_t \overline{\rho} \widetilde{A} + \frac{1}{r^2} \partial_r (r^2 [\overline{\rho} \widetilde{u}_r \widetilde{A}]) = - \frac{1}{r^2} \partial_r (r^2 [\overline{\rho} \widetilde{u''_r A''}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{237}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{A} = - \nabla_r f_A - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\rm nuc}}{A_\alpha}} \tag{238}$$

21.14 Mean charge per isotope (Z) equation

We begin by deriving 3D instantaneous Z equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$Z = +A \sum_\alpha \frac{Z_\alpha A_\alpha}{A_\alpha} \tag{239}$$

$$D_t Z = +D_t \left(A \sum_i \frac{Z_\alpha A_\alpha}{A_\alpha} \right) = \sum_\alpha \frac{Z_\alpha X_\alpha}{A_\alpha} D_t A + A \sum_\alpha D_t \frac{Z_\alpha X_\alpha}{A_\alpha} \tag{240}$$

$$D_t Z = -\sum_\alpha \frac{Z_\alpha X_\alpha}{A_\alpha} A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A \sum_\alpha \frac{A_\alpha D_t Z_\alpha X_\alpha - Z_\alpha X_\alpha D_t A_\alpha}{A_\alpha^2} \tag{241}$$

$$D_t Z = -ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A \sum_\alpha \frac{A_\alpha D_t Z_\alpha X_\alpha}{A_\alpha^2} \tag{242}$$

$$D_t Z = -ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A \sum_\alpha \frac{Z_\alpha D_t X_\alpha + X_\alpha D_t Z_\alpha}{A_\alpha^2} \tag{243}$$

$$D_t Z = -ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{244}$$

$$\rho D_t Z = -\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + \rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{245}$$

$$\partial_t \overline{\rho Z} = -\left(\overline{\frac{1}{r^2}\partial_r (r^2 [\rho u_r Z])} + \cancelto{0}{\overline{\frac{1}{r \sin\theta}\partial_\theta (\sin\theta [\rho u_\theta Z])}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi [\rho u_\phi Z]}} \right) - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{246}$$

$$\partial_t \overline{\rho Z} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho u_r Z}]) - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{247}$$

$$\partial_t \overline{\rho} \widetilde{Z} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho} \widetilde{u_r Z}]) - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{248}$$

$$\partial_t \overline{\rho} \widetilde{Z} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho} \widetilde{u}_r \widetilde{Z}]) - \frac{1}{r^2}\partial_r (r^2 [\overline{\rho} \widetilde{u''_r Z''}]) - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{249}$$

$$\partial_t \overline{\rho} \widetilde{Z} + \frac{1}{r^2}\partial_r (r^2 [\overline{\rho} \widetilde{u}_r \widetilde{Z}]) = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho} \widetilde{u''_r Z''}]) - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{250}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{Z} = -\nabla_r f_Z - \overline{\rho ZA \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A \sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{251}$$

21.15 Mean entropy equation

We can derive the mean entropy equation in the following way (Sect.1):

$$\rho D_t s = + (\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T \tag{252}$$

$$\partial_t \rho s + \nabla_r (\rho s u_r) = + (\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T \tag{253}$$

$$\partial_t \overline{\rho s} + \nabla_r \overline{\rho u_r s} = + \overline{(\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T} \tag{254}$$

$$\partial_t \overline{\rho} \widetilde{s} + \nabla_r (\overline{\rho} \widetilde{u}_r \widetilde{s}) = - \nabla_r (\overline{\rho} \widetilde{s'' u''_r}) + \overline{\nabla \cdot F_T / T} + \overline{\mathcal{S}/T} + \overline{\varepsilon_k / T} \tag{255}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{s} = - \nabla_r f_s + \overline{\nabla \cdot F_T / T} + \overline{\mathcal{S}/T} + \overline{\varepsilon_k / T} \tag{256}$$

22 General formula for second and third order moments and variances

22.1 Second-order moments

In order to calculate evolution equations for correlations of two arbitrary fluctuations, we can derive the following general formula.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} - \overline{\rho D_t c''d''} = \overline{\rho}\big(\partial_t\widetilde{c''d''} + \widetilde{u}_n\partial_n\widetilde{c''d''}\big) - \overline{\rho\big(\partial_t c''d'' + u_n\partial_n c''d''\big)} = \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \overline{\rho\partial_t c''d''} - \overline{\rho u_n\partial_n c''d''} =$$

$$= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\partial_t\rho c''d''} - \overline{c''d''\partial_t\rho}\big) - \overline{\rho u_n\partial_n c''d''} = \tag{257}$$

$$= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \partial_t\overline{\rho}\widetilde{c''d''} - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{258}$$

$$= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\rho}\partial_t\widetilde{c''d''} + \widetilde{c''d''}\partial_t\overline{\rho}\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{259}$$

$$= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\rho}\partial_t\widetilde{c''d''} - \widetilde{c''d''}\partial_n\overline{\rho}\widetilde{u}_n\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{260}$$

$$= \overline{\rho}\partial_t\widetilde{c''d''} - \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} + \widetilde{c''d''}\partial_n\overline{\rho}\widetilde{u}_n - \overline{\partial_n\rho u_n c''d''} = \tag{261}$$

$$= \partial_n\overline{\rho}\widetilde{u}_n\widetilde{c''d''} - \big(\overline{\partial_n\rho\widetilde{u}_n c''d''} + \overline{\partial_n\rho u''_n c''d''}\big) = -\overline{\partial_n\rho u''_n c''d''} \tag{262}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{\rho D_t c''d''} - \overline{\partial_n\rho u''_n c''d''} = \overline{c''\rho D_t d''} + \overline{d''\rho D_t c''} - \overline{\partial_n\rho u''_n c''d''} \tag{263}$$

$$\rho D_t c'' = \rho D_t c - \rho D_t\widetilde{c} = \rho D_t c - \rho\widetilde{D}_t\widetilde{c} - \rho u''_n\partial_n\widetilde{c} = \rho D_t c - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{c}\right] - \rho u''_n\partial_n\widetilde{c} \tag{264}$$

$$\rho D_t d'' = \rho D_t d - \rho D_t\widetilde{d} = \rho D_t d - \rho\widetilde{D}_t\widetilde{d} - \rho u''_n\partial_n\widetilde{d} = \rho D_t d - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{d}\right] - \rho u''_n\partial_n\widetilde{d} \tag{265}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\left(\rho D_t d - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{d}\right] - \rho u''_n\partial_n\widetilde{d}\right)} + \overline{d''\left(\rho D_t c - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{c}\right] - \rho u''_n\partial_n\widetilde{c}\right)} - \overline{\partial_n\rho c''d''u''_n} \tag{266}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = +\overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \tag{267}$$

22.2 Third-order moments

In order to calculate evolution equations for correlations of three arbitrary fluctuations, we can derive the following general formula.

$$\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{c''d''e''} - \overline{\rho D_t c''d''e''} &= \overline{\rho}\left(\partial_t\widetilde{c''d''e''} + \widetilde{u}_n\partial_n\widetilde{c''d''e''}\right) - \overline{\rho(\partial_t c''d''e'' + u_n\partial_n c''d''e'')} = && (268)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - \overline{\rho\partial_t c''d''e''} - \overline{\rho u_n\partial_n c''d''e''} = && (269)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - (\overline{\partial_t\rho c''d''e''} - \overline{c''d''e''\partial_t\rho}) - \overline{\rho u_n\partial_n c''d''e''} = && (270)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - \partial_t\overline{\rho}\widetilde{c''d''e''} - (\overline{c''d''e''\partial_n\rho u_n} + \overline{\rho u_n\partial_n c''d''e''}) = && (271)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - (\overline{\rho}\partial_t\widetilde{c''d''e''} + \widetilde{c''d''e''}\partial_t\overline{\rho}) - \overline{\partial_n c''d''e''\rho u_n} = && (272)\\
&= \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} + \widetilde{c''d''e''}\partial_n\overline{\rho}\widetilde{u}_n - \overline{\partial_n c''d''e''\rho\widetilde{u}_n} - \overline{\partial_n c''d''e''\rho u''_n} = && (273)\\
&= -\overline{\partial_n c''d''e''\rho u''_n} && (274)\\
& && (275)
\end{aligned}$$

$$\overline{\rho D_t c''d''e''} = \overline{\rho c''d''D_t e''} + \overline{\rho c''e''D_t d''} + \overline{\rho d''e''D_t c''} \qquad (276)$$

$$\begin{aligned}
\overline{c''d''\rho D_t e''} &= \overline{c''d''\rho D_t e} - \overline{c''d''\rho D_t\widetilde{e}} = \overline{c''d''\rho D_t e} - \overline{c''d''\rho(\partial_t\widetilde{e} + u_n\partial_n\widetilde{e})} = && (277)\\
&= \overline{c''d''\rho D_t e} - \overline{c''d''\rho\partial_t\widetilde{e}} - \overline{c''d''\rho u_n\partial_n\widetilde{e}} = \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\partial_t\widetilde{e} - (\overline{c''d''\rho\widetilde{u}_n\partial_n\widetilde{e}} + \overline{c''d''\rho u''_n\partial_n\widetilde{e}}) = && (278)\\
&= \overline{c''d''\rho D_t e} - (\overline{\rho}\widetilde{c''d''}\partial_t\widetilde{e} + \overline{\rho}\widetilde{c''d''}\widetilde{u}_n\partial_n\widetilde{e}) - \overline{\rho}\widetilde{c''d''u''_n}\partial_n\widetilde{e} = && (279)\\
&= \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\widetilde{D}_t\widetilde{e} - \overline{\rho}\widetilde{c''d''u''_n}\partial_n\widetilde{e} && (280)
\end{aligned}$$

$$\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{c''d''e''} = \;& \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\widetilde{D}_t\widetilde{e} - \overline{\rho}\widetilde{c''d''u''_n}\partial_n\widetilde{e} + && (281)\\
& + \overline{c''e''\rho D_t d} - \overline{\rho}\widetilde{c''e''}\widetilde{D}_t\widetilde{d} - \overline{\rho}\widetilde{c''e''u''_n}\partial_n\widetilde{d} + && (282)\\
& + \overline{d''e''\rho D_t c} - \overline{\rho}\widetilde{d''e''}\widetilde{D}_t\widetilde{c} - \overline{\rho}\widetilde{d''e''u''_n}\partial_n\widetilde{c} - && (283)\\
& - \overline{\partial_n c''d''e''\rho u''_n} && (284)
\end{aligned}$$

22.3 Reynolds and Favrian variance

The Reynolds variance can be derived in following way:

$$\widetilde{D}_t\overline{c'd'} - \overline{D_tc'd'} = +\partial_t\overline{c'd'} + \widetilde{u}_n\partial_n\overline{c'd'} - (\partial_t\overline{c'd'} + \overline{u_n\partial_n c'd'}) = \tag{285}$$
$$= +\partial_t\overline{c'd'} + \widetilde{u}_n\partial_n\overline{c'd'} - \partial_t\overline{c'd'} - \widetilde{u}_n\partial_n\overline{c'd'} - \overline{u''_n\partial_n c'd'} = \tag{286}$$
$$= -\overline{u''_n\partial_n c'd'} \tag{287}$$

Next step:

$$\widetilde{D}_t\overline{c'd'} = \overline{D_tc'd'} - \overline{u''_n\partial_n c'd'} = \overline{c'D_td'} + \overline{d'D_tc'} - \overline{u''_n\partial_n c'd'} \tag{288}$$

Next step:

$$D_tc' = D_tc - D_t\overline{c} = D_tc - \widetilde{D}_t\overline{c} - u''_n\partial_n\overline{c} \tag{289}$$
$$D_td' = D_td - D_t\overline{c} = D_td - \widetilde{D}_t\overline{d} - u''_n\partial_n\overline{d} \tag{290}$$

Now let's put these equations in the Equation 288:

$$\widetilde{D}_t\overline{c'd'} = +\overline{c'D_td'} + \overline{d'D_tc'} - \overline{u''_n\partial_n c'd'} = \tag{291}$$
$$= +\overline{c'(D_td - \widetilde{D}_t\overline{d} - u''_n\partial_n\overline{d})} + \overline{d'(D_tc - \widetilde{D}_t\overline{c} - u''_n\partial_n\overline{c})} - \overline{u''_n\partial_n c'd'} = \tag{292}$$
$$= +\overline{c'D_td} - \overline{c'u''_n}\partial_n\overline{d} + \overline{d'D_tc} - \overline{d'u''_n}\partial_n\overline{c} - \overline{u''_n\partial_n c'd'} = \tag{293}$$
$$= +\overline{c'D_td} - \overline{c'u''_n}\partial_n\overline{d} + \overline{d'D_tc} - \overline{d'u''_n}\partial_n\overline{c} - \partial_n\overline{u''_nc'd'} + \overline{c'd'\partial_n u''_n} \tag{294}$$

From this general formula by substituting $d = c$ we get:

$$\widetilde{D}_t\overline{c'c'} = +2\overline{c'D_tc} - 2\overline{c'u''_n}\partial_n\overline{c} - \overline{u''_n\partial_n c'c'} = \tag{295}$$
$$= +2\overline{c'D_tc} - 2\overline{c'u''_n}\partial_n\overline{c} - \partial_n\overline{u''_nc'c'} + \overline{c'c'\partial_n u''_n} \tag{296}$$

The Favrian variance can be easily derived from general equation for second-order moments Equation 267 ie.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = +\overline{c''\rho D_td} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_tc} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \tag{297}$$

Now, substitute $d = c$ and you'll get the equation for Favrian variance:

$$\overline{\rho}\widetilde{D}_t\widetilde{c''c''} = +2\overline{c''\rho D_tc} - 2\overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''c''u''_n} \tag{298}$$

23 Derivation of second-order moments equations

23.1 Reynolds stress equation

We can derive the Reynolds stress equation using the general formula for second order moments, where we substitute c'' with u''_i and d'' with u''_j.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{299}$$

$$\underbrace{\overline{\rho}\widetilde{D}_t\widetilde{u''_i u''_j}}_{\overline{\rho}\widetilde{D}_t(\widetilde{R}_{ij}/\overline{\rho})} = \overline{u''_i\rho D_t u_j} - \underbrace{\overline{\rho}\widetilde{u''_i u''_n}\partial_n\widetilde{u}_j}_{\widetilde{R}_{in}\partial_n\widetilde{u}_j} + \overline{u''_j\rho D_t u_i} - \underbrace{\overline{\rho}\widetilde{u''_j u''_n}\partial_n\widetilde{u}_i}_{\widetilde{R}_{jn}\partial_n\widetilde{u}_i} - \underbrace{\overline{\partial_n\rho u''_i u''_j u''_n}}_{\nabla\cdot\rho u''_i u''_j u''_n} \tag{300}$$

So, the general formula for Reynolds stress $\widetilde{R}_{ij}$ is:

$$\overline{\rho}\widetilde{D}_t(\widetilde{R}_{ij}/\overline{\rho}) = -\left(\widetilde{R}_{in}\partial_n\widetilde{u}_j + \widetilde{R}_{jn}\partial_n\widetilde{u}_i\right) - \left(\nabla_r\overline{\rho}\widetilde{u''_i u''_j u''_r} + \overline{G^R_{rr}}\right) + \overline{u''_i\rho D_t u_j} + \overline{u''_j\rho D_t u_i} \tag{301}$$

Mean equation for $\widetilde{R}_{rr}$

$$\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{rr}/\overline{\rho}\big) = -\left(\widetilde{R}_{rn}\partial_n\widetilde{u}_r + \widetilde{R}_{rn}\partial_n\widetilde{u}_r\right) - \left(\nabla_r 2f^r_k + \overline{G^R_{rr}}\right) + 2\overline{u''_r\rho D_t u_r} \tag{302}$$

where

$$\overline{u''_r\rho D_t u_r} = \overline{u''_r\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big) - G^M_r - \partial_r P + \rho g_r\right)} \tag{303}$$

$$\overline{u''_r\rho D_t u_r} = \overline{u''_r\left(\nabla_r\tau_{rr} + \nabla_\theta\tau_{r\theta} + \nabla_\phi\tau_{r\phi} - G^M_r - \partial_r P + \rho g_r\right)} \tag{304}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_r\nabla_r\tau_{rr}} = \overline{u''_r}\nabla_r\overline{\tau_{rr}} + \overline{u''_r\nabla_r\tau'_{rr}} = \cancelto{0}{\overline{u''_r}\nabla_r\overline{\tau_{rr}}} + \nabla_r(\overline{u''_r\tau'_{rr}}) - \overline{\tau'_{rr}\partial_r u''_r} \tag{305}$$

$$+\overline{u''_r\nabla_\theta\tau_{r\theta}} = \cancelto{0}{\overline{u''_r}\nabla_\theta\overline{\tau_{r\theta}}} + \overline{u''_r\nabla_\theta\tau'_{r\theta}} = \cancelto{0}{\overline{\nabla_\theta(u''_r\tau'_{r\theta})}} - \overline{\tau'_{r\theta}\frac{1}{r}\partial_\theta u''_r} \tag{306}$$

$$+\overline{u''_r\nabla_\phi\tau_{r\phi}} = \cancelto{0}{\overline{u''_r}\nabla_\phi\overline{\tau_{r\phi}}} + \overline{u''_r\nabla_\phi\tau'_{r\phi}} = \cancelto{0}{\overline{\nabla_\phi(u''_r\tau'_{r\phi})}} - \overline{\tau'_{r\phi}\frac{1}{r\sin\theta}\partial_\phi u''_r} \tag{307}$$

$$-\overline{u''_r G^M_r} = -\overline{u''_r G^M_r} \tag{308}$$

$$-\overline{u''_r\partial_r P} = -\overline{u''_r}\partial_r\overline{P} - \overline{u''_r\partial_r P'} = -\overline{u''_r}\partial_r\overline{P} - \nabla_r\overline{(u''_r P')} + \overline{P'\nabla_r u''_r} \tag{309}$$

$$+\overline{u''_r\rho g_r} \sim \overline{u''_r\rho}\overline{g}_r = 0 \tag{310}$$

Final equation is:

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -2\widetilde{R}_{rr}\partial_r\widetilde{u}_r - \nabla_r\widetilde{F}^R_{rrr} - \overline{G^R_{rr}} + 2\nabla_r(\overline{u''_r\tau'_{rr}}) - 2\overline{u''_r G^M_r} - 2\overline{u''_r}\partial_r\overline{P} - 2\nabla_r\overline{(u''_r P')} + 2\overline{P'\nabla_r u''_r} - \tag{311}$$

$$-2\left(\overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}\frac{1}{r}\partial_\theta u''_r} + \overline{\tau'_{r\phi}\frac{1}{r\sin\theta}\partial_\phi u''_r}\right)$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -\nabla_r(2f^r_k + 2f_P + 2f^r_\tau) + 2W_b - 2\widetilde{R}_{rr}\partial_r\widetilde{u}_r + 2\overline{P'\nabla_r u''_r} - 2\overline{u''_r G^M_r} - \overline{G^R_{rr}} - 2\varepsilon^r_k \tag{312}$$

Mean equation for $\widetilde{R}_{\theta\theta}$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\left(\widetilde{R}_{\theta n}\partial_n\widetilde{u}_\theta + \widetilde{R}_{\theta n}\partial_n\widetilde{u}_\theta\right) - \left(\nabla_r 2f^\theta_k - \overline{G^R_{\theta\theta}}\right) + 2\overline{u''_\theta\rho D_t u_\theta} \tag{313}$$

$$\tag{314}$$

where

$$\overline{u''_\theta\rho D_t u_\theta} = \overline{u''_\theta\left(\frac{1}{r^2}\partial_r\left(r^2[\tau_{\theta r}]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\tau_{\theta\theta}]\right) + \frac{1}{r\sin\theta}\partial_\phi[\tau_{\theta\phi}]\right) - G^M_\theta - \frac{1}{r}\partial_\theta P + \rho g_\theta\right)} \tag{315}$$

$$\overline{u''_\theta\rho D_t u_\theta} = \overline{u''_\theta\left(\nabla_r\tau_{\theta r} + \nabla_\theta\tau_{\theta\theta} + \nabla_\phi\tau_{\theta\phi} - G^M_\theta - \frac{1}{r}\partial_\theta P + \rho g_\theta\right)} \tag{316}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_\theta \nabla_r \tau_{\theta r}} = \cancelto{0}{\overline{u''_\theta}\nabla_r \overline{\tau_{\theta r}}} + \nabla_r(\overline{u''_\theta \tau'_{\theta r}}) - \overline{\tau'_{\theta r}\partial_r u''_\theta} \tag{317}$$

$$+\overline{u''_\theta \nabla_\theta \tau_{\theta\theta}} = -\overline{\tau'_{\theta\theta}\frac{1}{r}\partial_\theta u''_\theta} \tag{318}$$

$$+\overline{u''_\theta \nabla_\phi \tau_{\theta\phi}} = -\overline{\tau'_{\theta\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\theta} \tag{319}$$

$$-\overline{u''_\theta G^M_\theta} = -\overline{u''_\theta G^M_\theta} \tag{320}$$

$$-\overline{u''_\theta \frac{1}{r}\partial_\theta P} = -\overline{u''_\theta}\frac{1}{r}\partial_\theta \overline{P} - \overline{u''_\theta \frac{1}{r}\partial_\theta P'} = -\cancelto{0}{\overline{u''_\theta}\frac{1}{r}\partial_\theta\overline{P}} - \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, u''_\theta P')}} + \overline{P'\frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, u''_\theta)} \tag{321}$$

$$+\overline{u''_\theta \rho g_\theta} = 0 \tag{322}$$

Final equation is:

$$\begin{aligned}\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{\theta\theta}/\overline{\rho}\big) = & -2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta - \nabla_r 2f_k^\theta - \overline{G^R_{\theta\theta}} + 2\nabla_r(\overline{u''_\theta\tau'_{\theta r}}) - 2\overline{u''_\theta G^M_\theta} + 2\overline{P'\nabla_\theta u''_\theta} - \\ & -2\left(\overline{\tau'_{\theta r}\partial_r u''_\theta} - \overline{\tau'_{\theta\theta}\frac{1}{r}\partial_\theta u''_\theta} - \overline{\tau'_{\theta\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\theta}\right)\end{aligned} \tag{323}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\nabla_r(2f_k^\theta + 2f_\tau^\theta) - 2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + 2\overline{P'\nabla_\theta u''_\theta} - 2\overline{u''_\theta G^M_\theta} - \overline{G^R_{\theta\theta}} - 2\varepsilon_k^\theta \tag{324}$$

Mean equation for $\widetilde{R}_{\phi\phi}$

$$\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{\phi\phi}/\overline{\rho}\big) = -\left(\widetilde{R}_{\phi n}\partial_n\widetilde{u}_\phi + \widetilde{R}_{\phi n}\partial_n\widetilde{u}_\phi\right) - \left(\nabla_r 2f_k^\phi + \overline{G^R_{\phi\phi}}\right) + 2\overline{u''_\phi\rho D_t u_\phi} \tag{325}$$

where

$$\overline{u''_\phi\rho D_t u_\phi} = \overline{u''_\phi\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\phi r}]\big) + \frac{1}{r\sin\phi}\partial_\phi\big(\sin\phi[\tau_{\phi\theta}]\big) + \frac{1}{r\sin\phi}\partial_\phi[\tau_{\phi\phi}]\big) - G^M_\phi - \frac{1}{r\sin\theta}\partial_\phi P + \rho g_\phi\right)} \tag{326}$$

$$\overline{u''_\phi\rho D_t u_\phi} = \overline{u''_\phi\big(\nabla_r\tau_{\phi r} + \nabla_\theta\tau_{\phi\theta} + \nabla_\phi\tau_{\phi\phi} - G^M_\phi - \frac{1}{r\sin\theta}\partial_\phi P + \rho g_\phi\big)} \tag{327}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_\phi \nabla_r \tau_{\phi r}} = \xcancel{\overline{u''_\phi} \nabla_r \overline{\tau_{\phi r}}}^{\,0} + \nabla_r (\overline{u''_\phi \tau'_{\phi r}}) - \overline{\tau'_{\phi r} \partial_r u''_\phi} \tag{328}$$

$$+\overline{u''_\phi \nabla_\theta \tau_{\phi\theta}} = -\overline{\tau'_{\phi\theta} \frac{1}{r} \partial_\theta u''_\phi} \tag{329}$$

$$+\overline{u''_\phi \nabla_\phi \tau_{\phi\phi}} = -\overline{\tau'_{\phi\phi} \frac{1}{r\sin\theta} \partial_\phi u''_\phi} \tag{330}$$

$$-\overline{u''_\phi G^M_\phi} = -\overline{u''_\phi G^M_\phi} \tag{331}$$

$$-\overline{u''_\phi \frac{1}{r\sin\theta} \partial_\phi P} = -\overline{u''_\phi} \frac{1}{r\sin\theta} \partial_\phi \overline{P} - \overline{u''_\phi \frac{1}{r\sin\theta} \partial_\phi P'} = -\xcancel{\overline{u''_\phi} \frac{1}{r\sin\theta} \partial_\phi \overline{P}}^{\,0} - \xcancel{\overline{\frac{1}{r\sin\theta} \partial_\phi (u''_\phi P')}}^{\,0} + \overline{P' \frac{1}{r\sin\theta} \partial_\phi (u''_\phi)} \tag{332}$$

$$+\overline{u''_\phi \rho g_\phi} = 0 \tag{333}$$

Final equation is:

$$\begin{aligned} \overline{\rho} \widetilde{D}_t \big(\widetilde{R}_{\phi\phi} / \overline{\rho} \big) = &- 2\widetilde{R}_{\phi r} \partial_r \widetilde{u}_\phi - \nabla_r 2 f^\phi_k - \overline{G^R_{\phi\phi}} + 2\nabla_r (\overline{u''_\phi \tau'_{\phi r}}) - 2\overline{u''_\phi G^M_\phi} + 2\overline{P' \nabla_\phi u''_\phi} - \\ &- 2 \left(\overline{\tau'_{\phi r} \partial_r u''_\phi} - \overline{\tau'_{\phi\theta} \frac{1}{r} \partial_\theta u''_\phi} - \overline{\tau'_{\phi\phi} \frac{1}{r\sin\theta} \partial_\phi u''_\phi} \right) \end{aligned} \tag{334}$$

$$\overline{\rho} \widetilde{D}_t \left(\widetilde{R}_{\phi\phi} / \overline{\rho} \right) = -\nabla_r (2 f^\phi_k + 2 f^\phi_\tau) - 2\widetilde{R}_{\phi r} \partial_r \widetilde{u}_\phi + 2\overline{P' \nabla_\phi u''_\phi} - 2\overline{u''_\phi G^M_\phi} - \overline{G^R_{\phi\phi}} - 2\varepsilon^\phi_k \tag{335}$$

23.2 Turbulent kinetic energy equations

$$\widetilde{k} = + \frac{1}{2}\widetilde{R}_{ii}/\overline{\rho} \tag{336}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = - \nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{337}$$

$$\widetilde{k}^r = + \frac{1}{2}\widetilde{R}_{rr}/\overline{\rho} \tag{338}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = - \nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} \tag{339}$$

$$\widetilde{k}^h = + \widetilde{k}_\theta + \widetilde{k}_\phi = + \frac{1}{2}\big(\widetilde{R}_{\theta\theta} + \widetilde{R}_{\phi\phi}\big)/\overline{\rho} \tag{340}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = - \nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \tag{341}$$

23.3 Turbulent mass flux equation

The turbulent mass flux equation can be derived in the following way:

$$\rho D_t\widetilde{c} - \rho\widetilde{D}_t\widetilde{c} = \rho\partial_t\widetilde{c} + \rho u_n\partial_n\widetilde{c} - [\rho\partial_t\widetilde{c} + \rho\widetilde{u}_n\partial_n\widetilde{c}_n] = \rho(u_n - \widetilde{u}_n)\partial_n\widetilde{c} = \rho u''_n\partial_n\widetilde{c} \tag{342}$$

$$\rho D_t c'' = \rho D_t c - \rho D_t\widetilde{c} = \rho D_t c - \rho\widetilde{D}_t\widetilde{c} - \rho u''_n\partial_n\widetilde{c} = \rho D_t c - \frac{\rho}{\overline{\rho}}[\overline{\rho}\widetilde{D}_t\widetilde{c}] - \rho u''_n\partial_n\widetilde{c} \tag{343}$$

$$\rho D_t u''_r = + \left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\big([\tau_{r\phi}]\big) - G_r^M - \frac{\partial P}{\partial r}\right) + \rho g_r + \tag{344}$$

$$+ \frac{\rho}{\overline{\rho}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr} - \overline{\tau_{rr}}) + \overline{G_r^M} + \frac{\partial}{\partial r}\overline{P} - \overline{\rho}\widetilde{g}_r\right) - \rho u''_n\partial_n\widetilde{u_r} \tag{345}$$

$$\widetilde{D}_t\overline{u''_i} = \overline{D_t u''_i - u''_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{u_n\partial_n u''_i} - \overline{u''_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{u_n\partial_n u''_i} + \overline{\widetilde{u_n}\partial_n u''_i} - \overline{u_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{\widetilde{u_n}\partial_n u''_i} = \widetilde{D}_t\overline{u''_i} \tag{346}$$

$$\overline{\rho}\widetilde{D}_t\overline{u''_i} = \overline{\frac{\overline{\rho}}{\rho}[\rho D_t u''_i]} - \overline{\overline{\rho}u''_n\partial_n u''_i} \tag{347}$$

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = \overline{\frac{\overline{\rho}}{\rho}[\rho D_t u''_r]} - \overline{\rho}\overline{u''_n\partial_n u''_r} =$$

$$= \overline{\frac{\overline{\rho}}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) - G_r^M - \frac{\partial}{\partial r}P + \rho g_r + \frac{\rho}{\overline{\rho}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr} - \overline{\tau_{rr}}) + \overline{G_r^M} + \frac{\partial}{\partial r}\overline{P} - \overline{\rho}\widetilde{g}_r\right) - \rho u''_n\partial_n\widetilde{u}_r\right]} - \overline{\rho}\overline{u''_n\partial_n u''_r} = \tag{348}$$

$$= \overline{\frac{\overline{\rho}}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})\right]} - \overline{\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})\right]} - \overline{\frac{\overline{\rho}}{\rho}G_r^M} + \overline{G_r^M} - \overline{\frac{\overline{\rho}}{\rho}\left[\frac{\partial}{\partial r}P\right]} + \overline{\left[\frac{\partial}{\partial r}P\right]} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\overline{\rho}u''_n\partial_n\widetilde{u}_r} - \overline{\rho}\overline{u''_n\partial_n u''_r} \tag{349}$$

$$= \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})} - \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]G_r^M} - \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]\frac{\partial}{\partial r}P} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} = \tag{350}$$

$$= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} - \overline{\frac{\rho'}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})} + \overline{\frac{\rho'}{\rho}G_r^M} + \overline{\frac{\rho'}{\rho}\frac{\partial}{\partial r}P} = \tag{351}$$

$$= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} - \overline{\rho' v\frac{1}{r^2}\frac{\partial}{\partial r}(r^2(\overline{\tau_{rr}} + \tau'_{rr})} + \overline{\rho' v\frac{\partial}{\partial r}(\overline{P} + P')} + \overline{\rho' v G_r^M} = \tag{352}$$

$$= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} - \overline{\rho' v}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{\tau_{rr}}) - \frac{\partial}{\partial r}\overline{P}\right) - \overline{\rho' v\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau'_{rr}) - \frac{\partial}{\partial r}P'\right)} + \overline{\rho' v G_r^M} = \tag{353}$$

$$= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} + b\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{\tau_{rr}}) - \frac{\partial}{\partial r}\overline{P}\right) - \overline{\rho' v\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau'_{rr}) - \frac{\partial}{\partial r}P'\right)} + \overline{\rho' v G_r^M} = \tag{354}$$

$$= +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{\mathbf{u}''\cdot\nabla u_r} - b\nabla_r\overline{\tau}_{rr} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} - \overline{\rho' v\nabla_r\tau'_{rr}} + \overline{\rho' v G_r^M} \tag{355}$$

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n\partial_n u_r} - b\nabla_r\overline{\tau}_{rr} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} - \overline{\rho' v\nabla_r\tau'_{rr}} + \overline{\rho' v G_r^M} = \tag{356}$$

$$= +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n}\partial_n\overline{u}_r - \overline{\rho}\partial_n\overline{u''_n u'_r} + \overline{\rho}\overline{u'_r\partial_n u''_n} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} - \overline{\rho' v\nabla_r\tau'_{rr}} + \overline{\rho' v G_r^M} = \tag{357}$$

$$= (+\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\nabla_r\overline{u''_r u'_r}) - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} - \overline{\rho' v\nabla_r\tau'_{rr}} + \overline{\rho' v G_r^M} = \tag{358}$$

$$= -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} - \overline{\rho' v\nabla_r\tau'_{rr}} + \overline{\rho' v G_r^M} \tag{359}$$

23.4 Density-specific volume covariance equation

The density-specific volume ($v = 1/\rho$) covariance ($b = -\overline{\rho' v'}$) equation can be derived from the continuity equation in the following way.

$$\begin{aligned}
\partial_t \rho + \partial_n(\rho u_n) &= 0 && (360)\\
\partial_t \overline{\rho} + \widetilde{u}_n \partial_n \overline{\rho} &= -\overline{\rho}\partial_n \widetilde{u}_n && (361)\\
\partial_t \overline{\rho} + \overline{u}_n \partial_n \overline{\rho} - \overline{u''_n}\partial_n \overline{\rho} &= -\overline{\rho}\partial_n \overline{u}_n + \overline{\rho}\partial_n \overline{u''_n} && (362)\\
\overline{D}_t \overline{\rho} &= +\overline{u''_n}\partial_n \overline{\rho} - \overline{\rho}\partial_n \overline{u}_n + \overline{\rho}\partial_n \overline{u''_n} && (363)\\
\overline{D}_t \overline{\rho} &= +(\overline{u''_n}\partial_n \overline{\rho} + \overline{\rho}\partial_n \overline{u''_n}) - \overline{\rho}\partial_n \overline{u}_n && (364)\\
\overline{D}_t \overline{\rho} &= -\overline{\rho}\partial_n \overline{u}_n + \partial_n(\overline{u''_n}\overline{\rho}) && (365)
\end{aligned}$$

$$\begin{aligned}
\partial_t \rho + \partial_n(\rho u_n) &= 0 && (366)\\
\partial_t (1/v) + \partial_n(u_n/v) &= 0 && (367)\\
-\partial_t v + v\partial_n u_n - u_n \partial_n v &= 0 && (368)\\
\partial_t v - v\partial_n u_n - u_n\partial_n v + u_n \partial_n v + u_n\partial_n v &= 0 && (369)\\
\partial_t v - \partial_n(v u_n) + 2u_n\partial_n v &= 0 && (370)\\
\partial_t \overline{v} - \partial_n(\overline{v u_n}) + 2\overline{u_n \partial_n v} &= 0 && (371)\\
\overline{D}_t \overline{v} &= +\overline{v}\partial_n \overline{u}_n - \partial_n \overline{u'_n v'} + 2\overline{v' \partial_n u'_n} && (372)
\end{aligned}$$

$$\begin{aligned}
b = -\overline{\rho' v'} &= \overline{\rho}\overline{v} - 1 && (373)\\
\overline{D}_t b &= \overline{\rho}\overline{D}_t \overline{v} + \overline{v}\ \overline{D}_t \overline{\rho} && (374)
\end{aligned}$$

Using previously derived equations for $\overline{D}_t \overline{v}$ and $\overline{D}_t \overline{\rho}$ we get:

$$\overline{D}_t b = -\overline{\rho}\partial_n \overline{\rho' u'_n} + 2\overline{\rho}\overline{v' \partial_n u'_n} + \overline{v}\partial_n \overline{\rho}\overline{u''_n} \quad (375)$$

23.5 Mean internal energy flux equation

We can derive the internal energy flux equation using the general formula for second order moments, where we substitute c with ϵ_I and d with u_i.

$$\begin{aligned}
\overline{\rho}\widetilde{D}_t \widetilde{c'' d''} =& \overline{c'' \rho D_t d} - \overline{\rho}\widetilde{c'' u''_n}\partial_n \widetilde{d} + \overline{d'' \rho D_t c} - \overline{\rho}\widetilde{d'' u''_n}\partial_n \widetilde{c} - \overline{\partial_n \rho c'' d'' u''_n} && (376)\\
\overline{\rho}\widetilde{D}_t (f_I/\overline{\rho}) =& \mathcal{N}_{fI} - \nabla_r f_I^r - f_I \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{\epsilon}_I - \overline{\epsilon''_I}\partial_r \overline{P} - \overline{\epsilon''_I \partial_r P'} - \overline{u''_r\,(Pd)} + \overline{u''_r(\mathcal{S} + \nabla \cdot f_T)} + \mathcal{G}_I + \mathcal{N}_{fI} && (377)
\end{aligned}$$

23.6 Mean enthalpy flux equation

We can derive the enthalpy flux equation using the general formula for second order moments, where we substitute c with h and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \quad (378)$$

$$\overline{\rho}\widetilde{D}_t(f_h/\overline{\rho}) = -\nabla_r f_h^r - f_h\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{h} - \overline{h''}\partial_r\overline{P} - \overline{h''\partial_r P'} - \Gamma_1\overline{u''_r\,(Pd)} + \Gamma_3\overline{u''_r(\mathcal{S} + \nabla\cdot F_T)} + \mathcal{G}_h + \mathcal{N}_h \quad (379)$$

23.7 Mean entropy flux equation

We can derive the entropy flux equation using the general formula for second order moments, where we substitute c with s and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \quad (380)$$

$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u''_r(\mathcal{S} + \nabla\cdot f_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \quad (381)$$

23.8 Mean composition flux equation

We can derive the composition flux equation using the general formula for second order moments, where we substitute c with X_α and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \quad (382)$$

$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_\alpha - \overline{X''_\alpha}\partial_r\overline{P} - \overline{X''_\alpha\partial_r P'} + \overline{u''_r\rho\dot{X}_\alpha^{\mathrm{nuc}}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \quad (383)$$

$$(384)$$

Please note that the last term $\overline{\partial_n\rho X''_i u''_i u''_n} \equiv \overline{\nabla\cdot(\rho X''_i\mathbf{u''u''})}$ is div of 2nd order tensor and $\overline{\nabla\cdot(\rho X''_i\mathbf{u''u''})}(\mathbf{e_r}) = \nabla_r\overline{\rho X_i u''_r u''_r} - \overline{\rho X''_i u''_\theta u''_\theta/r} - \overline{\rho X''_i u''_\phi u''_\phi/r}$

Further calculation requires the following hydrodynamic equations:

$$\rho D_t(u_r) = +\left(\frac{1}{r^2}\partial_r\left(r^2[\tau_{rr}]\right) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\left([\tau_{r\phi}]\right) - G_r^M - \partial_r P\right) + \rho g_r \quad (385)$$

$$\rho D_t(X_i) = +\rho\dot{X}_i^n \qquad i = 1...N_{nuc} \quad (386)$$

For radial component of the flux, where $u_i = u_r$ and changing order of terms for clarity reasons we get:

$$\overline{\rho}\widetilde{D}_t\widetilde{X''_i u''_r} = \left(-\overline{\partial_n\rho X''_i u''_r u''_n}\right) - \overline{\rho}\widetilde{X''_i u''_n}\partial_n\widetilde{u_r} - \overline{\rho}\widetilde{u''_r u''_n}\partial_n\widetilde{X_i} + \overline{X''_i\rho D_t u_r} + \overline{u''_r\rho D_t X_i} \quad (387)$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = \left(-\nabla_r f_i^r - \overline{\rho X''_i u''_\theta u''_\theta/r} - \overline{\rho X''_i u''_\phi u''_\phi/r}\right) - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i + \overline{X''_i(\nabla\cdot\tau_r - G_r^M - \partial_r P + \rho g_r)} + \overline{u''_r\rho\dot{X}_i^{\mathrm{nuc}}} \quad (388)$$

Let's reorganize the terms again and split $P = \overline{P} + P'$ to get the equation to shape it has in the paper:

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \cancelto{0}{\overline{X_i''\rho g_r}} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (389)\\
& -\overline{\rho X_i'' u_\theta'' u_\theta''/r} - \overline{\rho X_i'' u_\phi'' u_\phi''/r} - \overline{X_i'' G_r^M} - \overline{X''\nabla\cdot\tau_r} && (390)\\
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (391)\\
& -\overline{\rho X_i'' u_\theta'' u_\theta''/r} - \overline{\rho X_i'' u_\phi'' u_\phi''/r} - \overline{X_i'' G_r^M} - \overline{\nabla\cdot X_i''\tau_r} + \overline{\tau_r\nabla X_i''} && (392)\\
& && (393)
\end{aligned}
$$

The viscoity τ is 2nd order tensor and $-\overline{\nabla\cdot X_i''\tau_r} = -\nabla_r\overline{X_i''\tau_{rr}} + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r}$

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (394)\\
& -\overline{\rho X_i'' u_\theta'' u_\theta''/r} - \overline{\rho X_i'' u_\phi'' u_\phi''/r} - \overline{X_i'' G_r^M} - \nabla_r(\overline{X_i''\tau_{rr}}) + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} && (395)\\
& -\overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} && (396)\\
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (397)\\
& -\overline{\rho X_i'' u_\theta'' u_\theta''/r} - \overline{\rho X_i'' u_\phi'' u_\phi''/r} - \overline{X_i'' G_r^M} - \nabla_r(\overline{X_i''\tau_{rr}}) - \varepsilon_i && (398)\\
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (399)\\
& -\overline{\rho X_i'' u_\theta'' u_\theta''/r} - \overline{\rho X_i'' u_\phi'' u_\phi''/r} - \overline{X_i'' G_r^M} + \mathcal{N}_{fi} && (400)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (401)\\
& +\overline{G_r^i} - \overline{X_i'' G_r^M} + \mathcal{N}_{fi} && (402)\\
\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) =& -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} && (403)\\
& +\mathcal{G}_i + \mathcal{N}_{fi} && (404)
\end{aligned}
$$

The exact formulation of the geometry and viscosity terms is:

$$
\begin{aligned}
\mathcal{G}_i =& \overline{G_r^i} - \overline{X_i'' G_r^M} && (405)\\
\mathcal{N}_{fi} =& -\nabla_r(\overline{X_i''\tau_{rr}}) - \overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} && (406)
\end{aligned}
$$

23.9 Mean A and Z flux equations

We can derive the composition flux equation using the general formula for second order moments, where we substitute c with A or Z and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c'' d'' u''_n} \tag{407}$$

$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = \mathcal{N}_{fA} - \nabla_r f_A^r - f_A\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{A} - \overline{A''}\partial_r\overline{P} - \overline{A''\partial_r P'} - \overline{u''_r \rho A^2 \Sigma_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha} + \mathcal{G}_A \tag{408}$$

$$\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = \mathcal{N}_{fZ} - \nabla_r f_Z^r - f_Z\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{Z} - \overline{Z''}\partial_r\overline{P} - \overline{Z''\partial_r P'} - \overline{u''_r \rho Z A \Sigma_\alpha (\dot{X}_\alpha^{\rm nuc}/A_\alpha)} -$$
$$- \overline{u''_r \rho A \Sigma_\alpha (Z_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z \tag{409}$$

23.10 Mean angular momentum flux equation

We can derive the angular momentum flux equation using the general formula for second order moments, where we substitute c with j_z and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c'' d'' u''_n} \tag{410}$$

$$\overline{\rho}\widetilde{D}_t(f_{jz}/\rho) = -\nabla_r f_{jz}^r - f_{jz}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{j}_z - \overline{j''_z}\partial_r\overline{P} - \overline{j''_z\partial_r P'} + \mathcal{G}_{jz} + \mathcal{N}_{jz} \tag{411}$$

23.11 Pressure flux equation

We can derive the pressure flux equation using the general formula for second order moments, where we substitute c with P and d with u_r.

$$\widetilde{D}_t\widetilde{c'd'} = \overline{c' D_t d} - \overline{c'u''_n}\partial_n\overline{d} + \overline{d' D_t c} + \overline{d'u''_n}\partial_n\overline{c} - \partial_n\overline{u''_n c' d'} + \overline{c' d'' \partial_n u''_n}$$
$$\widetilde{D}_t\overline{P'u'_r} = \overline{P' D_t u_r} - \overline{P'u''_n}\partial_n\overline{u}_r + \overline{u'_r D_t P} + \overline{u'_r u''_n}\partial_n\overline{P} - \partial_n\overline{u''_n P' u'_r} + \overline{P' u''_r \partial_n u''_n}$$
$$\widetilde{D}_t\overline{P'u'_r} = \overline{P' D_t u_r} - \overline{P'u''_r}\partial_r\overline{u}_r + \overline{u'_r D_t P} + \overline{u'_r u''_r}\partial_r\overline{P} - \nabla_r\overline{P' u''_r u'_r} + \overline{P' u''_r d''} \tag{412}$$

The evolution equation for $D_t u_r$ (see Sect.18) is

$$\rho D_t(u_r) = + \left(\frac{1}{r^2}\partial_r\left(r^2[\tau_{rr}]\right) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\left([\tau_{r\phi}]\right) - G_r^M - \partial_r P\right) + \rho g_r \tag{413}$$

By neglecting viscosity and dividing by ρ, it becomes:

$$D_t(u_r) = -G_r^M/\rho - \partial_r P/\rho + g_r \tag{414}$$

The evolution equation for $D_t P$ (see Sect.21.8) is

$$D_t P = -(1 - \Gamma_3 + \Gamma_1)Pd + (\Gamma_3 - 1)(-Pd + \mathcal{S} + \nabla\cdot F_T + \tau_{ij}\partial_i u_j) \tag{415}$$

By neglecting viscosity, it becomes:

$$D_t P = -(1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd+\mathcal{S}+\nabla\cdot F_T) \tag{416}$$

We continue here by putting everything together and neglecting heat flux due to conduction and radiation F_T:

$$\widetilde{D}_t \overline{P'u'_r} = \overline{P'\left(-G^M_r/\rho - \partial_r P/\rho + g_r\right)} - \overline{P'u''_r}\partial_r \overline{u}_r + \overline{u'_r\left(-(1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd+\mathcal{S}+\nabla\cdot F_T)\right)} + \overline{u'_r u''_r}\partial_r\overline{P} - \nabla_r \overline{P'u''_r u'_r} + \overline{P'u''_r d''} \tag{417}$$

$$\widetilde{D}_t \overline{P'u'_r} = -\overline{P'G^M_r/\rho} - \overline{P'\partial_r P/\rho} + \overline{P'g_r} - \overline{P'u''_r}\partial_r\overline{u}_r - (1-\Gamma_3-\Gamma_1)\overline{u'_r Pd} - (\Gamma_3-1)\overline{u'_r Pd} + (\Gamma_3-1)\overline{u'_r\rho\varepsilon_{nuc}} + \overline{u'_r u''_r}\partial_r\overline{P} - \nabla_r\overline{P'u''_r u'_r} + \overline{P'u''_r d''} \tag{418}$$

$$\widetilde{D}_t \overline{P'u'_r} = -\nabla_r f^r_p - f_p\partial_r\overline{u}_r + \overline{u'_r u''_r}\partial_r\overline{P} - (1-\Gamma_3-\Gamma_1)\overline{u'_r Pd} - (\Gamma_3-1)\overline{u'_r Pd} + (\Gamma_3-1)\overline{u'_r\rho\varepsilon_{nuc}} + \overline{P'u''_r d''} - \overline{P'G^M_r/\rho} - \overline{P'\partial_r P/\rho} \tag{419}$$

With a little more algebraic modifications, we get the final form:

$$\widetilde{D}_t \overline{P'u'_r} = -\nabla_r f^r_p - f_p\partial_r\overline{u}_r + \overline{u'_r u''_r}\partial_r\overline{P} + \Gamma_1\overline{u'_r Pd} + (\Gamma_3-1)\overline{u'_r\rho\varepsilon_{nuc}} + \overline{P'u''_r d''} - \overline{P'G^M_r/\rho} - \overline{P'\partial_r P/\rho} \tag{420}$$

24 Derivation of variance equations

The derivation of final variance equations can be achieved by utilization of general formula for variances (see Sect.22.3) and similar algebraic manipulation as shown in previous sections.

We show derivation of enthalpy variance equation only as an example

24.1 Enthalpy variance equation

We begin with general formula for Favrian variance equation:

$$\overline{\rho}\widetilde{D}_t\widetilde{c''c''} = +2\overline{c''\rho D_t c} - 2\overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''c''u''_n} \tag{421}$$

We substitute specific enthalpy $h = \epsilon_I + P/\rho$ for the c and get:

$$\overline{\rho}\widetilde{D}_t\widetilde{h''h''} = +\overline{2h''\rho D_t h} - 2\overline{\rho}\widetilde{h''u''_n}\partial_r\widetilde{h} - \nabla_r\overline{\rho h''h''u''_r} \tag{422}$$

Now, we have to derive evolution equation for specific enthalpy:

$$\rho D_t h = \rho \partial_t h + u_r \partial_r h = \partial_t \rho h + \nabla_r \rho u_r h = \partial_t (\rho \epsilon_I + P) + \nabla_r \rho u_r (\epsilon_I + P/\rho) \quad (423)$$
$$\rho D_t h = \partial_t \rho \epsilon_I + \nabla_r \rho u_r \epsilon_I + \partial_t P + \nabla_r u_r P \quad (424)$$
$$\rho D_t h = \rho D_t \epsilon_I + \partial_t P + u_r \partial_r P + P \nabla_r u_r \quad (425)$$
$$\rho D_t h = \rho D_t \epsilon_I + D_t P + P \nabla_r u_r \quad (426)$$

For the internal energy evolution (see Sect.21.5) and neglecting viscosity and thermal transport due to conduction and radiation, we have :

$$\rho D_t \epsilon_I = -Pd + \rho \varepsilon_{nuc} \quad (427)$$

The evolution equation for $D_t P$ (see Sect.21.8) is

$$D_t P = -(1 - \Gamma_3 + \Gamma_1) Pd + (\Gamma_3 - 1)(-Pd + \mathcal{S} + \nabla \cdot F_T + \tau_{ij} \partial_i u_j) \quad (428)$$

Now, let us combine everything together:

$$\rho D_t h = \ - Pd + \rho \varepsilon_{nuc} - (1 - \Gamma_3 + \Gamma_1) Pd - (\Gamma_3 - 1) Pd + (\Gamma_3 - 1) \rho \varepsilon_{nuc} + Pd \quad (429)$$
$$\rho D_t h = \ -(1 - \Gamma_3 + \Gamma_1) Pd - (\Gamma_3 - 1) Pd + \Gamma_3 \rho \varepsilon_{nuc} \quad (430)$$
$$\rho D_t h = (\Gamma_3 - 1) Pd - \Gamma_1 Pd - (\Gamma_3 - 1) Pd + \Gamma_3 \rho \varepsilon_{nuc} \quad (431)$$
$$\rho D_t h = (\Gamma_3 - 1) Pd - \Gamma_1 Pd - (\Gamma_3 - 1) Pd + \Gamma_3 \rho \varepsilon_{nuc} \quad (432)$$
$$\rho D_t h = \ - \Gamma_1 Pd + \Gamma_3 \rho \varepsilon_{nuc} \quad (433)$$

Substituing $\rho D_t h$ to our general equation for Favrian enthalpy variance, we get

$$\overline{\rho} \widetilde{D}_t \widetilde{h''h''} = \ + \overline{2h''(-\Gamma_1 Pd + \Gamma_3 \rho \varepsilon_{nuc})} - 2\overline{\rho} \widetilde{h'' u''_n} \partial_r \widetilde{h} - \nabla_r \overline{\rho h'' h'' u''_r} \quad (434)$$
$$\overline{\rho} \widetilde{D}_t \widetilde{h''h''} = \ -2\Gamma_1 \overline{h'' Pd} + 2\Gamma_3 \overline{h'' \rho \varepsilon_{nuc}} - 2 f_h \partial_r \widetilde{h} - \nabla_r f_h^r \quad (435)$$

After rearanging of these terms, we get final form or the equation to be:

$$\overline{\rho} \widetilde{D}_t \widetilde{h''h''} = \ - \nabla_r f_h^r - 2 f_h \partial_r \widetilde{h} - 2\Gamma_1 \overline{h'' Pd} + 2\Gamma_3 \overline{h'' \rho \varepsilon_{nuc}} \quad (436)$$

25 Divergence of tensors in spherical geometry up to third order

BACKGROUND READING:

CONTINUUM MECHANICS (Lecture Notes)
Zdenek Martinec, Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \quad : \text{nabla operator}$$

$$\mathbf{V} = \sum_i V_i \mathbf{e_i} \quad : \text{tensor of first order (vector)} \tag{437}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \quad : \text{tensor of second order} \tag{438}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{tensor of third order} \tag{439}$$

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{h_i}\left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m\right] \quad : \text{div of first order tensor (vector)} \tag{440}$$

$$\nabla \cdot \mathbf{S} = \sum_{ij} \frac{1}{h_i}\left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right] \mathbf{e_j} \quad : \text{div of second order tensor} \tag{441}$$

$$\nabla \cdot \mathbf{T} = \sum_{ijk} \frac{1}{h_i}\left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm}\right] (\mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{div of third order tensor} \tag{442}$$

$$x_1 = r \qquad x_2 = \theta \qquad x_3 = \phi \qquad \text{(coordinates)} \tag{443}$$

$$\mathbf{e_1} = \mathbf{e_r} \qquad \mathbf{e_2} = \mathbf{e}_\theta \qquad \mathbf{e_3} = \mathbf{e}_\phi \qquad \text{(unit base vectors)} \tag{444}$$

$$h_1 = h_r = 1 \qquad h_2 = h_\theta = r \qquad h_3 = h_\phi = r\sin\theta \qquad \text{(scale factors)} \tag{445}$$

$$\begin{pmatrix} \Gamma^\theta_{r\theta} = \ 1 & \Gamma^\phi_{r\phi} = \ \sin\theta & \Gamma^\phi_{\theta\phi} = \ \cos\theta \\ \Gamma^r_{\theta\theta} = -1 & \Gamma^r_{\phi\phi} = -\sin\theta & \Gamma^\theta_{\phi\phi} = -\cos\theta \end{pmatrix} \qquad \text{Christoffel symbols} \tag{446}$$

Divergence of first order tensor $\nabla \cdot \mathbf{V}$

$$\frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{447}$$

Divergence of second order tensor $\nabla \cdot \mathbf{S}$

$$S_r(\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{448}$$

$$S_\theta(\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{449}$$

$$S_\phi(\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{450}$$

Divergence of third order tensor $\nabla \cdot \mathbf{T}$

$$T_{rr}\ (\mathbf{e_r}\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rrr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta rr}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi rr}}{\partial\phi} - \frac{T_{\theta\theta r}}{r} - \frac{T_{\theta r\theta}}{r} - \frac{T_{\phi\phi r}}{r} - \frac{T_{\phi r\phi}}{r} \tag{451}$$

$$T_{r\theta}\ (\mathbf{e_r}\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\theta}}{\partial\phi} - \frac{T_{\theta\theta\theta}}{r} + \frac{T_{\theta rr}}{r} - \frac{T_{\phi\phi\theta}}{r} - \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{452}$$

$$T_{r\phi}\ (\mathbf{e_r}\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\phi}}{\partial\phi} + \frac{T_{\theta\theta\phi}}{r} - \frac{T_{\phi\phi\phi}}{r} + \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{453}$$

$$T_{\theta r}\ (\mathbf{e}_\theta\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta r}}{\partial\phi} + \frac{T_{\theta rr}}{r} - \frac{T_{\theta\theta\theta}}{r} - \frac{T_{\phi\phi r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}}{r} \tag{454}$$

$$T_{\theta\theta}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\theta}}{\partial\phi} + \frac{T_{\theta r\theta}}{r} + \frac{T_{\theta\theta r}}{r} - \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{455}$$

$$T_{\theta\phi}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\phi}}{\partial\phi} + \frac{T_{\theta r\phi}}{r} + \frac{T_{\phi\theta r}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} \tag{456}$$

$$T_{\phi r}\ (\mathbf{e}_\phi\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi r}}{\partial\phi} - \frac{T_{\theta\phi\theta}}{r} + \frac{T_{\phi rr}}{r} + \frac{T_{\phi\theta r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\phi\phi}}{r} \tag{457}$$

$$T_{\phi\theta}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\theta}}{\partial\phi} + \frac{T_{\theta\phi r}}{r} + \frac{T_{\phi r\theta}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{458}$$

$$T_{\phi\phi}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\phi}}{\partial\phi} + \frac{T_{\phi r\phi}}{r} + \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} + \frac{T_{\phi\phi r}}{r} + \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} \tag{459}$$

Bibliography

D. Arnett, C. Meakin, and P. A. Young. Convection Theory and Sub-Photospheric Stratification. *Apj*, 710:1619–1626, February 2010. doi: 10.1088/0004-637X/710/2/1619.

Didier Besnard, FH Harlow, RM Rauenzahn, and C Zemach. Turbulence transport equations for variable-density turbulence and their relationship to two-field models. Technical report, Los Alamos National Lab., NM (United States), 1992.

P. Chassaing, R.A. Antonia, F. Anselmet, L. Joly, and R. Sarkar. *Variable Density Fluid Turbulence*. Kluwer Academic Publishers, 2010.

C. A. Meakin and D. Arnett. Turbulent Convection in Stellar Interiors. I. Hydrodynamic Simulation. *Apj*, 667:448–475, September 2007. doi: 10.1086/520318.

M. Mocák, E. Müller, A. Weiss, and K. Kifonidis. The core helium flash revisited. II. Two and three-dimensional hydrodynamic simulations. *AaA*, 501:659–677, July 2009. doi: 10.1051/0004-6361/200811414.

M. Mocák, L. Siess, and E. Müller. Multidimensional hydrodynamic simulations of the hydrogen injection flash. *AaA*, 533:A53, September 2011. doi: 10.1051/0004-6361/201116940.

M. Viallet, C. Meakin, D. Arnett, and M. Mocák. Turbulent Convection in Stellar Interiors. III. Mean-field Analysis and Stratification Effects. *Apj*, 769:1, May 2013. doi: 10.1088/0004-637X/769/1/1.