Compressible Hydrodynamic Mean-Field Equations in Spherical Geometry and their Application to Turbulent Stellar Convection Data

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1 Introduction

We present a statistical analysis of turbulent convection in stars within our Reynolds-Averaged Navier Stokes (RANS) framework in spherical geometry which we derived from first principles (see §17 and further sections for details). The primary results reported in this document include (1) an extensive set of mean-field equations for compressible, multi-species hydrodynamics and (2) corresponding mean-field data computed from various simulation models. Some supplementary scale analysis data is also presented.

The simulation data which is presented includes (1) shell convection during oxygen burning in a 23 $M_\odot$ supernova progenitor, (2) envelope convection in a 5 $M_\odot$ red giant, (3) shell convection during the helium flash, and (4) a hydrogen injection flash in a 1.25 $M_\odot$ star. These simulations have been partially described previously in Meakin [2006], Meakin and Arnett [2007a,b, 2010], Arnett et al. [2009, 2010], Viallet et al. [2011, 2013a,b] and Mocák et al. [2009, 2011]. New data is also included in this document with several new domain and resolution configurations as well as some variations in the physical model such as convection zone depth and driving source term.

The long term goal of this work is to aid in the development of more sophisticated models for treating hydrodynamic phenomena (e.g., turbulent convection) in the field of stellar evolution by providing a direct link between 3D simulation data and the mean fields which are modeled by 1D stellar evolution codes. As such, this data can be used to test previously proposed turbulence models found in the literature and sometimes used in stellar modeling (see e.g. §11). This data can also serve to test basic physical principles for model building and inspire new prescriptions for use in 1D evolution codes.

1.1 Derivation of RANS Equations

We present an extensive set of mean-field evolution equations in the same spirit as those studied in classical turbulence research [e.g., Hinze, 1975] but extended to the fully compressible, multi-species treatment in spherical geometry relevant to stellar interiors. (A complimentary set of mean-field equations in the ideal magnetohydrodynamics (MHD) approximation will be released in the future.) Although the equations presented here are similar to those recently published in Canuto [2011a,b,c,d,e] we have adopted a different approach. In particular, we present our equation set completely unmodeled and unclosed and without approximation. Our intention is to provide exact evolution equations together with what is in essence raw data to be used to check approximations in models such as those presented in Canuto's and similar works (an example of this is presented in §11). Therefore, this work is largely pedagogical and includes lengthy derivations in later sections.

The equation set presented describes the time evolution of a variety of mean quantities. Examples include the mean specific internal energy, the mean specific entropy, the mean convective flux (enthalpy flux), the mean entropy flux, the mean specific kinetic energy (velocity variance), the mean composition, and the Reynolds stresses, to name just a few. The mean fields are all one-dimensional (a function of only radial coordinate or enclosed mass) and time-dependent such that they correspond to the quantities modeled in traditional 1D stellar evolution codes (see e.g. Kippenhahn and Weigert [1990], Paxton et al. [2011]). While we believe that this document is relatively complete and self-contained, we encourage readers that are not that familiar with RANS equation sets to peruse Viallet et al. [2013b] for additional background and discussion on the general approach as well as basic textbooks on turbulence modeling such as that by Pope [2000].

We obtain our 1D RANS equations by introducing two types of averaging: statistical averaging and horizontal averaging [Besnard et al., 1992, Viallet et al., 2013b]. In practice, statistical averages are computed by performing a time average (the ergodic hypothesis). Therefore, the combined average of a quantity q is defined as

$$\overline{q}(r,t) = \frac{1}{T\Delta\Omega}\int_{t-T/2}^{t+T/2} q(r,\theta,\phi,t')\ d\Omega\ dt' \tag{1}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle in spherical coordinates, T is the averaging time period, and $\Delta\Omega$ is total solid angle being averaged over. This type of average is referred to as a *Reynolds average.*

The flow variables are then decomposed into mean and fluctuation $q = \overline{q} + q'$, noting that $\overline{q'} = 0$ by construction. Similarly, we introduce Favre (or density weighted) averaged quantities by

$$\widetilde{q} = \frac{\overline{\rho q}}{\overline{\rho}} \tag{2}$$

which defines a complimentary decomposition of the flow into mean and fluctuations according to $q = \widetilde{q} + q''$. Here, q'' is the Favrian fluctuation and its mean is zero when Favre averaged $\widetilde{q''} = 0$. For a more complete elaboration on the algebra of these averaging procedures we refer the reader to §19 and 20 as well as Chassaing et al. [2010].

A note on domain geometry: All of the simulation data examined in this document has been modeled using a spherical coordinate system and all of the mean-fields are presented as one-dimensional mean fields in spherical coordinates. The choice of a spherical coordinate system results in several terms due solely to curvilinear coordinates such as the fictitious

Coriolis and centrifugal forces. For readability and concision we denote these so-called "geometric terms" with the super- and sub-scripted symbols G and $\mathcal{G}$ (see definitions in Tables 1 to 4).

On a practical note, all of the simulation data studied in this document use computational domains that are restricted to sectors (or wedges) in the spherical coordinate system and have periodic boundary conditions in both angular directions. Therefore the angular components of divergence terms vanish upon averaging (see §26).

Finally, it is worth mentioning that while we are able to present the current data in a 1D mean-field format, flows that involve rotation and/or large-scale magnetic field must be extended to at least a 2D mean-field description in order to retain self-consistency since these phenomena break the spherical symmetry of the system. This issue is discussed a bit further in Viallet et al. [2013b] (see their §§3 and 5).

A note on nomenclature: Mean fields are often referred to as *moment equations* and the order of the equation is simply the number of items in the product which defines the quantity. For example, the evolution equation for the mean velocity is a first-order moment equation since the mean velocity, e.g. $\widetilde{u_i}$, is not a product. The entropy flux is a second-order moment since it is defined by the covariance of two quantities, $f_s = \overline{\rho}\widetilde{s''u''_r}$, that of the Favrian entropy and velocity fluctuation (the mean density which multiplies this quantity is not counted by convention).

Another convention which we adopt in some places in this document is to refer to the evolution equations for mean-field quantities as *budget equations*.

1.2 Data Analysis and Presentation

We have computed the various terms appearing in a large number of the mean-field evolution equations that we have derived as well as the mean-fields themselves in several cases (e.g., mean turbulent kinetic energy and mean entropy profiles), all for a diverse set of stellar convection simulation data. In general, the mean fields are calculated from the simulation data according to the averaging operators defined in eqs. 1 and 2 above. Two averaging methods have been employed, one more accurate than the other. The most accurate method is to calculate averages during the course of the simulation so that every time-scale in the problem is sampled. This turns out to be an important issue because of the presence of acoustic phenomena that is often characterized by timescales smaller than the increment in simulation time used to store output data when using a compressible hydrodynamics simulation code.

For cases in which we do not have run-time averaged data stored, we find that we can filter troublesome acoustic phenomena (which is primarily radial in nature, including pressure waves trapped by the inner and outer boundaries) because it is generally of such a low amplitude that its impact on the other fields is negligible. In particular, as described previously in the appendix of Arnett et al. [2009], we are able to successfully identify the instantaneous fluctuations in any given snapshot by using a horizontally averaged mean rather than the full time-and-horizontally averaged mean.

A comparison with run-time averaged data shows that this procedure is very accurate for the mean-field data presented below. However, it should be kept in mind that if one were interested in studying the interaction of high frequency waves with turbulence or other physics (e.g., with nuclear burning as in an epsilon mechanism) then a more accurate run-time averaged data set would be needed if a fully sampled set of flow snapshot data were not available. In general, data sets which sample the acoustic time-scale are prohibitively large at present, even for modest resolution calculations and can easily exceed a petabyte.

A note on numerics: The data presented in this document was computed using three separate, but similar codes: MUSIC [Viallet et al., 2011, 2013a], PROMPI [Meakin and Arnett, 2007a], and Herakles [Mocák et al., 2009], all built around conservative, finite-volume solvers. PROMPI and Herakles both use piecewise parabolic method (PPM; see Colella and Woodward [1984], Colella and Glaz [1985]) solvers and have been used to simulate the oxygen-burning shell models (PROMPI; Tables 5), and the helium burning models (Herakles; Tables 7). The MUSIC code, which uses a finite-volume scheme on a staggered grid was used to simulate red giant envelope convection (Table 6).

The extremely large Reynolds numbers expected for stellar turbulence (larger than 10^{10}) makes it impossible to model all scales of the flow, from the stellar scale (or even the scale of a pressure or density scale height) down to the dissipation scale, on the current generation of computers. Therefore, as discussed in Viallet et al. [2013b] and Meakin and Arnett [2007a], we do not model viscosity explicitly and instead opt to solve the inviscid Euler equations and relegate the action of viscosity to the numerics, an approach referred to as implicit large eddy simulation (ILES; Grinstein et al. [2007]). This approach is to be distinguished from both direct numerical simulation (DNS), wherein the viscosity (and other relevant molecular properties of the gas) are directly simulated down to the scales on which the flow is smooth within the continuum approximation; as well as large eddy simulation (LES), which employs a model to account for the impact that sub-grid scales (SGS) have on resolved scales. LES methods are strongly dependent on the choice of SGS model and, despite interesting developments, remains an active area of research with difficult outstanding issues regarding flows under circumstances expected in stellar interiors (e.g., Sullivan and Patton [2011]). In fact, one of the main conclusions of this last reference was that LES was found to converged with resolution only once an inertial range was captured by the simulation, a condition which is in fact suited for ILES and thus obviating the need for an SGS model. Another motivation for choosing an ILES approach over LES is that, in some sense, ILES allows one to achieve the highest effective Reynolds number (or equivalently, the smallest effective Kolmogorov length scale) for a given number of grid zones since LES models generally result in additional mixing at the grid scale. This last point also applies to DNS.

While ILES provides arguably the highest effective Reynolds number for a given computational cost (see e.g. Benzi et al. [2008]) the role played by dissipative processes (such as viscosity and molecular diffusion) must be quantified in a manner indirect compared to DNS. Furthermore, these processes will depend on the specific flow and computational grid choice in a way that is not readily predicted prior to simulation. Therefore, ILES simulations are best suited to circumstances in which the flow is not dependent on the details at the dissipation scale, a circumstance typified by the Kolmogorov cascade found in high Reynolds number turbulent flow [Kolmogorov, 1941, Aspden et al., 2008].

Despite the indirect treatment of dissipative processes in ILES simulations one can use mean-field equations to quantify their effect to the extent that the hydrodynamics algorithm is formulated in a conservative manner. As an example, consider the turbulent kinetic energy (TKE) evolution equation (eq. 21). This budget equation will not balance for a dissipative momentum-conserving hydrodynamics scheme but will instead result in a residual term which we denote by $\mathcal{N}_k$ that can be recovered by summing all of the other explicitly modeled terms. In this example $\mathcal{N}_k$ quantifies the rate at which TKE is dissipated by the numerical scheme and provides a measure of the degree to which the solution deviates from that of truly inviscid hydrodynamics. (As discussed in Meakin [2008], Meakin and Arnett [2007a], Arnett et al. [2009], Viallet et al. [2013b], the TKE dissipation found by this method is consistent with fully developed turbulence and the presence of an inertial range). In our analysis, we have defined an analogous residual term for each of the mean-field budget equations that can be directly calculated from the simulation data and which provides insight into and quantitative information about the dissipation and transport phenomena taking place near the grid scale in ILES calculations. This technique provides a powerful tool for analyzing the turbulent phenomena being modeled (see super- and sub-scripted $\mathcal{N}$ terms in §2 and Tables 1 to 4 where they are referred to as "numerical effects").

A note on units: cgs units are used throughout. In addition, it should be noted that all calculated mean fields presented in this document are multiplied by the area factor $4\pi r^2$ to reflect the volume increment in spherical geometry which helps visualize the volume integral budgets of the individual mean fields which are then equivalent to the area below the corresponding mean field profiles. This area factor is *not* reflected in the legends for each figure but it *is* reflected by the units in each y-axis label.

1.3 Availability of Raw Data

We will continue to make the data presented in this document available for download. At present the subset of data discussed in Viallet et al. [2013b] is available at `http://www.stellarmodels.org`.

1.4 Structure of this Document

While this document is primarily meant as a reference for our research group, we are posting it publicly because we believe it has pedagogical significance for the field of stellar hydrodynamics and contains unique data and formulae that do not have a natural outlet for publishing and which can not be found elsewhere. We suggest that an interested reader simply browse the table of contents for orientation and then scroll through the remainder of the document in order to familiarize themselves with its content which we feel is fairly self explanatory. The authors are happy to address any questions that you might have or provide you with raw data if you are interested. Casey Meakin (casey.meakin@gmail.com) is a good first point of contact.

1.5 Acknowledgments

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2 Summary of Reynolds-averaged Navier Stokes equations in spherical geometry

2.1 Various mean fields equations (first-order moments)

$$\widetilde{D}_t \overline{\rho} = - \overline{\rho} \widetilde{d} + \mathcal{N}_\rho \tag{3}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{u}_r = - \nabla_r \widetilde{R}_{rr} - \overline{G^M_r} - \partial_r \overline{P} + \overline{\rho} \widetilde{g}_r + \mathcal{N}_{ur} \tag{4}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{u}_\theta = - \nabla_r \widetilde{R}_{\theta r} - \overline{G^M_\theta} - (1/r) \overline{\partial_\theta P} + \mathcal{N}_{u\theta} \tag{5}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{u}_\phi = - \nabla_r \widetilde{R}_{\phi r} - \overline{G^M_\phi} + \mathcal{N}_{u\phi} \tag{6}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{\epsilon}_I = - \nabla_r (f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \tag{7}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{\epsilon}_k = - \nabla_r (f_k + f_P) - \widetilde{R}_{ir} \partial_r \widetilde{u}_i + W_b + W_P + \overline{\rho} \widetilde{D}_t (\widetilde{u}_i \widetilde{u}_i / 2) + \mathcal{N}_{\epsilon k} \tag{8}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{\epsilon}_t = - \nabla_r (f_I + f_T + f_k + f_P) - \widetilde{R}_{ir} \partial_r \widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho} \widetilde{D}_t (\widetilde{u}_i \widetilde{u}_i / 2) + \mathcal{N}_{\epsilon t} \tag{9}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{h} = - \nabla_r f_h - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T + \mathcal{N}_h \tag{10}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{s} = - \nabla_r f_s - \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \tag{11}$$

$$\overline{D}_t \overline{P} = - \nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1 - \Gamma_1) W_P + (\Gamma_3 - 1) \mathcal{S} + (\Gamma_3 - 1) \nabla_r f_T + \mathcal{N}_P \tag{12}$$

$$\overline{D}_t \overline{T} = - \nabla_r f_T + (1 - \Gamma_3) \overline{T}\ \overline{d} + (2 - \Gamma_3) \overline{T'd'} + \overline{(\nabla \cdot F_T)/\rho c_v} + \overline{(\tau_{ij} \partial_i u_j)/\rho c_v} + \overline{\epsilon_{\mathrm{nuc}}/c_v} + \mathcal{N}_T \tag{13}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{X}_\alpha = - \nabla_r f_\alpha + \overline{\rho} \widetilde{\dot{X}}_\alpha^{\mathrm{nuc}} + \mathcal{N}_\alpha \tag{14}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{A} = - \nabla_r f_A - \overline{\rho A^2 \Sigma_\alpha (\dot{X}_\alpha^{\mathrm{nuc}} / A_\alpha)} + \mathcal{N}_A \tag{15}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{Z} = - \nabla_r f_Z - \overline{\rho Z A \Sigma_\alpha (\dot{X}_\alpha^{\mathrm{nuc}} / A_\alpha)} + \overline{\rho A \Sigma_\alpha (Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}} / A_\alpha)} + \mathcal{N}_Z \tag{16}$$

$$\overline{\rho} \widetilde{D}_t \widetilde{j}_z = - \nabla_r f_{jz} + \mathcal{N}_{jz} \tag{17}$$

2.2 Mean Reynolds stress equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -\nabla_r(2f_k^r + 2f_P) + 2W_b - 2\widetilde{R}_{rr}\partial_r\widetilde{u}_r + 2\overline{P'\nabla_r u''_r} + 2\mathcal{G}_k^r + \mathcal{N}_{Rrr} \tag{18}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\nabla_r(2f_k^\theta) - 2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + 2\overline{P'\nabla_\theta u''_\theta} + 2\mathcal{G}_k^\theta + \mathcal{N}_{R\theta\theta} \tag{19}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi\phi}/\overline{\rho}\right) = -\nabla_r(2f_k^\phi) - 2\widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi + 2\overline{P'\nabla_\phi u''_\phi} + 2\mathcal{G}_k^\phi + \mathcal{N}_{R\phi\phi} \tag{20}$$

2.3 Mean turbulent kinetic energy equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{21}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} \tag{22}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \tag{23}$$

$$\tag{24}$$

2.4 Mean turbulent mass flux and mean density-specific volume covariance equation (second-order moments)

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} + \mathcal{G}_a + \mathcal{N}_a \tag{25}$$

$$\overline{D}_t b = +\overline{v}\nabla_r\overline{\rho}\overline{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\overline{v' d'} + \mathcal{N}_b \tag{26}$$

2.5 Mean flux equations (second-order moments)

$$
\begin{align}
\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) &= -\nabla_r f_I^r - f_I\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{\epsilon}_I - \overline{\epsilon_I''}\partial_r\overline{P} - \overline{\epsilon_I''\partial_r P'} - \overline{u_r''\,(Pd)} + \overline{u_r''(\mathcal{S}+\nabla\cdot F_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \tag{27}\\
\overline{\rho}\widetilde{D}_t(f_h/\overline{\rho}) &= -\nabla_r f_h^r - f_h\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{h} - \overline{h''}\partial_r\overline{P} - \overline{h''\partial_r P'} - \Gamma_1\overline{u_r''\,(Pd)} + \Gamma_3\overline{u_r''(\mathcal{S}+\nabla\cdot F_T)} + \mathcal{G}_h + \mathcal{N}_h \tag{28}\\
\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) &= -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u_r''(\mathcal{S}+\nabla\cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \tag{29}\\
\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) &= -\nabla_r f_\alpha^r - f_\alpha\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_\alpha - \overline{X_\alpha''}\partial_r\overline{P} - \overline{X_\alpha''\partial_r P'} + \overline{u_r''\rho\dot{X}_\alpha^{\mathrm{nuc}}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \tag{30}\\
\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) &= -\nabla_r f_A^r - f_A\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{A} - \overline{A''}\partial_r\overline{P} - \overline{A''\partial_r P'} - \overline{u_r''\rho A^2\Sigma_\alpha\dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha} + \mathcal{G}_A + \mathcal{N}_{fA} \tag{31}\\
\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) &= -\nabla_r f_Z^r - f_Z\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{Z} - \overline{Z''}\partial_r\overline{P} - \overline{Z''\partial_r P'} - \overline{u_r''\rho ZA\Sigma_\alpha(\dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha)} - \overline{u_r''\rho A\Sigma_\alpha(Z_\alpha\dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha)} + \nonumber\\
&\quad + \mathcal{G}_Z + \mathcal{N}_{fZ} \tag{32}\\
\overline{\rho}\widetilde{D}_t(f_{jz}/\rho) &= -\nabla_r f_{jz}^r - f_{jz}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{j}_z - \overline{j_z''}\partial_r\overline{P} - \overline{j_z''\partial_r P'} + \mathcal{G}_{jz} + \mathcal{N}_{jz} \tag{33}\\
\widetilde{D}_t f_T &= -\nabla_r f_T^r - f_T\partial_r\overline{u}_r - \overline{u_r'u_r''}\partial_r\overline{T} - \overline{T'\partial_r P/\rho} - (\Gamma_3-1)(\overline{T}\ \overline{u_r'd''} + \widetilde{d}\ \overline{u_r'T'} + \overline{u_r'T'd''}) + \overline{T'u_r'd''} + \overline{u_r'\epsilon_{\mathrm{nuc}}/c_v} + \nonumber\\
&\quad + \overline{u_r'\nabla\cdot F_T/\rho c_v} + \mathcal{G}_T + \mathcal{N}_{fT} \tag{34}
\end{align}
$$

2.6 Mean Reynolds variance equations (second-order moments)

$$
\begin{align}
\widetilde{D}_t\sigma_\rho &= -\nabla_r\overline{(\rho'\rho'u_r'')} - 2\overline{\rho}\ \overline{\rho'd''} - 2\overline{\rho'u_r''}\partial_r\overline{\rho} - 2\widetilde{d}\ \sigma_\rho - \overline{\rho'\rho'd''} + \mathcal{N}_{\sigma_\rho} \tag{35}\\
\widetilde{D}_t\sigma_P &= -\nabla_r\overline{(P'P'u_r'')} - 2\Gamma_1\overline{P}\ W_P - 2f_P\partial_r\overline{P} - 2\Gamma_1\widetilde{d}\ \sigma_P - (2\Gamma_1-1)\overline{P'P'd''} + 2(\Gamma_3-1)\overline{P'\mathcal{S}} + \mathcal{N}_{\sigma_P} \tag{36}\\
\widetilde{D}_t\sigma_T &= -\nabla_r\overline{(T'T'u_r'')} - 2(\Gamma_3-1)\overline{T}\overline{T'd''} - 2\overline{T'u_r''}\partial_r\overline{T} - 2(\Gamma_3-1)\widetilde{d}\sigma_T + (3-2\Gamma_3)\overline{T'T'd''} + \overline{2T'\nabla\cdot F_T/\rho c_v} + \nonumber\\
&\quad + \overline{2T'\epsilon_{\mathrm{nuc}}/c_v} + \mathcal{N}_{\sigma T} \tag{37}
\end{align}
$$

2.7 Mean Favrian variance equations (second-order moments)

$$\overline{\rho}\widetilde{D}_t\sigma_{ur} = -\nabla_r(\overline{\rho u''_r u''_r u''_r}) + 2\nabla_r f_P + 2W_b - 2\widetilde{R}_{rr}\partial_r\widetilde{u}_r + 2\overline{P'\nabla_r u''_r} + \mathcal{G}_{\sigma_{ur}} + \mathcal{N}_{\sigma_{ur}} \tag{38}$$

$$\overline{\rho}\widetilde{D}_t\sigma_{\epsilon I} = -\nabla_r(\overline{\rho\epsilon''_I\epsilon''_I u''_r}) - 2f_I\partial_r\widetilde{\epsilon}_I - 2\overline{\epsilon''_I}\ \overline{P}\ \widetilde{d} - 2\overline{P}\ \overline{\epsilon''_I d''} - 2\widetilde{d}\ \overline{\epsilon''_I P'} - 2\overline{\epsilon''_I P' d''} + 2\overline{\epsilon''_I\mathcal{S}} + \mathcal{N}_{\sigma_{\epsilon I}} \tag{39}$$

$$\overline{\rho}\widetilde{D}_t\sigma_s = -\nabla_r(\overline{\rho s'' s'' u''_r}) - 2f_s\partial_r\widetilde{s} - 2\overline{s''\nabla\cdot F_T/T} + 2\overline{s''\mathcal{S}/T} + \mathcal{N}_{\sigma_s} \tag{40}$$

$$\overline{\rho}\widetilde{D}_t\sigma_\alpha = -\nabla_r(\overline{\rho X''_\alpha X''_\alpha u''_r}) - 2f_\alpha\partial_r\widetilde{X}_\alpha + 2\overline{X''_\alpha\rho\dot{X}^{\mathrm{nuc}}_\alpha} + \mathcal{N}_{\sigma_\alpha} \tag{41}$$

Table 1: Definitions

ρ density	g_r radial gravitational acceleration
T temperature	$\mathcal{S} = \rho\epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r \sin\theta\ u_\phi$ specific angular momentum	$F_T = \chi\partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d\ ln\ P/d\ ln\ \rho)\|_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d\ ln\ P/d\ ln\ T)\|_s$
h specific enthalpy	$\Gamma_3 - 1 = (d\ ln\ T/d\ ln\ \rho)\|_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho}\widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho}\widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho}\widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho}\widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P'd''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy
$f_P = \overline{P'u'_r}$ acoustic flux	$f_T = -\overline{\chi\partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued)

$f_I = \overline{\rho}\widetilde{\epsilon_I'' u_r''}$ internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X_\alpha'' u_r''}$ X_α flux
$f_s = \overline{\rho}\widetilde{s'' u_r''}$ entropy flux	$f_{jz} = \overline{\rho}\widetilde{j_z'' u_r''}$ angular momentum flux
$f_T = \overline{u_r' T'}$ turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u_r''}$ A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u_r''}$ enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u_r''}$ Z (mean charge per isotope) flux
$b = \overline{v' \rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f_\tau^r = -\overline{\tau_{rr}' u_r'}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f_\tau^\theta = -\overline{\tau_{\theta r}' u_\theta'}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f_\tau^\phi = -\overline{\tau_{\phi r}' u_\phi'}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_I^r = \overline{\rho}\widetilde{\epsilon_I'' u_r'' u_r''}$ radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_s^r = \overline{\rho}\widetilde{s'' u_r'' u_r''}$ radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$ numerical effect
$f_h^r = \overline{\rho}\widetilde{h'' u_r'' u_r''}$ radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$f_T^r = \overline{T' u_r' u_r'}$ radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect
$f_{jz}^r = \overline{\rho}\widetilde{j_z'' u_r'' u_r''}$ radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect
$f_\alpha^r = \overline{\rho}\widetilde{X_\alpha'' u_r'' u_r''}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f_A^r = \overline{\rho}\widetilde{A'' u_r'' u_r''}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect
$f_Z^r = \overline{\rho}\widetilde{Z'' u_r'' u_r''}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued)

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I\tau'_{rr}}) + \overline{u''_r\tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3-1)\tau_{ij}\partial_i u_j} - \overline{u''_r\nabla_i u_i\tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r\tau_{ij}\partial_i u_j/T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z\tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i\tau_{ri}/\rho} + \overline{u'_r\tau_{ij}\partial_i u_j/\rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	$\mathcal{N}_\alpha$ numerical effect
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	$\mathcal{N}_A$ numerical effect
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	$\mathcal{N}_Z$ numerical effect
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3-1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j/\rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s''s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i/T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I\epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I\tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued)

$$\varepsilon_k^r = \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r}$$
$$\varepsilon_k^\theta = \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta}$$
$$\varepsilon_k^\phi = \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi}$$
$$\varepsilon_k = \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi$$
$$\varepsilon_k^h = \varepsilon_k^\theta + \varepsilon_k^\phi$$
$$\varepsilon_a = \overline{\rho' v \nabla_r \tau'_{rr}}$$
$$\varepsilon_I = \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I}$$
$$\varepsilon_s = \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''}$$
$$\varepsilon_\alpha = \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha}$$
$$\varepsilon_A = \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''}$$
$$\varepsilon_Z = \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''}$$
$$\varepsilon_h = \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''}$$
$$\varepsilon_{jz} = \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}$$

$$\overline{G_r^M} = -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r}$$
$$\overline{G_\theta^M} = +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)}$$
$$\overline{G_\phi^M} = +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)}$$
$$\overline{G_{rr}^R} = -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r}$$
$$\overline{G_{\theta\theta}^R} = +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)}$$
$$\overline{G_{\phi\phi}^R} = +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)}$$
$$\overline{G_r^I} = -\overline{\rho \epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho \epsilon''_I u''_\phi u''_\phi/r}$$
$$\overline{G_r^s} = -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^\alpha} = -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r}$$
$$\overline{G_r^A} = -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^Z} = -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^h} = -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r}$$
$$\overline{G_r^T} = -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r}$$
$$\overline{G_r^{jz}} = -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}$$

$$\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2.) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta.) + \frac{1}{r\sin\theta}\partial_\phi(.)$$

3 Properties of oxygen shell burning and red giant simulation data

3.1 Summary of the Oxygen Burning Simulations

Parameter	ob.3D.lr	ob.3D.mr	ob.3D.hr	ob.3D.1hp	ob.3D.2hp	ob.3D.4hp	ob.3d.1hp.vc	ob.3d.1hp.vh
Grid zoning	192×128^2	384×256^2	786×512^2	200×50^2	400×100^2	320×50^2	200×50^2	400×100^2
$r_{\rm in}$, $r_{\rm out}$ (10^9 cm)	0.3, 1.0	0.3, 1.0	0.3, 1.0	0.3, 0.9	0.3, 1.0	0.3, 1.6	0.3, 0.9	0.3, 0.9
$r_{\rm b}^c$, $r_{\rm t}^c$ (10^9 cm)	0.43, 0.85	0.43, 0.85	0.43, 0.84	0.43, 0.68	0.43, 0.84	0.42, 1.4	0.43, 0.65	0.43, 0.68
$\Delta\theta$, $\Delta\phi$	45°	45°	45°	27.5°	27.5°	27.5°	27.5°	27.5°
CZ stratification (H_P)	1.9	1.9	1.9	1.2	1.9	4.1	1.1	1.2
$\Delta t_{\rm av}$ (s)	230	230	165	900	230	500	300	300
$v_{\rm rms}$ (10^6 cm/s)	10.7	10.9	10.9	5.28	9.15	4.88	4.66	4.97
$\tau_{\rm conv}$ (s)	78.2	77.1	75.6	94.7	89.2	403.	94.6	95.6
$P_{turb}/P_{gas}(10^{-4})$	3.88	4.05	4.03	0.96	3.01	1.73	0.79	0.96
L (10^{46} erg/s)	2.74	2.63	2.58	0.44	2.86	0.26	0.40	-0.42
$L_{\rm d}$ (10^{46} erg/s)	0.29	0.28	0.26	0.04	0.31	0.08	0.03	0.03
$l_{\rm d}$ (10^8 cm)	7.39	7.92	8.73	3.87	4.15	5.1	2.85	4.1
$\tau_{\rm d}$ (s)	34.48	36.47	39.98	36.72	22.64	52.04	30.56	41.1
$\tau_{\rm dr}$ (s)	38.06	39.27	-	75.65	46.59	130.73	90.20	90.8
$\tau_{\rm dh}$ (s)	30.77	32.14	-	22.91	14.21	28.42	18.4	25.24

Table 5: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r_{\rm b}^c$, $r_{\rm t}^c$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

3.2 Summary of Red Giant Simulations

Parameter	rg.3D.lr	rg.3D.mr	rg.3D.4hp
Grid zoning	216×128^2	432×256^2	176×128^2
$r_{\rm in}$, $r_{\rm out}$ (10^{12} cm)	0.82, 4.09	0.82, 4.09	0.82, 0.34
$r^c_{\rm in}$, $r^c_{\rm out}$ (10^{12} cm)	2.05, 3.86	2.07, 3.88	2.16, 3.33
$\Delta\theta$, $\Delta\phi$	45°	45°	45°
CZ stratification (H_p)	7.0	7.2	3.5
$\Delta t_{\rm av}$ (days)	800	800	800
$v_{\rm rms}$ (10^5 cm/s)	2.59	2.66	2.01
$\tau_{\rm conv}$ (days)	161.	158.	134.
$P_{turb}/P_{gas}(10^{-3})$	4.68	4.98	0.98
$L_{\rm cool}$ (10^{36} erg/s)	-8.57	-7.13	-9.2
$L_{\rm d}$ (10^{36} erg/s)	7.26	7.24	2.27
$l_{\rm d}$ (10^{11} cm)	9.95	10.4	11.6
$\tau_{\rm d}$ (days)	22.2	22.7	33.3
$\tau_{\rm dr}$ (days)	36.7	44.7	53.0
$\tau_{\rm dh}$ (days)	18.3	17.9	28.2

Table 6: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r^c_{\rm b}$, $r^c_{\rm t}$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

3.3 Snapshots of TKE in a meridional plane of the oxygen burning and red giant models

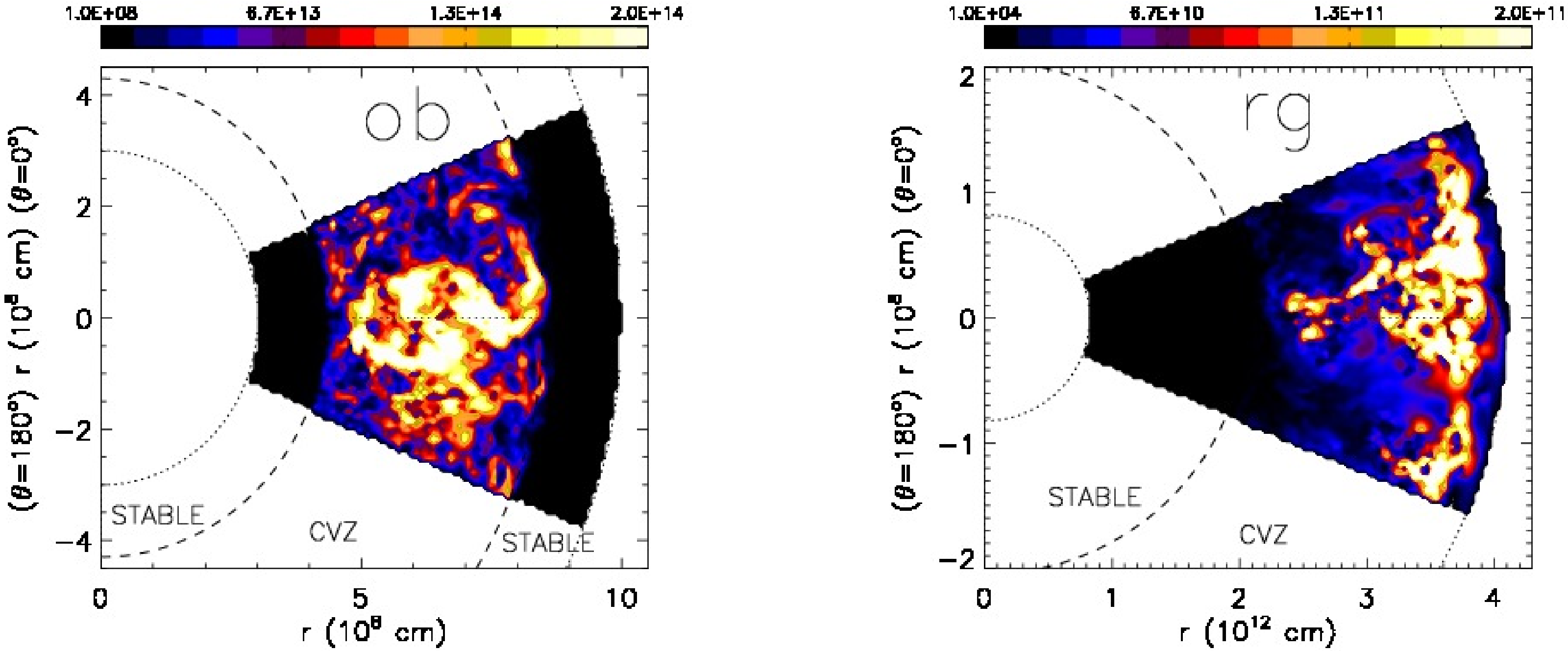

Figure 1: Snapshots of turbulent kinetic energy (in erg g^{-1}) in a meridional plane of 3D oxygen burning shell model ob.3D.mr (left) and red giant envelope convection model rg.3D.mr (right). Convectively unstable (CVZ) and stable layers (STABLE) are separated by dashed lines.

3.4 Background structure of oxygen burning and red giant models

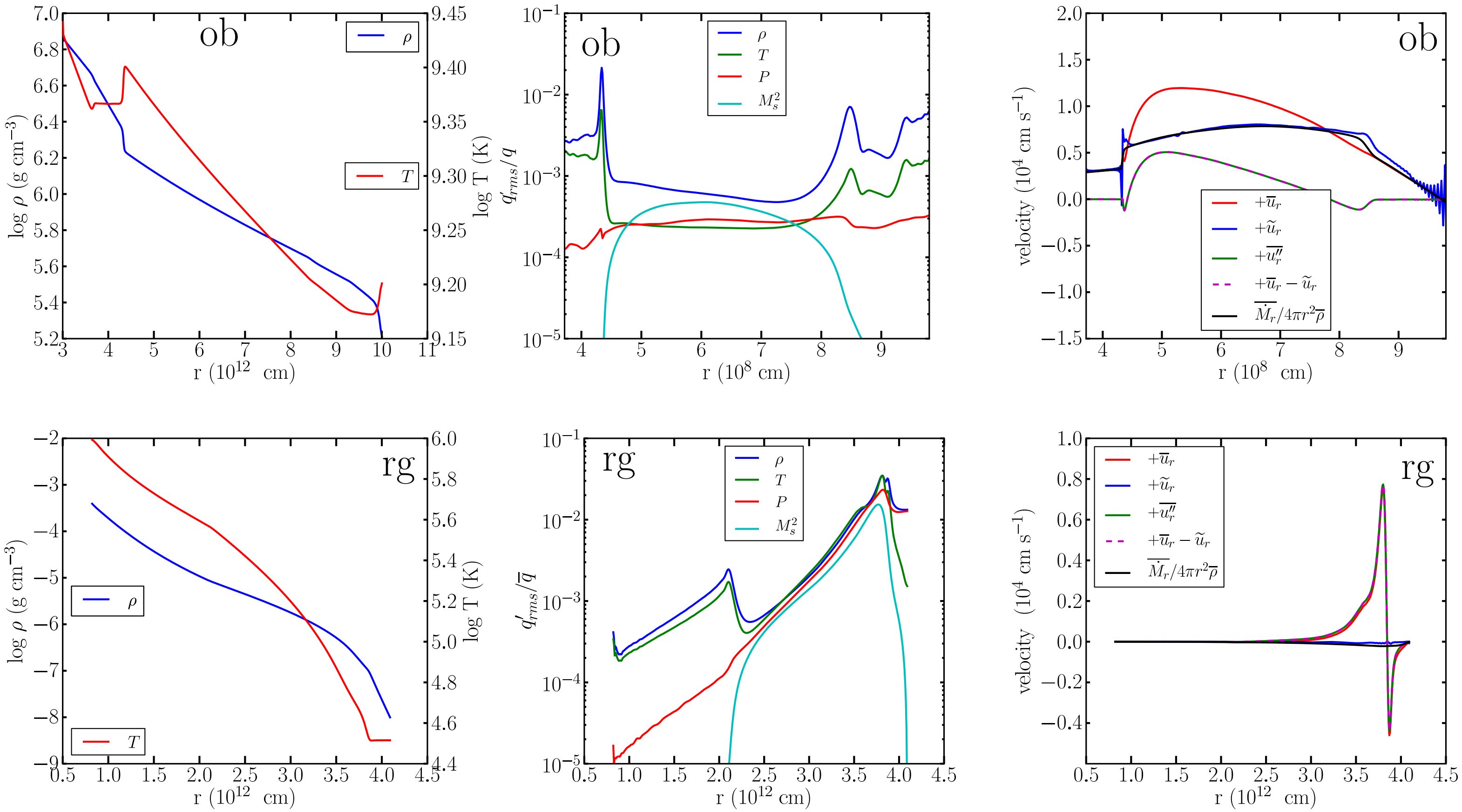

Figure 2: Properties of our data. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4 Profiles and integral budgets of mean fields

4.1 Mean continuity equation

$$\widetilde{D}_t \overline{\rho} = -\overline{\rho}\widetilde{d} + \mathcal{N}_\rho \tag{42}$$

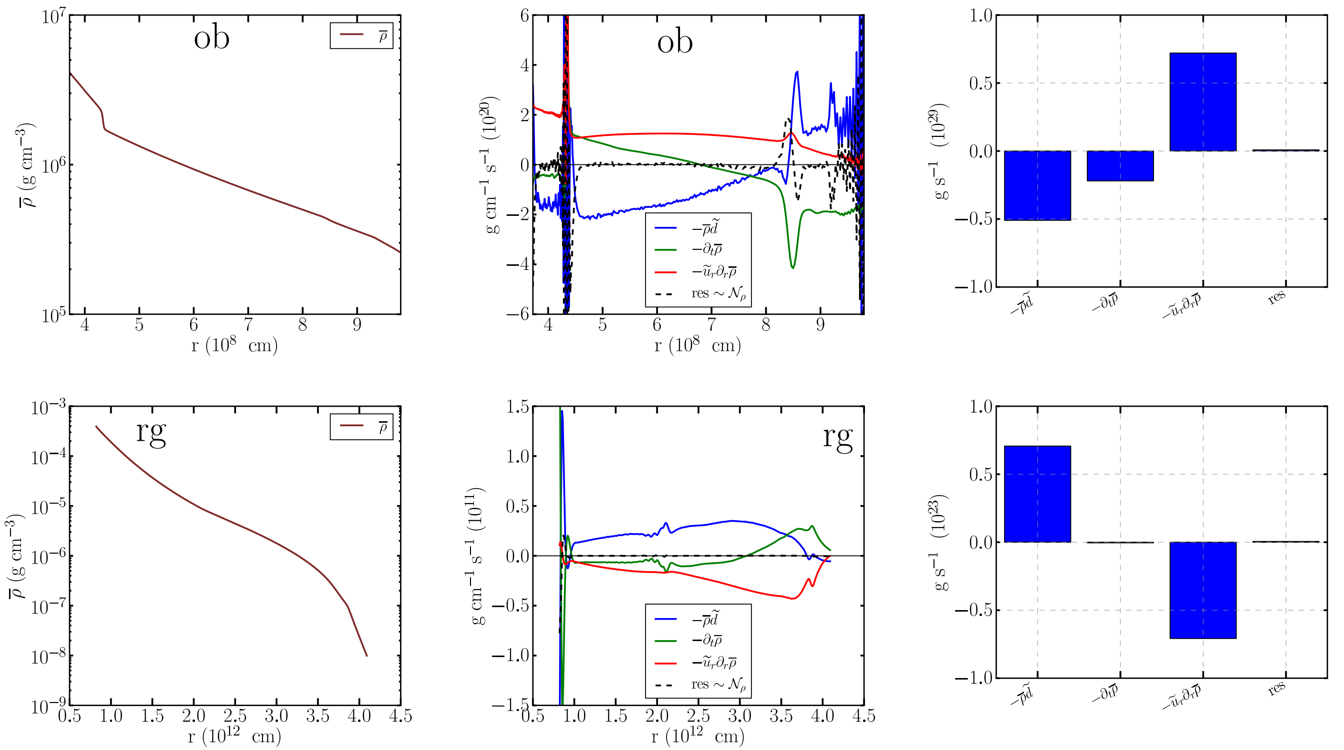

Figure 3: Mean continuity equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.2 Mean radial momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} \tag{43}$$

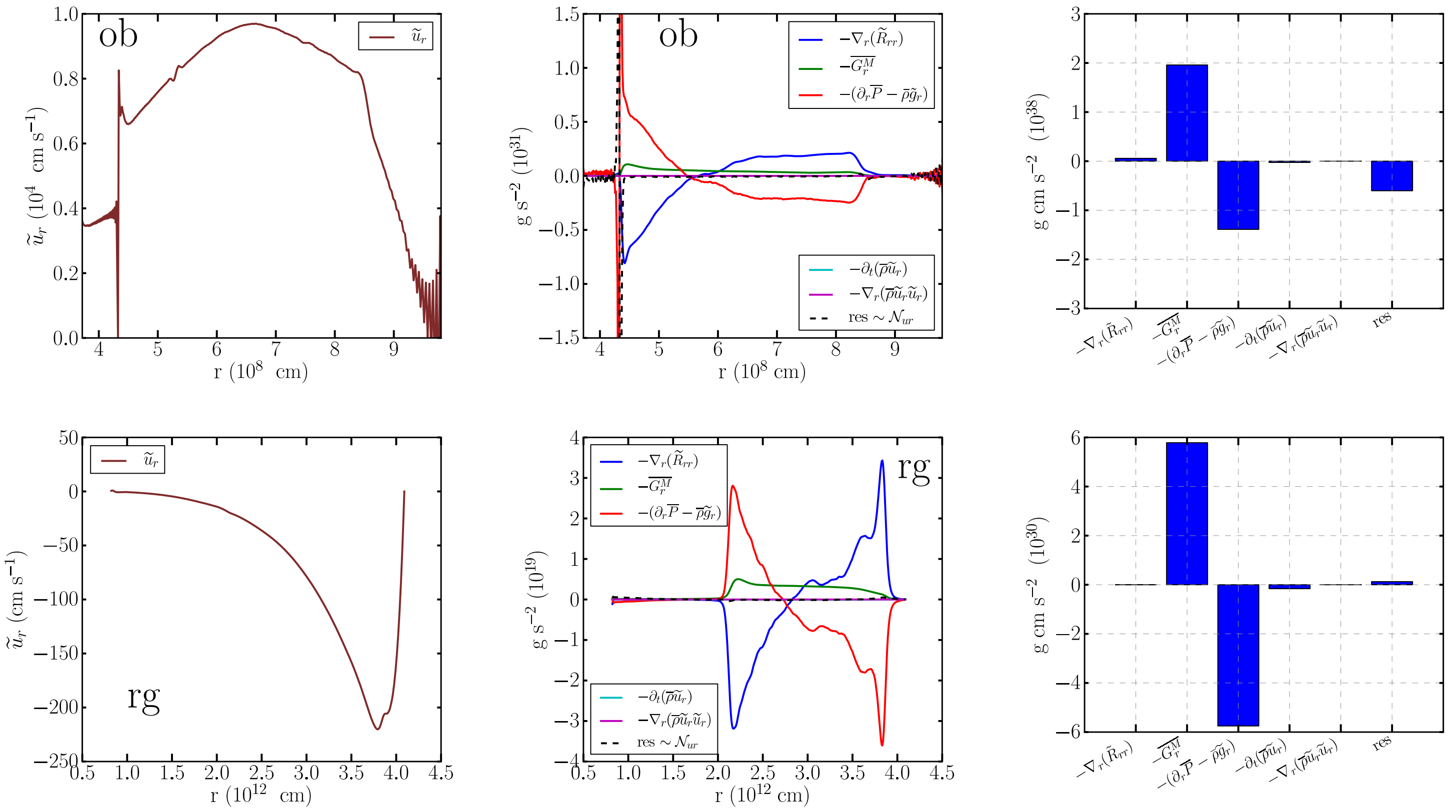

Figure 4: Mean radial momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.3 Mean azimutal momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G^M_\theta} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} \tag{44}$$

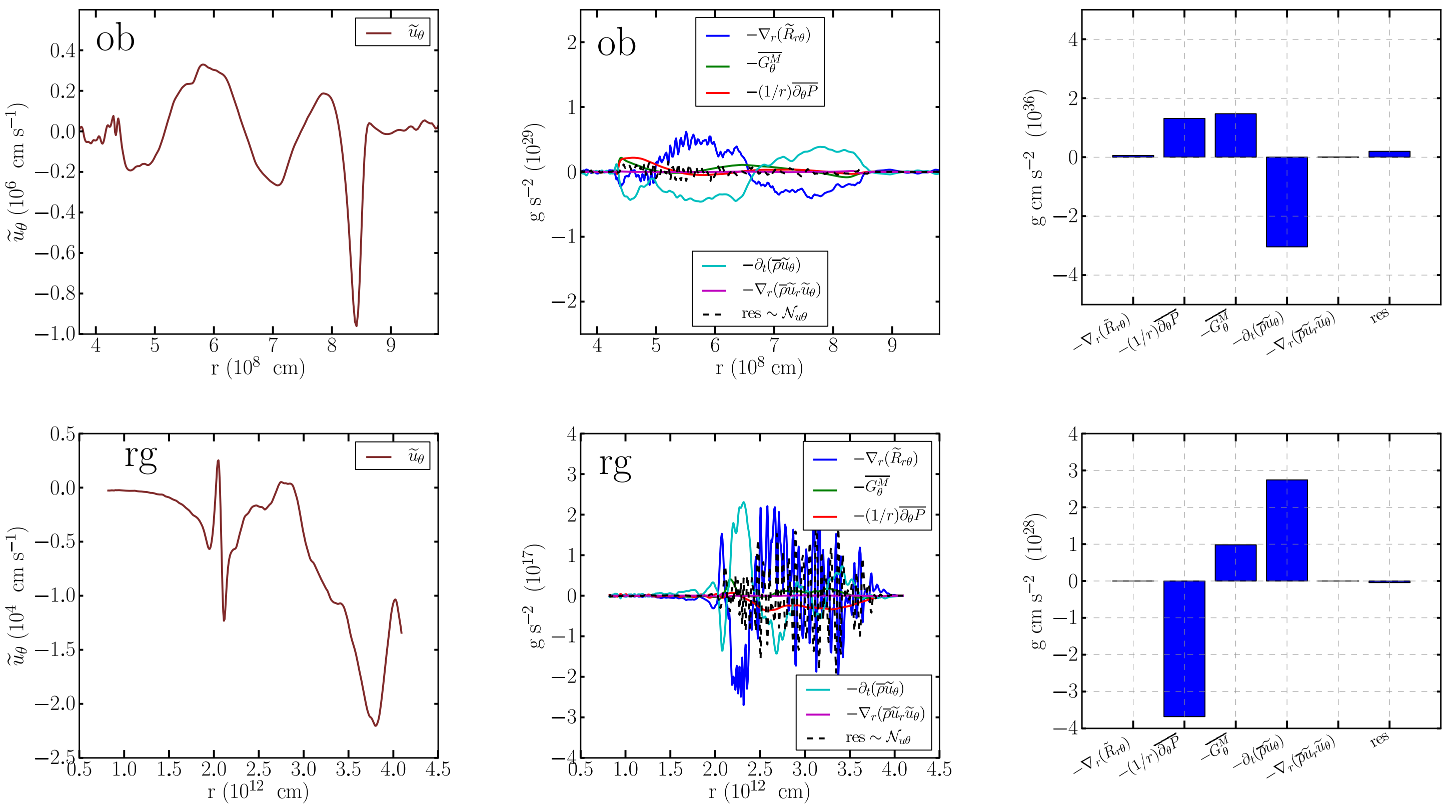

Figure 5: Mean azimuthal momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.4 Mean polar momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G_\phi^M} + \mathcal{N}_{u\phi} \tag{45}$$

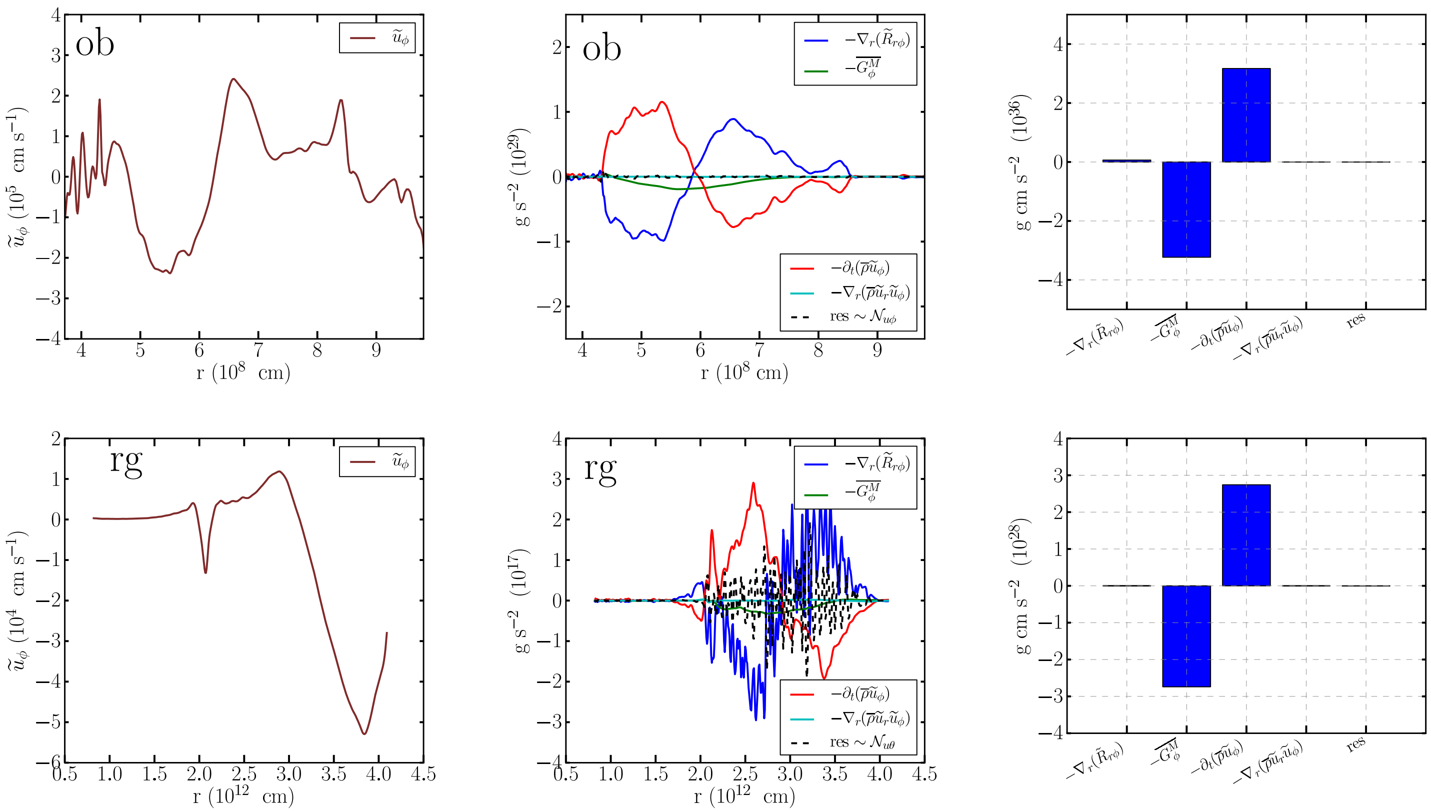

Figure 6: Mean polar omentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.5 Mean internal energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \tag{46}$$

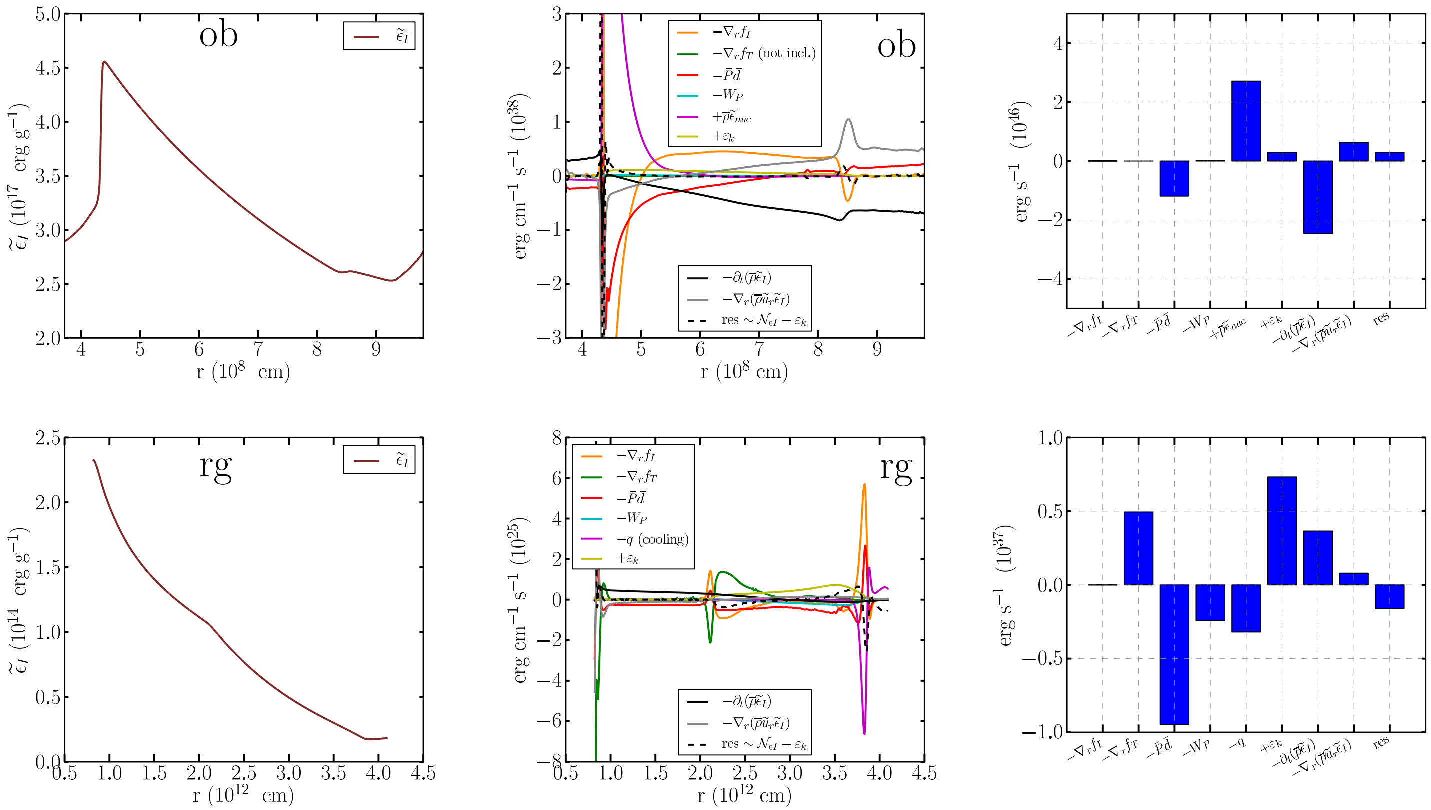

Figure 7: Mean internal energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.6 Mean kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_k = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} \quad (47)$$

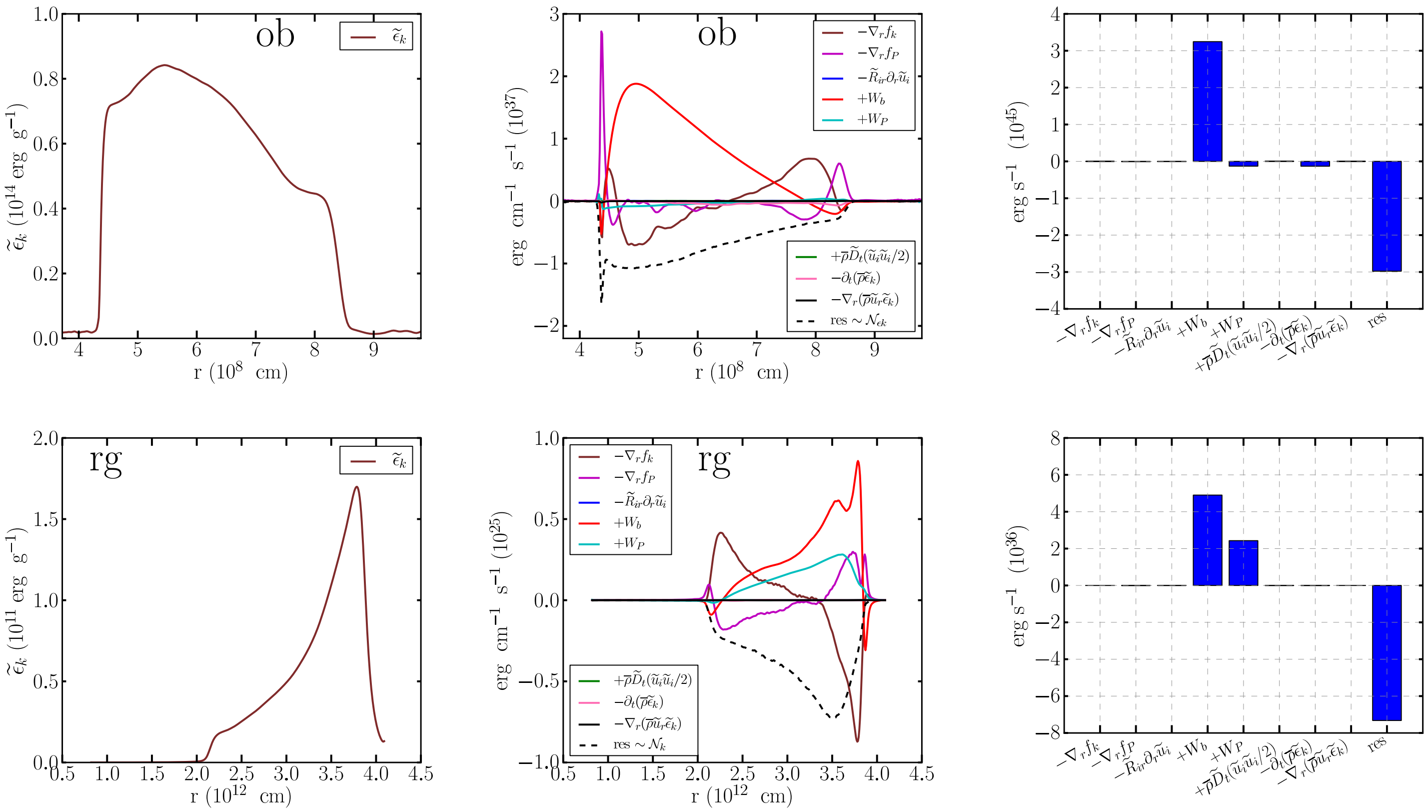

Figure 8: Mean kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.7 Mean total energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} \tag{48}$$

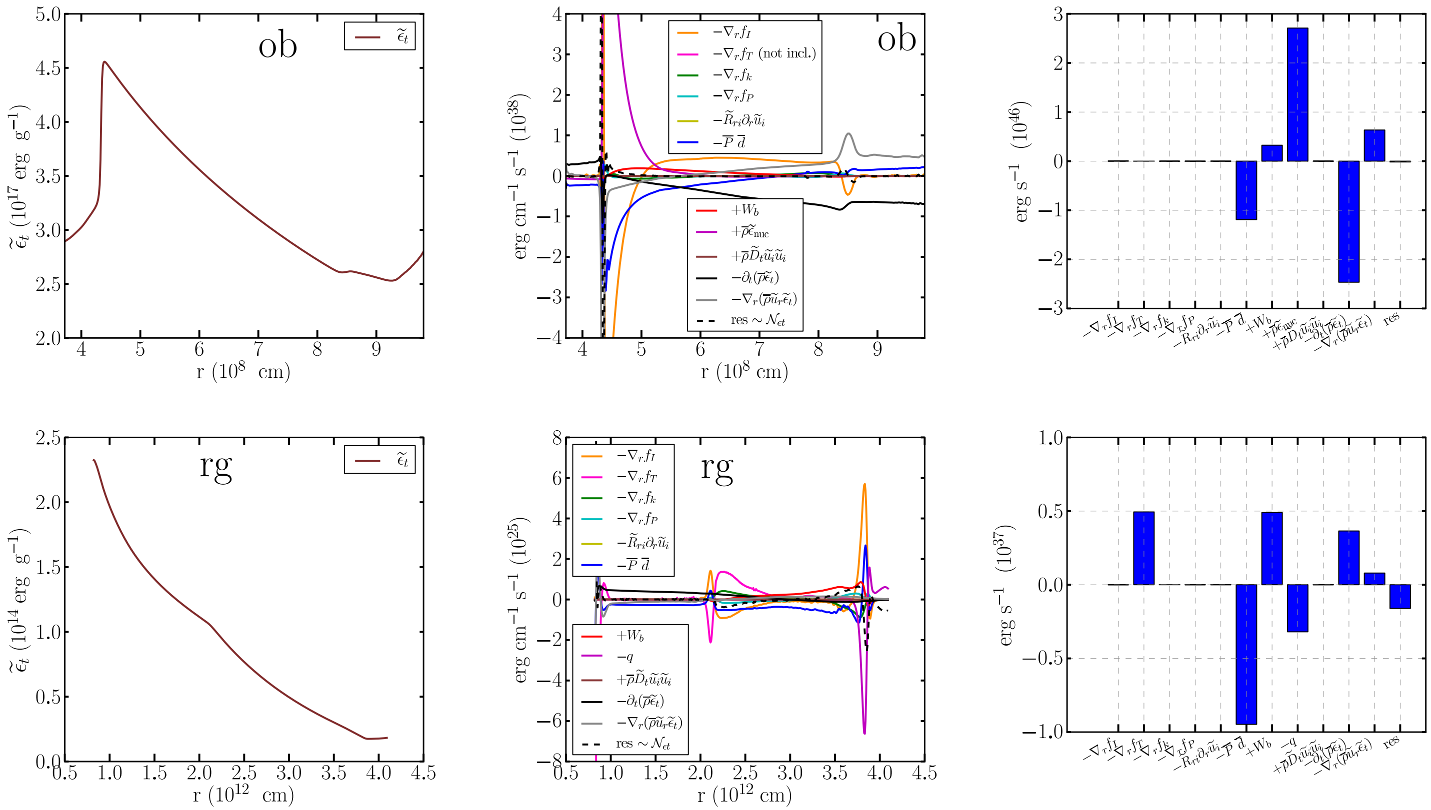

Figure 9: Mean total energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.8 Mean entropy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s - \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \tag{49}$$

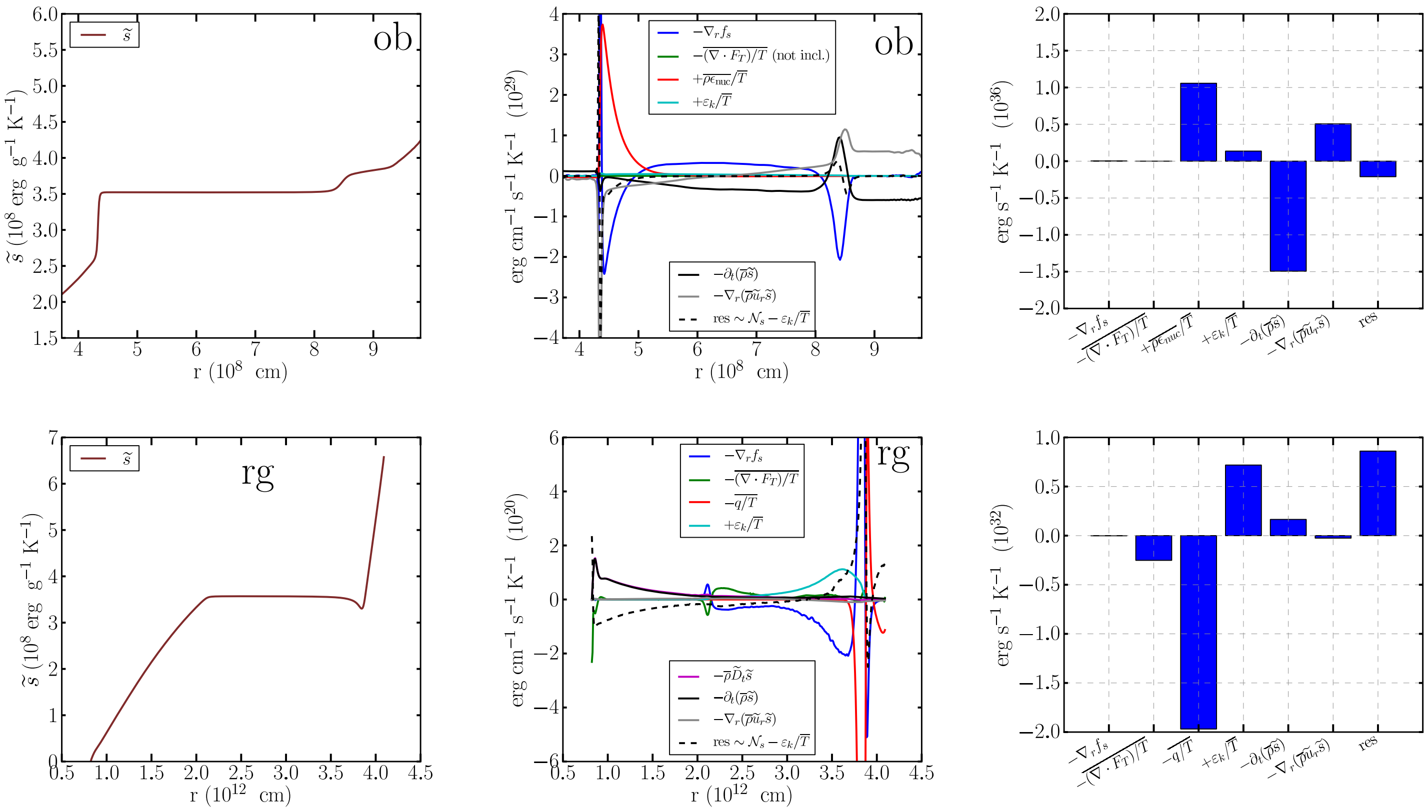

Figure 10: Mean entropy equation. Model ob.3D.2hp (upper panels) and model rg.3D.mr (lower panels).

4.9 Mean pressure equation

$$\overline{D}_t \overline{P} = -\nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1-\Gamma_1) W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P \tag{50}$$

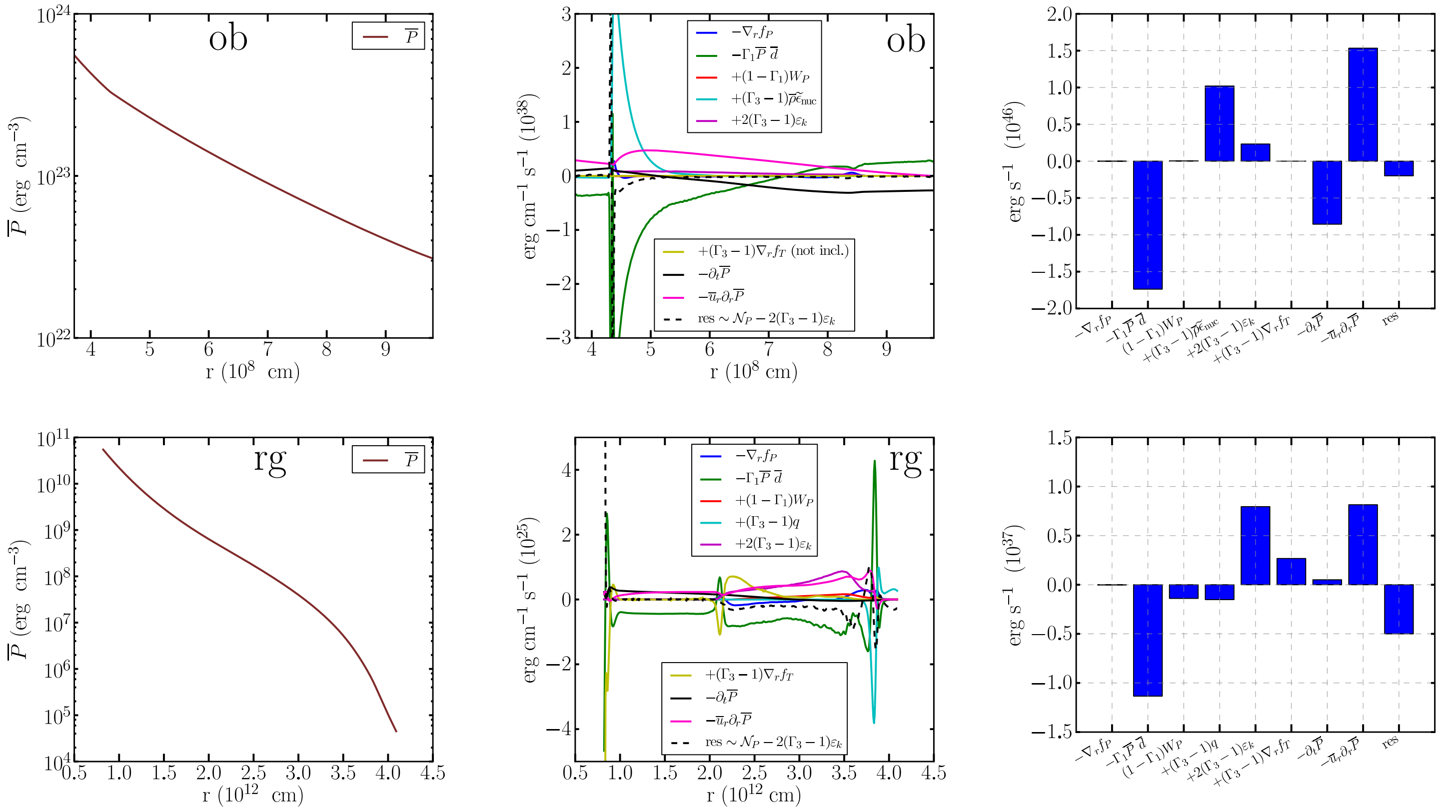

Figure 11: Mean pressure equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.10 Mean enthalpy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r f_h - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T + \mathcal{N}_h \tag{51}$$

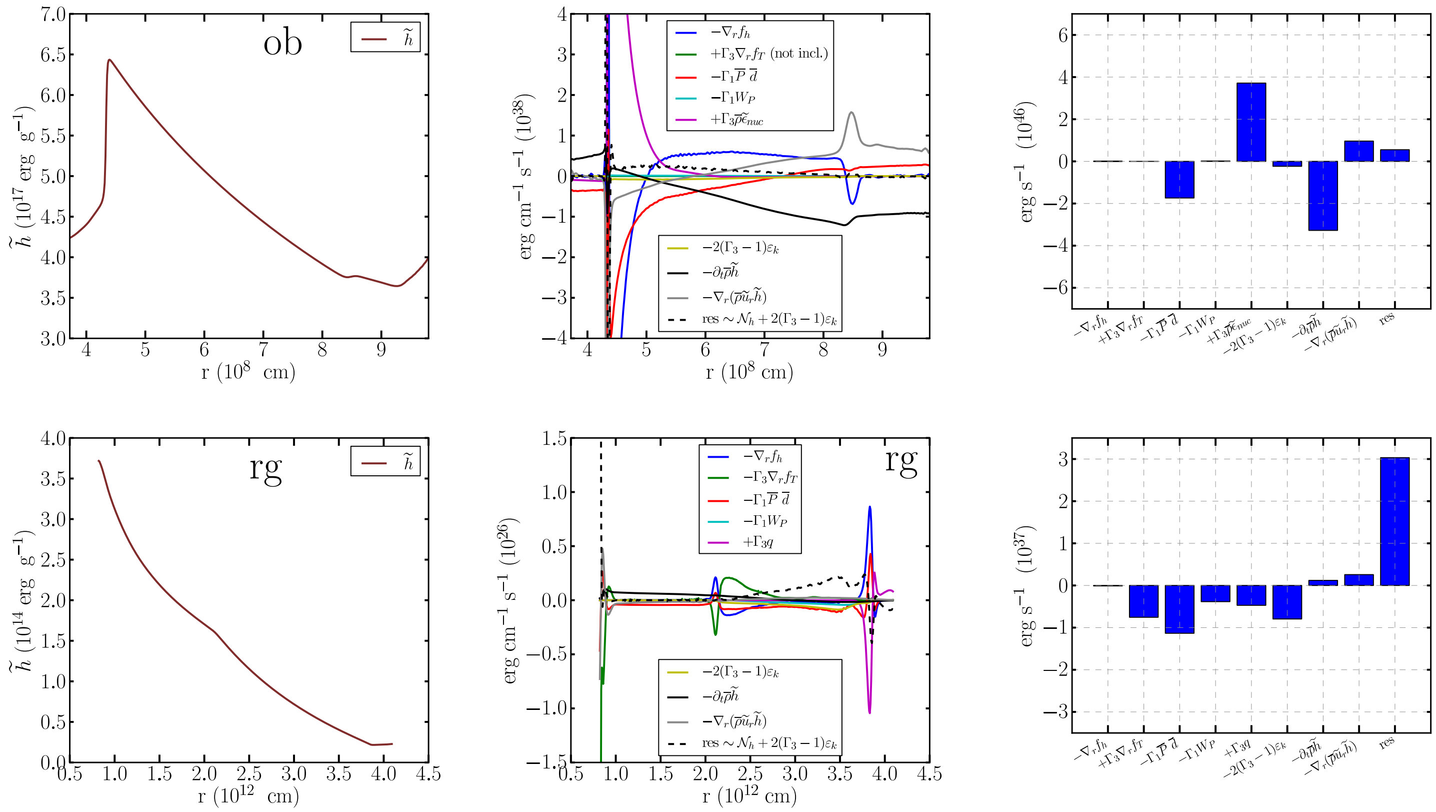

Figure 12: Mean enthalpy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.11 Mean angular momentum equation (z-component)

$$\overline{\rho}\widetilde{D}_t\widetilde{j}_z = -\nabla_r f_{jz} + \mathcal{N}_{jz} \tag{52}$$

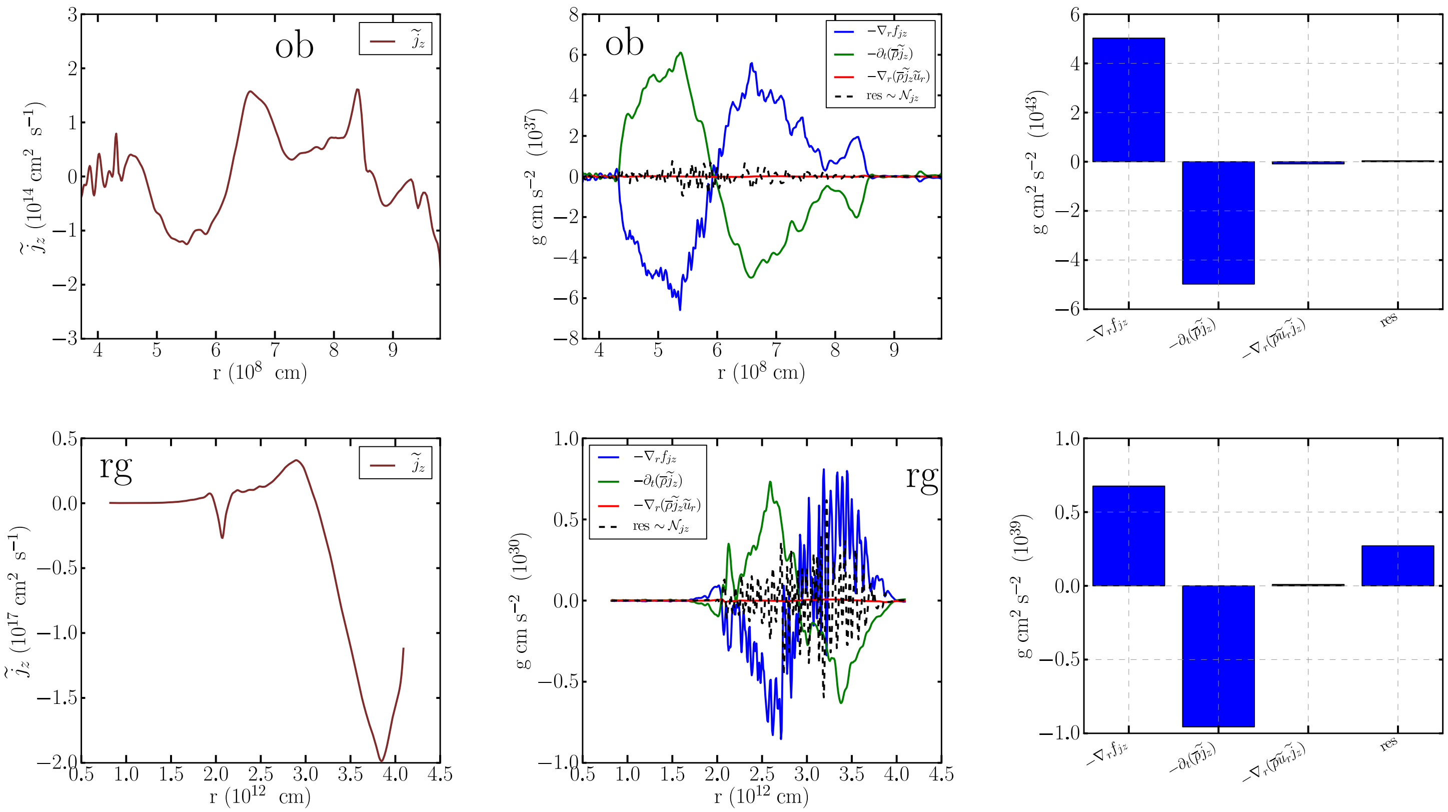

Figure 13: Mean angular momentum equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.12 Mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{53}$$

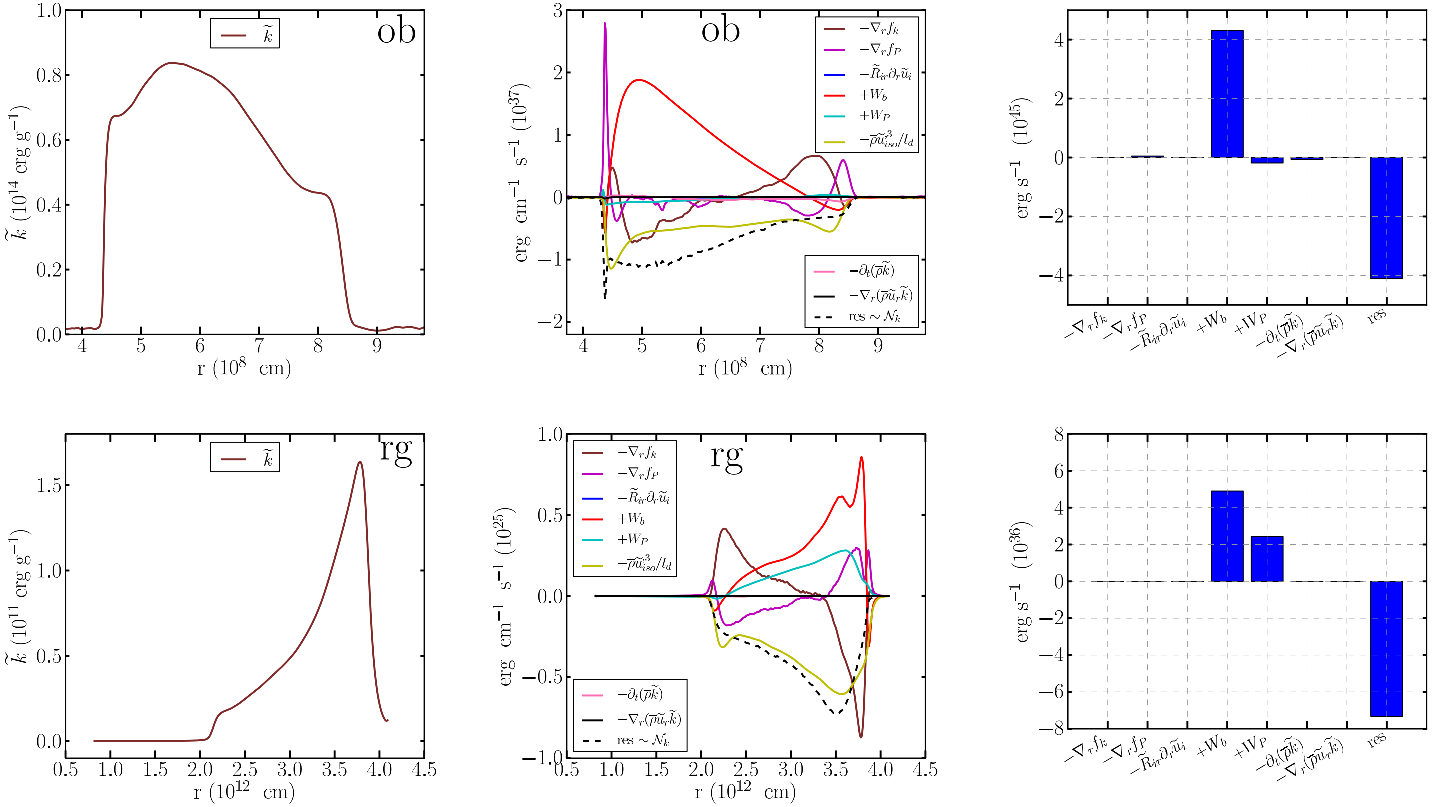

Figure 14: Turbulent kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.13 Mean turbulent kinetic energy equation (radial part)

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u_r''} + \mathcal{G}_k^r + \mathcal{N}_{kr} \tag{54}$$

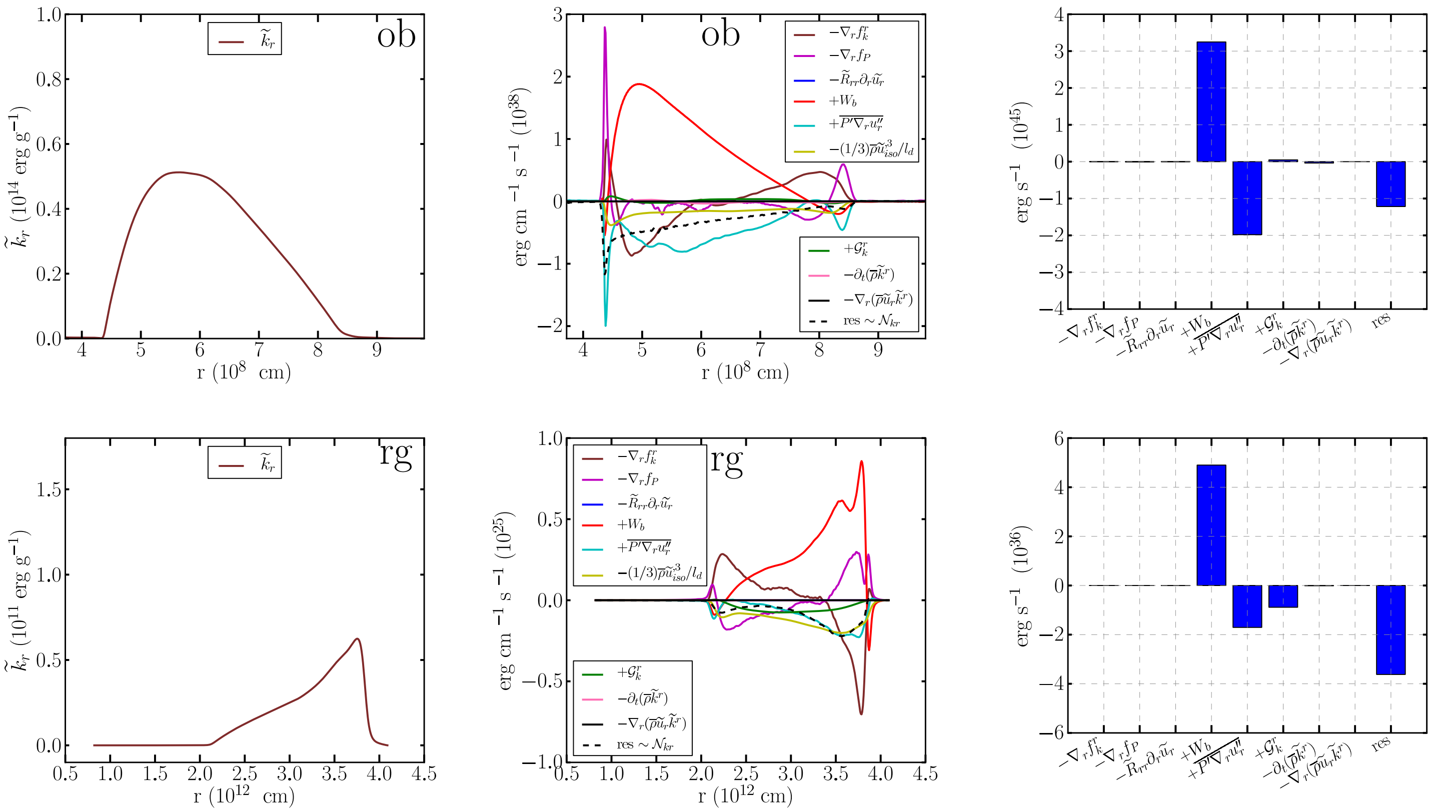

Figure 15: Turbulent radial kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.14 Mean turbulent kinetic energy equation (horizontal part)

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \quad (55)$$

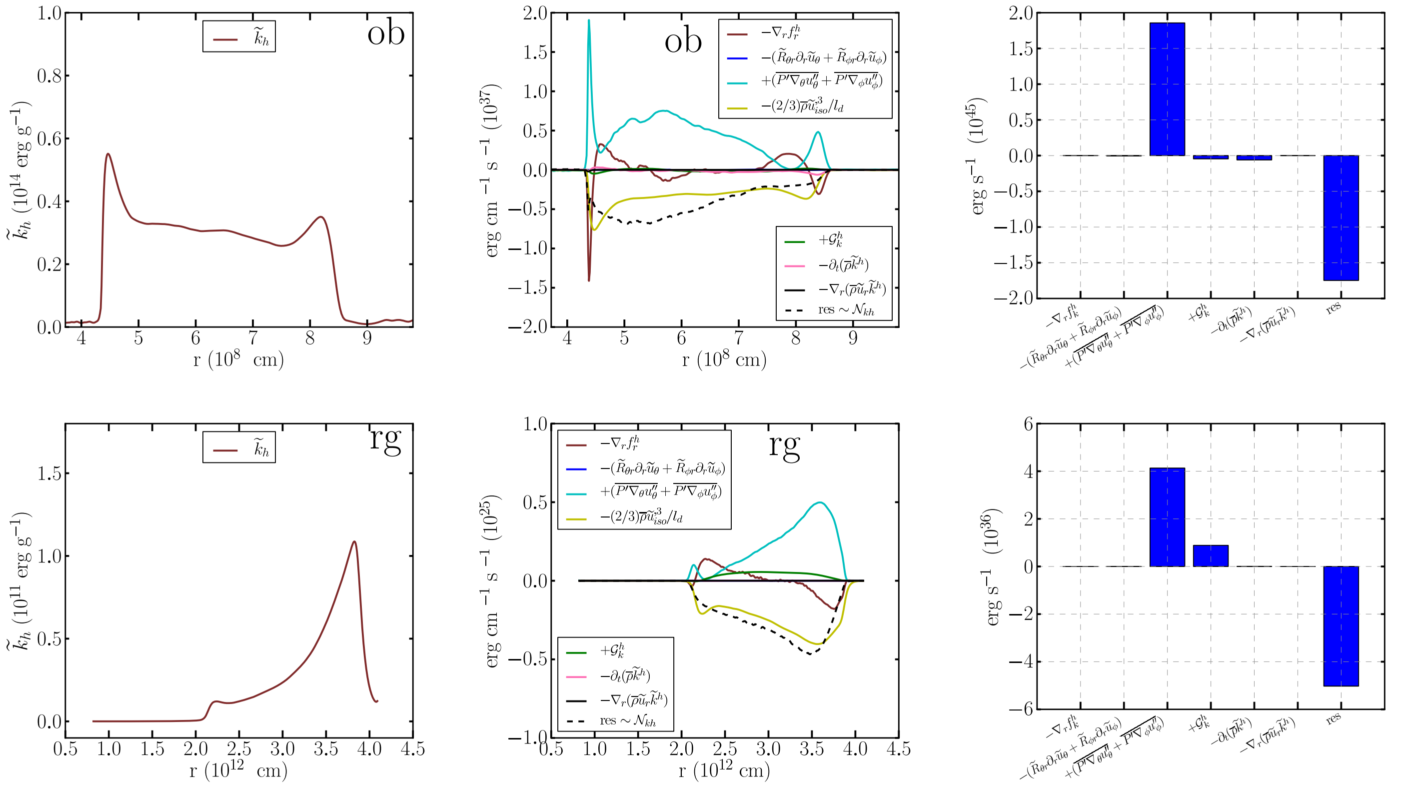

Figure 16: Turbulent horizontal kinetic energy equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.15 Mean turbulent mass flux equation

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} + \mathcal{G}_a + \mathcal{N}_a \quad (56)$$

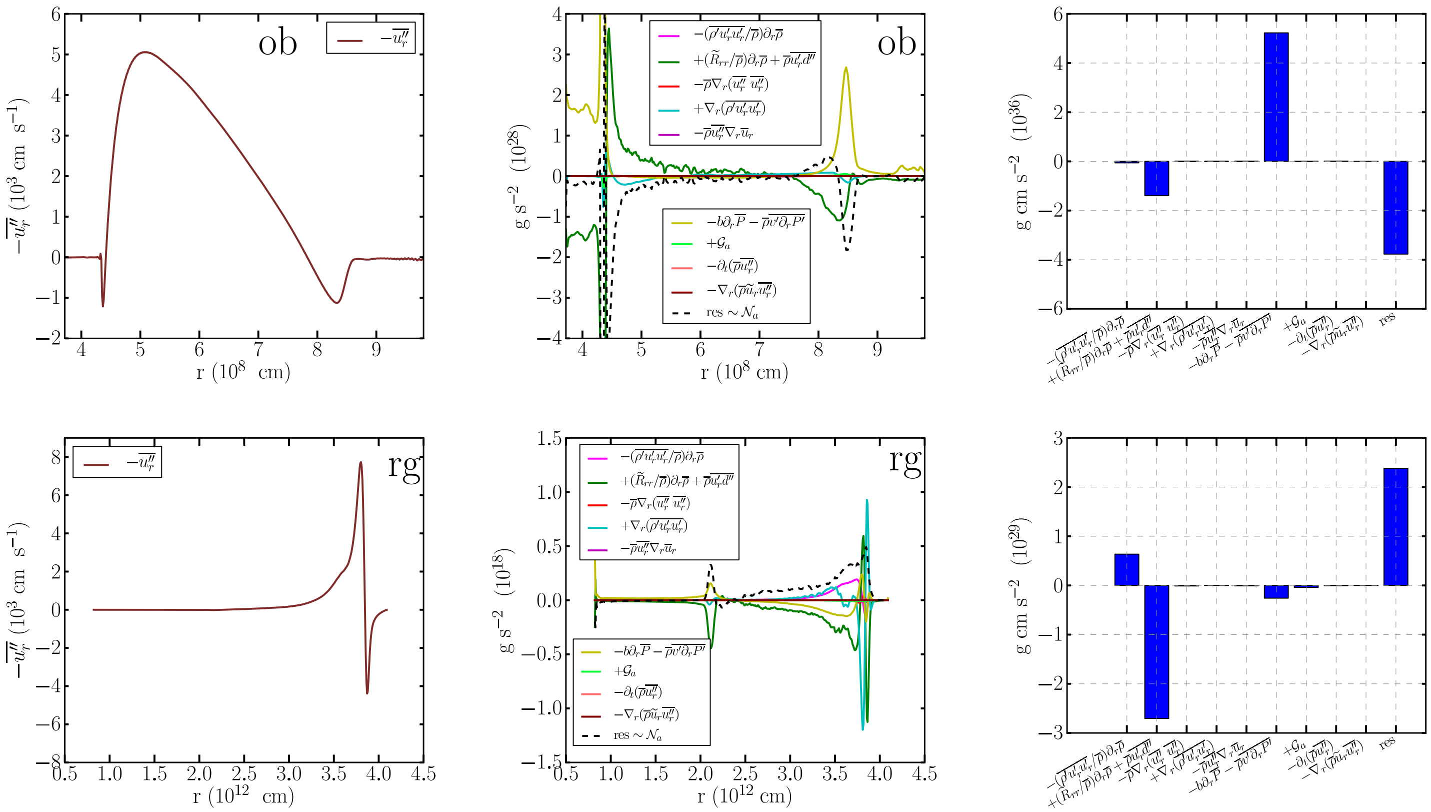

Figure 17: Turbulent mass flux equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.16 Mean density-specific volume covariance equation

$$\overline{D}_t b = + \overline{v}\nabla_r \overline{\rho}\overline{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\overline{v'd'} + \mathcal{N}_b \tag{57}$$

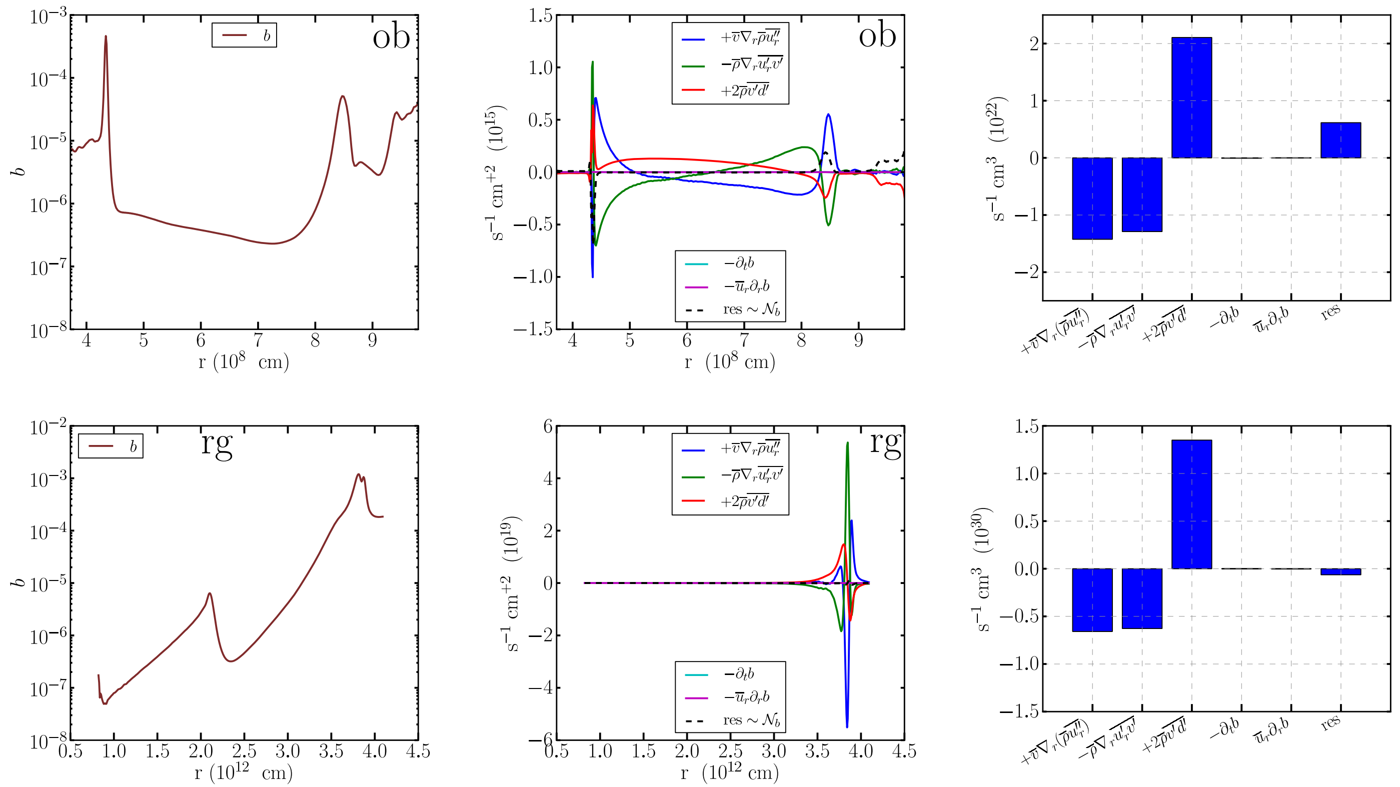

Figure 18: Density-specific volume covariance equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.17 Mean internal energy flux equation

$$\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) = \mathcal{N}_{fI} - \nabla_r f_I^r - f_I \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{\epsilon_I} - \overline{\epsilon_I''}\partial_r \overline{P} - \overline{\epsilon_I'' \partial_r P'} - \overline{u_r''\,(Pd)} + \overline{u_r''(\mathcal{S} + \nabla \cdot F_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \quad (58)$$

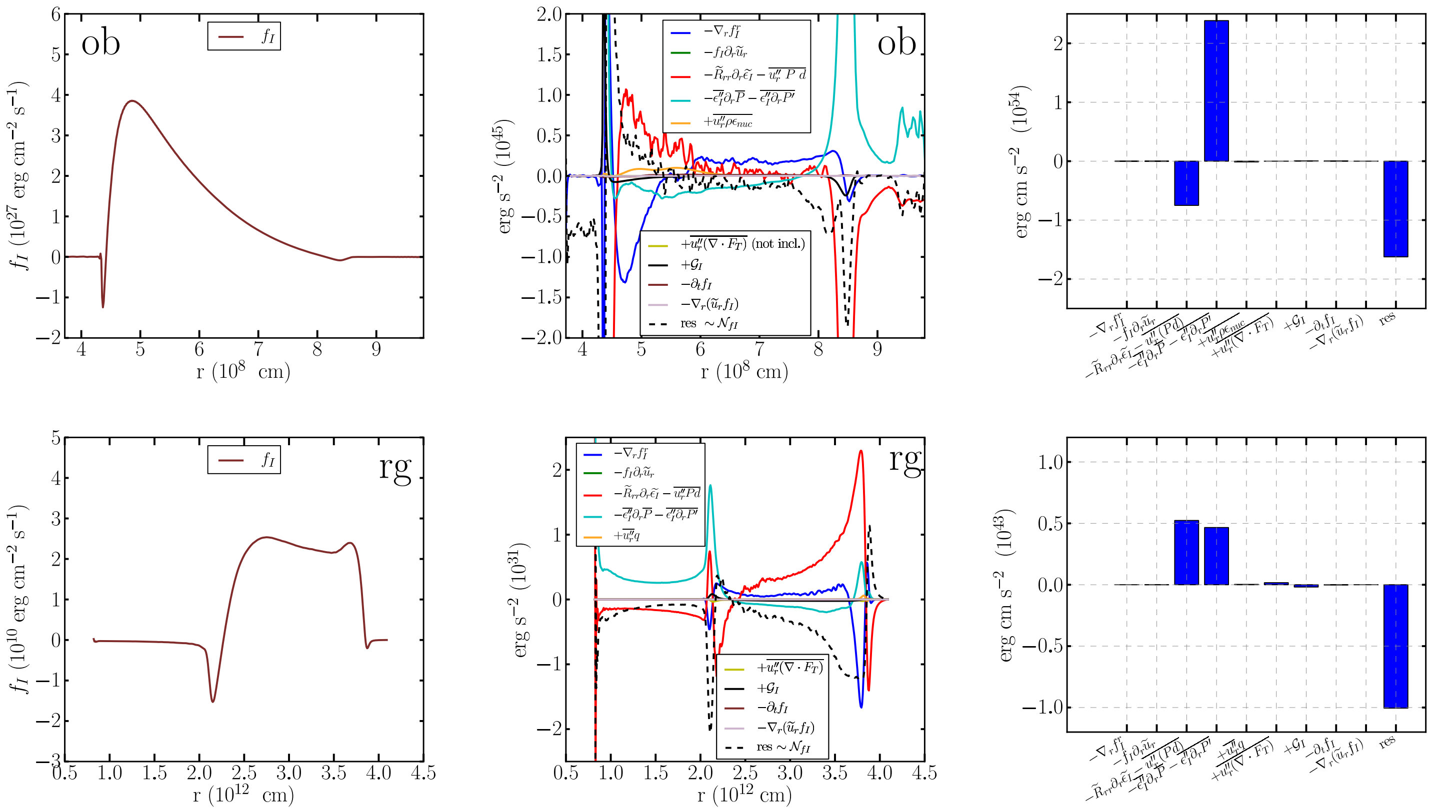

Figure 19: Mean internal energy flux equation. Model ob.3D.mr (upper panels) and model rg.3D.mr (lower panels).

4.18 Mean entropy flux equation

$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u_r''(\mathcal{S} + \nabla\cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \quad (59)$$

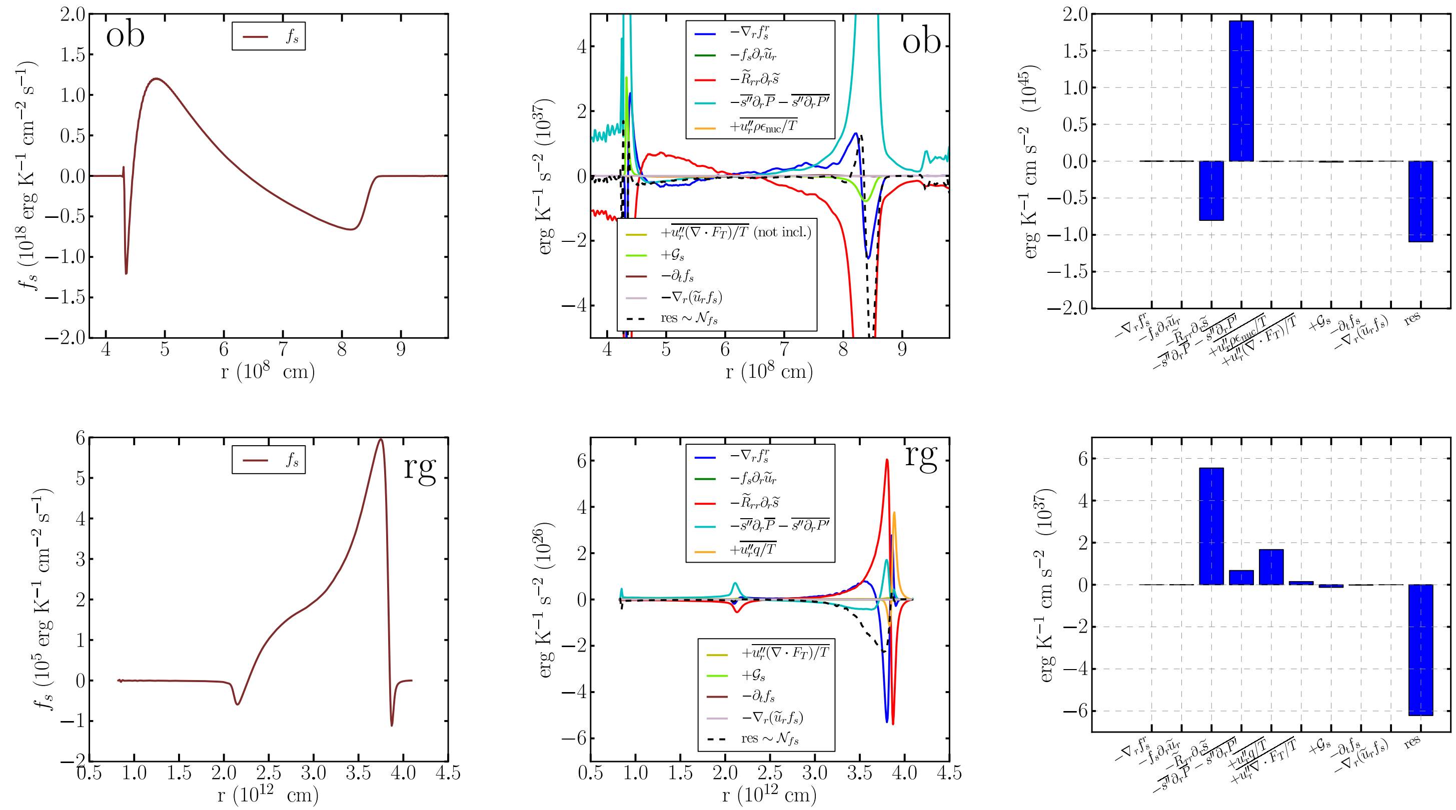

Figure 20: Mean entropy flux equation. Model ob.3D.2hp (upper panels) and model rg.3D.mr (lower panels).

4.19 Mean composition (^{16}O) flux equation

$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{X}_\alpha - \overline{X''_\alpha}\partial_r \overline{P} - \overline{X''_\alpha \partial_r P'} + \overline{u''_r \rho \dot{X}^{\mathrm{nuc}}_\alpha} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \tag{60}$$

$$\tag{61}$$

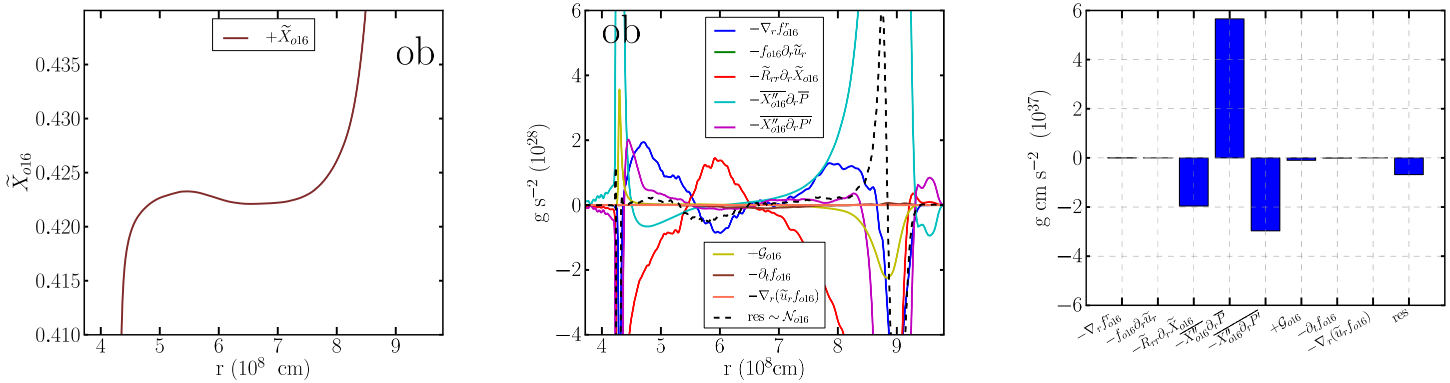

Figure 21: Mean composition (^{16}O) flux equation. Model ob.3D.2hp.

4.20 Mean A flux equation

$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = \mathcal{N}_{fA} - \nabla_r f_A^r - f_A \partial_r \widetilde{u}_r - \widetilde{R}_{rr}\partial_r \widetilde{A} - \overline{A''}\partial_r \overline{P} - \overline{A''\partial_r P'} - \overline{u_r'' \rho A^2 \Sigma_\alpha \dot{X}_\alpha^{\mathrm{nuc}}/A_\alpha} + \mathcal{G}_A \quad (62)$$

$$(63)$$

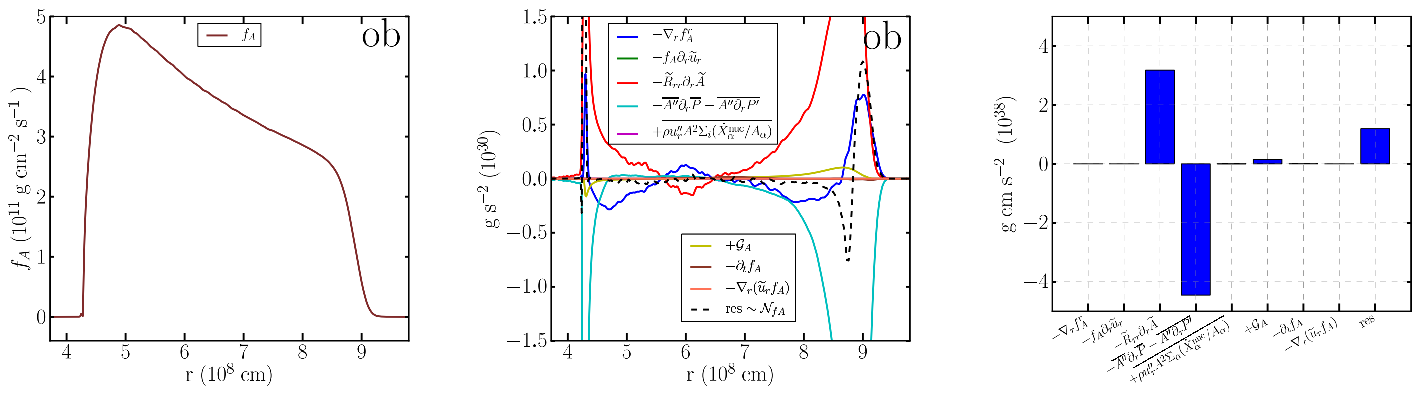

Figure 22: Mean A flux equation. Model ob.3D.2hp.

4.21 Mean Z flux equation

$$\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = \mathcal{N}_{fZ} - \nabla_r f_Z^r - f_Z\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{Z} - \overline{Z''}\partial_r\overline{P} - \overline{Z''\partial_r P'} - \overline{u''_r \rho Z A \Sigma_\alpha(\dot{X}_\alpha^{\rm nuc}/A_\alpha)} - \\ - \overline{u''_r \rho A \Sigma_\alpha(Z_\alpha \dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z \quad (64)$$

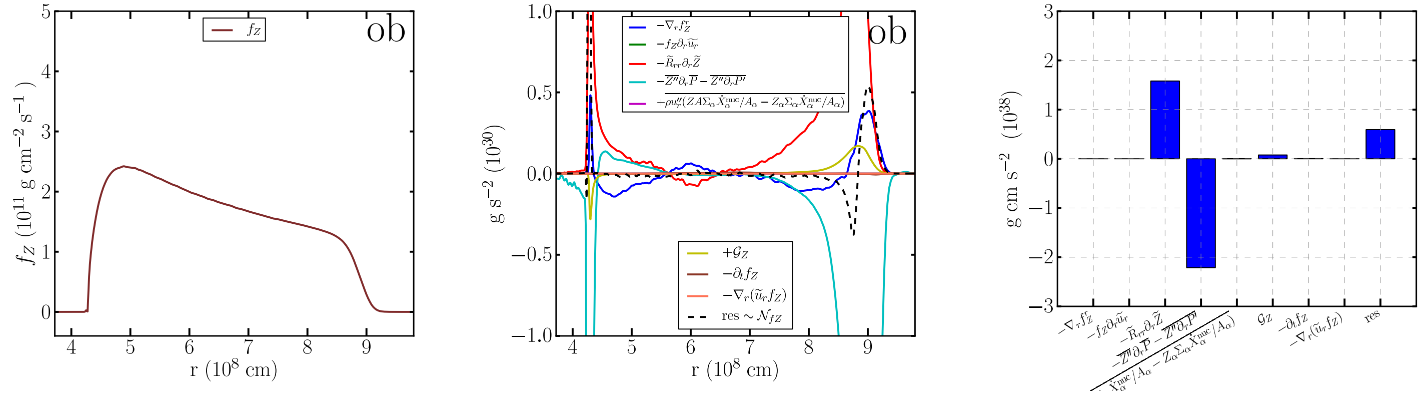

Figure 23: Mean Z flux equation. Model ob.3D.2hp.

5 Mean field composition data for the oxygen shell burning model ob.3D.2hp

5.1 Mean C^{12} and O^{16} equation

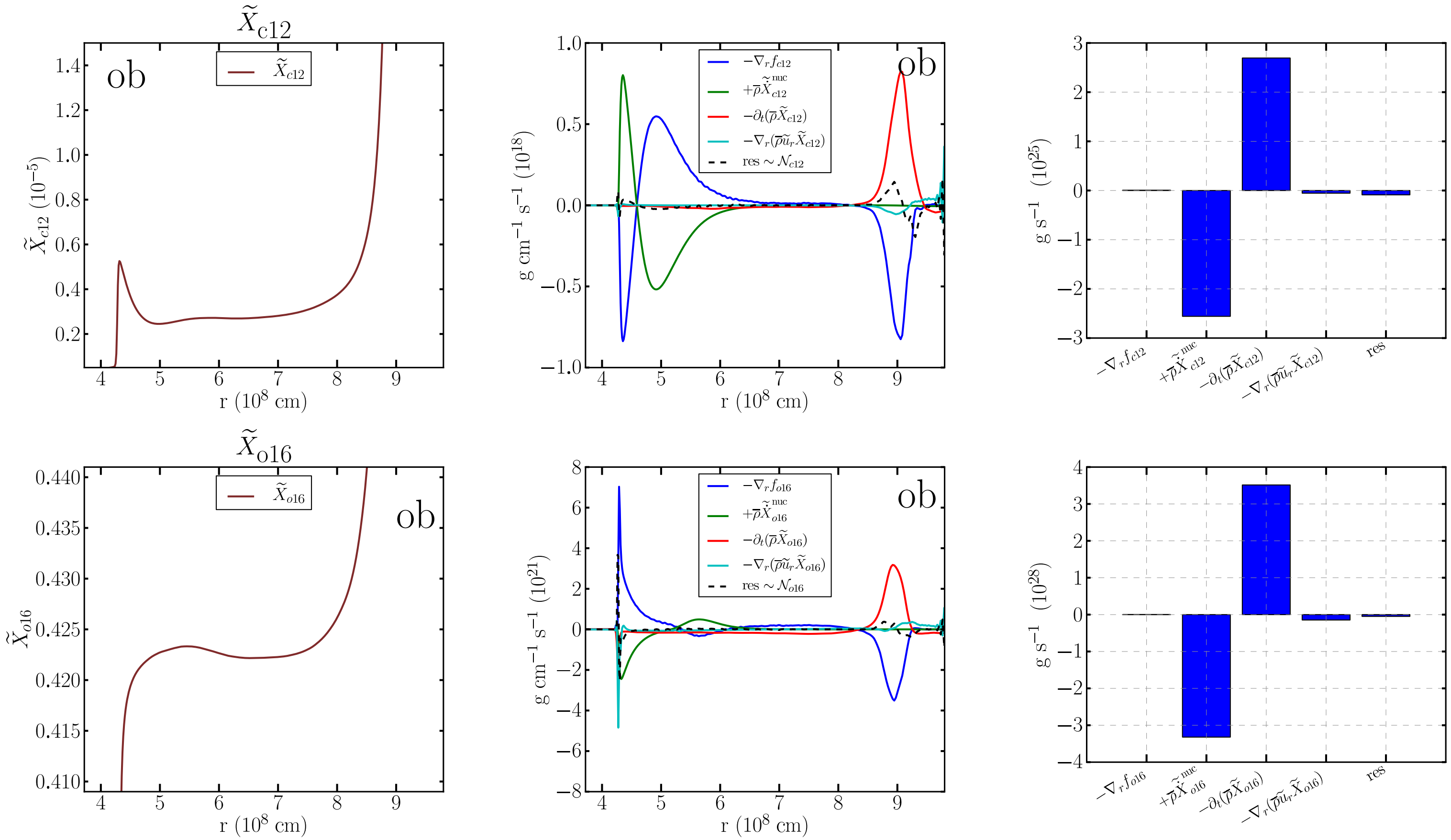

Figure 24: Mean composition equations. Model ob.3D.2hp.

5.2 Mean Ne^{20} and Na^{23} equation

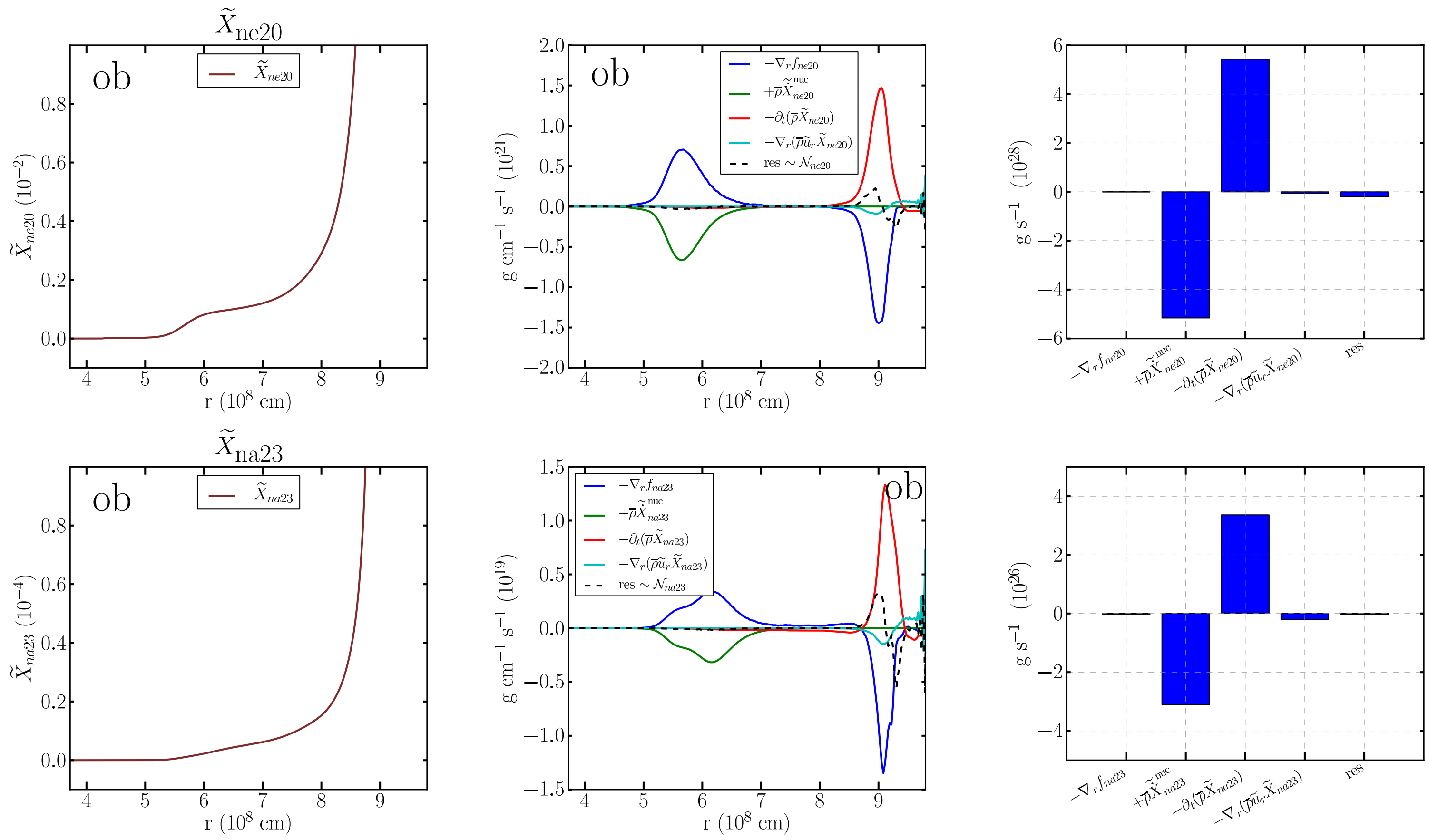

Figure 25: Mean composition equations. Model ob.3D.2hp.

5.3 Mean Mg^{24} and Si^{28} equation

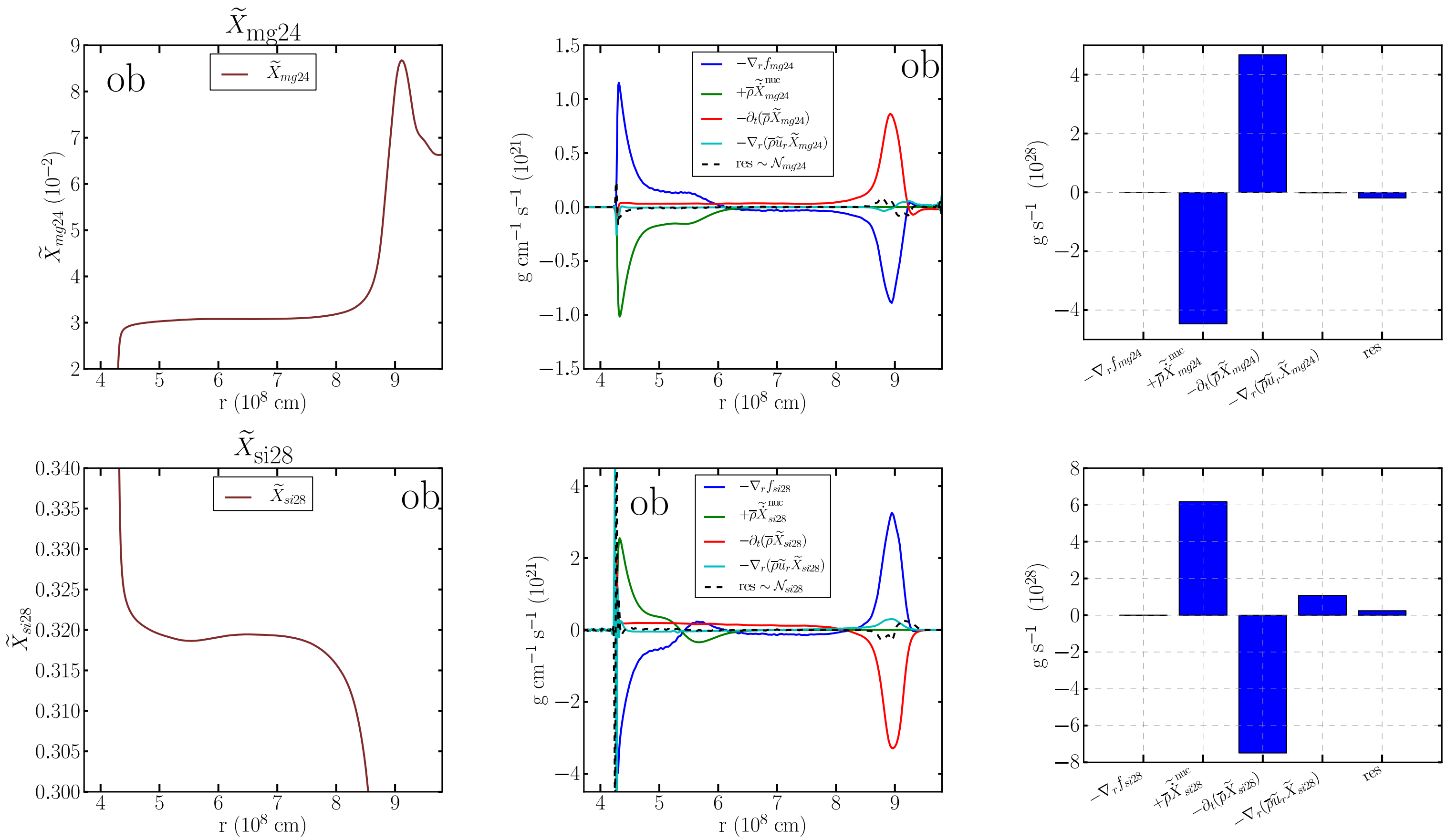

Figure 26: Mean composition equations. Model ob.3D.2hp.

5.4 Mean P^{31} and S^{32} equation

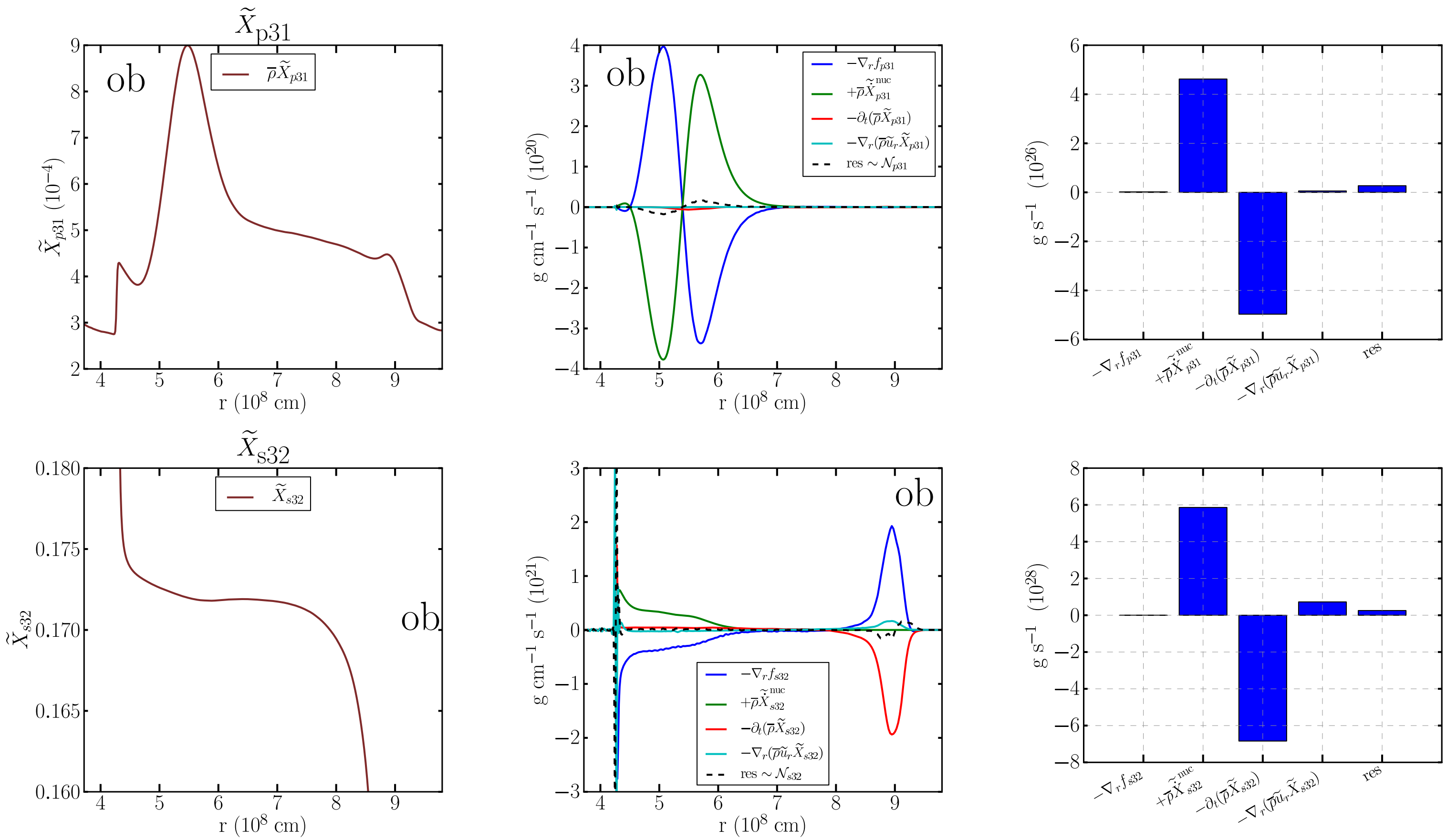

Figure 27: Mean composition equations. Model ob.3D.2hp.

5.5 Mean S^{34} and Cl^{35} equation

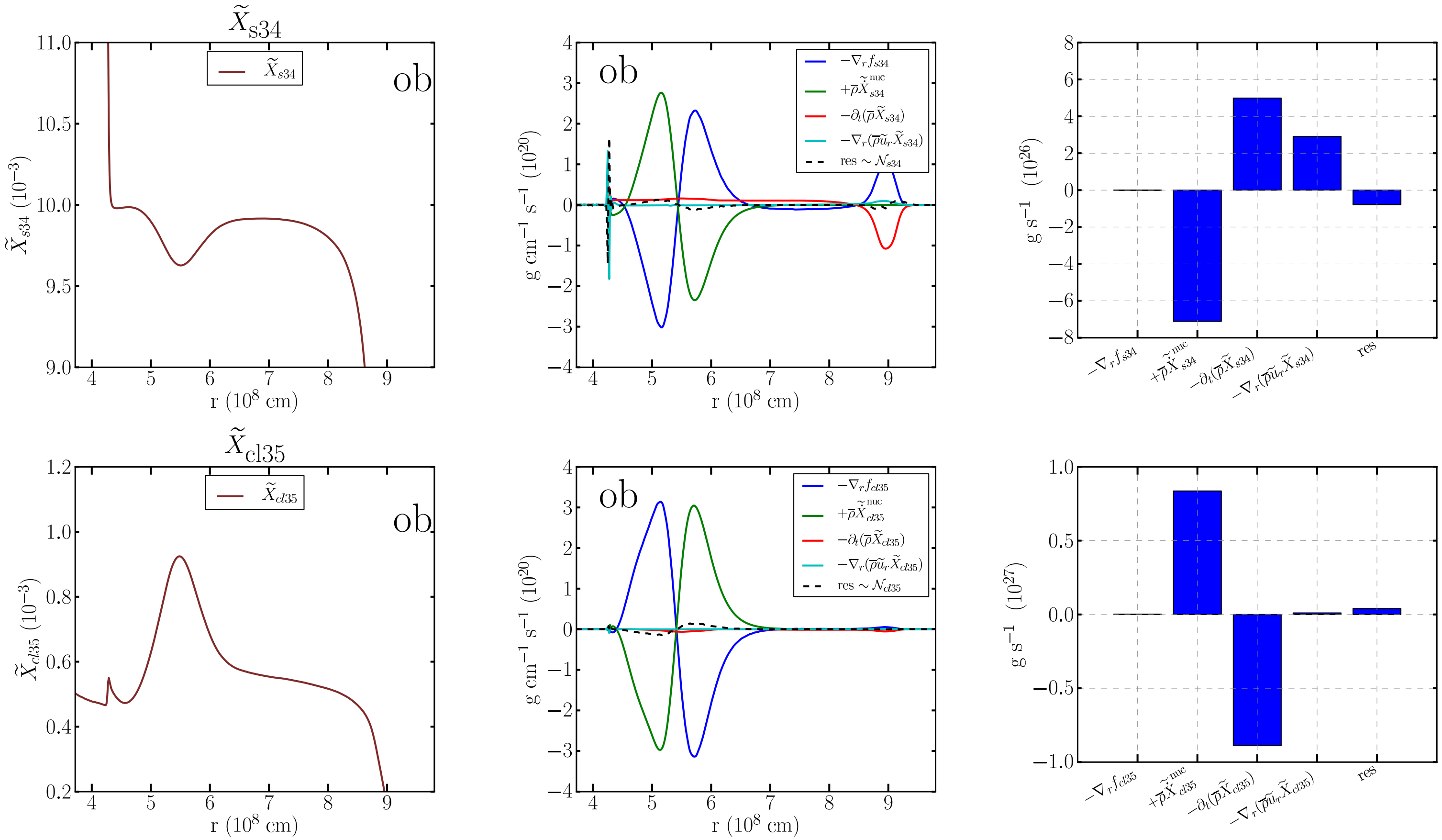

Figure 28: Mean composition equations. Model ob.3D.2hp.

5.6 Mean Ar^{36} and Ar^{38} equation

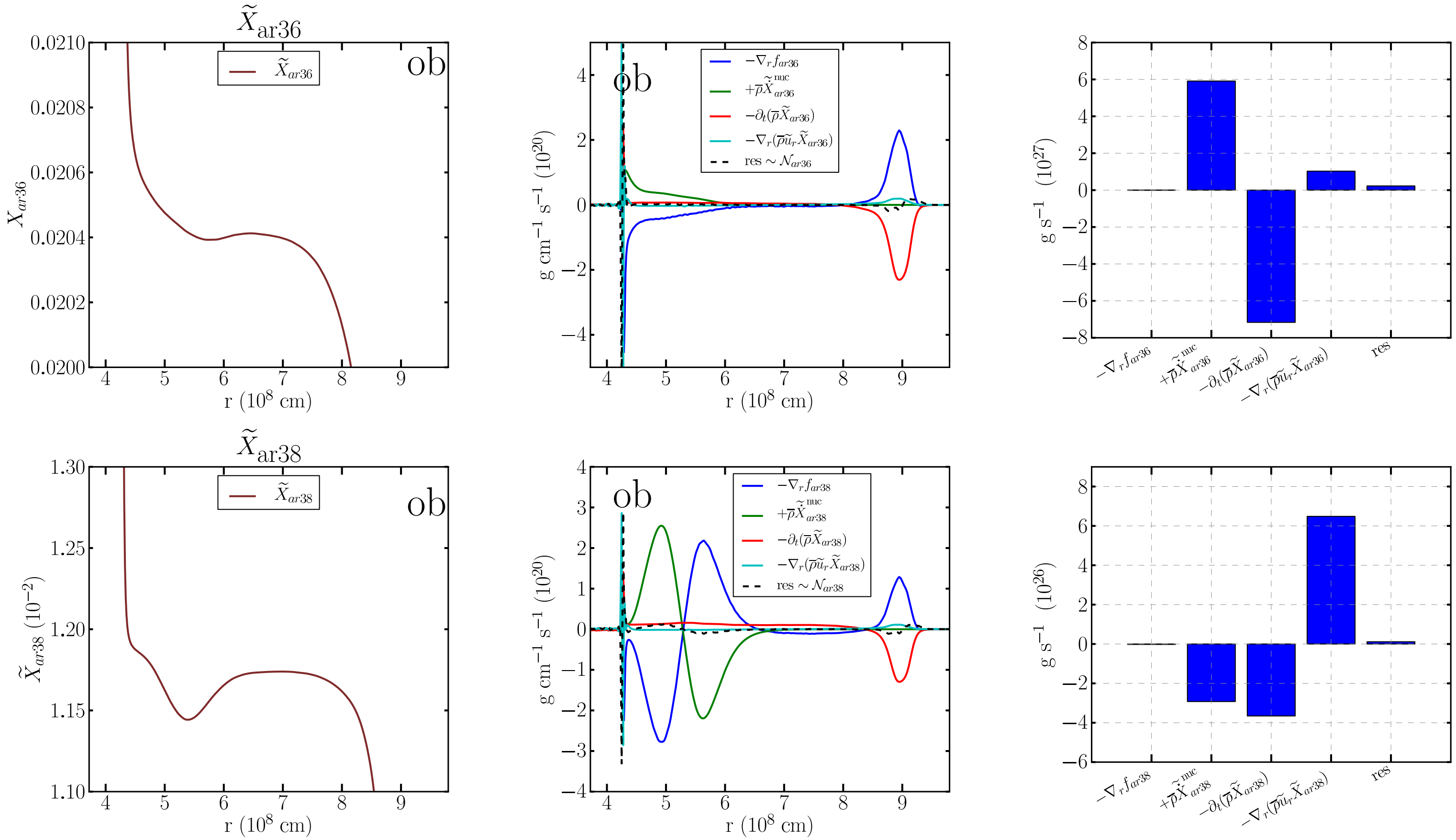

Figure 29: Mean composition equations. Model ob.3D.2hp.

5.7 Mean K^{39} and Ca^{40} equation

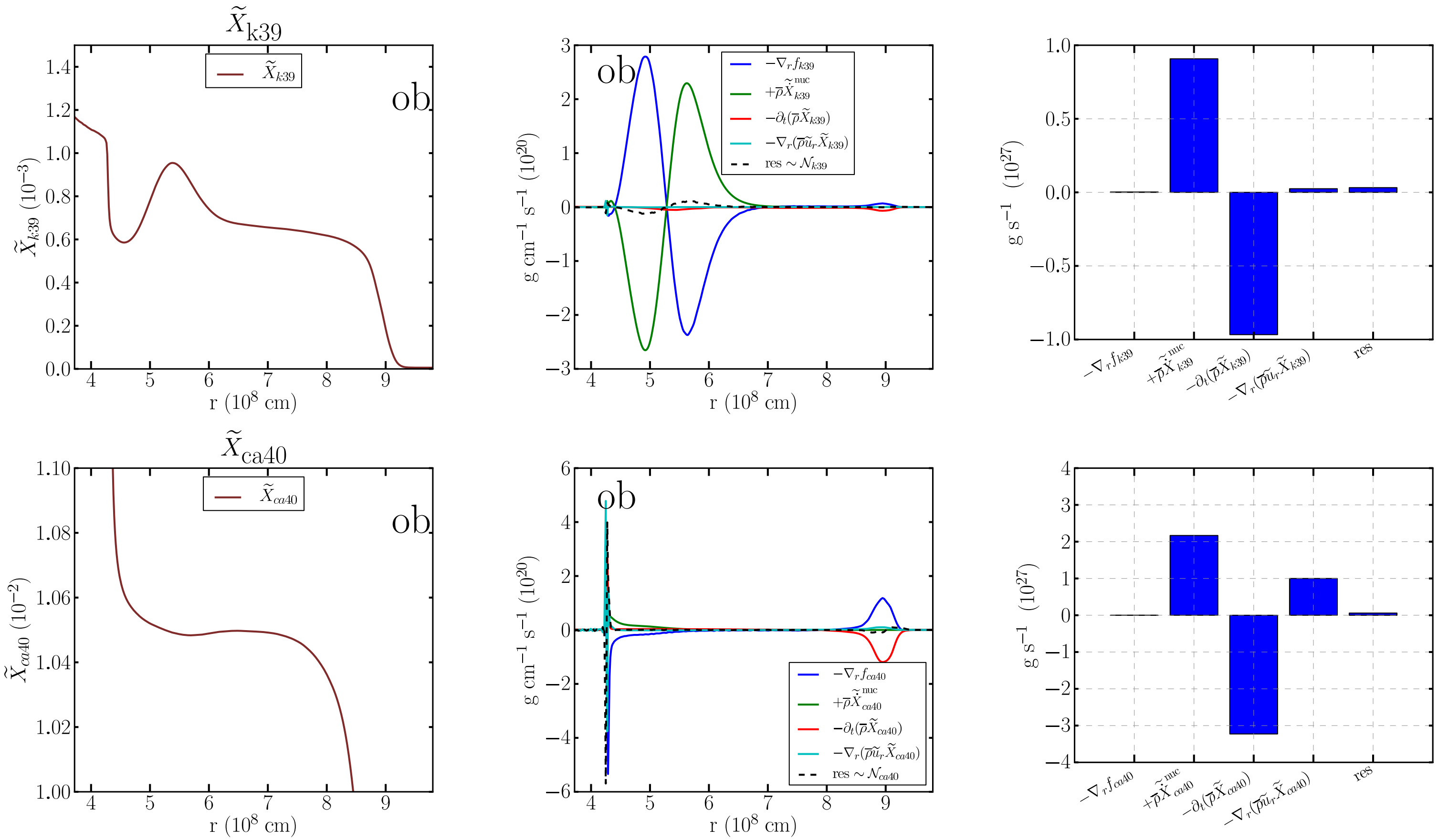

Figure 30: Mean composition equations. Model ob.3D.2hp.

5.8 Mean Ca^{42} and Ti^{44} equation

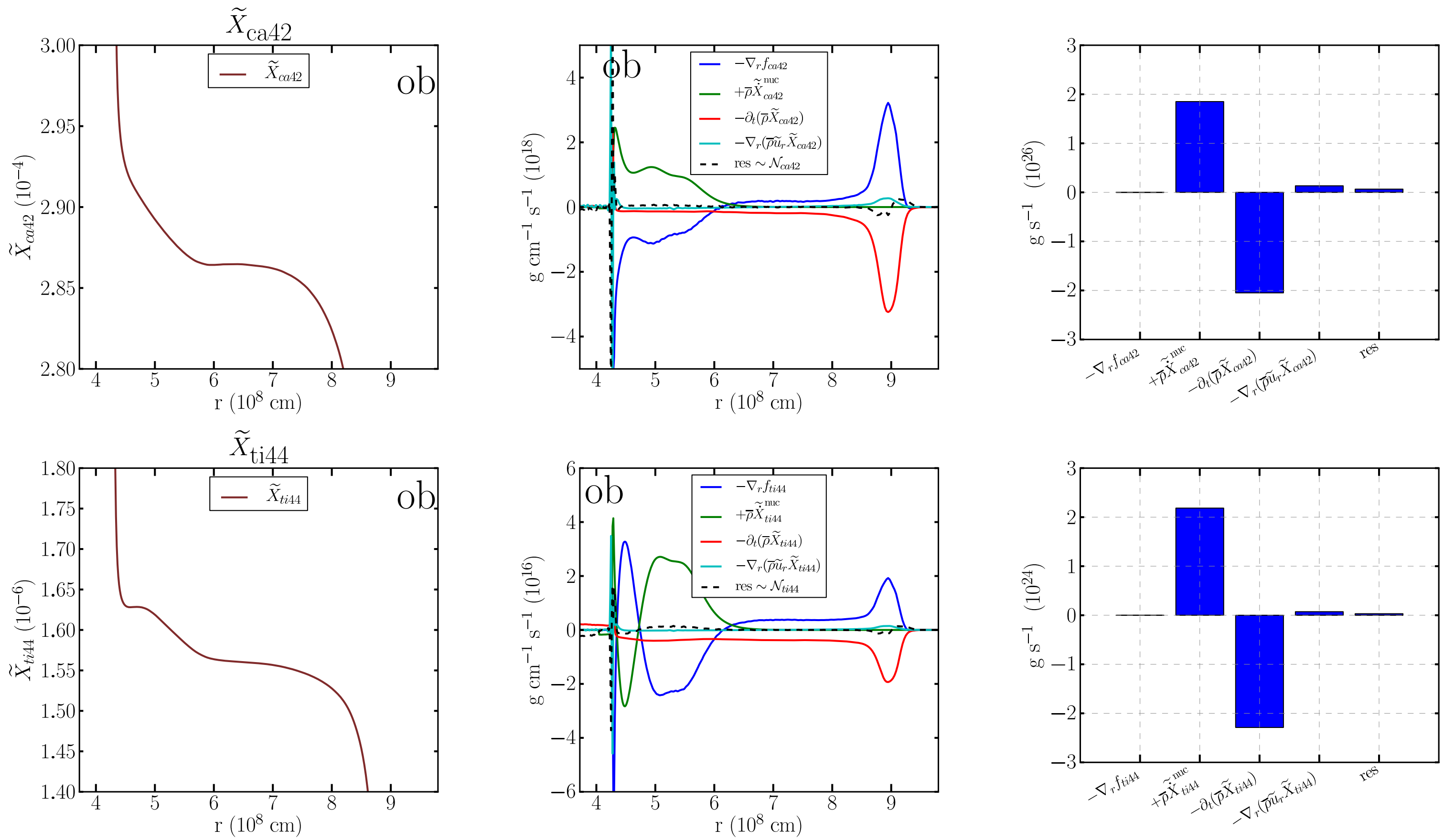

Figure 31: Mean composition equations. Model ob.3D.2hp.

5.9 Mean Ti^{46} and Cr^{48} equation

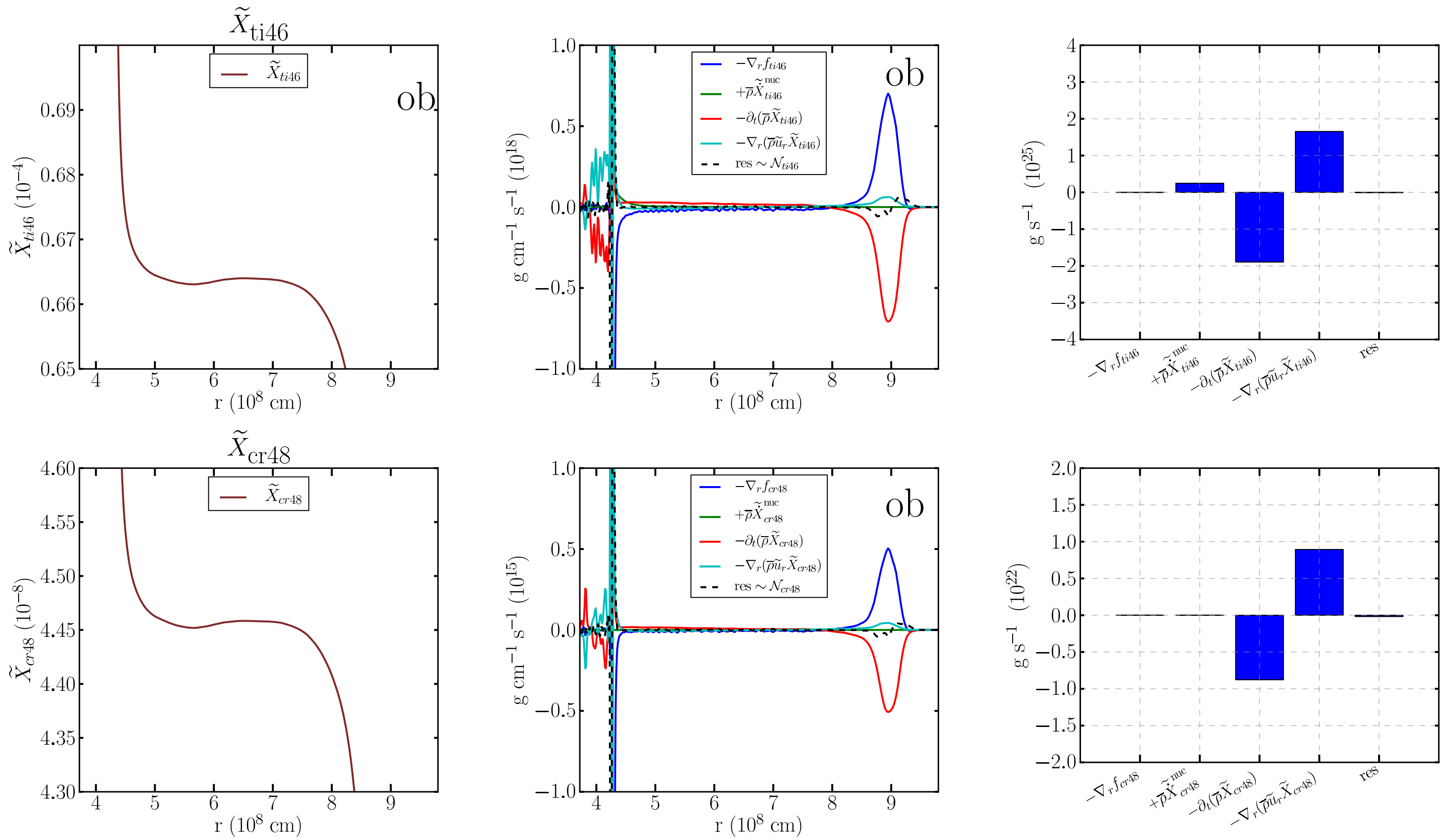

Figure 32: Mean composition equations. Model ob.3D.2hp.

5.10 Mean Cr^{50} and Fe^{52} equation

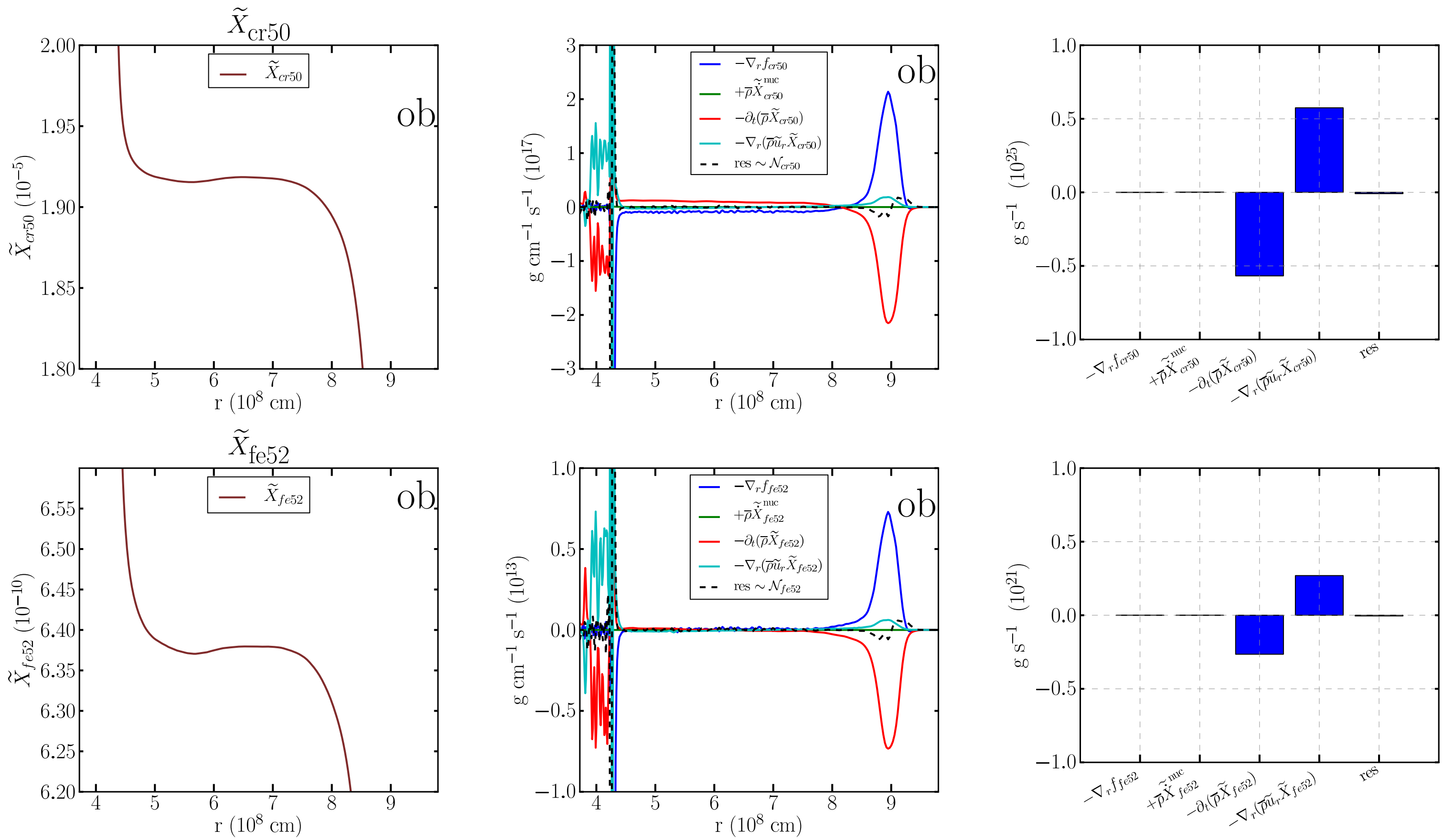

Figure 33: Mean composition equations. Model ob.3D.2hp.

5.11 Mean Fe^{54} and Ni^{56} equation

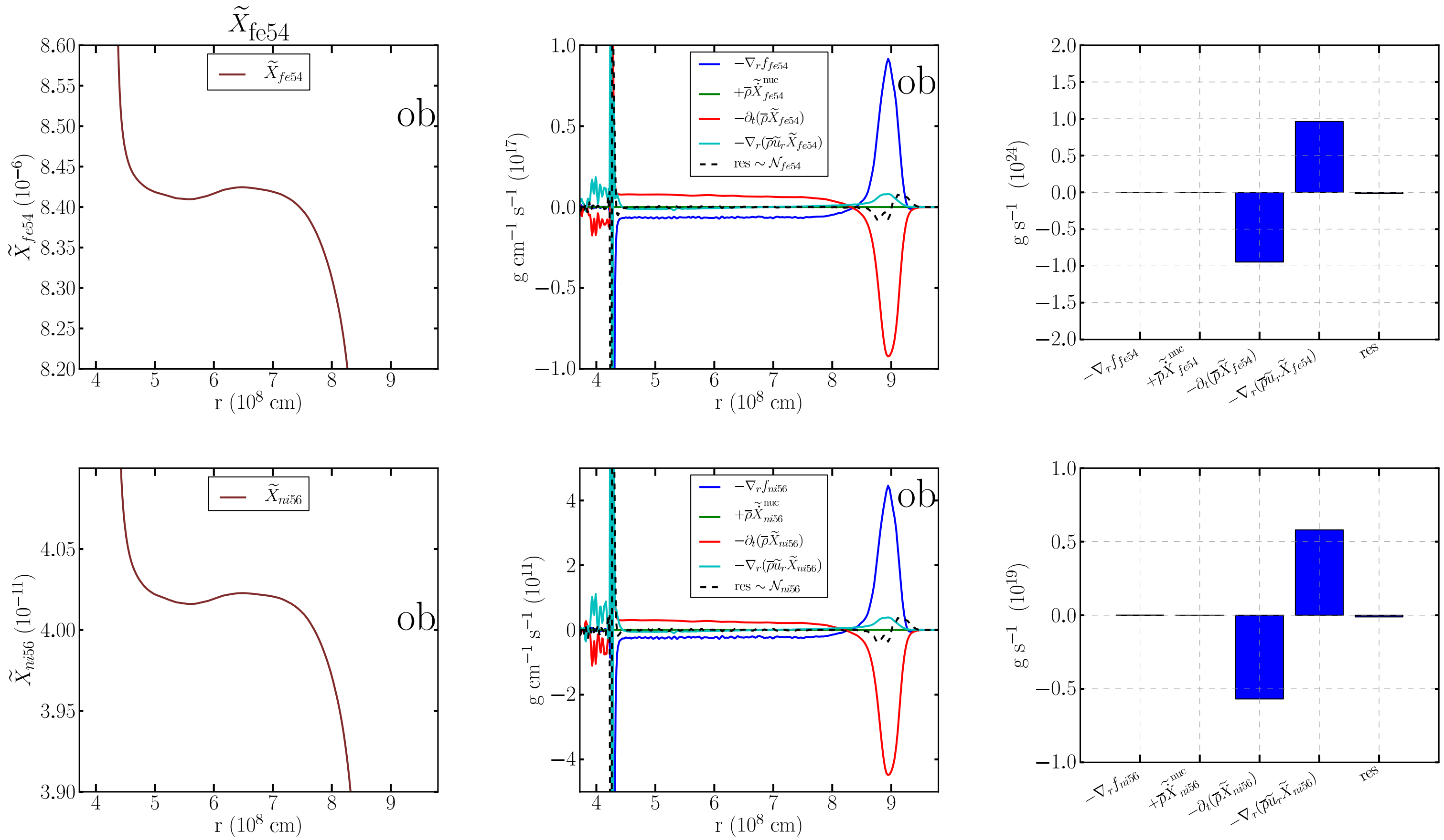

Figure 34: Mean composition equations. Model ob.3D.2hp.

6 Dependence on Numerical Resolution

6.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

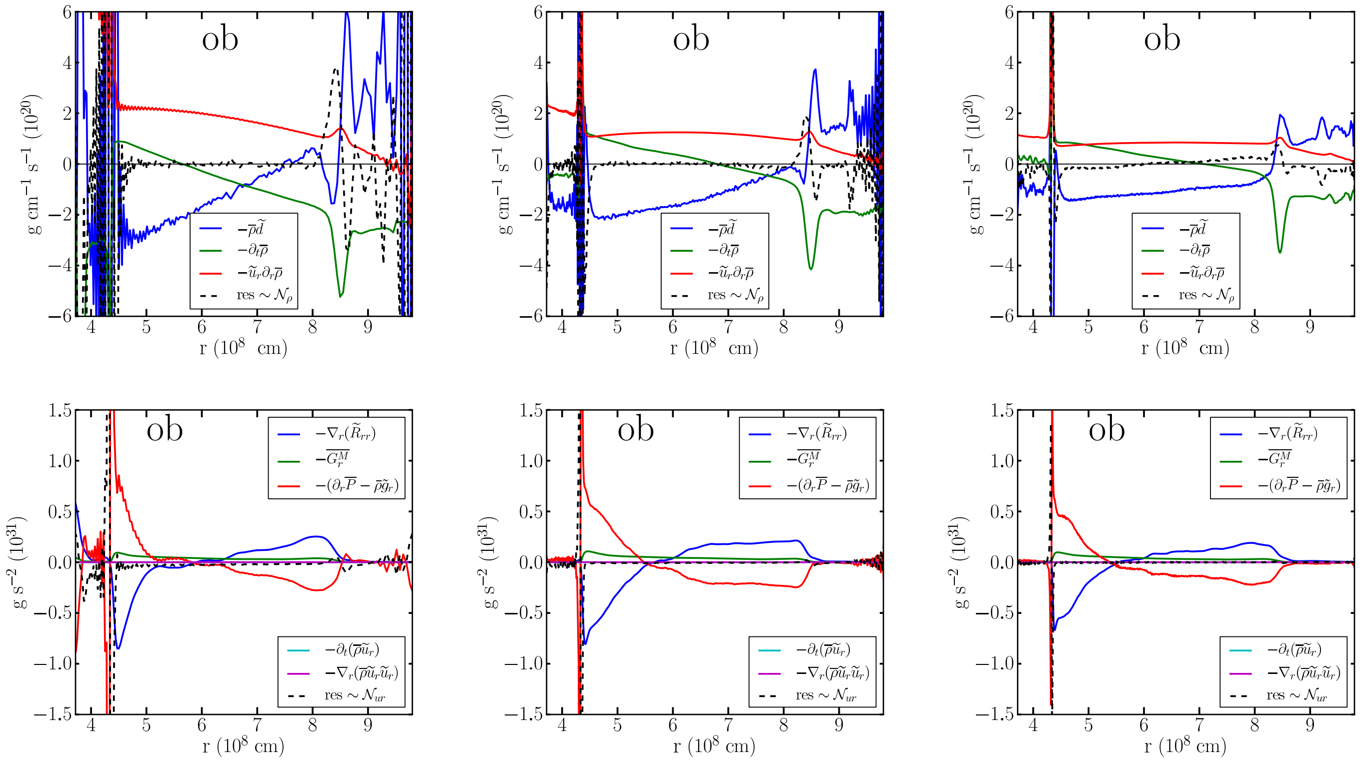

Figure 35: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean azimuthal and polar momentum equations

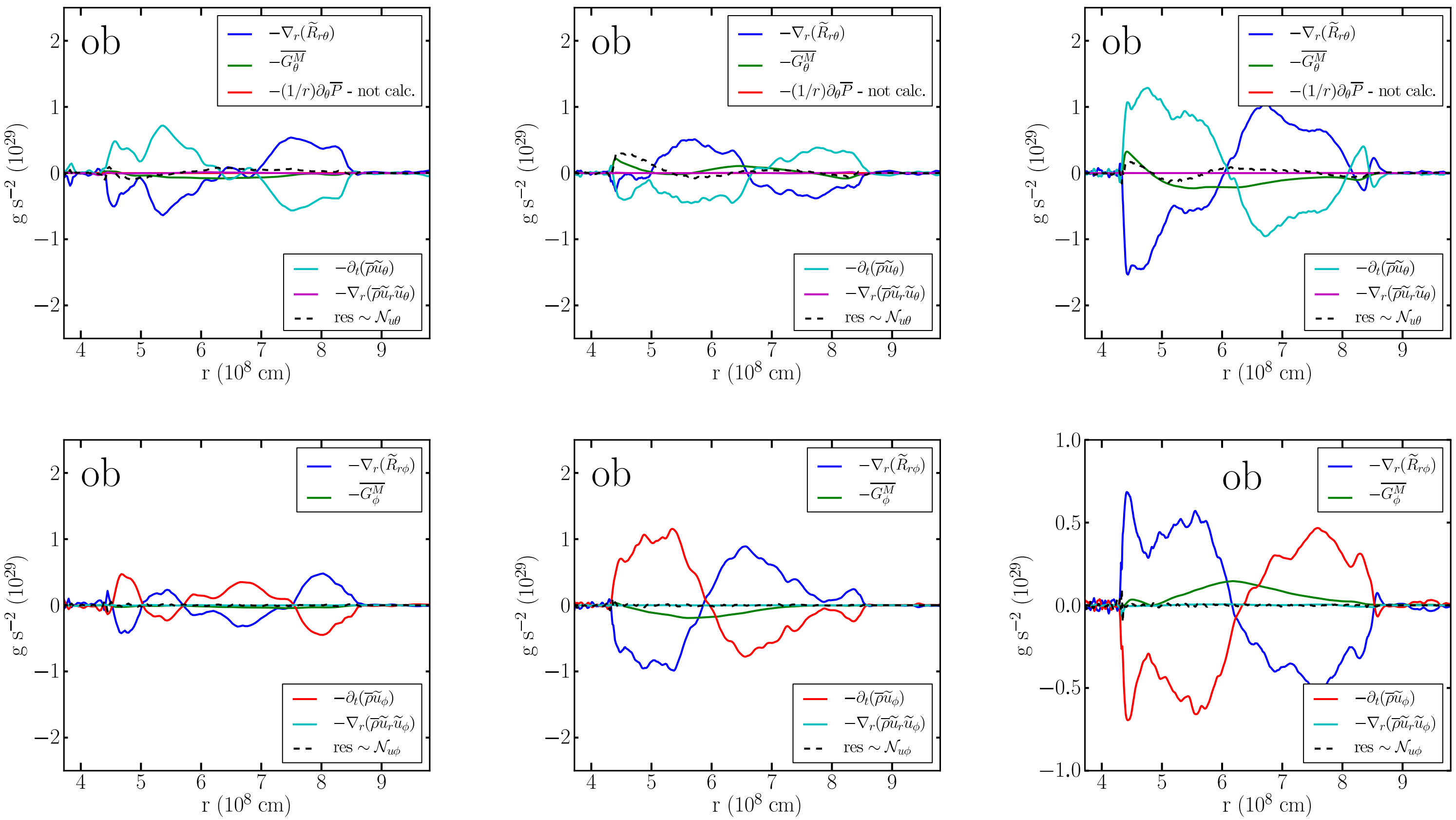

Figure 36: Mean azimuthal momentum equation (upper panels) and polar momentum equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean internal and kinetic energy equation

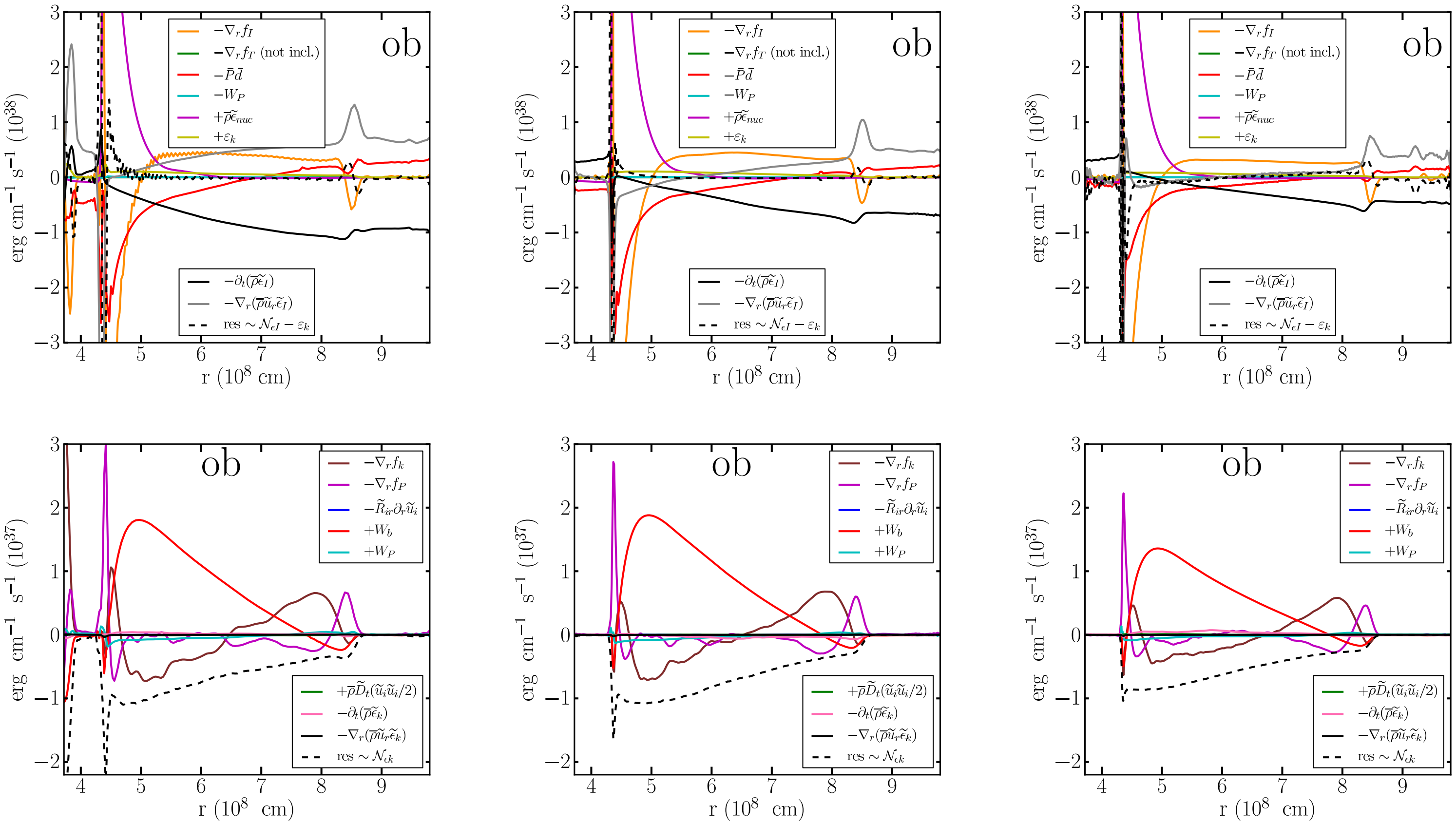

Figure 37: Mean internal energy equation (upper panels) and kinetic energy equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean total energy and entropy equation

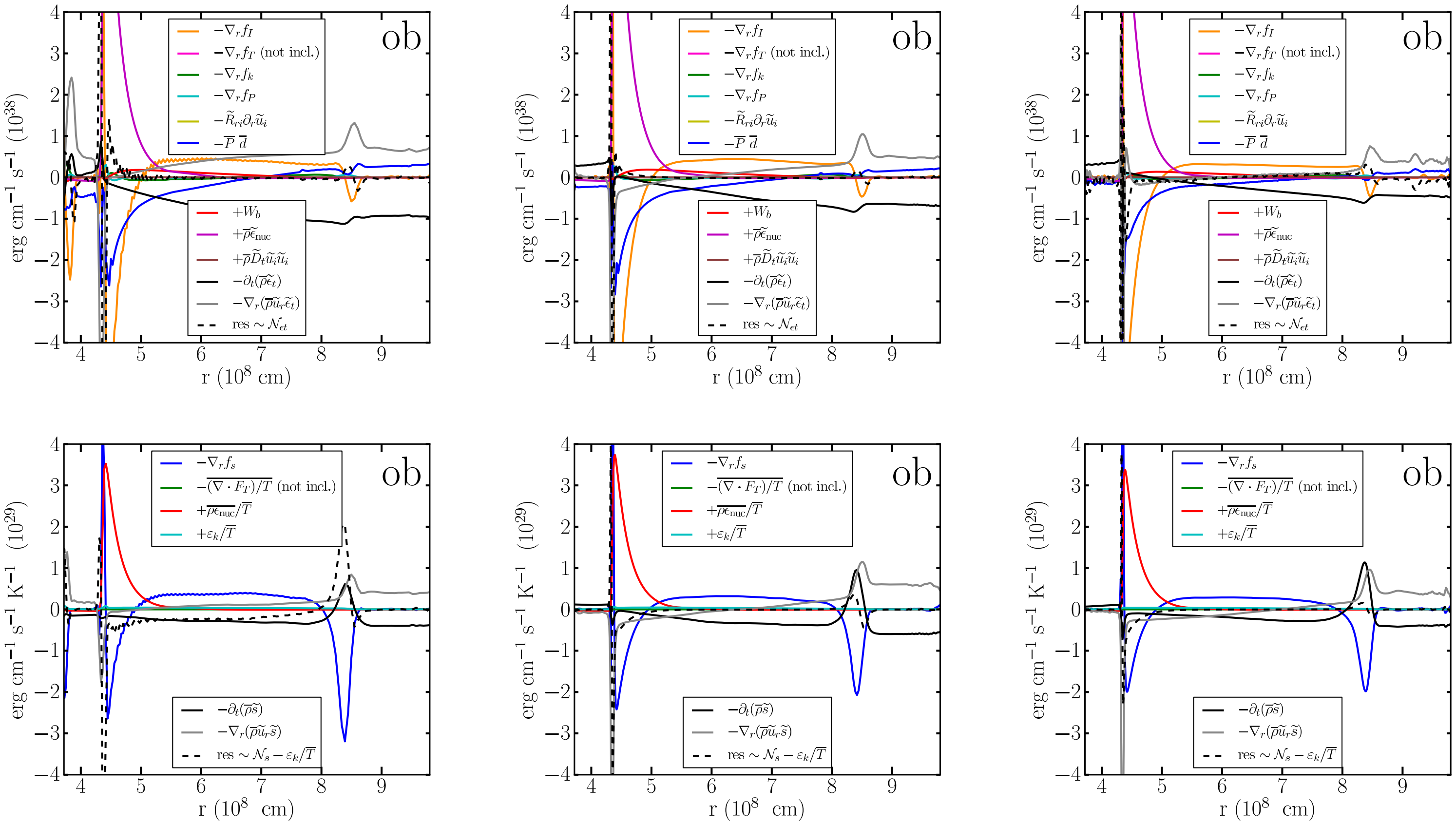

Figure 38: Mean total energy equation (upper panels) and mean entropy equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean density-specific volume covariance equation and mean number of nucleons per isotope equation

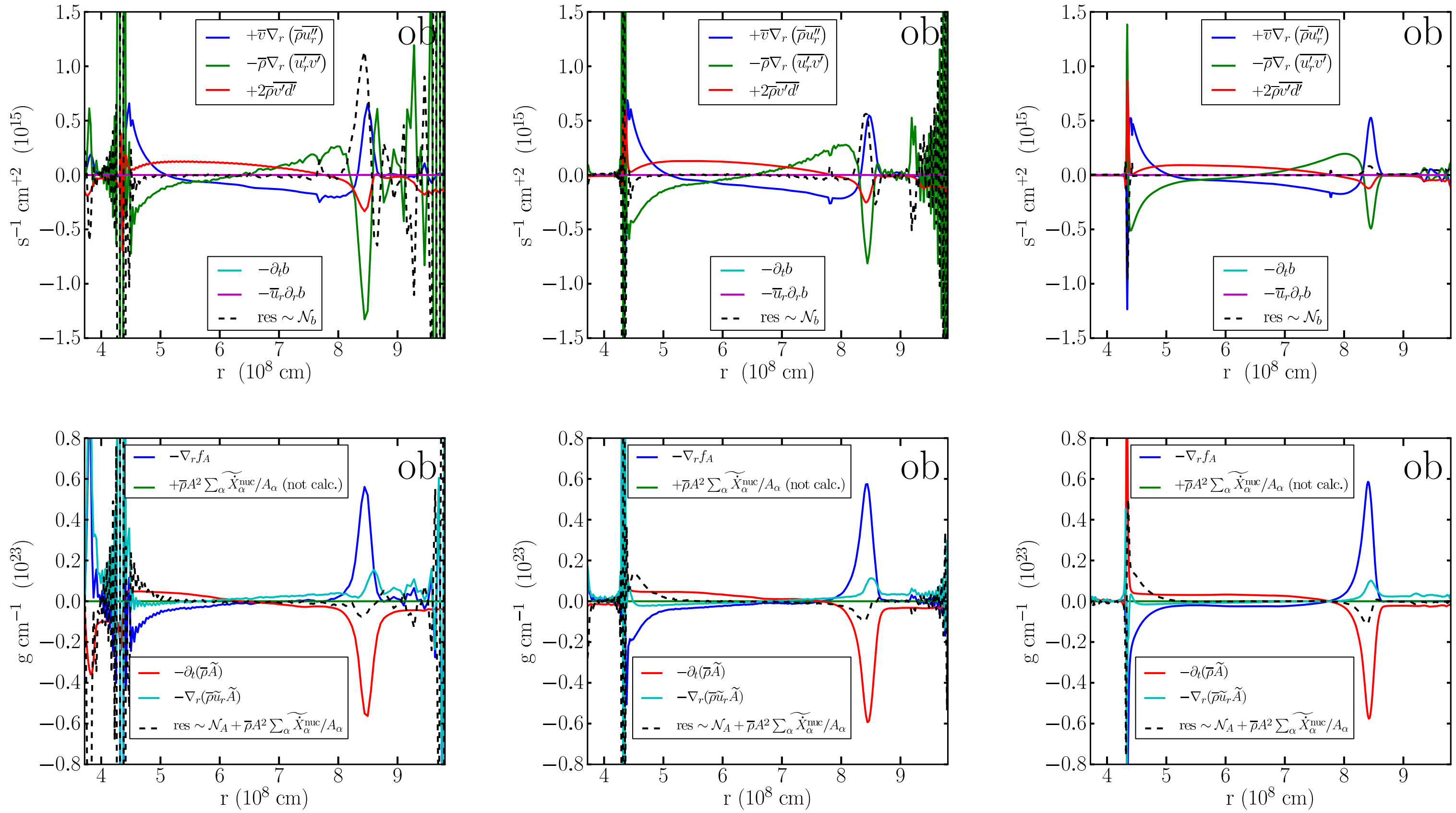

Figure 39: Mean density-specific volume covariance equation (upper panels) and mean number of nucleons per isotope equation (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

Mean turbulent kinetic energy equation and mean velocities

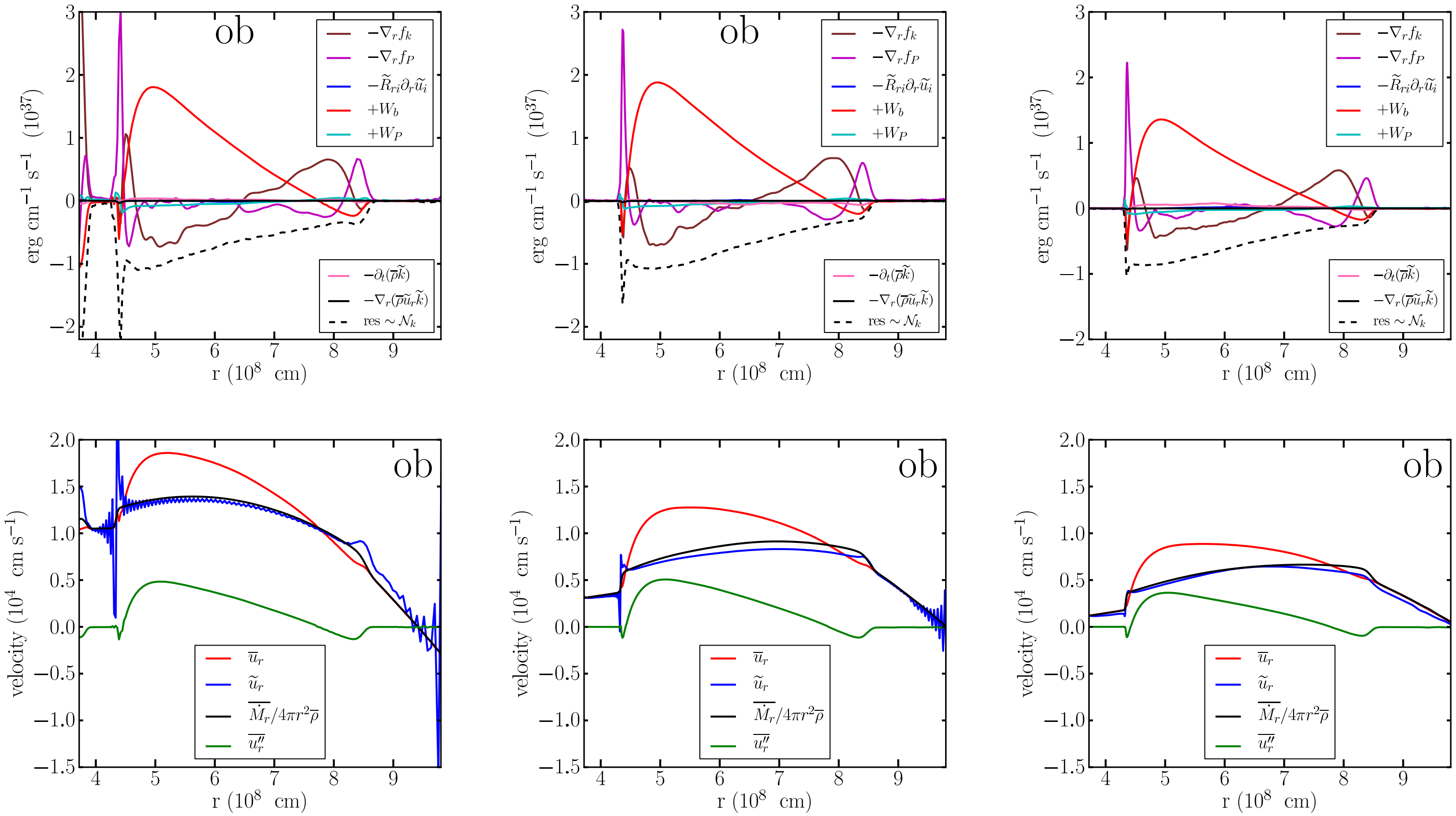

Figure 40: Mean turbulent kinetic energy equation (upper panels) and mean velocities (lower panels). Model ob.3D.lr (left), ob.3D.mr (middle), ob.3D.hr (right)

6.2 Red giant envelope convection

Mean continuity equation and mean radial momentum equation

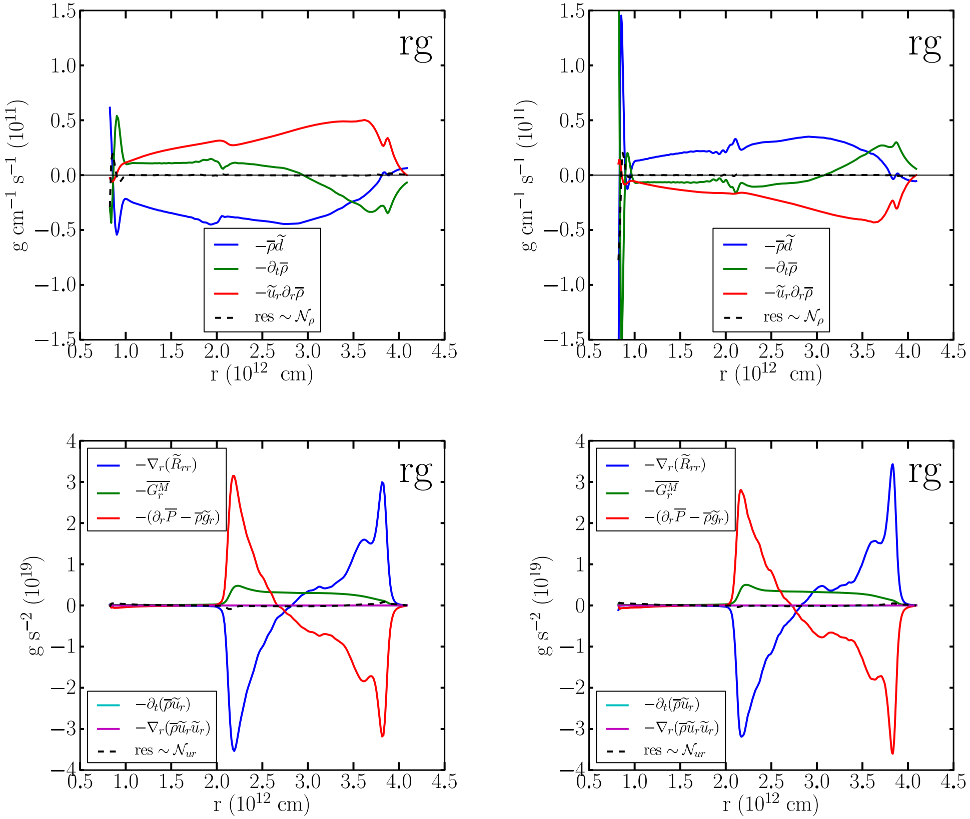

Figure 41: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean azimuthal and polar momentum equation

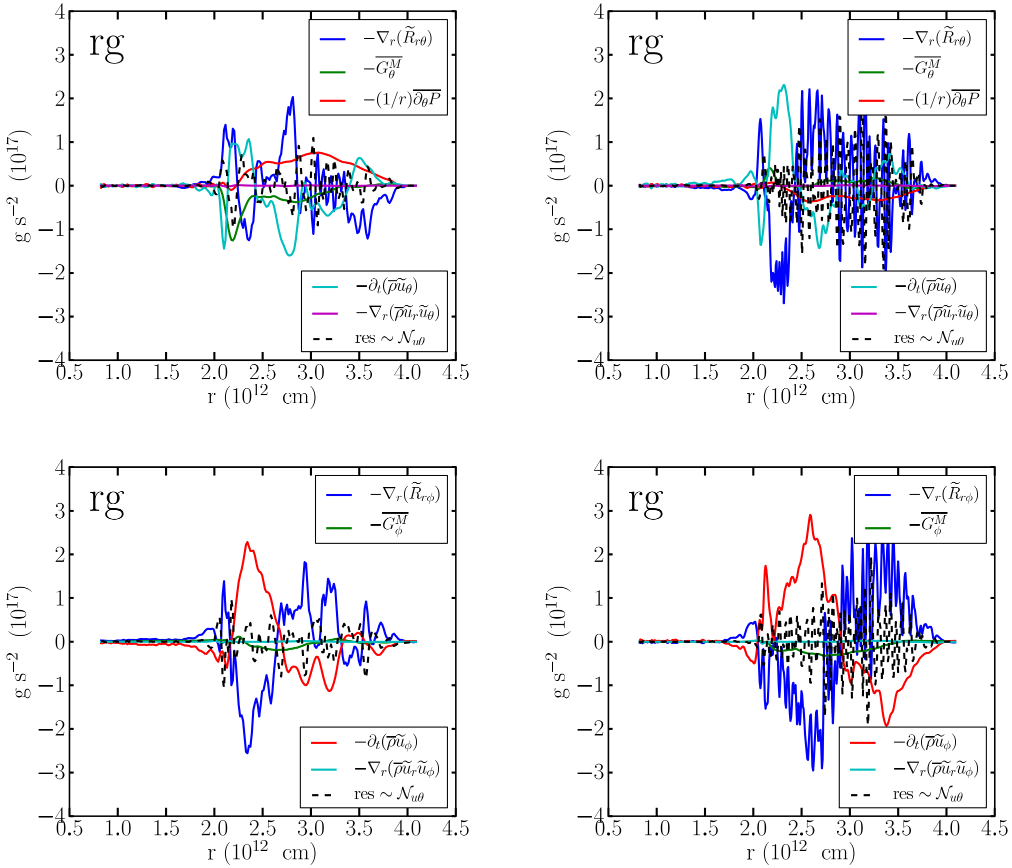

Figure 42: Mean azimuthal momentum (upper panels) and polar momentum equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean internal and kinetic energy equation

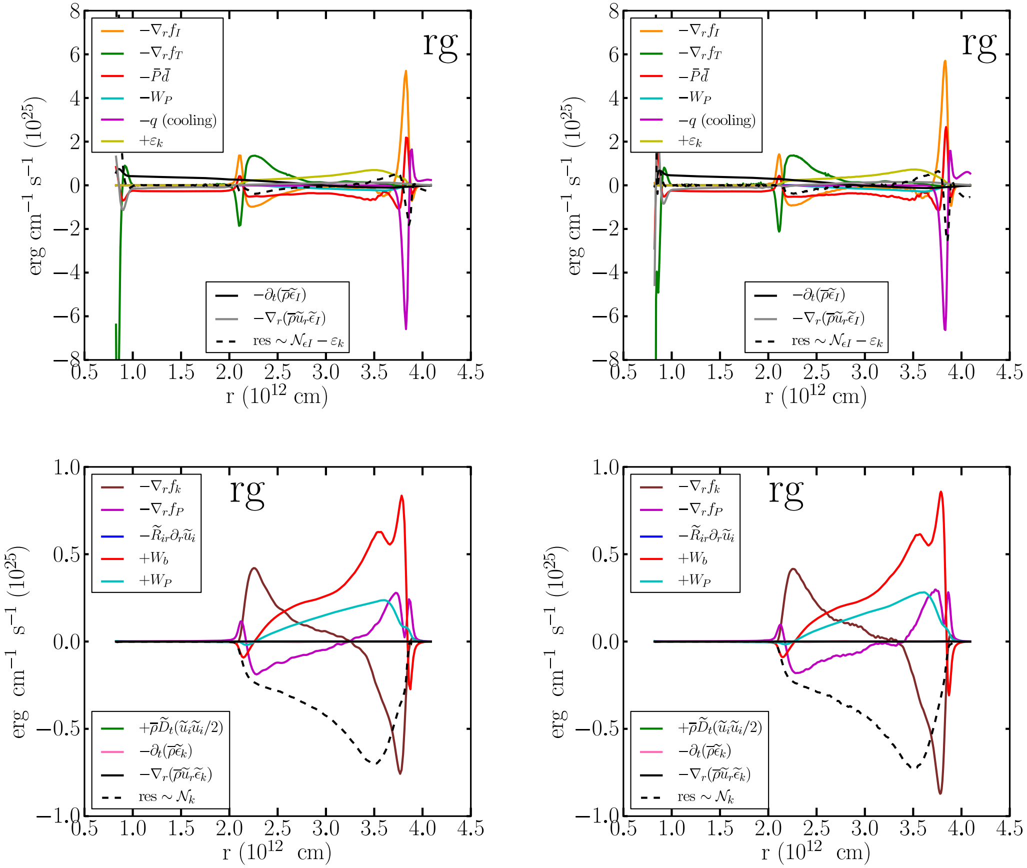

Figure 43: Mean internal energy equation (upper panels) and kinetic energy equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean total energy equation and mean entropy equation

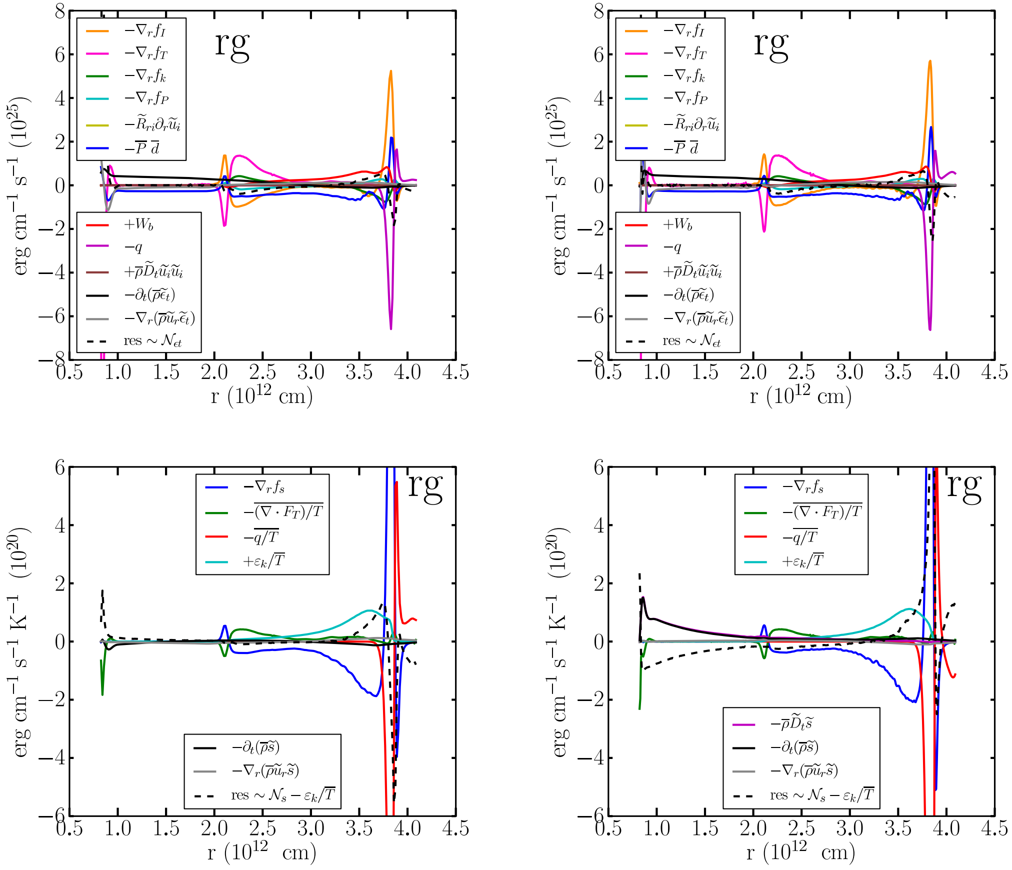

Figure 44: Mean total energy equation (upper panels) and mean entropy equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean density-specific volume covariance and entropy flux equation

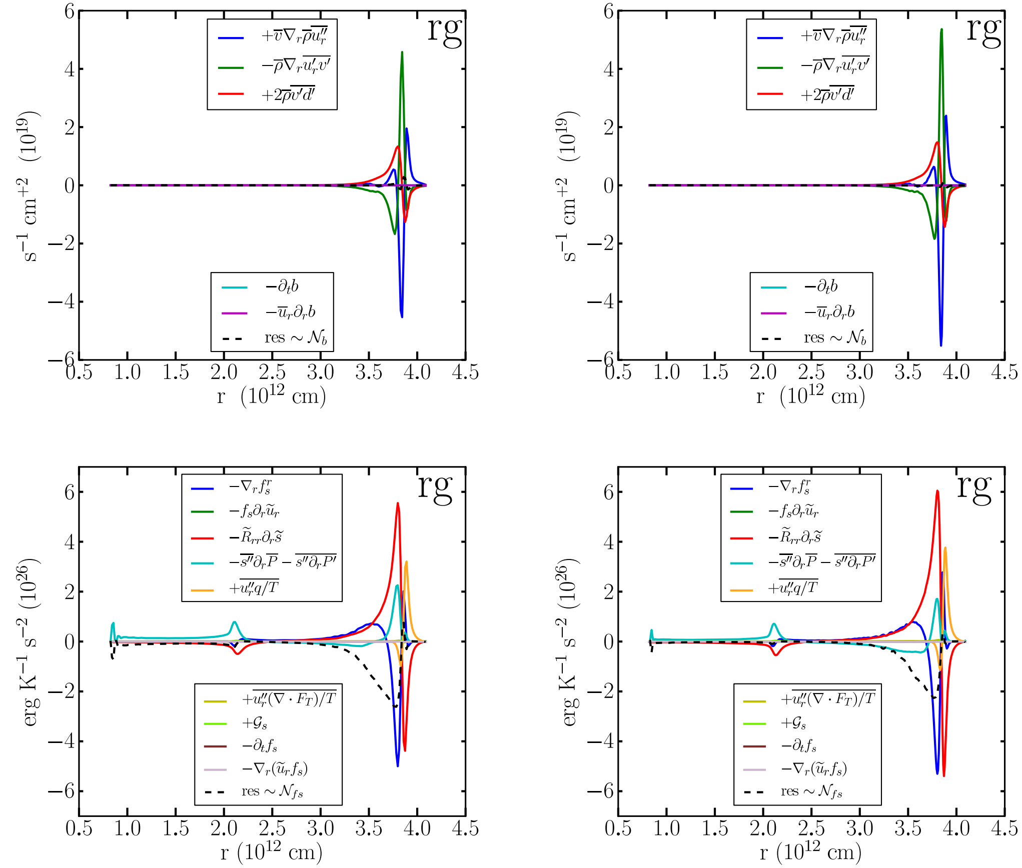

Figure 45: Mean density-specific volume covariance equation (upper panels) and entropy flux equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

Mean turbulent kinetic energy equation and mean turbulent mass flux equation

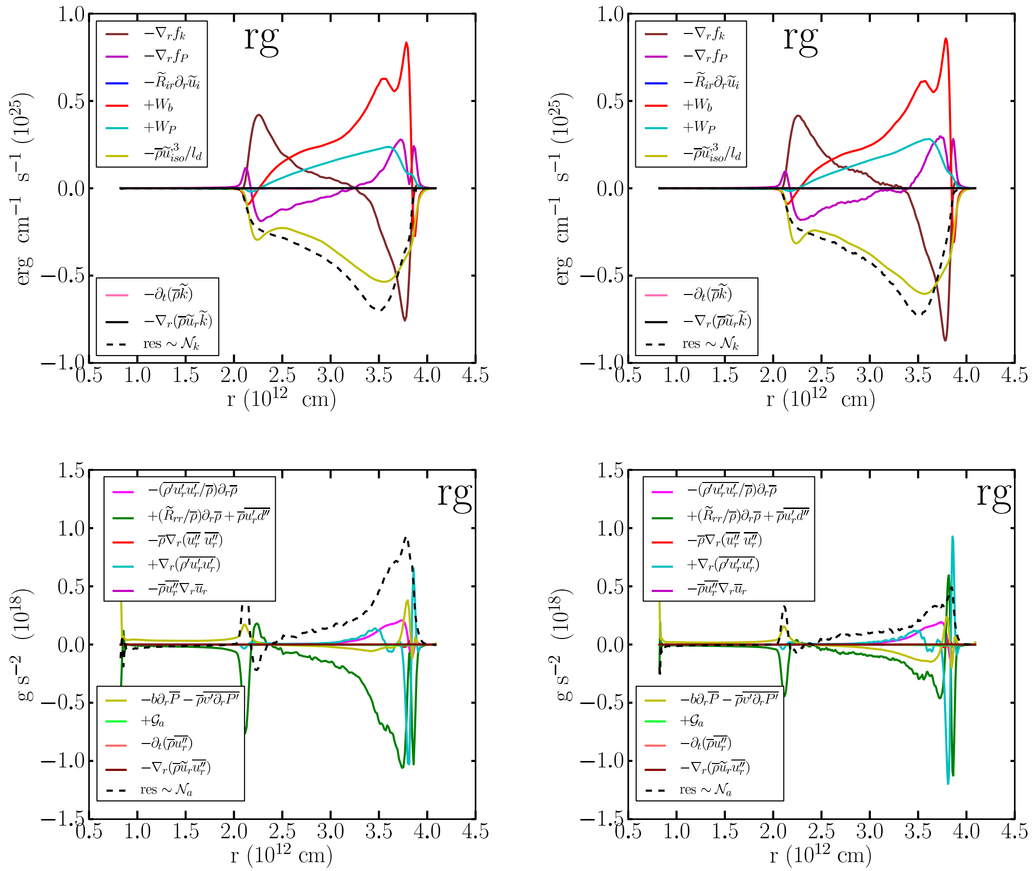

Figure 46: Mean turbulent kinetic energy equation (upper panels) and mean turbulent mass flux equation (lower panels). Model rg.3D.lr (left) and rg.3D.mr (right)

7 Dependence on Computational Domain Size

7.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

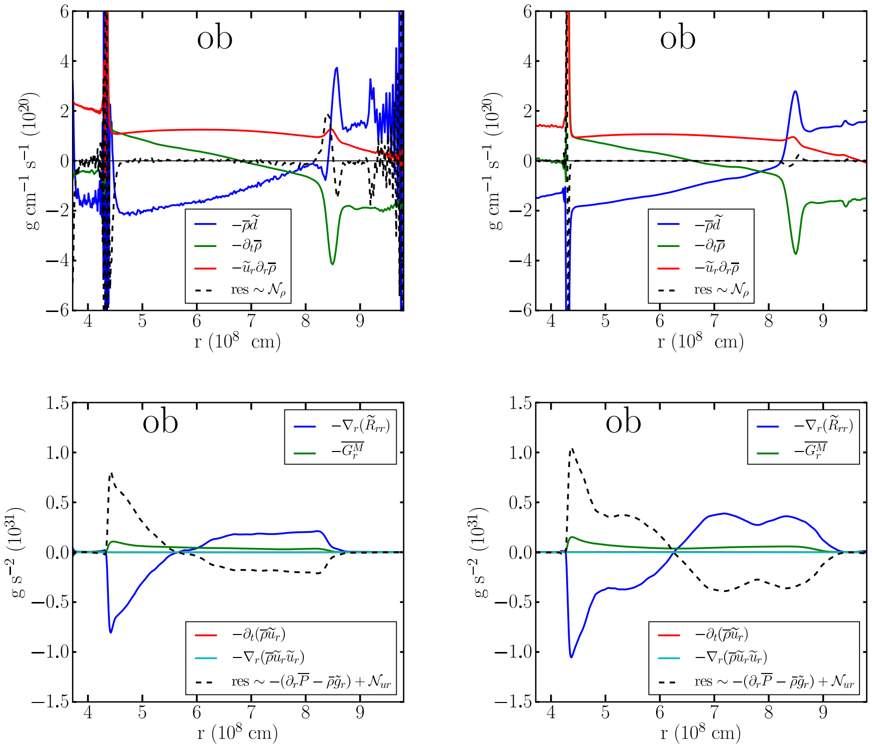

Figure 47: Continuity equation (upper panels) and radial momentum equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean azimuthal momentum and polar momentum equation

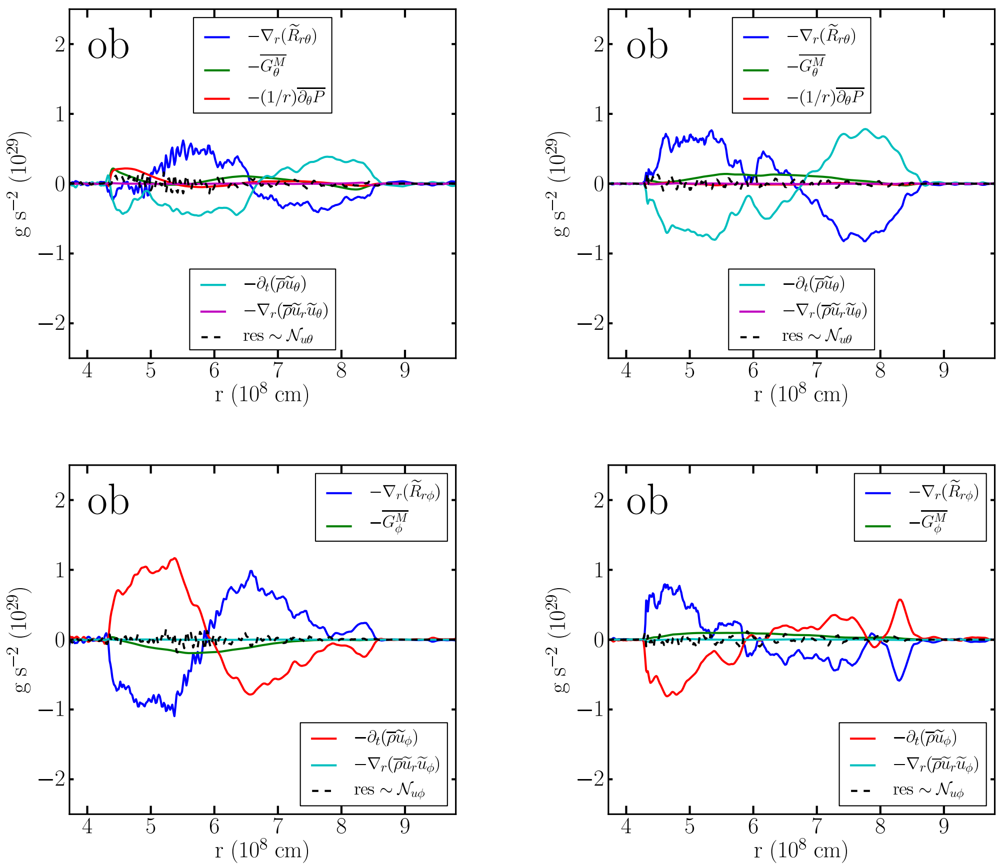

Figure 48: Mean azimuthal momentum (upper panels) and polar momentum equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean internal energy equation and total energy equation

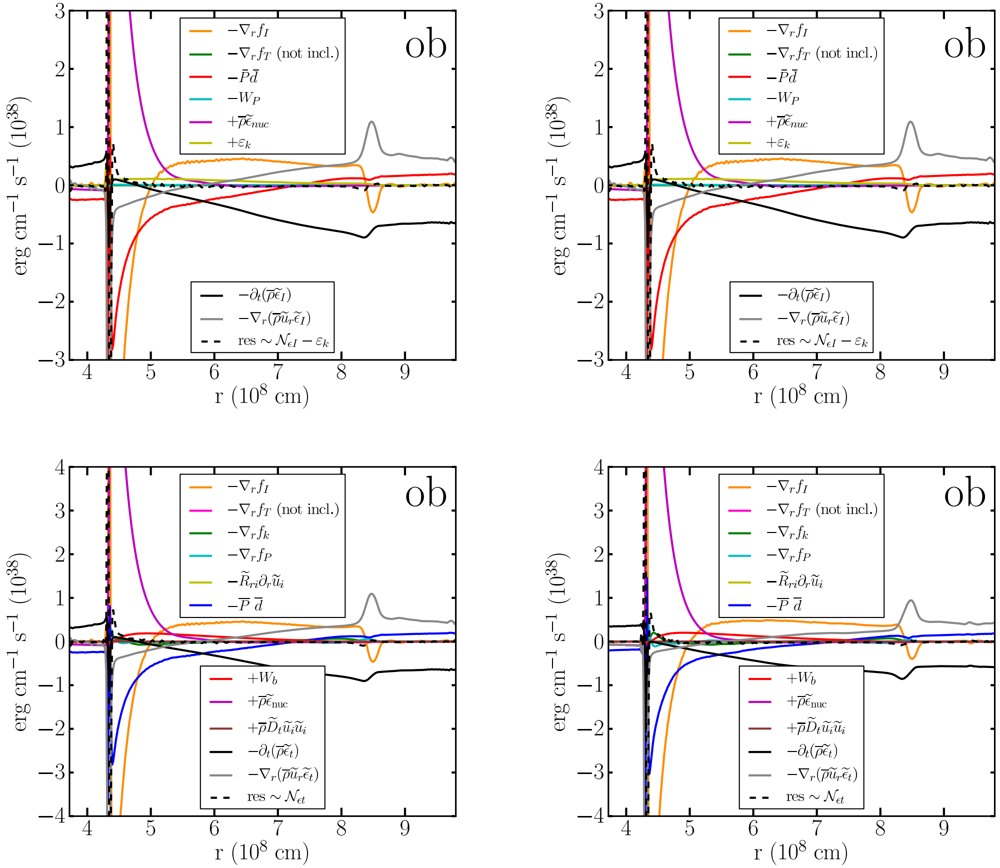

Figure 49: Mean internal energy (upper panels) equations and mean total energy equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean turbulent kinetic energy equation and turbulent mass flux equation

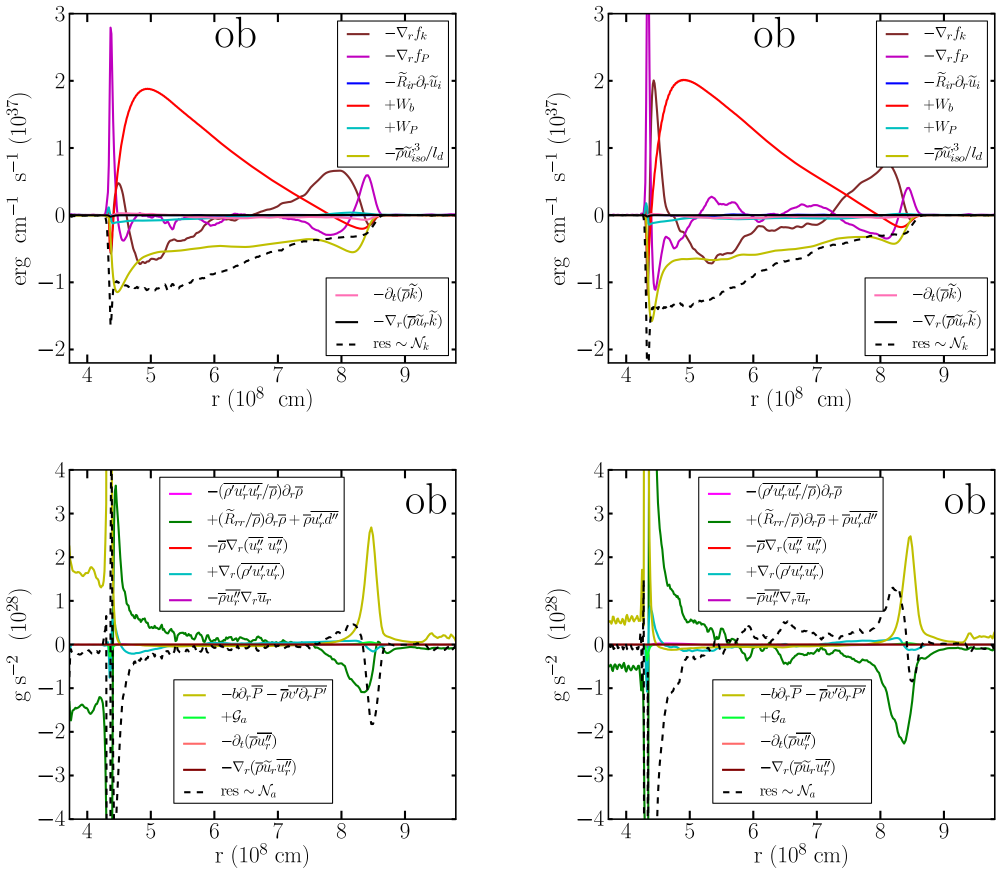

Figure 50: Mean turbulent kinetic energy equation and turbulent mass flux equation. Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean density-specific volume covariance and internal energy flux equation

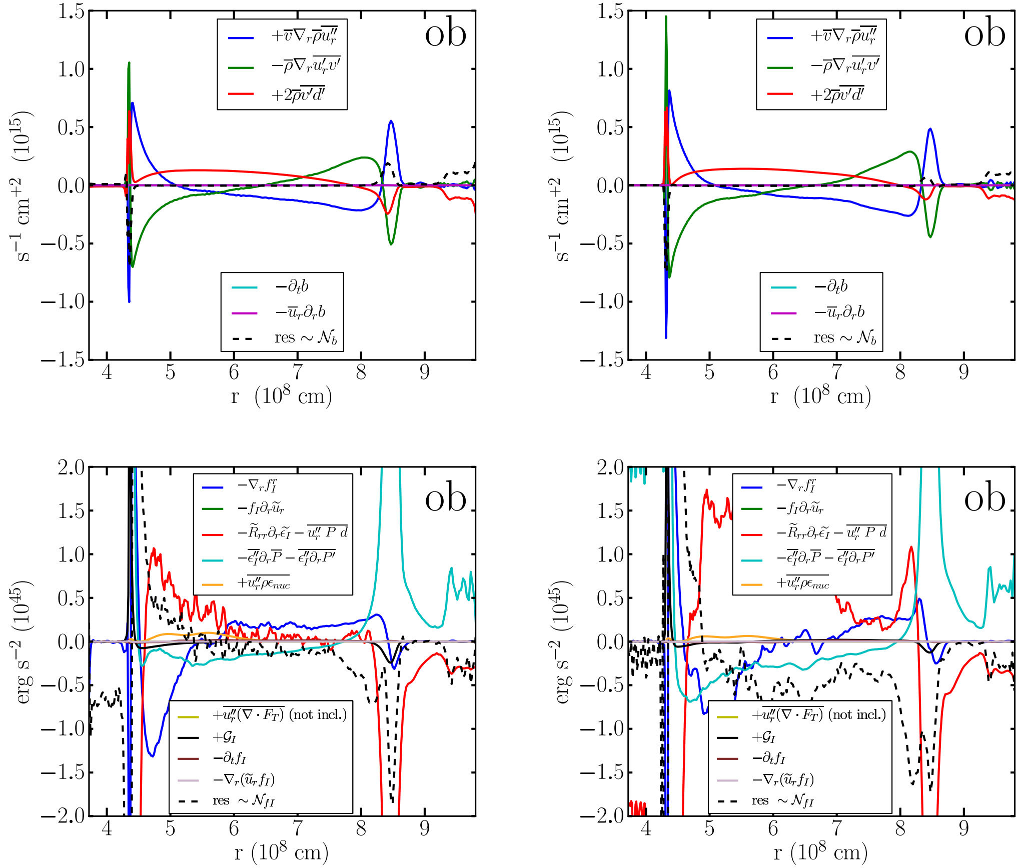

Figure 51: Mean density-specific volume covariance equation (upper panels) and mean internal energy flux equation (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

Mean kinetic energy equation and mean velocities

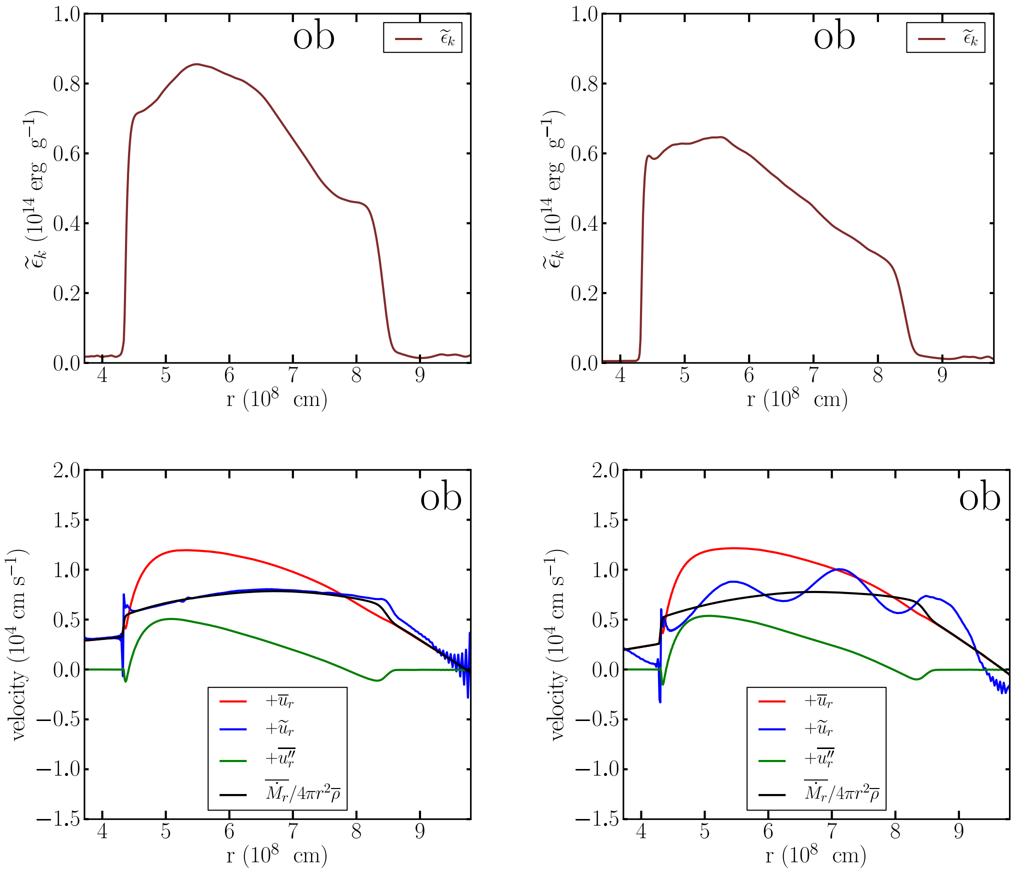

Figure 52: Mean kinetic energy equation (upper panels) mean velocities (lower panels). Model ob.3D.mr (45°wedge - left) and ob.3D.2hp (27.5°wedge - right).

8 Dependence on Time Averaging Window

8.1 Oxygen burning shell model

Mean continuity equation and mean radial momentum equation

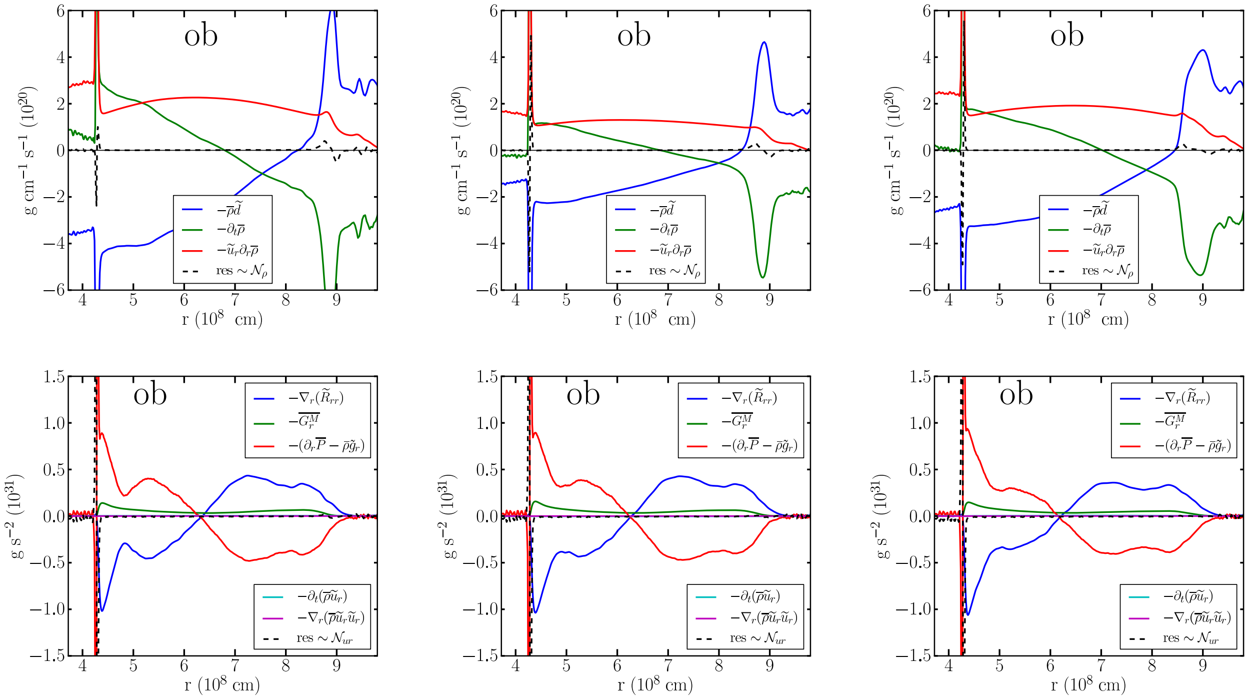

Figure 53: Mean continuity equation (upper panels) and radial momentum equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean azimuthal and polar momentum equations

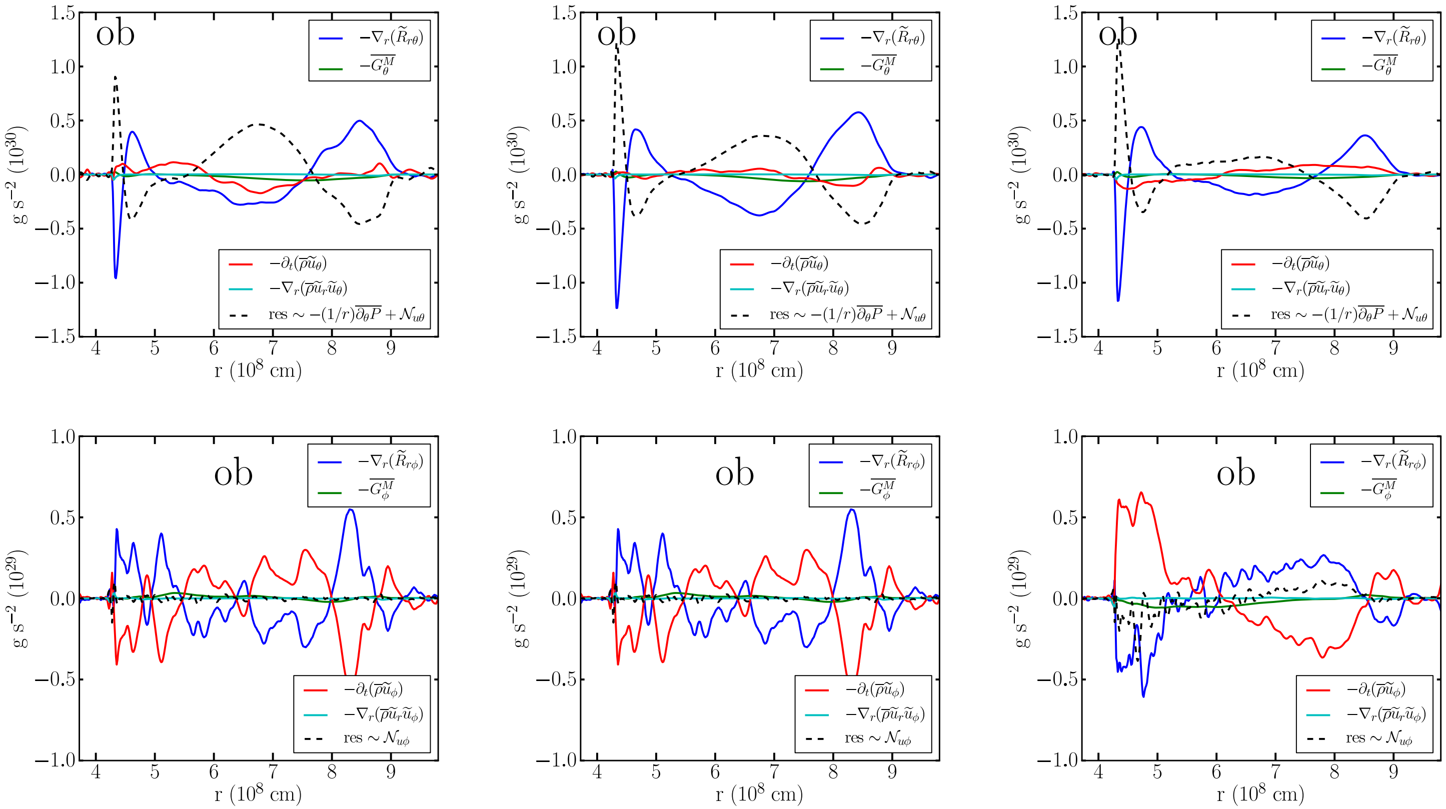

Figure 54: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean total energy equation and mean turbulent kinetic energy equation

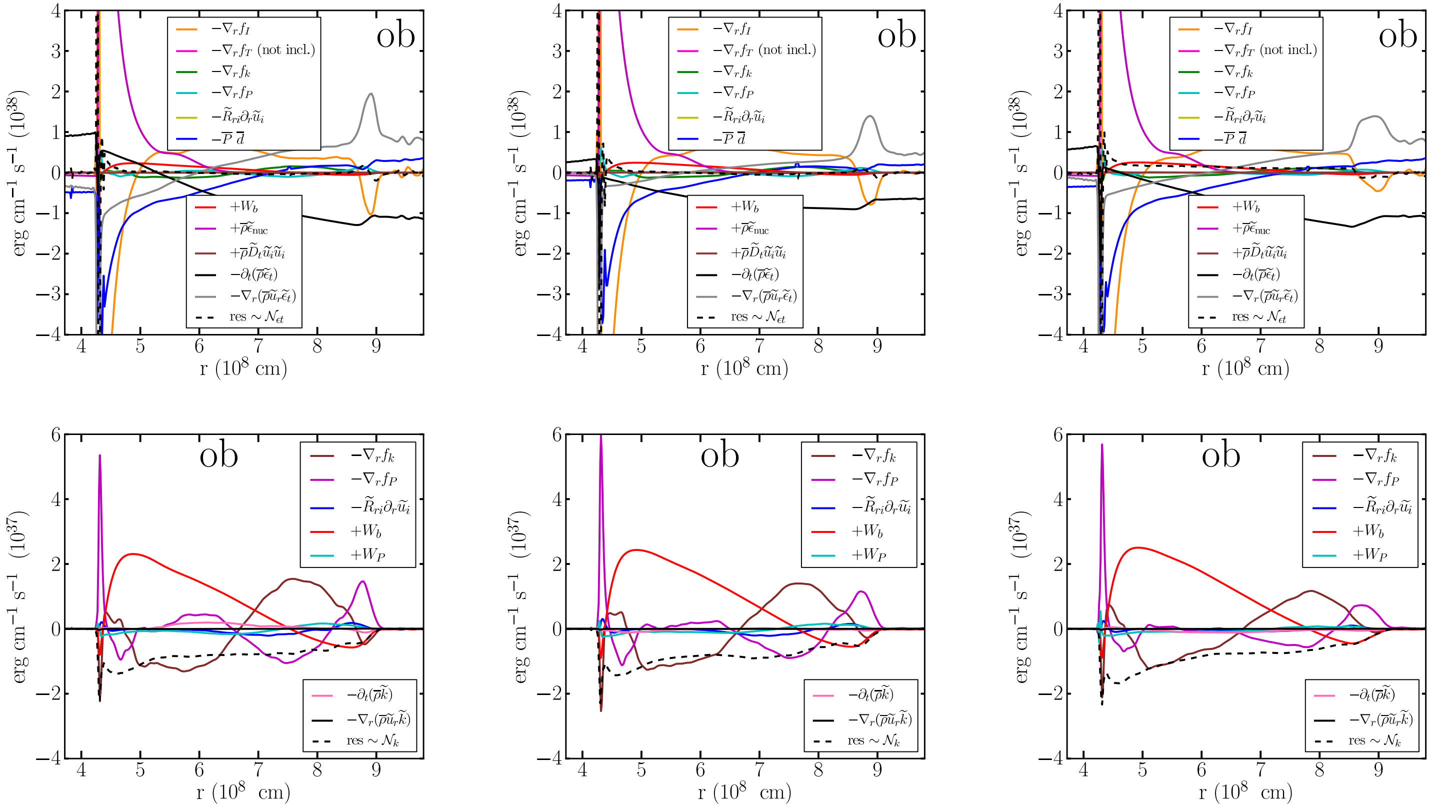

Figure 55: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean entropy equation and mean number of nucleons per isotope equation

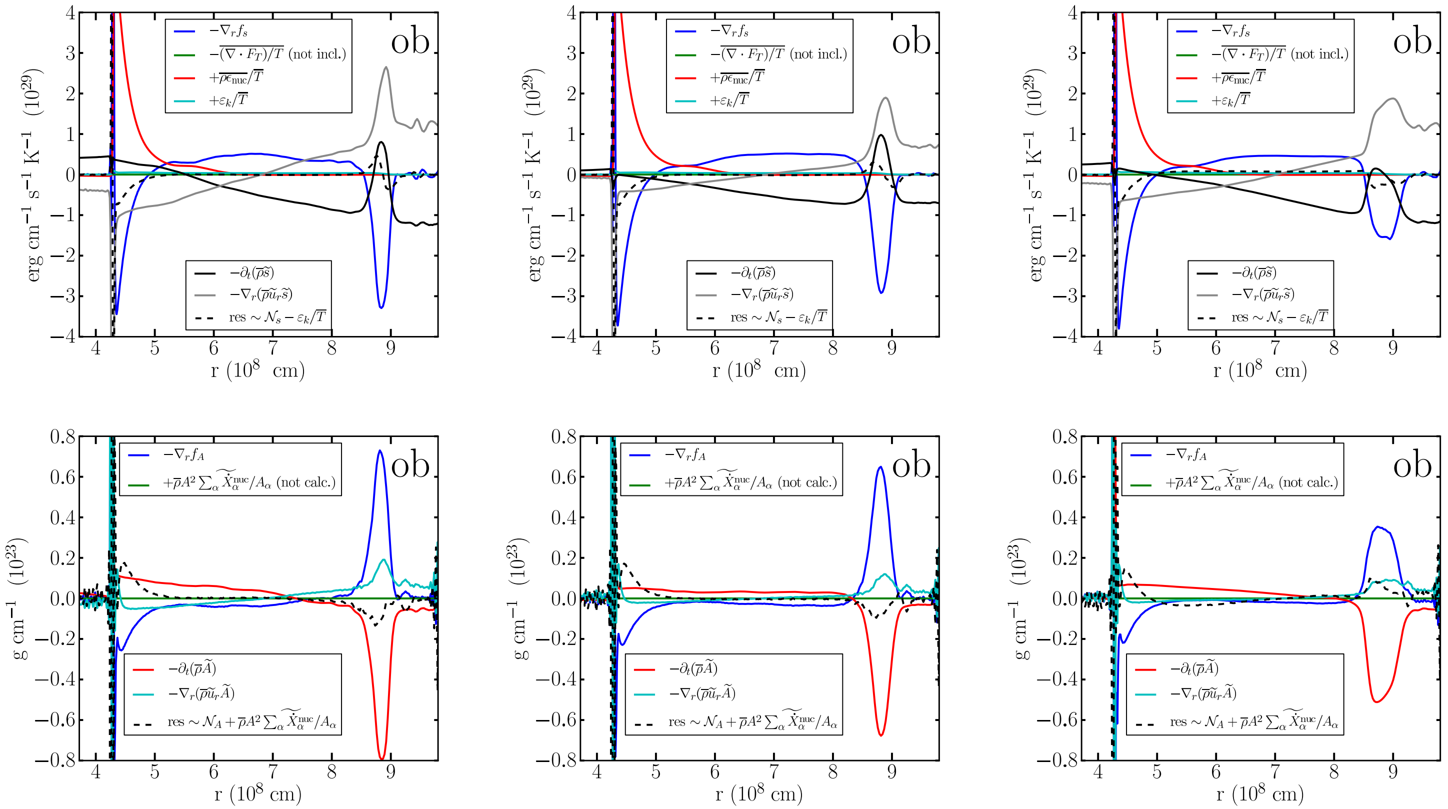

Figure 56: Mean entropy equation (upper panels) and mean number of nucleons per isotope (upper panels) from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

Mean turbulent kinetic energy and mean velocities

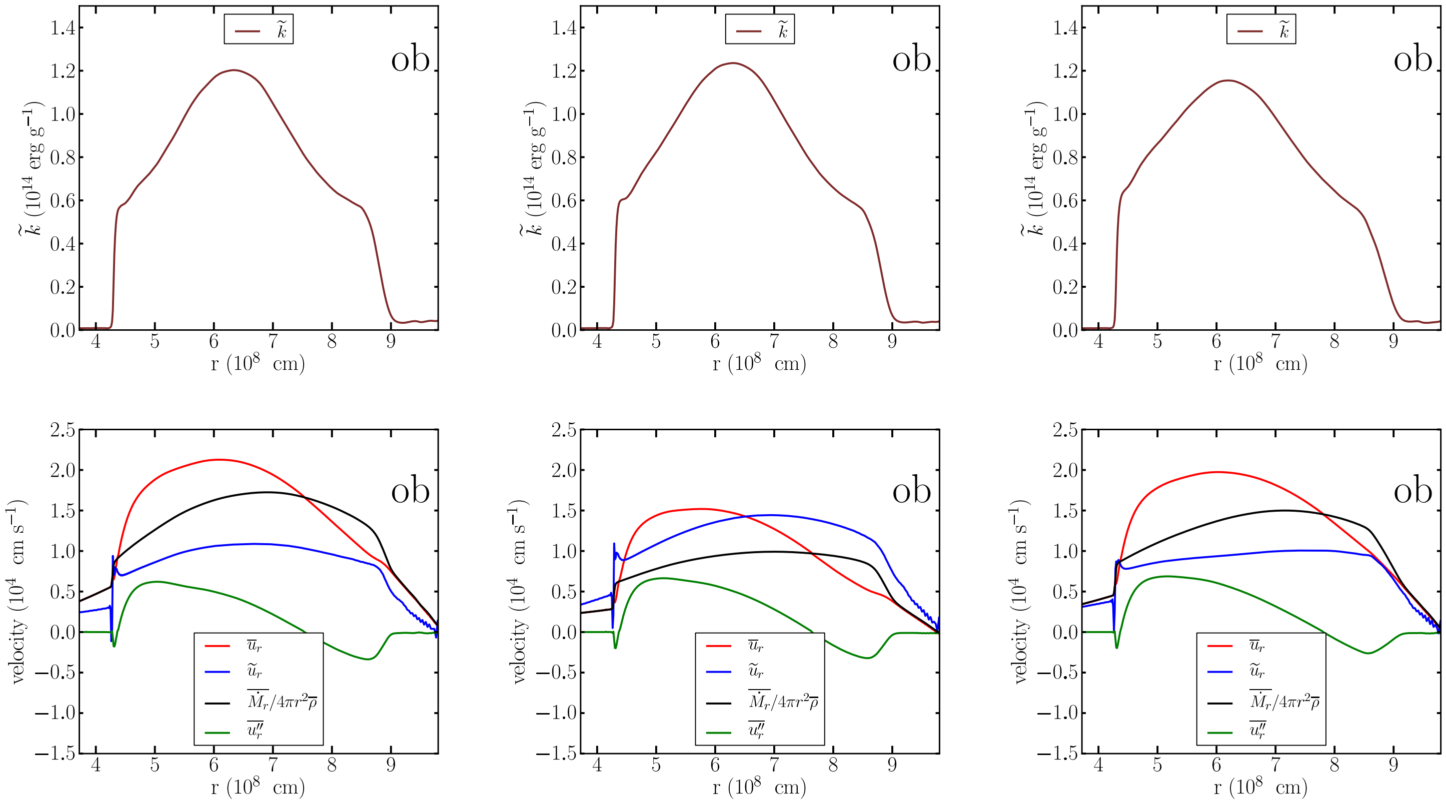

Figure 57: Mean turbulent kinetic energy equation (upper panels) and mean velocities from model ob.3D.2hp. Averaging window over roughly 2 convective turnover timescales 150 s (left), 3 convective turnover timescales 230 s (middle) and 4 convective turnover timescales 460 s (right).

9 Dependence on Convection Zone Depth

9.1 Oxygen burning shell model

Additional information about background structure of compared models

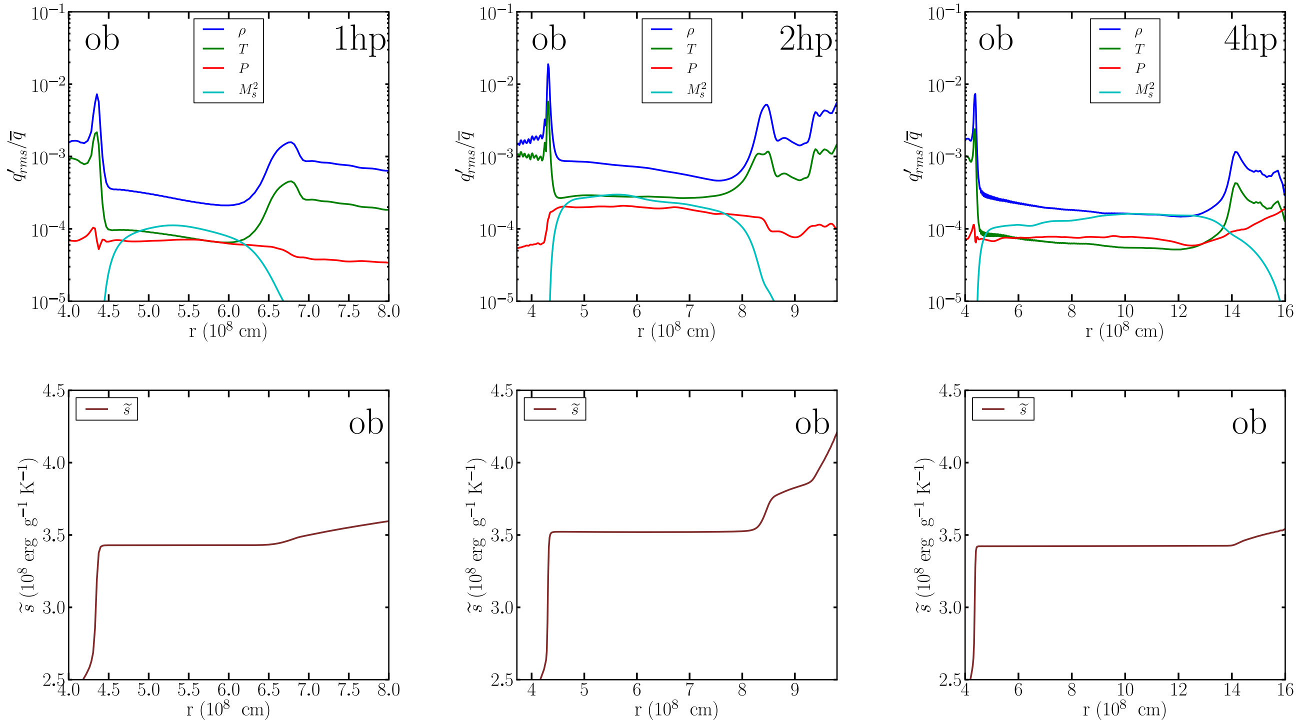

Figure 58: Relative root mean square of fluctuations of density ρ, temperature T and pressure P (upper panels) for the ob.3D.1hp (left), ob.3D.2hp (middle) and ob.3D.4hp (right) models togehter with square of the flow's Mach number M_s^2. Profiles of mean entropy (lower panels) in ob.3D.1hp (left), ob.3D.2hp (middle) and ob.3D.4hp (right).

Mean continuity equation and mean radial momentum equation

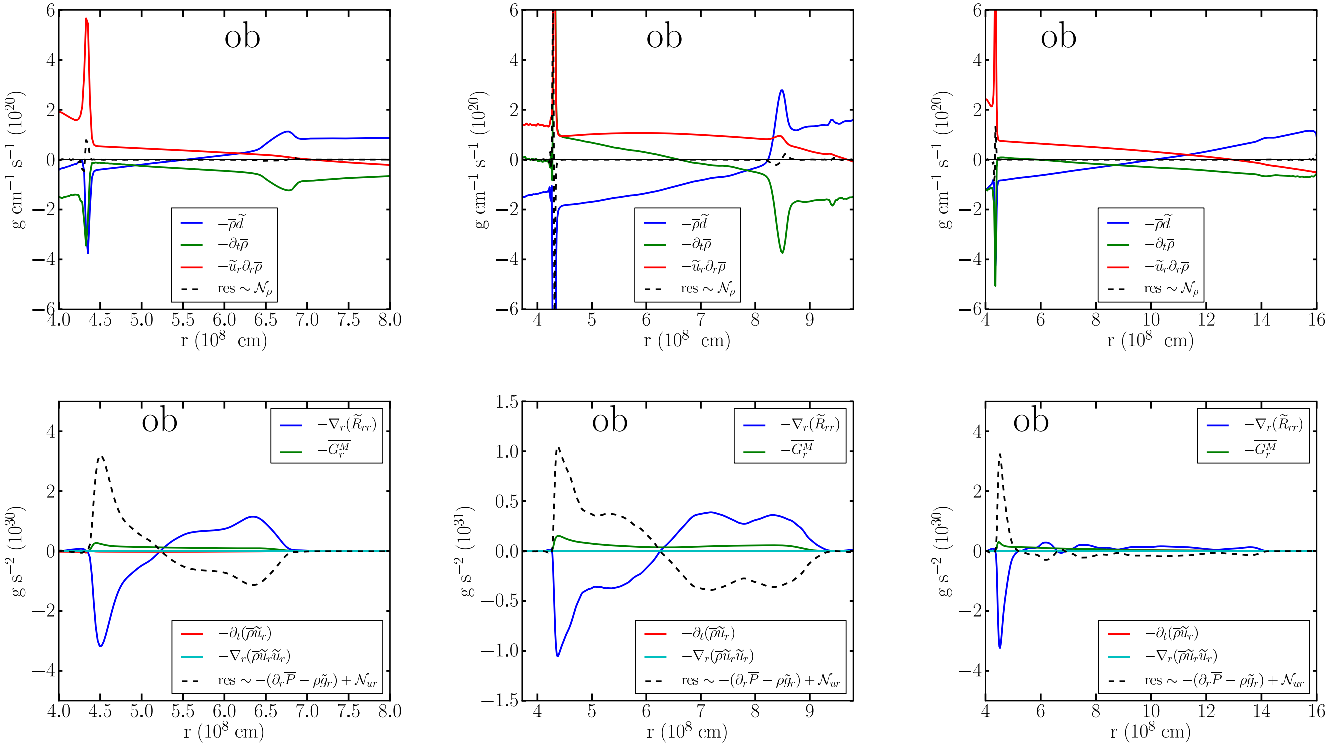

Figure 59: Mean continuity equation (upper panels) and radial momentum equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right)

Mean azimuthal and polar momentum equation

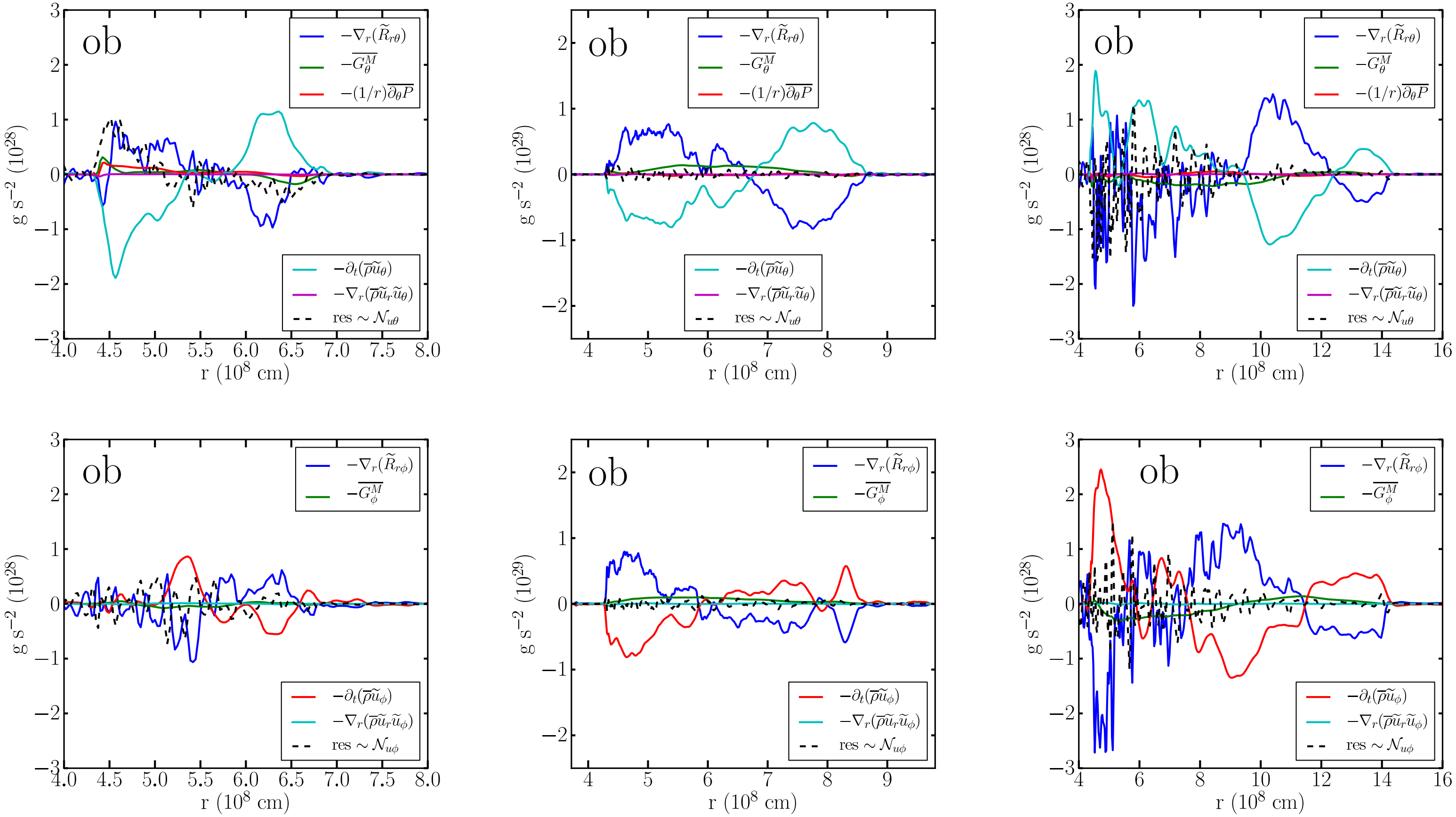

Figure 60: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean total energy equation and turbulent kinetic energy equation

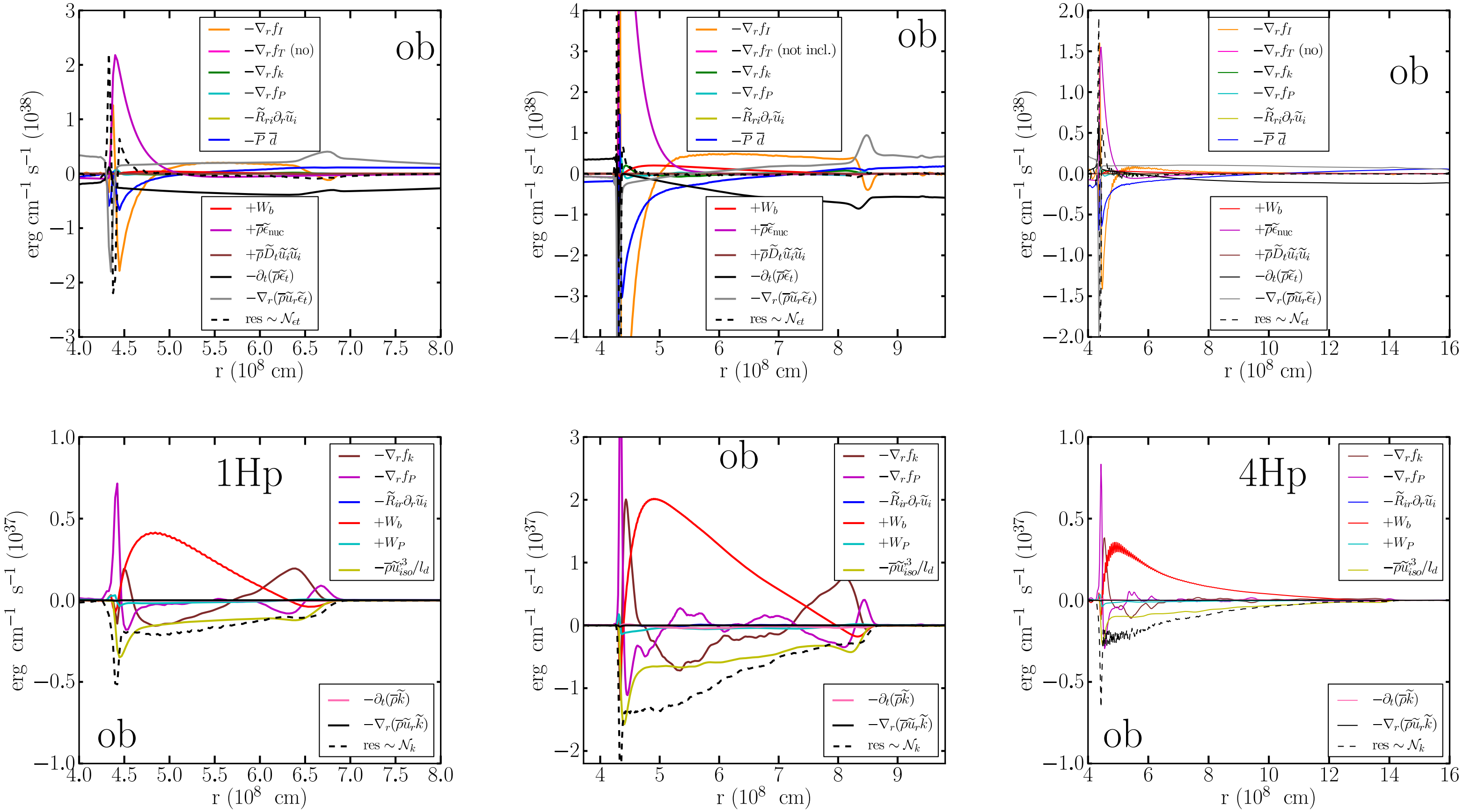

Figure 61: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent kinetic energy equation (radial + horizontal part)

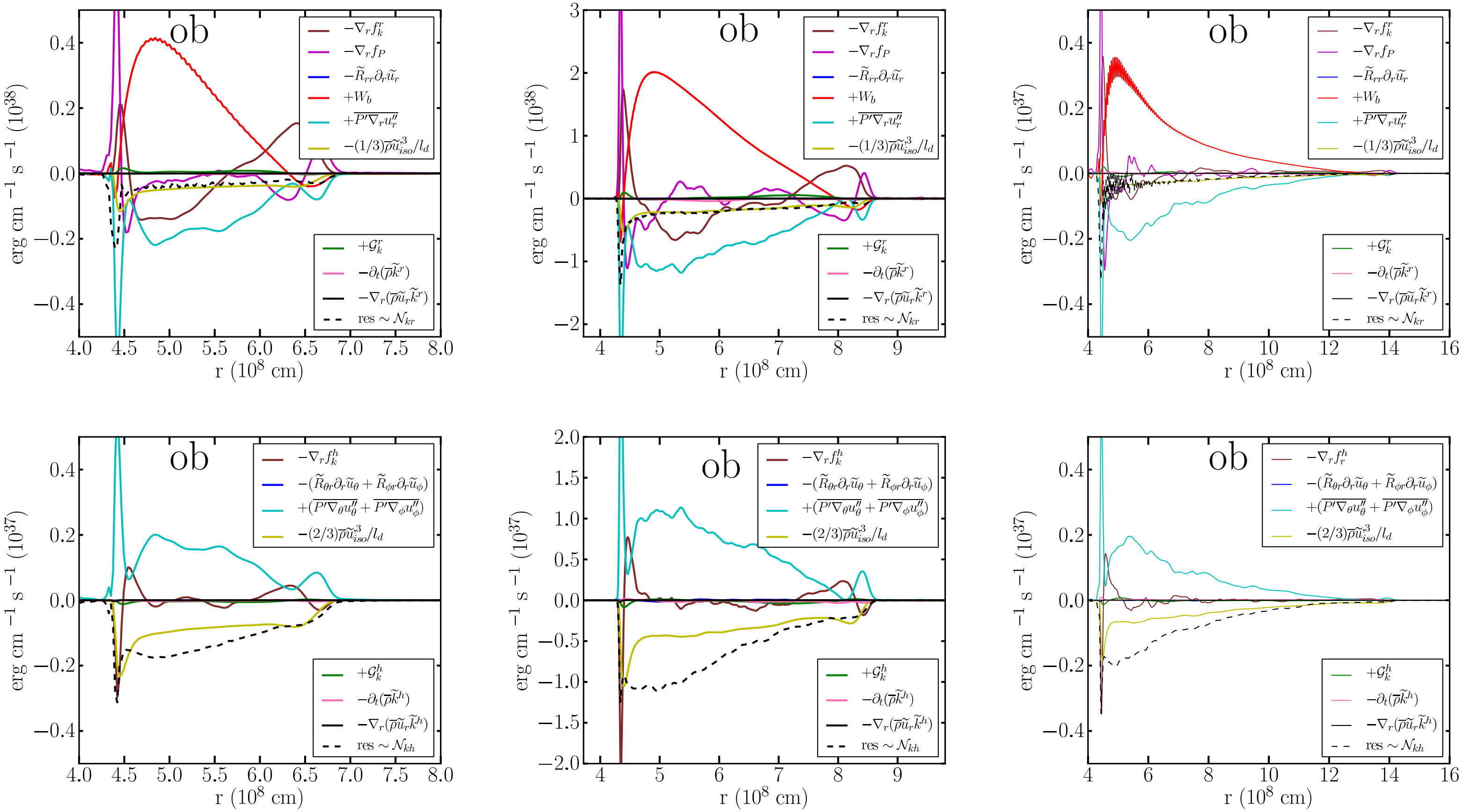

Figure 62: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent mass flux equation and mean density-specific volume covariance equation

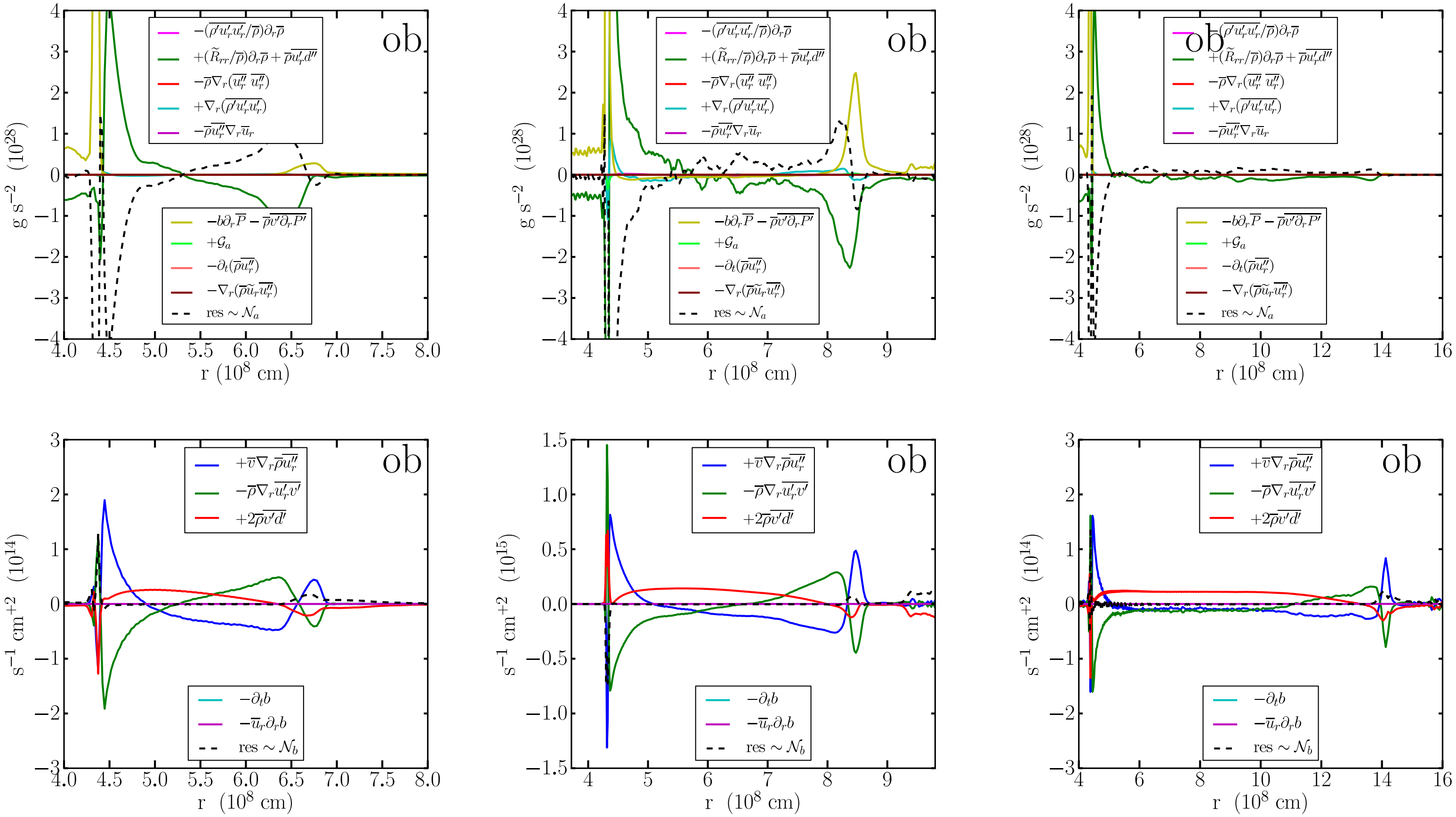

Figure 63: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean specific angular momentum equation and internal energy flux equation

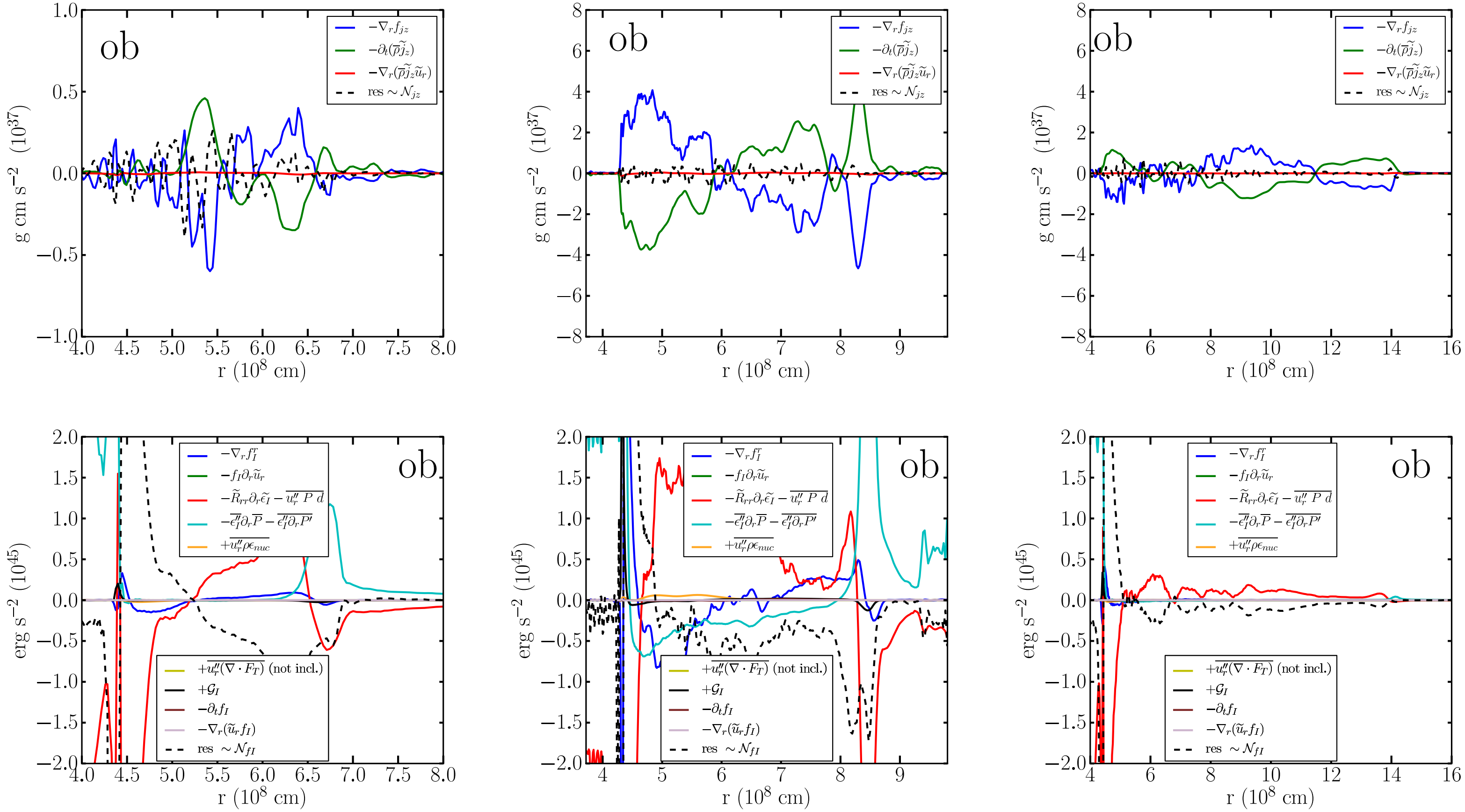

Figure 64: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean entropy equation and mean entropy flux equation

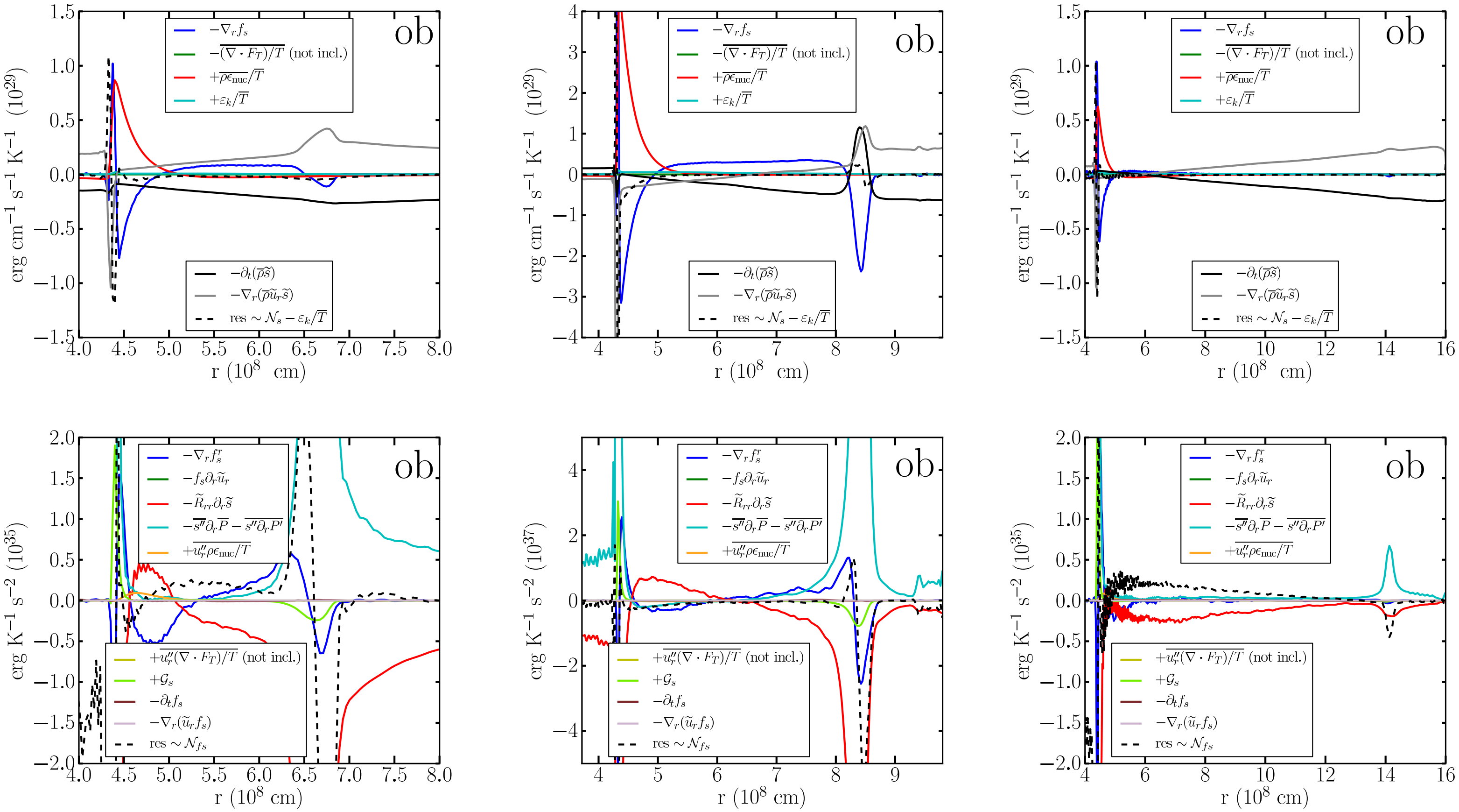

Figure 65: Mean entropy equation (upper panels) and mean entropy flux equation (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

Mean turbulent kinetic energy and mean velocities

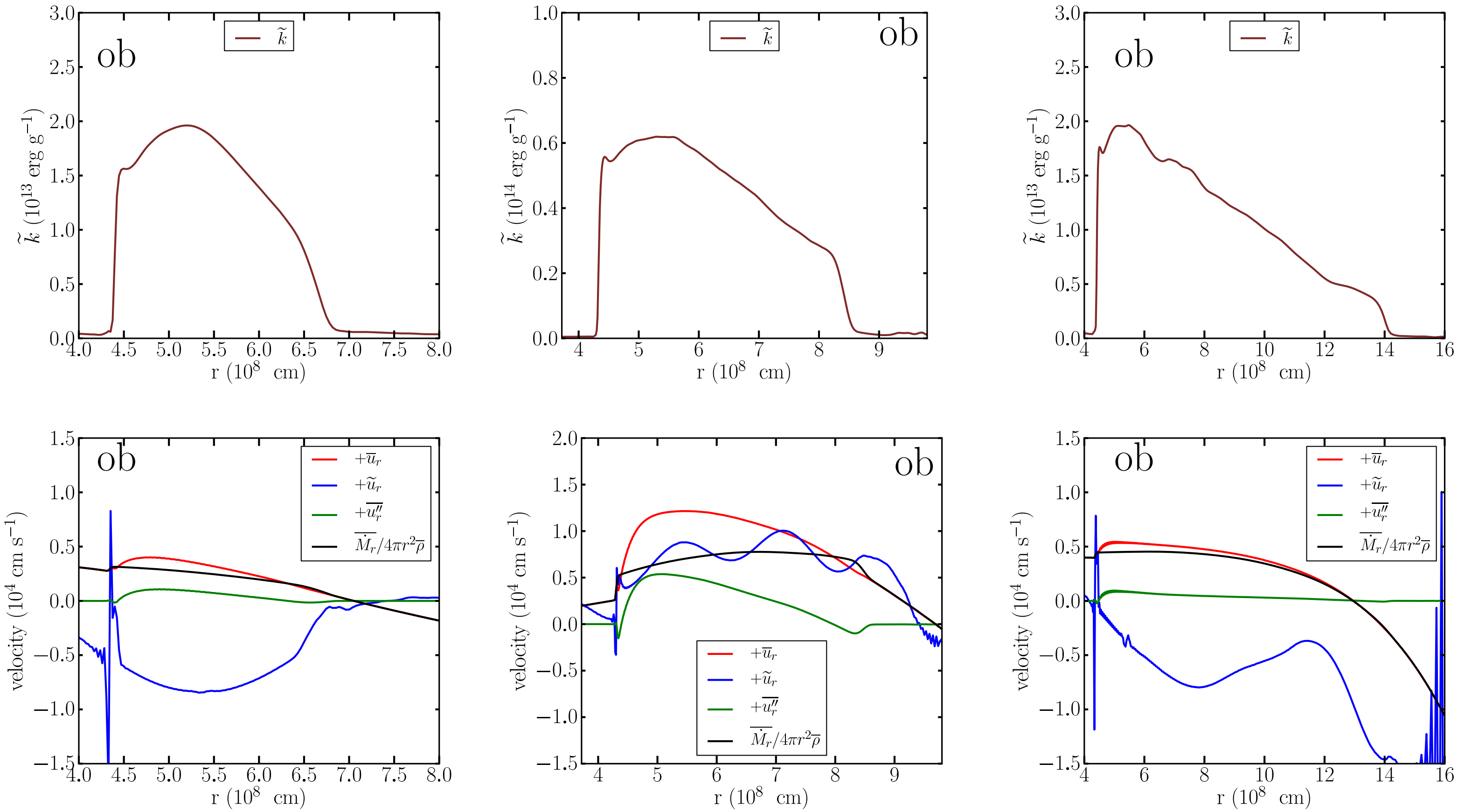

Figure 66: Mean turbulent kinetic energy (upper panels) and mean velocities (lower panels). 1 Hp model ob.3D.1hp (left), 2 Hp model ob.3D.2hp (middle) and 4 Hp model ob.3D.4hp (right).

9.2 Red giant convection envelope model

Mean continuity equation and mean radial momentum equation

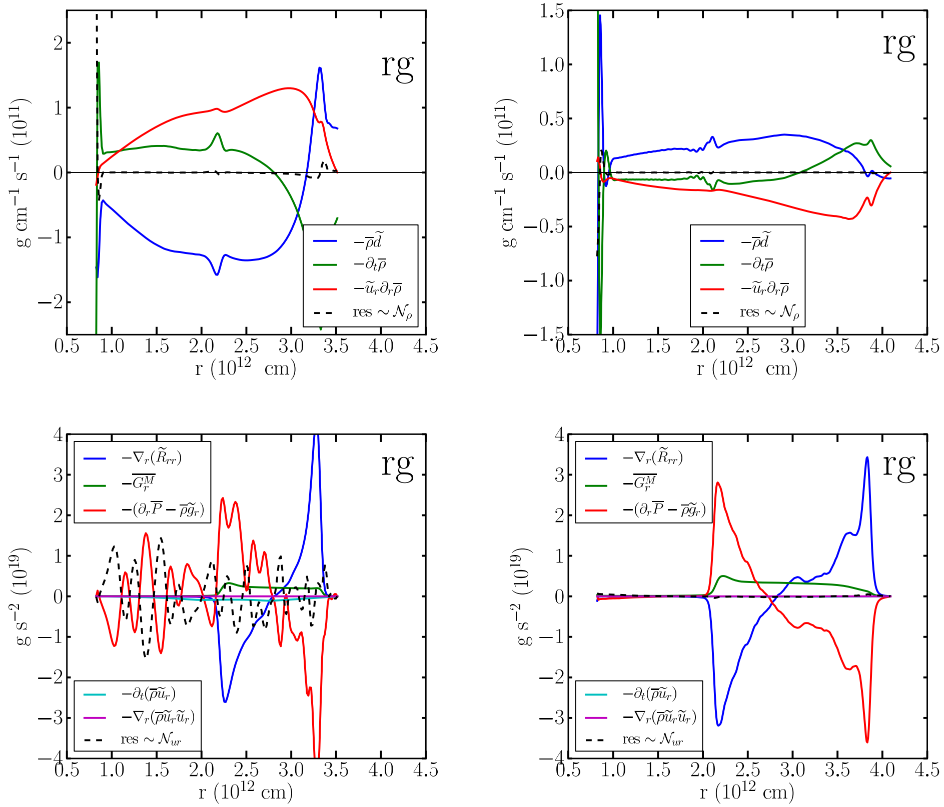

Figure 67: Mean continuity equation (upper panels) and radial momentum equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean azimuthal and polar momentum equation

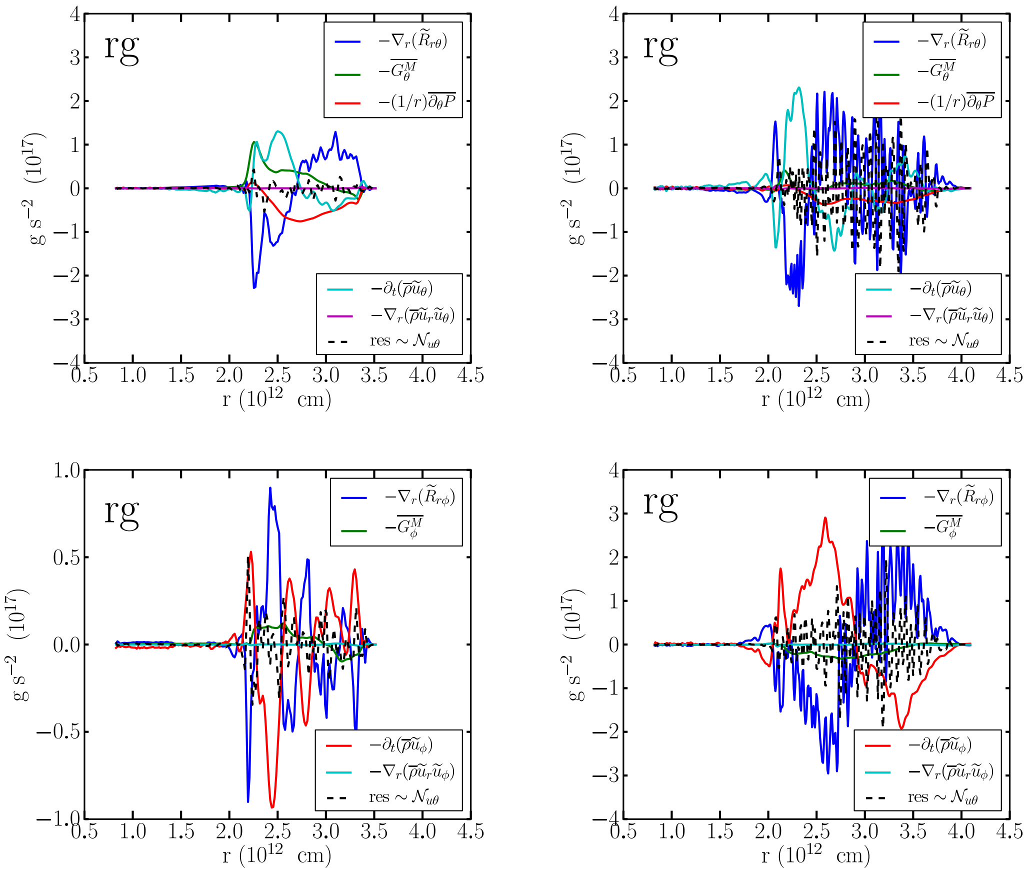

Figure 68: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean total energy equation and mean turbulent kinetic energy equation

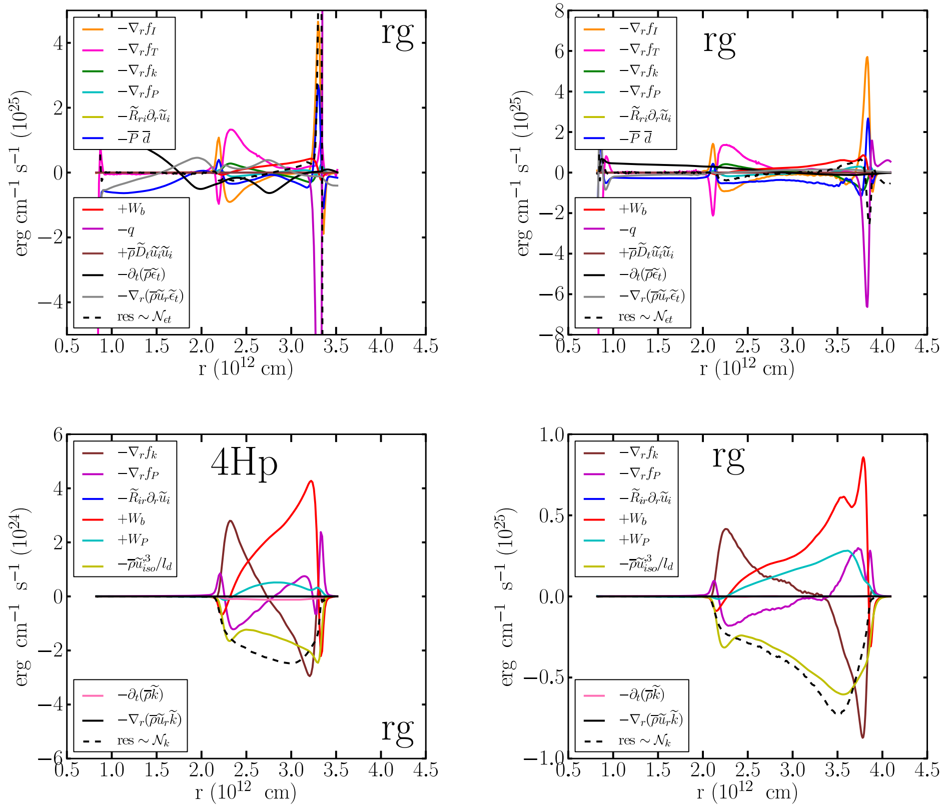

Figure 69: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent kinetic energy equation (radial + horizontal part)

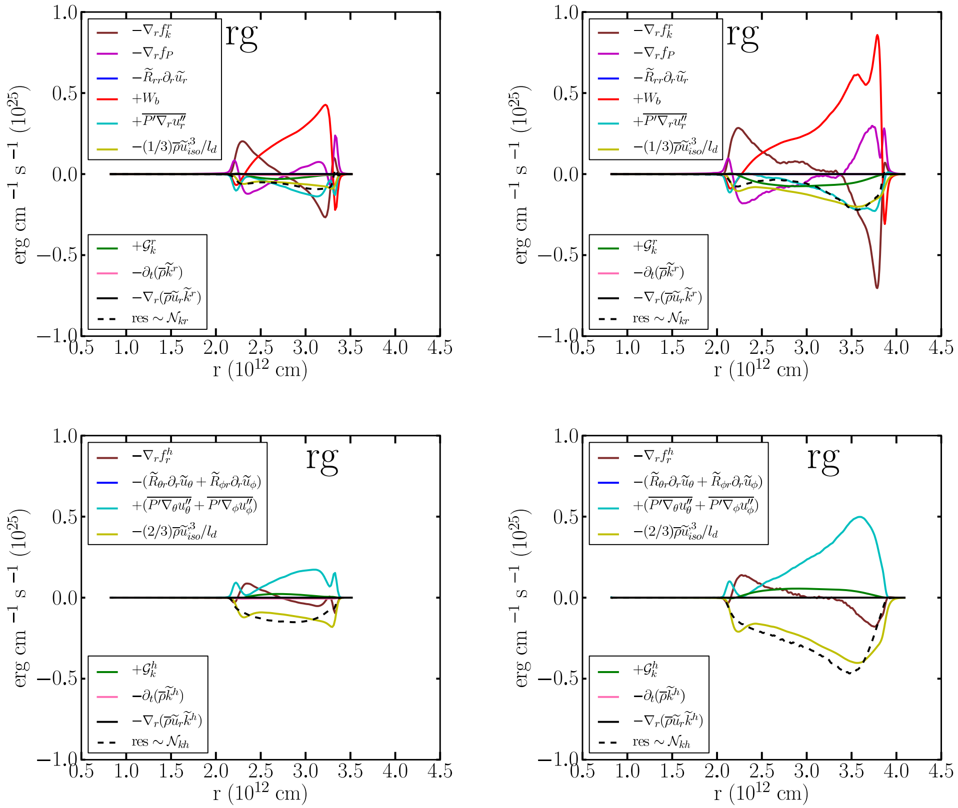

Figure 70: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent mass flux and mean density-specific volume covariance equation

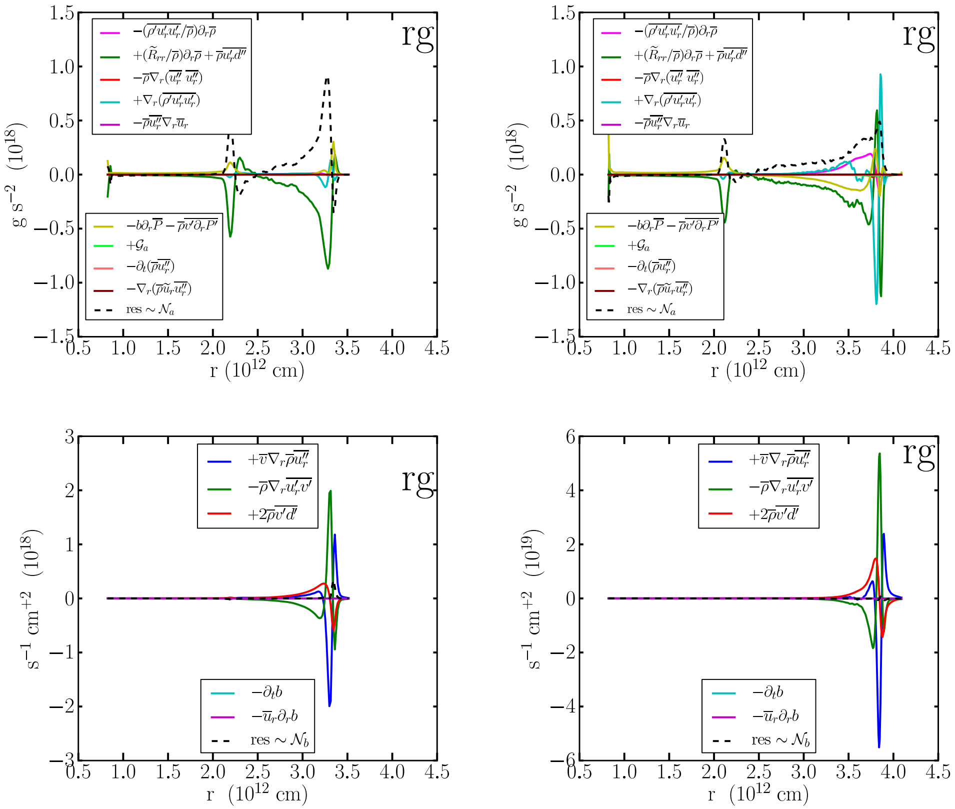

Figure 71: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean specific angular momentum equation and internal energy flux equation

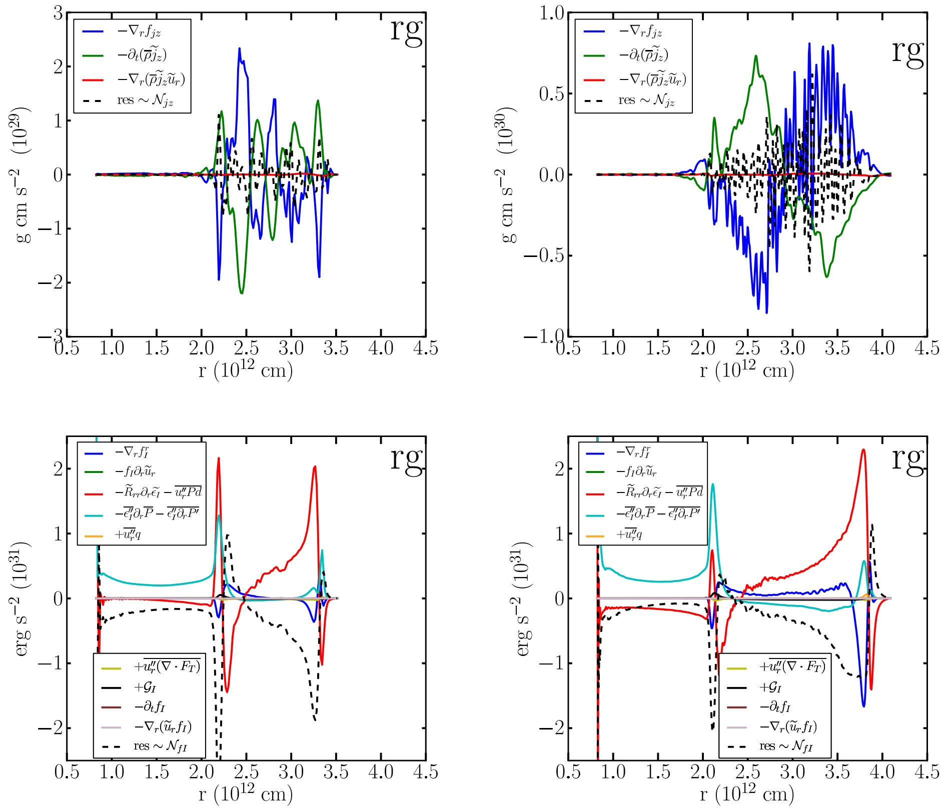

Figure 72: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean entropy equation and mean entropy flux equation

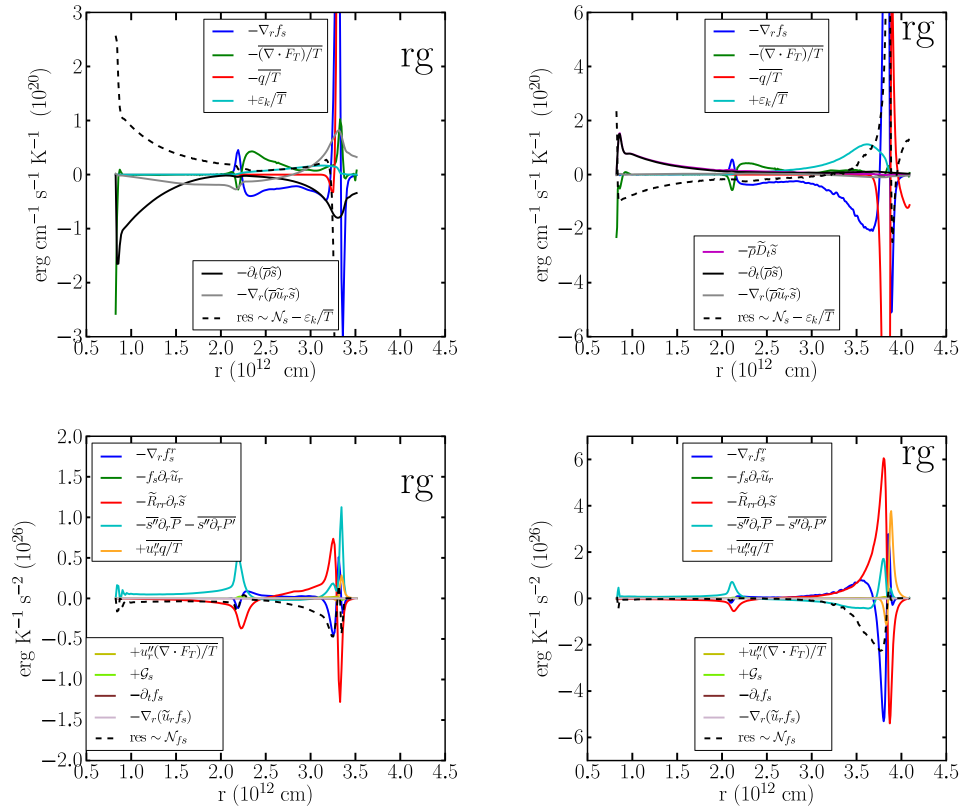

Figure 73: Mean entropy equation (upper panels) and mean entropy flux equation (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

Mean turbulent kinetic energy and mean velocities

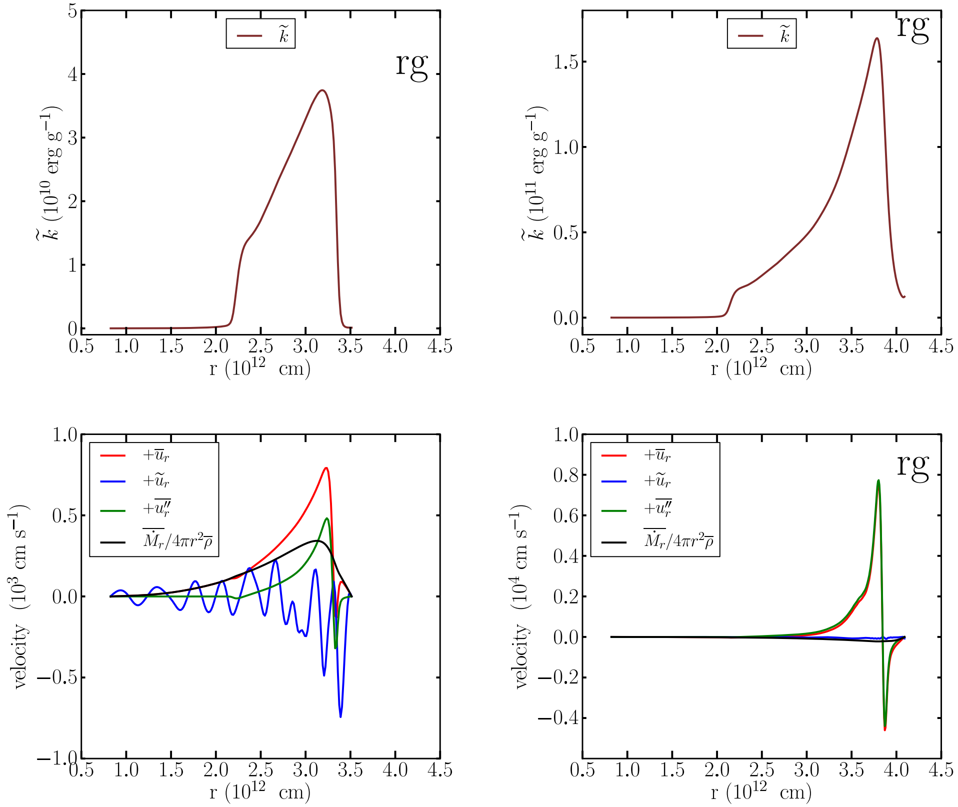

Figure 74: Mean turbulent kinetic energy (upper panels) and mean velocities (lower panels). 4 Hp model rg.3D.4hp (left) and 7 Hp model rg.3D.mrez (right).

10 Dependence on Convection Zone Driving Source Profile (Heating and Cooling)

10.1 Oxygen burning shell models

Mean continuity equation and mean radial momentum equation

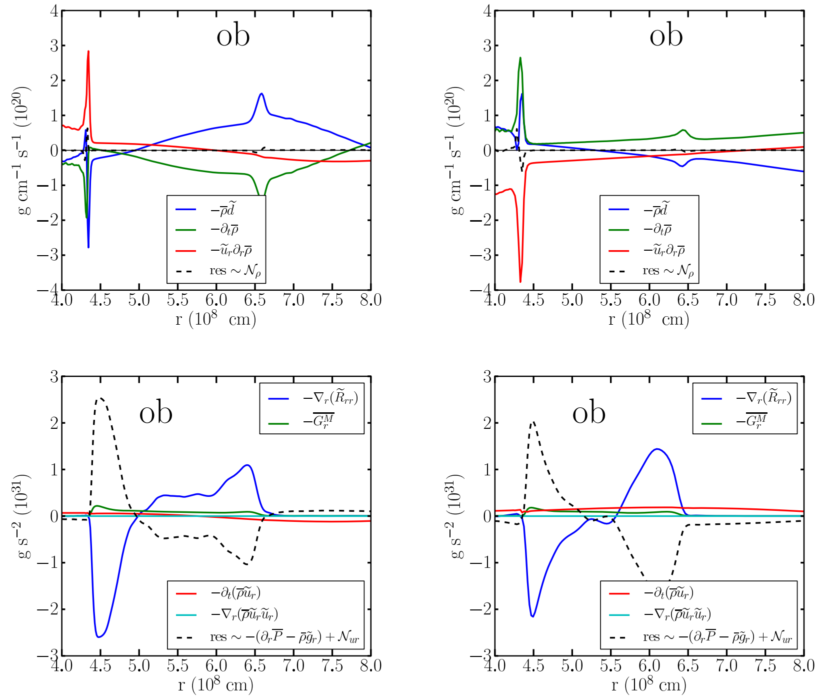

Figure 75: Mean continuity equation (upper panels) and radial momentum equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean azimuthal and polar momentum equation

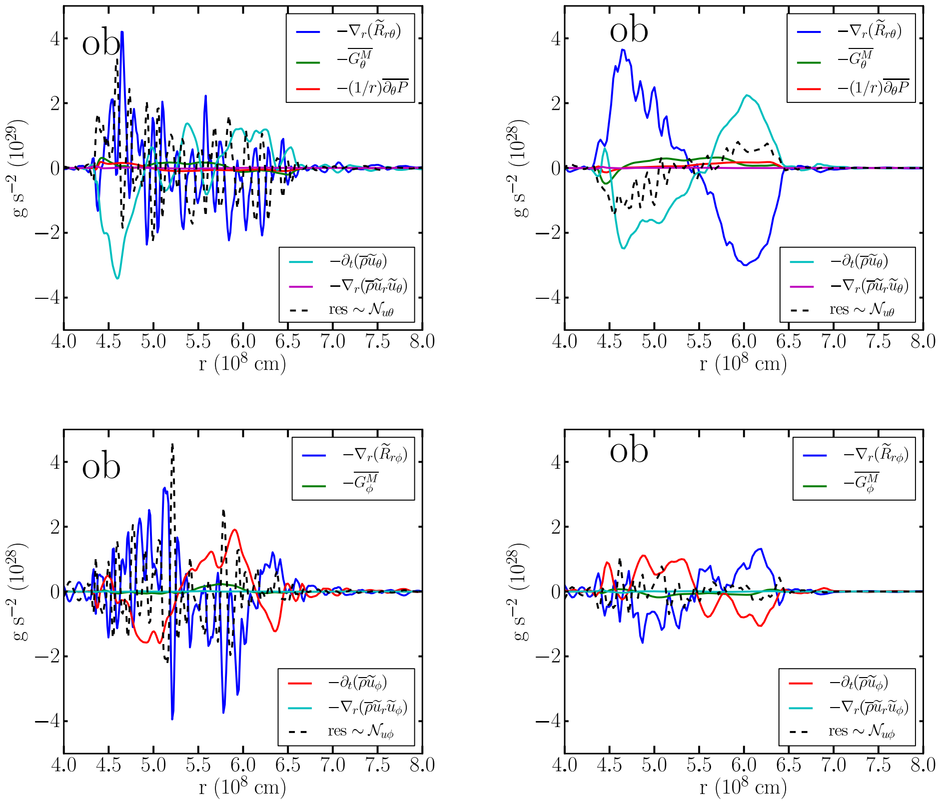

Figure 76: Mean azimuthal equation (upper panels) and mean polar momentum equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean total energy equation and mean turbulent kinetic energy equation

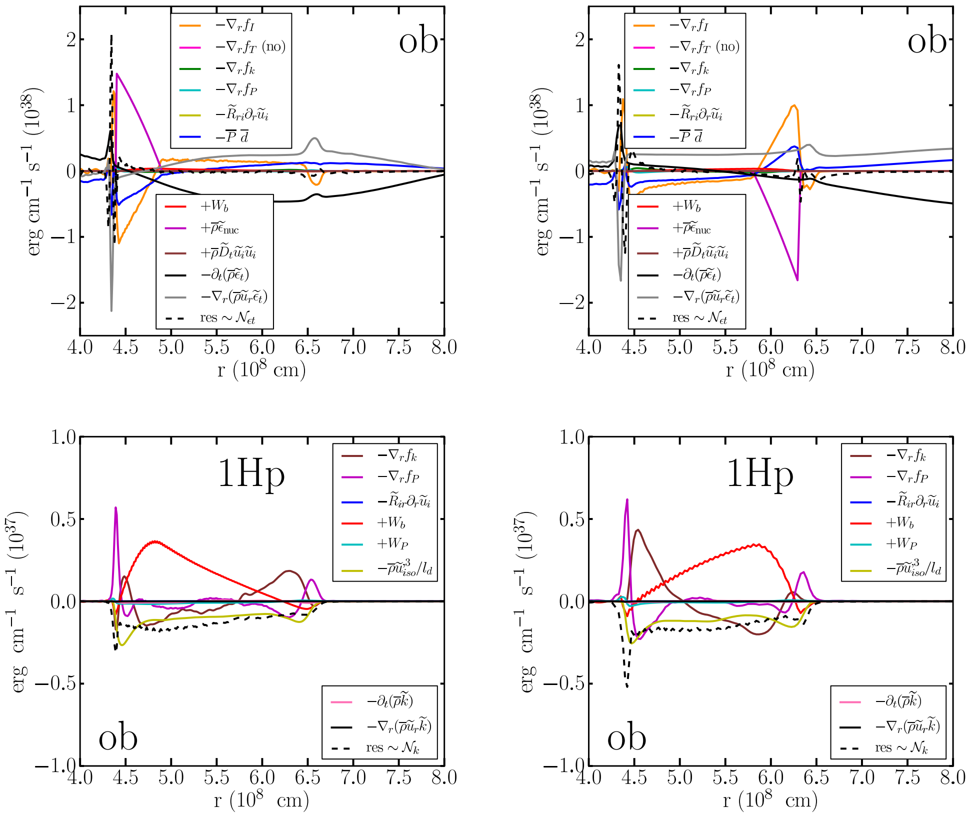

Figure 77: Mean total energy equation (upper panels) and mean turbulent kinetic energy equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent kinetic energy equations (radial + horizontal part)

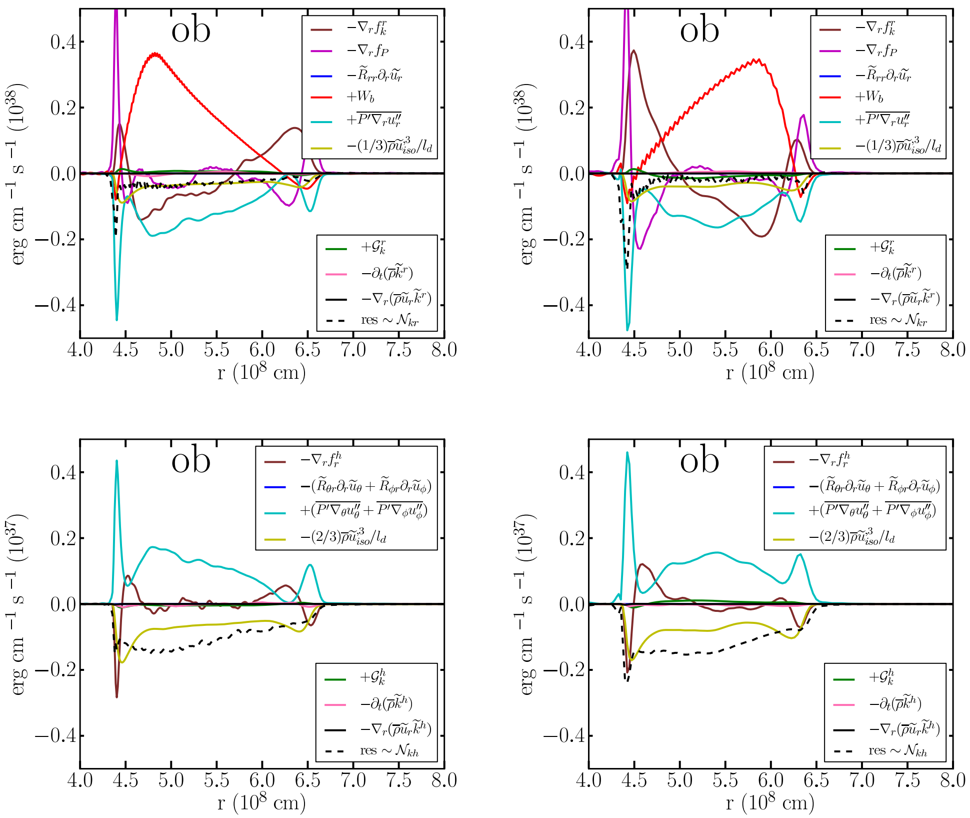

Figure 78: Radial (upper panels) and horizontal (lower panels) part of the mean turbulent kinetic energy equation. Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent mass flux and mean density-specific volume covariance equations

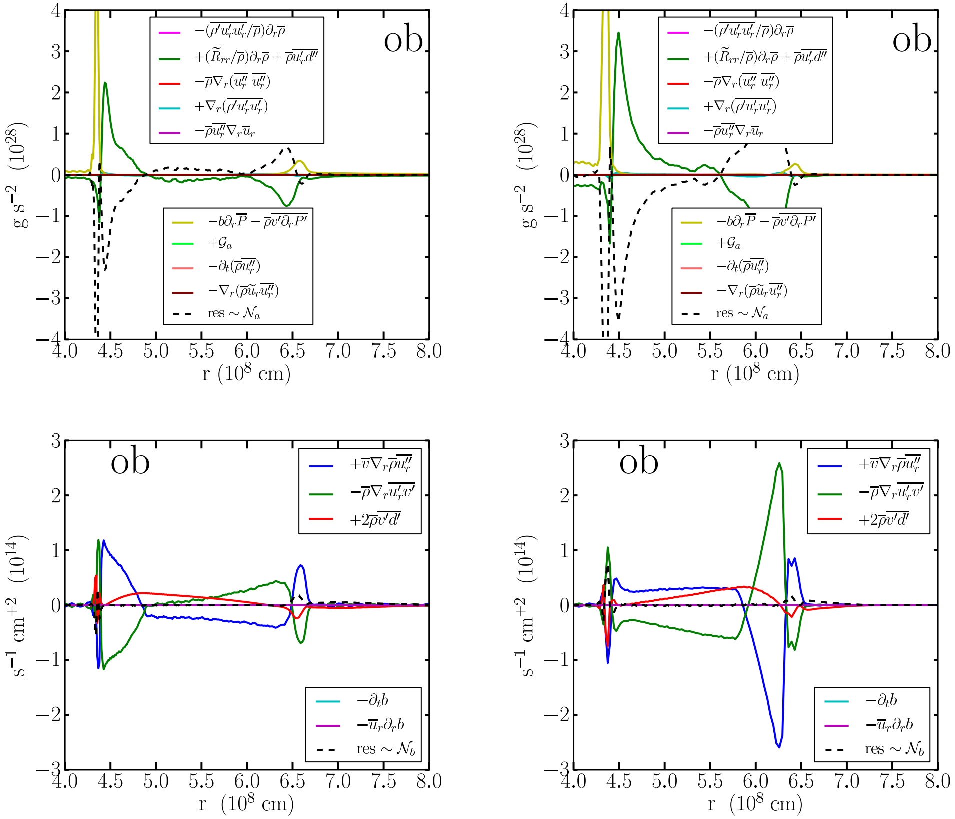

Figure 79: Mean turbulent mass flux equation (upper panels) and density-specific volume covariance equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean specific angular momentum equation and mean internal energy flux equation

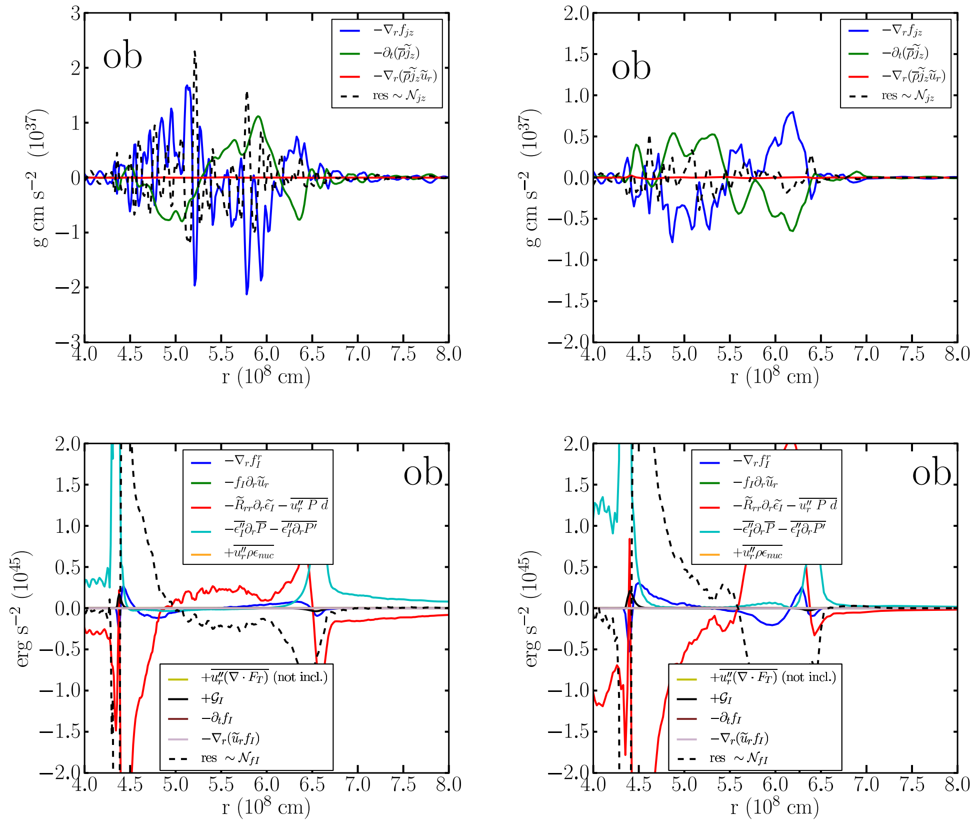

Figure 80: Mean specific angular momentum equation (upper panels) and mean turbulent internal energy flux equation (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

Mean turbulent kinetic energy and mean velocities

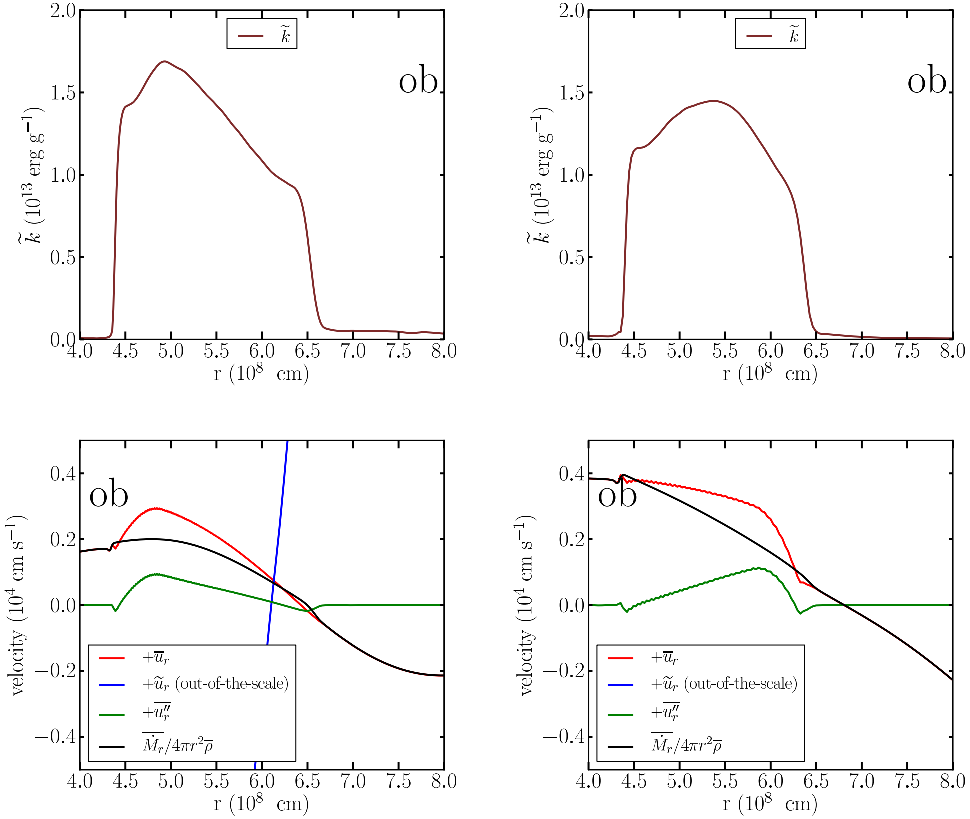

Figure 81: Mean turbulent kinetic energy (upper panels) and mean background velocities (lower panels). Model with volumetric heating at the bottom of convection zone ob.3D.1hp.vh (left) and model with volumetric cooling at the top of convection zone ob.3D.1hp.vc (right).

11 Assessment of Some One-Point Turbulence Closure Model Assumptions

11.1 Downgradient approximations

$$f_k = (C\ \rho\ \sqrt{\widetilde{k}}\ l_d)\ \partial_r \widetilde{k} \qquad f_I = (C\ \rho\ \sqrt{\widetilde{k}}\ l_d)\ \partial_r \widetilde{\epsilon}_I \qquad f_I = (C\ \rho\ \widetilde{k}^2/\varepsilon_k)\ \partial_r \widetilde{\epsilon}_I \tag{65}$$

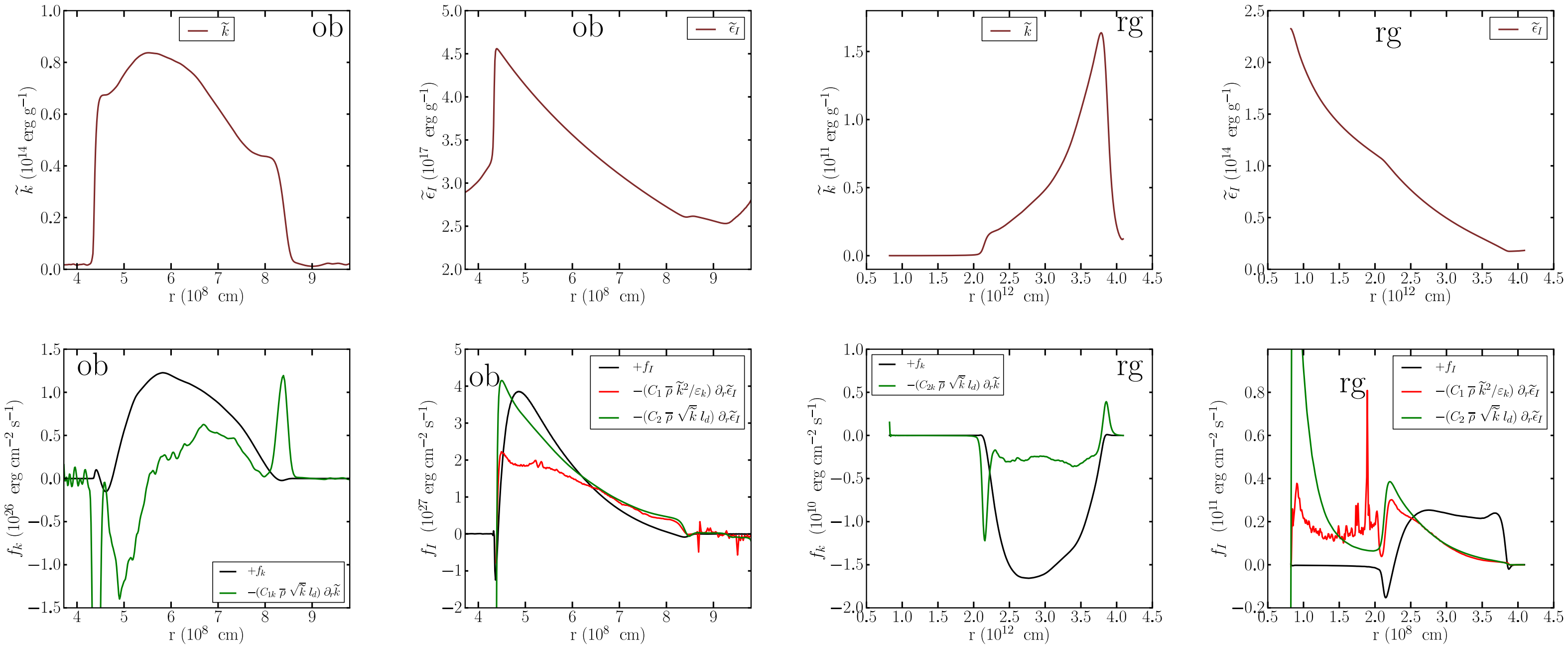

Figure 82: Profiles of mean turbulent kinetic energy and mean internal energy (upper panels) and various downgradient approximations to turbulent kinetic energy flux and internal energy flux (lower panels) for the oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg). l_d is dissipation length scale and C is model constant.

11.2 Various approximations from Besnard-Harlow-Rauenzahn (BHR)

$$\overline{\rho' u_r' u_r'} = C_{1a} \frac{l_d}{\sqrt{\widetilde{k}}} \widetilde{R}_{rr} \partial_r \overline{u_r''} \qquad \overline{v' \partial_r P'} = -C_{2a} \frac{\sqrt{k}}{l_d} \overline{u_r''} \qquad \overline{v' u_r'} = -C_{1b} \frac{l_d}{\sqrt{k}} \frac{\widetilde{R}_{rr}}{\overline{\rho}} \frac{\partial}{\partial r} \left(\frac{1+b}{\overline{\rho}} \right) \qquad \overline{v' d'} = -C_{2b} \frac{\sqrt{k}}{l_d} \frac{b}{\overline{\rho}}$$

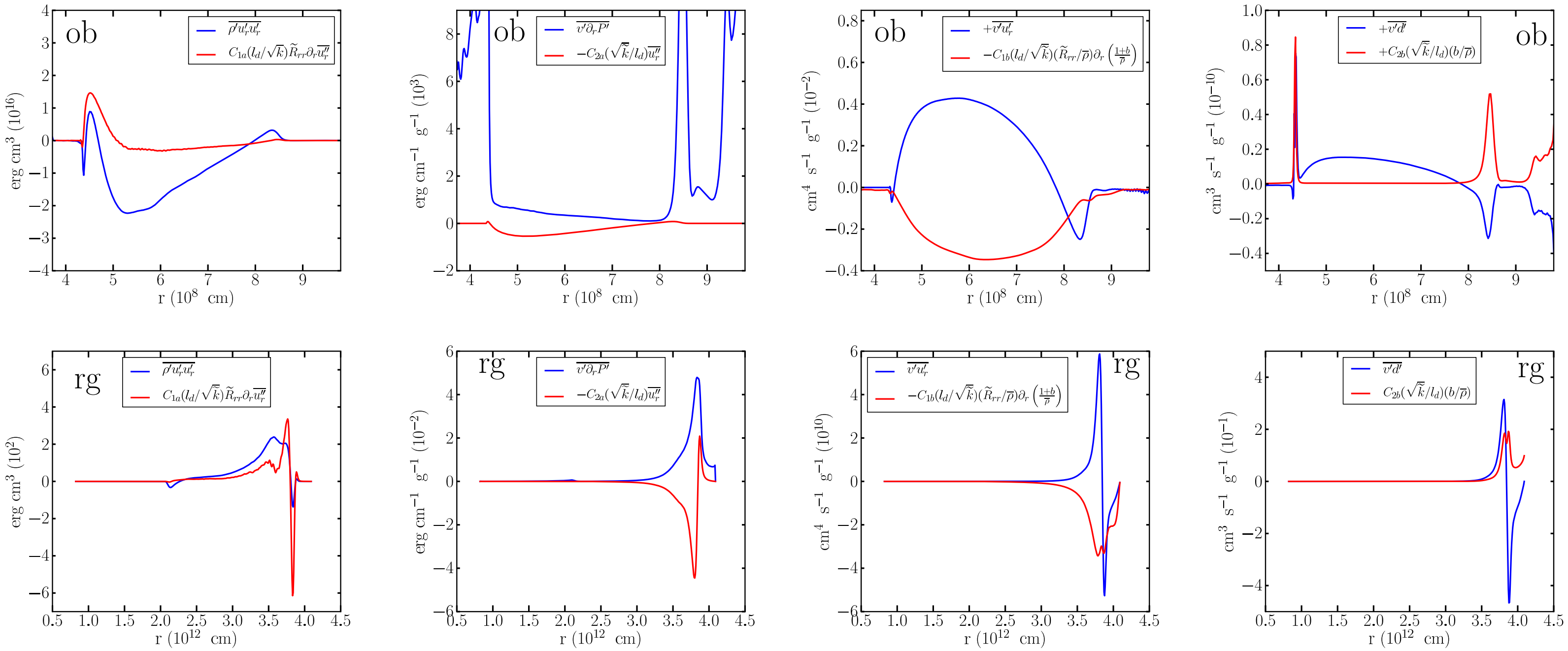

Figure 83: Various approximations taken from Besnard-Harlow-Rauenzahn (BHR) model for the oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg). l_d is dissipation length scale and C is model constant.

11.3 Quasi-normal approximation and decay-rate assumption model

$$\overline{a'b'c'd'} = \overline{a'b'}\ \overline{c'd'} + \overline{a'c'}\ \overline{b'd'} + \overline{a'd'}\ \overline{b'c'} \qquad \overline{a'b'} = (l_d^2/\Delta)\overline{\partial_k a'\partial_k b'} \qquad \text{(original formulations)} \quad (66)$$

$$\widetilde{a''b''c''d''} = \widetilde{a''b''}\ \widetilde{c''d''} + \widetilde{a''c''}\ \widetilde{b''d''} + \widetilde{a''d''}\ \widetilde{b''c''} \qquad \widetilde{a''b''} = (l_d^2/\Delta)\overline{\partial_k a''\partial_k b''} \qquad \text{(assumed Favre equivalents)} \quad (67)$$

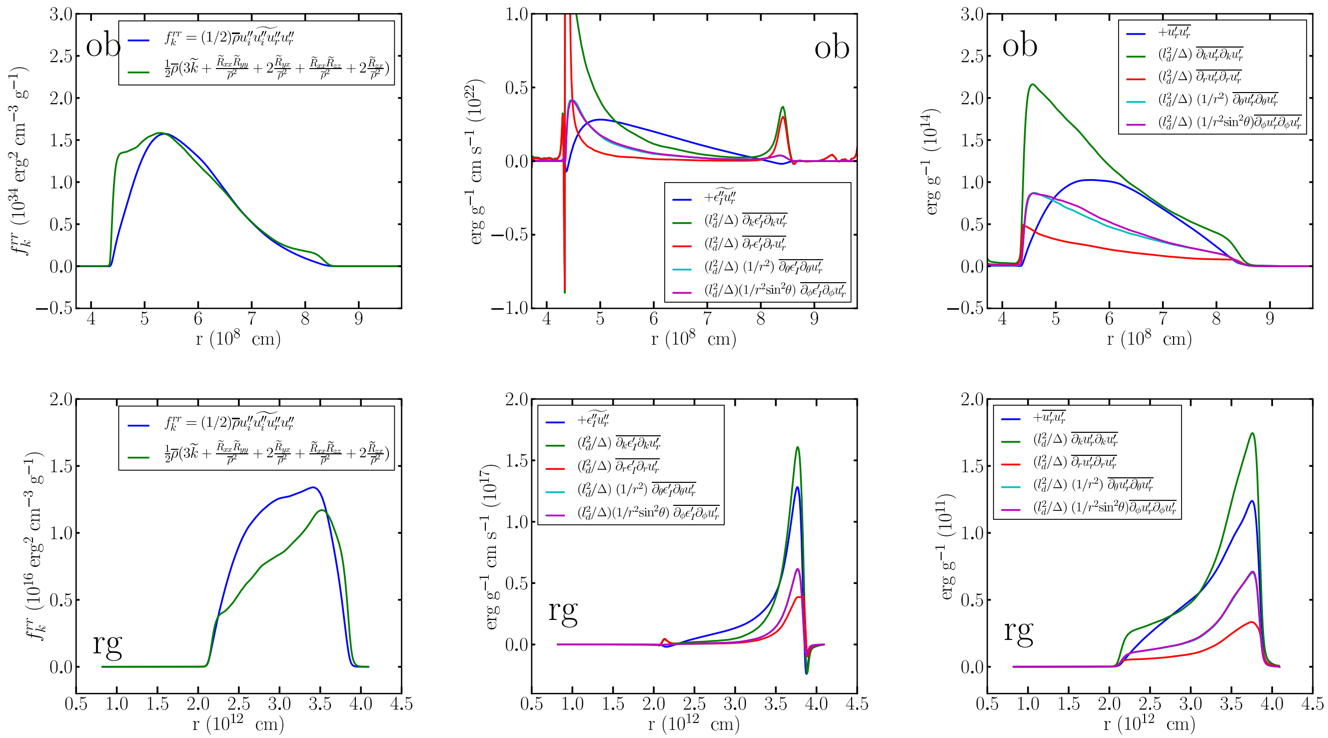

Figure 84: Quasi-normal approximations (left panels) and decay-rate assumption models (middle, right panels) for the oxygen burning shell model ob.3D.mr (ob) and red giant envelope convection rg.3D.mr (rg).

12 Fourier scale analysis

We present some preliminary scale analysis in this section including cumulative spectra for a variety of covariances at a location within the convection zone in §12.1 as well as radial profiles for mean-field data which has been separated into large and small components in §§12.2 to 12.4.

12.1 Cumulative Fourier spectra for Covariances

The mean "energy" spectrum $\overline{E_{ab}}(r, k, t_c)$ for a given covariance is found by taking the space-time average of the product of the fluctuations after writing each fluctuation as a Fourier series

$$\begin{aligned}
\overline{a'b'}(r, t_c) =& \frac{1}{\Delta\Omega T}\int a'(r,\theta,\phi,t)b'(r,\theta,\phi,t)d\Omega dt = \frac{1}{\Delta\Omega T}\int \sum_{\vec{l}}\sum_{\vec{m}} \widehat{a'}(r,\vec{l},t)\widehat{b'}(r,\vec{m},t)e^{i(\vec{l}+\vec{m})\cdot\vec{x}}d\Omega dt \\
=& \frac{1}{T}\int \sum_{\vec{l}} \widehat{a'}(r,\vec{l},t)\widehat{b'}^*(r,\vec{l},t)dt = \sum_{\vec{l}} \overline{\widehat{a'}(r,\vec{l},t)\widehat{b'}^*(r,\vec{l},t)} \\
=& \sum_{k=|\vec{l}|} \overline{E_{ab}}(r,k,t_c)
\end{aligned} \tag{68}$$

where $k = |\vec{l}| = (l_\phi^2 + l_\theta^2)^{1/2}$ is the horizontal wavenumber magnitude and $\vec{x} = (\theta, \phi)$ is the angular position vector and * indicates complex conjugation. By this definition, $E_{ab}(r, k, t)$ involves a sum of the terms $\overline{\widehat{a'}\widehat{b'}^*}$ over annuli of unit width in the plane of the wave vector $\vec{l}$ components l_θ and l_ϕ centered on k. Here, $d\Omega = d\theta d\phi$ rather than the solid angle increment. This choice greatly simplifies the algebra and results in a negligible difference compared to solid angle averages for the small wedges studied here. In the remainder of this section, our data is presented in terms of a normalized wavenumber $\hat{k} = k/k_0$ where $k_0 \equiv 2\pi/\mathcal{L}$ is the lowest wavenumber in the domain of width $\mathcal{L}$ so that an integer wavenumber $\hat{k}$ represents a Fourier mode of wavelength $\mathcal{L}/\hat{k}$. We drop the hat over the wavenumber in what follows.

Finally, the cumulative spectrum $E_c(k)$ is defined by the normalized partial sum

$$E_c(\overline{a'b'})(r, k, t) = \frac{\sum_{1\le l\le k} \overline{E}_{ab}(r, l, t)}{\overline{a'b'}}. \tag{69}$$

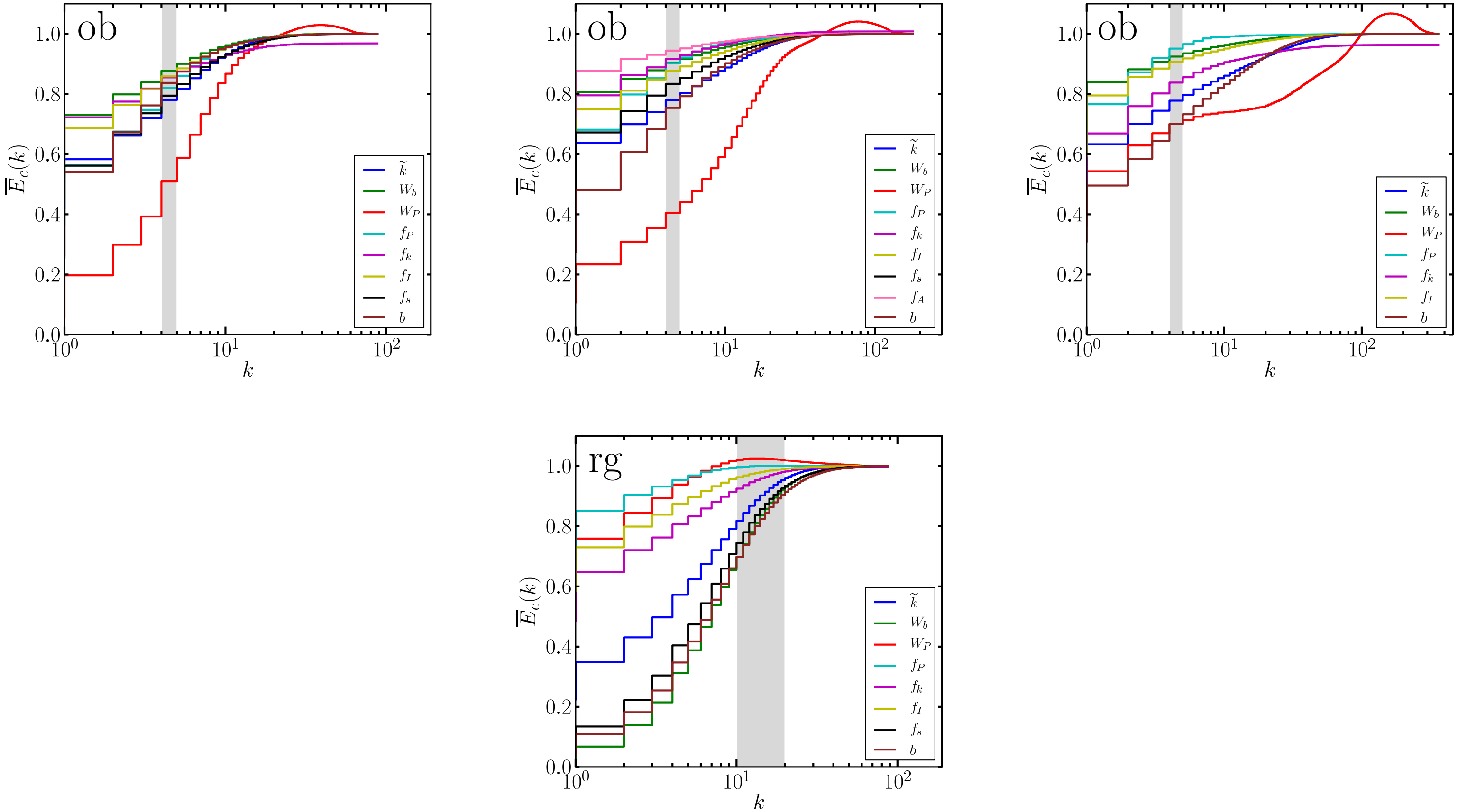

Figure 85: Cumulative Fourier spectra of relevant mean fields for the oxygen burning shell models (upper panels) ob.3D.lr (left), ob.3D.mr (middle) and ob.3D.hr (right) derived at radius where corresponding mean field has a maximum value. The same is done for red giant envelope convection model (lower panel) rg.3D.lr. The shaded vertical lines separates cumulative contributions from large and small scales.

12.2 Fourier scale decomposition of turbulent kinetic energy equation (ob.3D.mr)

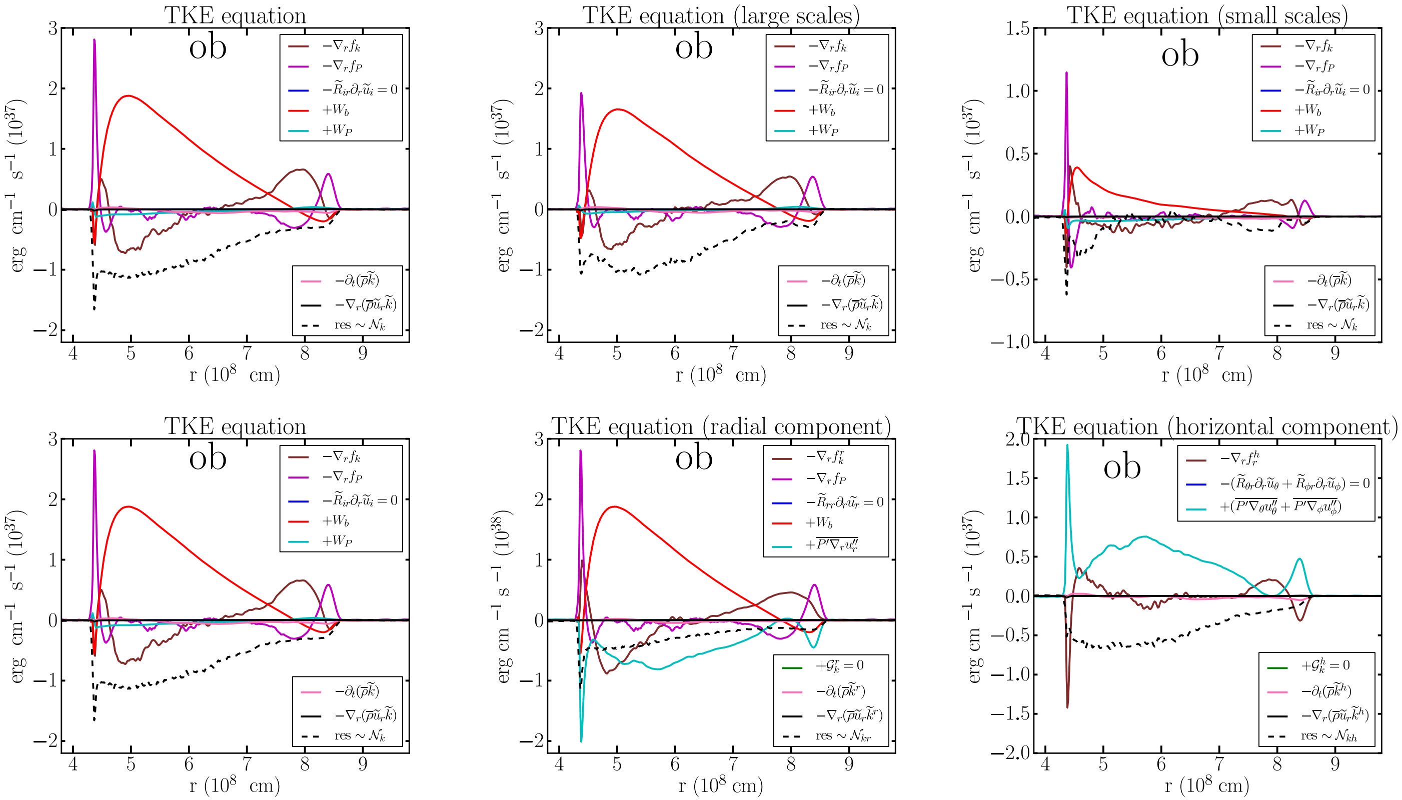

Figure 86: Upper panels: Fourier scale decomposition of mean fields in turbulent kinetic energy equation (left) into large (middle) and small (right) scale component. Scale separation wave number was taken to be 4. Averaging was performed over 230 s at around central time 505 s. Lower panels: Decomposition of mean fields in turbulent kinetic energy equation (left) into radial (middle) and horizontal components (right).

12.3 Fourier scale decomposition of radial and horizontal part of turbulent kinetic energy equation (ob.3D.mr)

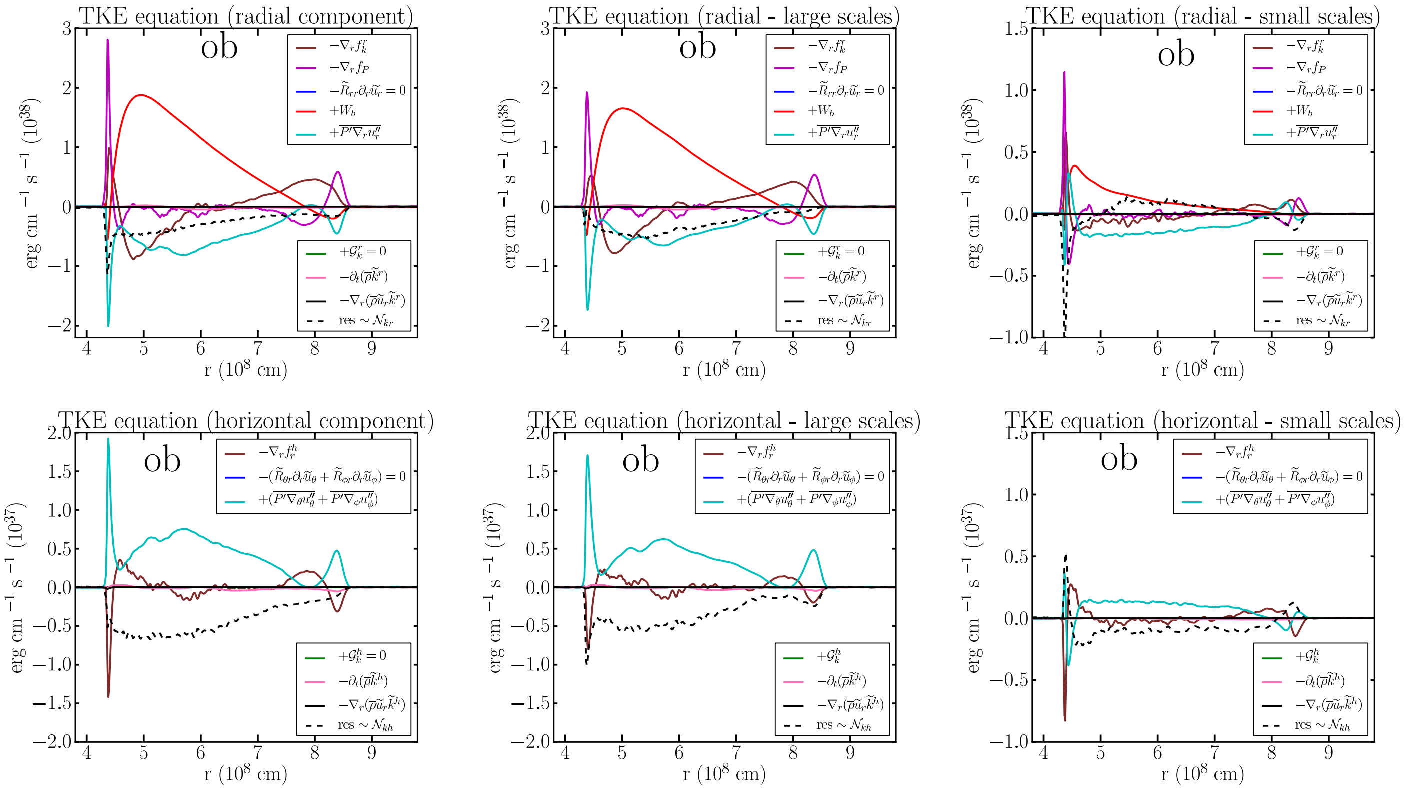

Figure 87: Upper panels: Fourier scale decomposition of radial part of mean fields in turbulent kinetic energy equation (left) into large (middle) and small (right) scale component. Lower panels: The same for the horizontal part of turbulent kinetic energy equation. Scale separation wave number was taken to be 4. Averaging was performed over 230 s at around central time 505 s.

12.4 Fourier scale decomposition of individual mean fields in turbulent kinetic energy equation (ob.3D.mr)

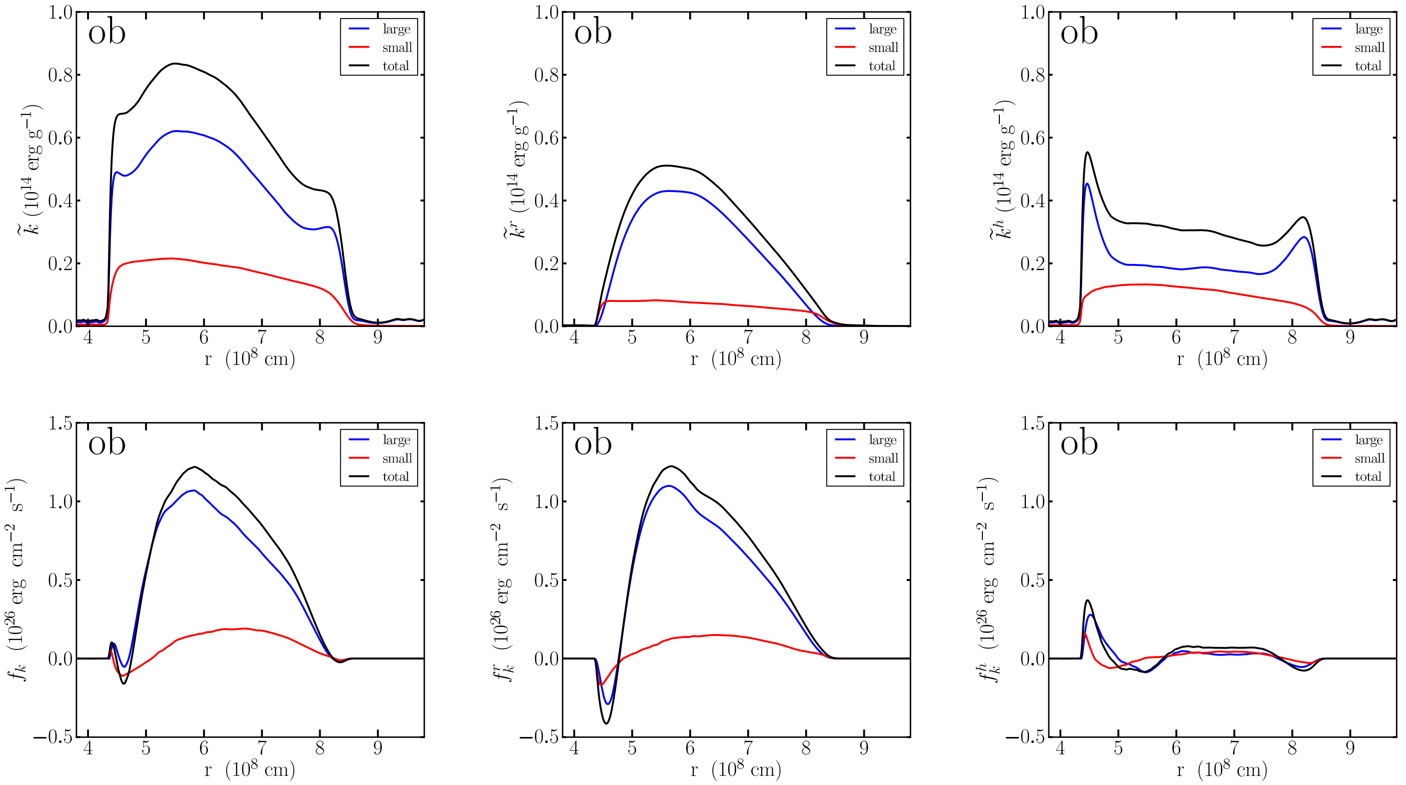

Figure 88: Upper panels: Fourier scale decoposition of turbulent kinetic energy (left), radial part of turbulent kinetic energy (middle) and horizontal part of turbulent kinetic energy (right). Lower panels: The same is done for the turbulent kinetic energy flux. Separation wave number was taken to be 4. Averaging was performed over 230 s at around central time 505 s.

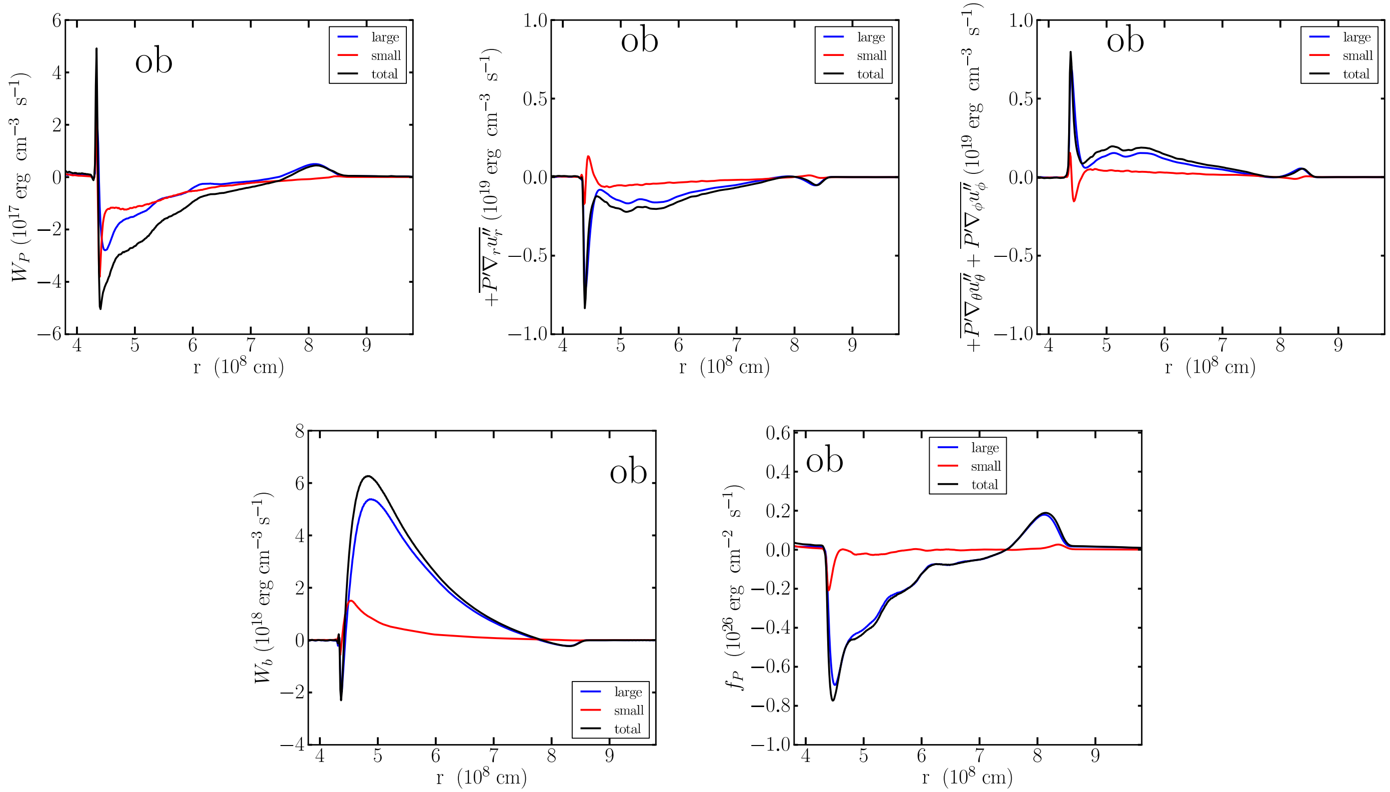

Figure 89: Upper panels: Fourier scale decomposition of the turbulent pressure dilataion (left), radial pressure strain (middle) and horizontal pressure strain (right). Lower panels: Fourier scale decomposition of the buoyancy work W_b and acoustic flux f_P. Separation wave number was taken to be 4. Averaging was performed over 230 s at around central time 505 s.

13 Properties of hydrogen injection flash and core helium flash simulation data

13.1 Summary of the hydrogen injection flash and core helium flash simulations and their properties

Parameter	hif.3D	hif.3D	chf.3D
Grid zoning	375×45^2	375×45^2	270×30^2
$r_{\rm in}$, $r_{\rm out}$ (10^8 cm)	2., 16.	2., 16.	2., 12.
$\Delta\theta$, $\Delta\phi$	45°		30°
Convection zone	He-burning (CVZ-1)	CNO-burning (CVZ-2)	He-burning (CVZ)
$r^c_{\rm in}$, $r^c_{\rm out}$ (10^8 cm)	4.7, 9.55	9.55, 11.3	4.7, 9.4
CZ stratification (H_p)	2.4	1.3	2.3
$\Delta t_{\rm av}$ (s)	4000	4000	18000
$v_{\rm rms}$ (10^5 cm/s)	8.2	11.8	8.5
$\tau_{\rm conv}$ (s)	1180	290	1100
$P_{turb}/P_{gas}(10^{-5})$	2.6	3.7	3.3
$L_{\rm heat}$ (10^{43} erg/s)	1.2	1.8	1.2
$L_{\rm d}$ (10^{41} erg/s)	5.3	6.1	6.
$l_{\rm d}$ (10^8 cm)	5.6	2.9	4.8
$\tau_{\rm d}$ (s)	337.4	123.6	281.9
$\tau_{\rm dr}$ (s)	719.7	175.8	659.7
$\tau_{\rm dh}$ (s)	211.2	116.1	142.5

Table 7: boundaries of computational domain $r_{\rm in}$, $r_{\rm out}$; boundaries of convection zone at bottom and top $r^c_{\rm b}$, $r^c_{\rm t}$; angular size of computational domain $\Delta\theta$, $\Delta\phi$; depth of convection zone "CZ stratification" in pressure scale height H_P; averaging timescale of mean fields analysis $\Delta t_{\rm av}$; global rms velocity $v_{\rm rms}$; convective turnover timescale $\tau_{\rm conv}$; average ratio of turbulent ram pressure and gas pressure p_{turb}/p_{gas}; total luminosity of the hydrodynamic model L; total rate of kinetic energy dissipation L_d; dissipation length-scale l_d; turbulent kinetic energy dissipation time-scale τ_d; radial turbulent kinetic energy dissipation time-scale τ_{dr}; horizontal turbulent kinetic energy dissipation time-scale τ_{dh}. The numerical values may vary in time up to 20% due to limited amount of data for averaging out the time dependence.

13.2 Snapshots of turbulent kinetic energy in a meridional plane

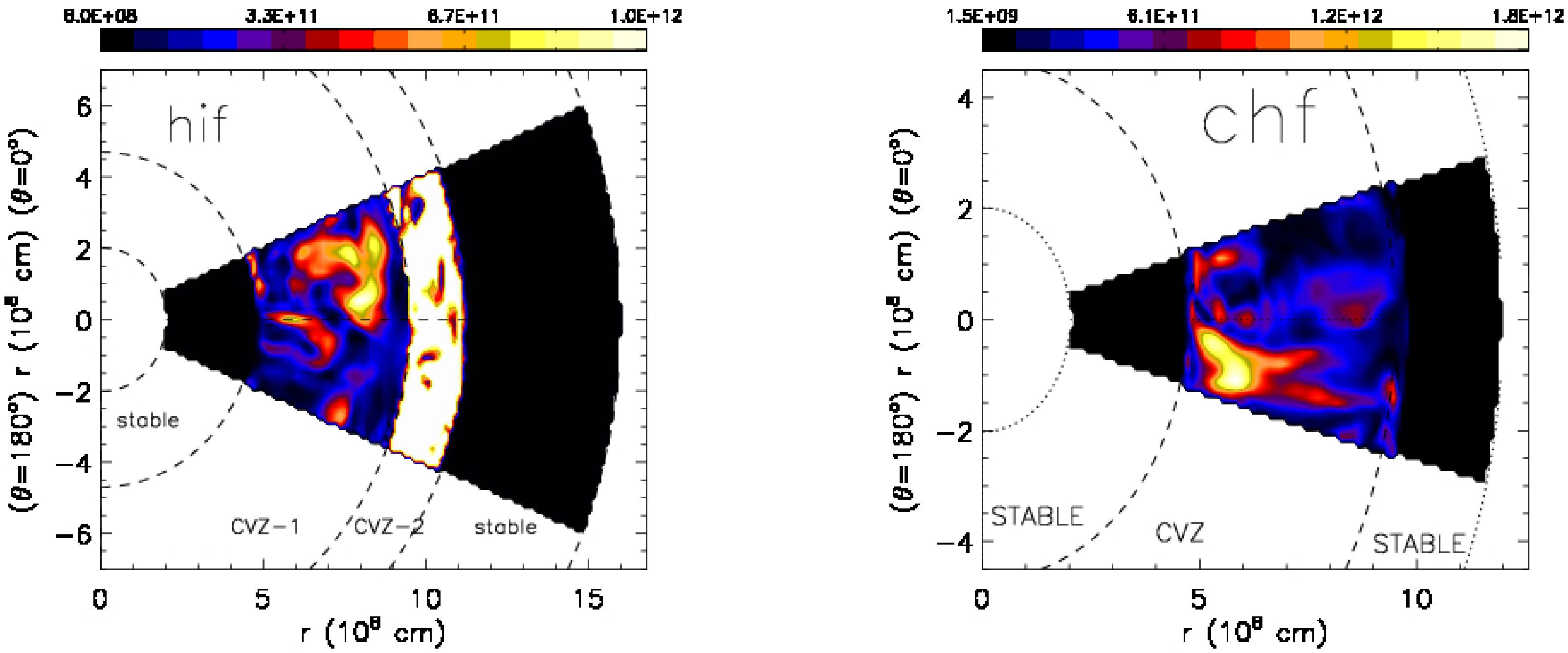

Figure 90: Snapshots of turbulent kinetic energy (in erg g^{-1}) in a meridional plane.

13.3 Background structure of our models

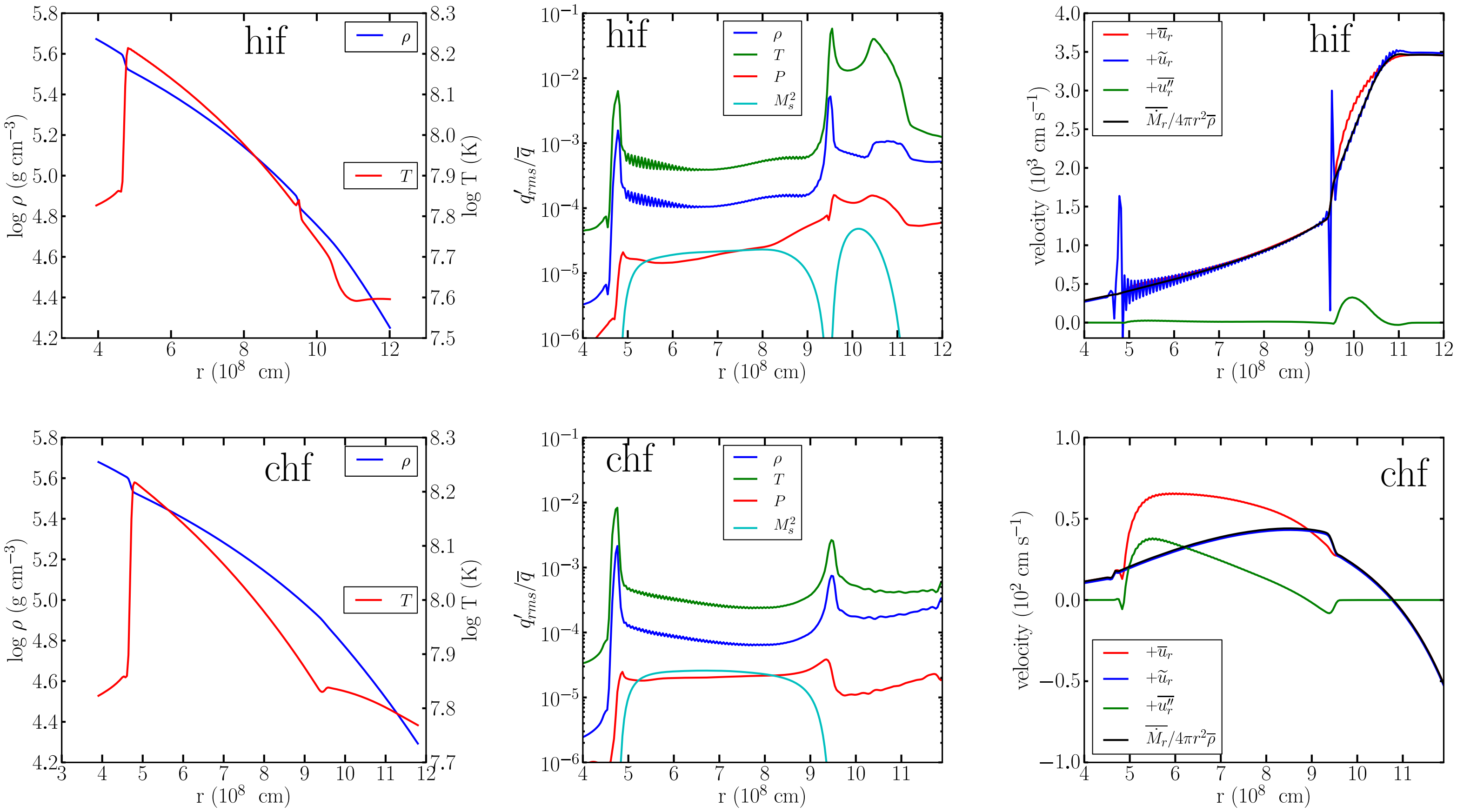

Figure 91: Properties of our data. Model hif.3D (upper panels) and model chf.3D (lower panels).

14 Profiles and integral budgets of mean fields

14.1 Mean continuity equation

$$\widetilde{D}_t \overline{\rho} = - \overline{\rho} \widetilde{d} + \mathcal{N}_\rho \tag{70}$$

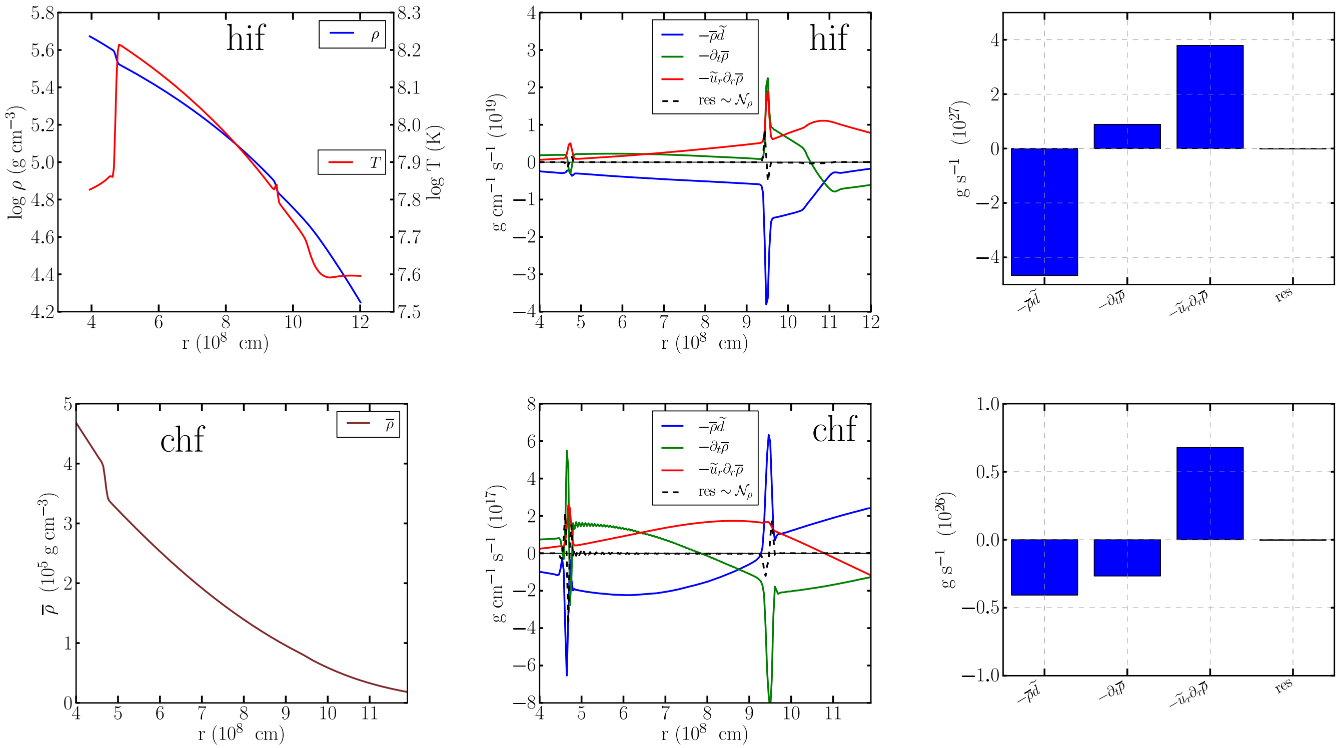

Figure 92: Mean continuity equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.2 Mean radial momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G^M_r} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} \quad (71)$$

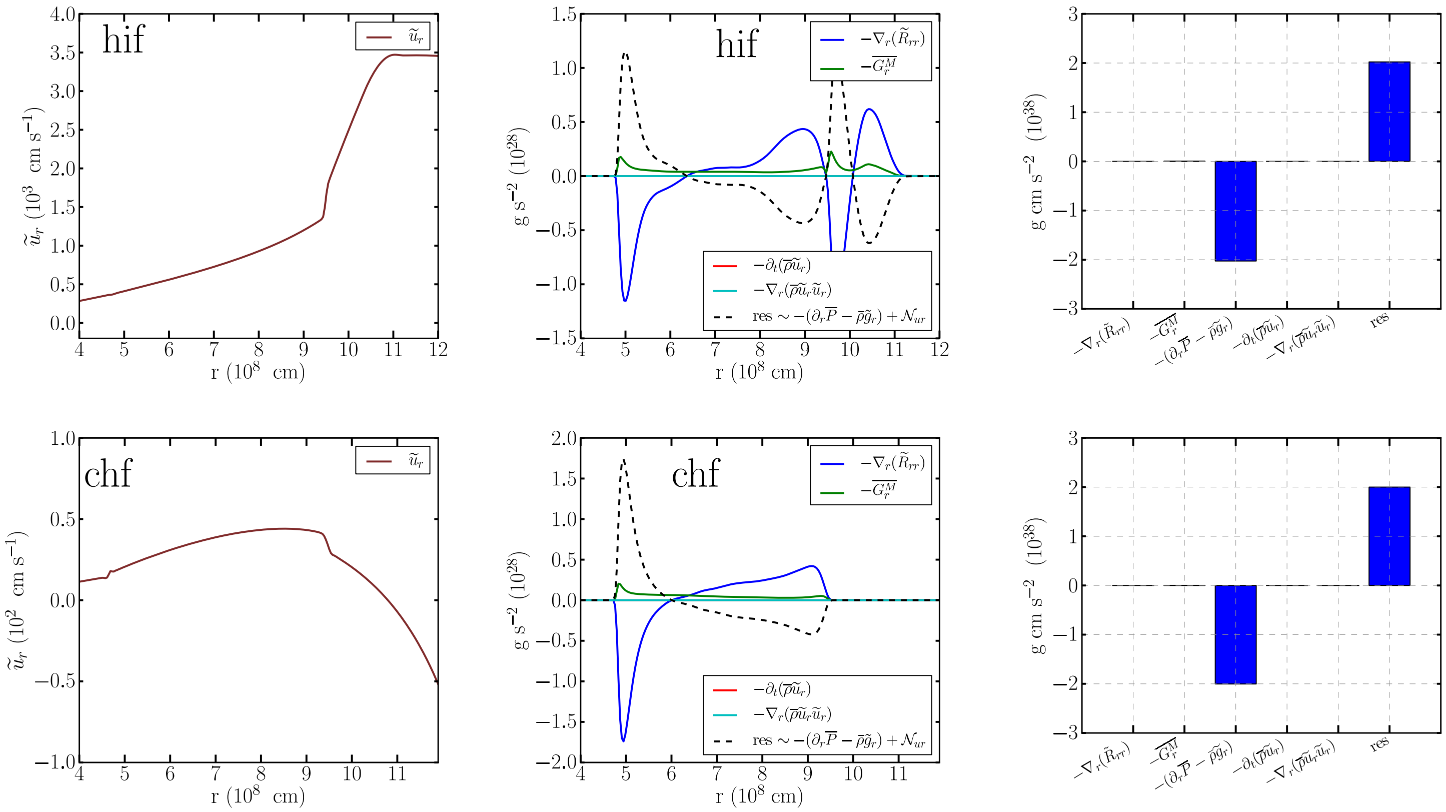

Figure 93: Mean radial momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.3 Mean azimuthal momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G_\theta^M} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} \tag{72}$$

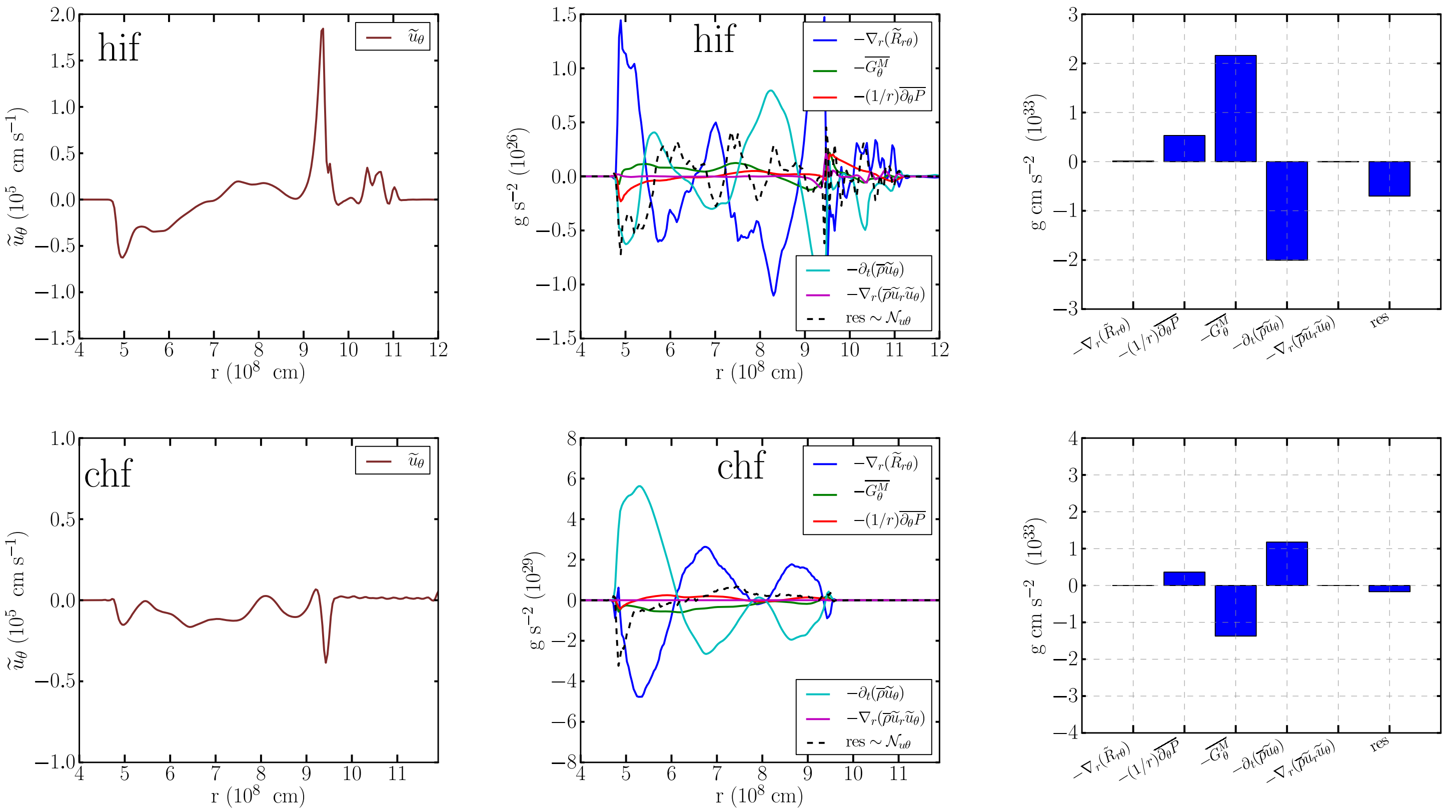

Figure 94: Mean azimuthal momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.4 Mean polar momentum equation

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G^M_\phi} + \mathcal{N}_{u\phi} \tag{73}$$

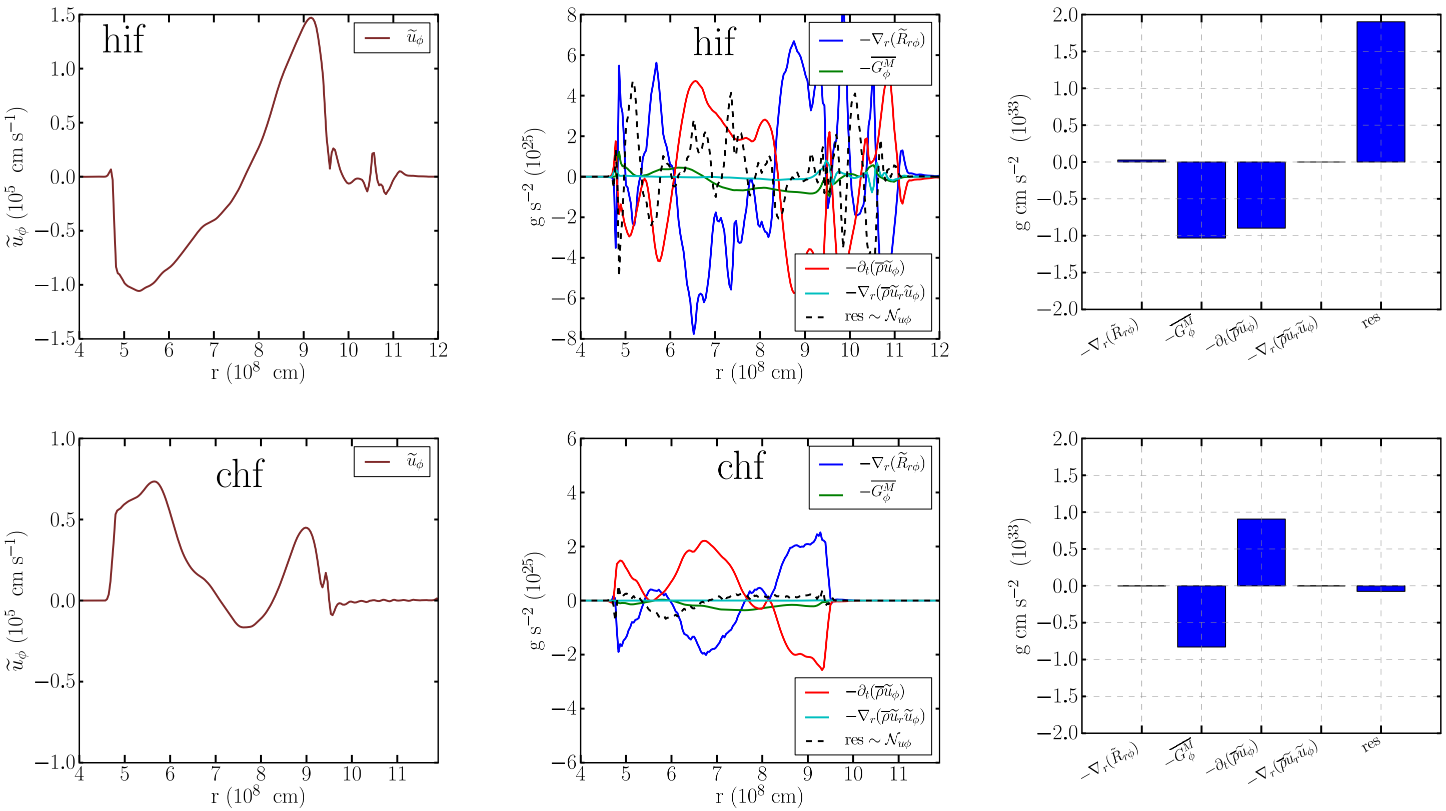

Figure 95: Mean polar momentum equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.5 Mean internal energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \tag{74}$$

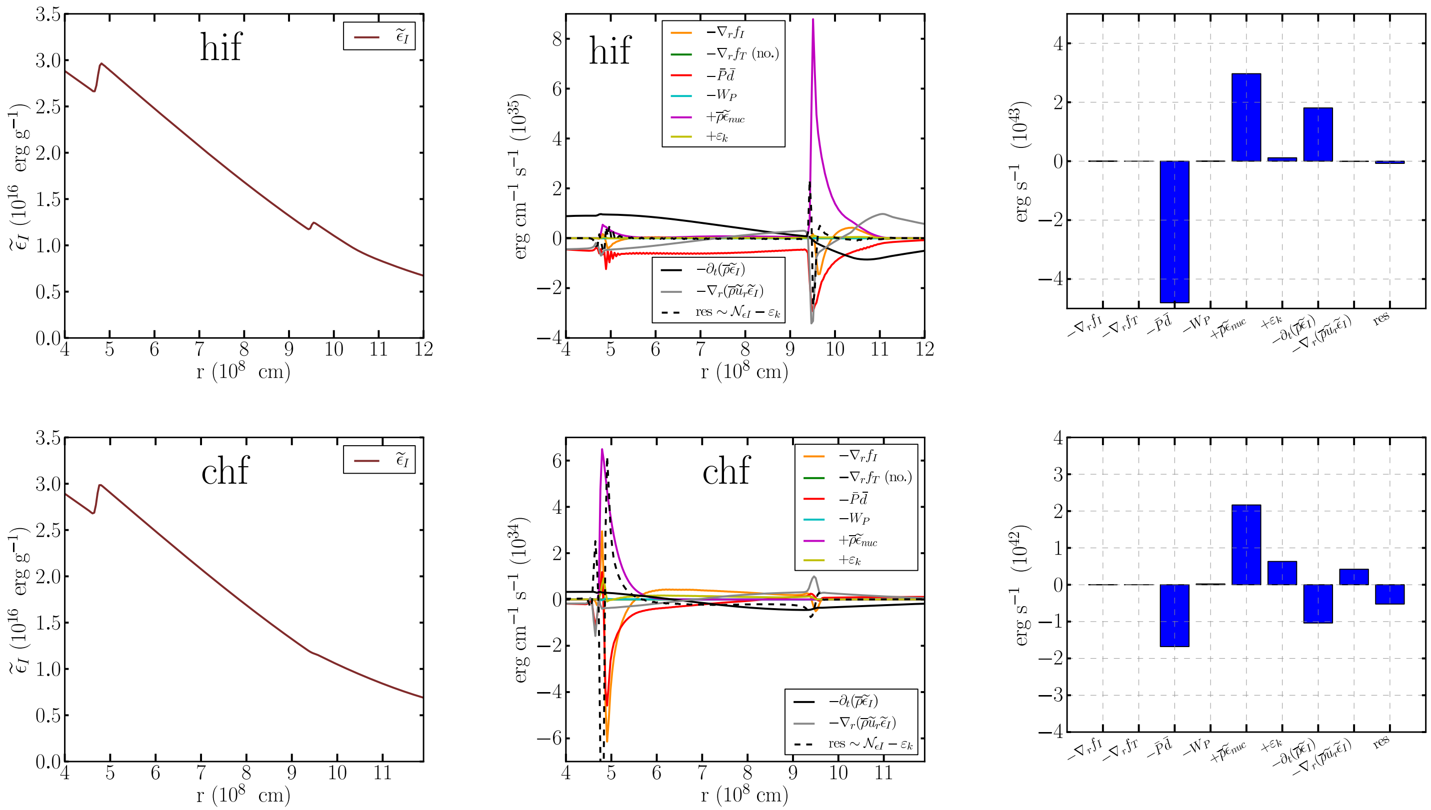

Figure 96: Mean internal energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.6 Mean kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_k = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} \tag{75}$$

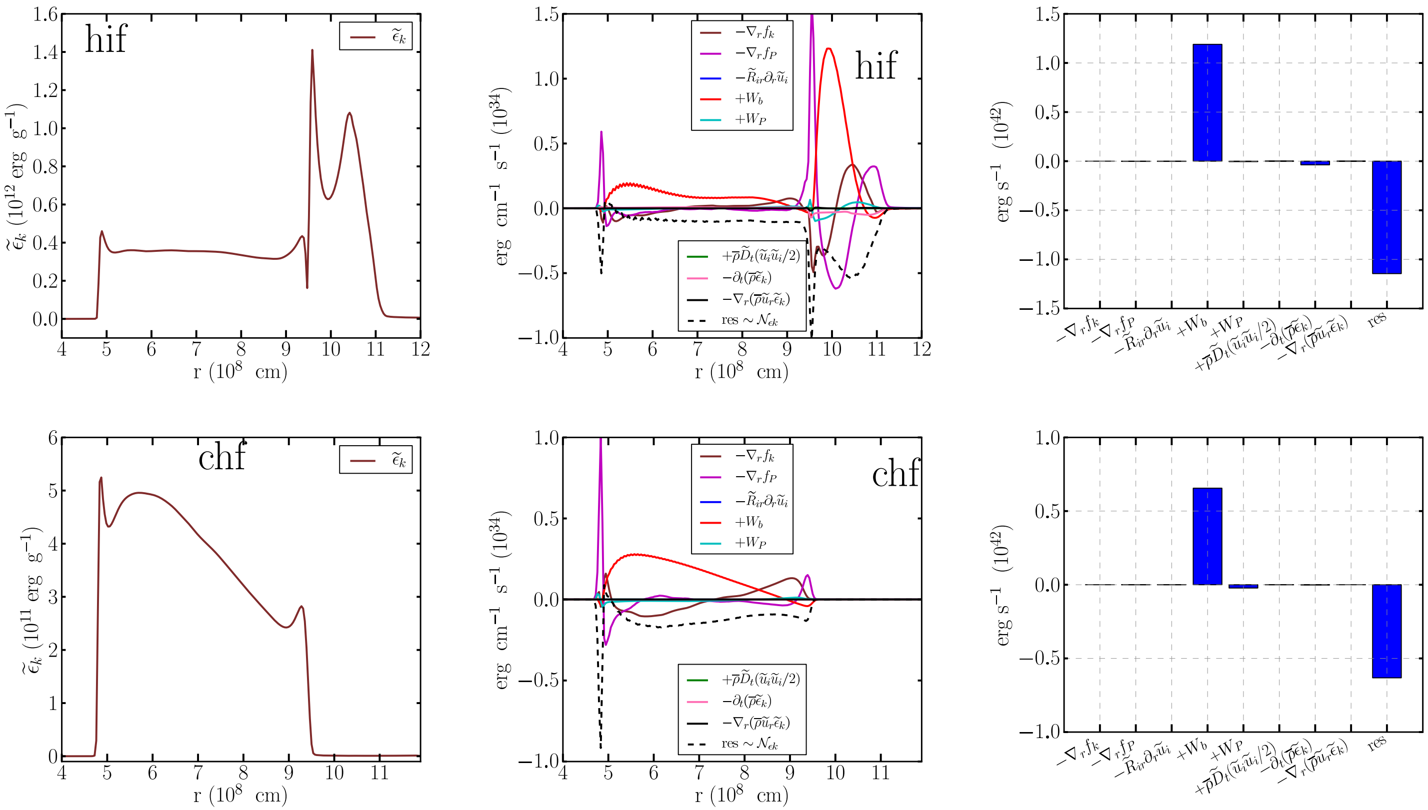

Figure 97: Mean kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.7 Mean total energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i - \overline{P}\ \overline{d} + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} \tag{76}$$

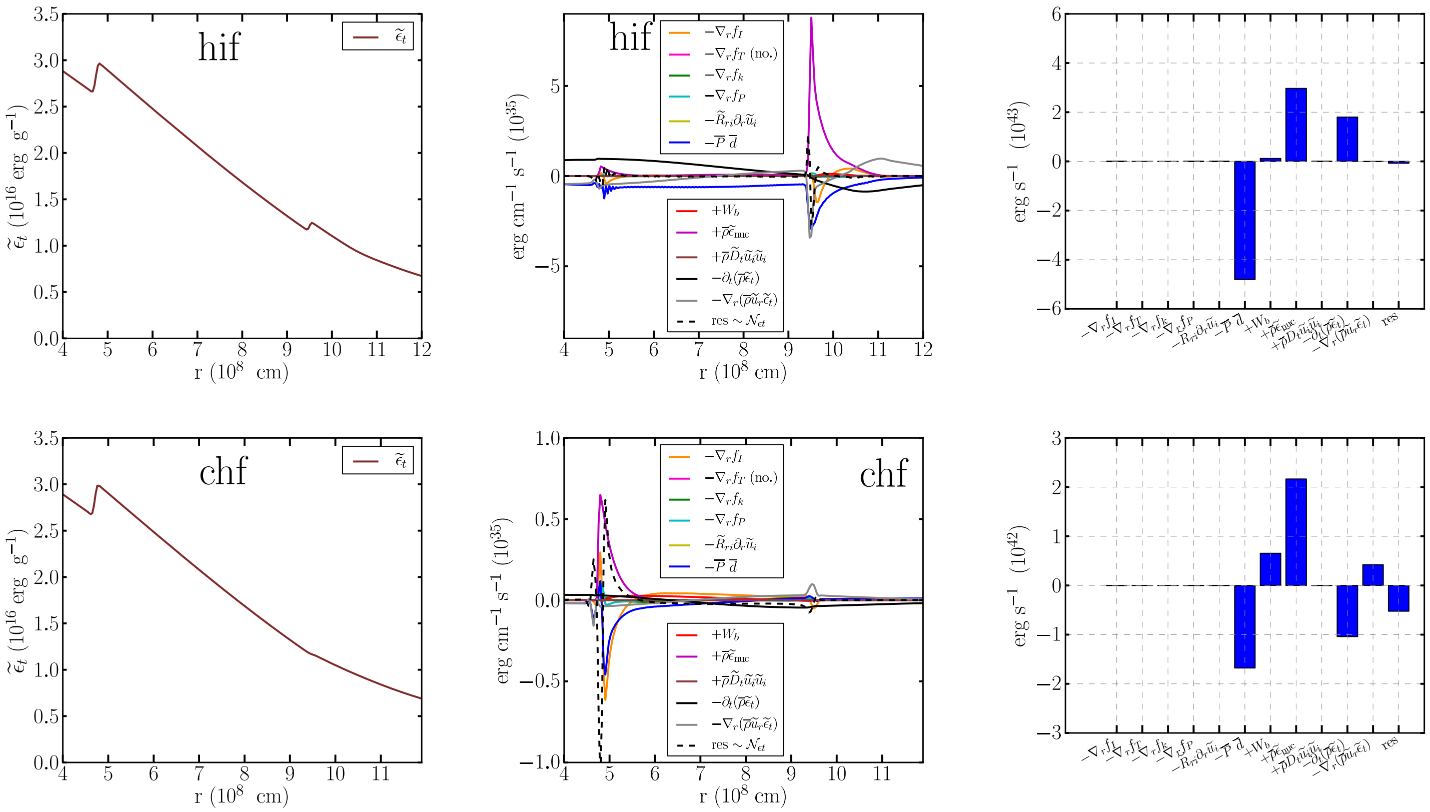

Figure 98: Mean total energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.8 Mean entropy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s - \overline{(\nabla \cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \tag{77}$$

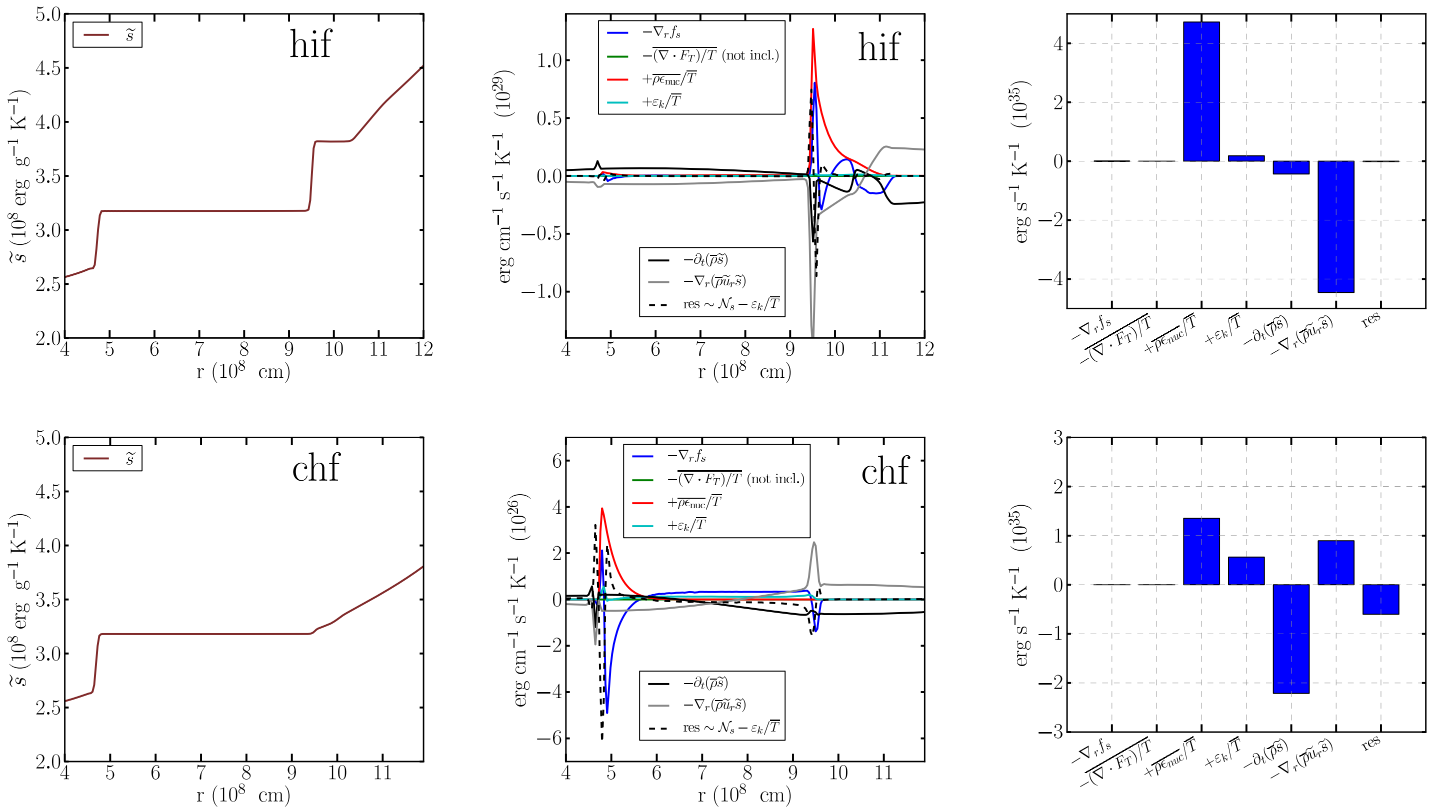

Figure 99: Mean entropy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.9 Mean pressure equation

$$\overline{D}_t \overline{P} = -\nabla_r f_P - \Gamma_1 \overline{P}\ \overline{d} + (1-\Gamma_1) W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P \tag{78}$$

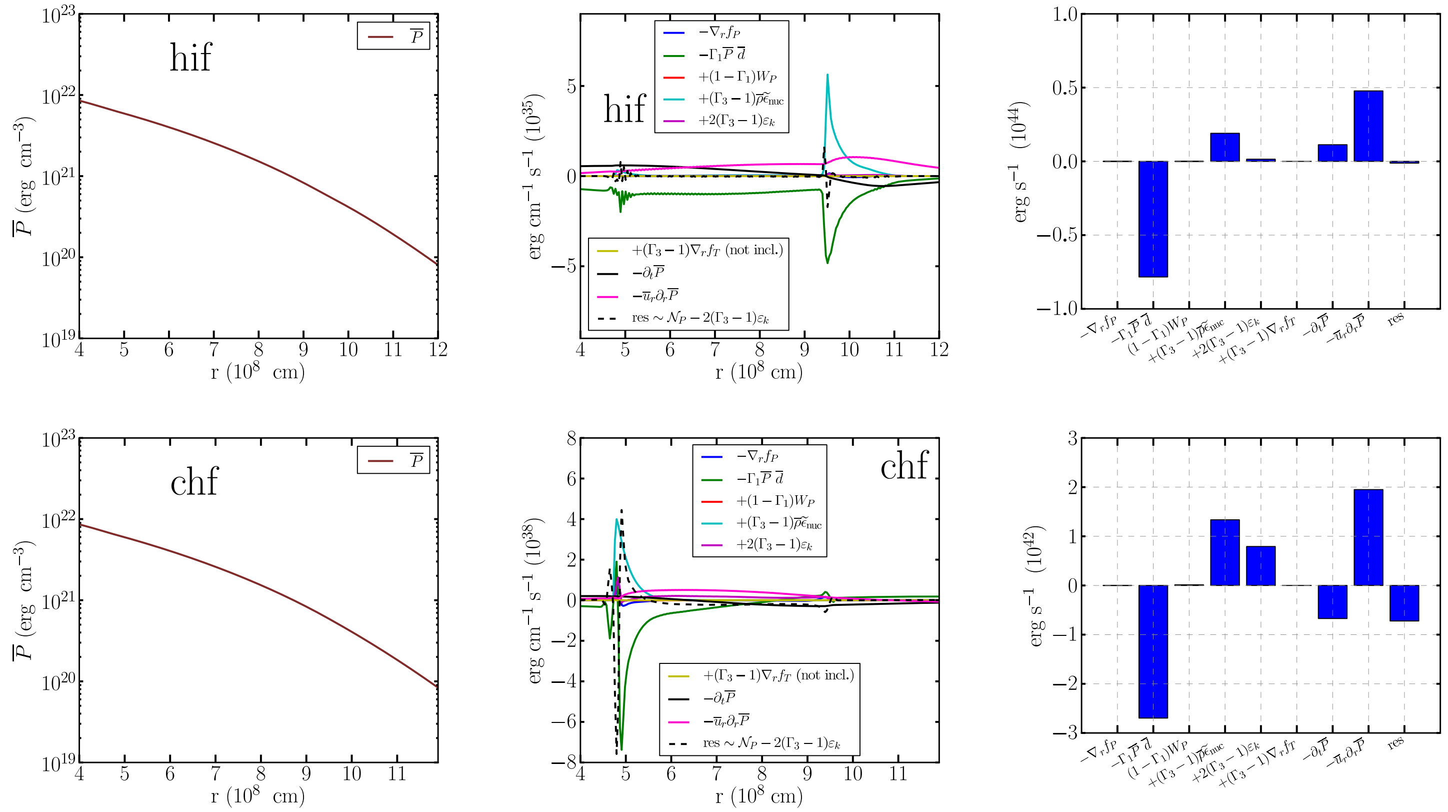

Figure 100: Mean pressure equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.10 Mean enthalpy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r f_h - \Gamma_1 \overline{P}\,\overline{d} - \Gamma_1 W_P + \Gamma_3 \mathcal{S} + \Gamma_3 \nabla_r f_T + \mathcal{N}_h \tag{79}$$

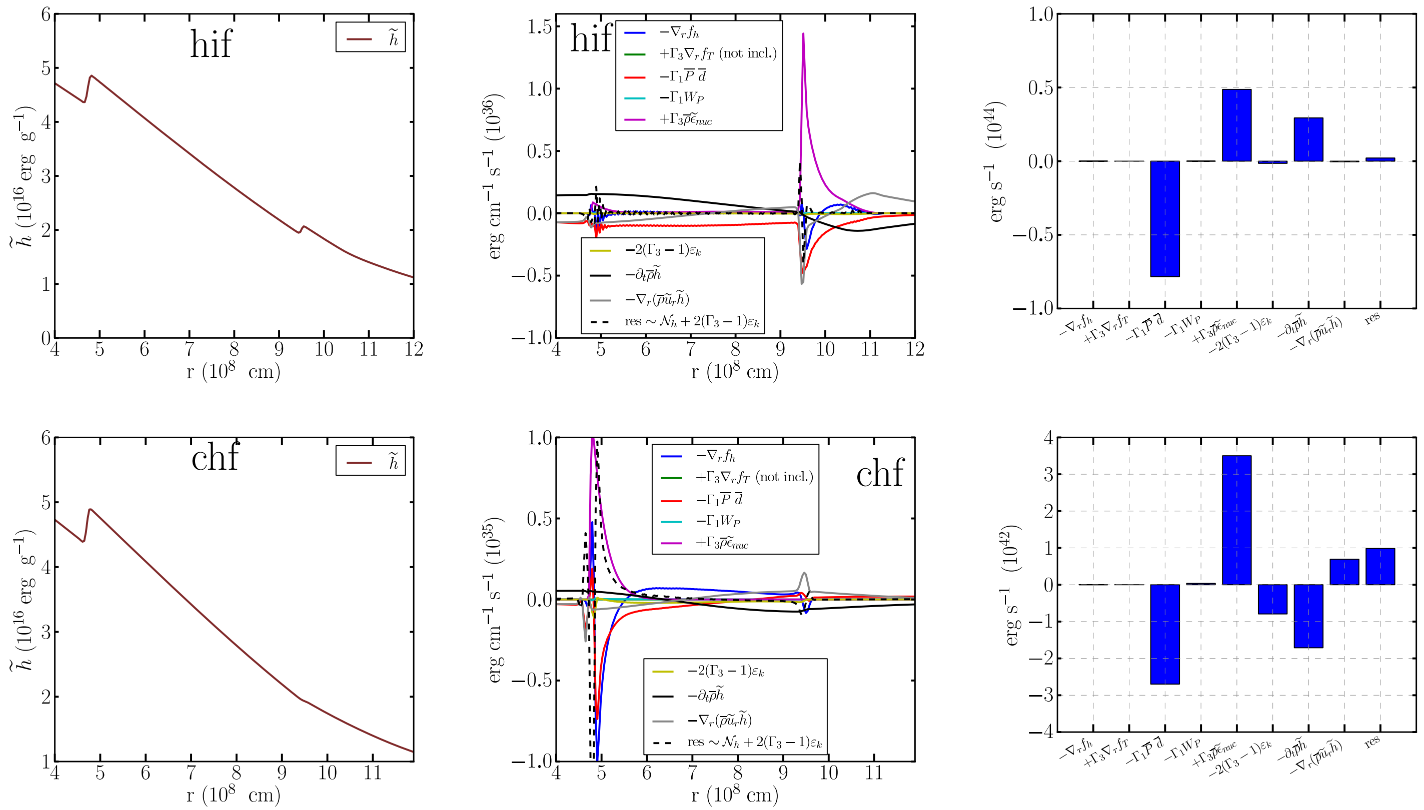

Figure 101: Mean enthalpy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.11 Mean angular momentum equation (z-component)

$$\overline{\rho}\widetilde{D}_t\widetilde{j}_z = -\nabla_r f_{jz} + \mathcal{N}_{jz} \tag{80}$$

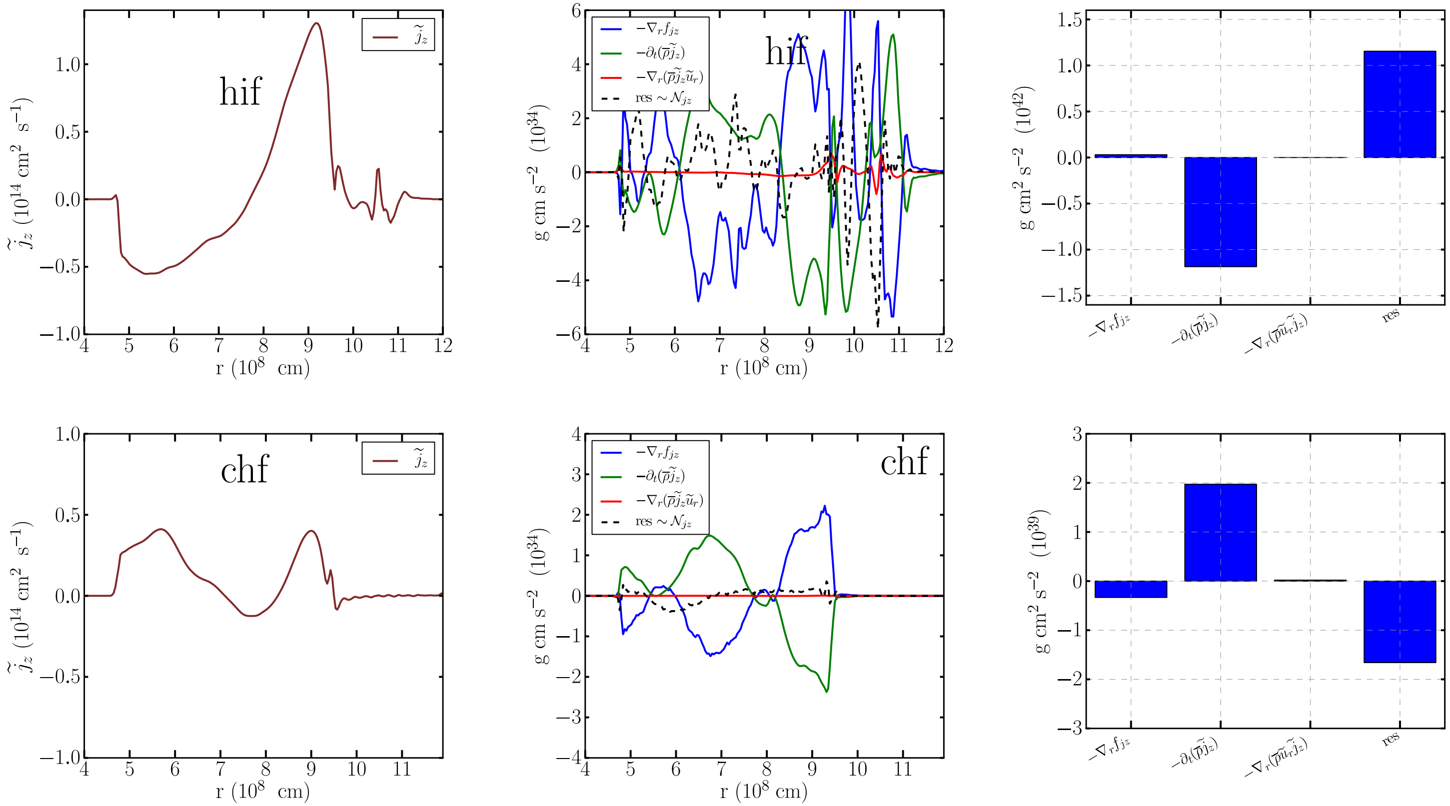

Figure 102: Mean angular momentum equation (z-component). Model hif.3D (upper panels) and model chf.3D (lower panels)

14.12 Mean composition equations

$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\,\mathrm{nuc}} + \mathcal{N}_\alpha \quad (81)$$

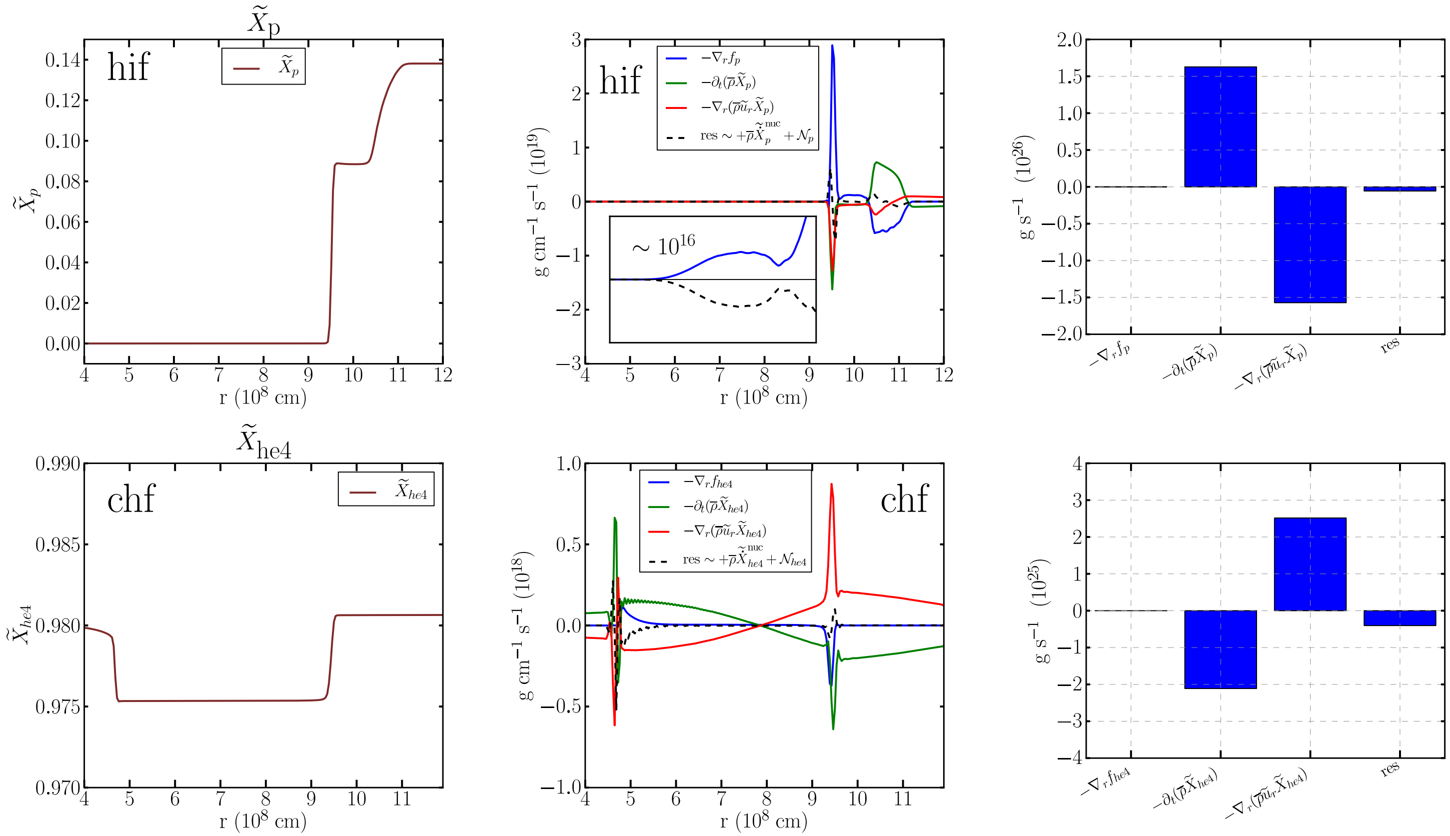

Figure 103: Mean composition equations. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.13 Mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = - \nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \tag{82}$$

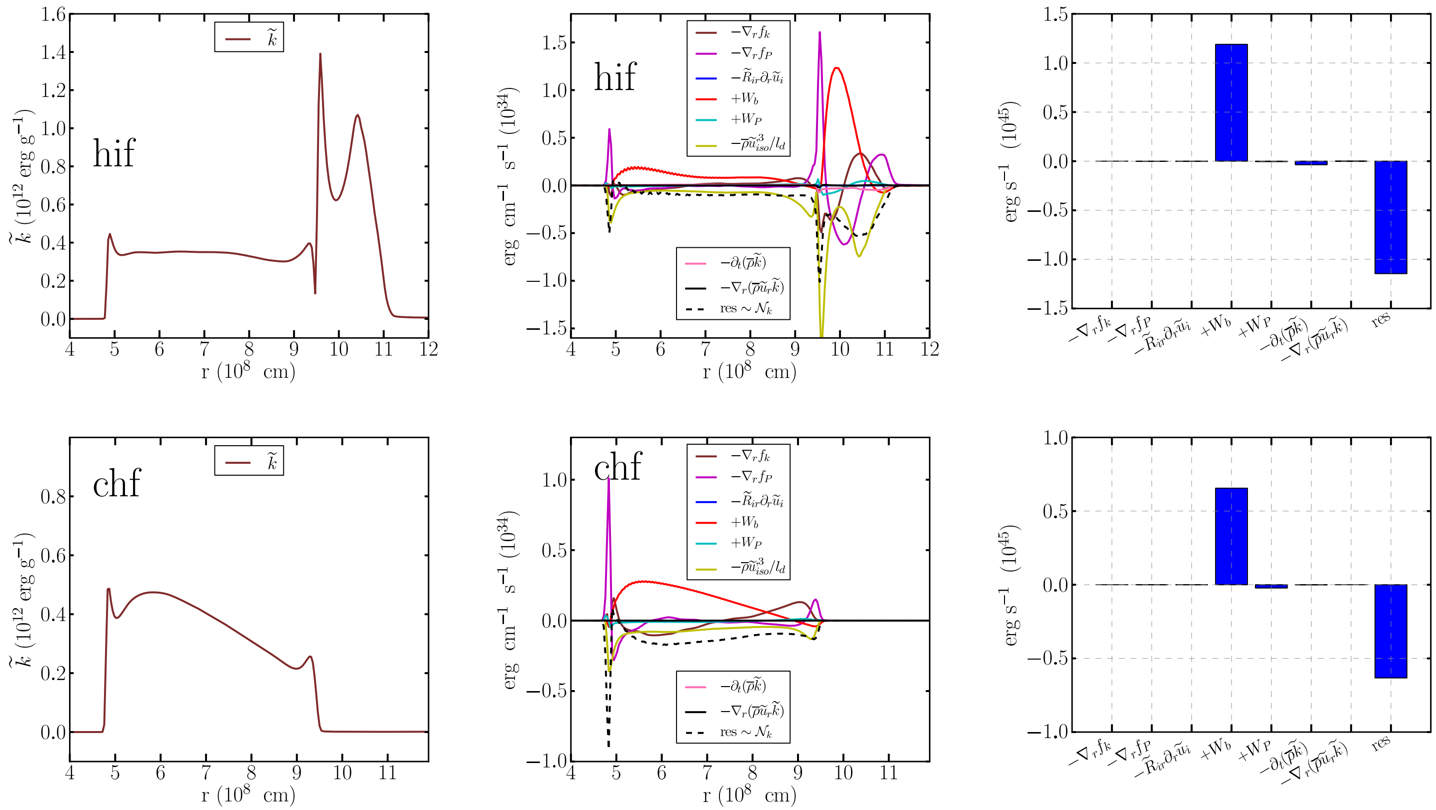

Figure 104: Mean turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.14 Radial part of mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^r = -\nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} \tag{83}$$

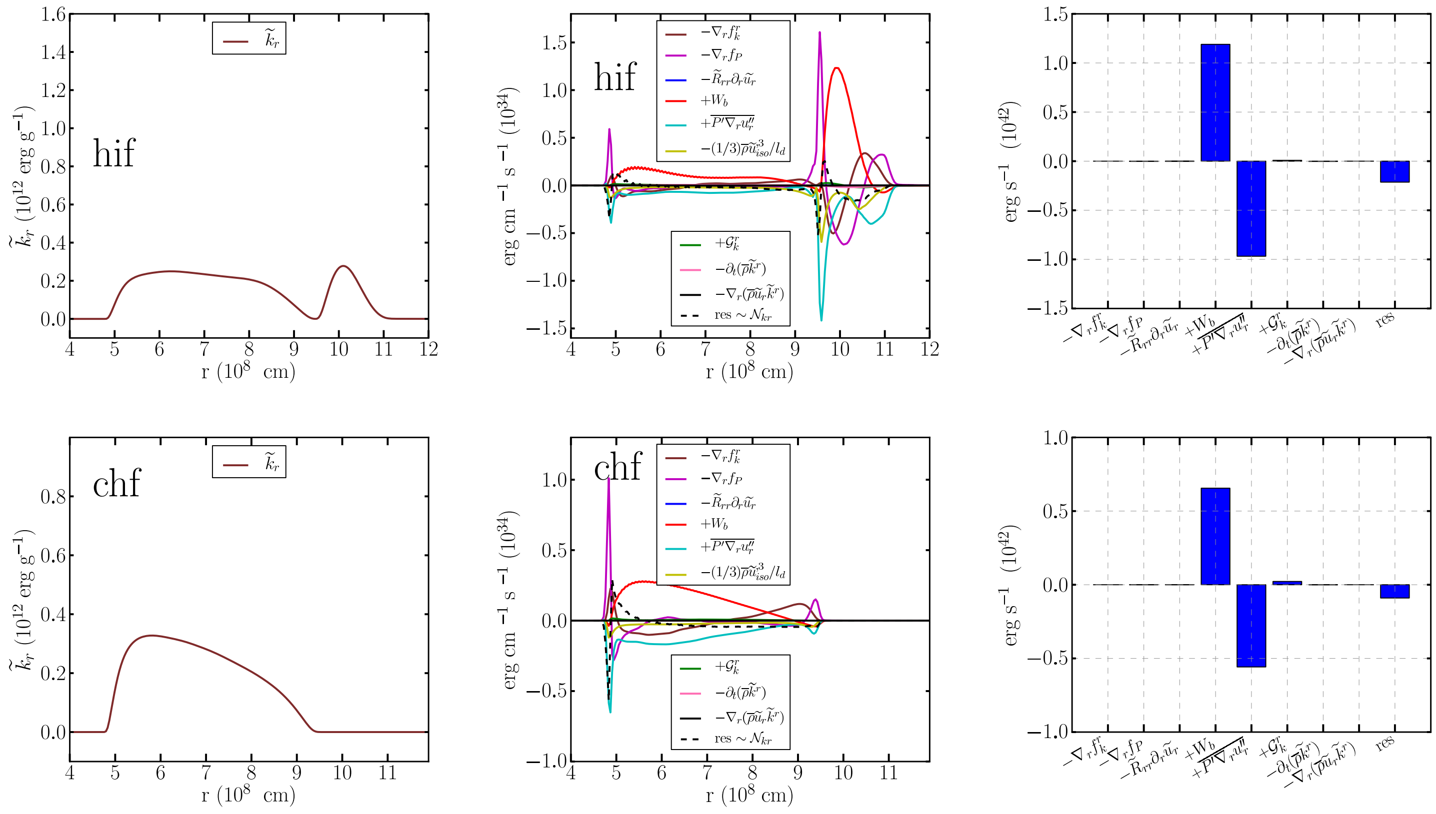

Figure 105: Radial turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

14.15 Horizontal part of mean turbulent kinetic energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{k}^h = -\nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} \quad (84)$$

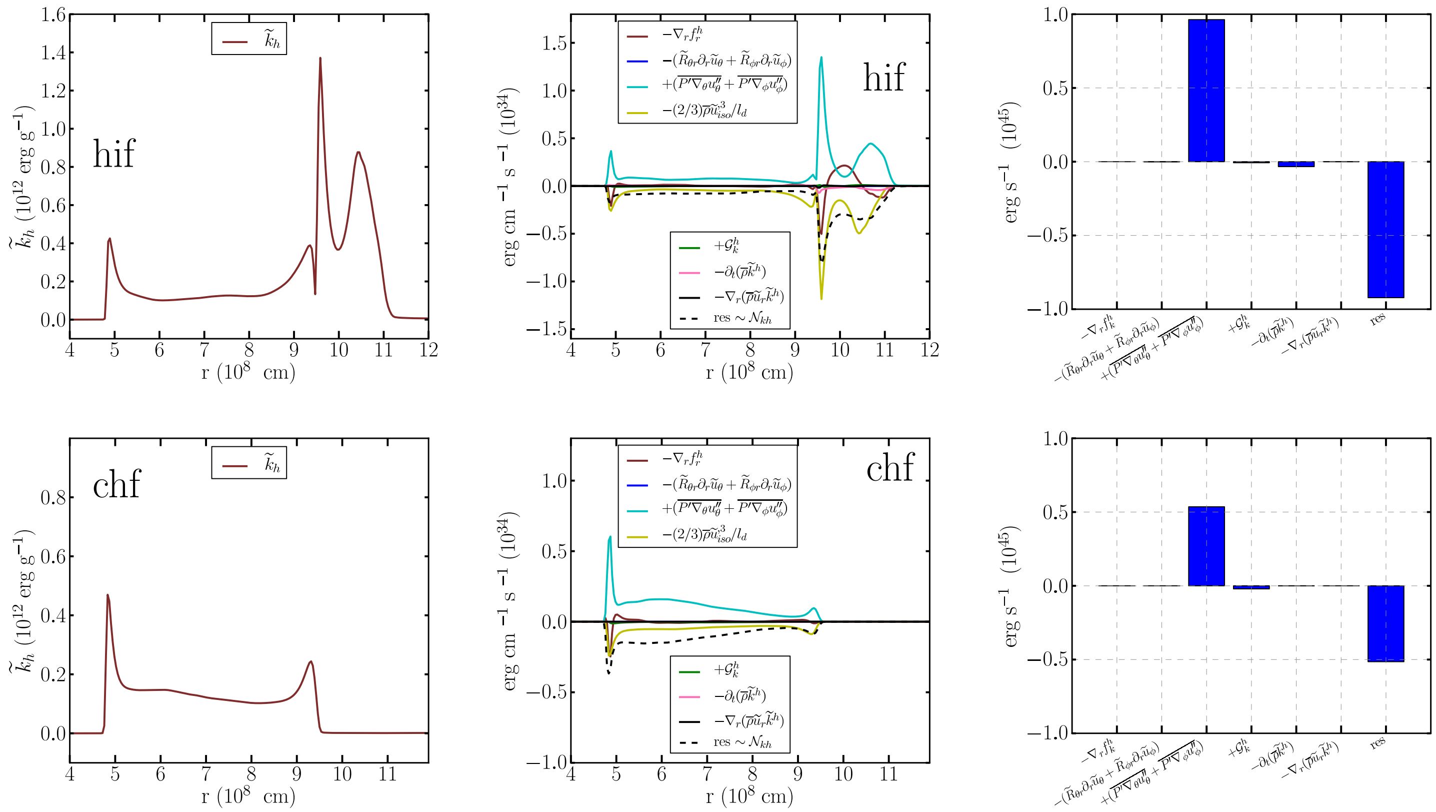

Figure 106: Horizontal turbulent kinetic energy equation. Model hif.3D (upper panels) and model chf.3D (lower panels)

15 Mean field composition data for the hydrogen injection flash model

15.1 Mean H^1 and He^3 equation

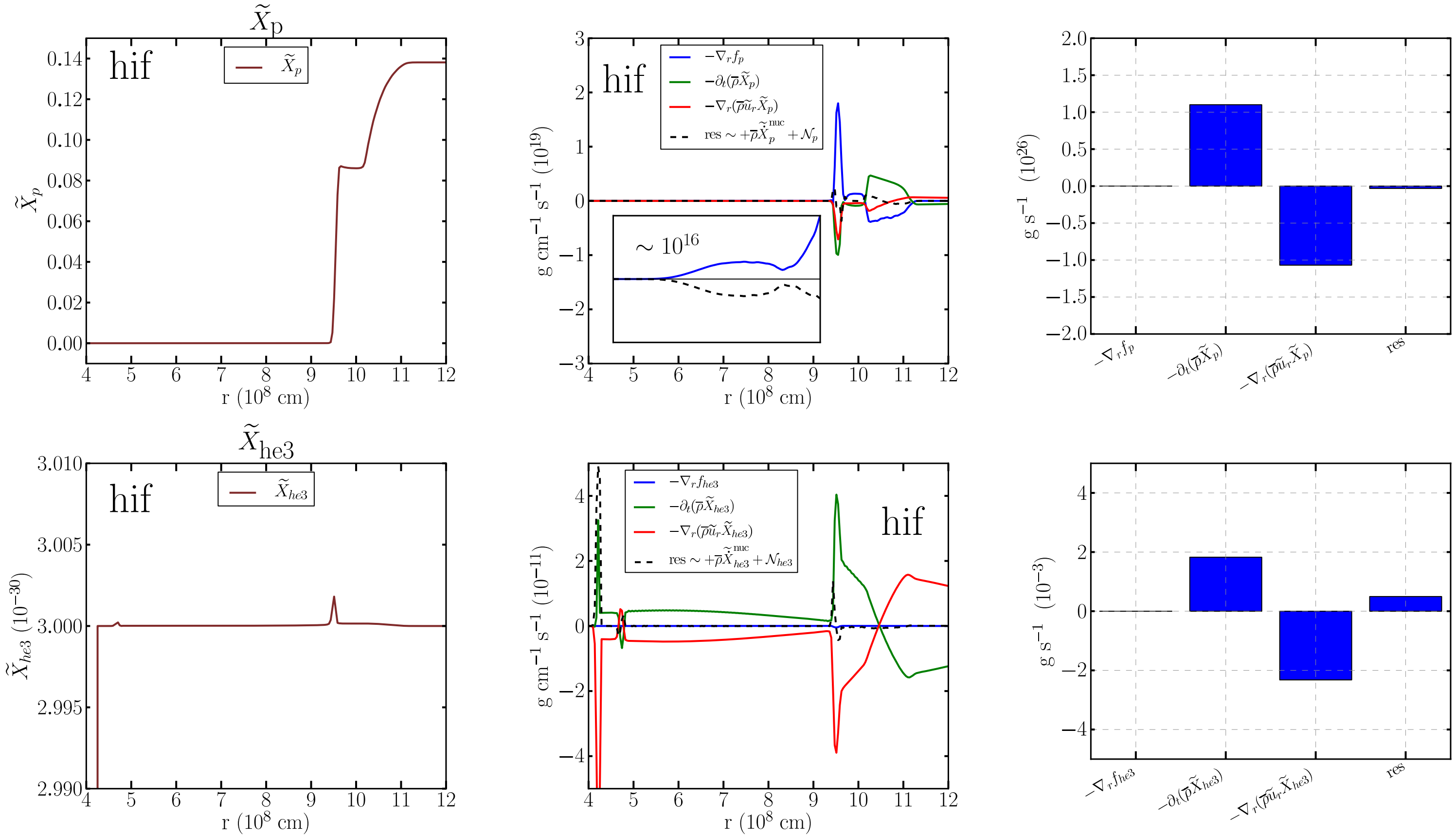

Figure 107: Mean composition equations for hif.3D.

15.2 Mean He^4 and C^{12} equation

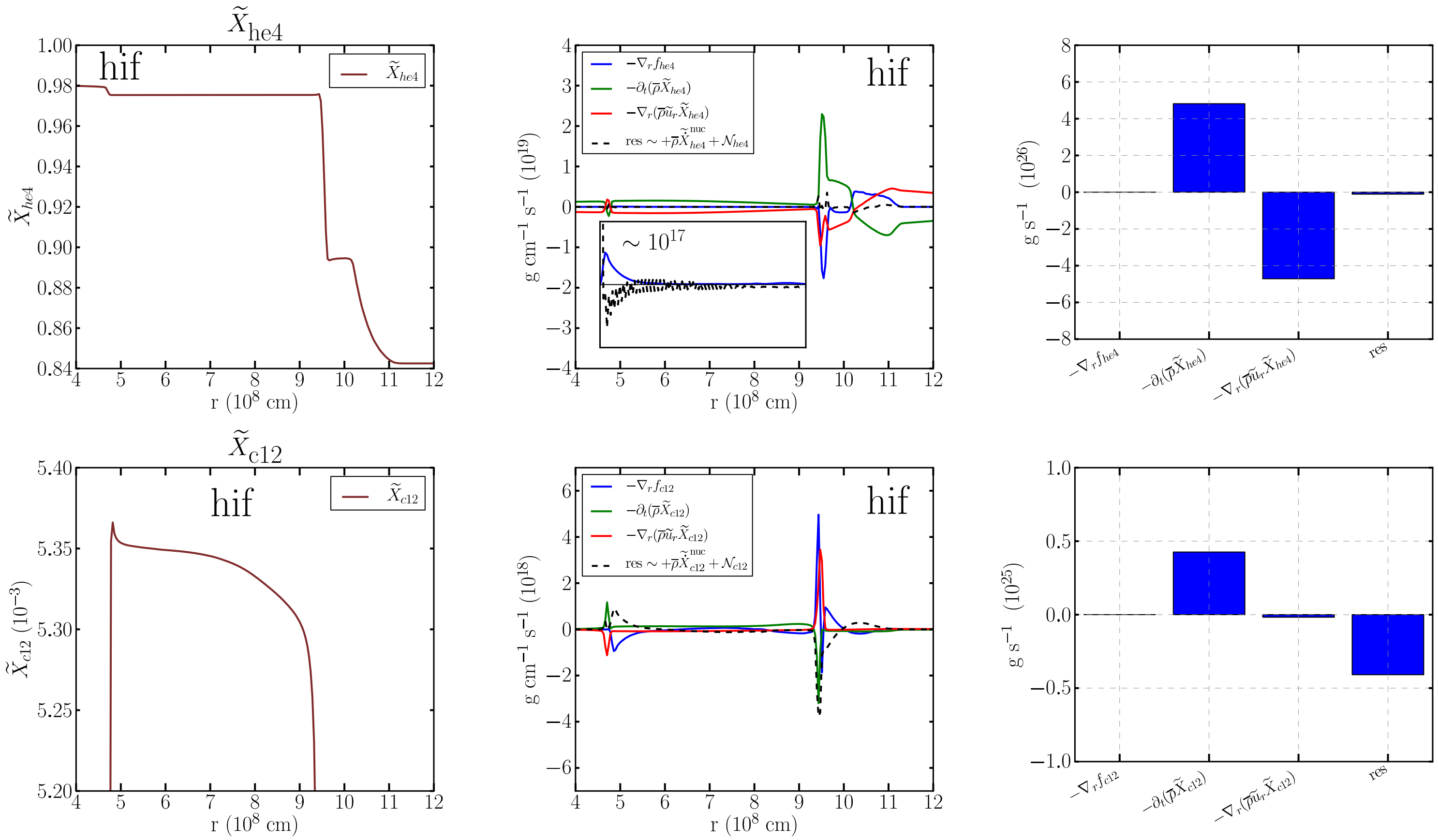

Figure 108: Mean composition equations for hif.3D.

15.3 Mean C^{13} and N^{13} equation

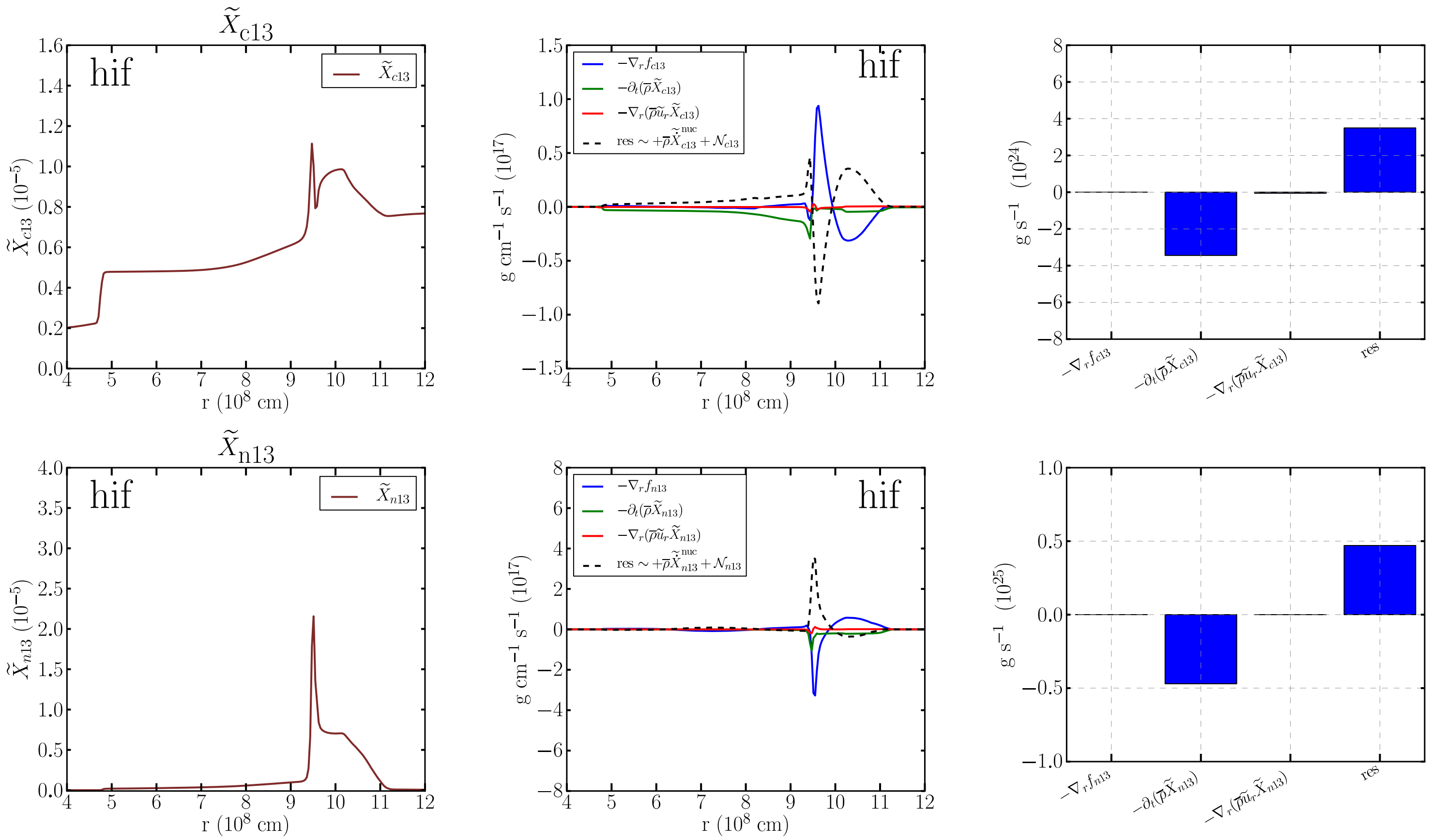

Figure 109: Mean composition equations for hif.3D.

15.4 Mean N^{14} and N^{15} equation

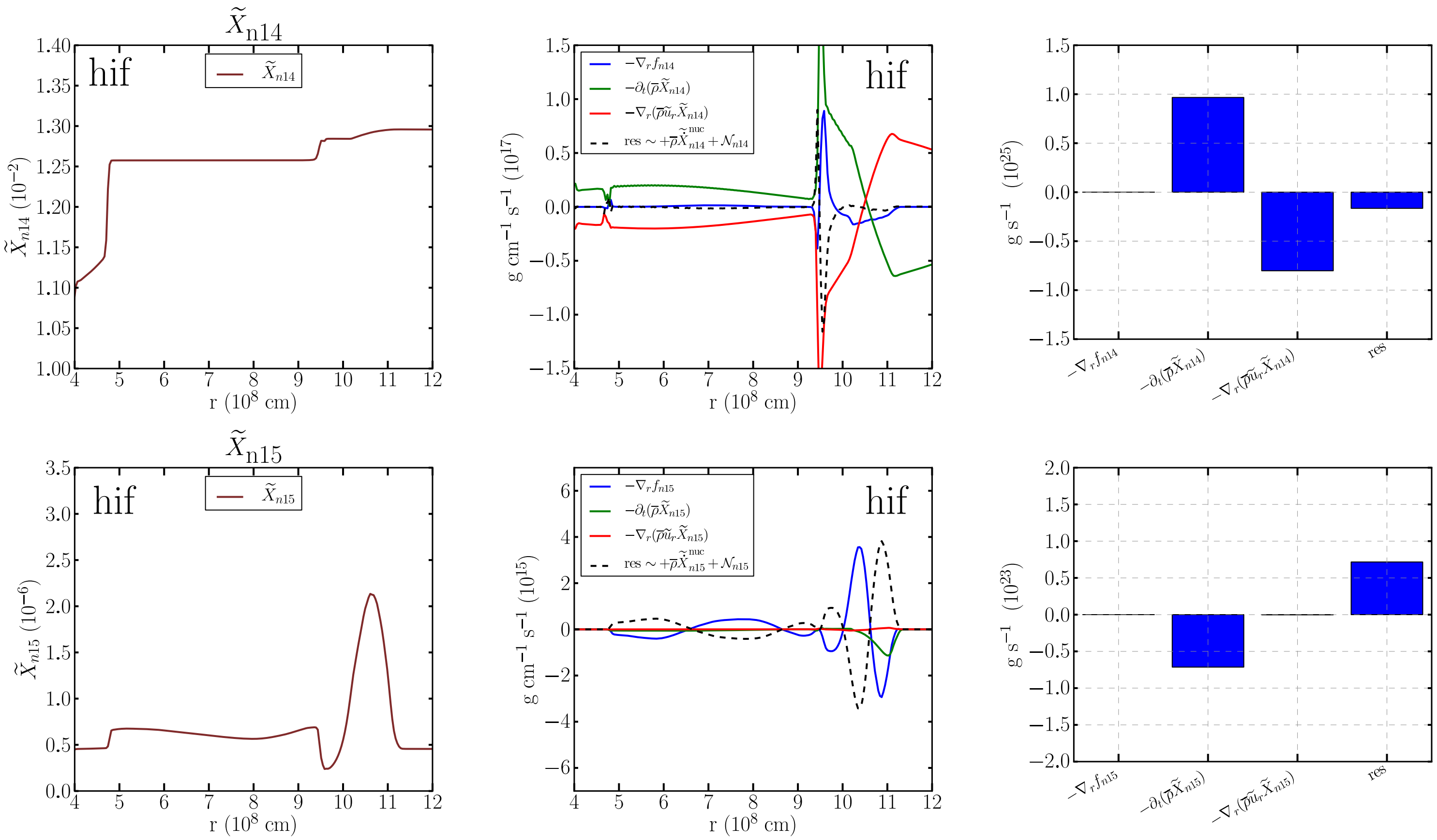

Figure 110: Mean composition equations for hif.3D.

15.5 Mean O^{15} and O^{16} equation

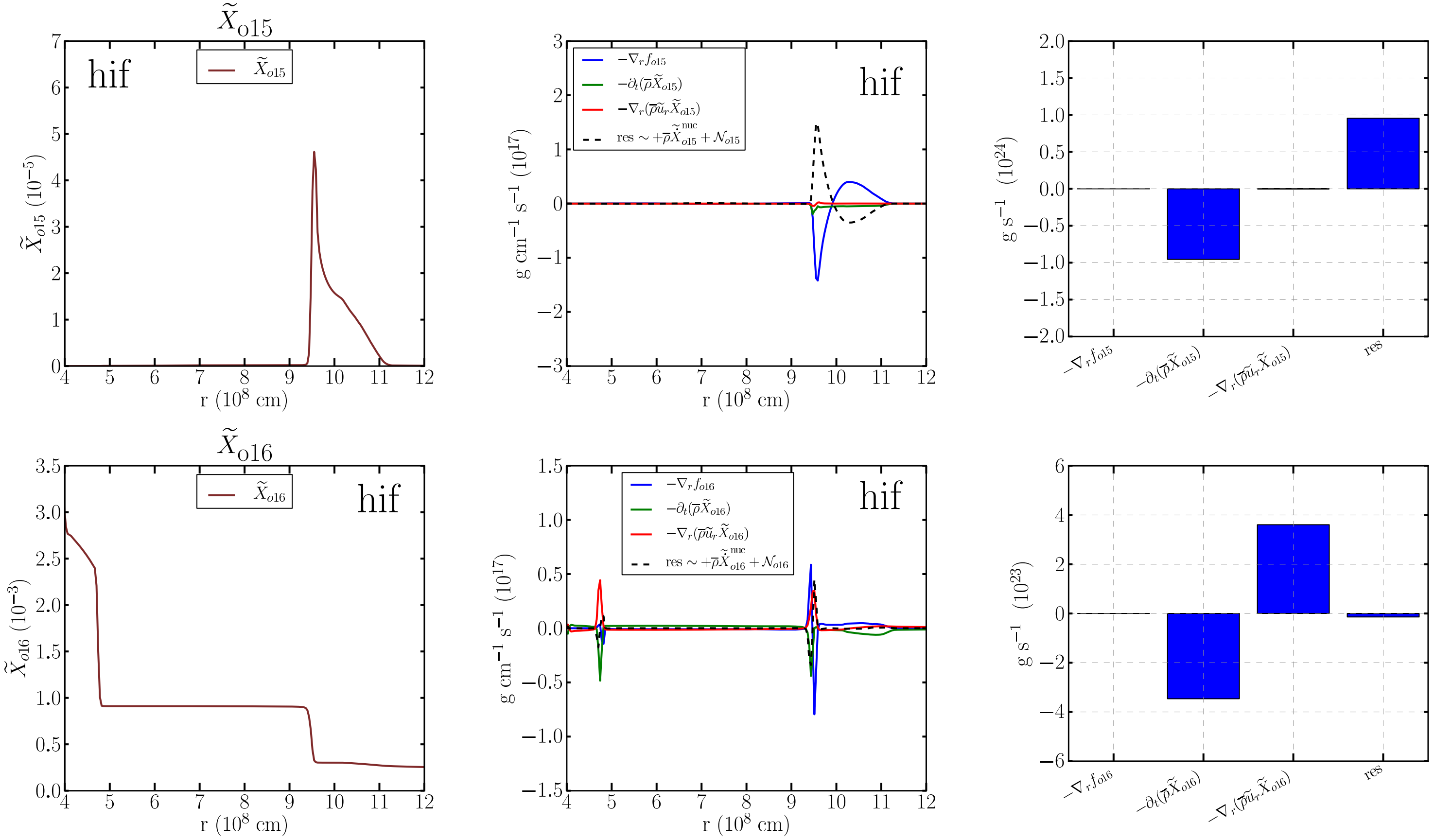

Figure 111: Mean composition equations for hif.3D.

15.6 Mean O^{17} and Ne^{20} equation

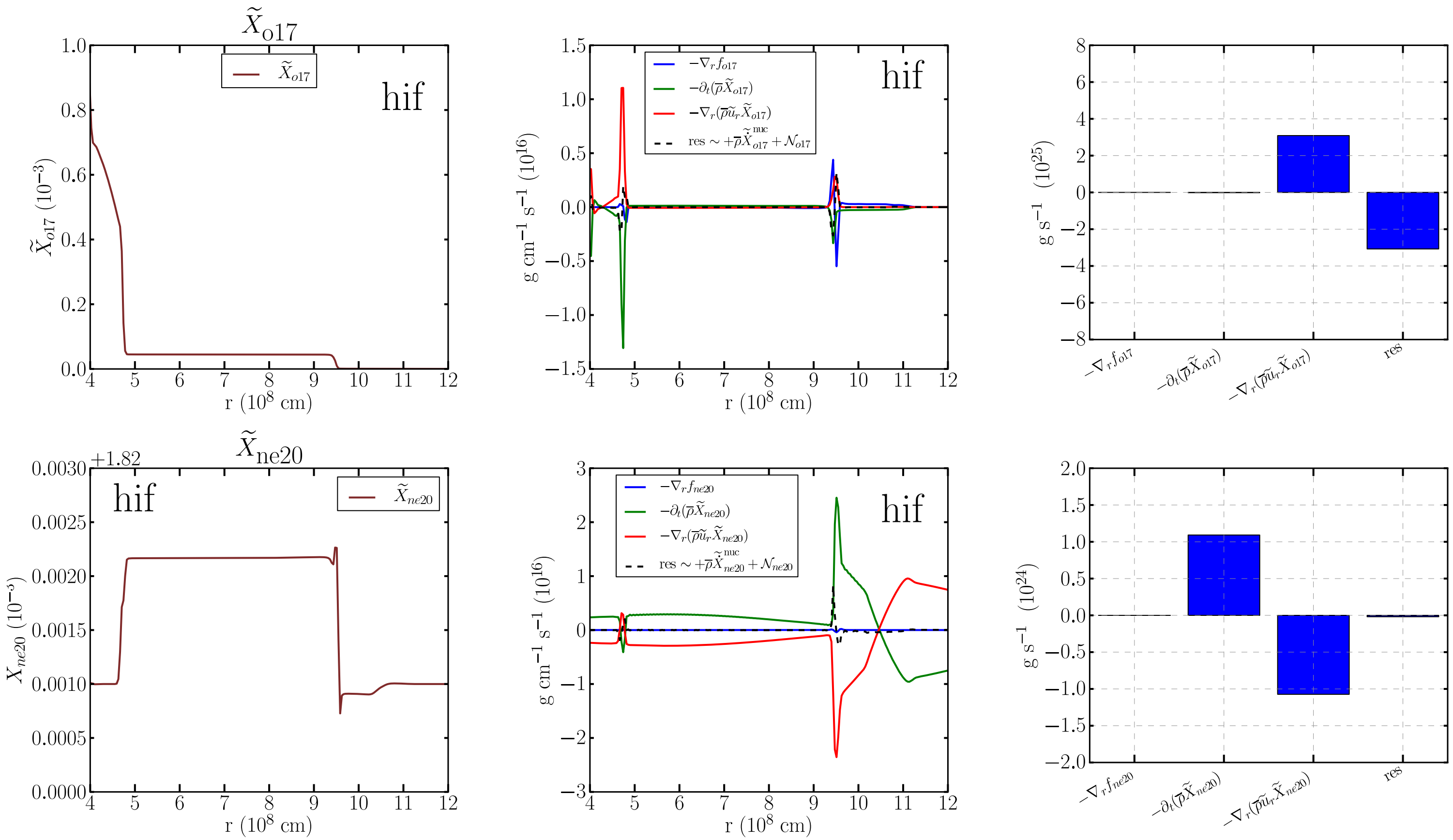

Figure 112: Mean composition equations for hif.3D.

15.7 Mean Mg^{24} and Si^{28} equation

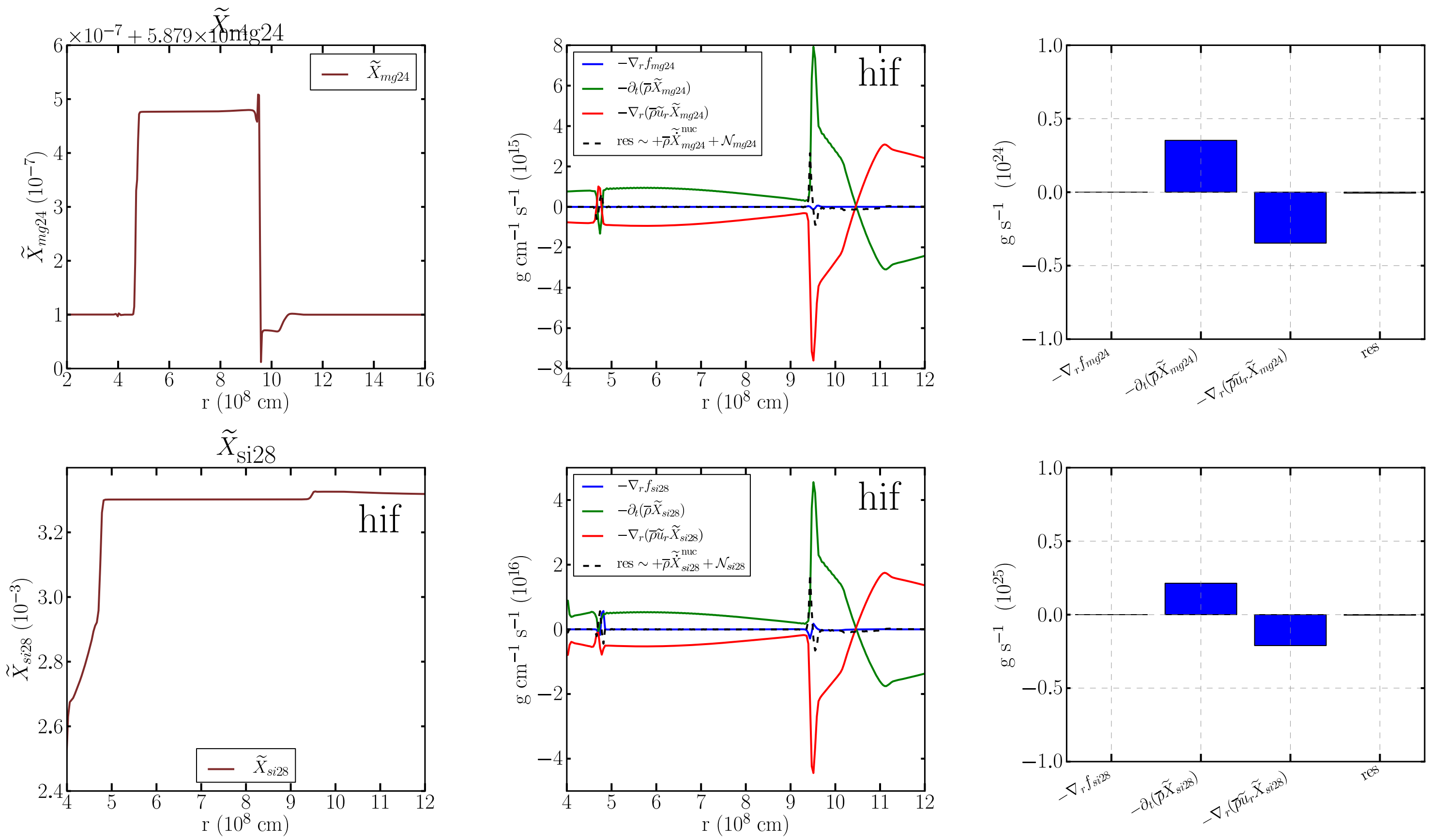

Figure 113: Mean composition equations for hif.3D

16 Mean field composition data for the core helium flash model

16.1 Mean He^4 and C^{12} equation

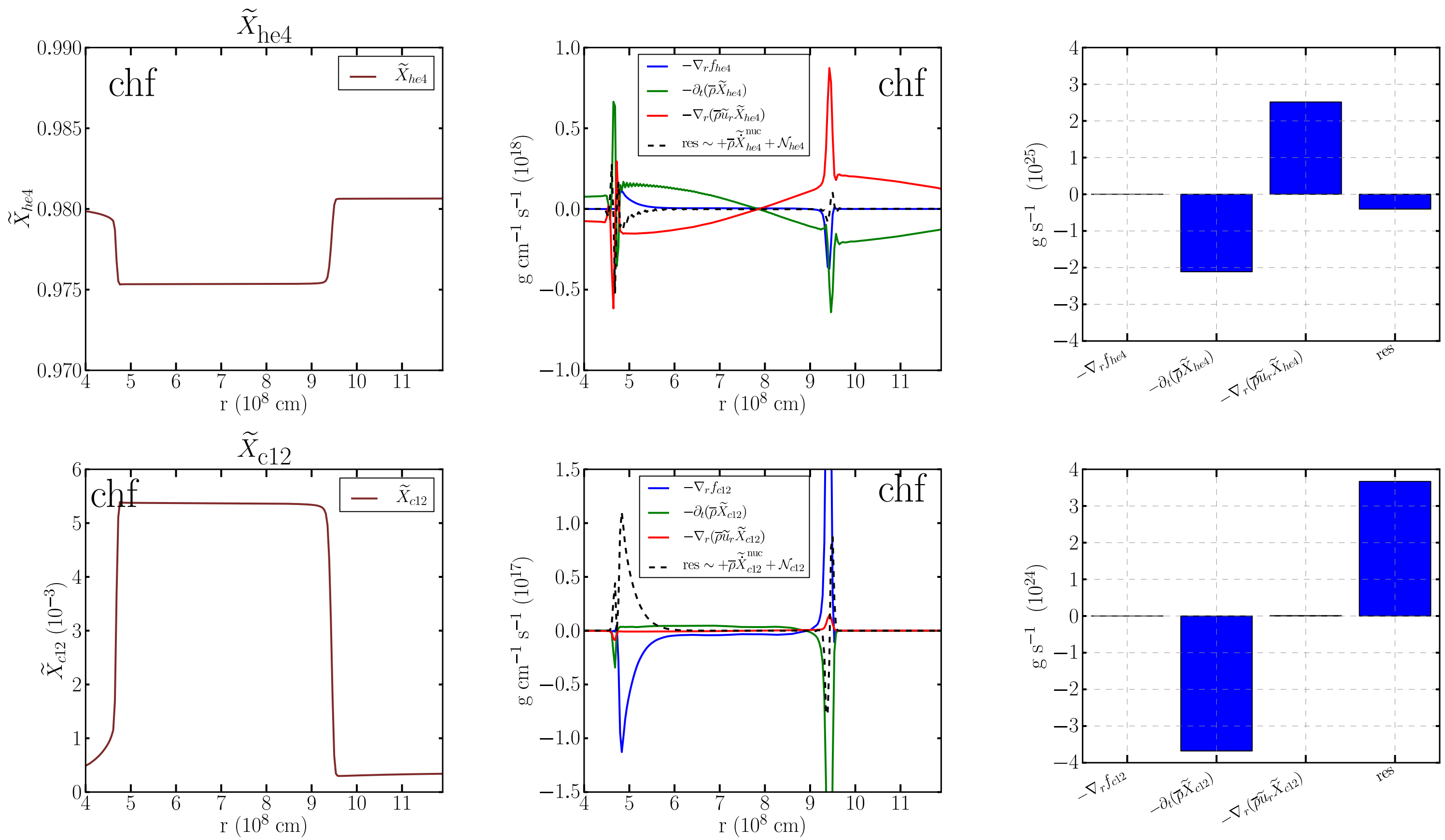

Figure 114: Mean composition equations for chf.3D.

16.2 Mean O^{16} and Ne^{20} equation

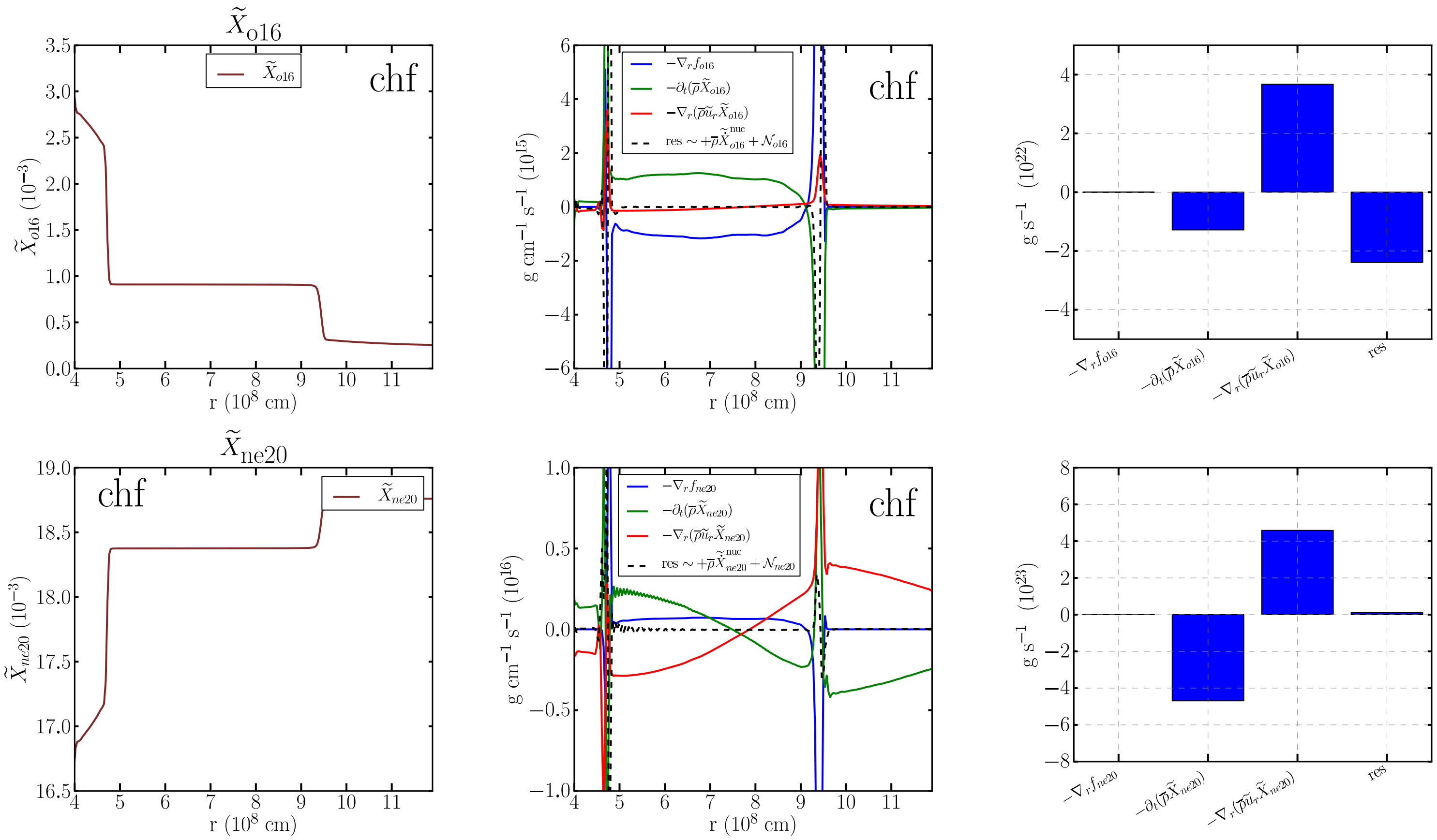

Figure 115: Mean composition equations for chf.3D.

17 Instantaneous hydrodynamic equations in spherical coordinates (Eulerian form)

The hydrodynamic equations of a viscous multi-component reactive gas subject to gravity and thermal transport in spherical coordinates (r, θ, ϕ):

$$\partial_t(\rho) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r\ ]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta]) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi]\right) \tag{85}$$

$$\partial_t(\rho u_r) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r^2 - \tau_{rr}]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi([\rho u_r u_\phi - \tau_{r\phi}]) + G_r^M + \partial_r P\right) - \rho\partial_r\Phi \tag{86}$$

$$\partial_t(\rho u_\theta) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_\theta u_r - \tau_{\theta r}]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]) + G_\theta^M + \frac{1}{r}\partial_\theta P\right) - \rho\frac{1}{r}\partial_\theta\Phi \tag{87}$$

$$\partial_t(\rho u_\phi) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_\phi u_r - \tau_{\phi r}]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]) + \frac{1}{r\sin\theta}\partial_\phi([\rho u_\phi^2 - \tau_{\phi\phi}]) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) - \rho\frac{1}{r\sin\theta}\partial_\phi\Phi \tag{88}$$

$$\begin{aligned}\partial_t(\rho\epsilon_T) = &-\left(\frac{1}{r^2}\partial_r(r^2[u_r(\rho\epsilon_T + P) - K\partial_r T]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[u_\theta(\rho\epsilon_T + P) - K\frac{1}{r}\partial_\theta T]) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi(\rho\epsilon_T + P) - K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \\ &+\left(\frac{1}{r^2}\partial_r(r^2[u_j\tau_{jr}]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[u_j\tau_{j\theta}]) + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}]\right) - \rho(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi) + \rho\epsilon_{nuc}\end{aligned} \tag{89}$$

$$\begin{aligned}\partial_t(\rho\epsilon_I) = &-\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r\epsilon_I]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta\epsilon_I]) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_I]\right) - P\left(\frac{1}{r^2}\partial_r(r^2[u_r\ ]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[u_\theta]) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) + \\ &+\left(\frac{1}{r^2}\partial_r(r^2[K\partial_r T]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[K\frac{1}{r}\partial_\theta T]) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho\epsilon_{nuc}\end{aligned} \tag{90}$$

$$\begin{aligned}\partial_t(\rho\epsilon_K) = &-\left(\frac{1}{r^2}\partial_r[r^2(\rho u_r\epsilon_K - u_j\tau_{jr})] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta(\rho u_\theta\epsilon_K - u_j\tau_{j\theta})] + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_K - u_j\tau_{j\phi}]\right) - \\ &-\left(\frac{1}{r^2}\partial_r(r^2[Pu_r]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[Pu_\theta]) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + P\left(\frac{1}{r^2}\partial_r(r^2[u_r]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[u_\theta]) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) - \\ &-\left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) - \rho(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi)\end{aligned} \tag{91}$$

$$\partial_t(\rho X_k) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r X_k]) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta X_k]) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_k]\right) + \rho\dot{X}_k^n \qquad k = 1...N_{nuc} \tag{92}$$

$$G_r^M = -\frac{(\rho u_\theta^2 - \tau_{\theta\theta})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})}{r} \quad G_\theta^M = +\frac{(\rho u_\theta u_r - \tau_{\theta r})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})\cos\theta}{r\sin\theta} \quad G_\phi^M = +\frac{(\rho u_\phi u_r - \tau_{\phi r})}{r} + \frac{(\rho u_\phi u_\theta - \tau_{\phi\theta})\cos\theta}{r\sin\theta} \tag{93}$$

18 Instantaneous hydrodynamic equations in spherical coordinates (Lagrangian form)

The hydrodynamic equations of a viscous multi-component reactive gas subject to gravity and thermal transport in spherical coordinates (r,θ,ϕ) are ($D_t(.) = \partial_t(.) + u_n\partial_n(.)$ is advective derivative):

$$
\begin{aligned}
D_t(\rho) &= -\rho \left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi] \right) && (94) \\
\rho D_t(u_r) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big) - G_r^M - \partial_r P \right) + \rho g_r && (95) \\
\rho D_t(u_\theta) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\theta r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{\theta\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi[\tau_{\theta\phi}]\big) - G_\theta^M - \frac{1}{r}\partial_\theta P \right) + \rho g_\theta && (96) \\
\rho D_t(u_\phi) &= +\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\phi r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{\phi\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{\phi\phi}]\big) - G_\phi^M - \frac{1}{r\sin\theta}\partial_\phi P \right) + \rho g_\phi && (97) \\
\rho D_t(\epsilon_T) &= -\left(\frac{1}{r^2}\partial_r\big(r^2[u_r P - K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta P - K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi P - K\frac{1}{r\sin\theta}\partial_\phi T] \right) - \\
&\quad + \left(\frac{1}{r^2}\partial_r[r^2\big(u_j\tau_{jr}\big)] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta\big(u_j\tau_{j\theta}\big)] + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}] \right) + \rho\big(u_r g_r + u_\theta g_\theta + u_\phi g_\phi\big) + \rho\epsilon_{nuc} && (98) \\
\rho D_t(\epsilon_I) &= -P \left(\frac{1}{r^2}\partial_r\big(r^2[u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi] \right) + \left(\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T] \right) + \\
&\quad + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j \right) + \rho\epsilon_{nuc} && (99) \\
\rho D_t(\epsilon_K) &= +\left(\frac{1}{r^2}\partial_r[r^2\big(u_j\tau_{jr}\big)] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta\big(u_j\tau_{j\theta}\big)] + \frac{1}{r\sin\theta}\partial_\phi[u_j\tau_{j\phi}] \right) - \left(\frac{1}{r^2}\partial_r\big(r^2[Pu_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Pu_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi] \right) + \\
&\quad + P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi] \right) - \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j \right) + \rho\big(u_r g_r + u_\theta g_\theta + u_\phi g_\phi\big) && (100) \\
\rho D_t(X_k) &= +\rho\dot{X}_k^n \qquad\qquad k = 1...N_{nuc} && (101)
\end{aligned}
$$

$$
G_r^M = -\frac{(\rho u_\theta^2 - \tau_{\theta\theta})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})}{r} \quad G_\theta^M = +\frac{(\rho u_\theta u_r - \tau_{\theta r})}{r} - \frac{(\rho u_\phi^2 - \tau_{\phi\phi})\cos\theta}{r\sin\theta} \quad G_\phi^M = +\frac{(\rho u_\phi u_r - \tau_{\phi r})}{r} + \frac{(\rho u_\phi u_\theta - \tau_{\phi\theta})\cos\theta}{r\sin\theta} \qquad (102)
$$

where ρ, u_r, u_θ, u_ϕ, P, ϵ_T, ϵ_I, ϵ_K, T, ϵ_{nuc}, X_k, and $\dot{X}_k^n$ are the density, the radial velocity, the θ-velocity, the rotation velocity, the pressure, the total specific energy, the specific internal energy, the specific kinetic energy, the temperature, the energy generation rate per mass due to reactions, the mass fraction of species k, and the change of this mass fraction due to reactions, respectively. N_{nuc} is the number of species the gas is composed. $\tau_{ij} = 2\mu S_{ij} - 2/3\mu\nabla \cdot \mathbf{u}\delta_{ij}$ is the viscous stress, where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is strain-rate, $\mu = \rho\nu$ is dynamic viscosity and ν is the kinematic viscosity. K is thermal conductivity. g_i is gravitational acceleration in r, θ, ϕ and Φ is gravitational potential. G are geometric terms.

19 Reynolds decomposition

Reynolds decomposition:

$$A(r,\theta,\phi) = \overline{A}(r) + A'(r,\theta,\phi) \tag{103}$$

Definition of the averaging space-time operator:

$$\overline{A(r)} = \frac{1}{\Delta T \Delta\Omega} \int_{\Delta T} \int_{\Delta\Omega} A(r,\theta,\phi)\ dt\ d\Omega \tag{104}$$

Some properties of the operator:

$$\overline{A'} = 0 \tag{105}$$

$$\overline{\overline{A}} = \overline{A} \tag{106}$$

$$(A')' = A' \tag{107}$$

$$\overline{\overline{A}B} = \overline{A}\ \overline{B} \tag{108}$$

$$\overline{AB} = \overline{A}\ \overline{B} + \overline{A'B'} \tag{109}$$

$$\overline{A'B'} = \overline{A'B} \tag{110}$$

20 Favre decomposition

Favre decomposition:

$$F = \widetilde{F}(r) + F''(r, \theta, \phi) \tag{111}$$

Definition of the averaging operator:

$$\widetilde{F} = \frac{\overline{\rho F}}{\overline{\rho}} \tag{112}$$

Some properties of the operator:

$$\overline{\rho F''} = \widetilde{F''} = 0 \tag{113}$$

$$\widetilde{F} = \overline{F} + \frac{\overline{\rho F'}}{\overline{\rho}} \tag{114}$$

$$F'' = F' - \frac{\overline{\rho F'}}{\overline{\rho}} \rightarrow \overline{F''} = -\frac{\overline{\rho F'}}{\overline{\rho}} \tag{115}$$

$$\overline{\rho F'' G''} = \overline{\rho}\widetilde{F'' G''} \tag{116}$$

21 Derivation of first order moments

21.1 Mean continuity equation

We begin by instantaneous 3D continuity equation and apply ”ensemble” (space-time) averaging (Sect.1):

$$\partial_t \rho = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r \;]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi] \right) \tag{117}$$

$$\partial_t \overline{\rho} = -\left(\overline{\frac{1}{r^2}\partial_r\big(r^2[\rho u_r \;]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi]}} \right) \quad \text{“ensemble” (space-time) averaging} \tag{118}$$

$$\partial_t \overline{\rho} = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_r \;]\big) \tag{119}$$

$$\partial_t \overline{\rho} = -\widetilde{u}_r\partial_r\overline{\rho} - \overline{\rho}\frac{1}{r^2}\partial_r(r^2\widetilde{u}_r) \tag{120}$$

$$\partial_t \overline{\rho} + \widetilde{u}_r\partial_r\overline{\rho} = -\overline{\rho}\frac{1}{r^2}\partial_r(r^2\widetilde{u}_r) \tag{121}$$

$$\widetilde{D}_t\overline{\rho} = -\overline{\rho}\widetilde{d} \tag{122}$$

21.2 Mean radial momentum equation

We begin by instantaneous 3D radial momentum equation and apply ”ensemble” (space-time) averaging (Sect.1):

$$\partial_t \rho u_r = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r^2 - \tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_r u_\phi - \tau_{r\phi}]\big) + G_r^M + \partial_r P \right) + \rho g_r \tag{123}$$

$$\partial_t \overline{\rho u_r} = -\left(\overline{\frac{1}{r^2}\partial_r\big(r^2[\rho u_r^2 - \tau_{rr}]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}])}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi\big([\rho u_r u_\phi - \tau_{r\phi}]\big)}} + \overline{G_r^M} + \partial_r\overline{P} \right) + \overline{\rho g_r} \tag{124}$$

$$\partial_t \overline{\rho u_r} = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho u_r u_r}]\big) - \cancelto{0}{\frac{1}{r^2}\partial_r\big(r^2\overline{\tau}_{rr}\big)} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho g_r} \quad \text{we neglect mean viscosity } \overline{\tau} \tag{125}$$

$$\partial_t \overline{\rho} \widetilde{u_r} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r u_r}]\right) - \overline{G_r^M} - \partial_r \overline{P} + \overline{\rho}\widetilde{g}_r \tag{126}$$

$$\partial_t \overline{\rho} \widetilde{u_r} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r}\widetilde{u_r} + \overline{\rho}\widetilde{u_r'' u_r''}]\right) - \overline{G_r^M} - \partial_r \overline{P} + \overline{\rho}\widetilde{g}_r \tag{127}$$

$$\partial_t \overline{\rho} \widetilde{u_r} + \frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_r}\widetilde{u_r}]\right) = -\frac{1}{r^2}\partial_r\left(\overline{\rho}\widetilde{u_r'' u_r''}\right) - \overline{G_r^M} - \partial_r \overline{P} + \overline{\rho}\widetilde{g}_r \tag{128}$$

$$\overline{\rho}\widetilde{D}_t \widetilde{u}_r = -\nabla_r \widetilde{R}_{rr} - \overline{G_r^M} - \partial_r \overline{P} + \overline{\rho}\widetilde{g}_r \tag{129}$$

21.3 Mean polar momentum equation

We begin by instantaneous 3D polar momentum equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t \rho u_\theta = -\left(\frac{1}{r^2}\partial_r\left(r^2[\rho u_\theta u_r - \tau_{\theta r}]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]\right) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]\right) + G_\theta^M + \frac{1}{r}\partial_\theta P\right) - \rho\frac{1}{r}\partial_\theta \Phi \tag{130}$$

$$\partial_t \overline{\rho u_\theta} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[\rho u_\theta u_r - \tau_{\theta r}]\right)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[\rho u_\theta^2 - \tau_{\theta\theta}]\right)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\theta u_\phi - \tau_{\theta\phi}]\right)}} + \overline{G_\theta^M} + \frac{1}{r}\overline{\partial_\theta P} + \overline{\rho g_\theta}\right) \tag{131}$$

$$\partial_t \overline{\rho u_\theta} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho u_\theta u_r}]\right) + \cancelto{0}{\frac{1}{r^2}\partial_r\left(r^2\overline{\tau_{\theta r}}\right)} - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} + \cancelto{0}{\overline{\rho g_\theta}} \quad \text{we neglect mean viscosity } \overline{\tau} \text{ and gravity in } \theta \tag{132}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u_\theta u_r}]\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{133}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta = -\frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_\theta\widetilde{u}_r + \overline{\rho}\widetilde{u_\theta'' u_r''}\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{134}$$

$$\partial_t \overline{\rho}\widetilde{u}_\theta + \frac{1}{r^2}\partial_r\left(r^2[\overline{\rho}\widetilde{u}_\theta\widetilde{u}_r]\right) = -\frac{1}{r^2}\partial_r\left(r^2\overline{\rho}\widetilde{u_\theta'' u_r''}\right) - \overline{G_\theta^M} - \frac{1}{r}\overline{\partial_\theta P} \tag{135}$$

$$\overline{\rho}\widetilde{D}_t \widetilde{u}_\theta = -\nabla_r \widetilde{R}_{\theta r} - \overline{G_\theta^M} - (1/r)\overline{\partial_\theta P} \tag{136}$$

21.4 Mean azimutal momentum equation

We begin by instantaneous 3D azimutal momentum equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t \big(\rho u_\phi\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \tau_{\phi r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \tau_{\phi\phi}]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) - \quad (137)$$

$$- \rho\frac{1}{r\sin\theta}\partial_\phi \Phi$$

$$\partial_t \overline{\rho u_\phi} = -\left(\overline{\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \tau_{\phi r}]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \tau_{\theta\phi}]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \tau_{\phi\phi}]\big)}} + \overline{G_\phi^M} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi P}}\right) +$$

$$+\overline{\rho g_\phi} \quad (138)$$

$$\partial_t \overline{\rho u_\phi} = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho u_\phi u_r}]\big) - \cancelto{0}{\frac{1}{r^2}\partial_r\big(r^2\overline{\tau_{\phi r}}\big)} + \overline{G_\phi^M} - \cancelto{0}{\overline{\rho g_\phi}} \text{ we neglect mean viscosity } \tau \text{ and gravity in } \phi \quad (139)$$

$$\partial_t \overline{\rho}\widetilde{u}_\phi = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u_\phi u_r}]\big) + \overline{G_\phi^M} \quad (140)$$

$$\partial_t \overline{\rho}\widetilde{u}_\phi = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_\phi\widetilde{u}_r - \overline{\rho}\widetilde{u''_\phi u''_r}]\big) - \overline{G_\phi^M} \quad (141)$$

$$\partial_t \overline{\rho}\widetilde{u}_\phi + \frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_\phi\widetilde{u}_r]\big) = -\frac{1}{r^2}\partial_r\big(r^2\overline{\rho}\widetilde{u''_\phi u''_r}\big) - \overline{G_\phi^M} \quad (142)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G_\phi^M} \quad (143)$$

21.5 Mean internal energy equation

We begin by instantaneous 3D internal energy equation and apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t\big(\rho\epsilon_I\big) = -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta\epsilon_I]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_I]\right) - P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right) +$$

$$+\left(\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big) + \frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]\right) + \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j\right) + \rho\epsilon_{\rm nuc} \quad (144)$$

$$
\begin{aligned}
\partial_t \overline{\rho \epsilon_I} = & - \left(\overline{\frac{1}{r^2}\partial_r\big(r^2[\rho u_r \epsilon_I]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta \epsilon_I]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi \epsilon_I]}} \right) - \overline{P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right)} + \\
& + \left(\overline{\frac{1}{r^2}\partial_r\big(r^2[K\partial_r T]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[K\frac{1}{r}\partial_\theta T]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[K\frac{1}{r\sin\theta}\partial_\phi T]}} \right) + \left(\overline{\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j} \right) + \overline{\rho\epsilon_{\mathrm{nuc}}} \qquad (145)
\end{aligned}
$$

$$\partial_t \overline{\rho\epsilon_I} = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho u_r \epsilon_I}]\big) - \overline{Pd} + \frac{1}{r^2}\partial_r\big(r^2[\overline{\chi\partial_r T}]\big) + \overline{\tau_{ji}\partial_i u_j} + \overline{\rho\epsilon_{\mathrm{nuc}}} \qquad (146)$$

$$\partial_t \overline{\rho}\widetilde{\epsilon}_I = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u_r\epsilon_I}]\big) - \overline{Pd} + \frac{1}{r^2}\partial_r\big(r^2[\overline{\chi\partial_r T}]\big) + \overline{\tau_{ji}\partial_i u_j} + \overline{\rho\epsilon_{\mathrm{nuc}}} \qquad (147)$$

$$\partial_t \overline{\rho}\widetilde{\epsilon}_I = -\frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u}_r\widetilde{\epsilon}_I]\big) - \frac{1}{r^2}\partial_r\big(r^2[\overline{\rho}\widetilde{u''_r\epsilon''_I}]\big) - \overline{Pd} + \frac{1}{r^2}\partial_r\big(r^2[\overline{\chi\partial_r T}]\big) + \cancelto{0}{\overline{\tau_{ji}}\partial_i\overline{u_j}} + \overline{\tau'_{ji}\partial_i u'_j} + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \quad \text{we neglect } \overline{\tau} \qquad (148)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r f_I - \overline{Pd} + \nabla_r f_T + \overline{\tau'_{ji}\partial_i u'_j} + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \qquad (149)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - \overline{P'd'} + \varepsilon_k + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \qquad (150)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \varepsilon_k + \overline{\rho}\widetilde{\epsilon}_{\mathrm{nuc}} \qquad (151)$$

21.6 Mean kinetic energy equation

We begin by instantaneous 3D kinetic energy equation and apply "ensemble" (space-time) averaging (Sect.1):

$$
\begin{aligned}
\partial_t\big(\rho\epsilon_K\big) = & - \left(\frac{1}{r^2}\partial_r[r^2\big(\rho u_r \epsilon_K - u_j\tau_{jr}\big)] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta\big(\rho u_\theta\epsilon_K - u_j\tau_{j\theta}\big)] + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_K - u_j\tau_{j\phi}] \right) - \\
& - \left(\frac{1}{r^2}\partial_r\big(r^2[Pu_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Pu_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi] \right) + P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi] \right) - \\
& - \left(\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j \right) - \rho\big(u_r\partial_r\Phi + u_\theta\frac{1}{r}\partial_\theta\Phi + u_\phi\frac{1}{r\sin\theta}\partial_\phi\Phi) \qquad (152)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \overline{\rho\epsilon_K} = & -\left(\overline{\frac{1}{r^2}\partial_r[r^2\big(\rho u_r \epsilon_K - u_j\tau_{jr}\big)]} + \overline{\cancelto{0}{\frac{1}{r\sin\theta}\partial_\theta[\sin\theta\big(\rho u_\theta\epsilon_K - u_j\tau_{j\theta}\big)]}} + \overline{\cancelto{0}{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi\epsilon_K - u_j\tau_{j\phi}]}} \right) - \\
& -\left(\overline{\frac{1}{r^2}\partial_r\big(r^2[Pu_r]\big)} + \overline{\cancelto{0}{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Pu_\theta]\big)}} + \overline{\cancelto{0}{\frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]}} \right) + \overline{P\left(\frac{1}{r^2}\partial_r\big(r^2[u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[u_\phi]\right)} - \\
& -\left(\overline{\tau_{jr}\partial_r u_j + \tau_{j\theta}\frac{1}{r}\partial_\theta u_j + \tau_{j\phi}\frac{1}{r\sin\theta}\partial_\phi u_j} \right) + \overline{\rho\big(u_r g_r + \cancelto{0}{u_\theta g_\theta} + \cancelto{0}{u_\phi g_\phi}\big)} && (153) \\
\partial_t \overline{\rho\epsilon_K} = & -\frac{1}{r^2}\partial_r[r^2\big(\overline{\rho u_r\epsilon_K}\big)] - \frac{1}{r^2}\partial_r[r^2\overline{(u_j\tau_{jr})}] - \frac{1}{r^2}\partial_r\big(r^2[\overline{Pu_r}]\big) + \overline{Pd} - \overline{\tau_{ji}\partial_i u_j} + \overline{\rho u_r g_r} && (154) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\frac{1}{r^2}\partial_r[r^2\big(\overline{\rho u''_r\epsilon''_K}\big)] - \cancelto{0}{\frac{1}{r^2}\partial_r[r^2(\overline{u_j}\ \overline{\tau_{jr}})]} - \frac{1}{r^2}\partial_r[r^2\overline{(u'_j\tau'_{jr})}] - \frac{1}{r^2}\partial_r\big(r^2[\overline{P}\,\overline{u_r}]\big) - \frac{1}{r^2}\partial_r\big(r^2[\overline{P'u'_r}]\big) + \overline{P}\ \overline{d} + \overline{P'd'} - \varepsilon_k - \overline{\rho u''_r}\widetilde{g}_r + \overline{\rho}\ \overline{u}_r\widetilde{g}_r && (155) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\frac{1}{r^2}\partial_r[r^2\big(\overline{\rho u''_r\epsilon''_K}\big)] - \frac{1}{r^2}\partial_r[r^2\overline{(u'_j\tau'_{jr})}] - \frac{1}{r^2}\partial_r\big(r^2[\overline{P}\,\overline{u_r}]\big) - \frac{1}{r^2}\partial_r\big(r^2[\overline{P'u'_r}]\big) + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r\widetilde{g}_r && (156) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\nabla_r\overline{\rho u''_r\epsilon''_K} - \nabla_r f_\tau - (\overline{P}\nabla_r\overline{u}_r + \overline{u}_r\partial_r\overline{P}) - \nabla_r f_P + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r\widetilde{g}_r && (157) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\nabla_r\overline{\rho u''_r\epsilon''_K} - \nabla_r f_\tau - \overline{P}\ \overline{d} - \overline{\rho}\ \overline{u}_r\widetilde{g}_r - \nabla_r f_P + \overline{P}\ \overline{d} + W_P - \varepsilon_k + W_b + \overline{\rho}\ \overline{u}_r\widetilde{g}_r && (158) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\nabla_r\overline{\rho u''_r\epsilon''_K} - \nabla_r(f_\tau + f_P) + W_P - \varepsilon_k + W_b && (159)
\end{aligned}
$$

Second way:

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & +\overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i + \overline{\rho}\widetilde{D}_t\widetilde{u''_i u''_i} && (160) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & +\overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i + \overline{\rho}\widetilde{D}_t\widetilde{k} && (161) \\
\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = & -\nabla_r(f_k + f_P + f_\tau) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P - \varepsilon_k + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) && (162)
\end{aligned}
$$

where equation for the $\widetilde{k}$ is derived later.

21.7 Mean total energy equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = +\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I + \overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K \quad (163)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P + f_\tau) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) \quad (164)$$

21.8 Mean pressure equation

We begin by deriving 3D instantaneous pressure equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$dP = \frac{\partial P}{\partial \rho}\Big|_{\epsilon_I} d\rho + \frac{\partial P}{\partial \epsilon_I}\Big|_{\rho} d\epsilon_I = \frac{P}{\rho}(1-\Gamma_3+\Gamma_1)d\rho + \rho(\Gamma_3-1)d\epsilon_I \quad (165)$$

$$D_t P = +\frac{P}{\rho}(1-\Gamma_3+\Gamma_1)D_t\rho + (\Gamma_3-1)\rho D_t\epsilon_I \quad (166)$$

$$D_t P = -(1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd + \mathcal{S} + \nabla\cdot F_T + \tau_{ij}\partial_i u_j) \quad (167)$$

$$\partial_t P = -u_n\partial_n P - (1-\Gamma_3+\Gamma_1)Pd + (\Gamma_3-1)(-Pd + \mathcal{S} + \nabla\cdot F_T + \tau_{ij}\partial_i u_j) \quad (168)$$

$$\partial_t P = -\left(\frac{1}{r^2}\partial_r\left(r^2[Pu_r\ ]\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[Pu_\theta]\right) + \frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]\right) + (1-\Gamma_1)Pd + (\Gamma_3-1)(\mathcal{S} + \nabla\cdot F_T + \tau_{ij}\partial_i u_j) \quad (169)$$

$$\partial_t\overline{P} = -\left(\overline{\frac{1}{r^2}\partial_r\left(r^2[Pu_r\ ]\right)} + \overline{\cancel{\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta[Pu_\theta]\right)}}^{\,0} + \overline{\cancel{\frac{1}{r\sin\theta}\partial_\phi[Pu_\phi]}}^{\,0}\right) + (1-\Gamma_1)\overline{Pd} + (\Gamma_3-1)(\mathcal{S} + \overline{\nabla\cdot F_T} + \overline{\tau_{ij}\partial_i u_j}) \quad (170)$$

$$\partial_t\overline{P} = -\frac{1}{r^2}\partial_r\left(r^2[\overline{P}\overline{u}_r]\right) - \frac{1}{r^2}\partial_r\left(r^2[\overline{P'u'_r}]\right) + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)\overline{P'd'} + (\Gamma_3-1)(\mathcal{S} + \frac{1}{r^2}\partial_r\left(r^2\overline{\chi\partial_r T}\right) + \cancel{\overline{\tau_{ij}}\partial_i\overline{u'_j}}^{\,0} + \overline{\tau'_{ij}\partial_i u'_j}) \quad (171)$$

$$\partial_t\overline{P} = -\nabla_r\overline{P}\overline{u}_r - \nabla_r f_P + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S} + \nabla_r f_T + \varepsilon_k) \quad (172)$$

$$\partial_t\overline{P} = -\overline{u}_r\partial_r\overline{P} - \overline{P}\ \overline{d} - \nabla_r f_P + (1-\Gamma_1)\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\overline{\mathcal{S}} + \nabla_r f_T + \varepsilon_k) \quad (173)$$

$$\partial_t\overline{P} + \overline{u}_r\partial_r\overline{P} = -\nabla_r f_P - \Gamma_1\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S} + \nabla_r f_T + \varepsilon_k) \quad (174)$$

$$\overline{D}_t\overline{P} = -\nabla_r f_P - \Gamma_1\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3-1)(\mathcal{S} + \nabla_r f_T + \varepsilon_k) \quad (175)$$

21.9 Mean enthalpy equation

We start from the total energy equation, where we can subsitute $\rho\epsilon_t = \rho h + \rho\epsilon_k - P$ (for clarity reasons we use here more compact vector notation):

$$\partial_t \epsilon_t + \vec{\nabla} \cdot \big((\rho\epsilon_t + P)\vec{u} \big) = \rho\vec{u} \cdot \vec{g} + \vec{\nabla} \cdot F_T + \mathcal{S} \tag{176}$$

$$\partial_t(\rho h + \rho\epsilon_k - P) + \vec{\nabla} \cdot (\rho h\vec{u} + \rho\epsilon_k\vec{u}) = \rho\vec{u} \cdot \vec{g} + \vec{\nabla} \cdot F_T + \mathcal{S} \tag{177}$$

$$\partial_t\rho h + \partial_t\rho\epsilon_k - \partial_t P = -\vec{\nabla} \cdot (\rho h\vec{u} + \rho\epsilon_k\vec{u}) + \rho\vec{u} \cdot \vec{g} + \vec{\nabla} \cdot F_T + \mathcal{S} \tag{178}$$

$$\partial_t\rho h + \Big[-\vec{\nabla} \cdot (\rho\epsilon_k\vec{u}) - \vec{\nabla} \cdot (P\vec{u}) + P(\vec{\nabla} \cdot \vec{u}) + \rho\vec{u} \cdot \vec{g} + \nabla_i u_j \tau_{ji} \Big] - \Big[-\vec{\nabla} \cdot (P\vec{u}) + (1-\Gamma_1)P\vec{\nabla} \cdot \vec{u} + (\Gamma_3 - 1)(\mathcal{S} + \vec{\nabla} \cdot F_T + \tau_{ij}\partial_j u_i) \Big] = \tag{179}$$

$$= -\vec{\nabla} \cdot (\rho h\vec{u} + \rho\epsilon_k\vec{u}) + \rho\vec{u} \cdot \vec{g} + \vec{\nabla} \cdot F_T + \mathcal{S} \tag{180}$$

$$\partial_t\rho h + \Big[-\cancel{\vec{\nabla} \cdot (\rho\epsilon_k\vec{u})} - \cancel{\vec{\nabla} \cdot (P\vec{u})} + \cancel{P(\vec{\nabla} \cdot \vec{u})} + \cancel{\rho\vec{u} \cdot \vec{g}} + \nabla_i u_j \tau_{ji} \Big] + \cancel{\vec{\nabla} \cdot (P\vec{u})} - \cancel{P\vec{\nabla} \cdot \vec{u}} + \Gamma_1 P\vec{\nabla} \cdot \vec{u} - \Gamma_3(\mathcal{S} + \vec{\nabla} \cdot F_T) + (\cancel{\mathcal{S}} + \cancel{\vec{\nabla} \cdot F_T}) - = \tag{181}$$

$$-(\Gamma_3 - 1)\tau_{ij}\partial_j u_i = -\vec{\nabla} \cdot (\rho h\vec{u} + \cancel{\rho\epsilon_k\vec{u}}) + \cancel{\rho\vec{u} \cdot \vec{g}} + \cancel{\vec{\nabla} \cdot F_T} + \cancel{\mathcal{S}} \tag{182}$$

So, from the above we have:

$$\partial_t\rho h + \nabla_i u_j \tau_{ji} + \Gamma_1 P\vec{\nabla} \cdot \vec{u} - \Gamma_3\mathcal{S} - \Gamma_3\vec{\nabla} \cdot F_T - (\Gamma_3 - 1)\tau_{ij}\partial_j u_i = -\vec{\nabla} \cdot (\rho h\vec{u}) \tag{183}$$

$$\rho D_t h = -\Gamma_1 P\vec{\nabla} \cdot \vec{u} + \Gamma_3\mathcal{S} + \Gamma_3\vec{\nabla} \cdot F_T - \nabla_i u_j \tau_{ji} + (\Gamma_3 - 1)\tau_{ij}\partial_j u_i \tag{184}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r f_h - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3\mathcal{S} + \Gamma_3\nabla_r f_T - \nabla_r \overline{u_j \tau_{jr}} + (\Gamma_3 - 1)\overline{\tau_{ij}\partial_j u_i} \tag{185}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{h} = -\nabla_r(f_h + f_\tau) - \Gamma_1 \overline{P}\ \overline{d} - \Gamma_1 W_P + \Gamma_3\mathcal{S} + \Gamma_3\nabla_r f_T + (\Gamma_3 - 1)\varepsilon_k \tag{186}$$

21.10 Mean temperature equation

We begin by deriving 3D instantaneous temperature equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$
\begin{aligned}
dT =& \frac{\partial T}{\partial \rho}\Big|_{\epsilon_I} d\rho + \frac{\partial T}{\partial \epsilon_I}\Big|_{\rho} d\epsilon_I = \left(\frac{T}{\rho}(\Gamma_3 - 1) - \frac{P}{\rho^2}\frac{1}{c_v}\right) d\rho + \frac{1}{c_v} d\epsilon_I && (187) \\
D_t T =& + \frac{T}{\rho}(\Gamma_3 - 1) D_t \rho - \frac{P}{\rho^2}\frac{1}{c_v} D_t \rho + \frac{1}{c_v} D_t \epsilon_I && (188) \\
D_t T =& - \frac{T}{\rho}(\Gamma_3 - 1)(\rho d) + \frac{P}{\rho^2}\frac{1}{c_v}(\rho d) + \frac{1}{c_v}\left(-\frac{Pd}{\rho} + \frac{\nabla \cdot F_T}{\rho} + \frac{\tau_{ij}\partial_j u_i}{\rho} + \frac{\mathcal{S}}{\rho}\right) && (189) \\
D_t T =& - (\Gamma_3 - 1) T d + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij}\partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} && (190) \\
\partial_t T =& - u_n \partial_n T - (\Gamma_3 - 1) T d + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij}\partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} && (191) \\
\partial_t T =& - \left(\frac{1}{r^2}\partial_r\big(r^2[Tu_r\ ]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Tu_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi[Tu_\phi]\right) + Td - (\Gamma_3 - 1)Td + \frac{\nabla \cdot F_T}{c_v \rho} + \frac{\tau_{ij}\partial_j u_i}{c_v \rho} + \frac{\mathcal{S}}{c_v \rho} && (192) \\
\partial_t \overline{T} =& - \left(\overline{\frac{1}{r^2}\partial_r\big(r^2[Tu_r\ ]\big)} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[Tu_\theta]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[Tu_\phi]}}\right) + \overline{Td} - (\Gamma_3 - 1)\overline{Td} + \overline{\frac{\nabla \cdot F_T}{c_v \rho}} + \overline{\frac{\tau_{ij}\partial_j u_i}{c_v \rho}} + \overline{\frac{\mathcal{S}}{c_v \rho}} && (193) \\
\partial_t \overline{T} =& - \nabla_r \overline{T}\overline{u}_r - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (194) \\
\partial_t \overline{T} + \overline{u}_r \partial_r \overline{T} =& - \overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (195) \\
\overline{D}_t \overline{T} =& - \overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{Td} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (196) \\
\overline{D}_t \overline{T} =& - \overline{T}\ \overline{d} - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{T}\ \overline{d} + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (197) \\
\overline{D}_t \overline{T} =& - \nabla_r f_T + \overline{Td} - \Gamma_3 \overline{Td} + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (198) \\
\overline{D}_t \overline{T} =& - \nabla_r f_T + \overline{T}\ \overline{d} + \overline{T'd'} - \Gamma_3(\overline{T}\ \overline{d} + \overline{T'd'}) + \overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (199) \\
\overline{D}_t \overline{T} =& -\nabla_r f_T + (1 - \Gamma_3)\overline{T}\ \overline{d} + (2 - \Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot F_T)/(c_v \rho)} + \overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)} + \overline{\mathcal{S}/(c_v \rho)} && (200)
\end{aligned}
$$

21.11 Mean angular momentum equation (z-component)

Z component of the specific angular momentum is defined as $j_z = r \sin\theta u_\phi$. We begin by multiplying the instantaneous 3D azimutal momentum equation by $r \sin\theta$, neglect viscosity and ϕ component of gravity and obtain (Sect.1):

$$r\sin\theta\partial_t\big(\rho u_\phi\big) = -r\sin\theta\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r - \xcancel{\tau_{\phi r}}^{0}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi - \xcancel{\tau_{\theta\phi}}^{0}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2 - \xcancel{\tau_{\phi\phi}}^{0}]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) -$$

$$-r\sin\theta\rho\xcancel{\frac{1}{r\sin\theta}\partial_\phi\Phi}^{0} \tag{201}$$

$$r\sin\theta\partial_t\big(\rho u_\phi\big) = -r\sin\theta\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_\phi u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta u_\phi]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_\phi^2]\big) + G_\phi^M + \frac{1}{r\sin\theta}\partial_\phi P\right) \tag{202}$$

$$\partial_t\rho j_z = -\frac{1}{r^2}\partial_r(r^2\rho j_z u_r) - \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\rho j_z u_\theta) - \frac{1}{r\sin\theta}\partial_\phi(\rho j_z u_\phi) - \partial_\phi P \tag{203}$$

$$\partial_t\overline{\rho j_z} = -\overline{\frac{1}{r^2}\partial_r(r^2\rho j_z u_r)} - \xcancel{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta\rho j_z u_\theta)}}^{0} - \xcancel{\overline{\frac{1}{r\sin\theta}\partial_\phi(\rho j_z u_\phi)}}^{0} - \xcancel{\overline{\partial_\phi P}}^{0} \tag{204}$$

$$\partial_t\overline{\rho j_z} = -\frac{1}{r^2}\partial_r(r^2\overline{\rho j_z u_r}) \tag{205}$$

$$\partial_t\overline{\rho}\widetilde{j_z} = -\frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z}\widetilde{u_r}) - \frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z'' u_r''}) \tag{206}$$

$$\partial_t\overline{\rho}\widetilde{j_z} + \frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z}\widetilde{u_r}) = -\frac{1}{r^2}\partial_r(r^2\overline{\rho}\widetilde{j_z'' u_r''}) \tag{207}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{j_z} = -\nabla_r f_{jz} \tag{208}$$

21.12 Mean α equation

We begin by 3D instantaneous composition equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$\partial_t\big(\rho X_\alpha\big) = -\left(\frac{1}{r^2}\partial_r(r^2[\rho u_r X_\alpha]) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta X_\alpha]\big) + \frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_\alpha]\right) + \rho\dot{X}_\alpha^{\rm nuc} \qquad \alpha = 1...N_{nuc} \tag{209}$$

$$\partial_t\overline{\rho X_\alpha} = -\left(\overline{\frac{1}{r^2}\partial_r(r^2[\rho u_r X_\alpha])} + \xcancel{\overline{\frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_\theta X_\alpha]\big)}}^{0} + \xcancel{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi X_\alpha]}}^{0}\right) + \overline{\rho\dot{X}_\alpha^{\rm nuc}} \tag{210}$$

$$\partial_t\overline{\rho}\widetilde{X}_\alpha = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{X}_\alpha + \overline{\rho}\widetilde{u_r''X_\alpha''}]) + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \tag{211}$$

$$\partial_t\overline{\rho}\widetilde{X}_\alpha + \frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{X}_\alpha]) = -\frac{1}{r^2}\partial_r(\overline{\rho}\widetilde{u_r''X_\alpha''}) + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \tag{212}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} \tag{213}$$

21.13 Mean number of nucleons per isotope (A) equation

We begin by deriving 3D instantaneous A equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$A = + \left(\sum_\alpha \frac{X_\alpha}{A_\alpha} \right)^{-1} \tag{214}$$

$$D_t A = + D_t \left(\sum_\alpha \frac{X_\alpha}{A_\alpha} \right)^{-1} = +D_t \frac{1}{\sum_\alpha (X_\alpha / A_\alpha)} = -\frac{D_t \sum_\alpha (X_\alpha/A_\alpha)}{[\sum_\alpha (X_\alpha/A_\alpha)]^2} = -A^2 D_t \sum_\alpha \frac{X_\alpha}{A_\alpha} \tag{215}$$

$$D_t A = -A^2 D_t \sum_\alpha \frac{X_\alpha}{A_\alpha} = -A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha - X_\alpha D_t A_\alpha}{A_\alpha^2} = -A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha}{A_\alpha^2} \tag{216}$$

$$D_t A = -A^2 \sum_\alpha \frac{A_\alpha D_t X_\alpha}{A_\alpha^2} = -A^2 \sum_\alpha \frac{A_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha^2} = -A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{217}$$

$$\rho D_t A = -\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{218}$$

$$\partial_t \rho A = -\left(\frac{1}{r^2}\partial_r (r^2 [\rho u_r A]) + \frac{1}{r \sin\theta}\partial_\theta \big(\sin\theta [\rho u_\theta A]\big) + \frac{1}{r\sin\theta}\partial_\phi [\rho u_\phi A] \right) - \rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{219}$$

$$\partial_t \overline{\rho A} = -\left(\overline{\frac{1}{r^2}\partial_r (r^2 [\rho u_r A])} + \cancelto{0}{\overline{\frac{1}{r \sin\theta}\partial_\theta \big(\sin\theta [\rho u_\theta A]\big)}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi [\rho u_\phi A]}} \right) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{220}$$

$$\partial_t \overline{\rho A} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho u_r A}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{221}$$

$$\partial_t \overline{\rho}\widetilde{A} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho}\widetilde{u_r A}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{222}$$

$$\partial_t \overline{\rho}\widetilde{A} = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho}\widetilde{u}_r\widetilde{A}]) - \frac{1}{r^2}\partial_r (r^2 [\overline{\rho}\widetilde{u''_r A''}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{223}$$

$$\partial_t \overline{\rho}\widetilde{A} + \frac{1}{r^2}\partial_r (r^2 [\overline{\rho}\widetilde{u}_r\widetilde{A}]) = -\frac{1}{r^2}\partial_r (r^2 [\overline{\rho}\widetilde{u''_r A''}]) - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{224}$$

$$\overline{\rho}\widetilde{D}_t \widetilde{A} = -\nabla_r f_A - \overline{\rho A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{225}$$

21.14 Mean charge per isotope (Z) equation

We begin by deriving 3D instantaneous Z equation and then apply "ensemble" (space-time) averaging (Sect.1):

$$Z = +A\sum_\alpha \frac{Z_\alpha A_\alpha}{A_\alpha} \tag{226}$$

$$D_t Z = +D_t\left(A\sum_i \frac{Z_\alpha A_\alpha}{A_\alpha}\right) = \sum_\alpha \frac{Z_\alpha X_\alpha}{A_\alpha} D_t A + A\sum_\alpha D_t \frac{Z_\alpha X_\alpha}{A_\alpha} \tag{227}$$

$$D_t Z = -\sum_\alpha \frac{Z_\alpha X_\alpha}{A_\alpha} A^2 \sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A\sum_\alpha \frac{A_\alpha D_t Z_\alpha X_\alpha - Z_\alpha X_\alpha D_t A_\alpha}{A_\alpha^2} \tag{228}$$

$$D_t Z = -ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A\sum_\alpha \frac{A_\alpha D_t Z_\alpha X_\alpha}{A_\alpha^2} \tag{229}$$

$$D_t Z = -ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A\sum_\alpha \frac{Z_\alpha D_t X_\alpha + X_\alpha D_t Z_\alpha}{A_\alpha^2} \tag{230}$$

$$D_t Z = -ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{231}$$

$$\rho D_t Z = -\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} + \rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha} \tag{232}$$

$$\partial_t\overline{\rho Z} = -\left(\overline{\frac{1}{r^2}\partial_r(r^2[\rho u_r Z])} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\rho u_\theta Z])}} + \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi[\rho u_\phi Z]}}\right) - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{233}$$

$$\partial_t\overline{\rho Z} = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho u_r Z}]) - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{234}$$

$$\partial_t\overline{\rho}\widetilde{Z} = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u_r Z}]) - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{235}$$

$$\partial_t\overline{\rho}\widetilde{Z} = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{Z}]) - \frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u''_r Z''}]) - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{236}$$

$$\partial_t\overline{\rho}\widetilde{Z} + \frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u}_r\widetilde{Z}]) = -\frac{1}{r^2}\partial_r(r^2[\overline{\rho}\widetilde{u''_r Z''}]) - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{237}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{Z} = -\nabla_r f_Z - \overline{\rho ZA\sum_\alpha \frac{\dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} + \overline{\rho A\sum_\alpha \frac{Z_\alpha \dot{X}_\alpha^{\mathrm{nuc}}}{A_\alpha}} \tag{238}$$

21.15 Mean entropy equation

We can derive the mean entropy equation in the following way (Sect.1):

$$\rho D_t s = +(-\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T \tag{239}$$

$$\partial_t \rho s + \nabla_r(\rho s u_r) = +(-\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T \tag{240}$$

$$\partial_t \overline{\rho s} + \nabla_r \overline{\rho u_r s} = +\overline{(-\nabla \cdot F_T + \mathcal{S} + \varepsilon_k)/T} \tag{241}$$

$$\partial_t \overline{\rho}\widetilde{s} + \nabla_r(\overline{\rho}\widetilde{u}_r\widetilde{s}) = -\nabla_r(\overline{\rho}\widetilde{s''u''_r}) - \overline{\nabla \cdot F_T/T} + \overline{\mathcal{S}/T} + \overline{\varepsilon_k/T} \tag{242}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s - \overline{\nabla \cdot F_T/T} + \overline{\mathcal{S}/T} + \overline{\varepsilon_k/T} \tag{243}$$

22 General formula for second and third order moments and variances

22.1 Second-order moments

In order to calculate evolution equations for correlations of two arbitrary fluctuations, we can derive the following general formula.

$$\bar\rho\widetilde{D}_t\widetilde{c''d''} - \overline{\rho D_t c''d''} = \bar\rho\big(\partial_t\widetilde{c''d''} + \tilde{u}_n\partial_n\widetilde{c''d''}\big) - \overline{\rho\big(\partial_t c''d'' + u_n\partial_n c''d''\big)} = \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} - \overline{\rho\partial_t c''d''} - \overline{\rho u_n\partial_n c''d''} =$$

$$= \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\partial_t\rho c''d''} - \overline{c''d''\partial_t\rho}\big) - \overline{\rho u_n\partial_n c''d''} = \tag{244}$$

$$= \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} - \partial_t\bar\rho\widetilde{c''d''} - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{245}$$

$$= \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} - \big(\bar\rho\partial_t\widetilde{c''d''} + \widetilde{c''d''}\partial_t\bar\rho\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{246}$$

$$= \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} - \big(\bar\rho\partial_t\widetilde{c''d''} - \widetilde{c''d''}\partial_n\bar\rho\tilde{u}_n\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \tag{247}$$

$$= \bar\rho\partial_t\widetilde{c''d''} - \bar\rho\partial_t\widetilde{c''d''} + \bar\rho\tilde{u}_n\partial_n\widetilde{c''d''} + \widetilde{c''d''}\partial_n\bar\rho\tilde{u}_n - \overline{\partial_n\rho u_n c''d''} = \tag{248}$$

$$= \partial_n\bar\rho\tilde{u}_n\widetilde{c''d''} - \big(\overline{\partial_n\rho\tilde{u}_n c''d''} + \overline{\partial_n\rho u_n''c''d''}\big) = -\overline{\partial_n\rho u_n''c''d''} \tag{249}$$

$$\bar\rho\widetilde{D}_t\widetilde{c''d''} = \overline{\rho D_t c''d''} - \overline{\partial_n\rho u_n''c''d''} = \overline{c''\rho D_t d''} + \overline{d''\rho D_t c''} - \overline{\partial_n\rho u_n''c''d''} \tag{250}$$

$$\rho D_t c'' = \rho D_t c - \rho D_t\tilde{c} = \rho D_t c - \rho\widetilde{D}_t\tilde{c} - \rho u_n''\partial_n\tilde{c} = \rho D_t c - \frac{\rho}{\bar\rho}\left[\bar\rho\widetilde{D}_t\tilde{c}\right] - \rho u_n''\partial_n\tilde{c} \tag{251}$$

$$\rho D_t d'' = \rho D_t d - \rho D_t\tilde{d} = \rho D_t d - \rho\widetilde{D}_t\tilde{d} - \rho u_n''\partial_n\tilde{d} = \rho D_t d - \frac{\rho}{\bar\rho}\left[\bar\rho\widetilde{D}_t\tilde{d}\right] - \rho u_n''\partial_n\tilde{d} \tag{252}$$

$$\bar\rho\widetilde{D}_t\widetilde{c''d''} = \overline{c''\left(\rho D_t d - \frac{\rho}{\bar\rho}\left[\bar\rho\widetilde{D}_t\tilde{d}\right] - \rho u_n''\partial_n\tilde{d}\right)} + \overline{d''\left(\rho D_t c - \frac{\rho}{\bar\rho}\left[\bar\rho\widetilde{D}_t\tilde{c}\right] - \rho u_n''\partial_n\tilde{c}\right)} - \overline{\partial_n\rho c''d''u_n''} \tag{253}$$

$$\bar\rho\widetilde{D}_t\widetilde{c''d''} = +\overline{c''\rho D_t d} - \bar\rho\widetilde{c''u_n''}\partial_n\tilde{d} + \overline{d''\rho D_t c} - \bar\rho\widetilde{d''u_n''}\partial_n\tilde{c} - \overline{\partial_n\rho c''d''u_n''} \tag{254}$$

22.2 Third-order moments

In order to calculate evolution equations for correlations of three arbitrary fluctuations, we can derive the following general formula.

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{c''d''e''} - \overline{\rho D_t c''d''e''} &= \overline{\rho}\left(\partial_t\widetilde{c''d''e''} + \widetilde{u}_n\partial_n\widetilde{c''d''e''}\right) - \overline{\rho(\partial_t c''d''e'' + u_n\partial_n c''d''e'')} = && (255)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - \overline{\rho\partial_t c''d''e''} - \overline{\rho u_n\partial_n c''d''e''} = && (256)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - (\overline{\partial_t\rho c''d''e''} - \overline{c''d''e''\partial_t\rho}) - \overline{\rho u_n\partial_n c''d''e''} = && (257)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - \partial_t\overline{\rho}\widetilde{c''d''e''} - (\overline{c''d''e''\partial_n\rho u_n} + \overline{\rho u_n\partial_n c''d''e''}) = && (258)\\
&= \overline{\rho}\partial_t\widetilde{c''d''e''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} - (\overline{\rho}\partial_t\widetilde{c''d''e''} + \widetilde{c''d''e''}\partial_t\overline{\rho}) - \overline{\partial_n c''d''e''\rho u_n} = && (259)\\
&= \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''e''} + \widetilde{c''d''e''}\partial_n\overline{\rho}\widetilde{u}_n - \overline{\partial_n c''d''e''\rho\widetilde{u}_n} - \overline{\partial_n c''d''e''\rho u''_n} = && (260)\\
&= -\overline{\partial_n c''d''e''\rho u''_n} && (261)\\
& && (262)
\end{aligned}
$$

$$
\overline{\rho D_t c''d''e''} = \overline{\rho c''d''D_t e''} + \overline{\rho c''e''D_t d''} + \overline{\rho d''e''D_t c''} \tag{263}
$$

$$
\begin{aligned}
\overline{c''d''\rho D_t e''} &= \overline{c''d''\rho D_t e} - \overline{c''d''\rho D_t\widetilde{e}} = \overline{c''d''\rho D_t e} - \overline{c''d''\rho(\partial_t\widetilde{e} + u_n\partial_n\widetilde{e})} = && (264)\\
&= \overline{c''d''\rho D_t e} - \overline{c''d''\rho\partial_t\widetilde{e}} - \overline{c''d''\rho u_n\partial_n\widetilde{e}} = \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\partial_t\widetilde{e} - (\overline{c''d''\rho\widetilde{u}_n\partial_n\widetilde{e}} + \overline{c''d''\rho u''_n\partial_n\widetilde{e}}) = && (265)\\
&= \overline{c''d''\rho D_t e} - (\overline{\rho}\widetilde{c''d''}\partial_t\widetilde{e} + \overline{\rho}\widetilde{c''d''}\widetilde{u}_n\partial_n\widetilde{e}) - \overline{\rho}\widetilde{c''d''u''_n}\partial_n\widetilde{e} = && (266)\\
&= \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\widetilde{D}_t\widetilde{e} - \overline{\rho}\widetilde{c''d''}u''_n\partial_n\widetilde{e} && (267)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{c''d''e''} = &\ \overline{c''d''\rho D_t e} - \overline{\rho}\widetilde{c''d''}\widetilde{D}_t\widetilde{e} - \overline{\rho}\widetilde{c''d''u''_n}\partial_n\widetilde{e}+ && (268)\\
&+\overline{c''e''\rho D_t d} - \overline{\rho}\widetilde{c''e''}\widetilde{D}_t\widetilde{d} - \overline{\rho}\widetilde{c''e''u''_n}\partial_n\widetilde{d}+ && (269)\\
&+\overline{d''e''\rho D_t c} - \overline{\rho}\widetilde{d''e''}\widetilde{D}_t\widetilde{c} - \overline{\rho}\widetilde{d''e''u''_n}\partial_n\widetilde{c}- && (270)\\
&-\overline{\partial_n c''d''e''\rho u''_n} && (271)
\end{aligned}
$$

22.3 Reynolds and Favrian variance

The Reynolds variance can be derived in following way:

$$
\begin{aligned}
\widetilde{D}_t\overline{c'd'} - \overline{D_t c'd'} &= +\partial_t\overline{c'd'} + \widetilde{u}_n\partial_n\overline{c'd'} - (\partial_t\overline{c'd'} + \overline{u_n\partial_n c'd'}) = && (272)\\
&= +\partial_t\overline{c'd'} + \widetilde{u}_n\partial_n\overline{c'd'} - \partial_t\overline{c'd'} - \widetilde{u}_n\partial_n\overline{c'd'} - \overline{u''_n\partial_n c'd'} = && (273)\\
&= -\overline{u''_n\partial_n c'd'} && (274)
\end{aligned}
$$

Next step:

$$
\widetilde{D}_t\overline{c'd'} = \overline{D_t c'd'} - \overline{u''_n\partial_n c'd'} = \overline{c'D_t d'} + \overline{d'D_t c'} - \overline{u''_n\partial_n c'd'} \quad (275)
$$

Next step:

$$
\begin{aligned}
D_t c' =& D_t c - D_t\overline{c} = D_t c - \widetilde{D}_t\overline{c} - u''_n\partial_n\overline{c} && (276)\\
D_t d' =& D_t d - D_t\overline{c} = D_t d - \widetilde{D}_t\overline{d} - u''_n\partial_n\overline{d} && (277)
\end{aligned}
$$

Now let's put these equations in the Equation 275:

$$\widetilde{D}_t\overline{c'd'} = +\overline{c'D_td'} + \overline{d'D_tc'} - \overline{u''_n\partial_n c'd'} = \tag{278}$$

$$= +\overline{c'(D_td - \widetilde{D}_t\overline{d} - u''_n\partial_n\overline{d})} + \overline{d'(D_tc - \widetilde{D}_t\overline{c} - u''_n\partial_n\overline{c})} - \overline{u''_n\partial_n c'd'} = \tag{279}$$

$$= +\overline{c'D_td} - \overline{c'u''_n}\partial_n\overline{d} + \overline{d'D_tc} - \overline{d'u''_n}\partial_n\overline{c} - \overline{u''_n\partial_n c'd'} = \tag{280}$$

$$= +\overline{c'D_td} - \overline{c'u''_n}\partial_n\overline{d} + \overline{d'D_tc} - \overline{d'u''_n}\partial_n\overline{c} - \partial_n\overline{u''_n c'd'} + \overline{c'd'\partial_n u''_n} \tag{281}$$

From this general formula by substituting $d = c$ we get:

$$\widetilde{D}_t\overline{c'c'} = +2\overline{c'D_tc} - 2\overline{c'u''_n}\partial_n\overline{c} - \overline{u''_n\partial_n c'c'} = \tag{282}$$

$$= +2\overline{c'D_tc} - 2\overline{c'u''_n}\partial_n\overline{c} - \partial_n\overline{u''_n c'c'} + \overline{c'c'\partial_n u''_n} \tag{283}$$

The Favrian variance can be easily derived from general equation for second-order moments Equation 254 ie.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = + \overline{c''\rho D_td} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_tc} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''d''u''_n} \tag{284}$$

Now, substitute $d = c$ and you'll get the equation for Favrian variance:

$$\overline{\rho}\widetilde{D}_t\widetilde{c''c''} = +2\overline{c''\rho D_tc} - 2\overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{c} - \overline{\partial_n\rho c''c''u''_n} \tag{285}$$

23 Derivation of second-order moments equations

23.1 Reynolds stress equation

We can derive the Reynolds stress equation using the general formula for second order moments, where we substitute c'' with u''_i and d'' with u''_j.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{286}$$

$$\underbrace{\overline{\rho}\widetilde{D}_t\widetilde{u''_i u''_j}}_{\overline{\rho}\widetilde{D}_t(\widetilde{R}_{ij}/\overline{\rho})} = \overline{u''_i\rho D_t u_j} - \underbrace{\overline{\rho}\widetilde{u''_i u''_n}\partial_n\widetilde{u}_j}_{\widetilde{R}_{in}\partial_n\widetilde{u}_j} + \overline{u''_j\rho D_t u_i} - \underbrace{\overline{\rho}\widetilde{u''_j u''_n}\partial_n\widetilde{u}_i}_{\widetilde{R}_{jn}\partial_n\widetilde{u}_i} - \underbrace{\overline{\partial_n \rho u''_i u''_j u''_n}}_{\nabla\cdot\overline{\rho u''_i u''_j u''_n}} \tag{287}$$

So, the general formula for Reynolds stress $\widetilde{R}_{ij}$ is:

$$\overline{\rho}\widetilde{D}_t(\widetilde{R}_{ij}/\overline{\rho}) = -\left(\widetilde{R}_{in}\partial_n\widetilde{u}_j + \widetilde{R}_{jn}\partial_n\widetilde{u}_i\right) - \left(\nabla_r\overline{\rho}\widetilde{u''_i u''_j u''_r} + \overline{G^R_{rr}}\right) + \overline{u''_i\rho D_t u_j} + \overline{u''_j\rho D_t u_i} \tag{288}$$

Mean equation for $\widetilde{R}_{rr}$

$$\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{rr}/\overline{\rho}\big) = -\left(\widetilde{R}_{rn}\partial_n\widetilde{u}_r + \widetilde{R}_{rn}\partial_n\widetilde{u}_r\right) - \left(\nabla_r 2f^r_k + \overline{G^R_{rr}}\right) + 2\overline{u''_r\rho D_t u_r} \tag{289}$$

where

$$\overline{u''_r\rho D_t u_r} = \overline{u''_r\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big) - G^M_r - \partial_r P + \rho g_r\right)} \tag{290}$$

$$\overline{u''_r\rho D_t u_r} = \overline{u''_r\left(\nabla_r\tau_{rr} + \nabla_\theta\tau_{r\theta} + \nabla_\phi\tau_{r\phi} - G^M_r - \partial_r P + \rho g_r\right)} \tag{291}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_r\nabla_r\tau_{rr}} = \overline{u''_r}\nabla_r\overline{\tau_{rr}} + \overline{u''_r\nabla_r\tau'_{rr}} = \cancelto{0}{\overline{u''_r}\nabla_r\overline{\tau_{rr}}} + \nabla_r(\overline{u''_r\tau'_{rr}}) - \overline{\tau'_{rr}\partial_r u''_r} \tag{292}$$

$$+\overline{u''_r\nabla_\theta\tau_{r\theta}} = \cancelto{0}{\overline{u''_r}\nabla_\theta\overline{\tau_{r\theta}}} + \overline{u''_r\nabla_\theta\tau'_{r\theta}} = \cancelto{0}{\nabla_\theta(\overline{u''_r\tau'_{r\theta}})} - \overline{\tau'_{r\theta}\frac{1}{r}\partial_\theta u''_r} \tag{293}$$

$$+\overline{u''_r\nabla_\phi\tau_{r\phi}} = \cancelto{0}{\overline{u''_r}\nabla_\phi\overline{\tau_{r\phi}}} + \overline{u''_r\nabla_\phi\tau'_{r\phi}} = \cancelto{0}{\nabla_\phi(\overline{u''_r\tau'_{r\phi}})} - \overline{\tau'_{r\phi}\frac{1}{r\sin\theta}\partial_\phi u''_r} \tag{294}$$

$$-\overline{u''_r G^M_r} = -\overline{u''_r G^M_r} \tag{295}$$

$$-\overline{u''_r\partial_r P} = -\overline{u''_r}\partial_r\overline{P} - \overline{u''_r\partial_r P'} = -\overline{u''_r}\partial_r\overline{P} - \nabla_r\overline{(u''_r P')} + \overline{P'\nabla_r u''_r} \tag{296}$$

$$+\overline{u''_r\rho g_r} \sim \overline{u''_r\rho}\overline{g}_r = 0 \tag{297}$$

Final equation is:

$$\begin{aligned}
\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = & -2\widetilde{R}_{rr}\partial_r\widetilde{u}_r - \nabla_r\widetilde{F}^R_{rrr} - \overline{G^R_{rr}} + 2\nabla_r(\overline{u''_r\tau'_{rr}}) - 2\overline{u''_r G^M_r} - 2\overline{u''_r}\partial_r\overline{P} - 2\nabla_r\overline{(u''_r P')} + 2\overline{P'\nabla_r u''_r} - \\
& -2\left(\overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}\frac{1}{r}\partial_\theta u''_r} + \overline{\tau'_{r\phi}\frac{1}{r\sin\theta}\partial_\phi u''_r}\right)
\end{aligned} \tag{298}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{rr}/\overline{\rho}\right) = -\nabla_r(2f^r_k + 2f_P + 2f^r_\tau) + 2W_b - 2\widetilde{R}_{rr}\partial_r\widetilde{u}_r + 2\overline{P'\nabla_r u''_r} - 2\overline{u''_r G^M_r} - \overline{G^R_{rr}} - 2\varepsilon^r_k \tag{299}$$

Mean equation for $\widetilde{R}_{\theta\theta}$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\left(\widetilde{R}_{\theta n}\partial_n\widetilde{u}_\theta + \widetilde{R}_{\theta n}\partial_n\widetilde{u}_\theta\right) - \left(\nabla_r 2f^\theta_k - \overline{G^R_{\theta\theta}}\right) + 2\overline{u''_\theta\rho D_t u_\theta} \tag{300}$$

$$\tag{301}$$

where

$$\overline{u''_\theta \rho D_t u_\theta} = \overline{u''_\theta \left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\theta r}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{\theta\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi[\tau_{\theta\phi}]\big) - G^M_\theta - \frac{1}{r}\partial_\theta P + \rho g_\theta \right)} \tag{302}$$

$$\overline{u''_\theta \rho D_t u_\theta} = \overline{u''_\theta\big(\nabla_r\tau_{\theta r} + \nabla_\theta\tau_{\theta\theta} + \nabla_\phi\tau_{\theta\phi} - G^M_\theta - \frac{1}{r}\partial_\theta P + \rho g_\theta\big)} \tag{303}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_\theta\nabla_r\tau_{\theta r}} = \cancelto{0}{\overline{u''_\theta}\nabla_r\tau_{\theta r}} + \nabla_r(\overline{u''_\theta\tau'_{\theta r}}) - \overline{\tau'_{\theta r}\partial_r u''_\theta} \tag{304}$$

$$+\overline{u''_\theta\nabla_\theta\tau_{\theta\theta}} = -\overline{\tau'_{\theta\theta}\frac{1}{r}\partial_\theta u''_\theta} \tag{305}$$

$$+\overline{u''_\theta\nabla_\phi\tau_{\theta\phi}} = -\overline{\tau'_{\theta\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\theta} \tag{306}$$

$$-\overline{u''_\theta G^M_\theta} = -\overline{u''_\theta G^M_\theta} \tag{307}$$

$$-\overline{u''_\theta\frac{1}{r}\partial_\theta P} = -\overline{u''_\theta}\frac{1}{r}\partial_\theta\overline{P} - \overline{u''_\theta\frac{1}{r}\partial_\theta P'} = -\cancelto{0}{\overline{u''_\theta}\frac{1}{r}\partial_\theta\overline{P}} - \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\theta(\sin\theta u''_\theta P')}} + \overline{P'\frac{1}{r\sin\theta}\partial_\theta(\sin\theta u''_\theta)} \tag{308}$$

$$+\overline{u''_\theta\rho g_\theta} = 0 \tag{309}$$

Final equation is:

$$\begin{aligned}\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{\theta\theta}/\overline{\rho}\big) = & -2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta - \nabla_r 2f^\theta_k - \overline{G^R_{\theta\theta}} + 2\nabla_r(\overline{u''_\theta\tau'_{\theta r}}) - 2\overline{u''_\theta G^M_\theta} + 2\overline{P'\nabla_\theta u''_\theta} - \\ & -2\left(\overline{\tau'_{\theta r}\partial_r u''_\theta} - \overline{\tau'_{\theta\theta}\frac{1}{r}\partial_\theta u''_\theta} - \overline{\tau'_{\theta\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\theta}\right)\end{aligned} \tag{310}$$

$$\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\theta\theta}/\overline{\rho}\right) = -\nabla_r(2f^\theta_k + 2f^\theta_\tau) - 2\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + 2\overline{P'\nabla_\theta u''_\theta} - 2\overline{u''_\theta G^M_\theta} - \overline{G^R_{\theta\theta}} - 2\varepsilon^\theta_k \tag{311}$$

Mean equation for $\widetilde{R}_{\phi\phi}$

$$\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{\phi\phi}/\overline{\rho}\big) = -\Big(\widetilde{R}_{\phi n}\partial_n\widetilde{u}_\phi + \widetilde{R}_{\phi n}\partial_n\widetilde{u}_\phi\Big) - \Big(\nabla_r 2f_k^\phi + \overline{G^R_{\phi\phi}}\Big) + 2\overline{u''_\phi\rho D_t u_\phi} \tag{312}$$

where

$$\overline{u''_\phi\rho D_t u_\phi} = \overline{u''_\phi\left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{\phi r}]\big) + \frac{1}{r\sin\phi}\partial_\phi\big(\sin\phi[\tau_{\phi\theta}]\big) + \frac{1}{r\sin\phi}\partial_\phi[\tau_{\phi\phi}]\big) - G^M_\phi - \frac{1}{r\sin\theta}\partial_\phi P + \rho g_\phi\right)} \tag{313}$$

$$\overline{u''_\phi\rho D_t u_\phi} = \overline{u''_\phi\big(\nabla_r\tau_{\phi r} + \nabla_\theta\tau_{\phi\theta} + \nabla_\phi\tau_{\phi\phi} - G^M_\phi - \frac{1}{r\sin\theta}\partial_\phi P + \rho g_\phi\big)} \tag{314}$$

Some terms can be further manipulated in following way:

$$+\overline{u''_\phi\nabla_r\tau_{\phi r}} = \cancelto{0}{\overline{u''_\phi}\nabla_r\overline{\tau_{\phi r}}} + \nabla_r\big(\overline{u''_\phi\tau'_{\phi r}}\big) - \overline{\tau'_{\phi r}\partial_r u''_\phi} \tag{315}$$

$$+\overline{u''_\phi\nabla_\theta\tau_{\phi\theta}} = -\overline{\tau'_{\phi\theta}\frac{1}{r}\partial_\theta u''_\phi} \tag{316}$$

$$+\overline{u''_\phi\nabla_\phi\tau_{\phi\phi}} = -\overline{\tau'_{\phi\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\phi} \tag{317}$$

$$-\overline{u''_\phi G^M_\phi} = -\overline{u''_\phi G^M_\phi} \tag{318}$$

$$-\overline{u''_\phi\frac{1}{r\sin\theta}\partial_\phi P} = -\overline{u''_\phi}\frac{1}{r\sin\theta}\partial_\phi\overline{P} - \overline{u''_\phi\frac{1}{r\sin\theta}\partial_\phi P'} = \cancelto{0}{-\overline{u''_\phi}\frac{1}{r\sin\theta}\partial_\phi\overline{P}} - \cancelto{0}{\overline{\frac{1}{r\sin\theta}\partial_\phi(u''_\phi P')}} + \overline{P'\frac{1}{r\sin\theta}\partial_\phi(u''_\phi)} \tag{319}$$

$$+\overline{u''_\phi\rho g_\phi} = 0 \tag{320}$$

Final equation is:

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\big(\widetilde{R}_{\phi\phi}/\overline{\rho}\big) = & - 2\widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi - \nabla_r 2f_k^\phi - \overline{G_{\phi\phi}^R} + 2\nabla_r(\overline{u''_\phi\tau'_{\phi r}}) - 2\overline{u''_\phi G_\phi^M} + 2\overline{P'\nabla_\phi u''_\phi} - \\
& - 2\left(\overline{\tau'_{\phi r}\partial_r u''_\phi} - \overline{\tau'_{\phi\theta}\frac{1}{r}\partial_\theta u''_\phi} - \overline{\tau'_{\phi\phi}\frac{1}{r\sin\theta}\partial_\phi u''_\phi}\right)
\end{aligned}
\tag{321}
$$

$$
\overline{\rho}\widetilde{D}_t\left(\widetilde{R}_{\phi\phi}/\overline{\rho}\right) = - \nabla_r(2f_k^\phi + 2f_\tau^\phi) - 2\widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi + 2\overline{P'\nabla_\phi u''_\phi} - 2\overline{u''_\phi G_\phi^M} - \overline{G_{\phi\phi}^R} - 2\varepsilon_k^\phi
\tag{322}
$$

23.2 Turbulent kinetic energy equations

$$
\begin{aligned}
\widetilde{k} &= + \frac{1}{2}\widetilde{R}_{ii}/\overline{\rho} && (323)\\
\overline{\rho}\widetilde{D}_t\widetilde{k} &= - \nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k && (324)\\
\widetilde{k}^r &= + \frac{1}{2}\widetilde{R}_{rr}/\overline{\rho} && (325)\\
\overline{\rho}\widetilde{D}_t\widetilde{k}^r &= - \nabla_r(f_k^r + f_P) - \widetilde{R}_{rr}\partial_r\widetilde{u}_r + W_b + \overline{P'\nabla_r u''_r} + \mathcal{G}_k^r + \mathcal{N}_{kr} && (326)\\
\widetilde{k}^h &= + \widetilde{k}_\theta + \widetilde{k}_\phi = + \frac{1}{2}\big(\widetilde{R}_{\theta\theta} + \widetilde{R}_{\phi\phi}\big)/\overline{\rho} && (327)\\
\overline{\rho}\widetilde{D}_t\widetilde{k}^h &= - \nabla_r f_k^h - (\widetilde{R}_{\theta r}\partial_r\widetilde{u}_\theta + \widetilde{R}_{\phi r}\partial_r\widetilde{u}_\phi) + (\overline{P'\nabla_\theta u''_\theta} + \overline{P'\nabla_\phi u''_\phi}) + \mathcal{G}_k^h + \mathcal{N}_{kh} && (328)
\end{aligned}
$$

23.3 Turbulent mass flux equation

The turbulent mass flux equation can be derived in the following way:

$$
\rho D_t\widetilde{c} - \rho\widetilde{D}_t\widetilde{c} = \rho\partial_t\widetilde{c} + \rho u_n\partial_n\widetilde{c} - [\rho\partial_t\widetilde{c} + \rho\widetilde{u}_n\partial_n\widetilde{c}_n] = \rho(u_n - \widetilde{u}_n)\partial_n\widetilde{c} = \rho u''_n\partial_n\widetilde{c} \tag{329}
$$

$$
\rho D_t c'' = \rho D_t c - \rho D_t\widetilde{c} = \rho D_t c - \rho\widetilde{D}_t\widetilde{c} - \rho u''_n\partial_n\widetilde{c} = \rho D_t c - \frac{\rho}{\overline{\rho}}[\overline{\rho}\widetilde{D}_t\widetilde{c}] - \rho u''_n\partial_n\widetilde{c} \tag{330}
$$

$$
\rho D_t u''_r = + \left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\big([\tau_{r\phi}]\big) - G_r^M - \frac{\partial P}{\partial r}\right) + \rho g_r + \tag{331}
$$

$$
+ \frac{\rho}{\overline{\rho}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr} - \overline{\tau_{rr}}) + \overline{G_r^M} + \frac{\partial}{\partial r}\overline{P} - \overline{\rho}\widetilde{g}_r\right) - \rho u''_n\partial_n\widetilde{u_r} \tag{332}
$$

$$
\widetilde{D}_t\overline{u''_i} = \overline{D_t u''_i - u''_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{u_n\partial_n u''_i} - \overline{u''_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{u_n\partial_n u''_i} + \overline{\widetilde{u_n}\partial_n u''_i} - \overline{u_n\partial_n u''_i} = \overline{\partial_t u''_i} + \overline{\widetilde{u_n}\partial_n u''_i} = \widetilde{D}_t\overline{u''_i} \tag{333}
$$

$$
\overline{\rho}\widetilde{D_t}\overline{u''_i} = \overline{\frac{\overline{\rho}}{\rho}[\rho D_t u''_i]} - \overline{\overline{\rho}u''_n\partial_n u''_i} \tag{334}
$$

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\overline{u''_r} &= \overline{\frac{\overline{\rho}}{\rho}[\rho D_t u''_r]} - \overline{\overline{\rho} u''_n \partial_n u''_r} = \\
&= \overline{\frac{\overline{\rho}}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau_{rr}\big) - G^M_r - \frac{\partial}{\partial r}P + \rho g_r + \frac{\rho}{\overline{\rho}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr} - \overline{\tau_{rr}}) + \overline{G^M_r} + \frac{\partial}{\partial r}\overline{P} - \overline{\rho}\widetilde{g}_r\right) - \rho u''_n \partial_n \widetilde{u_r}\right]} - \overline{\overline{\rho} u''_n \partial_n u''_r} = && (335) \\
&= \overline{\frac{\overline{\rho}}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau_{rr}\big)\right] - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau_{rr}\big)\right]} - \overline{\frac{\overline{\rho}}{\rho}G^M_r + \overline{G^M_r}} - \overline{\frac{\overline{\rho}}{\rho}\left[\frac{\partial}{\partial r}P\right] + \left[\frac{\partial}{\partial r}P\right]} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\overline{\rho} u''_n \partial_n \widetilde{u_r}} - \overline{\overline{\rho} u''_n \partial_n u''_r} && (336) \\
&= \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau_{rr}\big)} - \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]G^M_r} - \overline{\left[\frac{\overline{\rho}}{\rho} - 1\right]\frac{\partial}{\partial r}P} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} = && (337) \\
&= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} - \overline{\frac{\rho'}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau_{rr}\big)} + \overline{\frac{\rho'}{\rho}G^M_r} + \overline{\frac{\rho'}{\rho}\frac{\partial}{\partial r}P} = && (338) \\
&= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} - \overline{\rho' v\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2(\overline{\tau_{rr}} + \tau'_{rr}\big)} + \overline{\rho' v\frac{\partial}{\partial r}(\overline{P} + P')} + \overline{\rho' v G^M_r} = && (339) \\
&= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} - \overline{\rho' v}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\overline{\tau_{rr}}\big) - \frac{\partial}{\partial r}\overline{P}\right) - \overline{\rho' v\left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau'_{rr}\big) - \frac{\partial}{\partial r}P'\right)} + \overline{\rho' v G^M_r} = && (340) \\
&= +\frac{1}{r^2}\frac{\partial}{\partial r}r^2(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} + b\left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\overline{\tau_{rr}}\big) - \frac{\partial}{\partial r}\overline{P}\right) - \overline{\rho' v\left(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\tau'_{rr}\big) - \frac{\partial}{\partial r}P'\right)} + \overline{\rho' v G^M_r} = && (341) \\
&= +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{\mathbf{u}'' \cdot \nabla u_r} - b\nabla_r\overline{\tau}_{rr} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} - \overline{\rho' v \nabla_r \tau'_{rr}} + \overline{\rho' v G^M_r} && (342)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\overline{u''_r} &= +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n \partial_n u_r} - b\nabla_r\overline{\tau}_{rr} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} - \overline{\rho' v \nabla_r \tau'_{rr}} + \overline{\rho' v G^M_r} = && (343) \\
&= +\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\overline{u''_n}\partial_n \overline{u}_r - \overline{\rho}\partial_n \overline{u''_n u'_r} + \overline{\rho}\overline{u'_r \partial_n u''_n} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} - \overline{\rho' v \nabla_r \tau'_{rr}} + \overline{\rho' v G^M_r} = && (344) \\
&= (+\nabla_r(\widetilde{R}_{rr}) - \overline{\rho}\nabla_r \overline{u''_r u'_r}) - \overline{\rho}\overline{u''_r}\nabla_r \overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} - \overline{\rho' v \nabla_r \tau'_{rr}} + \overline{\rho' v G^M_r} = && (345) \\
&= -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r \overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v \partial_r P'} - \overline{\rho' v \nabla_r \tau'_{rr}} + \overline{\rho' v G^M_r} && (346)
\end{aligned}
$$

23.4 Density-specific volume covariance equation

The density-specific volume ($v = 1/\rho$) covariance ($b = -\overline{\rho' v'}$) equation can be derived from the continuity equation in the following way.

$$\partial_t \rho + \partial_n(\rho u_n) = 0 \tag{347}$$
$$\partial_t \overline{\rho} + \widetilde{u}_n \partial_n \overline{\rho} = -\overline{\rho}\partial_n \widetilde{u}_n \tag{348}$$
$$\partial_t \overline{\rho} + \overline{u}_n \partial_n \overline{\rho} - \overline{u''_n}\partial_n \overline{\rho} = -\overline{\rho}\partial_n \overline{u}_n + \overline{\rho}\partial_n \overline{u''_n} \tag{349}$$
$$\overline{D}_t \overline{\rho} = +\overline{u''_n}\partial_n \overline{\rho} - \overline{\rho}\partial_n \overline{u}_n + \overline{\rho}\partial_n \overline{u''_n} \tag{350}$$
$$\overline{D}_t \overline{\rho} = +(\overline{u''_n}\partial_n \overline{\rho} + \overline{\rho}\partial_n \overline{u''_n}) - \overline{\rho}\partial_n \overline{u}_n \tag{351}$$
$$\overline{D}_t \overline{\rho} = -\overline{\rho}\partial_n \overline{u}_n + \partial_n(\overline{u''_n}\overline{\rho}) \tag{352}$$

$$\partial_t \rho + \partial_n(\rho u_n) = 0 \tag{353}$$
$$\partial_t(1/v) + \partial_n(u_n/v) = 0 \tag{354}$$
$$-\partial_t v + v\partial_n u_n - u_n \partial_n v = 0 \tag{355}$$
$$\partial_t v - v\partial_n u_n - u_n\partial_n v + u_n \partial_n v + u_n \partial_n v = 0 \tag{356}$$
$$\partial_t v - \partial_n(v u_n) + 2u_n \partial_n v = 0 \tag{357}$$
$$\partial_t \overline{v} - \partial_n(\overline{v u_n}) + 2\overline{u_n \partial_n v} = 0 \tag{358}$$
$$\overline{D}_t \overline{v} = +\overline{v}\partial_n \overline{u}_n - \partial_n \overline{u'_n v'} + 2\overline{v' \partial_n u'_n} \tag{359}$$

$$b = -\overline{\rho' v'} = \overline{\rho v} - 1 \tag{360}$$
$$\overline{D}_t b = \overline{\rho}\overline{D}_t \overline{v} + \overline{v}\ \overline{D}_t \overline{\rho} \tag{361}$$

Using previously derived equations for $\overline{D}_t \overline{v}$ and $\overline{D}_t \overline{\rho}$ we get:

$$\overline{D}_t b = -\overline{\rho}\partial_n \overline{\rho' u'_n} + 2\overline{\rho}\overline{v' \partial_n u'_n} + \overline{v}\partial_n \overline{\rho}\overline{u''_n} \tag{362}$$

23.5 Mean internal energy flux equation

We can derive the internal energy flux equation using the general formula for second order moments, where we substitute c with ϵ_I and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{363}$$

$$\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) = \mathcal{N}_{fI} - \nabla_r f_I^r - f_I\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{\epsilon_I} - \overline{\epsilon''_I}\partial_r\overline{P} - \overline{\epsilon''_I\partial_r P'} - \overline{u''_r\,(Pd)} + \overline{u''_r(\mathcal{S} + \nabla\cdot F_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \tag{364}$$

23.6 Mean enthalpy flux equation

We can derive the enthalpy flux equation using the general formula for second order moments, where we substitute c with h and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{365}$$

$$\overline{\rho}\widetilde{D}_t(f_h/\overline{\rho}) = -\nabla_r f_h^r - f_h\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{h} - \overline{h''}\partial_r\overline{P} - \overline{h''\partial_r P'} - \Gamma_1\overline{u''_r\,(Pd)} + \Gamma_3\overline{u''_r(\mathcal{S} + \nabla\cdot F_T)} + \mathcal{G}_h + \mathcal{N}_h \tag{366}$$

23.7 Mean entropy flux equation

We can derive the entropy flux equation using the general formula for second order moments, where we substitute c with s and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{367}$$

$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u''_r(\mathcal{S} + \nabla\cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \tag{368}$$

23.8 Mean composition flux equation

We can derive the composition flux equation using the general formula for second order moments, where we substitute c with X_α and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{369}$$

$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_\alpha - \overline{X''_\alpha}\partial_r\overline{P} - \overline{X''_\alpha\partial_r P'} + \overline{u''_r\rho\dot{X}_\alpha^{\rm nuc}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \tag{370}$$

$$\tag{371}$$

23.9 Mean A and Z flux equations

We can derive the composition flux equation using the general formula for second order moments, where we substitute c with A or Z and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{372}$$

$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = \mathcal{N}_{fA} - \nabla_r f_A^r - f_A\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{A} - \overline{A''}\partial_r\overline{P} - \overline{A''\partial_r P'} - \overline{u''_r\rho A^2\Sigma_\alpha\dot{X}_\alpha^{\rm nuc}/A_\alpha} + \mathcal{G}_A \tag{373}$$

$$\begin{aligned}\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = &\ \mathcal{N}_{fZ} - \nabla_r f_Z^r - f_Z\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{Z} - \overline{Z''}\partial_r\overline{P} - \overline{Z''\partial_r P'} - \overline{u''_r\rho ZA\Sigma_\alpha(\dot{X}_\alpha^{\rm nuc}/A_\alpha)} - \\ & - \overline{u''_r\rho A\Sigma_\alpha(Z_\alpha\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z\end{aligned} \tag{374}$$

23.10 Mean angular momentum flux equation

We can derive the angular momentum flux equation using the general formula for second order moments, where we substitute c with j_z and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{375}$$

$$\overline{\rho}\widetilde{D}_t(f_{jz}/\rho) = -\nabla_r f_{jz}^r - f_{jz}\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{j}_z - \overline{j''_z}\partial_r\overline{P} - \overline{j''_z\partial_r P'} + \mathcal{G}_{jz} + \mathcal{N}_{jz} \tag{376}$$

24 Derivation of Reynolds and Favrian variance equations

The variance evolution equations can be found by using the general formula for second-order moments and similar algebraic manipulation presented in previous sections.

25 Divergence of tensors in spherical geometry up to third order

BACKGROUND READING:

CONTINUUM MECHANICS (Lecture Notes)
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$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \quad : \text{nabla operator} \qquad \mathbf{V} = \sum_i V_i \mathbf{e_i} \quad : \text{tensor of first order (vector)} \tag{377}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \quad : \text{tensor of second order} \tag{378}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{tensor of third order} \tag{379}$$

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{h_i}\left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m\right] \quad : \text{div of first order tensor (vector)} \tag{380}$$

$$\nabla \cdot \mathbf{S} = \sum_{ij} \frac{1}{h_i}\left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right]\mathbf{e_j} \quad : \text{div of second order tensor} \tag{381}$$

$$\nabla \cdot \mathbf{T} = \sum_{ijk} \frac{1}{h_i}\left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm}\right](\mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{div of third order tensor} \tag{382}$$

$$x_1 = r \qquad x_2 = \theta \qquad x_3 = \phi \qquad \text{(coordinates)} \tag{383}$$

$$\mathbf{e_1} = \mathbf{e_r} \qquad \mathbf{e_2} = \mathbf{e_\theta} \qquad \mathbf{e_3} = \mathbf{e_\phi} \qquad \text{(unit base vectors)} \tag{384}$$

$$h_1 = h_r = 1 \qquad h_2 = h_\theta = r \qquad h_3 = h_\phi = r\sin\theta \qquad \text{(scale factors)} \tag{385}$$

$$\begin{pmatrix} \Gamma^\theta_{r\theta} = \ 1 & \Gamma^\phi_{r\phi} = \ \sin\theta & \Gamma^\phi_{\theta\phi} = \ \cos\theta \\ \Gamma^r_{\theta\theta} = -1 & \Gamma^r_{\phi\phi} = -\sin\theta & \Gamma^\theta_{\phi\phi} = -\cos\theta \end{pmatrix} \qquad \text{Christoffel symbols} \tag{386}$$

Divergence of first order tensor $\nabla \cdot \mathbf{V}$

$$\frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{387}$$

Divergence of second order tensor $\nabla \cdot \mathbf{S}$

$$S_r(\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{388}$$

$$S_\theta(\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{389}$$

$$S_\phi(\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{390}$$

Divergence of third order tensor $\nabla \cdot \mathbf{T}$

$$T_{rr}\ (\mathbf{e_r}\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rrr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta rr}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi rr}}{\partial\phi} - \frac{T_{\theta\theta r}}{r} - \frac{T_{\theta r\theta}}{r} - \frac{T_{\phi\phi r}}{r} - \frac{T_{\phi r\phi}}{r} \tag{391}$$

$$T_{r\theta}\ (\mathbf{e_r}\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\theta}}{\partial\phi} - \frac{T_{\theta\theta\theta}}{r} + \frac{T_{\theta rr}}{r} - \frac{T_{\phi\phi\theta}}{r} - \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{392}$$

$$T_{r\phi}\ (\mathbf{e_r}\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\phi}}{\partial\phi} + \frac{T_{\theta\theta\phi}}{r} - \frac{T_{\phi\phi\phi}}{r} + \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{393}$$

$$T_{\theta r}\ (\mathbf{e}_\theta\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta r}}{\partial\phi} + \frac{T_{\theta rr}}{r} - \frac{T_{\theta\theta\theta}}{r} - \frac{T_{\phi\phi r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}}{r} \tag{394}$$

$$T_{\theta\theta}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\theta}}{\partial\phi} + \frac{T_{\theta r\theta}}{r} + \frac{T_{\theta\theta r}}{r} - \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{395}$$

$$T_{\theta\phi}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\phi}}{\partial\phi} + \frac{T_{\theta r\phi}}{r} + \frac{T_{\phi\theta r}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} \tag{396}$$

$$T_{\phi r}\ (\mathbf{e}_\phi\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi r}}{\partial\phi} - \frac{T_{\theta\phi\theta}}{r} + \frac{T_{\phi rr}}{r} + \frac{T_{\phi\theta r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\phi\phi}}{r} \tag{397}$$

$$T_{\phi\theta}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\theta}}{\partial\phi} + \frac{T_{\theta\phi r}}{r} + \frac{T_{\phi r\theta}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{398}$$

$$T_{\phi\phi}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\phi}}{\partial\phi} + \frac{T_{\phi r\phi}}{r} + \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} + \frac{T_{\phi\phi r}}{r} + \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} \tag{399}$$

26 Periodic boundary conditions and properties of divergence angular components

All of the simulation data studied in this document use domains that are wedges in a spherical coordinate system and have period boundary conditions in the θ and ϕ directions. Therefore, the angular components of divergence terms vanish upon averaging.

$$\theta \in [\theta_L, \theta_R] \tag{400}$$

$$\phi \in [\phi_L, \phi_r] \tag{401}$$

$$\langle \nabla \cdot \mathbf{q}\ (r,t) \rangle = \frac{1}{\Delta\Omega} \int \nabla \cdot \mathbf{q}\ (r,\theta,\phi,t')\ d\Omega \qquad \mathbf{q} = q(q_r, q_\theta, q_\phi) \tag{402}$$

Proof:

$$\begin{aligned}\langle \nabla_\theta q_\theta \rangle =& \int_{\Delta\Omega} \frac{1}{r\sin\theta} \partial_\theta(\sin\theta[q_\theta])\ d\Omega = \int_{\Delta\theta}\int_{\Delta\phi} \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[q_\theta])\ \sin\theta\ d\theta\ d\phi = \\ =& \frac{1}{r}\int_{\Delta\theta}\int_{\Delta\phi} \partial_\theta(\sin\theta[q_\theta])\ d\theta\ d\phi = \frac{1}{r}\int_{\Delta\theta} \partial_\theta \langle \sin\theta[q_\theta]\rangle_\phi\ d\theta = \frac{1}{r}(\langle \sin\theta[q_\theta]\rangle_\phi)^{\theta_L}_{\theta_R} = 0\end{aligned} \tag{403}$$

$$\tag{404}$$

$$\begin{aligned}\langle \nabla_\phi q_\phi \rangle =& \int_{\Delta\Omega} \frac{1}{r\sin\theta} \partial_\phi[q_\phi] = \int_{\Delta\theta}\int_{\Delta\phi} \frac{1}{r\sin\theta}\partial_\theta[q_\phi]\ \sin\theta\ d\theta\ d\phi = \\ =& \frac{1}{r}\int_{\Delta\theta} \partial_\phi[q_\phi]\ d\theta d\phi = \frac{1}{r}\int_{\Delta\theta} \partial_\phi \langle \rho U_\phi \rangle_\theta\ d\phi = \frac{1}{r}(\langle q_\phi \rangle_\theta)^{\phi_L}_{\phi_R} = 0\end{aligned} \tag{405}$$

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