10M Geometry Solutions

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IMO 2002/2

Simple angle chasing reveals that triangles ODA and AIO are congruent, and thus $AI = AO = r$, where r is the radius of the circle. Furthermore, $AF = FO = AO = AE$, so F, I, E lie on a circle with center A. Lastly, notice that A is the arc midpoint of arc FE, so the incenter-excenter lemma applies.

IMO 2003/4

Notice that P, Q, R lie on the Simson line.

Let X be the second intersection of line DQ with the circle ω circumscribing $ABCD$. We claim that $\overline{BX} \parallel \overline{RQ}$. Using directed angles,

$$\angle BXD = \angle BAD = \angle RAD = \angle RQD,$$

and thus, $\overline{BX}$ is parallel to the Simson line.

Next, note that by the angle bisector theorem, the bisector condition is equivalent to $ABCD$ being harmonic. Additionally, we have

$$(A, C; B, D) \stackrel{X}{=} (A, C; \overline{BX} \cap \overline{AC}, Q) \stackrel{B}{=} (R, P; P_\infty, Q),$$

and since $(R, P; P_\infty, Q) = -1 \iff PQ = QR$, we are done.

IMO 2010/4

Let D be the intersection point of the other tangent from S, so $ABCD$ is harmonic. Let N be the second intersection point of line DP with the circle. Then, projecting through a conic, we have

$$-1 = (A, B; C, D) \stackrel{P}{=} (K, L; M, N).$$

It suffices to show $\overline{MN}$ is a diameter. Using the fact that D, P, C lie on a circle centered at S,

$$\begin{aligned}\angle MON &= \angle MOD + \angle DOC + \angle CON \\ &= 2\angle MPD + \angle DOC \\ &= 2\angle CPD + \angle DOC \\ &= \angle CSD + \angle DOC \\ &= 0.\end{aligned}$$

IMO 2018/1

Let M_B be the midpoint of arc AC and let M_C be the midpoint of arc AB. We first claim that M_BM_C is parallel to DE. If we let X be the intersection of lines AB and M_BM_C and Y be the intersection of lines AC and M_BM_C, this follows by noticing triangles AXM_C and AYM_B are similar.

Let F' be the reflection of F over the perpendicular bisector of AB, and define G' similarly. Notice that F' and G' lie on (ABC) and that $FF'AD$ and $GG'AE$ are parallelograms. It then follows that arcs FF' and GG' have the same measure. Finally, since M_C is the arc midpoint of FF' and M_B is the arc midpoint of GG', lines FG and M_BM_C are parallel, and we are done.

IMO 2020/1

Let O be the circumcenter of triangle PAB. Then, through angle chasing, $DPOA$ and $CPOB$ are cyclic. Furthermore, in $(DPOA)$, O is the arc midpoint of arc PA and similarly for the other circle. Thus, by the incenter-excenter lemma, O is the desired intersection point.

USAMO 2000/5

Let O_k denote the center of w_k. In order to fulfill the conditions, we need that O_1, O_4, and O_7 be on the perpendicular bisector of AB, O_2 and O_5 be on the perpendicular bisector of BC, and O_3 and O_6 be on the perpendicular bisector of AC. Furthermore, so that the circles are tangent to each other, we need the following groups of points to be collinear: O_1BO_2, O_2CO_3, O_3AO_4, O_4BO_5, O_5CO_6, and O_6AO_7. These conditions are enough to uniquely determine all of the points from O_1.

It suffices to show O_1, A, and O_6 are collinear. We have

$$\begin{aligned}
\angle O_1AO &= -\angle O_1BO \\
&= -\angle O_2BO \\
&= \angle O_2CO \\
&= \angle O_3CO \\
&= -\angle O_3AO \\
&= -\angle O_4AO \\
&= \angle O_4BO \\
&= \angle O_5BO \\
&= -\angle O_5CO \\
&= -\angle O_6CO \\
&= \angle O_6AO,
\end{aligned}$$

so we are done.

USA TSTST 2017/1

By homothety, $PA^2 = PM \cdot PN$, so P is on the radical axis of the circumcircle and the nine-point circle. Since $RQ \cdot RA = RF \cdot RE$, R is also on this radical axis. Thus, $\overline{PR} \perp \overline{ON_9} = \overline{OH}$.