5M Geometry Solutions

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IMO 2004/1

Let K be the intersection of lines AR and BC. We show that $BMRK$ and $CNRK$ are cyclic, leading to the desired result.

We will show that $BMRK$ is cyclic, and a symmetrical argument can be used to show $CNRK$ is cyclic.

Claim: $AMRN$ is cyclic. Let R' be the phantom point corresponding to the second intersection of (AMN) and line AK. Since $\angle R'AM = \angle R'AN$, we know that R' must be the arc midpoint of arc MN, meaning that it is the intersection of the perpendicular bisector of MN and line AK. However, since triangle OMN is isosceles, R is also the intersection of the perpendicular bisector of MN and line AK. Thus, $R' = R$ and $AMRN$ is cyclic.

Simple angle chasing gives us that $\angle RMN = \frac{1}{2}\angle A$ and $\angle OMN = \angle A$, and thus $\angle OMR = \frac{1}{2}\angle A = \angle BAK$. Putting things together, we have (using directed angles)

$$\angle BMR = \angle BMO + \angle OMR = \angle OBM + \angle BAK = \angle KBA + \angle BAK = \angle BKA = \angle BKR.$$

Thus, we are done.

IMO 2007/4

We proceed by barycentric coordinates. Compute $R = (a^2+ab : b^2+ab : -c^2)$, $K = (0 : 1 : 1)$, and $L = (1 : 0 : 1)$. We can compute P and Q using the equation for the perpendicular bisector. The perpendicular bisector of BC is given by $a^2(z-y) + x(c^2 - b^2) = 0$. Then, we compute $P = (a^2 : ab : ab + b^2 - c^2)$. By symmetry, $Q = (ab : b^2 : ab + a^2 - c^2)$.

Finally, we compute the areas using the determinant and see that they are equal.

USAMO 2001/4

We proceed with vectors. Let $\mathbf{p} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{BA}$, and $\mathbf{c} = \overrightarrow{CA}$. We wish to prove $\mathbf{b} \cdot \mathbf{c} > 0$. The obtuse triangle condition can be written as

$$\|\mathbf{p} - \mathbf{b}\|^2 + \|\mathbf{p} - \mathbf{c}\|^2 < \|\mathbf{p}\|^2.$$

Using the identity $\|\mathbf{v}\| = \mathbf{v} \cdot \mathbf{v}$, we have

$$\begin{aligned}(\mathbf{p} - \mathbf{b}) \cdot (\mathbf{p} - \mathbf{b}) + (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) &= 2(\mathbf{p} \cdot \mathbf{p}) - 2\mathbf{p} \cdot (\mathbf{b} + \mathbf{c}) + \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} \\ &< \mathbf{p} \cdot \mathbf{p}.\end{aligned}$$

Simplifying and adding $2(\mathbf{b} \cdot \mathbf{c})$ to both sides, we get

$$\mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot (\mathbf{b} + \mathbf{c}) + (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c}) < 2(\mathbf{b} \cdot \mathbf{c}).$$

The left side factors as $(\mathbf{p} - (\mathbf{b} + \mathbf{c})) \cdot (\mathbf{p} - (\mathbf{b} + \mathbf{c}))$. That is nonnegative, so we conclude that $\mathbf{b} \cdot \mathbf{c} > 0$, which is what we wanted to prove.

USAMO 2020/1

We claim $\triangle OO_1O_2 \sim \triangle CBA$.

Since OO_2 is the perpendicular bisector of XB and OO_1 is the perpendicular bisector of XA, we have

$$\angle O_2OO_1 = \angle O_2OX + \angle XOO_1 = \frac{1}{2}(\angle BOX + \angle XOA) = \frac{1}{2}\angle BOA = \angle C.$$

The other angle is a bit more tricky. For clarification, all denoted arcs are with respect to the circle with center O_1. Since O_1O_2 is the perpendicular bisector of XD, we see that $\angle XO_1O_2 = \frac{1}{2}\widehat{XD}$. Similarly, $\angle XO_1O = \frac{1}{2}\widehat{XA}$. Thus,

$$\angle O_2O_1O = \angle XO_1O - \angle XO_1O_2 = \frac{1}{2}(\widehat{XA} - \widehat{XD}) = \frac{1}{2}\widehat{AD}.$$

Furthermore,

$$\frac{1}{2}\widehat{AD} = \angle DXA = \angle CXA = \angle B.$$

Thus, we have AA similarity. This means minimizing the area is equivalent to minimizing the length of O_1O_2.