Test 1 Solutions

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Problem 1

Problem 2

We claim only $n = 7$ works. We can verify manually up to $n = 11$:

$$\begin{aligned}
4? &= 6 \neq 2(4) + 16, \\
5? &= 6 \neq 2(5) + 16, \\
6? &= 30 \neq 2(6) + 16, \\
8? &= 210 \neq 2(8) + 16, \\
&\vdots \\
11? &= 210 \neq 2(11) + 16.
\end{aligned}$$

Then, assume $n \geq 12$. By Bertrand's postulate, there exists p satisfying $5 < \frac{n}{2} < p < n$. Thus, since 2, 3, 5, and p are distinct primes less than n, we have

$$n? \geq 2 \cdot 3 \cdot 5 \cdot p > 30 \cdot \frac{n}{2} > 2n + 16,$$

and hence it is impossible for any $n \geq 12$ to satisfy the condition.

Problem 3

The answer is no. Rewrite the equation as $\frac{x!}{y+1} = (y!)^2$. As x gets large enough, Bertrand's postulate tells us that there is a prime p satisfying $\frac{x}{2} < p < x$. Note that p can only divide $x!$ once. Thus, in order for the LHS to be a perfect square, p must divide $y + 1$ exactly once. If $y + 1 \geq 2p > x$, the original equation obviously cannot be satisfied because the RHS is larger. Thus, we must have $y + 1 = p$.

This holds for any p satisfying $\frac{x}{2} < p < x$. Thus, there must only be one prime between $\frac{x}{2}$ and x. Nagura's result tells us that if $x > 25$, there will always be a prime between $\frac{x}{2}$ and $\frac{6}{5} \cdot \frac{x}{2}$ and a prime between $\frac{5}{6} \cdot x$ and x. Thus, the original equation cannot hold after x exceeds a finite value, and since there is obviously at most one solution for y for each x, there must be finitely many solutions to the original equation.

Note: I was unfortunately unable to solve the problem without resorting to this somewhat obscure result. :(

Problem 4

Problem 5

I got that $\varphi(n)$ is a power of 2 iff $n = 2^a p_1 p_2 \dots p_k$, where $a \geq 0$ and each p_i is a distinct prime that is one more than a power of 2. Not sure how to continue...

Problem 6

Problem 7

Problem 8