Symmetric Polynomials Solutions

Michael Middlezong

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Problem 1 (USAMO 1973/4)

Consider the cubic polynomial $(t-x)(t-y)(t-z)$. From Newton's sums and Vieta's, this cubic polynomial must equal $t^3 - 3t^2 + 3t - 1$. The only factorization of this is $(t-1)^3$, so the only solution must be $(x, y, z) = (1, 1, 1)$.

Problem 6

Consider the polynomial with roots $r + s$, $s + t$, and $r + t$. We will find its coefficients and show that it is the desired polynomial. Using Vieta's, we can see that

$$A = -2(r + s + t) = -14.$$

We can also see that

$$B = (r + s)(s + t) + (s + t)(r + t) + (r + s)(r + t).$$

Expanding and simplifying with Vieta's and Newton sums, we get $B = 52$.

The C term is slightly more involved, but we can use a combination of Newton sums, Vieta's, and grouping of terms to get $C = -23$.

All these terms are rational, so overall, our answer is $A+B+C = -14+52-23 = 15$.

Problem 13 (HMMT 2023/T2)

We can rearrange the equation $a^3 - bcd = b^3 - cda$ to get

$$(a - b)(a^2 + ab + b^2) = cd(b - a).$$

If we assume to the contrary that a, b, c, and d are pairwise distinct, this means

$$a^2 + ab + b^2 = -cd \implies a^2 + ab + b^2 + cd = 0.$$

Here, the variables a and b can be replaced with any two of a, b, c, or d. Thus, we also have:

$$c^2 + cd + d^2 + ab = 0.$$

We can conclude from these two equations that $a^2 + b^2 = c^2 + d^2$.

Notice that there was nothing special about our choices of a, b, c, and d. Using symmetry, we can deduce that $a^2 + c^2 = b^2 + d^2$.

Thus, $b^2 = c^2$. Similarly, $a^2 = b^2 = c^2 = d^2$. Therefore, we can see that a, b, c, and d cannot be pairwise distinct.

Problem 15 (SMT 2011)

We can notice that the polynomial $P(2x) - P(x) - 1$ has roots $x = 2^i$ for $0 \leq i \leq 2010$. Thus, we can write

$$P(2x) - P(x) - 1 = c(x - 2^0)(x - 2^1) \cdots (x - 2^{2010}).$$

Plugging in $x = 0$, we can find $\frac{1}{c} = 1 + 2 + \cdots + 2010$ (denote by S this sum).

Now, let a be the coefficient of the linear term in $P(x)$. Then, the linear term of $P(2x) - P(x) - 1$ is $2ax - ax = ax$. So, it suffices to find the linear coefficient of $c(x - 2^0)(x - 2^1) \cdots (x - 2^{2010})$.

For this, we can use Vieta's. We end up with

$$a = 2^S + 2^{S-1} + \cdots + 2^{S-2010}.$$

We can simplify this to $a = 2 - \frac{1}{2^{2010}}$.

Problem 18 (SMT 2013)

Putting the three terms over a common denominator and factoring the numerator, we can find that the expression equals

$$a^2 + b^2 + c^2 + ab + bc + ca.$$

We can rewrite this as $(a + b + c)^2 - (ab + bc + ca)$.

Let $x = \sqrt{3}$, $y = \sqrt{5}$, $z = \sqrt{7}$, and $S = a + b + c = x + y + z$. Then, our desired expression is

$$S^2 - [(S - 2x)(S - 2y) + (S - 2y)(S - 2z) + (S - 2z)(S - 2x)].$$

We can simplify this to get the answer of

$$2S^2 - 4(xy + yz + zx) = 30.$$