Submission for DCW-GLOBAL

OTIS (internal use)

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Example (Canada 2009/2, 0♣). Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

09CAN2

Walkthrough.

(a) Solve the problem by linearity of expectation.

(b) Write the proof in a "low-tech" way that doesn't quote the linearity of expectation, by considering all 200 rotations at once. This gives a "politically correct solution".

Example (OTIS Mock AIME 2024, by Joshua Liu, 0♣). Perry the Panda is eating some bamboo over a five-day period from Monday to Friday (inclusive). On Monday, he eats 14 pieces of bamboo. Each following day, Perry eats either one less than three times the previous day or one more than the previous day, with equal probability. Compute the expected number of pieces of bamboo Perry has eaten throughout the week after the end of Friday.

24OIME3

Walkthrough. This is another easy problem with linearity of expectation. Again, the intention of this walkthrough is to make you work through both the short technical solution (using linearity of expectation) as well as the longer elementary solution. The lesson from this walkthrough is again to see that these two solutions are equivalent so that you understand how the expected value solution is capturing the "work" of the problem for you.

First, here's the linearity solution. Let X_i denote the random variable for the number of pieces of bamboo on the ith day, for $i = 1, \dots, 5$.

(a) Show that $\mathbb{E}[X_i] = 7 \cdot 2^i$.

(b) Extract the answer.

Now, if we wanted to show this to a small toddler who would cry foul if you used the words "random variable", what should you do instead? Well, let's imagine we drew the following tree to represent all outcomes.

Label the bottom entries of the tree by 1, 2, ..., 16 from left to right and let s_i denote the sum of the entries from the root the i^{th} leaf. (For example, $s_1 = 14+15+16+17+18$ while $s_6 = 14 + 15 + 44 + 45 + 134$) The problem asks us to compute the value of

$$A = \frac{s_1 + s_2 + \cdots + s_{16}}{16}.$$

However, for this large sum it's easier to sum across the rows of the tree instead.

(c) The numerator of A contains $5 \cdot 16 = 80$ terms total when you expand out the definition of each s_i. For each node in the tree, how many times does it appear among these 80 terms?

(d) Show that moreover, the sum of each row in the tree is four times the sum of the row above it.

(e) Using these two key observations, show that

$$A = \frac{2^4 \cdot 14 + 2^3 \cdot 4^1 \cdot 14 + 2^2 \cdot 4^2 \cdot 14 + 2^1 \cdot 4^3 \cdot 14 + 2^0 \cdot 4^4 \cdot 14}{2^4}.$$

(f) Extract the final answer.

When we compare the two solutions, we see that the elementary solution had two main "insights": first to sum across the rows instead of by the branches, and then that the average value of each row is doubled compared to the preceding one. The advanced solution shows that *both* insights are actually each a special case of linearity of expectation. This demonstrates how powerful the linearity of expectation theory is: it "automagically" erases both main hurdles of the elementary solution.

> **Example** (ELMO 2013/1, 0♣)**.** Let $a_1, a_2, \dots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

13ELMO1

Walkthrough. We say a triple $t = (a_i, a_j, a_k)$ is *large* if $a_i + a_j + a_k \ge 3m$.

(a) Show that among any three disjoint triples, at least one triple is large. Give a heuristic argument why we expect $A \ge 28$ as a result.

(b) Give a construction for $A = 28$. (Try making one element large.)

(c) We now proceed to the "global" idea of looking at every possible partition in (a) at once. Show that there are

$$C = \frac{1}{3!}\binom{9}{3,3,3} = 280$$

ways to partition the 9 elements into three disjoint triples.

(d) How many of the C partitions does each triple t appear in?

(e) Use your answer to (d) to prove $A \ge 28$, thereby solving the problem.

(f) Optionally, for an alternate solution, explicitly construct a partition of the $\binom{9}{3} = 84$ triples into 28 disjoint triples. This would give another proof that $A \ge 28$.

When doing this calculation for the first time, you might be surprised that the division of seemingly random constants ends up with 28 in the end. It's important to recognize that the argument in (e) is "guaranteed" to work in a sense.

To elaborate: we constructed in (b) an example of an equality case, and every estimate we used was sharp. At the end of (e) we get some number again. The existence of the equality case means that this number *must* match the corresponding constant in (a), namely 28. This point is one of the key ideas in the Equality unit; the so-called "Sharpness Principle".

> **Example** (Romania 2004, 0♣). Prove that for any complex numbers z_1, z_2, …, z_n, satisfying $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1$, one can select $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$ such that
> $$\left|\sum_{k=1}^{n} \varepsilon_k z_k\right| \le 1.$$

04ROU

Walkthrough.

(a) Homogenize the inequality in the z_i's to eliminate the given condition. (You will need a square root.)

(b) Square both sides to eliminate the square root, and expand the left-hand side in a way that eliminates the absolute value.

(c) Show that the desired conclusion can now be rewritten as
$$\sum_{i \neq j} \varepsilon_i \varepsilon_j z_i \overline{z_j} \le 0.$$

(d) Intuitively we don't expect any reason for the left-hand side to be positive versus negative for a random choice of ε_i's. Use this idea to show that there exists a choice of ε_i's fulfilling the inequality.

(e) Is it okay to use these inequalities even though we are working over complex numbers? (For example, why is the left-hand side of (c) even real?)

> **Example** (Online Math Open, Ray Li, 0♣). Kevin has $2^n - 1$ cookies, each labeled with a unique nonempty subset of $\{1, 2, \dots, n\}$. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie. Determine the expected value of the number of days that Kevin eats a cookie before all cookies are gone.

13OMOF29

Walkthrough.

(a) How does the answer to the problem change if the cookie $\varnothing$ is added? This shows the omission of the cookie $\varnothing$ is sort of a red herring.

(b) What is the probability that the cookie $\{1, \dots, n\}$ is *chosen* on some day? How about $\{1, \dots, n-1\}$?

(c) Solve (b) for a general set S. Your answer should only depend on $|S|$.

(d) How is the work in (c) related to the expected number of days?

(e) Using linearity of expectation, show that the answer is $(3/2)^n - (1/2)^n$.

> **Example** (Buffon's needle/noodle, 0♣). You are about to drop a needle of length 1 onto a lined floor; the lines of the floor are spaced 1 unit apart. Show that the needle hits a line with probability $\frac{2}{\pi}$.

BUFFON

Walkthrough. This walkthrough is a bit unconventional, but I couldn't resist including it because it's so slick and yet not really known at all. I was first shown this argument by Qiaochu Yuan — it is more elegant, more general, and doesn't use calculus.

Rather than solving Buffon's needle, we will solve the general **Buffon's noodle** instead, where we drop a noodle of arbitrary shape.

> Given a noodle Γ of length ℓ, we let $E(\Gamma)$ denote the expected *number* of intersections of Γ with the lined floor. Determine $E(\Gamma)$.

We won't worry exactly how to define the "length" of a noodle, since arc length is a bit finnicky to define. So you have permission to handwave a bit on this walkthrough.

(a) Show that $E(\Gamma) = c \cdot \ell$ for some absolute constant c, by approximating Γ by a bunch of tiny equal straight line segments and using linearity of expectation. (You can ignore subtle technical issues with this "approximating".)

(b) Consider the special case where Γ is a circle of diameter 1. Check that $E(\Gamma) = 2$.

(c) Use the special case in (b) to extract the value of c. This solves Buffon's noodle.

(d) Use the fact that a needle of length 1 intersects the floor's ruling at most once (with probability 1) in order to extract the answer for Buffon's needle.

Practice problems

Instructions: Solve [40♣]. If you have time, solve [54♣].

> By this construction, Yahweh's work was indicated, and Yahweh's work was concealed. Thus would men know their place.
>
> Ted Chiang in *Tower of Babylon*

Problem 1 (HMMT 2013, 2♣)

Values $a_1, \dots, a_{2013}$ are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

13HMMTC6

Let's count the expected number of values that don't appear, since the answer is 2023 minus that. For a given value, the chance it doesn't appear is $\left(\frac{2012}{2013}\right)^{2013}$. Linearity of expectation gives us that the expected number of values that don't appear is just $2013 \cdot \left(\frac{2012}{2013}\right)^{2013}$. Therefore, the answer is

$$2013 - 2013 \cdot \left(\frac{2012}{2013}\right)^{2013}.$$

Required Problem 2 (Russia 1996, 3♣)

In the Duma there are 1600 delegates, who have formed 16000 committees of 80 people each. Prove that one can find two committees having no fewer than four common members.

96RUS94

First, notice that the average number of committees a delegate is in is 800 by simply double counting.

Pick two committees at random. Then, if D is the set of delegates, and n_d represents the number of committees delegate d is in, we have

$$\begin{aligned}
\mathbb{E}(\#\text{ of people in both committees}) &= \sum_{d \in D} \mathbb{P}(d \text{ is in both}) \\
&= \sum_{d \in D} \frac{\binom{n_d}{2}}{\binom{16000}{2}} \\
&\geq \sum_{d \in D} \frac{\binom{800}{2}}{\binom{16000}{2}} && \text{(by Jensen's inequality)} \\
&= 1600 \cdot \frac{800 \cdot 799}{16000 \cdot 15999} \\
&> 3.
\end{aligned}$$

There must then be a set of two committees who have at least 4 delegates in common, so we're done.

Problem 3 (EGMO 2019/5, 5♣)
Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \ldots, b_n$ satisfying the following three conditions:

(a) $a_i \leq b_i$ for $i = 1, 2, \ldots, n$;

(b) the remainders of b_1, b_2, $\ldots$, b_n on division by n are pairwise different,

(c) $b_1 + \cdots + b_n \leq n\left(\frac{n-1}{2} + \left\lfloor\frac{a_1+\cdots+a_n}{n}\right\rfloor\right)$.

19EGMO5

Problem 4 (BAMO 2017/4, 3♣)
Let $\mathcal{P}$ be a convex n-gon, and let $h > 0$ be a real number. On each of the n sides of $\mathcal{P}$ we erect internally a rectangle of height h (meaning the rectangle shares a side with $\mathcal{P}$ and moreover the interiors overlap). Prove that it's possible to pick a h such that the n rectangles together cover the interior of $\mathcal{P}$, and moreover the sum of their areas is at most twice the area of $\mathcal{P}$.

17BAMO4

Choose h so that the sum of the areas is exactly twice the area of $\mathcal{P}$, which we denote by A. Then, we claim every point is covered.

Suppose FTSOC that some point X is not covered. In this case, we claim that the distance from X to any side is greater than h. Consider a side s_i which is closer (or as close) to X than any other side, and suppose it is a distance h_i away from X. The foot of the altitude from X to s_i must be contained in s_i, because otherwise, the altitude must pass through a side closer to X than s_i by convexity. So, if $h_i \leq h$, the rectangle on $\mathcal{S}$ would contain it, so we must have $h_i > h$.

However, by splitting the polygon into triangles and adding up their areas,

$$2A = \sum_{i=1}^{n} s_i h < \sum_{i=1}^{n} s_i h_i = 2A,$$

where s_i denotes the ith side, and h_i denotes the distance from X to s_i. This gives us a contradiction.

Problem 5 (ELMO 2015/2, 3♣)
Let m, n, and x be positive integers. Prove that

$$\sum_{i=1}^{n} \min\left(\left\lfloor\frac{x}{i}\right\rfloor, m\right) = \sum_{i=1}^{m} \min\left(\left\lfloor\frac{x}{i}\right\rfloor, n\right).$$

15ELMO2

Write the left side as

$$\sum_{i=1}^{n} \sum_{j=1}^{\min\left(\left\lfloor\frac{x}{i}\right\rfloor, m\right)} 1$$

and swap the order of summation. We just need to use the fact that $\left\lfloor\frac{x}{i}\right\rfloor \geq j$ if and only if $\left\lfloor\frac{x}{j}\right\rfloor \geq i$ (both are equivalent to $ij \leq x$).

Note: this problem nicely represents a $m \times n$ multiplication table.

Problem 6 (BAMO 2014/5, 3♣)

Let n be a positive integer. There are $2n+1$ ranked chess players in a tournament, and each player played every other player exactly once, with no ties. It turns out that in exactly k games, the lower-rated player beat the higher-rated player. Prove that some player won between $n-\sqrt{2k}$ and $n+\sqrt{2k}$ games (inclusive).

14BAMO5

Problem 7 (JBMO 2007/3, 2♣)

A set of 50 points in the plane is given, no three collinear. Each point is colored one of four colors. Prove that there exists a color for which at least 130 scalene triangles have all three vertices of that color.

07JBMO3

Pigeonhole tells us that there must be a color used for 13 or more points. We claim that among any 13 points with no three collinear, there are at least 130 scalene triangles among them.

Take any pair of points. Notice that the line segment formed can be the base of at most two isosceles triangles, because otherwise, there would be three points collinear. This means there are at most $2\binom{13}{2} = 156$ isosceles triangles out of $\binom{13}{3} = 286$ total triangles, giving us at least $286 - 156 = 130$ scalene triangles.

Problem 8 (Anthony Wang, 2♣)

Bob has n stacks of rocks in a row, each with heights randomly and uniformly selected from the set $\{1,2,3,4,5\}$. In each move, he picks a group of one or more consecutive stacks with positive heights and removes 1 rock from each stack. Find, in terms of n, the expected value of the minimum number of moves he must execute to remove all rocks.

20JJCA4

Let $a_1, a_2, \dots, a_n$ be the sizes of the stacks, from left to right, and set $a_0 = a_{n+1} = 0$ for convenience. Consider the monovariant

$$\sum_{i=0}^{n} |a_{i+1} - a_i|.$$

This quantity is 0 if and only if all the rocks are gone. Also, if $a_i = a_j = 0$, $i < j-1$, and $a_{i+1}, a_{i+2}, \dots, a_{j-1} > 0$, Bob can remove one rock from the stacks $a_{i+1}, a_{i+2}, \dots, a_{j-1}$, and one can check that the monovariant decreases by 2. At any point in time before all the rocks are gone, a pair of indices i, j exists (remember how we set $a_0 = a_{n+1} = 0$). So, Bob can remove all rocks in $\frac{1}{2}\sum_{i=0}^{n} |a_{i+1} - a_i|$ moves.

We show this is the minimum. Notice that one move only can affect the value of $|a_{i+1} - a_i|$ for two indices i. And at each of those indices, the value cannot be changed by more than 1. So, the monovariant cannot decrease by more than 2 in one move.

The answer extraction involves finding the expected value of $\frac{1}{2}\sum_{i=0}^{n} |a_{i+1} - a_i|$. By linearity of expectation, this is just

$$\frac{1}{2}(3 + 3 + \frac{8}{5}(n-1)) = \boxed{\frac{4n+11}{5}}.$$

Problem 9 (Putnam 2008 B3, 3♣)
Let $n \geq 2$ be a positive integer. What is the largest possible radius of a circle inside a n-dimensional hypercube of side length 2?

08PTNMB3

Problem 10 (USEMO 2022/1, 3♣)
A *stick* is defined as a $1 \times k$ or $k \times 1$ rectangle for any integer $k \geq 1$. We wish to partition the cells of a 2022×2022 chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible.

22USEMO1

Required Problem 11 (TSTST 2023/4, 3♣)
Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

23TSTST4

Let C be the number of triangles with exactly two edges of the same color. Notice that $A + B + C = \binom{n}{3}$.

The inequality can be rewritten as

$$\binom{n}{3} - \frac{n(n-1)}{3} \leq 3A + C.$$

Let r_i be the number of red edges emanating from vertex i, and define g_i and b_i similarly. Then, the key observation is that, by counting per vertex,

$$\begin{aligned} 3A + C &= \sum \binom{r_i}{2} + \sum \binom{g_i}{2} + \sum \binom{b_i}{2} \\ &= \sum \left(\binom{r_i}{2} + \binom{g_i}{2} + \binom{b_i}{2} \right) \\ &\geq 3n \binom{(n-1)/3}{2} && \text{(by Jensen's inequality)} \\ &= \binom{n}{3} - \frac{n(n-1)}{3}, \end{aligned}$$

which concludes the proof.

Problem 12 (St Petersburg 2021/10.5, 3♣)
The vertices of a convex 2550-gon are colored black and white as follows: black, white, two black, two white, three black, three white, ..., 50 black, 50 white. Dania divides the polygon into 1274 quadrilaterals by drawing diagonals that do not intersect inside the polygon. Prove that there exists a quadrilateral among these, in which two adjacent vertices are black and the other two are white.

21SPBRG105

Problem 13 (IMO 2015/1, 3♣)

We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there are no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

15IMO1

For part (a), drop one of the points anywhere, and consider a circle centered at that point. We can then add pairs or triplets of points that are separated by an arc measuring 60°. Any points A and B on the circle will satisfy the balanced condition, because the center of the circle is equidistant from the two points. Also, the center and any point A on the circle will also satisfy the condition, because there is at least one point a 60° arc away from A, which will be equidistant from A and the center of the circle.

We claim the answer to part (b) is all odd n. For any odd n, the set of vertices of a regular n-gon is a balanced center-free set. This fact can easily be verified.

It remains to show that even n is impossible. There are $\binom{n}{2}$ pairs of points, so there must be a point P that is equidistant from at least $\frac{n-1}{2}$ pairs of points. However, this is not an integer, so P must be equidistant from at least $\frac{n}{2}$ pairs of points. These pairs cannot all be disjoint, so P is the desired center.

Problem 14 (USAMO 2012/6, 3♣)

For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

12AMO6

Required Problem 15 (Russia 1999, 5♣)

In a certain finite nonempty school, every boy likes at least one girl. Prove that we can find a set S of strictly more than half the students in the school such that each boy in S likes an odd number of girls in S.

99RUS

Pick a subset G of the girls randomly such that each girl has a $\frac{1}{2}$ chance of being in the set. Then, construct the subset B of the boys consisting of all the boys who like an odd number of girls in G. The desired set will be $B \cup G$. We have $\mathbb{E}(|G|) = \frac{g}{2}$ (where b and g are the number of boys and girls, respectively), so it remains to show that $\mathbb{E}(|B|) \geq \frac{b}{2}$. Let n_i be the number of girls liked by boy i. Then, by counting, the probability that boy i is in B is

$$\frac{\binom{n_i}{1} + \binom{n_i}{3} + \cdots}{2^{n_i}}.$$

The numerator is always at least 2^{n_i-1}, so the probability is at least $\frac{1}{2}$. Summing over all boys yields the desired conclusion.

(Note: to get a strict inequality, note that the empty set satisfies the condition.)

Problem 16 (Shortlist 1999 C4, 2♣)

Let n be a positive integer and let $Z = \mathbb{Z}/n^2\mathbb{Z}$ denote the set of integers modulo n^2. Suppose $A \subseteq Z$ with $|A| = n$. Prove that there exists $B \subseteq Z$ with $|B| = n$ such that $|A + B| \geq \frac{1}{2}n^2$.

99SLC4

Choose B randomly out of all n element subsets of Z. The probability that any given number i appears in $A + B$ is the probability that $i - a \in B$ for some $a \in A$. This is equivalent to B overlapping with an arbitrary subset of size n, which has probability

$$1 - \frac{\binom{n^2-n}{n}}{\binom{n^2}{n}}.$$

Thus,

$$\mathbb{E}[|A + B|] = n^2 \left(1 - \frac{\binom{n^2-n}{n}}{\binom{n^2}{n}}\right).$$

Since

$$\begin{aligned} \frac{\binom{n^2-n}{n}}{\binom{n^2}{n}} &\leq \left(\frac{n-1}{n}\right)^n \\ &< \frac{1}{e} < \frac{1}{2}, \end{aligned}$$

we have $\mathbb{E}[|A + B|] \geq \frac{1}{2}n^2$, so there must exist some B satisfying $|A + B| \geq \frac{1}{2}n^2$.

Problem 17 (Ireland 1994/10, 3♣)

Fix an integer $n \geq 1$. A square is partitioned into n convex polygons, A line segment which joins two vertices of polygons in the dissection, and does not contain any other vertices of the polygons in its interior, is called a *basic* segment.

Determine the maximum number of basic segments which could be present in such a dissection, in terms of n.

94IRL10

Problem 18 (Extension of IMO 1970, due to Ravi Boppana, 9♣)

Prove that for all sufficiently large positive integers n, within any n points on the plane in general position, at most 66.67% of the triangles with vertices among the points are acute.

70IMO6

Problem 19 (Iran TST 2008/6, 3♣)

Suppose 799 teams participate in a round-robin tournament. Prove that one can find two disjoint groups A and B of seven teams each such that all teams in A defeated all teams in B.

08IRNTST6

Pick a random group of seven teams and call it A. Then, let S be the set of teams that lost to all teams in A. Let l_i denote the number of losses of team i. The expected value of the size of S is then

$$\sum_{i=1}^{799} \frac{\binom{l_i}{7}}{\binom{799}{7}} \ge 799 \cdot \frac{\binom{399}{7}}{\binom{799}{7}} > 6,$$

so there must exist some choice of A such that $|S| \ge 7$. By letting $B = S$ in this case, we are done.

Required Problem 20 (Shortlist 2010 C5, 5♣)

Suppose $n \ge 4$ players participate in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament.

Let w_i and ℓ_i be respectively the number of wins and losses of the i-th player. Prove that

$$\sum_{i=1}^{n} (w_i - \ell_i)^3 \ge 0.$$

10SLC5

Companies of four players can be classified by the multiset of outdegrees into four groups:

A. Outdegrees $\{3, 2, 1, 0\}$.

B. Outdegrees $\{3, 1, 1, 1\}$.

C. Outdegrees $\{2, 2, 1, 1\}$.

D. Outdegrees $\{2, 2, 2, 0\}$, which we call bad.

Let a be the number of companies of four players of type A, and define b, c, and d similarly. By double counting,

$$\begin{aligned}
\sum \binom{w_i}{3} &= a + b, \\
\sum \binom{l_i}{3} &= a + d, \\
\sum \binom{w_i}{2} \cdot l_i &= 3d + a + 2c, \\
\sum \binom{l_i}{2} \cdot w_i &= 3b + a + 2c.
\end{aligned}$$

By expanding,

$$(w_i - l_i)^3 = 6\left(\binom{w_i}{3} - \binom{l_i}{3} - \binom{w_i}{2} l_i + \binom{l_i}{2} w_i\right) + 3w_i^2 - 2w_i - 3l_i^2 + 2l_i.$$

This means

$$\begin{aligned}\sum(w_i - l_i)^3 &= 24(b-d) + 3\sum(w_i^2 - l_i^2) + 2\sum(l_i - w_i)\\ &= 24(b-d)\\ &= 24b\\ &\geq 0.\end{aligned}$$

Problem 21 (USAMO 2020/2, 5♣)

An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

20AMO2

Required Problem 22 (USAMO 2010/6, 9♣)

There are 68 ordered pairs (not necessarily distinct) of nonzero integers on a blackboard. It's known that for no integer k does both (k, k) and $(-k, -k)$ appear. A student erases some of the 136 integers such that no two erased integers have sum zero, and scores one point for each ordered pair with at least one erased integer. What is the maximum possible score the student can guarantee?

10AMO6

The answer is 43. To show that this is always attainable, notice that for each integer $k > 0$, we can erase all the k's or all the $-k$'s. Suppose WLOG that all pairs of the form (k, k) satisfy $k > 0$. Then, for each $k > 0$, we choose at random whether to erase all the k's or all the $-k$'s. Specifically, we erase all the k's with probability p, where $p = 1 - p^2$. Then,

- If an ordered pair has a positive number in it, one of its elements is erased with probability at least p.
- If an ordered pair consists of two different negative numbers, then by basic counting, one of its elements is erased with probability $1 - p^2 = p$.

In either case, the probability is at least p. By linearity of expectation, the expected value of points scored is $68p \approx 42.024$. Thus, there must exist a scenario where we achieve 43 points scored.

A construction to show that 44 is not always attainable is to write 5 copies of each ordered pair (k, k) for $1 \leq k \leq 8$, and then write the $\binom{8}{2} = 28$ pairs (i, j) where $1 \leq i < j \leq 8$. If we choose to erase the positive numbers for m of the 8 different values, then the total number of points scored is

$$5m + \left(\binom{8}{2} - \binom{m}{2}\right) \leq 43.$$

Problem 23 (RMM 2017/5, 9♣)

Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any integer $k \geq 1$. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

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