OTIS Application Problems

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Geometry Problems

Problem A.1

The angle conditions imply $\overline{AB}$ is tangent to (PSR) and $\overline{AC}$ is tangent to (QRS). Assume (PSR) and (QRS) are not the same circle. Then, since A has the same power with respect to both circles, A must lie on the radical axis, $\overline{SR}$, which is just $\overline{BC}$, so that is impossible. Hence, they are the same circle, and we are done.

Problem A.2

It suffices to show that P and Q have the same power with respect to (ABC). We claim triangles APQ and MLK are similar. We have

$$\angle MKL = \angle PML = \angle QML = \angle QPC = \angle QPA,$$

and likewise for the other angle.

Finally,

$$QB \cdot QA = 2MK \cdot QA = 2PA \cdot ML = PA \cdot PC,$$

so P and Q have the same power with respect to (ABC).

Problem A.3

Let P, Q, R, and S be the feet of the perpendiculars from E to AB, BC, CD, and DA, respectively. By a homothety with scale factor 2 at E, it suffices to prove $PQRS$ is cyclic.

Notice that $DSER$, $ASEP$, $BPEQ$, and $CQER$ are cyclic. Then,

$$\begin{aligned}
\angle RSP &= \angle RSE + \angle ESP \\
&= \angle CDE + \angle EAB \\
&= 180 - (\angle ECD + \angle ABE) \\
&= \angle DCE + \angle EBA \\
&= \angle RQE + \angle EQP \\
&= \angle RQP,
\end{aligned}$$

so we are done.

Inequalities Problems

Problem B.1

We claim the answer is $\sqrt{3}$. This answer is achieved when $a = b = c$.

First, we homogenize the inequality:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq \sqrt{3(a^2 + b^2 + c^2)}.$$

Clearing denominators, we get

$$a^2b^2 + b^2c^2 + a^2c^2 \geq abc\sqrt{3(a^2 + b^2 + c^2)}.$$

For convenience, perform the substitutions $x = a^2$, $y = b^2$, and $z = c^2$. Then, after squaring both sides, we have

$$(xy + yz + xz)^2 \geq 3(x + y + z)xyz.$$

This reduces to

$$x^2y^2 + y^2z^2 + z^2x^2 \geq x^2yz + xy^2z + xyz^2.$$

Using Muirhead with $(2, 2, 0) \succ (2, 1, 1)$ gives us the result.

Problem B.3

First, we homogenize the inequality by multiplying the RHS by $(a + c)(b + d) = ab + bc + cd + da = 1$:

$$\sum_{cyc} \frac{a^3}{b + c + d} \geq \frac{1}{3}(ab + bc + cd + da)$$

Then, Holder gives us

$$\left(\sum_{cyc} \frac{a^3}{b + c + d}\right)\left(\sum_{cyc} a(b + c + d)\right) \geq (a^2 + b^2 + c^2 + d^2)^2.$$

Noting that $\sum_{cyc} a(b + c + d) = 2(ab + ac + ad + bc + bd + cd)$, it suffices to prove

$$3(a^2 + b^2 + c^2 + d^2)^2 \geq 2(ab + bc + cd + da)(ab + ac + ad + bc + bd + cd).$$

It is not hard to show by AM-GM that

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da.$$

Also, using Muirhead with $(2, 0, 0, 0) \succ (1, 1, 0, 0)$ yields

$$6(a^2 + b^2 + c^2 + d^2) \geq 4(ab + ac + ad + bc + bd + cd).$$

Putting these together yields the desired result.

Additional Problems

Problem C.1

The answer is 1270.

Python source code:

```
# Find all primes up to 10^5
sieve = [0] * 100000
for i in range(2, 50001):
  if sieve[i - 1] != 0:
    continue
  for j in range(2 * i, 100001, i):
    sieve[j - 1] = 1
primes = []
for i in range(2, 100001):
  if sieve[i - 1] == 0:
    primes.append(i)

# Loop through primes^2
count = 0
for i in range(len(primes)):
  p = primes[i]
  for j in range(len(primes)):
    q = primes[j]
    N = p**2 + q**3
    if len(str(N)) > 10:
      break
    if len(str(N)) < 10:
      continue
    fails = False
    for c in range(10):
      fails |= not str(c) in str(N)
    if not fails:
      count += 1
print(count)
```

Problem C.2

Let $P(x,y)$ denote the assertion. First, we show $f(0)=0$. Plugging in $y=-f(x)^2$ for any x, we see that

$$f(xf(x)+f(-f(x)^2)))=0.$$

For simplicity, let u be any value satisfying $f(u)=0$. Then, $P(0,u)$ gives

$$f(0)=f(0)^2+u,$$

and $P(u,u)$ gives

$$f(0)=u.$$

Thus, $u=u^2+u \implies u=0$.

Then, $P(0,x)$ gives

$$f(f(x))=x,$$

and thus f is an involution, and therefore bijective. Next, $P(f(t),0)$ gives

$$f(tf(t))=t^2,$$

and $P(t, 0)$ gives
$$f(tf(t)) = f(t)^2.$$
Thus, $f(t) \in \{-t, t\}$ for all t.

Assume for the sake of contradiction that $f(a) = a$ and $f(b) = -b$ for nonzero a, b. Then, $P(a, b)$ yields
$$f(a^2 - b) = a^2 + b,$$
which implies either a or b is zero. Therefore, the possible functions for $f(x)$ are $f(x) = x$ and $f(x) = -x$. These can be easily checked to work.

Problem C.3

The answer is 16. This is achievable; simply take $P(x) = (x + 1)^4$. Now, we show it is the minimum.

We have
$$\begin{aligned}
\prod_{j=1}^{4}(x_j + 1) &= \prod_{j=1}^{4}(x_j - i)(x_j + i) \\
&= \prod_{j=1}^{4}(x_j - i)\prod_{j=1}^{4}(x_j + i) \\
&= [(d - b + 1) + (c - a)i][(d - b + 1) - (c - a)i] \\
&= (d - b + 1)^2 + (c - a)^2.
\end{aligned}$$

Since $|d - b + 1| + |-1| \geq |d - b| \geq 5$ by the triangle inequality, we know $|d - b + 1| \geq 4$. Thus, the minimum value of the desired expression is $4^2 + 0^2 = 16$.

Problem C.4

Given a word, define a streak to be a maximal contiguous subsequence with all characters equal. We claim that all words with a streak of length 1 work. Indeed, if Banana chooses k, Ana can simply supply the word created by replacing any streak of length 1 in Ana's original word with k copies of the streak. Clearly, this word has exactly k subsequences matching Ana's original word. So, it suffices to show all other words do not work.

Let Ana's original word be $w = \{w_i\}_{1 \leq i \leq L}$, where L is the length of her word. If w does not have a streak of length 1, then for all $1 \leq i \leq L$, either $w_{i-1} = w_i$ or $w_i = w_{i+1}$. We claim that it is impossible for Ana to supply a word with exactly 2 subsequences equal to w.

Assume FTSOC that Ana can supply a word $\{x_i\}_{1 \leq i \leq M}$ of length M with exactly two subsequences equal to w. We will get a contradiction by constructing another subsequence equal to w.

Denote the two distinct subsequences by $\{x_{a_i}\}$ and $\{x_{b_i}\}$, where $\{a_i\}_{1 \leq i \leq L}$ and $\{b_i\}_{1 \leq i \leq L}$ are indexing sequences (i.e., increasing and with terms in $[1, M]$).

First, we claim that $\{a_i\}$ and $\{b_i\}$ differ by exactly one element. If not, then let $n > m$ be any two indices in which they differ.

Assume WLOG that $a_n > b_n$. Then, we claim the sequence
$$b_1, b_2, \ldots, b_{n-1}, a_n, a_{n+1}, \ldots, a_L$$
is an indexing sequence for a subsequence of $\{x_i\}$ equal to w. It suffices to show $a_n > b_{n-1}$, but this is true because $a_n > b_n > b_{n-1}$. Furthermore, this sequence differs from $\{a_i\}$

at index m and from $\{b_i\}$ at index n, so this is a third valid subsequence, giving us a contradiction.

Next, let m be the index at which the indexing subsequences differ. Assume WLOG that $a_m > b_m$. Then, we have two cases.

Case 1: $w_{m-1} = w_m$. In this case, we replace a_{m-1} with b_m to get another indexing sequence. Specifically, we have $x_{b_m} = w_m = w_{m-1}$ and $a_m > b_m > b_{m-2} = a_{m-2}$, so

$$a_1, a_2, \ldots, a_{m-2}, b_m, a_m, a_{m+1}, a_{m+2}, a_{m+3}, \ldots, a_L$$

is an indexing sequence for a third valid subsequence.

Case 2: $w_{m+1} = w_m$. In this case, we replace a_m with b_m and a_{m+1} with a_m to get another indexing sequence. Specifically, we have $x_{a_m} = w_m = w_{m+1}$ and $a_m > b_m > b_{m-1} = a_{m-1}$, so

$$a_1, a_2, \ldots, a_{m-2}, a_{m-1}, b_m, a_m, a_{m+2}, a_{m+3}, \ldots, a_L$$

is an indexing sequence for a third valid subsequence.

Thus, we have a contradiction in either case, and we are done.