Shortlist 2010

Michael Middlezong

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Geometry

G1

Let P_1 and P_2 be the two intersection points of line EF with the circumcircle, and WLOG assume P_2 is closest out of the two to B. Let Q_1 and Q_2 be defined as usual. We claim that Q_1 and P_2 are reflections over line AB, and Q_2 and P_1 are also reflections over line AB.

First, notice that $AP_1 = AP_2$; this follows from the fact that $\overline{OA} \perp \overline{EF}$ (trivial by angle chasing). Thus,

$$\angle P_2BA = \angle P_2P_1A = \angle AP_2P_1 = \angle ABP_1.$$

Furthermore, we have

$$\angle Q_2FB = \angle DFA = \angle DCA = \angle BCE = \angle BFE = \angle BFP_1.$$

We can now conclude that triangles BFQ_2 and BFP_1 are congruent, and thus Q_2 and P_1 are reflections over AB.

We also see that triangles P_2FQ_2 and Q_1FP_1 are congruent, and thus Q_1 and P_2 are reflections over AB.

Finally,

$$AQ_1 = AP_2 = AP_1 = AQ_2,$$

and we are done.

Number Theory

N1

First, we prove the bound $n \geq 39$. This is obvious as $\frac{1}{39} > \frac{51}{2010} = \frac{17}{670}$.

What remains is the construction. After some experimentation, we see

$$\left(\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33}\right)\left(\frac{34}{35} \cdots \frac{39}{40}\right)\left(\frac{66}{67}\right) = \frac{17}{670}.$$

(To speed up the construction, start by including $\frac{66}{67}$ and prioritize minimizing values of s_i.)

N2

First, arrive at $m \mid 2 \cdot 3^n$ by either noticing that it is a quadratic equation in m or taking the whole equation mod m. Then, writing $m = 2 \cdot 3^a$, the equation simplifies to

$$2 \cdot 3^a + 3^{n-a} = 2^{n+1} - 1.$$

Solving this is actually the bulk of the problem.

To solve this, first notice that if either $a < 3$ or $n - a < 3$, we can manually solve the equation using basic size arguments to get the only solutions:

$$(m, n) \in \{(6, 3), (9, 3), (9, 5), (54, 5)\}.$$

Next, the goal is to show there are no more solutions. There are two approaches here.

- **Mod chasing using orders.** Taking mod 27 yields $n \equiv -1 \pmod{18}$. This leads to an array of opportunities, as taking mods 7 and 19 gives us lots of information. It turns out to be enough to produce a contradiction.
- **Bounding ν_3.** Intuitively, the LHS of the equation has too many powers of 3. A size argument using LTE can solve this problem.