Submission for BAY-ALGMANIP

OTIS (internal use)

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> **Example** (0♣)**.** Solve over real numbers the system of equations
>
> $$\begin{aligned} a + 2 &= b^2 \\ b + 2 &= c^2 \\ c + 2 &= a^2. \end{aligned}$$

ZC329504

Walkthrough. This is the archetypal trig problem.

(a) Optionally, if you don't know how $\cos z$ is defined for $z \in \mathbb{C}$, first prove that $|a| \le 2$.

(b) Thus, we may let $a = 2\cos x$, $b = 2\cos y$, $c = 2\cos z$ where x, y, z are real numbers if you did part (a), or complex numbers if you skipped part (a). Show that $\cos 2y = \cos x$, etc.

(c) There's a lemma that whenever $\cos\theta = \cos\theta'$ we have $\cos 2\theta = \cos 2\theta'$. Prove this lemma if you have not seen it; it's easy (the simplest proof is by using the double angle formula).

(d) Show that $\cos x = \cos 8x$.

(e) Use this to find eight solutions to the system of equations.

(f) Conversely, show there are at most eight possible values of a, and hence at most eight solutions.

> **Example** (Czech Polish Slovak 2005/1, 0♣)**.** Let n be a positive integer. Solve the system of equations
>
> $$\begin{aligned} x_1 + x_2^2 + x_3^3 + \cdots + x_n^n &= n \\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n &= \frac{n(n+1)}{2} \end{aligned}$$
>
> over nonnegative real numbers.

05CPS1

Walkthrough. It shouldn't take too much to convince you $x_1 = x_2 = \cdots = x_n = 1$ is the only solution. But since this has n variables and 2 equations, the only way there could be only one solution is if some inequality was taking place.

(a) Prove that if $\sum_k x_k^k = n$ then $\sum_k kx_k \le \frac{1}{2}n(n+1)$.

(b) Finish by extracting the equality case.

Example (CMIMC 2020 A7, 0♣)**.** Solve over $\mathbb{R}$ the equation

$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7) = -48\sqrt{3}.$$

20CMIMCA7

Walkthrough. The solution proceeds with just suitable substitutions.

(a) Let $y = x^2 - 9x + 14$. Rewrite everything in terms of y.

(b) Let $z = y/\sqrt{3}$. Rewrite everything in terms of z. What motivated this?

(c) You should have a cubic in z. Solve it; you should find it has integer solutions.

(d) Use this to extract the answer for $x = \frac{9 \pm \sqrt{25+8\sqrt{3}}}{2}$.

Practice problems

Instructions: Solve [35♣]. If you have time, solve [50♣].

Problem 1 (IIT JEE, 2♣)
Find all real numbers x such $4^x + 6^x = 9^x$.

IITJEE

Dividing both sides by 6^x and setting $u = \left(\frac{2}{3}\right)^x$, we get

$$u + 1 = \frac{1}{u} \implies u = \frac{1+\sqrt{5}}{2},$$

taking only the positive solution as u must be positive. This means $x = \log_{2/3}\left(\frac{1+\sqrt{5}}{2}\right)$ is the only solution, and we can check that our steps are reversible.

Problem 2 (CMIMC 2018 A5, 2♣)
Suppose a, b, c are nonzero real numbers satisfying

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a + b + c$.

18CMIMCA5

Let the common value be x. Then

$$ax + bx + cx = abc + 1 + abc + 2 + abc + 7 = (a + b + c)\frac{1}{a+b+c} = 1.$$

This means

$$abc = -3.$$

The rest is simple. The final answer is $-\frac{\sqrt[3]{3}}{2}$.

Problem 3 (EGMO 2019/1, 3♣)
Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

19EGMO1

Homogenize to get

$$a^2b + bc^2 + c^2a = a^2b + a^2c + b^2c = ac^2 + ab^2 + b^2c.$$

Taking pairs of equations at a time, we get

$$\begin{aligned} c^2(a + b) &= c(a^2 + b^2), \\ a^2(b + c) &= a(b^2 + c^2), \\ b^2(a + c) &= b(a^2 + c^2). \end{aligned}$$

Assume one of the variables is zero, and WLOG $a = 0$. Then, the condition that $ab + bc + ca = 1$ implies $b, c \neq 0$. Since $c^2b = cb^2$, we must have $b = c$.

If all variables are nonzero, then we have

$$\begin{aligned} c(a+b) &= a^2+b^2, \\ a(b+c) &= b^2+c^2, \\ b(a+c) &= a^2+c^2. \end{aligned}$$

Adding them up, we get the equality case of repeated AM-GM, from which we can conclude that $a=b=c$.

Putting everything together, the solutions are $a=b=c=\frac{1}{\sqrt{3}}$, $a=b=c=-\frac{1}{\sqrt{3}}$, and permutations of $(1,1,0)$ and $(-1,-1,0)$.

Required Problem 4 (Vietnam 2014/1, 3♣)

Let $(x_n)_{n\ge 1}$ and $(y_n)_{n\ge 1}$ be two sequences of positive real numbers with $x_1=1$ and $y_1=\sqrt{3}$, satisfying the recursions

$$\begin{aligned} x_{n+1}y_{n+1}-x_n &= 0 \\ x_{n+1}^2+y_n &= 2. \end{aligned}$$

Show that $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and determine their values.

14VNM1

We claim that $x_n = 2\sin\left(\frac{30^\circ}{n}\right)$ and $y_n = 2\cos\left(\frac{30^\circ}{n}\right)$. We proceed by induction. The base case $n=1$ clearly works, so assuming n works, we conclude

$$x_{n+1}^2 = 2-y_n = 2-2\cos\left(\frac{30^\circ}{n}\right).$$

Using either the half-angle or double-angle formulas, it follows that

$$x_{n+1} = 2\sin\left(\frac{30^\circ}{n+1}\right).$$

Also,

$$\begin{aligned} x_{n+1}^2+y_{n+1}^2 &= x_{n+1}^2+\left(\frac{x_n}{x_{n+1}}\right)^2 \\ &= \frac{x_n^2+(2-y_n)^2}{x_{n+1}^2} \\ &= \frac{8-4y_n}{2-y_n} = 4, \end{aligned}$$

so we must have $y_{n+1} = 2\cos\left(\frac{30^\circ}{n+1}\right)$ as well. Finally it is now clear that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 2$.

Problem 5 (IMO 2014/1, 3♣)

Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0+a_1+a_2+\cdots+a_n}{n} \le a_{n+1}.$$

14IMO1

Problem 6 (AIME 2014/14, 3♣)
Find the largest real number x satisfying

$$\frac{3}{x-3}+\frac{5}{x-5}+\frac{17}{x-17}+\frac{19}{x-19}=x^2-11x-4.$$

14AIME14

First, add 4 to both sides and use it to make the numerators on the left side all equal to x. Then, substitute $u = x - 11$ for symmetry purposes and the rest is easy. The answer is $x = 11 + \sqrt{52 + 10\sqrt{2}}$.

Problem 7 (EGMO 2015/4, 5♣)
A sequence $a_1, a_2, a_3, \dots, a_N$ of positive integers (where $N \geq 3$) satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every $1 \leq n \leq N - 2$. Determine the largest possible value of N, or prove that no such maximum exists.

15EGMO4

Problem 8 (OTIS Mock AIME 2024, by Joshua Liu and Ashvin Sinha, 3♣)
For each real number $k > 0$, let $S(k)$ denote the set of real numbers x satisfying

$$\lfloor x \rfloor \cdot (x - \lfloor x \rfloor) = kx.$$

The set of positive real numbers k such that $S(k)$ has exactly 24 elements is a half-open interval of length ℓ. Compute $1/\ell$.

24OIME6

Graph the left hand side, it's just a union of line segments. We see that if $k \leq 1$, then we get infinitely many solutions. And if $k > 1$, we get no solutions where $x > 0$. So, we count the number of lines in the third quadrant that we have to intersect, and eventually get an answer of

$$\frac{1}{\frac{23}{22}-\frac{24}{23}} = \boxed{506}.$$

Problem 9 (AIME II 2006/15, 3♣)
Solve over real numbers the system of equations

$$\begin{aligned} x &= \sqrt{y^2-\frac{1}{16}}+\sqrt{z^2-\frac{1}{16}} \\ y &= \sqrt{z^2-\frac{1}{25}}+\sqrt{x^2-\frac{1}{25}} \\ z &= \sqrt{x^2-\frac{1}{36}}+\sqrt{y^2-\frac{1}{36}}. \end{aligned}$$

06AIMEII15

Problem 10 (Baltic Way 2020, added by Pedro Rosalba, 3♣)
Find all real numbers x, y, z so that

$$x^2y + y^2z + z^2 = 0$$
$$z^3 + z^2y + zy^3 + x^2y = \frac{1}{4}(x^4 + y^4).$$

20BWAY5

Problem 11 (AIME II 2024/11, 2♣)
Compute the number of triples of nonnegative integers (a, b, c) satisfying $a+b+c = 300$ and

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b = 6000000.$$

24AIMEII11

Required Problem 12 (Mathematical Reflections J479, 3♣)
Let a, b, c be nonzero real numbers, not all equal, such that

$$\left(\frac{a^2}{bc} - 1\right)^3 + \left(\frac{b^2}{ca} - 1\right)^3 + \left(\frac{c^2}{ab} - 1\right)^3 = 3\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} - \frac{bc}{a^2} - \frac{ca}{b^2} - \frac{ab}{c^2}\right).$$

Prove that $a + b + c = 0$.

MRJ479

Let $x = \frac{a^2}{bc} - 1$, $y = \frac{b^2}{ca} - 1$, and $z = \frac{c^2}{ab} - 1$. Then, the equation simplifies to

$$x^3 + y^3 + z^3 - 3xyz = 0,$$

Since $x^3 + y^3 + z^3 - 3xyz$ factors as $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$, and $x^2 + y^2 + z^2 - xy - yz - zx \neq 0$ (because otherwise $a = b = c$), we have $x + y + z = 0$.

Also, we have

$$\begin{aligned} x + y + z &= \frac{a^2 - bc}{bc} + \frac{b^2 - ac}{ac} + \frac{c^2 - ab}{ab} \\ &= \frac{(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}{abc}, \end{aligned}$$

so we must have $a + b + c = 0$ as desired.

Problem 13 (AIME II 2013/15, 3♣)
In obtuse triangle ABC with $\angle B > 90^\circ$ we have

$$\cos^2 A + \cos^2 B + 2\sin A \sin B \cos C = \frac{15}{8}$$
$$\cos^2 B + \cos^2 C + 2\sin B \sin C \cos A = \frac{14}{9}.$$

Compute

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B.$$

13AIMEII15

Problem 14 (Canadian training camp, added by Haozhe Yang, 3♣)
The sequences a_n and b_n are such that, for every positive integer n,

$$a_n > 0, \qquad b_n > 0, \qquad a_{n+1} = a_n + \frac{1}{b_n}, \qquad b_{n+1} = b_n + \frac{1}{a_n}$$

Prove that $a_{50} + b_{50} > 20$.

ZCB390FF

Problem 15 (Germany 2008, added by Joel Gerlach, 2♣)
Solve over real numbers:

$$(x+y)(x^2-y^2) = 675$$
$$(x-y)(x^2+y^2) = 351.$$

08GER33

Expanding, we get

$$x^3 + x^2y - xy^2 - y^3 = 675$$
$$x^3 - x^2y + xy^2 - y^3 = 351.$$

Subtract the first equation from twice the second to get

$$x^3 - 3x^2y + 3xy^2 - y^3 = 27 \implies (x-y)^3 = 27.$$

This means $x - y = 3$. Thus, the original second equation yields $x^2 + y^2 = 117$. We can solve this system of equations using substitution, and the only possible solutions are

$$\{(9,6), (-6,-9)\},$$

which can be checked to work.

Problem 16 (ARML Local 2021, added by Qiao Zhang, 2♣)
A sequence $a_1, a_2, \ldots$ of real numbers satisfies

$$a_n = na_{n-1} + (n-1)(n!(n-1)! - 1)$$

for integers $n \geq 2$. Given that $a_{2021} = (2021! + 1)^2 + 2020!$, compute a_1.

21ARMLOCI10

We find that

$$a_n = (n! + 1)^2 + \frac{n!}{2021}$$

satisfies the recurrence and the given value for a_{2021}, so the answer is $4 + \frac{1}{2021}$.

Problem 17 (Summer Mock AIME 2020/14, 5♣)
Let $P(x) = x^3 - 3x^2 + 3$. For how many positive integers $n < 1000$ does there not exist a pair (a, b) of positive integers such that the equation

$$\underbrace{P(P(\ldots P}_{a \text{ times}}(x)\ldots)) = \underbrace{P(P(\ldots P}_{b \text{ times}}(x)\ldots))$$

has exactly n distinct real solutions?

20SIME14

Required Problem 18 (IMO 2018/2, 9♣)

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo n.

18IMO2

If $3 \mid n$, then the repeating sequence $(2, -1, -1, 2, -1, -1, \dots)$ works. Otherwise, multiply the given equation by a_{i+2} and rearrange to get

$$a_i a_{i+1} a_{i+2} = a_{i+2}^2 - a_{i+2}.$$

Since $a_{i+3} = a_{i+1}a_{i+2} + 1$, we can rewrite the equation as

$$a_i a_{i+3} - a_i = a_{i+2}^2 - a_{i+2}.$$

Summing over all i, the degree 1 terms cancel out and we are left with

$$a_1 a_4 + a_2 a_5 + \cdots = a_1^2 + a_2^2 + \dots.$$

Since n is not divisible by 3, none of the terms on the left side repeat. Thus, a repeated application of AM-GM yields

$$a_1 = a_2 = \cdots = a_n,$$

and at this point, it is obvious that no solution can exist.

Problem 19 (EGMO 2020/2, 9♣)

Find all lists $(x_1, x_2, \dots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- $x_1 \leq x_2 \leq \cdots \leq x_{2020}$;
- $x_{2020} \leq x_1 + 1$;
- there is a permutation $(y_1, y_2, \dots, y_{2020})$ of $(x_1, x_2, \dots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

20EGMO2

Required Problem 20 (Iberoamerican 2021/4, 5♣)

Let a, b, c, x, y, z be real numbers such that

$$\begin{aligned} a^2 + x^2 = b^2 + y^2 = c^2 + z^2 &= (a+b)^2 + (x+y)^2 \\ &= (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2 \end{aligned}$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

21IBERO4

Let $u = a + xi$, $v = b + yi$, and $w + c + zi$. Then, we have

$$|u| = |v| = |w| = |u + v| = |v + w| = |w + u|.$$

Using the law of cosines, this means that u, v, and w must lie on a circle centered at the origin and form an equilateral triangle. Thus, the complex numbers u^2, v^2, and w^2 also form an equilateral triangle, so $u^2 + v^2 + w^2 = 0$. Expanding and taking the real part of both sides yields the desired conclusion.

Problem 21 (IMC 2023/2, 9♣)

Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \text{ and } B^3 = ABC + 2\,\mathrm{id}\,.$$

Prove that $A^6 = \mathrm{id}$.

23IMC2