Submission for BAY-CYCLOTOMIC

OTIS (internal use)

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> **Example** (Product of diagonals of n-gon, 0♣)**.** Let $n \geq 3$ be a positive integer. Let $A_1A_2 \dots A_n$ be a regular polygon inscribed in a circle of radius 1. Prove that
>
> $$A_nA_1 \times A_nA_2 \times A_nA_3 \times \cdots \times A_nA_{n-1} = n.$$

Z6EF432B

Walkthrough. Let $\zeta = \exp(2\pi i/n)$. Toss on the complex plane, with A_k at ζ^k,

(a) Show that the desired quantity equals $\prod_{k=1}^{n-1} \left|1 - \zeta^k\right|$.

(b) More generally, identify the coefficients of the polynomial $(X - \zeta)(X - \zeta^2) \dots (X - \zeta^{n-1})$. Possible hint: what are the roots of $X^n - 1$?

(c) Show that the sum of coefficients of the polynomial in (b) is n.

(d) Finish the problem.

> **Example** (Morrie's Law, 0♣)**.** Evaluate $\cos(20°)\cos(40°)\cos(80°)$.

Z45144C5

Walkthrough. We define $\zeta = \cos(20°) + i\sin(20°)$.

(a) Write $\cos(20°)$ in terms of ζ.

(b) Write $\cos(40°)$ in terms of ζ.

(c) Write $\cos(80°)$ in terms of ζ.

(d) Multiply out the three preceding answers.

(e) Show that $1 + \zeta^2 + \zeta^4 + \zeta^6 + \cdots + \zeta^{16} = 0$.

(f) Use this to simplify the expanded expression that you got in (d).

(g) Compute the answer (it should be a rational number).

> **Example** (Putnam 2015 A3, 0♣)**.** Compute
>
> $$\log_2\left(\prod_{a=1}^{2015}\prod_{b=1}^{2015}\left(1 + e^{2\pi iab/2015}\right)\right)$$

> Here i is the imaginary unit (that is, $i^2 = -1$).

15PTNMA3

Walkthrough. Not too bad.

(a) Let m be an odd integer and let ζ be a primitive mth root of unity. Compute

$$\prod_{1 \le b \le m} (1 + \zeta^b).$$

(b) For each integer a, evaluate

$$\prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right)$$

using the result in (a). Your answer should involve $\gcd(a, 2015)$.

(c) Conclude that the final answer equals $\sum_{a=1}^{2015} \gcd(a, 2015)$.

(d) As $2015 = 5 \cdot 13 \cdot 31$, factor the expression in (c) as $(5 + 1 + \dots + 1)(13 + 1 + \dots + 1)(31 + \dots + 1)$ (figure out how many 1's are in each of the ellipses).

(e) Extract the numerical answer 13725.

> **Example** (Gauss sum, 0♣)**.** Let p be an odd prime. Show that
>
> $$\sum_{x=0}^{p-1} \exp\left(\frac{2\pi i x^2}{p}\right) = \begin{cases} \pm\sqrt{p} & p \equiv 1 \pmod 4 \\ \pm i\sqrt{p} & p \equiv 3 \pmod 4. \end{cases}$$

18099PS

Walkthrough. For brevity, we define

$$e(x) = \exp(2\pi i x).$$

Thus

$$S = \sum_x e\left(\frac{x^2}{p}\right).$$

We first show that $|S| = \sqrt{p}$. The usual way we handle absolute values in complex numbers is to square both sides: since $|S|^2 = S \cdot \overline{S}$ where $\overline{S}$ is the complex conjugate of S.

(a) Show that we have

$$|S|^2 = S \cdot \overline{S} = \sum_{x=0}^{p-1}\sum_{y=0}^{p-1} e\left(\frac{x^2 - y^2}{p}\right).$$

(b) Using the substitution $(x-y)(x+y) = ab$, conclude that

$$S \cdot \overline{S} = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{ab}{p}\right).$$

Note that we use here that p is an odd prime.

At this point we imagine we are back in grade school and learning multiplication tables, except our school is teaching them modulo p. For example, here is the multiplication table modulo 5.

$\mathbb{F}_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(c) For each residue $r \pmod p$, how many times does r appear in the $p \times p$ multiplication table?

(d) Conclude that

$$|S|^2 = (2p-1) \cdot 1 + (p-1) \cdot (e(1/p) + e(2/p) + e(3/p) + \cdots + e((p-1)/p)) .$$

(e) Finish the calculation of right-hand side.

To finish the problem we then need to use the following fact. You can either view it as a corollary of Fermat's Christmas theorem, or one of the parts of the quadratic reciprocity theorem.

> Let p be an odd prime. There exists an integer u with $u^2 \equiv -1 \pmod p$ if and only if $p \equiv 1 \pmod 4$.

If you have not seen this fact before, then you should take it on faith now; this is a useful fact, though. (The fully general quadratic reciprocity theorem gives you a way to describe whether q is a quadratic residue modulo p for any q, but we will only need the special case $t = -1$ for this walkthrough. Wikipedia calls this the "first supplement" on the Wikipedia page.)

(f) Use this fact to show that $\sum_x e(-x^2/p) = \sum_x e(x^2/p)$ if $p \equiv 1 \pmod 4$.

(g) Conclude that $S = \overline{S}$, i.e. that S equals its own complex conjugate, and therefore S is a real number.

(h) On the other hand, for $p \equiv 3 \pmod 4$, show that

$$\sum_x e(-x^2/p) + \sum_x e(x^2/p) = 2\,(1 + e(1/p) + e(2/p) + \cdots + e((p-1)/p)) = 0.$$

(i) Prove that for $p \equiv 3 \pmod 4$, we have $S \in i\mathbb{R}$, as needed.

In fact it turns out that the sum mentioned in the problem always takes the sign $+$; but this is very difficult to prove.

> **Example** (ELMO 2009/5, 0♣**).** Let $ABCDEFG$ be a regular heptagon with center O. Let M be the centroid of $\triangle ABD$. Prove that $\cos^2(\angle GOM)$ is rational and determine its value.

09ELMO5

Walkthrough. We use complex numbers setting G as 1, A as $\omega = \exp(2\pi i/7)$, etc.

(a) Find the complex number corresponding to M.

(b) By quoting Gauss sum show that $\omega + \omega^4 + \omega^9 = \frac{\sqrt{7}i - 1}{2}$.

(c) Use this to prove that M is located at $\frac{\sqrt{7}i-1}{6}$.

(d) Show that $\cos^2 \angle GOM = 1/8$.

> **Example** (Math Prize 2010/20, 0♣). Evaluate
> $$\sum_z \frac{1}{|1-z|^2}$$
> where z ranges over all 7 complex solutions to $z^7 = -1$.

10MP4G20

Walkthrough. We present an approach based on differentiation due to Serena An.

(a) Eliminate the absolute values by showing $|1-z|^2 = (1-z) \cdot \overline{1-z} = (1-z)(1-\frac{1}{z})$ for any $|z| = 1$. (Here the bar denotes the complex conjugate).

(b) Conclude that the desired sum equals $-\sum_z \frac{z}{(1-z)^2}$.

Define $f(X) = X^7 + 1 = \prod_z (X - z)$. The idea is to consider the so-called *logarithmic derivative* which lets us push the terms into denominators.

(c) Prove that
$$\frac{f'(X)}{f(X)} = \sum_z \frac{1}{X - z}.$$

(d) Differentiate both sides of (c) to get another equation which has $(X-z)^2$ in the denominator.

(e) Substitute $X = 1$ in (c) and (d) and add. Show that the RHS is exactly the quantity in (b).

(f) Verify that $\frac{f'(1)}{f(1)} = \frac{7}{2}$.

(g) Show that the derivative of $\frac{f'(X)}{f(X)}$ takes on the value $\frac{35}{4}$ when $X = 1$.

(h) Extract the final answer.

Practice problems

Instructions: Solve [36♣]. If you have time, solve [50♣].

> All humans will, without exception, eventually die.
> When they die, the place they go is MU.
>
> From the rules of the *Death Note*.

Problem 1 (2♣) (a) Find all complex numbers z with $z^2 = i$.

(b) Express $\cos(5\theta)$ as a function of $\cos(\theta)$.

(c) Is $\cos 1^\circ$ rational?

No PUID

(a) We must have $z^8 = 1$ but not $z^4 = 1$. This leaves 4 solutions, but two of them satisfy $z^2 = -i$. The ones we want are $z = \pm e^{\frac{2\pi i}{8}}$.

(b) We have

$$\begin{aligned}\cos(5\theta) + i\sin(5\theta) &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\ &\quad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta.\end{aligned}$$

Looking at the real part of both sides, we get the desired

$$\begin{aligned}\cos(5\theta) &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

(c) No, because $\cos 45^\circ = \frac{\sqrt{2}}{2}$ is an integer polynomial of $\cos 1^\circ$.

Problem 2 (2♣)
Compute the coefficients of $\Phi_{30}(X)$.

No PUID

Factor $x^{30} - 1 = (x^{15} - 1)(x^{15} + 1)$. Our desired polynomial can't be in the first factor, so factor $x^{15} + 1$ to get

$$(x^5 + 1)(x^{10} - x^5 + 1).$$

Again, our desired polynomial (which has degree $\varphi(30) = 8$) can't be in the first factor, so it must divide $x^{10} - x^5 + 1$.

At this point, we notice that if ω is a primitive 6th root of unity, then

$$\omega^{10} - \omega^5 + 1 = \omega^4 + \omega^2 + 1 = 0.$$

This means $x^{10} - x^5 + 1$ is divisible by $\Phi_6(x) = x^2 - x + 1$, so the final step is to perform long division to get

$$\frac{x^{10} - x^5 + 1}{x^2 - x + 1} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1.$$

(This long division took a while, so there might be a faster way.)

Problem 3 (2♣)

A monic cubic polynomial has roots $\cos(2\pi/7)$, $\cos(4\pi/7)$, $\cos(6\pi/7)$. Show that its coefficients are rational and determine their values.

No PUID

Set $z = e^{\frac{2\pi i}{7}}$. Then, the roots are

$$\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2}\left(z^2+\frac{1}{z^2}\right), \frac{1}{2}\left(z^3+\frac{1}{z^3}\right),$$

and now we can just bash Vieta's formulas or expand to get the coefficients. The answer is

$$x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8}.$$

Required Problem 4 (Inspired by past mistakes, 3♣)

Find seven distinct 30^{th} roots of unity whose sum is 0, but for which no nonempty proper subset has sum 0.

Z01B65DA

Let $\omega = e^{2\pi i/30}$. Notice that by taking two triangles and a pentagon,

$$(\omega^1 + \omega^{11} + \omega^{21}) + (\omega^5 + \omega^{15} + \omega^{25}) + (\omega^4 + \omega^{10} + \omega^{16} + \omega^{22} + \omega^{28}) = 0.$$

Also,

$$(\omega^1 + \omega^{16}) + (\omega^{10} + \omega^{25}) = 0,$$

so we subtract the two equations to get

$$\omega^4 + \omega^5 + \omega^{11} + \omega^{15} + \omega^{21} + \omega^{22} + \omega^{28} = 0.$$

To show that no proper subset of this sums to 0, notice that if that were the case, it would partition the set into two sets each summing to 0. One of these sets must have at most three elements. However, the only sets with two or three elements that sum to 0 are pairs of diametrically opposite points and equilateral triangles, which aren't present here.

Problem 5 (2♣)

Find the real number $0 < x < 90$ for which

$$\tan(x^\circ) = \sin(1^\circ) + \sin(2^\circ) + \sin(3^\circ) + \cdots + \sin(179^\circ).$$

SINSUM

The answer is $x = \frac{179}{2}$. The solution is a bunch of algebraic manipulation after setting $z = \operatorname{cis}(1^\circ)$ and using the complex number definitions for sine and tangent.

Problem 6 (AIME 2023, added by Hansen Shieh, 2♣)

Let $\omega = e^{\frac{2\pi i}{7}}$. Compute

$$\prod_{k=0}^{6}\left(\omega^{3k} + \omega^k + 1\right).$$

23AIMEII8

Notice that the term in the product for $k = 0$ is equal to 3, and for each i, the terms for $k = i$ and $k = 7 - i$ can be paired with each pair having a product of 2. By this pairing argument, the answer is

$$3 \cdot 2^3 = 24.$$

These problems are in chronological order, not in difficulty order.

Problem 7 (Math Prize 2012/13, 3♣)

For how many integers n with $1 \le n \le 2012$ is the product

$$\prod_{k=0}^{n-1} \left(\left(1 + e^{2\pi ik/n}\right)^n + 1 \right)$$

equal to zero?

12MP4G13

The condition is equivalent to there existing a solution to $x^n + 1$ such that when it is shifted left by 1, it becomes a solution to $x^n - 1$. So, we are talking about a point on the unit circle that remains on the unit circle when shifted left by 1. Geometrically, there are only two points satisfying this property, and both are 6th roots of unity. It's not hard to conclude that the only n that work satisfy $3 \mid n$ and $6 \nmid n$.

Problem 8 (Math Prize 2012/16, 3♣)

Say that a complex number z is *three-presentable* if there is a complex number w of absolute value 3 such that $z = w - \frac{1}{w}$. Let T be the set of all three-presentable complex numbers. The set T forms a closed curve in the complex plane. What is the area inside T?

12MP4G16

Problem 9 (Math Prize 2014/20, 9♣)

Determine how many complex numbers z satisfy $|z| < 30$ and

$$e^z = \frac{z-1}{z+1}.$$

14MP4G20

Problem 10 (Math Prize 2015/15, 3♣)

Let z_1, z_2, z_3, and z_4 be the four distinct complex solutions of the equation

$$z^4 - 6z^2 + 8z + 1 = -4(z^3 - z + 2)i.$$

Find the sum of the six pairwise distances between z_1, z_2, z_3, and z_4.

15MP4G15

Problem 11 (Math Prize 2016/12, 2♣)

Let b_1, b_2, b_3, c_1, c_2, and c_3 be real numbers such that for every real number x, we have

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)(x^2 + b_3x + c_3).$$

Compute $b_1c_1 + b_2c_2 + b_3c_3$.

16MP4G12

Problem 12 (Math Prize 2016/18, 3♣)

Let $T = \{1, 2, 3, \dots, 14, 15\}$. Say that a subset S of T is *handy* if the sum of all the elements of S is a multiple of 5. For example, the empty set is handy (because its sum is 0) and T itself is handy (because its sum is 120). Compute the number of handy subsets of T.

16MP4G18

Consider the polynomial $F(x) = (x+1)(x^2+1)\dots(x^{15}+1)$. Letting a_i be the coefficient of the ith degree term, we want to find the sum of a_i for all i that is a multiple of 5.

This is done by roots of unity filter; letting $z = e^{\frac{2\pi i}{5}}$, we want to find

$$\frac{1}{5}\sum_{i=0}^{4} F(z^i).$$

When $i = 0$, we have $F(z^i) = 2^{15}$. For all $i \neq 0$,

$$F(z^i) = (1 + z^i)(1 + z^{2i})\dots(1 + z^{15i}) = Q(1),$$

where $Q(x)$ is a polynomial with roots

$$\{-1, -z, -z^2, -z^3, -z^4\},$$

all with multiplicity 3.

Clearly, $Q(x) = (x^5 + 1)^3$, so our final answer is

$$\frac{1}{5}(2^{15} + 4 \cdot 2^3) = 6560.$$

Required Problem 13 (Math Prize 2016/20, 9♣)

Let a_1, a_2, a_3, a_4, and a_5 be random integers chosen independently and uniformly from the set $\{0, 1, 2, \dots, 23\}$. (Note that the integers are not necessarily distinct.) Find the probability that

$$\sum_{k=1}^{5} \exp\left(\frac{a_k \pi i}{12}\right) = 0.$$

16MP4G20

The problem is asking for the probability that five random 24th roots of unity sum to 0. The main idea is that the sum has to consist of a diameter and an equilateral triangle which each sum to 0.

To prove this, consider $f(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5}$. If this choice of a_i satisfies the condition, then $f(x)$ has $\omega = e^{\frac{2\pi i}{24}}$ as a root. However, $\Phi_{24}(x) = x^8 - x^4 + 1$ is the minimal polynomial with this root, and by this result, we must have $\Phi_{24}(x) \mid f(x)$. Taking $f(x)$ mod $x^{12} + 1$ and looking at the quotient $q(x) = \frac{f(x)}{x^8 - x^4 + 1}$, we see that it has degree 3.

However, if we rewrite $f(x)$ as $q(x)(x^8 - x^4 + 1)$, we see that the sum of the absolute values of the coefficients of $f(x)$ mod $x^{12} + 1$ must be a multiple of 3, which is only possible if there existed i and j with $a_i = 12 + a_j$. So, these two form a diameter, and it's obvious that the remaining three must form an equilateral triangle.

The rest is just basic counting (split into two cases: all a_i distinct and otherwise), and the final answer is $\frac{35}{27648}$.

Required Problem 14 (Math Prize 2017/19, 5♣)

Give an example of an equilateral 13-gon, convex and nondegenerate, whose internal angle measures are all multiples of 20°. (Specify the polygon by giving the angle measures.)

17MP4G19

Let $\omega = e^{\frac{2\pi i}{18}}$. We have

$$(\omega^0 + \omega^6 + \omega^{12}) + (\omega^1 + \omega^{10}) + (\omega^4 + \omega^{13}) + (\omega^5 + \omega^{14}) + (\omega^7 + \omega^{16}) + (\omega^8 + \omega^{17}) = 0.$$

These represent 13 unit vectors that sum to 0, so they can be arraged to form an equilateral 13-gon. Also, the arguments are all multiples of 20°, and since 180° is a multiple of 20°, we are done.

Problem 15 (Math Prize 2017/20, 9♣)
Determine the value of the sum

$$\sum_{k=1}^{11} \frac{\sin\left(2^{k+4}\frac{\pi}{89}\right)}{\sin\left(2^k\frac{\pi}{89}\right)}.$$

17MP4G20

Problem 16 (Math Prize 2018/18, 5♣)
Evaluate the expression

$$\left|\prod_{k=0}^{15}\left(1 + e^{2\pi i k^2/31}\right)\right|.$$

18MP4G18

Problem 17 (Math Prize 2019/16, 5♣)
Let $ABCDEFG$ be a regular heptagon of side length 1 and construct rhombus $CDEP$. Find the distance from P to the midpoint of $\overline{AG}$.

19MP4G16

Problem 18 (Math Prize 2019/20, 5♣)
Evaluate

$$\prod_{k=2}^{\infty}\left(1 - 4\sin^2\left(\frac{\pi}{3\cdot 2^k}\right)\right).$$

19MP4G20

Problem 19 (Putnam 2008 A5, 5♣)
Let f and g be polynomials with real coefficients, and let $n \geq 3$ be an integer. Suppose that the n points $(f(0), g(0))$, $(f(1), g(1))$, …, $(f(n-1), g(n-1))$ are the vertices of a nondegenerate regular n-gon, in counterclockwise order. Prove that $\max(\deg f, \deg g) \geq n-1$.

08PTNMA5

Required Problem 20 (IMO 1990/6, 9♣)
Prove that there exists a convex equiangular 1990-gon whose side lengths are 1^2, 2^2, 3^2, …, 1990^2 in some order.

90IMO6

This problem is asking us to assign distinct weights to the 1990th roots of unity so that each weight is one of $\{1^2, 2^2, \ldots, 1990^2\}$ and the weighted sum equals zero.

Suppose we put consecutive squares on opposite ends of the circle. Then, using difference of squares, the problem is reduced to assigning the weights

$$\{3, 7, 11, \ldots, 995 \cdot 4 - 1\}$$

to the 995th roots of unity.

Since the sum of the 995th roots of unity is 0, and scaling doesn't matter, we can add one copy of each 995th root of unity and then scale down by 4, reducing the desired weights to just

$$\{0, 1, 2, \ldots, 994\}.$$

For every 995th root of unity $e^{\frac{2k\pi i}{995}}$, assign to it the weight $k \mod 199$. Notice that this makes the weighted sum 0, because we are just adding a bunch of regular pentagons.

Then, take any regular 199-gon with vertices among the 995th roots of unity, and consider the numbers assigned to the vertices. Since $\gcd(199, 5) = 1$, the vertices together must have all of the residues mod 199 exactly once. Each of these 199-gons can then incremented by different multiples of 199 to get the desired weights. The weighted sum remains 0, because we are just adding a bunch of regular 199-gons.

Thus, the desired weighted sum is possible, and we are done.

Problem 21 (Added by Robert Yang, 3♣)

For any positive integers n and k prove that

$$\frac{\prod_{i=n-k+1}^{n}(x^i - 1)}{\prod_{i=1}^{k}(x^i - 1)}$$

is a polynomial with integer coefficients.

ZD708DAD

Problem 22 (Indonesia TST 2021, added by Tilek Askerbekov, 3♣)

Let p be an odd prime. Determine the number of nonempty subsets from $\{1, 2, \ldots, p-1\}$ for which the sum of its elements is divisible by p.

21IDNTST132

Consider the polynomial $F(x) = (x+1)(x^2+1)\ldots(x^{p-1}+1)$. The coefficient of x^k is just the number of subsets of $\{1, 2, \ldots, p-1\}$ with sum k. Therefore, we want to find the sum of the coefficients of x^k for all k that is a multiple of p.

This is done by roots of unity filter. It suffices to find

$$\frac{1}{p}\sum_{k=0}^{p-1} F(\omega^k),$$

where $\omega = e^{2\pi i/p}$. However, when $k \neq 0$, we have (by expanding)

$$F(\omega^k) = P(-1),$$

where P is a polynomial with roots $\{\omega^k, \omega^{2k}, \ldots, \omega^{(p-1)k}\} = \{\omega, \omega^2, \ldots, \omega^{p-1}\}$. A polynomial satisfying this is $P(x) = 1 + x + \cdots + x^{p-1}$, and clearly, $P(-1) = 1$. So, this gives us

$$\frac{1}{p}\sum_{k=0}^{p-1} F(\omega^k) = \frac{1}{p}(2^{p-1} + \sum_{k=1}^{p-1} P(-1)) = \frac{1}{p}(2^{p-1} + p - 1).$$

This counts the empty set, so we subtract one for a final answer of $\frac{2^{p-1}-1}{p}$.