Submission for DCW-HALL

OTIS (internal use)

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August 23, 2024

> **Example** (PUMaC 2016, Alex Song, 0♣)**.** There are twelve candies arranged in a circle, four of which are rare candies. Chad and Eric want to collaborate on a strategy for the following act. First, Eric comes and is told which four candies are rare candies, then removes four non-rare candies from the circle. Then Eric leaves, and Chad comes and must determine which of the four candies (of the eight remaining candies) are rare. Decide whether this is possible or not.

16PUMACFA1

Walkthrough. Let $|S| = 12$. Most of this problem is trying to think about the problem in the right way; afterwards, Hall will destroy it.

(a) Phrase the problem as trying to match $\binom{S}{4}$ to itself.

(b) What are the constraints on this matching?

(c) Apply regular Hall to finish.

Practice problems

Instructions: Solve [24♣]. If you have time, solve [32♣].

> I'm looking for something that can deliver a 50-pound payload of snow on a small feminine target. Can you suggest something? Hello. . . ?
>
> Calvin on the phone in *Calvin and Hobbes*

Problem 1 (Another useful-later special case of Hall, 2♣)
Let $G = A \cup B$ be a bipartite graph on $2n$ vertices with minimum degree $n/2$ and $|A| = |B| = n$. Show that G has a perfect matching.

ZB9281DC

If $k \leq \frac{n}{2}$, then the desired result is trivial. Otherwise, we claim that any set of $k > \frac{n}{2}$ gifts is compatible with all n boxes. Assume not; then, some box is only compatible with less than $\frac{n}{2}$ gifts, which is a contradiction.

Problem 2 (Infinite Hall, added by Edward Yu, 2♣)
Show that Hall's theorem does not hold for infinite graphs, even if they are countable. That is, find an infinite bipartite graph between countably infinite sets A and B for which every finite subset of $S \subseteq A$ has at least $|S|$ neighbors in B, but there is no way of matching A to a subset of B.

ZB1FC962

Let $A = \{0, 1, 2, \dots\}$ and $B = \{1, 2, \dots\}$. Match 0 in A with all numbers in B, and for any $i \neq 0 \in A$, match it with only i in B. Clearly, there cannot be a perfect matching, even though the two sets have the same cardinality.

Problem 3 (Birkoff von Neumann, 2♣)
Let A be a square $n \times n$ matrix whose entries are nonnegative and whose rows and columns have sum 1 (such a matrix is *doubly stochastic*). Show that it's possible to write

$$A = c_1 P_1 + \cdots + c_m P_m$$

where each P_i is a permutation matrix, and each c_i is a positive real.

ZFD257C4

Problem 4 (Baltic Way 2013, 2♣)
Santa Claus has some gifts for n children. For $1 \leq i \leq n$, the i-th child considers $x_i > 0$ of these items to be desirable. Assume that

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \leq 1.$$

Prove that Santa Claus can give each child a gift that this child likes.

13BWAY

We claim that in any set of k children, at least one of them likes at least k of Santa's gifts. This is clearly sufficient to prove Hall's condition, and thus, the desired conclusion.

Suppose not. Then, summing over the k children,

$$\sum_{i\in S}\frac{1}{x_i} > \sum_{i\in S}\frac{1}{k} = 1,$$

which, when combined with the problem statement's condition, gives us a contradiction.

Required Problem 5 (3♣)

On a 1000×1000 chessboard, we delete some squares from the board so that each row and each column has at most k deleted squares. For which values of k is it always still possible to place 1000 non-attacking rooks on the board? (Rooks may still attack across deleted squares.)

ZF13143F

We form a relation (bipartite graph) between the columns and the rows. Specifically, we connect a row with a column if that row has a cell in that column (i.e. it wasn't deleted). Also, placing 1000 non-attacking rooks is just a perfect matching between the rows and columns.

The condition of removing at most k squares from each row and column is equivalent to saying that each vertex has degree greater than or equal to $1000-k$. If $1000-k\geq 1000/2=500$, meaning $k\leq 500$, then by the result of Problem 1, we are done.

If $k>500$, then delete the bottom k cells from the k leftmost columns and delete the leftmost k cells from the bottom k rows. It can be easily checked that every row and column has either k or $1000-k$ deleted cells, which satisfies the condition. Also, the first k columns are only connected with $1000-k<k$ rows, so Hall's condition is not satisfied.

Problem 6 (Tuymaada 2018/7, 3♣)

A school has three senior classes of M students each. Every student knows at least $\frac{3}{4}M$ people in each of the other two classes. Prove that the school can send M non-intersecting teams to the olympiad so that each team consists of 3 students from different classes who know each other.

18TMD7

The solution is to do Hall twice. First, we consider only two of the classes. By the result of problem 1, we can pair the students into M pairs where each pair consists of 2 students from different classes who know each other.

Then, consider the M pairs on one side of a bipartite graph, and the M students of the third senior class on the other side. We connect a pair with a student from the third class if the student knows both students in the pair. Notice that this graph has minimum degree $\frac{1}{2}M$, because otherwise, some student would not know at least $\frac{3}{4}M$ students from each of the other two classes. So, we use the result of problem 1 again to create M teams of 3.

Problem 7 (Putnam 2012 B3, 3♣)

A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

12PTNMB3

Problem 8 (Halls Chessboard, 3♣)

An $n \times n$ chessboard has some of its squares painted blue. Assume that for every n squares chosen, no two in the same row or column, at least one of the squares is blue. Prove that one can find a rows and b columns whose intersection contains only blue squares, so that $a + b \geq n + 1$.

Z685102E

Problem 9 (3♣)

A table has m rows and n columns with $m, n > 1$. The following permutations of its mn elements are permitted: any permutation leaving each element in the same row (a "horizontal move"), and any permutation leaving each element in the same column (a "vertical move"). Find the smallest integer k in terms of m and n such that any permutation of the mn elements can be realized by at most k permitted moves.

WOOTP046

Problem 10 (3♣)

Let G be a graph. Show that G has an orientation in which every indegree is at most

$$\left\lceil \max_H \frac{\#E(H)}{\#V(H)} \right\rceil$$

where the maximum is taken over nonempty subgraphs of H.

Z5F11887

Problem 11 (Russia 2021, added by Rohan Goyal, 3♣)

Each gopher among 100 gophers has 100 balls; there are in total 10000 balls in 100 colors, from each color there are 100 balls. On a move, two gophers can exchange a ball (the first gives the second one of her balls, and vice versa). The operations can be made in such a way, that in the end, each gopher has 100 balls, colored in the 100 distinct colors.

Prove that there is a sequence of operations, in which each ball is exchanged no more than 1 time, and at the end, each gopher has 100 balls, colored in the 100 colors.

21RUS118

Problem 12 (5♣)

Let G be a bipartite graph on $A \cup B$ with no isolated vertices. Assume that for each edge ab with $a \in A$ and $b \in B$, we have $\deg a \geq \deg b$. Prove that G contains a matching using all vertices in A.

Z043CFE1

Problem 13 (Vietnam TST 2001/3, added by Milind Pattanik, 3♣)

A club has 42 members, each of which is either a boy or girl. Itâ€™s known that among 31 arbitrary club members, we can find a boy and a girl who know each other. Show that there exist 12 disjoint pairs of club members, each of which is a boy and a girl that know each other.

01VNMTST3

Problem 14 (Shortlist 2006 C6, 9♣)

An upward equilateral triangle of side length n is divided into n^2 cells which are equilateral triangles of unit length. A *holey triangle* is such a triangle with n upward unit triangular holes cut out along gridlines. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

06SLC6

Problem 15 (Bulgaria 1997, 3♣)

Let $n \ge 2$ be a positive integer, and consider ordered n-tuples of distinct integers in the set $\{1, \dots, n+1\}$. Two such tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are called *disjoint* if there exists $1 \le i, j \le n$ such that $i \ne j$ and $a_i = b_j$. What is the maximum possible number of pairwise disjoint n-tuples?

97BGR

Required Problem 16 (Shortlist 2010 C2, 5♣)

Let $n \ge 4$ be an integer. A *flag* is a binary string of length n. We say that a set of n flags is *diverse* if these flags can be the rows of an $n \times n$ binary matrix with the entries in its main diagonal all equal. Determine the smallest positive integer M such that among any M distinct flags, there exist n flags forming a diverse set.

10SLC2

We claim the answer is $M = 2^{n-2} + 1$. First, 2^{n-2} doesn't work; just consider all the 2^{n-2} flags starting with 01.

Now, we show that $M = 2^{n-2} + 1$ works. Let D be either 0 or 1, and construct a bipartite graph between the set of M flags (arranged into an M by n grid) and the n columns. We connect a row with column i if that row has D in column i. It suffices to find a perfect matching from the columns to the rows (flags).

We claim that this graph satisfies Hall's condition for some choice of D, first starting with $k = 1$. Hall's condition for $k = 1$ is only false for some D if a column has the binary digit other than D in every row. But this can't be true for both choices of D, as that would restrict the number of unique flags to $2^{n-2} < M$. So, at least one choice of D satisfies Hall's condition for $k = 1$.

We do casework on how many choices of D satisfy Hall's condition for $k = 1$. If both choices of D satisfy it, then we claim that Hall's condition for $k = 2$ must be true for some choice of D. Suppose not. Then, $M - 1$ rows must have 0's in two fixed columns, and $M - 1$ rows must have 1's in some two other fixed columns. These conditions overlap in at least $M - 2$ rows, but the number of distinct flags that satisfy both is $2^{n-4} < M - 2$, giving us a contradiction.

If only one choice of D satisfies Hall's condition for $k - 1$, then WLOG let that choice be $D = 0$. We claim that Hall's condition for $k = 2$ must be true for $D = 0$. Suppose not. Then, at least $M - 1$ rows have 1's in two fixed columns. These rows also have 0 in another fixed column, since $D = 1$ fails Hall's condition for $k = 1$. The number of distinct flags satisfying these conditions is $2^{n-3} < M - 1$, so we have a contradiction. Therefore, Hall's condition must be true for $k = 1, 2$, for some choice of D.

Again WLOG that choice to be $D = 0$, and in one final sweep, we show Hall's condition for $k \ge 3$. Again, suppose not. Then, at least $M - k + 1$ flags have 1 in some k fixed columns. But 2^{n-k} distinct flags satisfy that condition, and $k \ge 3 \implies 2^{n-k} < M - k + 1$, so we are done.

Problem 17 (RMM 2017/5, 5♣)

Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any integer $k \geq 1$. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

17RMM5

Problem 18 (Shortlist 2012 C5, 5♣)

The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

12SLC5

Problem 19 (Harder 2012 C5 by Ji Min Kim, 5♣)

The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 1$ from its original position, and each square contains a token with the same color as the square.

12SLC5GEN

Remark. The "Harder 2012 C5" is 2012 C5 with $d + 2$ improved to $d + 1$. Solving this problem also gives you the 5♣ for the original.

Required Problem 20 (Math Prize 2022/4, 9♣)

Let $n > 1$ be an integer. Let A denote the set of divisors of n that are less than $\sqrt{n}$. Let B denote the set of divisors of n that are greater than $\sqrt{n}$. Prove that there exists a bijective function $f : A \to B$ such that a divides $f(a)$ for all $a \in A$.

22MP04

Split the divisors of n into equivalence classes based on the following relation: divisors a and b are related if for every prime p dividing n, either $\nu_p(a) + \nu_p(b) = \nu_p(n)$ or

$\nu_p(a) = \nu_p(b)$. This relation is partially motivated by the representation of the divisors of n as lattice points in a hypercuboid.

We claim that in each equivalence class C, we can construct a bijection $f\colon A\cap C \to B\cap C$ satisfying $d \mid f(d)$ for all d.

Notice that within each equivalence class C and for a fixed prime $p \mid n$, there are at most two possible values of $\nu_p(d)$ for $d \in C$. If a prime has only one possible value of $\nu_p(d)$ for all $d \in C$, then the divisibility condition is trivial for that prime (i.e. we might as well divide all numbers in C by that prime raised to the maximal exponent). So, call a prime p *significant* when there are two possible values of $\nu_p(d)$, and let e_p be the smaller value and E_p be the greater value.

Our bijection f can be reframed as a bijection g from $A \cap C$ to itself, by reframing our condition $d \mid f(d)$ as

$$\min\{\nu_p(d), \nu_p(g(d))\} = e_p$$

for all significant primes p. From this, we can construct f as $f(d) = \frac{n}{g(d)}$.

Now, we reframe the problem again. Let X be the set of prime divisors of n, and associate each element $d \in A \cap C$ with the set containing all significant primes p such that d is divisible by p^{E_p}. It is not hard to check that no two elements can be associated with the same set.

Consider the collection $\mathcal{F}$ of sets associated with elements in d. This collection satisfies the property that for all $A \in \mathcal{F}$, all subsets of A are also in $\mathcal{F}$ (i.e. $\mathcal{F}$ is closed downward). The central claim of the problem is that there exists a bijection $\sigma : \mathcal{F} \to \mathcal{F}$ such that $\sigma(A) \cap A = \varnothing$ for every set $A \in \mathcal{F}$. This is indeed equivalent to the condition for the bijection g that we stated earlier. It is also a problem from an earlier Iran olympiad: https://aops.com/community/p20560.

We proceed by induction on $|X|$. The base case $|X| = 0$ is trivial. For the inductive step, fix an element $x \in X$. We divide the sets in $\mathcal{F}$ into three categories:

- **Type 1 sets:** Sets that contain x.
- **Type 2 sets:** Sets that don't contain x but would become a type 1 set if x were added to it.
- **Type 3 sets:** All other sets (not containing x).

Notice that the set of type 2 sets and the set of type 3 sets are each closed downward. We construct σ as follows:

- Let σ_1 be a bijection from the set of type 2 sets to itself, satisfying the condition $\sigma_1(A) \cap A = \varnothing$. It exists because of the inductive hypothesis.
- Similarly, let σ_2 be a bijection from the set of type 3 sets to itself, satisfying the condition.
- If A is a type 1 set, then σ takes A to $\sigma_1(A \setminus \{x\})$.
- If A is a type 2 set, then σ takes A to $\sigma_1(A) \cup \{x\}$.
- If A is a type 3 set, then $\sigma(A) = \sigma_2(A)$.

One can check that this is indeed a bijection that satisfies our criterion.