2024 AIME I Solutions

Michael Middlezong

31 December 2024

Problem 11

We do casework on the number of blue vertices. If there are 0, 1, or 2 blue vertices, then any arrangement is valid, giving us 1, 8, and $\binom{8}{2} = 28$ cases, respectively.

If there are 3 blue vertices, then suppose no two of them are adjacent. Then, we can just rotate by one in either direction. Suppose two of them are adjacent. Then, it is clear that no matter where the third is, it will still work out. So, every arrangement works, giving us $\binom{8}{3} = 56$ cases.

The hardest case to deal with is if there are 4 blue vertices. For this, we do casework on the maximal contiguous blocks of blue vertices.

- One block of 4 yields 8 cases.
- A block of 3 with a separate block of 1 yields 0 cases.
- Two separate blocks of 2 yields 4 cases.
- A block of 2 and two blocks of 1 yields 8 cases.
- Four blocks of 1 yields 2 cases.

Adding up yields 22 cases.

There cannot be more than 4 blue vertices. So, the final probability is

$$\frac{1+8+28+56+22}{256} = \frac{115}{256}.$$

This gives an answer of $115 + 256 = \boxed{371}$.

Remark. I sillied in-contest by forgetting to count the case of "A block of 2 and two blocks of 1."

Problem 12

After graphing each function, we see that intersections can only happen in the square $[0, 1] \times [0, 1]$. Furthermore, the first graph looks like a line that oscillates, changing direction 16 times, while the second graph looks like a line that oscillates 24 times. The first graph starts at $(0, 1)$ and ends at $(1, 1)$, and the second graph starts at $(1, 1)$ and ends at $(1, 0)$. So, there will be an intersection at $(1, 1)$, but not in any other corner.

Trying small examples and conjecturing a pattern, we expect this sort of oscillating line configuration to generate $16 \cdot 24 = 384$ intersection points. However, we must be careful around the $(1, 1)$ corner, since these are not actually lines. The first graph's derivative is some positive value at $(1, 1)$, and the second graph has a derivative approaching infinity. Thus, the two graphs must cross twice, once at exactly $(1, 1)$ and once right before. This gives an extra intersection, so the answer is $384 + 1 = \boxed{385}$.

Problem 13

Basic orders yields $8 \mid p - 1$, so let's assume $p = 17$ for now. Based on the existence of primitive roots mod 17^2, there has to be something of order 8, so $p = 17$ definitely works.

Using Hensel's lemma, we will lift a root mod 17 to a root mod 17^2. It is not hard to see that the roots mod 17 of $x^4 + 1$ are $x = \pm 2, \pm 8$. Since $4x^3 \neq 0$ in all of those cases, Hensel's will work.

At this point, one can expand $(17k + x)^4$ mod 17^2 using the binomial theorem for each of the possible x values, and find that the minimum is $m = \boxed{110}$.

Problem 14

Notice that $4^2 + 5^2 = 41$, $5^2 + 8^2 = 89$, and $4^2 + 8^2 = 80$. We want to find a way to use this niceness/symmetry to our advantage, and we can by introducing a coordinate system.

Let $A = (0, 0, 0)$, $B = (4, 5, 0)$, $C = (4, 0, 8)$, and $D = (0, 5, 8)$. We want to find the inradius r, so we will prove a formula analogous to $A = sr$ but for tetrahedrons. Consider splitting the tetrahedron into four mini tetrahedrons $IABC$, $IBCD$, $IABD$, and $IACD$. Each of these tetrahedrons has height r, so the sum of the volumes is

$$V = \frac{1}{3}r(b_1 + b_2 + b_3 + b_4) = \frac{1}{3}Ar,$$

where A is the surface area.

Thankfully, we can find the surface area and the volume easily, because of our coordinate system. The volume of the tetrahedron is $\frac{1}{6}$ the volume of the associated parallepiped, or

$$\frac{1}{6}\begin{vmatrix} 4 & 4 & 0 \\ 5 & 0 & 5 \\ 0 & 8 & 8 \end{vmatrix} = \frac{160}{3}.$$

To find the surface area, notice that all the sides are congruent. The area of one of the sides is half the area of the associated parallelogram, or

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2\sqrt{189}.$$

So, the surface area is $8\sqrt{189}$. Plugging this into the formula, we get

$$\frac{160}{3} = \frac{1}{3}8r\sqrt{189} \implies r = \frac{20\sqrt{21}}{63}.$$

So, our answer is $20 + 21 + 63 = \boxed{104}$.

Problem 15

After noticing that two of the sides must be equal, it becomes a simple algebra exercise.

Otherwise, we proceed with Lagrange multipliers.