Submission for DGW-HARMONIC

OTIS (internal use)

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> **Example** (Lemma 4.26 from my book, added by Catherine Xu, 0♣)**.** Let $\triangle ABC$ be a triangle, and let the tangents to its circumcircle at B and C meet at point X. Let M be the midpoint of $\overline{BC}$. Show that AX is the A-symmedian of $\triangle ABC$, meaning that $\measuredangle BAX = \measuredangle MAC$.

EGIMO426

Walkthrough. Let D be the intersection of BC and AX, and let the tangent of the circumcircle of $\triangle ABC$ at A intersect BC at the point Y.

(a) Show that $(BC; DY) = -1$.

(b) Show that the reflection of line $\overline{AY}$ across the $\angle A$-bisector is parallel to $\overline{BC}$.

(c) Take the reflection lines AB, AC, AD, AY in (a) across the $\angle A$-bisector; these four lines are still harmonic. Use (b) to deduce the problem statement.

> **Example** (IMO 2014/4, 0♣)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

14IMO4

Walkthrough. We give a walkthrough for the harmonic bundles solution.

(a) Show that the tangent to B is parallel to $\overline{APM}$.

(b) Find a natural harmonic bundle using the answer to (a).

Let $\overline{BM}$ intersect the circumcircle again at X.

(c) Projecting the answer to (b) onto the circumcircle gives a harmonic quadrilateral. Which one?

(d) Deduce that $\overline{CN}$ passes through X as well.

> **Example** (Brazil 2011/5, 0♣)**.** Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

11BRA5

Walkthrough. There are two general approaches, one by harmonic quadrilaterals and one by spiral similarity. Both begin the same way.

(a) Show that the condition is equivalent to $FB/FC = HB/HC$.

If you are working on the harmonic route:

(b) Let X be the intersection of ray AH with the circumcircle of $\triangle ABC$. Prove that the problem is equivalent to $FBXC$ being harmonic.

(c) Show that lines AF, DE, BC are concurrent.

(d) Use (b) and (c) together to solve the problem.

Here is a spiral similarity route:

(e) Identify F as a Miquel point of a quadrilateral, and write down the two pairs of similar triangles.

(f) Use this to express FB/FC as a ratio not involving the point F.

(g) Show that the ratios you found in (f) are equal.

Added bonus: the line FH bisects $\overline{BC}$ and passes through the A-antipode.

> **Example** (IMO 2010/4, 0♣)**.** Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M, respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

10IMO4

Walkthrough. Here is a walkthrough of a projective solution.

Let D denote the other tangency point from S. Let $\overline{DP}$ meet (ABC) again at N.

(a) Show that $KMLN$ is a harmonic quadrilateral.

(b) We want to prove $MK = ML$. What does that suggest should be true about $MLNK$?

In light of (b), our goal is to show that $\overline{MN}$ is a diameter.

It will make more sense actually to let N' be the antipode of M on Γ, and let D' be the second intersection of $N'P$ with Γ. We will show $D' = D$.

(c) Show that $(CD'P)$ is orthogonal to Γ. (Possible hint: "three tangents" lemma in §1 of EGMO.)

(d) Identify the circumcenter of $\triangle CD'P$.

(e) Show that $\overline{SD'}$ is tangent to Γ, so $D' = D$.

So we now know have everything we need: we know both that $\overline{MN}$ is a diameter and $\overline{SD}$ is tangent.

(f) Deduce that $KLNM$ is a kite, and thus $NL = NK$.

Practice problems

Instructions: Solve [45♣]. If you have time, solve [55♣].

Given three statements, two of them are equivalent.

Boris Alexeev, MOP 2003

Required Problem 1 (Useful in later problems, 2♣)

Let A, X, B, Y be points on a line in this order. Let M be the midpoint of $\overline{AB}$. Show that the following are equivalent:

- $(AB;XY) = -1$.
- $MA^2 = MX \cdot MY$ and $AX > XB$.
- $YA \cdot YB = YX \cdot YM$.

(Try to find "synthetic" solutions involving circles. This is a useful lemma in many problems, so keep it in mind!)

Z79810E7

Let ω be the circle with diameter $\overline{AB}$, and let γ be the circle with diameter $\overline{XY}$.

The first condition is equivalent to X mapping to Y under inversion around ω. This means the circles are orthogonal, and thus,

$$MA^2 = MP^2 = MX \cdot MY.$$

Also, we have

$$YA \cdot YB = YM^2 - MA^2 = YM^2 - XM \cdot YM = YM(YM - XM) = YM \cdot YX,$$

as desired.

Problem 2 (EGMO Lemma 9.27: Self-Polar Orthogonality, 2♣)

Let ω be a circle and suppose P and Q are points such that P lies on the polar of Q (and hence Q lies on the polar of P). Prove that the circle γ with diameter $\overline{PQ}$ is orthogonal to ω.

EGIMO927

Let D be the foot of the perpendicular from P to line OQ. Notice that since $\triangle DPQ$ is right, D must lie on circle γ. Moreover, an inversion around ω takes D to Q, since DP is the polar of Q. Thus, γ maps to itself under this inversion, so it is orthogonal to ω.

Problem 3 (Canada 1994, 2♣)

Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively.

Prove that $\angle EDH = \angle FDH$.

94CAN5

Intersect line EF with line BC, and the problem becomes trivial using the right angles and bisectors lemma.

Problem 4 (PAGMO 2022/3, 2♣)

Let ABC be an acute triangle with $AB < AC$. Denote by P and Q points on the segment BC such that $\angle BAP = \angle CAQ < \frac{\angle BAC}{2}$. Point B_1 lies on segment AC, and BB_1 intersects AP and AQ at P_1 and Q_1, respectively. The angle bisectors of $\angle BAC$ and $\angle CBB_1$ intersect at M. If $PQ_1 \perp AC$ and $QP_1 \perp AB$, prove that AQ_1MPB is cyclic.

22PAGMO3

Required Problem 5 (ELMO SL 2012 G3, 3♣)

Let ABC be a triangle with incenter I. The foot of the perpendicular from I to $\overline{BC}$ is D, and the foot of the perpendicular from I to $\overline{AD}$ is P. Prove that $\angle BPD = \angle DPC$.

12ESLG3

If $AB = AC$, then we are done by symmetry. Otherwise, let K be the intersection point of lines IP and BC. Notice that K is the inverse of P with respect to the incircle, and thus, A lies on the polar of K. By La Hire's theorem, we know that K lies on the polar of A. In other words, if E and F are the contact points of the incircle with sides AC and AB, respectively, then K lies on line EF.

It is well known that the cevians AD, BE, and CF concur, so we can use the concurrent cevians lemma to deduce that $(ED; BC) = -1$. Finally, since $\angle EPD = 90$, the right angles and bisectors lemma tells us that $\angle BPD = \angle DPC$.

Problem 6 (JMO 2011/5, 2♣)

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

11JMO5

Letting $M = \overline{BE} \cap \overline{AC}$ and F be the second intersection point of line DM with ω, we have

$$-1 = (CA; DB) \stackrel{M}{=} (AC; FE) \stackrel{D}{=} (AC; M\infty),$$

and thus, M is the midpoint of AC.

Problem 7 (MOP 2013, 2♣)

Let ABC be an acute scalene triangle, and let H be a point inside it such that $\overline{AH} \perp \overline{BC}$. Rays BH and CH meet $\overline{AC}$ and $\overline{AB}$ at E, F. Prove that if quadrilateral $BFEC$ is cyclic then H is in fact the orthocenter of ABC.

13MOPHWR8

Let X be the intersection point of line EF with line BC. Let M be the center of $(BFEC)$. Brocard's theorem tells us that M is the orthocenter of triangle AXH. This means that $\overline{XM} \perp \overline{AD}$, and thus, M must be the midpoint of BC. The desired result easily follows from here.

Problem 8 (PAGMO 2021/6, 3♣)

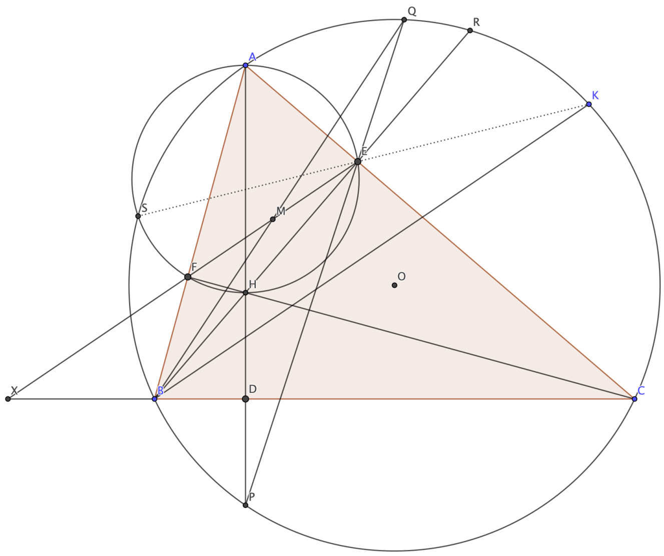
Let ABC be a triangle with incenter I, and A-excircle Γ. Let A_1, B_1, C_1 be the points of tangency of Γ with BC, AC and AB, respectively. Suppose IA_1, IB_1 and IC_1 intersect Γ for the second time at points A_2, B_2, C_2, respectively. M is the midpoint of segment AA_1. If the intersection of A_1B_1 and A_2B_2 is X, and the intersection of A_1C_1 and A_2C_2 is Y, prove that $MX = MY$.

21PAGMO6

Required Problem 9 (ELMO Shortlist 2019 G1, added by Kevin Wang, 5♣)
Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

19ESLG1

Let D be the foot of the altitude from A to BC, let $M = \overline{BQ} \cap \overline{EF}$, let S be the second intersection of (AEF) with Γ, let $X = \overline{EF} \cap \overline{BC}$, let R be the second intersection of line BE and Γ, and let K be the point on Γ satisfying $\overline{BK} \parallel \overline{EF}$.



First, we show S, E, K are collinear. This is trivial by angle chasing:

$$\angle ASE = \angle AFE = \angle ABK = \angle ASK.$$

Now, notice that A, S, and X are collinear. This is well known and the proof is by radical axis. Finally,

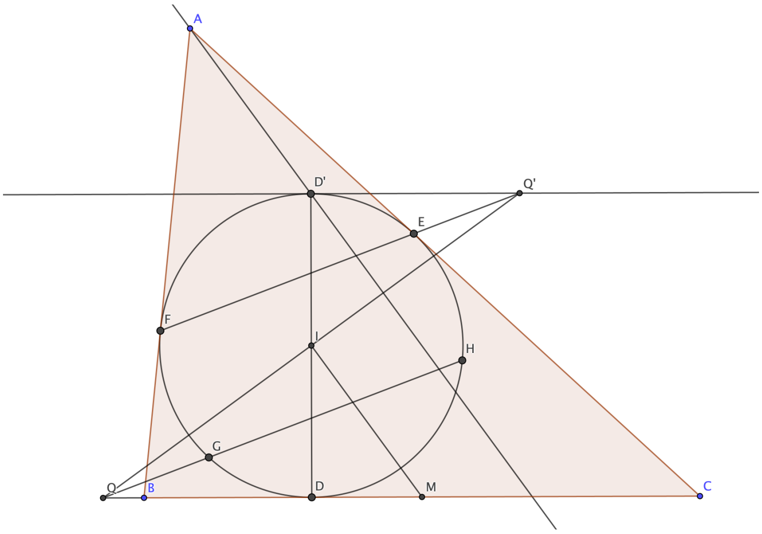
$$-1 = (XD; BC) \stackrel{A}{=} (SP; BC) \stackrel{E}{=} (KQ; RA) \stackrel{B}{=} (\infty M; EF),$$

and we are done.

Problem 10 (Taiwan TST 2014/1J/3, 3♣)
In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

14TWNTST1J3

Let D be the contact point of the incircle with BC, let Q' be the intersection of lines QI and EF, and let D' be the antipode of D.



By symmetry, $\overline{Q'D'} \parallel \overline{BC}$, and thus, $\overline{Q'D'}$ is tangent to the incircle. Then, by La Hire's theorem, Q' is the pole of line AD'. Finish by noticing that $\overline{AD} \parallel \overline{IM}$ (well-known; proof is by homothety).

Required Problem 11 (Sloth blocking ruler, 3♣)

You have a large sheet of paper in which three marked points A, B, C are collinear in that order. You want to construct line ABC with your straightedge, but a cute sloth is sleeping peacefully on the paper and obstructing the segment BC. Determine how to extend ray AB past C without disturbing the sloth (with straightedge alone).

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First, if $AB \leq BC$, we can extend ray BA and move point A so that $AB > BC$.

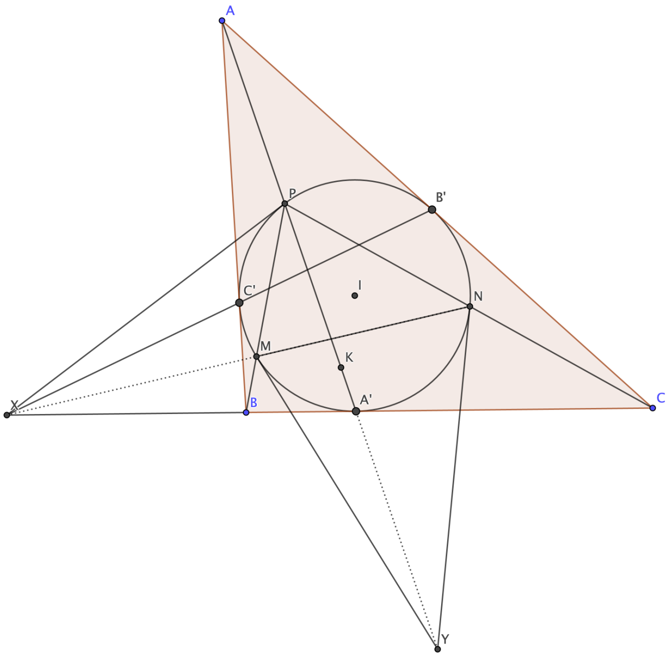
Draw point D not on line ABC. Draw lines DA, DB, and DC. Pick a point P on DB. Now, we create the Ceva/Menelaus configuration as follows:

Extend ray AP to hit DC at E and ray CP to hit DA at F. Notice that the intersection point with line EF and line AC, which we want, is just the harmonic conjugate of $\overline{DB} \cap \overline{EF}$ with respect to EF. However, this line is actually accessible to us, so we can just repeat this configuration again to obtain the desired point.

Problem 12 (Iran 2002, 5♣)

Let ABC be a triangle. The incircle of triangle ABC touches the side BC at A', and the line AA' meets the incircle again at a point P. Let the lines CP and BP meet the incircle of triangle ABC again at N and M, respectively. Prove that the lines AA', BN and CM are concurrent.

02IRN



Let B' and C' be the other two incircle contact points, and let X be the intersection point of line $B'C'$ and line BC. It is well known that X is the pole of line PA', and thus, line XP is tangent to the incircle.

Notice that by the Ceva/Menelaus configuration, the problem is equivalent to showing that X, M, and N are collinear. Let Y be the intersection point of the tangents to the incircle at M and N. By La Hire's theorem, it suffices to prove that P, A', and Y are collinear. We have

$$-1 = (XA'; BC) \stackrel{P}{=} (PA'; MN),$$

so by the symmedian config, line PY intersects the incircle at A', so we are done.

Problem 13 (Kazakhstan 2011/9.5, 3♣)

Given a non-degenerate triangle ABC, let A_1, B_1, C_1 be the points of tangency of the incircle to the sides BC, CA, AB. Let Q and L be the intersection of the segment AA_1 with the incircle and the segment B_1C_1 respectively. Let M be the midpoint of B_1C_1. Let T be the point of intersection of BC and B_1C_1. Let P be the foot of the perpendicular from the point L on the line AT. Prove that the points A_1, M, Q, P lie on a circle.

11KAZ95

Problem 14 (TSTST 2015/2, 5♣)

Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

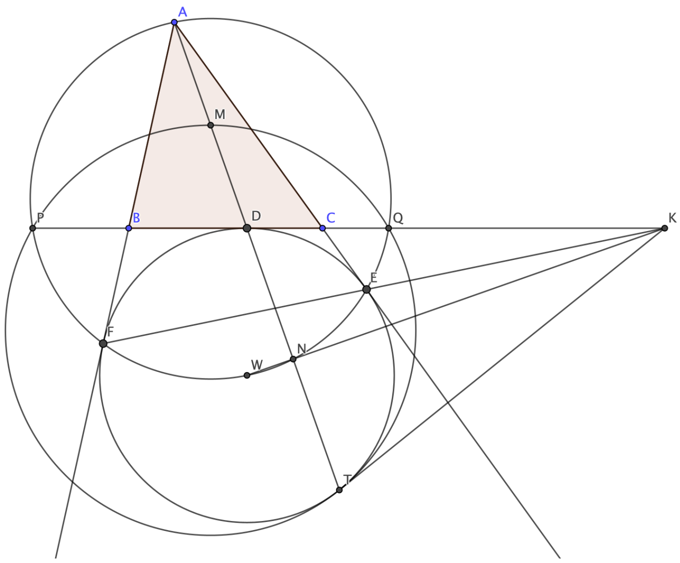
15TSTST2

Problem 15 (Shortlist 2017 G4, 5♣)

Let ABC be a triangle and let ω be the A-excircle, tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. The circumcircle of $\triangle AEF$ intersects line BC at P and Q. Let M be the midpoint of $\overline{AD}$. Prove that the circumcircle of $\triangle MPQ$ is tangent to ω.

17SLG4

Let W be the center of ω, let K be the intersection of lines EF and BC, and let T be the second intersection of line AD with ω. We claim that T is the desired point of tangency.



First, notice that $FDET$ is harmonic, so line KT is tangent to ω.

Let N be the midpoint of DT. Notice that W lies on (AEF) by angle chasing, and N lies on (AEF) because $WN \perp AN$.

Also, T lies on (MPQ) because $DT \cdot DM = DN \cdot DA = DP \cdot DQ$.

Finally, we have

$$KP \cdot KQ = KF \cdot KE = KT^2,$$

so by power of a point, we are done.

Problem 16 (Iran Geo Olympiad 2019, added by Tilek Askerbekov, 3♣)

Circles ω_1 and ω_2 have centers O_1 and O_2, respectively. These two circles intersect at points X and Y. Line AB is a common tangent of these two circles such that A lies on ω_1 and B lies on ω_2. Let the tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 again at M and AK intersects ω_1 again at N. Prove that AM, BN and O_1O_2 concur.

19IGOA3

Problem 17 (Sharygin 2018, added by Kevin Wang, 3♣)

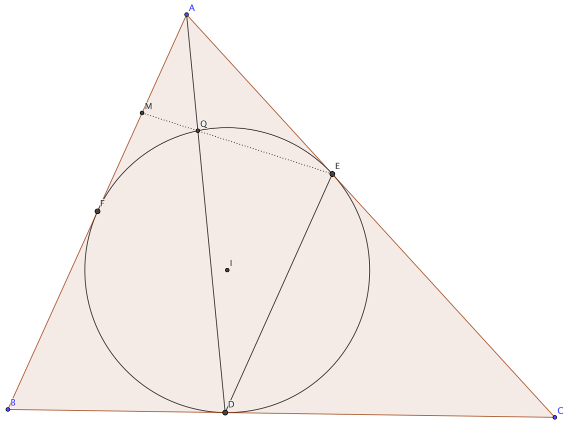
Let scalene triangle ABC have intouch triangle XYZ. The A-excircle touches the side BC at point N. Let T be the common point of AN and the incircle, closest to N, and $K = \overline{YZ} \cap \overline{XT}$. Prove that $\overline{AK} \parallel \overline{BC}$.

18SHRG20

Problem 18 (South Africa 2005/4, 2♣)

The inscribed circle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. Let Q denote the other point of intersection of AD and the inscribed circle. Prove that EQ extended passes through the midpoint of AF if and only if $AC = BC$.

05SAF4



Note that $(EF; DQ) = -1$. The problem is finished by considering the projection at E onto line AB.

Problem 19 (Shortlist 2005 G6, 5♣)

Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to $\overline{BC}$, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

05SLG6

Problem 20 (China Southeast MO 2018, 5♣)

Let ABC be an isosceles triangle with $AB = AC$. A circle Γ centered at the midpoint M of $\overline{BC}$ is tangent to lines AB and AC at F and E, respectively. Point G is chosen on Γ with $\angle AGE = 90°$. The tangents to Γ at G and F meet at K. Prove that $\overline{CK}$ bisects $\overline{EF}$.

18CSMO6

Problem 21 (Added by William Zhao, 3♣)

Let ABC be a right triangle with $\angle A = 90°$, and let D be a point lying on the side AC. Denote by E reflection of A into the line BD, and by F the intersection point of CE with the perpendicular in D to the line BC. Prove that AF, DE and BC are concurrent.

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Problem 22 (Added by Krishna Pothapragada, 3♣)

In $\triangle ABC$ with incenter I and A-excenter I_A, let G be the centroid of $\triangle BIC$. Define $E = \overline{BI} \cap \overline{AC}$ and $F = \overline{CI} \cap \overline{AB}$. Prove that $\angle BGC + \angle EI_AF = 180°$.

H2648084

Problem 23 (USA TST 2017, Danielle Wang, 9♣)

Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

17USATST5

Problem 24 (HMMT 2018, added by Ram Goel, 5♣)

Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K. Show that $KX \cdot KQ$ is an integer and determine its value.

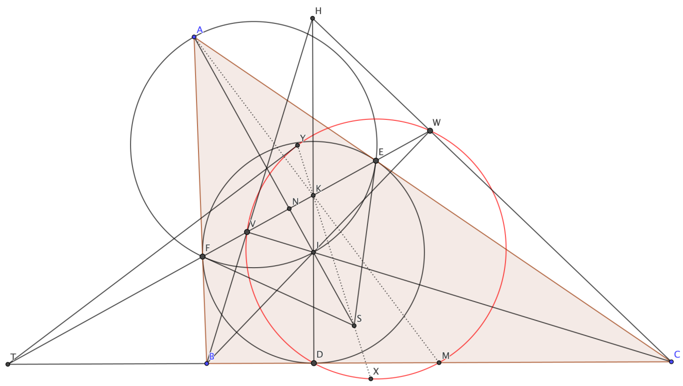
18HMMTG8

Required Problem 25 (Taiwan TST Quiz 2015, by me, 9♣)

In scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of $\triangle AEF$ at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter $\overline{ST}$ is orthogonal to the nine-point circle of triangle BIC.

15TWNQ3J6

By the self-polar orthogonality lemma, it suffices to show that S lies on the polar of T with respect to the nine-point circle.



To complete the nine-point circle configuration, let H be the orthocenter of $\triangle BIC$, let D be the remaining intouch point, and let M be the midpoint of BC. Note that D and M lie on the nine-point circle, and I is the orthocenter of $\triangle HBC$.

Let V and W be the feet of the perpendiculars from C and B to their respective sides. By the Iran lemma (proof: angle chasing), V and W lie on line EF. Also, V and W lie on the nine-point circle.

Now, let K be the intersection of lines ID and EF, let N be the midpoint of EF, and let X and Y be the points of tangency of T with the nine-point circle. We want to show that X, Y, and S are collinear.

First, we have

$$-1 = (TD;BC) \overset{I}{=} (TK;WV),$$

so K lies on line XY. Also, K lies on line AM by an incircle concurrency lemma. So then,

$$-1 = (AI;NS) \overset{K}{=} (M,D;T,\overline{KS} \cap \overline{BC}),$$

which means $\overline{KS} \cap \overline{BC}$ is on the polar of T. Thus, S is collinear with two points on the polar of T, so S must be on the polar of T, and we are done.