Shortlist 2012

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Combinatorics

C1

Call the act of replacing (x, y) by $(y+1, x)$ a *sum-increasing swap*, and call the act of replacing (x, y) by $(x-1, x)$ a *lifting swap*.

Let M be the maximum number written on the board initially. We claim that the maximum number written on the board never exceeds M. If it did exceed M, then some action must have replaced (x, y) by (a, x), where $a > x > y$. This is impossible given the two types of actions we have.

Thus, if there are n numbers on the board, the sum of the numbers is bounded above by nM. It follows that there may only be a finite number of sum-increasing swaps.

Assume FTSOC that Alice performs an infinite number of lifting swaps. If $y = x - 1$ prior to performing a lifting swap on (x, y), then we call it a type 1 lifting swap (or just a swap, as $(x, x-1) \to (x-1, x)$). Otherwise, we call it a type 2 lifting swap. Notice that type 2 lifting swaps increase the total sum, so there can be only finitely many of them. Therefore, Alice must perform infinitely many type 1 lifting swaps.

Now, consider a point in Alice's infinite sequence of actions such that no more sum-increasing swaps or type 2 lifting swaps are to be done. Such a point must exist because there are finitely many such actions.

From this point onward, every action is a type 1 lifting swap. However, this cannot continue infinitely, so we are done. (To show the final statement, consider the monovariant ia_i where i is the number in the ith position.)

C2

The answer is $\left\lfloor \frac{2n-1}{5} \right\rfloor$. Let k be the number of pairs. First, we show that $k > \left\lfloor \frac{2n-1}{5} \right\rfloor$ is impossible.

Consider S, the sum of the sums of all k pairs. By double counting, we have that the maximum possible value of S is

$$n + (n-1) + (n-2) + \cdots + (n-(k-1))$$

and that the minimum possible value of S is

$$1 + 2 + \cdots + 2k.$$

Thus, we have

$$n + (n-1) + (n-2) + \cdots + (n-(k-1)) \geq 1 + 2 + \cdots + 2k.$$

Simplifying this gives $k \leq \frac{2n-1}{5}$, and thus, $k > \left\lfloor \frac{2n-1}{5} \right\rfloor$ is impossible.

Next, we provide a construction for when $k = \left\lfloor \frac{2n-1}{5} \right\rfloor$. If $n = 5s + 1$ for an integer s, then we have $k = 2s$, and the desired k pairs are the union of

$$\{(i, 3s + i + 1), (s + i, 2s + i)\}$$

for $i = 1, 2, \ldots, s$.

If $n = 5s + 2$ for an integer s, then we still have $k = 2s$ and hence, we can use the same construction as above.

If $n = 5s + 3$ for an integer s, then $k = 2s + 1$, and the desired k pairs are the union of

$$\{(i, 3s + i + 2), (s + j, 2s + j + 1)\}$$

for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s + 1$.

Finally, if $n = 5s + 4$ or $n = 5s + 5$ for an integer s, then we still have $k = 2s + 1$ and hence, we can use the same construction as above.

These constructions can be easily shown to work, concluding the proof.

Geometry

G1

The difficulty in this problem mainly lies in algebraic manipulation.

We start by computing $J = (-a : b : c)$ and $M = (0 : s - b : s - c)$. Notice that $KB = s - c$ and $KA = s$. From this, we can deduce $K = (c - s : s : 0)$. Similarly, $L = (b - s : 0 : s)$.

Now, we set out to compute F. Since F lies on line BJ, we know that it can be written in the form $(-a : t : c)$ for some t. We also know F, M, and L are collinear, so we have the equation

$$\begin{vmatrix} -a & t & c \\ 0 & s-b & s-c \\ b-s & 0 & s \end{vmatrix} = 0 \implies t = \frac{-as + c(s-b)}{s-c}.$$

At this point, continuing with the computation leads to very messy expressions. We wonder if the expression for t can be simplified. Indeed, after some algebra:

$$\frac{-as + c(s-b)}{s-c} = -(a+c).$$

So, we have $F = (-a : -(a+c) : c) = (a : a+c : -c)$. Similarly, $G = (a : -b : a+b)$.

Now, we have pretty much finished the problem. Computing S and T and then the midpoint of ST gives M, so we are done.

G2

The reflection is arbitrary, so we get rid of point H and instead note that it suffices to show $\angle FED + \angle FGD = 180$.

First, we claim that lines FE and FG are isogonal with respect to $\angle CFD$. This can be shown using the parallelogram isogonality lemma. Alternatively, we can prove this by showing $\triangle FAE \sim \triangle FCG$ by SAS similarity. Indeed, using directed angles,

$$\angle FAE = \angle DAE = \angle DBC = \angle GCF,$$

and furthermore,

$$\frac{FA}{FC} = \frac{AB}{CD} = \frac{AE}{ED} = \frac{AE}{CG},$$

so the claim is proven.

Then, easy angle chasing yields $\angle FEB = \angle DGF$, and the desired result shortly follows.

Number Theory

N1

All nonzero integers m, n such that $\gcd(m, n) = 1$ work. First, we claim that if integers a, b are in an admissible set and $|a - b| = 1$, then that set must contain all integers. WLOG let $b = a + 1$. Then, since

$$a^2 - 2a(a + 1) + (a + 1)^2 = 1,$$

we have that 1 is also in the set. From there, it easily follows that the set contains all integers.

Next, suppose nonzero integers m, n with $\gcd(m, n) = 1$ are in the set. Then, $\gcd(m^2, n^2) = 1$ and by Bezout's lemma,

$$am^2 - bn^2 = 1$$

has an integer solution (a, b). However, am^2 and bn^2 must be in the set, because by setting $x = y = m$ or $x = y = n$, we can obtain all multiples of m^2 and n^2 respectively. So, there are two consecutive integers in the set, and thus, the set contains all integers.

It remains to prove that when $\gcd(m, n) > 1$, the set need not contain all integers. Consider the set of all multiples of $\gcd(m, n)$. This set does not contain all integers. Also, the integers m and n are in this set, and it clearly satisfies the condition for an admissible set, so we are done.