# Symmetric Polynomials Solutions

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## Problem 1 (USAMO 1973/4)

Consider the cubic polynomial $(t-x)(t-y)(t-z)$. From Newton's sums and Vieta's, this cubic polynomial must equal $t^3-3t^2+3t-1$. The only factorization of this is $(t-1)^3$, so the only solution must be $(x,y,z)=(1,1,1)$.

## Problem 2 (Canada 1996)

It is easy to show that $1-\alpha$, $1-\beta$, and $1-\gamma$ are the roots of the polynomial

$$x^3-3x^2+2x+1$$

using Vieta's.

Then, we can easily calculate the desired expression using Vieta's as well. The answer is $-7$.

## Problem 3 (HMMT Nov 2016 Guts)

By Newton's sums, the sum of the squares of the roots is 0. This means each term in the requested expression is $-1$, giving us a total answer of $-4$.

## Problem 5 (USAMO 1984/1)

Notice that we can write the polynomial as

$$(x^2+ax-32)(x^2+bx+62)$$

for constants $a$ and $b$. Expanding this and matching coefficients, we get the system of equations

$$\begin{aligned} a+b&=-18 \\ 62a-32b&=200. \end{aligned}$$

We can solve this system to get $a=-4$, $b=-14$. We also know $k=30+ab$ from the earlier expansion, so $k=86$.

## Problem 6

Consider the polynomial with roots $r+s$, $s+t$, and $r+t$. We will find its coefficients and show that it is the desired polynomial. Using Vieta's, we can see that

$$A = -2(r+s+t) = -14.$$

We can also see that

$$B = (r+s)(s+t) + (s+t)(r+t) + (r+s)(r+t).$$

Expanding and simplifying with Vieta's and Newton sums, we get $B = 52$.

The $C$ term is slightly more involved, but we can use a combination of Newton sums, Vieta's, and grouping of terms to get $C = -23$.

All these terms are rational, so overall, our answer is $A+B+C = -14+52-23 = 15$.

## Problem 9

We notice that the polynomial vanishes whenever $a = b$, $a = c$, or $b = c$. So, the polynomial is divisible by $(a-b)(a-c)(b-c)$. We know the last factor must be a multiple of $a+b+c$. We can match the coefficient of $ab^3$ to get that the factored form is

$$(a-b)(a-c)(b-c)(-a-b-c).$$

## Problem 13 (HMMT 2023/T2)

We can rearrange the equation $a^3 - bcd = b^3 - cda$ to get

$$(a-b)(a^2+ab+b^2) = cd(b-a).$$

If we assume to the contrary that $a$, $b$, $c$, and $d$ are pairwise distinct, this means

$$a^2+ab+b^2 = -cd \implies a^2+ab+b^2+cd = 0.$$

Here, the variables $a$ and $b$ can be replaced with any two of $a$, $b$, $c$, or $d$. Thus, we also have:

$$c^2+cd+d^2+ab = 0.$$

We can conclude from these two equations that $a^2+b^2 = c^2+d^2$.

Notice that there was nothing special about our choices of $a$, $b$, $c$, and $d$. Using symmetry, we can deduce that $a^2+c^2 = b^2+d^2$.

Thus, $b^2 = c^2$. Similarly, $a^2 = b^2 = c^2 = d^2$. Therefore, we can see that $a$, $b$, $c$, and $d$ cannot be pairwise distinct.

## Problem 15 (SMT 2011)

We can notice that the polynomial $P(2x) - P(x) - 1$ has roots $x = 2^i$ for $0 \le i \le 2010$. Thus, we can write

$$P(2x) - P(x) - 1 = c(x - 2^0)(x - 2^1) \cdots (x - 2^{2010}).$$

Plugging in $x = 0$, we can find $\frac{1}{c} = 1 + 2 + \cdots + 2010$ (denote by $S$ this sum).

Now, let $a$ be the coefficient of the linear term in $P(x)$. Then, the linear term of $P(2x) - P(x) - 1$ is $2ax - ax = ax$. So, it suffices to find the linear coefficient of $c(x - 2^0)(x - 2^1) \cdots (x - 2^{2010})$.

For this, we can use Vieta's. We end up with

$$a = 2^S + 2^{S-1} + \cdots + 2^{S-2010}.$$

We can simplify this to $a = 2 - \frac{1}{2^{2010}}$.

## Problem 18 (SMT 2013)

Putting the three terms over a common denominator and factoring the numerator, we can find that the expression equals

$$a^2 + b^2 + c^2 + ab + bc + ca.$$

We can rewrite this as $(a + b + c)^2 - (ab + bc + ca)$.

Let $x = \sqrt{3}$, $y = \sqrt{5}$, $z = \sqrt{7}$, and $S = a + b + c = x + y + z$. Then, our desired expression is

$$S^2 - [(S - 2x)(S - 2y) + (S - 2y)(S - 2z) + (S - 2z)(S - 2x)].$$

We can simplify this to get the answer of

$$2S^2 - 4(xy + yz + zx) = 30.$$

## Problem 20 (Black MOP 2012)

Let $\sqrt{a + h_B}$, $\sqrt{b + h_C}$, and $\sqrt{c + h_A}$ be the roots of a polynomial.

Then, we claim this polynomial also has roots $\sqrt{a + h_C}$, $\sqrt{b + h_A}$, and $\sqrt{c + h_B}$. This can be shown with Vieta's and Newton sums, along with the fact that

$$(a + h_B)(b + h_C)(c + h_A) = (a + h_C)(b + h_A)(c + h_B),$$

which can be shown by expanding and simplifying using the triangle area formula.

Thus, we have three cases:

1. $\sqrt{a + h_B} = \sqrt{a + h_C}$. Let $A$ be the area of the triangle. Then it is obvious that $b = c$.

2. $\sqrt{a + h_B} = \sqrt{b + h_A}$. We can derive that $a = b$ or $ab = -1$, the latter of which is impossible.

3. $\sqrt{a + h_B} = \sqrt{c + h_B}$. Obviously $a = c$.

In any case, the triangle is isosceles.