2022 AIME II Solutions

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Problem 1

The total number of people before the bus arrived must be a multiple of 12 so that the number of adults is an integer. Let this amount be $12x$ for an integer x.

After the bus arrives, the total number of people must be a multiple of 25. Call this amonunt $25y$ for an integer y. We have the equation

$$12x + 50 = 25y.$$

Taking mod 25, we see that $25 \mid x$. So, the least possible value of x is 25. This gives $y = 14$, so there are 350 people in total after the bus arrived. This means that the least possible number of adults is $\frac{11}{25} \cdot 350 = \boxed{154}$. (Note that minimizing the number of adults is equivalent to minimizing the total number of people.)

Problem 2

There are three cases with equal probability: A faces C, A faces J, or A faces S. The last two are symmetric so we only consider two cases.

In the first case, there is a $\frac{1}{3}$ chance C wins the semifinals and no matter the outcome of the other semifinal, there is a $\frac{3}{4}$ chance C wins the finals, giving us an overall chance of $\frac{1}{4}$.

In the second case, C will be facing either J or S, and he has a $\frac{3}{4}$ chance of winning that match. A will be facing either J or S, and we must consider two sub-cases. If A wins his match, which happens with probability $\frac{3}{4}$, then C will have a $\frac{1}{3}$ chance of winning the finals. If A loses, which happens with probability $\frac{1}{4}$, then C will have a $\frac{3}{4}$ chance of winning the finals. This gives us an overall probability of

$$\frac{3}{4} \cdot \left(\frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{3}{4}\right) = \frac{21}{64}.$$

So, our final probability is

$$\frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{21}{64} = \frac{29}{96},$$

giving us an answer of $29 + 96 = \boxed{125}$.

Problem 3

Use $V = \frac{1}{3}bh$ to get $h = \frac{9}{2}$. Then, it is obvious by symmetry that the center of the sphere lies on the central altitude. So, placing it a distance of r away from the apex, we get the following by the Pythagorean theorem:

$$\left(\frac{9}{2} - r\right)^2 + 18 = r^2 \implies r = \frac{17}{4}.$$

So, our answer is $17 + 4 = \boxed{21}$.

Problem 4

Change of base yields

$$\frac{\log(22x)}{\log(20x)} = \frac{\log(202x)}{\log(2x)}.$$

However, it is a useful fact that whenever $\frac{a}{b} = \frac{c}{d}$, the quantity $\frac{a+c}{b+d}$ is equal to both of them too. Since we have

$$\frac{\log(22x)}{\log(20x)} = \frac{-\log(202x)}{-\log(2x)} = \frac{\log(22x) - \log(202x)}{\log(20x) - \log(2x)} = \frac{\log\left(\frac{22x}{202x}\right)}{\log 10} = \log\left(\frac{11}{101}\right),$$

we are done and our answer is $11 + 101 = \boxed{112}$.

Problem 5

Suppose a triangle is formed with vertices at a, b, and c, with $a < b < c$.

Then, $b - a = p$ for some prime p, $c - b = q$ for some prime q, and $c - a = r$ for some prime r. Furthermore, $p + q = r$.

We can easily find all the solutions to $p + q = r$ satisfying $r \leq 19$. Since one of p and q must be 2, we simply list the twin primes, giving us $(2, 3, 5)$, $(2, 5, 7)$, $(2, 11, 13)$, $(2, 17, 19)$, and those but with p and q swapped.

Now, in the case that $(p, q, r) = (2, 3, 5)$, a can be anything from 1 to 15, giving us 15 cases. In the other cases, we get 13, 7, and 1 triangles, respectively.

This adds up to 36 triangles, but we must multiply by two to account for swapping p and q. This gives $36 \cdot 2 = \boxed{72}$ in total.

Problem 6

Clearly, there must be some index i such that x_i is the first nonnegative number. Also, the sum of the absolute values of the negative numbers is $\frac{1}{2}$, and the sum of the positive numbers is also $\frac{1}{2}$. There are $i - 1$ negative numbers and $101 - i$ positive numbers.

In order to maximize $x_{76} - x_{16}$, we claim x_{76} must be positive and x_{16} must be negative. Suppose they were both positive. Then, we could get a bigger value by setting x_{16} and any positive value below it to 0 and compensating in the other direction with x_{100}. A similar argument works in the case that they are both negative.

So, i must be between 17 and 76, inclusive. Consider fixing i. Then, the maximum difference is attained by setting $x_1 = x_2 = \cdots = x_{16}$ and $x_{76} = x_{77} = \cdots = x_{100}$. Otherwise, one could "smooth out" the values and obtain a higher difference. Furthermore, every other value should be 0. Otherwise, any negative values could be absorbed by x_0 and smoothed out, and any positive values could be absorbed by x_{100} and smoothed out, creating a larger difference. Using the sum restriction we got earlier, we get an answer of

$$\frac{1}{32} + \frac{1}{50} = \frac{41}{800},$$

so we bubble in $\boxed{841}$.

Problem 7

One-minute solve by similar triangles and PoP. The answer is $\boxed{192}$.

Problem 8

It evidently repeats every 60, so we just create a table with the first 60 numbers and manually count the ones that are uniquely identifiable. We get a total of 8 for the first 60, so the answer is $10 \cdot 8 = \boxed{80}$.

Problem 9

Start with 5 points on one side and 1 point on the other. This gives us 4 regions. Now, add another point to the side with 1 point. As we connect this new point with the 5 points on the other side, we create $1 + 2 + 3 + 4 + 5 = 15$ more regions. Add yet another point to the side with 2 points. We create $1 + 3 + 5 + 7 + 9 = 25$ more regions. Do it again, and we create $1 + 4 + 7 + 10 + 13 = 35$ more regions. Continuing the pattern, we get an answer of

$$4 + 15 + 25 + 35 + 45 + 55 + 65 = \boxed{244}.$$

Problem 10

Algebraic manipulation yields

$$\binom{\binom{n}{2}}{2} = \frac{(n-2)(n-1)(n)(n+1)}{8} = 3\binom{n+1}{4}.$$

So, a simple application of the hockey stick identity yields a final answer of

$$3\binom{42}{5} \equiv \boxed{4} \pmod{1000}.$$

Problem 11

Extend rays AB and DC to meet at X, and note that the midpoint M of BC is the incenter.

Since XM bisects $\angle X$, we know by the angle bisector theorem that $XB = XC$. Let this length be x.

Now, let E, F, and G be the tangency points of the incircle with AD, XA, and XB, respectively. The semiperimeter s is $6 + x$. We know $AF = s - XD = 3$, so $BF = 1$. By the Pythagorean theorem, the inradius is $\sqrt{x-1}$.

Now, Heron's formula gives that the area of the triangle is

$$A = \sqrt{12(x+6)(x-1)},$$

but this is also equal to sr where $r = \sqrt{x-1}$. Solving this yields $x = 6$, so the inradius is $\sqrt{5}$.

The rest is easy, and the answer is $(6\sqrt{5})^2 = \boxed{180}$.

Problem 12

Interpret as two ellipses having an intersection. One of the ellipses has foci at $A(-4,0)$ and $B(4,0)$, with a semimajor axis a. The other one has foci at $C(20,10)$ and $D(20,12)$, with a semimajor axis b.

Consider any point P on both ellipses. We have

$$PA + PB = 2a$$

and

$$PC + PD = 2b.$$

So, we are effectively trying to minimize $a+b = \frac{1}{2}(PA+PB+PC+PD)$. However, because of the triangle inequality,

$$\frac{1}{2}(PA+PB+PC+PD) \le \frac{1}{2}(AC+BD) = \frac{1}{2}(26+20) = \boxed{23},$$

and this bound is achievable with P as the intersection point of AC and BD.

Remark. I sillied because I wrote $PA+PB=a$ instead of $PA+PB=2a$. Remember, a is the semimajor axis!

Problem 13

Interpret using geometric series formula:

$$P(x) = (x^{2310}-1)^2(1+x^{105}+x^{210}+\cdots+x^{2310-105})(1+x^{70}+\cdots+x^{2310-70})(1+\dots)(1+\dots).$$

For finding the coefficient of x^{2022}, the $(x^{2310}-1)^2$ part doesn't matter. The rest is counting the number of nonnegative integer solutions to

$$105a + 70b + 42c + 30d = 2022.$$

Mod 5 and mod 7 analysis reveals $c \equiv 1 \pmod 5$ and $d \equiv 3 \pmod 7$.

Now, we can rewrite the LHS to get

$$210\left(\frac{a}{2}+\frac{b}{3}+\frac{c}{5}+\frac{d}{7}\right) = 2022.$$

Letting $c' = c-1$ and $d' = d-3$, notice that c' and d' are still guaranteed to be nonnegative. After simplifying, we get

$$\frac{a}{2}+\frac{b}{3}+\frac{c'}{5}+\frac{d'}{7} = 9.$$

Our mod analysis from earlier reveals that these are all nonnegative integers. After a quick check that our steps were reversible, we get an answer using stars-and-bars of $\binom{12}{3} = \boxed{220}$.

Remark. I sillied by saying $\binom{12}{3} = 660$ for some reason.

Problem 14

Note that a must be 1. It can be shown that

$$\left\lceil \frac{1000}{c} \right\rceil \leq f(a,b,c) \leq \left\lfloor \frac{1000}{c} \right\rfloor + \left\lfloor \frac{c}{b} \right\rfloor + (b-1).$$

Bash to get the solutions $(1,7,11)$, $(1,86,88)$, and $(1,86,89)$, yielding a total of $11+88+89=\boxed{188}$.

Remark. I sillied because I assumed $n_c = \left\lfloor \frac{1000}{c} \right\rfloor$ in the $c=87$ case. In fact, $n_c = 10$ is more optimal.

Problem 15

Reflect AB across the perpendicular bisector of O_1O_2 to create symmetry. Note that this does not change the area of $ABCD$. Then, $ABCD$ is an isosceles trapezoid. Also, AO_1DO_2 and BO_1CO_2 are also isosceles trapezoids, so $AD = BC = O_1O_2 = 15$.

By dropping an altitude from A to CD, we find that $\cos(\angle ADC) = \frac{3}{5}$. Furthermore, the height of trapezoid $ABCD$ is 12. So, the area of trapezoid $ABCD$ is

$$12 \cdot 9 = 108.$$

It remains to find the area of $\triangle AO_1C \cong \triangle DO_2B$.

This area is equal to $\frac{1}{2}r_1r_2\sin(\angle AO_1C)$. And, $\sin(\angle AO_1C) = \sin(\angle ADC) = \frac{4}{5}$, so we just have to find r_1r_2.

For this, we use the law of cosines on $\triangle AO_1C$. This gives

$$AC^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\angle AO_1C).$$

We know from the altitude from earlier that

$$AC^2 = 7^2 + 12^2 = 193.$$

We also know

$$r_1^2 + r_2^2 = (r_1+r_2)^2 - 2r_1r_2 = 225 - 2r_1r_2.$$

Finally, we know (from cyclic quads)

$$\cos(\angle AO_1C) = -\cos(\angle ADC) = -\frac{3}{5}.$$

Putting everything together, we get $r_1r_2 = 40$. The final area is $\boxed{140}$.