Shortlist 2011

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Combinatorics

C1

The answer is $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1$. Let a_n be the answer to the problem when there are n weights. We will show the recursive formula $a_n = (2n-1)a_{n-1}$, from which the desired result follows.

Consider the placement of the 1-gram weight. It cannot be placed on the right pan at step 1, but it can be placed on either pan in any of the other steps. There are thus a total of $2n-1$ ways to place the 1-gram weight.

Then, fixing the placement of the 1-gram weight, there are a_{n-1} ways to place the remaining weights. This is because as long as there are other weights on the balance, the 1-gram weight cannot influence which side is heavier (think about binary representation). So, it does not affect the placement of the other $n-1$ weights, and we can just scale by a factor of 2 to get a problem identical to the one with $n-1$ weights.

Thus, multiplying the two, we have $a_n = (2n-1)a_{n-1}$, and we're done.

Geometry

G2

Notice that expression in the denominators is the power of A_i with respect to the circle with center O_i, which we will denote by $\operatorname{pow}_{\omega_i}(A_i)$.

We proceed with barycentric coordinates. Our reference triangle will be $A_2A_3A_4$, and we will let $A_1 = (d, e, f)$, where $d+e+f = 1$. Then, let

$$P = \operatorname{pow}_{\omega_1}(A_1) = -a^2ef - b^2fd - c^2de.$$

By plugging in points, we find that the equation of w_2 is

$$-a^2yz - b^2zx - c^2xy - \frac{P}{d}x(x+y+z) = 0,$$

and the equations of w_3 and w_4 can be found by symmetry.

It follows that

$$\begin{aligned}\operatorname{pow}_{\omega_2}(A_2) &= -\frac{P}{d},\\ \operatorname{pow}_{\omega_3}(A_3) &= -\frac{P}{e},\\ \operatorname{pow}_{\omega_4}(A_4) &= -\frac{P}{f},\end{aligned}$$

and plugging in what we have into the desired equation gives the result.

Number Theory

N2

Assume for some x that $P(x)$ is only divisible by primes less than 20. Let $p < 20$ be a prime, and let $M = \prod_{i>j}(d_i - d_j)$. Then, for distinct indices i, j,

$$\min\{\nu_p(x+d_i), \nu_p(x+d_j)\} = \nu_p(\gcd(x+d_i, x+d_j)).$$

Since $\gcd(x+d_i, x+d_j) \mid d_i - d_j$, we have

$$\nu_p(\gcd(x+d_i, x+d_j)) \leq \nu_p(d_i - d_j) \leq \nu_p(M).$$

So,

$$\min\{\nu_p(x+d_i), \nu_p(x+d_j)\} \leq \nu_p(M)$$

for all $i \neq j$. It follows that there is at most one j such that $\nu_p(x+d_j) > \nu_p(M)$.

If we repeat this for all eight of the primes less than 20, we will find that since there are nine possible indices j, at least one index j will be left over. More precisely, there exists j such that $\nu_p(x+d_j) \leq \nu_p(M)$ for all primes $p < 20$, and by our assumption, this extends to all primes p. So,

$$x + d_j \mid M \implies x + d_j \leq M \implies x \leq M - d_j,$$

and thus, x is bounded. The desired result follows.