Submission for DGW-ELEMGEO

OTIS (internal use)

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> **Example** (Monge's theorem, 0♣)**.** Let ω_1, ω_2, ω_3 be three pairwise incongruent circles, and let X_{12}, X_{23}, X_{31} be the pairwise exsimilicenters. Show that X_{12}, X_{23}, X_{31} are collinear.

ZABF24E3

Walkthrough. Let h_{12} denote the homothety sending ω_1 to ω_2, which is a function from the plane to itself. Define h_{23} and h_{13} similarly.

(a) The function composition $h_{23} \circ h_{12}$ is a homothety too. Which is it?

(b) Show that $h_{13}(X_{12})$ is collinear with X_{12} and X_{31}.

(c) Show that $h_{23}(X_{12})$ is collinear with X_{12} and X_{23}.

(d) Conclude.

> **Example** (USEMO 2021/4, Sayandeep Shee, 0♣)**.** Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC.
>
> Can line EF share a point with the circumcircle of triangle AMN?

21USEMO4

Walkthrough. There are lots of different approaches. We'll give the most classical one presented by the author, but there are plenty of other things that could work too!

(a) Let P be the midpoint of $\overline{AD}$. Where else does P lie?

(b) Show that $FB^2 = FP \cdot FA$.

(c) Show that $EB^2 = EN \cdot EA$.

(d) Using radical axis with a circle of radius zero at B, prove that line EF is disjoint from (AMN).

> **Example** (JMO 2018/3, Ray Li, 0♣)**.** Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over $\overline{BA}$ and $\overline{BC}$, respectively, and let P be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of EPD and FPD meet ω at Q and R different from D. Show that

> $EQ = FR$.

18JMO3

Walkthrough. Most of this problem is about realizing where the points P, Q, R are.

(a) Using what you know about the Simson line, figure out where point P is.

(b) Determine the circumcenters of $\triangle EPD$ and $\triangle FPD$.

(c) Figure out where the points Q and R are.

(d) Finish the problem.

> **Example** (TSTST 2017/5, Ray Li, 0♣)**.** Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

17TSTST5

Walkthrough. Let ω denote the incircle of $\triangle ABC$.

(a) Identify the point $Z = \overline{EX} \cap \overline{FY}$ in a good diagram. (This was worth a point! Despite this, many contestants were unable to find it.)

(b) Consider the positive homothety sending ω to ω_C. Determine its center.

(c) Consider the negative homothety sending ω_C to ω_B. Determine its center.

(d) The composition of the previous two homotheties in (b) and (c) is a negative homothety sending ω to ω_B. Determine with proof the center of this homothety.

This is not as simple as the previous two parts; you will need to *use* (b) and (c) to do this part, as well as the simple observation that the center should lie on the $\angle B$ bisector.

(e) Conclude that $\overline{FY}$ passes through the point you claimed in (a).

Experts may notice that this walkthrough gives what is essentially a proof of Monge d'Alembert theorem.

> **Example** (TSTST 2016/2, Evan Chen, 0♣)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

16TSTST2

Walkthrough. Let DEF be the orthic triangle of ABC.

(a) Show that P is really just the intersection of the tangents to γ at A and G (and thus the line $\overline{OM}$ is just a distraction).

(b) Show that lines $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent, say at R.

(c) Prove that $(PAMG)$, (MBC), $(MFDNE)$ are concurrent at a point $T \neq M$.

(d) Show that $T = \overline{PN} \cap \overline{MR}$.

(e) Show that $R \in \overline{HQ}$.

(f) Show that R, G, T, Q, N are concyclic, completing the proof.

Practice problems

Instructions: Solve [42♣]. If you have time, solve [56♣].

Life is full of surprises, but never when you need one.

Calvin in *Calvin and Hobbes*

Problem 1 (India TST 2015/1, 3♣)
Diagonals $\overline{AC}$ and $\overline{BD}$ of convex quadrilateral $ABCD$ meet at P. Prove that the incenters of the triangles $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PDA$ are concyclic if and only if their P-excenters are also concyclic.

15INDTST1

Problem 2 (Added by Atul Shatavart Nadig, 3♣)
Let ABC be a triangle with $\angle A = 120°$. The angle bisectors of ABC meet the opposite sides at A_1, B_1, C_1. Find the measure of $\angle B_1A_1C_1$.

Z284BC3E

Problem 3 (Iran TST 2020, added by Leonardo Wang, 2♣)
Let ABC be an isosceles triangle with $AB = AC$ and incenter I. Circle ω passes through C and I and is tangent to AI. Circle ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent.

20IRNTST10

Problem 4 (Shortlist 2021 G1, 5♣)
Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of triangle ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

21SLG1

Required Problem 5 (IMO 2000/1, 3♣)
Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

00IMO1

Considering a homothety at N taking PQ to AB. The point M is taken to the midpoint of AB by radical axis, so thus, M is the midpoint of PQ.

It then suffices to show that $EM \perp CD$. Draw radii from the centers of the circles to the tangency points. Then, we see that $CD = 2AB$, and the desired result shortly follows.

Problem 6 (APMO 2000/3, 3♣)

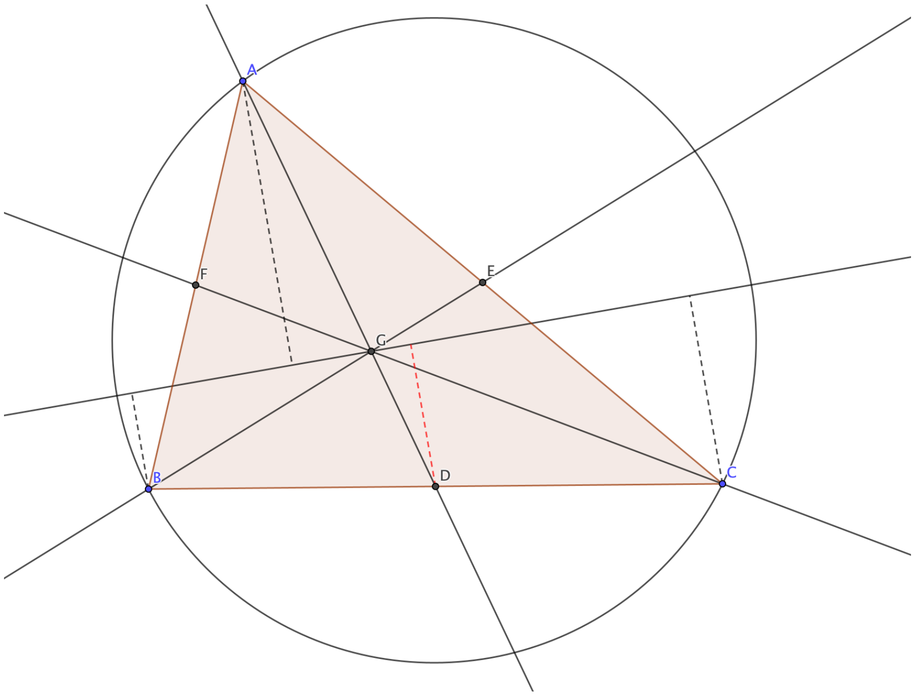
Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets $\overline{MA}$ and $\overline{BA}$, respectively. Let O be the point in which the perpendicular at P to BA meets ray AN. Prove that $\overline{QO} \perp \overline{BC}$.

00APMO3

Problem 7 (APMO 2004/2, 2♣)

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

04APMO2



Note that line OH passes through G, so it suffices to prove the stronger statement that this works for any line l passing through G.

WLOG assume A is on one side of l, while B and C are on the other side. Then, consider a homothety at G with scale factor $-\frac{1}{2}$. The distance from A to l is transformed into the distance from D to l, which is clearly the average of the distance from B to l and the distance from C to l, so we are done.

Problem 8 (China 2021/4, added by Leonardo Wang, 5♣)

In acute triangle ABC with $AB > AC$, point M is the midpoint of minor arc BC, O is the circumcenter of (ABC) and AK is its diameter. The line parallel to AM through O meets segment AB at D, and CA extended at E. Lines BM and CK meet at P, lines BK and CM meet at Q. Prove that $\angle OPB + \angle OEB = \angle OQC + \angle ODC$.

21CHN4

Problem 9 (Germany 2015, added by Joel Gerlach, 2♣)

Fix a real number $\ell > 0$ and two rays in the plane with common endpoint S. Point A moves along one ray, while point B moves along the other ray, such that $AS + SB = \ell$. Prove that the perpendicular bisector of AB passes through a fixed point in the plane.

15GER36

Problem 10 (GaussJMO 2022/1, by Qiao Zhang, 3♣)

Let $ABCDE$ be a cyclic pentagon with $AB = CD$ and $BC = DE$. Let P and Q be points on $\overline{CB}$ and $\overline{CD}$, respectively, such that $BPQD$ is cyclic. Let M be the midpoint of $\overline{BD}$. Prove that lines CM, AP, and EQ concur.

22GAUSSMOJ1

Problem 11 (Mexico 2023, added by Alan Alejandro LÃ³pez Grajales, 3♣)

Let $ABCD$ be a convex quadrilateral. Let M, N, K be the midpoints of the segments AB, BC, and CD, respectively. Suppose there is also a point P inside the quadrilateral $ABCD$ such that $\angle BPN = \angle PAD$ and $\angle CPN = \angle PDA$. Show that $AB \cdot CD = 4PM \cdot PK$.

23MEX3

Problem 12 (CGMO 2007/5, 3♣)

Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment BC. Point F lies on segment AC with $AF = 2FC$. Prove that $\overline{DE} \perp \overline{EF}$.

07CGMO5

Required Problem 13 (EGMO 2016/4, 5♣)

Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

16EGMO4

Required Problem 14 (BrMO 2013/2, 3♣)

Let ABC be a triangle and let P be a point inside it satisfying $\angle ABP = \angle PCA$. Let Q be the reflection of P across the midpoint of $\overline{BC}$. Prove that $\angle BAP = \angle CAQ$.

13BRMO2

Problem 15 (Poland 2018, added by Kevin Wang, 3♣)

An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circumcircle of ABC crosses BC at P. The line through A parallel to BC crosses EF at Q. Prove that PQ is perpendicular to the median from A of triangle ABC.

18POL5

Problem 16 (Korea Junior 2014, added by Kevin Wang, 5♣)

Let ABC be a triangle with incenter I. Line AI meets BC at D. The incenters of $\triangle ABD$ and $\triangle ADC$ are E and F, respectively. Line DE meets the circumcircle of $\triangle BCE$ again at P, while line DF meets the circumcircle of $\triangle BCF$ again at Q. Show that the midpoint of BC lies on the circumcircle of $\triangle DPQ$.

14KOR2J1

Problem 17 (HMMT 2017, 5♣)

Let ABC be an acute triangle. The altitudes BE and CF intersect at the orthocenter H, and point O denotes the circumcenter. Point P is chosen so that $\angle APH = \angle OPE = 90^\circ$, and point Q is chosen so that $\angle AQH = \angle OQF = 90^\circ$. Lines EP and FQ meet at point T. Prove that points A, T, O are collinear.

17HMMTT5

Problem 18 (Shortlist 2004 G3, 5♣)

Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$ and let $D = \overline{AO} \cap \overline{BC}$. Let E and F denote the circumcenters of triangles ABD and ACD. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

04SLG3

Problem 19 (ARML 2019 T-10, 5♣)

Triangle ABC with $AB = 14$, $AC = 30$, $BC = 40$ is inscribed in a circle ω. The tangents to ω at B and C meet at a point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Compute the area of triangle PBC.

19ARMLT10

Problem 20 (Shortlist 2012 G2, 3♣)

Let $ABCD$ be a cyclic quadrilateral and let $E = \overline{AC} \cap \overline{BD}$. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that the points D, H, F, G are concyclic.

12SLG2

Problem 21 (China 2019/3, 5♣)

Let ABC be a triangle with circumcenter O and circumcircle Γ. Point D lies on the internal $\angle A$-bisector. Point E is chosen on line BC such that $\overline{DE} \perp \overline{BC}$ and $\overline{AD} \parallel \overline{OE}$. Point K lies on ray EB with $AE = KE$. The circumcircle of triangle AKD meets line BC again at P, and meets Γ again at Q. Show that $\overline{PQ}$ is tangent to Γ.

19CHN3

Required Problem 22 (Shortlist 2007 G3, 5♣)

Let $ABCD$ be a trapezoid whose diagonals meet at P. Point Q lies between parallel lines BC and AD, and line CD separates points P and Q. Given that $\angle AQD = \angle CQB$, prove that $\angle BQP = \angle DAQ$.

07SLG3

Problem 23 (Shortlist 2020 G5, 9♣)

Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that $KLMN$ is a rhombus with $\overline{KL} \parallel \overline{AC}$ and $\overline{LM} \parallel \overline{BD}$. Let ω_A, ω_B, ω_C, and ω_D be the incircles of $\triangle ANK$, $\triangle BKL$, $\triangle CLM$, and $\triangle DMN$. Prove that the common internal tangents to ω_A and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

20SLG5

Problem 24 (Shortlist 2011 G3, 9♣)

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendicular from E to the lines AB, BC, CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

11SLG3

Required Problem 25 (Shortlist 2020 G8, added by Guanjie Lu, 9♣)

Let ABC be a triangle with incenter I and circumcircle Γ. Circles ω_B passing through B and ω_C passing through C are tangent at I. Let ω_B meet minor arc AB of Γ at P and AB at $M \neq B$, and let ω_C meet minor arc AC of Γ at Q and AC at $N \neq C$. Rays PM and QN meet at X. Let Y be a point such that YB is tangent to ω_B and YC is tangent to ω_C.

Show that A, X, Y are collinear.

20SLG8

Problem 26 (Added by Lasitha Vishwajith Jayasinghe and Shreya Sharma, 3♣)

Let ABC be a triangle with circumcircle γ. Let ω be a circle passing through B intersecting AB at D, γ at E, and line BC at F. Let G be the intersection of AF and ω. Let M and N be the intersections of lines DF and DG with the tangent to γ at A. Finally, let L be the second intersection of MC and (ABC). Prove that M, L, D, E and N are concyclic.

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