Submission for DCW-LOCAL

OTIS (internal use)

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> **Example** (0♣). Suppose 4951 distinct points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

ZEBE7889

Walkthrough. Following RUST, keep grabbing points until we cannot take any more. Suppose at this point we have n points.

(a) Show that $4951 - n \le \binom{n}{2}$.

(b) Prove that $n \ge 100$.

So this is an example of a greedy algorithm of the most direct sort.

> **Example** (Putnam 1979, 0♣). Given n red points and n blue points in the plane, no three collinear, prove that we can draw n segments, each joining a red point to a blue point, such that no segments intersect.

79PTNM

Walkthrough. Starting from an arbitrary configuration, we will use the algorithm "given a crossing, un-cross it". This is a very natural algorithm to come up with, and playing with some simple examples one finds that it always work. So we just have to prove that.

(a) Show that a step of this algorithm does *not* necessarily decrease the total number of intersections. (But this is the first thing we *should* try, given that our goal is to get zero intersections at the end.)

(b) Find a different monovariant M which *does* decrease at each step of the algorithm.

(c) Remark on the finiteness of the configuration space, and complete the problem using (b).

I want to say a few words about why I chose this example. This problem is touted in olympiad cultures as an example of "extremal principle", with "choose the minimal M" as the poster description. In my humble opinion, I think this is hogwash. The motivation should be the natural algorithm we used; the monovariant comes after the fact.

Indeed, the fact that the natural guess of the monovariant in (a) fails is what makes this problem a little interesting (and not completely standard). However, it doesn't change the fact that the algorithm comes before the monovariant in our thought process.

Example (USAMO 1999/4, 0♣)**.** Let a_1, a_2, ..., a_n be a sequence of $n > 3$ real numbers such that

$$a_1 + \cdots + a_n \ge n \quad \text{and} \quad a_1^2 + \cdots + a_n^2 \ge n^2.$$

Prove that $\max(a_1, \dots, a_n) \ge 2$.

99AM04

Walkthrough. We will proceed by contradiction, and assume that there exists *some* sequence of (a_i) satisfying all three of the following conditions:

(1) $\sum a_i \ge n$,

(2) $\sum a_i^2 \ge n^2$,

(3) but also $a_i < 2$ for all i.

Our game plan is then to take this starting sequence, and smooth the variables to obtain a new simpler sequence satisfying (1)-(3). We'll then derive a contradiction in the simplified setting.

(a) If $a_i \le 0$ and $a_j \le 0$, how can we smooth these two terms while preserving (1), (2), and (3)?

(b) If $a_i \ge 0$, how can we smooth the a_i while preserving (1), (2) and (3)?

(c) Combining (a) and (b), reduce the case where one term is negative and the rest are $2 - \varepsilon$ for some small ε.

(d) Solve the problem in the situation of (c).

One can avoid the issue of ε in (c) as well if one notes that increasing $2 - \varepsilon$ to 2 makes the inequalities (1) and (2) strict (while relaxing (3) to be non-strict). This makes the calculations a little less technical.

Example (USA TST 2018/3, 0♣)**.** At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

18USATST3

Walkthrough. This is a rather detail-oriented problem which requires a lot of care in order to get all the arguments to be correct. In my opinion, being able to work out a flawless solution in the time limit of the exam is an impressive feat.

The original proposed solution is pretty combinatorial, and the walkthrough for that is below; it uses a *local* smoothing idea in the first half and then a *global* idea in the second part. (There is also an algebraic solution using inequality smoothing instead.)

Viewing the problem in graph theory terms, let

$$S(G) := \sum_{e=vw} \min(\deg v, \deg w)$$

and thus we want to maximize $S(G)$ over simple graphs G with 2017 edges.

Let's get started.

(a) Find an example of a graph achieving $S(G) = 127009$. We will prove this is the maximum value.

(b) Make a conjecture what the answer is if 2017 is replaced by $\binom{k}{2} + 1$ where $k \geq 4$.

(c) Show that the natural extension of (b) does *not* hold when $k = 3$. This edge case will haunt us briefly later on again.

Here is the local half:

(d) Suppose G is a graph with $\binom{k}{2}$ edges for $k \geq 3$. Assume G has no universal vertex, and let w denote the vertex of minimal degree.

Prove that one can delete w, and rewire the edges originally adjacent to w, to obtain a graph G^* with fewer vertices and $S(G^*) \geq S(G)$.

(e) Repeat part (d) where G has $\binom{k}{2} + 1$ edges and $k \geq 4$, and $\#V(G) > k + 1$. Then check the existence of G^* by hand for $\#V(G) = k + 1$.

(f) Show that (e) is false if $k = 3$.

So, we can reduce to the case where G has a universal vertex. We will now prove by induction on $k \geq 3$ the following claim:

For any graph G,

- If G has at most $\binom{k}{2}$ edges for $k \geq 3$, then $S(G) \leq \binom{k}{2} \cdot (k - 1)$.
- If G has at most $\binom{k}{2} + 1$ edges for $k \geq 4$, then $S(G) \leq \binom{k}{2} \cdot (k - 1) + 1$.

This is the global half.

(g) Check the base case $k = 3$. (We won't need a base case for the second bullet; it will reduce to the first one.)

(h) WLOG, let v be the universal vertex of G. Show that if v is deleted, the resulting graph H has at most $\binom{k-1}{2}$ edges (in both cases) and thus we can apply the induction hypothesis to the resulting graph H.

(i) Prove that $S(G) = S(H) + 3\#E(H) + \#V(H)$.

(j) Combine the previous two parts to complete the problem.

Practice problems

Instructions: Solve [36♣]. If you have time, solve [50♣].

> Mario, I will follow you to the end, I swear it! I feel bad for the princess, but that queen must fall before us! And when she does, you and I can. . . Well, anyway, let's take this fight to her!
>
> Vivian in the Shadow Queen battle, from *Paper Mario: The Thousand Year Door*

Required Problem 1 (PUMaC 2013, 2♣)
Let G be a graph and let k be a positive integer. A *k-star* is a set of k edges with a common endpoint and a *k-matching* is a set of k edges such that no two have a common endpoint. Prove that if G has more than $2(k-1)^2$ edges then it either has a k-star or a k-matching.

13PUMACFA3

Assume there is no k-star. This means every vertex has a degree which is at most $k-1$. Then, for the sake of contradiction, take a maximal matching of size $m < k$. Every edge in the graph must either be in the matching or share an endpoint with an edge in the matching. The maximum number of edges not in the matching is then

$$2m(k-2).$$

Counting up the total number of edges in the graph, we get

$$2m(k-2) + m \le 2(k-1)(k-2) + (k-1) = 2k^2 - 5k + 3 \le 2(k-1)^2,$$

contradicting the assumption that G has more than $2(k-1)^2$ edges.

Problem 2 (3♣)
Prove that if n is a sufficiently large positive integer then it is possible to find a collection $\mathcal{F}$ of subsets of $\{1, \dots, n\}$, such that $|\mathcal{F}| > 1.001^n$ and whenever X and Y are distinct elements of $\mathcal{F}$, we have

$$|X \cup Y| - |X \cap Y| \ge \frac{n}{3}.$$

GOWERSE3

Required Problem 3 (IMO 2003/1, 3♣)
Let A be a 101-element subset of $S = \{1, 2, \dots, 10^6\}$. Prove that there exist numbers $t_1, t_2, \dots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \dots, 100$$

are pairwise disjoint.

03IMO1

Problem 4 (BAMO 2017/3, 2♣)

Let n be a positive integer and consider an $n \times n$ multiplication table as shown below, where the entry in the ith column and the jth row is ij.

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 4 & 6 & \dots & 2n-2 & 2n \\ 3 & 6 & 9 & \dots & 3n-3 & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 2n & 3n & \dots & n^2-n & n^2 \end{bmatrix}$$

An ant starts on the cell labeled 1, and walks towards the cell labeled n^2 by taking $2n-2$ steps each either right or down. Determine the minimum and maximum possible sum of the labels in such a path, as a function of n.

17BAMO3

Consider the set of elements ij for $i+j=k$; call this "diagonal k." Clearly, in any path the ant takes, the number of the diagonal it is in increments by 1 with each step. Thus, if we could get the ant to always choose the maximum number in each diagonal, that would produce the maximum sum (and vice versa for minimum).

It is not hard to show that the element(s) in the middle of each diagonal are the greatest and the elements on the ends of each diagonal are the smallest. This means that it is possible to always choose the maximum/minimum element in each diagonal. The maximum sum is achieved by taking a zigzag path down the main diagonal of the matrix, and the minimum sum is achieved by going right $n-1$ times and then going down $n-1$ times.

The minimum sum is then

$$\begin{aligned}(1+2+\dots+(n-1))+(n+2n+\dots+n^2) &= \frac{n(n-1)}{2}+\frac{n^2(n+1)}{2} \\ &= \frac{n(n^2+2n-1)}{2}.\end{aligned}$$

Finally, the maximum sum is

$$\begin{aligned}\sum_{i=1}^{n}(i^2+i(i-1)) &= \sum_{i=1}^{n}(2i^2-i) \\ &= 2\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2} \\ &= \frac{n(n+1)(4n-1)}{6}.\end{aligned}$$

Problem 5 (Putnam 2016 A5, 5♣)

Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2}\dots g^{m_r}h^{n_r}$$

with $1 \le r \le |G|$ and $m_1, n_1, m_2, n_2, \dots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of G.)

16PTNMA5

Problem 6 (MP4G 2010, 3♣)
Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x-coordinate or y-coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

10MP04

Problem 7 (Russia 2004, 5♣)
A country has 1001 cities, and each two cities are connected by a one-way street. From each city exactly 500 roads begin, and in each city 500 roads end. Now an independent republic splits itself off the country, which contains 668 of the 1001 cities. Prove that one can reach every other city of the republic from each city of this republic without being forced to leave the republic.

04RUS106

Problem 8 (Shortlist 2019 C2, 3♣)
You're given n blocks each with weight at least 1 and total weight $2n$. Prove that for every real number $r \in [0, 2n-2]$ there is a subset of the blocks whose total weight is between r and $r+2$ inclusive.

19SLC2

Problem 9 (Shortlist 2001 C1, added by Pranav Choudhary, 3♣)
Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \le i < j < k \le 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

01SLC1

Required Problem 10 (IMO 2014/5, 9♣)
For every positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

14IMO5

Problem 11 (USAMO 2022/2, 5♣)
Let $b \ge 2$ and $w \ge 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.

Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

22AMO2

Problem 12 (IMO 1999/2, 3♣)

Find the least constant C such that for any integer $n > 1$ the inequality

$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$

holds for all real numbers $x_1, \dots, x_n \ge 0$. Determine the cases of equality.

99IMO2

Required Problem 13 (HMMT 2018 T6, 5♣)

Let $n \ge 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \le x < n$, there is a subset of S (possibly empty) whose sum has remainder x when divided by n. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

18HMMTT6

Problem 14 (3♣)

Prove that in any tournament, either

- there exists a directed Hamiltonian cycle, or
- there exists a nontrivial partition $A \sqcup B$ of the vertices such that A and B are nonempty and whenever $a \in A$ and $b \in B$, there is an edge from a to b.

GRTH6

Problem 15 (USA TST 2017/1, 5♣)

In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned *any* signature color of a different team in S. For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

17USATST1

(This TST1 is easy to misread and quite easy to mess up, so be careful.)

Problem 16 (Moscow 1957, added by Tilek Askerbekov, 3♣)

Let $1 = a_1 \le a_2 \le \cdots \le a_n$ be integers. Assume $a_{i+1} \le 2a_i$ for all $i = 1, \dots, n-1$ and $a_1 + \cdots + a_n$ is even. Prove that the n integers can be split into two piles with equal sum.

57MOSCOW

Problem 17 (Canada 2023, added by Haozhe Yang, 9♣)

Let $n \ge 1$ and $k \ge 1$ be fixed positive integers. A simple graph on n vertices has the property that whenever it its vertices are partitioned into two sets, then at most kn edges have endpoints in different sets. What is the largest integer m (in terms of n and k) such that there is guaranteed to be a set of m vertices with no edge between them?

23CAN5

> **Remark.** It's debatable whether these problems really count as "local": some find them suitable, while others find them unconnected. In my opinion, these feel different because they are not algorithmic.

Problem 18 (Korea 1995, also Putnam 2016 B3, 3♣)

Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Show that all these points lie within the interior of some triangle with area 4.

16PTNMB3

Problem 19 (Iran TST 2010/7, 3♣)

Let $\mathcal{P} = P_1P_2 \dots P_{2n}$ be a polygon, possibly concave but not self-intersecting. Suppose there is a line ℓ such that for each $k = 1, 2, \dots, n$, ray $P_{2k-1}P_{2k}$ intersects ℓ and is perpendicular to it. Prove that ℓ intersects $\mathcal{P}$.

10IRNTST7

Problem 20 (Sylvester-Gallai, 3♣)

Finitely many points in the plane are given, not all collinear. Prove that there exists a line passing through exactly two of them.

ZC62C56E