EGMO Solutions

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Chapter 4

Problem 4.48 (Japanese Olympiad 2009)

Notice $APOQ$ is cyclic. This can be proven using the homothety at Q. Then, notice POQ is isosceles and the result shortly follows.

Problem 4.49

Let ray AE intersect the circumcircle at W. Because $\angle BAT = \angle CAE = \angle CAW$, we know arc BT has the same measure as arc CW.

Now, extend ray TD to hit the circumcircle at V. Line TV is just the reflection of line WA across the perpendicular bisector of BC, because of the fact that $BD = CE$ and that arc BT equals arc CW.

Thus, arcs BA and CV have the same measure, and the result follows.

Problem 4.50 (Vietnam TST 2003/2)

Let I_A, I_B, I_C denote the excenters. We know from a lemma in this chapter that line A_0D is just line DI_A, and so forth. Also, we can see that line DF is parallel to line I_AI_C. Let Z be the intersection point of lines DI_A and FI_C. Then, a homothety at Z takes F to I_C and D to I_A. This homothety also takes E to I_B for the same reason. So, lines DI_A, FI_C, and EI_B concur at Z. For the OI part, notice that O is the nine-point center of triangle $I_AI_BI_C$, and Euler line leads to the result.

Problem 4.51 (Sharygin 2013)

Let M be the midpoint of AB. From a previous lemma, we know CM, $A'B'$, and $C'I$ are concurrent at a point X. Notice that X is also the orthocenter of triangle CIK. Thus, line IX is perpendicular to CK. However, line IX is also perpendicular to AB, so $AB \parallel CK$.

Problem 4.52 (APMO 2012/4)

Let H' be H reflected over D, and H'' be H reflected over M. It is well known that H' and H'' lie on the circumcircle of ABC. By PoP, $HE \cdot HH'' = HA \cdot HH'$. Dividing both sides by two, we obtain the equation $HE \cdot HM = HA \cdot HD$. In other words, $AEDM$ is cyclic.

Now, we claim triangle ABF is similar to triangle AMC. We know $\angle ACM = \angle ACB = \angle AFB$.

Also, $\angle AMC = \angle AMD = \angle AED = \angle AEF = \angle ABF$ (using directed angles). Thus, the two triangles are similar, and it follows that AF is a symmedian. Finally, the desired result is a well-known consequence of AF being a symmedian.

Problem 4.53 (Shortlist 2002/G7)

As always, we can remove M from our diagram by noting that line MK is the same as line KI_A. Let Q be the midpoint of KI_A. We claim $BNCQ$ is cyclic. Let S be the midpoint of NK. Since $\angle ISI_A = \angle IBI_A = 90$ (well known), we know S lies on the circle containing B, I, C, and I_A (this circle being from a common configuration). By PoP, $KS \cdot KI_A = KB \cdot KC$. However, we know $KS \cdot KI_A = KN \cdot KQ$. Thus, $BNCQ$ is cyclic.

Let P be the circumcenter of BCN. Notice that since $BK = XC$, we have $QB = QC$ and thus QP is the perpendicular bisector of BC. In other words, Q is the arc midpoint of arc BC on the circumcircle of BCN. Consider a homothety at N that takes K to Q. This homothety must also take I to P, finishing the proof.

Chapter 5

Problem 5.16 (Star Theorem)

Using the Law of Sines, we write

$$\prod_{i=1}^{5} X_iA_{i+2} = \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+2}A_{i+3}X_i$$

and

$$\begin{aligned}\prod_{i=1}^{5} X_iA_{i+3} &= \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+3}A_{i+2}X_i \\ &= \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+1}A_{i+2}X_{i-1}.\end{aligned}$$

Notice that this is the same expression by re-indexing. Thus, we are done.

Problem 5.17

We know the length of the exradius r_A is $\frac{sr}{s-a}$. Then, simply use Heron's formula and $A = sr$.

Problem 5.18 (APMO 2013/1)

WLOG we will just prove triangles ODB and OAE have the same area, and then we can get three pairs from symmetry. We note that OB and OA have the same length, so we just need to compare the heights of the altitudes from D and E to their respective sides. So, using some angle chasing and trigonometry, we can reduce what we are trying to prove to

$$AE \sin(90 - B) = BD \sin(90 - A).$$

Then, we notice that $AE = AB \sin(90 - A)$ and $BD = AB \sin(90 - B)$ by drawing altitudes, giving us the result.

Problem 5.19 (EGMO 2013/1)

Let a, b, c denote the side lengths of ABC in their usual way. We can compute

$$\begin{aligned} AD^2 &= c^2 + 4a^2 - 4ac \cos B \\ BE^2 &= c^2 + 4b^2 + 4bc \cos A. \end{aligned}$$

(The + is not a mistake in the second line there!) Equating the two, we get $a^2 - ac \cos B = b^2 + bc \cos A$. Using the Law of Cosines but solving for angles, we get

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos A &= \frac{b^2 + c^2 - a^2}{2bc}. \end{aligned}$$

Plugging these back in, we can simplify to get $a^2 = b^2 + c^2$. Thus, triangle ABC is right-angled.

Problem 5.20 (HMMT 2013)

Let E be the contact point of the incircle with AB, and let M be the midpoint of BC. Also, let a, b, and c mean the usual side lengths. The condition $2a = b + c$ can also be written as $s - a = \frac{a}{2}$, where s is the semiperimeter. Since $AE = s - a$ and $MC = \frac{a}{2}$, we know $AE = MC$.

We also know $\angle DCM = \angle IAE$. So, by AAS congruence, we have that triangle AIE is congruent to triangle CDM. Therefore, $DC = AI = DI$ (by another lemma), and we are done.

Problem 5.21 (USAMO 2010/4)

Notice that I is the incenter. Law of Cosines tells us

$$BC^2 = BI^2 + CI^2 - 2 \cdot BI \cdot CI \cos \angle BIC.$$

Angle chasing gives us $\angle BIC = 135$. So, we have

$$BC^2 = BI^2 + CI^2 + \sqrt{2} \cdot BI \cdot CI.$$

Assume BI and CI have integer lengths. Then $BC^2 = AB^2 + AC^2$ is not an integer. Thus, the six segments cannot all have integer lengths.

Problem 5.22 (Iran Olympiad 1999)

We can rewrite the condition as $ID \cdot (\sin B + \sin C) = \frac{1}{2}AD$ (using some angle chasing). Since $ID = BD = CD$, we now use Ptolemy's theorem to get

$$(AB + AC) \cdot ID = AD \cdot BC.$$

However, we know that $ID = \frac{AD}{2(\sin B + \sin C)}$, so we can plug that in and simplify to get

$$BC = \frac{AB + AC}{2(\sin B + \sin C)}.$$

Using the Extended Law of Sines again, we can write $\sin B = \frac{AC}{2R}$ and $\sin C = \frac{AB}{2R}$ where R is the circumradius. Then, the above equation simplifies to

$$BC = R.$$

Using the Extended Law of Sines, this means that $\sin A = \frac{1}{2}$, so $\angle A = 30$ or $\angle A = 150$.

Problem 5.23 (CGMO 2002/4)

Using the Law of Sines,

$$\frac{AH}{HF} = \frac{EA \sin \angle HEA}{EF \sin \angle HEF}.$$

Note that $EC = EF$ because chord CF is perpendicular to diameter AB. So, we rewrite our expression as

$$\frac{EA \sin \angle HEA}{EC \sin \angle HEF}.$$

Simple angle chasing and trig finishes this proof:

$$\begin{aligned}
\frac{EA\sin\angle HEA}{EC\sin\angle HEF} &= \frac{EA\sin\angle GCB}{EC\sin\angle CBD} \\
&= \frac{EA\sin(90-\angle CBD)}{EC\sin\angle CBD} \\
&= \frac{EA}{EC\tan\angle CBD} \\
&= \frac{\tan\angle ECA}{\tan\angle CBD} \\
&= \frac{\tan\angle CBA}{\tan\angle CBD} \\
&= \frac{AC}{CD}.
\end{aligned}$$

Chapter 6

Problem 6.29

We scale down to the unit circle and center our arc on the real axis. Let our arc have endpoints at a and $\overline{a} = \frac{1}{a}$, where a is on the unit circle. Let the other point on the circle be b, and the center of the circle is obviously 0. Then, the inscribed angle theorem is equivalent to

$$\arg\left(\frac{a-b}{\frac{1}{a}-b}\right) = \frac{1}{2}\arg\left(\frac{a}{\frac{1}{a}}\right).$$

Notice that with some manipulation, this is equivalent to proving that $\frac{a-b}{1-ab}$ is real, or equal to its conjugate. Indeed, we have

$$\begin{aligned}
\overline{\frac{a-b}{1-ab}} &= \frac{\overline{a}-\overline{b}}{1-\overline{ab}} \\
&= \frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{ab}} \\
&= \frac{\frac{b-a}{ab}}{\frac{ab-1}{ab}} \\
&= \frac{b-a}{ab-1} \\
&= \frac{a-b}{1-ab}.
\end{aligned}$$

So, we are done.

Lemma 6.30

If P is on chord AB, then

$$\frac{p-a}{p-b} = \overline{\left(\frac{p-a}{p-b}\right)} = \frac{\overline{p}-\frac{1}{a}}{\overline{p}-\frac{1}{b}}.$$

With enough algebraic manipulation, we can get to the result.

Problem 6.31

Let a, b, c, and d be on the unit circle. Then, we have

$$\begin{aligned} h_a &= b+c+d \\ h_b &= a+c+d \\ h_c &= a+b+d \\ h_d &= a+b+c. \end{aligned}$$

We can now see that the point $\frac{1}{2}(a+b+c+d)$ is the midpoint of AH_A, BH_B, CH_C, and DH_D, and thus the lines concur at this point.

Problem 6.32

Let x be the point of tangency of the incircle with AB, y be that of BC, z be that of CD, and w be that of AD. Also, we scale down such that w, x, y, and z are on the unit circle. Then, using the intersection of tangents formula, we get

$$\begin{aligned} a &= \frac{2wx}{w+x} \\ b &= \frac{2xy}{x+y} \\ c &= \frac{2yz}{y+z} \\ d &= \frac{2wz}{w+z}. \end{aligned}$$

Then, the midpoint of AC is

$$m_1 = \frac{wx}{w+x} + \frac{yz}{y+z} = \frac{wxy+wxz+wyz+xyz}{(w+x)(y+z)}.$$

The midpoint of BD is

$$m_2 = \frac{xy}{x+y} + \frac{wz}{w+z} = \frac{wxy+wxz+wyz+xyz}{(x+y)(w+z)}.$$

Since we have placed I at the origin, we seek to prove $\frac{m_1}{m_2}$ is real. Indeed:

$$\frac{m_1}{m_2} = \frac{(x+y)(w+z)}{(w+x)(y+z)}$$

is equal to its conjugate (through enough algebraic manipulation).

Problem 6.33 (Chinese TST 2011)

Let $a = A$, $b = B$, and $c = C$ in complex numbers. We can derive

$$\begin{aligned}
d &= \frac{1}{2}(b + c + p - bc\overline{p}) \\
e &= \frac{1}{2}(a + c + p - ac\overline{p}) \\
f &= \frac{1}{2}(a + b + p - ab\overline{p}) \\
x &= 2d + a \\
y &= 2e + b \\
z &= 2f + c.
\end{aligned}$$

Plugging in the expressions for d, e, and f into the last three equations and simplifying, we get

$$\begin{aligned}
x &= a + b + c + p - bc\overline{p} \\
y &= a + b + c + p - ac\overline{p} \\
z &= a + b + c + p - ab\overline{p}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{z - y}{z - x} &= \frac{ac\overline{p} - ab\overline{p}}{bc\overline{p} - ab\overline{p}} \\
&= \frac{ac - ab}{bc - ab} \\
&= \frac{\frac{1}{b} - \frac{1}{c}}{\frac{1}{a} - \frac{1}{c}} \\
&= \frac{\overline{b} - \overline{c}}{\overline{a} - \overline{c}}.
\end{aligned}$$

Thus, triangles XYZ and ABC are oppositely similar.

Problem 6.34 (Napoleon's Theorem)

We will compute o_b and then derive the rest using symmetry. Notice that the magnitude of $o_b - a$ is $\frac{\sqrt{3}}{3}$ times the magnitude of $c - a$. Also, the arguments of $o_b - a$ and $c - a$ differ by $\frac{\pi}{6}$. Assume WLOG that A, B, C are arranged in a counterclockwise order (like in the diagram). Then,

$$o_b - a = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(\frac{\sqrt{3}}{3}\right)(c - a).$$

We can simplify this to get

$$o_b = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right)a + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right)c.$$

So by symmetry,

$$\begin{aligned} o_c &= \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a \\ o_a &= \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b. \end{aligned}$$

Next, we prove this triangle is equilateral. We have

$$\begin{aligned} o_b - o_c &= \left(-\frac{\sqrt{3}}{3}i\right) a - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c \\ o_b - o_a &= -\left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a + \left(\frac{\sqrt{3}}{3}\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c. \end{aligned}$$

Notice that $\frac{o_b - o_a}{o_b - o_c} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, which is just a $60°$ rotation. By symmetry, the other angles must also be 60 degrees. Thus, the triangle is equilateral.

Also,

$$\frac{o_a + o_b + o_c}{3} = \frac{a + b + c}{3},$$

so the center of $O_A O_B O_C$ coincides with the centroid of ABC.