EGMO Solutions

Michael Middlezong

29 July 2024

Chapter 4

Problem 4.48 (Japanese Olympiad 2009)

Notice $APOQ$ is cyclic. This can be proven using the homothety at Q. Then, notice POQ is isosceles and the result shortly follows.

Problem 4.49

Let ray AE intersect the circumcircle at W. Because $\angle BAT = \angle CAE = \angle CAW$, we know arc BT has the same measure as arc CW.

Now, extend ray TD to hit the circumcircle at V. Line TV is just the reflection of line WA across the perpendicular bisector of BC, because $BD = CE$ and that arc BT equals arc CW.

Thus, arcs BA and CV have the same measure, and the result follows.

Problem 4.50 (Vietnam TST 2003/2)

Let I_A, I_B, I_C denote the excenters. We know from a lemma in this chapter that line A_0D is just line DI_A, and so forth. Also, we can see that line DF is parallel to line I_AI_C. Let Z be the intersection point of lines DI_A and FI_C. Then, a homothety at Z takes F to I_C and D to I_A. This homothety also takes E to I_B for the same reason. So, lines DI_A, FI_C, and EI_B concur at Z. For the OI part, notice that O is the nine-point center of triangle $I_AI_BI_C$, and Euler line leads to the result.

Problem 4.51 (Sharygin 2013)

Let M be the midpoint of AB. From a previous lemma, we know CM, $A'B'$, and $C'I$ are concurrent at a point X. Notice that X is also the orthocenter of triangle CIK. Thus, line IX is perpendicular to CK. However, line IX is also perpendicular to AB, so $AB \parallel CK$.

Problem 4.52 (APMO 2012/4)

Let H' be H reflected over D, and H'' be H reflected over M. It is well known that H' and H'' lie on the circumcircle of ABC. By PoP, $HE \cdot HH'' = HA \cdot HH'$. Dividing both sides by two, we obtain the equation $HE \cdot HM = HA \cdot HD$. In other words, $AEDM$ is cyclic.

Now, we claim triangle ABF is similar to triangle AMC. We know $\angle ACM = \angle ACB = \angle AFB$.

Also, $\angle AMC = \angle AMD = \angle AED = \angle AEF = \angle ABF$ (using directed angles). Thus, the two triangles are similar, and it follows that AF is a symmedian. Finally, the desired result is a well-known consequence of AF being a symmedian.

Problem 4.53 (Shortlist 2002/G7)

As always, we can remove M from our diagram by noting that line MK is the same as line KI_A. Let Q be the midpoint of KI_A. We claim $BNCQ$ is cyclic. Let S be the midpoint of NK. Since $\angle ISI_A = \angle IBI_A = 90$ (well known), we know S lies on the circle containing B, I, C, and I_A (this circle being from a common configuration). By PoP, $KS \cdot KI_A = KB \cdot KC$. However, we know $KS \cdot KI_A = KN \cdot KQ$. Thus, $BNCQ$ is cyclic.

Let P be the circumcenter of BCN. Notice that since $BK = XC$, we have $QB = QC$ and thus QP is the perpendicular bisector of BC. In other words, Q is the arc midpoint of arc BC on the circumcircle of BCN. Consider a homothety at N that takes K to Q. This homothety must also take I to P, finishing the proof.

Chapter 5

Problem 5.16 (Star Theorem)

Using the Law of Sines, we write

$$\prod_{i=1}^{5} X_iA_{i+2} = \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+2}A_{i+3}X_i$$

and

$$\begin{aligned}\prod_{i=1}^{5} X_iA_{i+3} &= \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+3}A_{i+2}X_i \\ &= \prod_{i=1}^{5} \frac{A_{i+2}A_{i+3}}{\sin \angle A_{i+2}X_iA_{i+3}} \sin \angle A_{i+1}A_{i+2}X_{i-1}.\end{aligned}$$

Notice that this is the same expression by re-indexing. Thus, we are done.

Problem 5.17

We know the length of the exradius r_A is $\frac{sr}{s-a}$. Then, simply use Heron's formula and $A = sr$.

Problem 5.18 (APMO 2013/1)

WLOG we will just prove triangles ODB and OAE have the same area, and then we can get three pairs from symmetry. We note that OB and OA have the same length, so we just need to compare the heights of the altitudes from D and E to their respective sides. So, using some angle chasing and trigonometry, we can reduce what we are trying to prove to

$$AE \sin(90 - B) = BD \sin(90 - A).$$

Then, we notice that $AE = AB \sin(90 - A)$ and $BD = AB \sin(90 - B)$ by drawing altitudes, giving us the result.

Problem 5.19 (EGMO 2013/1)

Let a, b, c denote the side lengths of ABC in their usual way. We can compute

$$AD^2 = c^2 + 4a^2 - 4ac\cos B$$
$$BE^2 = c^2 + 4b^2 + 4bc\cos A.$$

(The + is not a mistake in the second line there!) Equating the two, we get $a^2 - ac\cos B = b^2 + bc\cos A$. Using the Law of Cosines but solving for angles, we get

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Plugging these back in, we can simplify to get $a^2 = b^2 + c^2$. Thus, triangle ABC is right-angled.

Problem 5.20 (HMMT 2013)

Let E be the contact point of the incircle with AB, and let M be the midpoint of BC. Also, let a, b, and c mean the usual side lengths. The condition $2a = b + c$ can also be written as $s - a = \frac{a}{2}$, where s is the semiperimeter. Since $AE = s - a$ and $MC = \frac{a}{2}$, we know $AE = MC$.

We also know $\angle DCM = \angle IAE$. So, by AAS congruence, we have that triangle AIE is congruent to triangle CDM. Therefore, $DC = AI = DI$ (by another lemma), and we are done.

Problem 5.21 (USAMO 2010/4)

Notice that I is the incenter. Law of Cosines tells us

$$BC^2 = BI^2 + CI^2 - 2 \cdot BI \cdot CI \cos\angle BIC.$$

Angle chasing gives us $\angle BIC = 135$. So, we have

$$BC^2 = BI^2 + CI^2 + \sqrt{2} \cdot BI \cdot CI.$$

Assume BI and CI have integer lengths. Then $BC^2 = AB^2 + AC^2$ is not an integer. Thus, the six segments cannot all have integer lengths.

Problem 5.22 (Iran Olympiad 1999)

We can rewrite the condition as $ID \cdot (\sin B + \sin C) = \frac{1}{2}AD$ (using some angle chasing). Since $ID = BD = CD$, we now use Ptolemy's theorem to get

$$(AB + AC) \cdot ID = AD \cdot BC.$$

However, we know that $ID = \frac{AD}{2(\sin B + \sin C)}$, so we can plug that in and simplify to get

$$BC = \frac{AB + AC}{2(\sin B + \sin C)}.$$

Using the Extended Law of Sines again, we can write $\sin B = \frac{AC}{2R}$ and $\sin C = \frac{AB}{2R}$ where R is the circumradius. Then, the above equation simplifies to

$$BC = R.$$

Using the Extended Law of Sines, this means that $\sin A = \frac{1}{2}$, so $\angle A = 30$ or $\angle A = 150$.

Problem 5.23 (CGMO 2002/4)

Using the Law of Sines,
$$\frac{AH}{HF} = \frac{EA \sin\angle HEA}{EF \sin\angle HEF}.$$
Note that $EC = EF$ because chord CF is perpendicular to diameter AB. So, we rewrite our expression as
$$\frac{EA \sin\angle HEA}{EC \sin\angle HEF}.$$
Simple angle chasing and trig finishes this proof:
$$\begin{aligned}\frac{EA \sin\angle HEA}{EC \sin\angle HEF} &= \frac{EA \sin\angle GCB}{EC \sin\angle CBD}\\ &= \frac{EA \sin(90 - \angle CBD)}{EC \sin\angle CBD}\\ &= \frac{EA}{EC \tan\angle CBD}\\ &= \frac{\tan\angle ECA}{\tan\angle CBD}\\ &= \frac{\tan\angle CBA}{\tan\angle CBD}\\ &= \frac{AC}{CD}.\end{aligned}$$

Problem 5.28 (IMO 2001/1)

Let M be the midpoint of BC, and consider right triangle OMC. Since $\angle COM = \angle A$, it suffices to prove that $\angle PCO > \angle COP$, or $CP < PO$. We claim that $CP < PM$. This is equivalent to $CP < 3PB$, or
$$c \cos B \geq 3b \cos C.$$
This simplifies to
$$\tan C \geq 3 \tan B.$$
Using the angle condition and some algebra, we can see that this is true. Finally, $PM < PO$ is obvious, so we are done.

Chapter 6

Problem 6.29

We scale down to the unit circle and center our arc on the real axis. Let our arc have endpoints at a and $\overline{a} = \frac{1}{a}$, where a is on the unit circle. Let the other point on the circle be b, and the center of the circle is obviously 0. Then, the inscribed angle theorem is equivalent to
$$\arg\left(\frac{a-b}{\frac{1}{a}-b}\right) = \frac{1}{2}\arg\left(\frac{a}{\frac{1}{a}}\right).$$
Notice that with some manipulation, this is equivalent to proving that $\frac{a-b}{1-ab}$ is real, or equal to its conjugate. Indeed, we have
$$\overline{\frac{a-b}{1-ab}} = \frac{\overline{a}-\overline{b}}{1-\overline{ab}}$$

$$
\begin{aligned}
&= \frac{\frac{1}{a} - \frac{1}{b}}{1 - \frac{1}{ab}} \\
&= \frac{\frac{b-a}{ab}}{\frac{ab-1}{ab}} \\
&= \frac{b-a}{ab-1} \\
&= \frac{a-b}{1-ab}.
\end{aligned}
$$

So, we are done.

Lemma 6.30

If P is on chord AB, then

$$\frac{p-a}{p-b} = \overline{\left(\frac{p-a}{p-b}\right)} = \frac{\overline{p} - \frac{1}{a}}{\overline{p} - \frac{1}{b}}.$$

With enough algebraic manipulation, we can get to the result.

Problem 6.31

Let a, b, c, and d be on the unit circle. Then, we have

$$
\begin{aligned}
h_a &= b + c + d \\
h_b &= a + c + d \\
h_c &= a + b + d \\
h_d &= a + b + c.
\end{aligned}
$$

We can now see that the point $\frac{1}{2}(a+b+c+d)$ is the midpoint of AH_A, BH_B, CH_C, and DH_D, and thus the lines concur at this point.

Problem 6.32

Let x be the point of tangency of the incircle with AB, y be that of BC, z be that of CD, and w be that of AD. Also, we scale down such that w, x, y, and z are on the unit circle. Then, using the intersection of tangents formula, we get

$$
\begin{aligned}
a &= \frac{2wx}{w+x} \\
b &= \frac{2xy}{x+y} \\
c &= \frac{2yz}{y+z} \\
d &= \frac{2wz}{w+z}.
\end{aligned}
$$

Then, the midpoint of AC is

$$m_1 = \frac{wx}{w+x} + \frac{yz}{y+z} = \frac{wxy + wxz + wyz + xyz}{(w+x)(y+z)}.$$

The midpoint of BD is

$$m_2 = \frac{xy}{x+y} + \frac{wz}{w+z} = \frac{wxy + wxz + wyz + xyz}{(x+y)(w+z)}.$$

Since we have placed I at the origin, we seek to prove $\frac{m_1}{m_2}$ is real. Indeed:

$$\frac{m_1}{m_2} = \frac{(x+y)(w+z)}{(w+x)(y+z)}$$

is equal to its conjugate (through enough algebraic manipulation).

Problem 6.33 (Chinese TST 2011)

Let $a = A$, $b = B$, and $c = C$ in complex numbers. We can derive

$$\begin{aligned} d &= \frac{1}{2}(b + c + p - bc\overline{p}) \\ e &= \frac{1}{2}(a + c + p - ac\overline{p}) \\ f &= \frac{1}{2}(a + b + p - ab\overline{p}) \\ x &= 2d + a \\ y &= 2e + b \\ z &= 2f + c. \end{aligned}$$

Plugging in the expressions for d, e, and f into the last three equations and simplifying, we get

$$\begin{aligned} x &= a + b + c + p - bc\overline{p} \\ y &= a + b + c + p - ac\overline{p} \\ z &= a + b + c + p - ab\overline{p}. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{z-y}{z-x} &= \frac{ac\overline{p} - ab\overline{p}}{bc\overline{p} - ab\overline{p}} \\ &= \frac{ac - ab}{bc - ab} \\ &= \frac{\frac{1}{b} - \frac{1}{c}}{\frac{1}{a} - \frac{1}{c}} \\ &= \frac{\overline{b} - \overline{c}}{\overline{a} - \overline{c}}. \end{aligned}$$

Thus, triangles XYZ and ABC are oppositely similar.

Problem 6.34 (Napoleon's Theorem)

We will compute o_b and then derive the rest using symmetry. Notice that the magnitude of $o_b - a$ is $\frac{\sqrt{3}}{3}$ times the magnitude of $c - a$. Also, the arguments of $o_b - a$ and $c - a$ differ by $\frac{\pi}{6}$. Assume WLOG that A, B, C are arranged in a counterclockwise order (like in the diagram). Then,

$$o_b - a = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(\frac{\sqrt{3}}{3}\right)(c - a).$$

We can simplify this to get

$$o_b = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c.$$

So by symmetry,

$$\begin{aligned} o_c &= \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a \\ o_a &= \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b. \end{aligned}$$

Next, we prove this triangle is equilateral. We have

$$\begin{aligned} o_b - o_c &= \left(-\frac{\sqrt{3}}{3}i\right) a - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c \\ o_b - o_a &= -\left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a + \left(\frac{\sqrt{3}}{3}\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c. \end{aligned}$$

Notice that $\frac{o_b - o_a}{o_b - o_c} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, which is just a 60° rotation. By symmetry, the other angles must also be 60 degrees. Thus, the triangle is equilateral. Also,

$$\frac{o_a + o_b + o_c}{3} = \frac{a + b + c}{3},$$

so the center of $O_A O_B O_C$ coincides with the centroid of ABC.

Problem 6.35 (USAMO 2015/2)

The first step is to notice that the center is the midpoint of AO, where O is the midpoint of AB. We compute using $a = -1$, s, and t as free variables. In our world, the center of the circle on which M travels on is $-\frac{1}{2}$. We have

$$x = \frac{1}{2}\left(-1 + s + t + \frac{s}{t}\right).$$

Also, the magnitude we want to compute is

$$\left|\frac{s+t}{2} - \left(-\frac{1}{2}\right)\right| = \frac{1}{2}\left|s + t + 1\right|.$$

Notice that

$$\begin{aligned} \left|s + t + 1\right|^2 &= (s + t + 1)\overline{(s + t + 1)} \\ &= 3 + s + t + \frac{1}{s} + \frac{1}{t} + \frac{s}{t} + \frac{t}{s}. \end{aligned}$$

Computing the real component of x, which is $\frac{x + \overline{x}}{2}$, we can see that this only depends on the real component of x, which gives us the result.

Problem 6.36 (MOP 2006)

I initially solved this problem by encoding the parallel condition as $ad = be = cf$, but a nicer way to solve it is to rotate the diagram such that $d = \overline{a}$, $e = \overline{b}$, and $f = \overline{c}$. This encodes the parallel condition and makes the computation much easier.

Problem 6.37 (USA January TST for IMO 2014)

Notice that W is the midpoint of A and the orthocenter of triangle ABD. Using this, we can compute

$$\begin{aligned}w &= a + \frac{b+d}{2}\\ x &= b + \frac{a+c}{2}\\ y &= c + \frac{b+d}{2}\\ z &= d + \frac{a+c}{2}.\end{aligned}$$

Then, we can also compute the conjugates:

$$\begin{aligned}\overline{w} &= \frac{1}{a} + \frac{b+d}{2bd}\\ &\vdots\end{aligned}$$

Shoelace bash gives us the desired result. (The computation takes around 10 minutes, but be sure to take advantage of cyclic symmetry.)

Chapter 7

Problem 7.32

We have $I = (a : b : c)$ and $G = (1 : 1 : 1)$. Then, we compute N. Let D be the contact point of the incircle with BC. Then, we know $BD = s - b$ and $CD = s - c$. Let D' be the contact point of the A-excircle with BC. We know D' is the reflection of D over the midpoint of BC, so $D' = (0 : s - b : s - c)$. Similarly, $E' = (s - a : 0 : s - c)$ and $F' = (s - a : s - b : 0)$. We can now see that $N = (s - a : s - b : s - c)$ falls on all three cevians. Computing the determinant of the appropriate matrix easily gets us the fact that I, G, and N are collinear.

Now, we prove $NG = 2GI$. Normalizing coordinates, we have $G = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $I = (\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s})$, and $N = (1 - \frac{a}{s}, 1 - \frac{b}{s}, 1 - \frac{c}{s})$. We can see that $N = 3G - 2I$, so we are done.

Problem 7.33 (IMO 2014/4)

We use similar triangles to compute P and Q, and then it is quite straightforward to compute the intersection point as $(-a^2 : 2b^2 : 2c^2)$ which satisfies the equation of the circumcircle.

Problem 7.34 (EGMO 2013/1)

The points are easy to compute. Then, use displacement vectors to find

$$\begin{aligned}|AD|^2 &= 2a^2 + 2b^2 - c^2,\\ |BE|^2 &= -2a^2 + 6b^2 + 3c^2.\end{aligned}$$

Setting them equal, we get $a^2 = b^2 + c^2$, so ABC is a right triangle.

Problem 7.35 (ELMO Shortlist 2013)

Set $D = (0, m, n)$ where $m + n = 1$. Use the general form of a circle and compute everything. The result is straightforward.

Problem 7.36 (IMO 2012/1)

The difficulty in this problem mainly lies in algebraic manipulation.

We start by computing $J = (-a : b : c)$ and $M = (0 : s - b : s - c)$. Notice that $KB = s - c$ and $KA = s$. From this, we can deduce $K = (c - s : s : 0)$. Similarly, $L = (b - s : 0 : s)$.

Now, we set out to compute F. Since F lies on line BJ, we know that it can be written in the form $(-a : t : c)$ for some t. We also know F, M, and L are collinear, so we have the equation

$$\begin{vmatrix} -a & t & c \\ 0 & s-b & s-c \\ b-s & 0 & s \end{vmatrix} = 0 \implies t = \frac{-as + c(s-b)}{s-c}.$$

At this point, continuing with the computation leads to very messy expressions. We wonder if the expression for t can be simplified. Indeed, after some algebra:

$$\frac{-as + c(s-b)}{s-c} = -(a+c).$$

So, we have $F = (-a : -(a+c) : c) = (a : a + c : -c)$. Similarly, $G = (a : -b : a + b)$.

Now, we have pretty much finished the problem. Computing S and T and then the midpoint of ST gives M, so we are done.

Problem 7.37 (Shortlist 2001/G1)

Start by taking a homothety so that the squares are outside the triangle. Suppose this homothety takes A_1 to P. Then, we can compute P using Conway's formula. We end up getting that points on AP can be parametrized as

$$(t_1 : S_C + S : S_B + S).$$

Similarly, points on BB_1 can be written as

$$(S_C + S : t_2 : S_A + S)$$

and points on CC_1 can be written as

$$(S_B + S : S_A + S : t_3).$$

It is clear that the point of concurrency is

$$\left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

Problem 7.38 (USA TST 2008/7)

We want to prove that the intersection of (AQR) and the isogonal of AG does not depend on the choice of P.

Let $P = (0, m, n)$ where $m + n = 1$. Then, it is easy to see that $Q = (m, 0, n)$ and $R = (n, m, 0)$. Next, we find the equation of (AQR). Using the general form of a circle and plugging in values, we get that the desired equation is

$$-a^2yz - b^2zx - c^2xy + (c^2ny + b^2mz)(x + y + z) = 0.$$

Now, we find that the isogonal of AG can be parametrized as $(t : 3b^2 : 3c^2)$ using Lemma 7.6. Plugging this into the equation for the circle, we notice that m and n cancel out, and the resulting expression does not depend on the choice of P, so we are done.

Note: after the fact, I realized that the isogonal of AG is just the A-symmedian, which we already know can be parametrized as $(t : b^2 : c^2)$.

Problem 7.39 (USAMO 2001/2)

It is well known that $D_2 = (0 : s - b : s - c)$ and $E_2 = (s - a : 0 : s - c)$. We can then deduce that $P = (s - a : s - b : s - c)$. Also, using a lemma from Chapter 4, we know QD_1 is a diameter of the incircle, i.e., Q is the reflection of D_1 over I.

Since we know $D_1 = (0 : s - c : s - b)$ and $I = (a : b : c)$, we can calculate $Q = \left(\frac{a}{s}, \frac{b}{s} - \frac{s-c}{a}, \frac{c}{s} - \frac{s-b}{a}\right)$ and verify that $\overrightarrow{AQ} = \overrightarrow{PD_2}$, finishing the problem.

Problem 7.40 (USA TSTST 2012/7)

Through angle chasing, we can reduce this problem to trying to show $\overline{AD}$ is parallel to $\overline{NM}$. Because AD passes through the incenter $(a : b : c)$, it is easy to see that $D = (0 : b : c)$. Also, we know $M = (0 : 1 : 1)$. Now, we turn our attention to computing N.

Through some work, we find that the equation of (ADM) is

$$-a^2yz - b^2zx - c^2xy + \left(\frac{a^2c}{2(b+c)}y + \frac{a^2b}{2(b+c)}z\right)(x + y + z) = 0.$$

We can solve for Q and P to get, after a lot of algebra:

$$Q = (a^2 : 2c(b + c) - a^2 : 0),$$
$$P = (a^2 : 0 : 2b(b + c) - a^2).$$

We can then calculate $N = (a^2(b + c) : 2bc(b + c) - a^2b : 2bc(b + c) - a^2c)$.

The displacement vector $\overrightarrow{NM} = (-a^2(b + c) : a^2b : a^2c)$. Also, the displacement vector $\overrightarrow{AD} = (-(b + c) : b : c)$. It is clear that $\overline{AD}$ and $\overline{NM}$ are parallel, so we are done.

Problem 7.41

Using properties of angle bisectors, we can compute $P = (a : 0 : b - a) = (\frac{a}{b} : 0 : 1 - \frac{a}{b})$ and $Q = (a : c - a : 0) = (\frac{a}{c} : 1 - \frac{a}{c} : 0)$. Then, using the theorem for generalized perpendicularity, we can obtain the result.

Lemma 7.42

Using the mixtilinear incircle configuration, we find that the concurrency point of AT_A, BT_B, and CT_C is the isogonal conjugate of the Nagel point. Since the Nagel point has coordinates $(s - a : s - b : s - c)$, the point of concurrency is $(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}) = (a^2(s - b)(s - c) : b^2(s - a)(s - c) : c^2(s - a)(s - b))$.

All that remains is to show that this point is collinear with O and I, which is relatively easy to do by showing the determinant of the matrix is 0.

Chapter 8

Problem 8.23

Simply invert around C. The four points become a rectangle.

Problem 8.24

Inverting around A, we get a structure with two lines and two circles in between. Similar triangles finishes the problem.

Problem 8.25

Using the inverting the circumcenter lemma, we invert around P. The resulting problem is solvable by noticing the homothety + Simson line, or complex bashing.

Problem 8.26 (BAMO 2008/4)

Inverting around D, the problem becomes equivalent to a simple problem from Chapter 3, which I already solved.

Problem 8.27 (Iran Olympiad 1996)

Invert around the circle. Then, AC and BD intersect at K^*, and we wish to prove that $\angle K^*M^*O = 90$, where M^* is the second intersection point of (COD) with line AB.

Let M' be the phantom point which is the foot of the perpendicular from K^* to line AB. We wish to show $COM'D$ is cyclic. Notice through angle chasing that C is the foot from B to AK^*, and similarly for D. Thus, $CM'D$ is the orthic triangle of triangle K^*AB, and we have (through angle chasing):

$$\angle OM'D = \angle AM'D = \angle AKD = \angle AKM' + \angle M'KB = \angle OCB + \angle BCD = \angle OCD.$$

(One of the steps in that equation is significantly more involved than the rest.) Therefore, we are done.

Alternate ending: notice that (COD) is the nine-point circle of triangle K^*AB. So, M^* must be the foot of the altitude.

Problem 8.28 (Shortlist 2003/G4)

After inverting around P, it is a trivial computational problem using the inversive distance formula. Specifically, begin by noticing that the image of $ABCD$ is a parallelogram.

Problem 8.29

Inverting about the incircle takes (ABC) to the nine-point circle of DEF. Thus, O, I, and the nine-point center of DEF are collinear. This means that line OI is the Euler line of triangle DEF. Since G_1 also lies on this line, we are done.

Problem 8.30 (NIMO 2014)

Notice that Q is just the antipode of A on (ABC). Let line QI intersect (ABC) again at X. Notice that $AXFIE$ is cyclic. Now, consider an inversion around the incircle. The circle $(AXFIE)$ is mapped to line EF, and (ABC) is mapped to the nine-point circle of DEF. Since X^* must be on line EF, it must coincide with point P since X, P, and I

are collinear. But since X^* also lies on the nine-point circle of DEF, and $X \neq A$, we have that $X^* = P$ must be the foot of the altitude from D to EF, and so we are done.

Chapter 9

Problem 9.42 (USA TSTST 2012/4)

Let H be the orthocenter. Brocard's theorem applied to quadrilaterial BCB_1C_1 yields that D is the orthocenter of triangle AA_2H, meaning that line DH is perpendicular to line AA_2. Similarly, we can see that all the perpendicular lines pass through H, so they are concurrent.

Problem 9.43

Let F be the reflection of B over O. Notice that $ABCF$ is a rectangle, E is the intersection point of lines AF and CD, and lines BF and AC intersect at O. Therefore, Pascal's theorem on $BDCAAF$ gives the result.

Problem 9.44 (Canada 1994/5)

Trivial using the Right Angles and Bisectors lemma (Lemma 9.18).

Problem 9.45 (Bulgarian Olympiad 2001)

Let F be the midpoint of AB, and let X be the intersection point of the tangents to k through C and E. Let G be the intersection point of lines BD and EC.

Notice that $BEDC$ is a harmonic quadrilateral, and in particular, $(B, D; G, X) = -1$. Projecting through C onto line AB, we have $(B, A; F, \overline{CX} \cap \overline{AB}) = -1$. Since F is the midpoint of AB, we must have that lines CX and AB are parallel, which quickly leads to the desired conclusion.

Problem 9.46 (ELMO Shortlist 2012)

If $AB = AC$, then we are done by symmetry. Otherwise, let K be the intersection point of lines IP and BC. Notice that K is the inverse of P with respect to the incircle, and thus, A lies on the polar of K. By La Hire's theorem, we know that K lies on the polar of A. In other words, if E and F are the contact points of the incircle with sides AC and AB, respectively, then K lies on line EF.

It is well known that the cevians AD, BE, and CF concur, so we can use the concurrent cevians lemma to deduce that $(E, D; B, C) = -1$. Finally, since $\angle EPD = 90$, the right angles and bisectors lemma tells us that $\angle BPD = \angle DPC$.

Problem 9.47 (IMO 2014/4)

Let X_1 be the intersection point of (ABC) with line BM, and let X_2 be that intersection point with line CN. We want to show $X_1 = X_2$.

Some trivial angle chasing reveals that line OB is perpendicular to line AP and line OC is perpendicular to line AQ. Projecting through B, we see

$$-1 = (A, M; P, P_\infty) = (A, X_1; C, B),$$

since $\overline{BP_\infty}$ is the tangent at B.

Similarly, we have $(A, X_2; B, C) = -1$. Thus, $X = X_1 = X_2$ is the point that makes $ABXC$ harmonic, and we are done.

Problem 9.48 (Shortlist 2004/G8)

Clearly, N and M must be on opposite sides of chord AB. Let N' be the intersection point of line EF and (ABM) which is not on the same side of chord AB as M. Then, we wish to prove $AMBN'$ is harmonic, and by the uniqueness of harmonic conjugates, we will be done.

Let $P = \overline{EF} \cap \overline{CD}$ and $G = \overline{AB} \cap \overline{CD}$. Using the midpoint lengths lemma and Power of a Point, we see that P lies on (ABM). We also know that $(G, P; C, D) = -1$. Thus,

$$-1 = (G, P; C, D) \stackrel{E}{=} (G, \overline{EF} \cap \overline{AB}; B, A) \stackrel{P}{=} (M, N'; B, A).$$

Problem 9.49 (Sharygin 2013)

Let M be the midpoint of AB, and let D be the foot of the perpendicular from I to CM. Notice that since K lies on the polar of C, by La Hire's theorem, C lies on the polar of K. In other words, CM is the polar of K. Then, we know

$$-1 = (B', A'; \overline{A'B'} \cap \overline{CM}, K) \stackrel{C}{=} (A, B; M, \overline{CK} \cap \overline{AB}).$$

Since M is the midpoint of AB, $\overline{CK} \cap \overline{AB}$ must be the point at infinity, and we are done.

Chapter 10

Problem 10.17 (NIMO 2014)

We will show that R, M, and S are collinear, from which the result follows easily (perhaps by congruent triangles). We know (using directed angles)

$$\angle SBM = \angle SHM = \angle QHM = \angle QCM = \angle C.$$

Since $\angle MBH = 90 - \angle C$, we know $\angle SBH$ is right. Similarly, $\angle RCH$ is right. Thus, $\angle SMH = \angle SBH = \angle RCH = \angle RMH$, so we are done.

Problem 10.18 (USAMO 2013/1)

Draw the Miquel point M. Through angle chasing, we get $\triangle MYX \sim \triangle MBP$ and $\triangle MYZ \sim \triangle MBC$ with the same scale factor MY/MB. So, we are done.

Problem 10.19 (Shortlist 1995/G8)

Let ω_{XY} denote the circle with diameter XY. Notice that the orthocenter of triangle EAD is the radical center of circles ω_{AD}, ω_{AB}, and ω_{CD}. Thus, it lies on the radical axis of ω_{AB} and ω_{CD}. Notice that the orthocenter of triangle EBC is the radical center of circles ω_{AB}, ω_{BC}, and ω_{CD}. Thus, it also lies on the radical axis of ω_{AB} and ω_{CD}. Point F also lies on this radical axis because of Power of a Point. So, F and the two orthocenters lie on the same line.

Problem 10.20 (USA TST 2007/1)

Take the quadrilateral $APDX$. We know that Q is the Miquel point of this quadrilateral, since Q is the second intersection of (BPD) and (CAP). Thus, $AXQB$ is cyclic.

Then, we have (using directed angles)

$$\angle QYP = \angle QAP = \angle QAB = \angle QXB.$$

Since $\overline{XB} \parallel \overline{YP}$, we must have that Q, X, and Y are collinear.

Similarly,

$$\angle QZP = \angle QBP = \angle QBA = \angle QXA,$$

so points Q, X, and Z are collinear.

Problem 10.21 (USAMO 2013/6)

This problem is 100000 MOHS so I can't really write it up

Problem 10.22 (USA TST 2007/5)

The length conditions can be interpreted by drawing a circle centered at T passing through B.

We show that A is the Miquel point of BB_1C_1C. Let $Q = \overline{BB_1} \cap \overline{CC_1}$. Then, by the three tangents configuration, Q lies on (ABC). Fixing BB_1C_1C, only one point A satisfies $\angle BAC$ being acute, A being on (QBC), and $\angle TAS = 90$. The Miquel point also satisfies these criteria whenever BB_1C_1C is a quadrilateral following from the problem statement (based on various properties listed in EGMO), so A must be the Miquel point.

Note: for me, the motivation for A being a Miquel point came from the fact that $E = \overline{B_1C} \cap \overline{C_1B}$ lies on (ABC) when drawn and looks like the inverse of A (orthogonal circles).

Problem 10.23 (IMO 2005/5)

Let M be the Miquel point of self-intersecting quadrilateral $BCAD$. In other words, it is the second intersection of (PAD) and (PBC). We claim that all of the circumcircles of triangles PQR pass through M. Note that M is the center of the spiral similarity taking A, F, D to C, E, B. Then, M is also the center of the spiral similarity taking AF to CE, so M is the second intersection of (AFR) and (CER). This means that $RMEC$ is cyclic. Similarly, $QMBE$ is cyclic. A simple directed angle chase finishes the proof:

$$\angle RMQ = \angle RME + \angle EMQ = \angle RCE + \angle EBQ = \angle PCB + \angle CBP = \angle CPB = \angle RPQ.$$

Problem 10.24 (USAMO 2006/6)

Consider M, the center of the spiral similarity taking AD to BC. Since $\frac{AE}{ED} = \frac{BF}{FC}$, this spiral similarity also takes E to F. Thus, M is Miquel point of complete quadrilaterals $ABEF$ and $EFDC$, which all four circles must pass through.

Chapter 11

Problem 11.1 (Canada 2000/4)

Let $\alpha = \angle ADB$ and $\beta = \angle CDB$. Through angle chasing, we compute $\angle BAD = 180 - \alpha - 2\beta$ and $\angle BCD = 180 - 2\alpha - \beta$. Using the law of sines on $\triangle ABD$:

$$\frac{AB}{\sin\alpha} = \frac{BD}{\sin(180 - 2\beta - \alpha)} = \frac{BD}{\sin(2\beta + \alpha)}.$$

Similarly,

$$\frac{BC}{\sin\beta} = \frac{BD}{\sin(2\alpha + \beta)}.$$

Since $AB = BC$, we conclude that

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin(2\beta + \alpha)}{\sin(2\alpha + \beta)}.$$

Cross multiplying and using product-to-sum, this simplifies to

$$\cos(3\alpha + \beta) = \cos(3\beta + \alpha).$$

For a non-degenerate quadrilateral, we must have $\alpha, \beta > 0$ and $\alpha + \beta < 90$. So, we see that the only solution is $\alpha = \beta$, and by symmetry, $AD = CD$.

Problem 11.2 (EGMO 2012/1)

First, notice through angle chasing that $\angle FDE = 180 - 2\angle A$. Let I be the incenter of triangle DEF. Since it is easy to see that line ID is perpendicular to side BC, we wish to show that K, I, and D are collinear.

More angle chasing reveals that $AFIE$ is cyclic. Then, by the incenter-excenter lemma on $\triangle DEF$, we are done.

Problem 11.3 (ELMO 2013/4)

First, we claim that $BE = BC$. Since

$$\angle AEB = \angle ACE = \angle RCS = \angle RBS = \angle RBA,$$

we know that S lies on the angle bisector of $\angle B$ in isosceles $\triangle BER$. Thus, by symmetry,

$$\angle BCS = \angle BRS = -\angle BES,$$

so $BE = BC$.

Next, we claim K is the incenter of $\triangle ELD$. It suffices to show that $\angle REL = \angle DER$. We have

$$\angle REL = \angle REB + \angle BEL = \angle BRE + \angle ECB = \angle DRE + \angle EDR = \angle DER.$$

Finally,

$$\angle ELK = \frac{1}{2}\angle ELD = \angle BLC,$$

and it is clear that $\triangle BLC \sim \triangle BCD$, so we are done.

Problem 11.4 (Sharygin 2012)

First, it is easy to see that C_1 is the midpoint of BC and A_1 is the midpoint of BA.

Notice that $\overline{C_1C_2}$ bisects $\angle CC_1A_1$ using basic angle chasing. Similarly, $\overline{A_1A_2}$ bisects $\angle AA_1C_1$.

Let X be the intersection point of lines C_1C_2 and A_1A_2. Then, a homothety at B with scale factor 2 takes X to the B-excenter, so the result is obvious.

Problem 11.5 (USAMTS)

Draw I, the incenter of triangle ABD. The key step is to notice that $IBCD$ is cyclic; the rest of the problem is easy.

Problem 11.6 (MOP 2012)

First, we see that H lies on γ. Then, notice that inverting around B, this problem inverts to itself. We see that under this inversion, P is sent to Q and vice versa. Thus, B, P, and Q are collinear.

Problem 11.7 (Sharygin 2013)

Let K be the midpoint of BC. We wish to show $DKEN$ is cyclic. Let A' be the reflection of A over K. We claim that $DA'EM$ is cyclic.

First, we show $FA'TM$ to be cyclic. Because of the parallel condition and symmetry, $DFA'A$ is an isosceles trapezoid and

$$\angle FA'A = \angle A'FD = -\angle ADF = -\angle ETF = -\angle MTF = \angle FTM,$$

so $FA'TM$ is cyclic. Then, the radical axis theorem tells us that $DA'EM$ is cyclic.

Finally, we have $AD \cdot AE = AA' \cdot AM = AK \cdot AN$, so we are done.

Problem 11.8 (ELMO 2012/1)

Let M be the midpoint of BC. It is well known that B, F, E, and C lie on a circle centered at M and lines ME and MF are tangent to ω. Thus, line ME is tangent to w_1 and line MF is tangent to w_2, so M has the same power with respect to w_1 and w_2. D also has the same power 0 with respect to both circles, so $\overline{DM} = \overline{BC}$ must be the radical axis of the two circles, making the result obvious.

Problem 11.9 (Sharygin 2013)

Let L be the midpoint of BC and P be the midpoint of QR. Set $AB = 2x$, $CD = 2y$, and $BC = 2l$. Chasing lengths using power of a point, we get

$$PL = \frac{y^2 - x^2}{2l}$$

and $KL = \frac{x+y}{2}$.

Let E be the foot of the perpendicular from B to DC. Then, we see by SAS similarity that $\triangle KPL \sim \triangle BEC$. Thus, line KP is perpendicular to line BC. Finally, SAS congruence finishes up the problem.