# Submission for BAW-SYMPOLY

**OTIS (internal use)**

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> **Example** (0♣)**.** Show that $a^4 + b^4 \ge a^3b + ab^3$ for $a, b > 0$.

Z68B4C1C

**Walkthrough.** For the purposes of this example, assume you don't know AM-GM or Muirhead, etc. The goal is to show how to solve the problem with "bare hands".

We will prove $a^4 + b^4 - a^3b - ab^3 \ge 0$.

**(a)** Noting that equality holds when $a = b$, what factor must divide the left-hand side?

**(b)** Imagine fixing $b$, and treating the left-hand side as a polynomial $P(a)$. It has a root at $a = b$. If you also know that $P \ge 0$ everywhere, what kind of root must that root be?

**(c)** Use this to factor the left-hand side completely. (There should be three factors.)

The condition $a, b > 0$ isn't actually used here but makes things simpler to think about.

> **Example** (AIME 2010, 0♣)**.** Compute the maximum possible value of $a^3 + b^3 + c^3$ over all real numbers $(a, b, c)$ satisfying
>
> $$\begin{aligned} a^3 &= abc + 2 \\ b^3 &= abc + 6 \\ c^3 &= abc + 20. \end{aligned}$$

10AIME9

**Walkthrough.**

**(a)** Express $a^3 + b^3 + c^3$ in terms of $abc$. Thus it suffices to compute $abc$.

**(b)** Find a way to get a quadratic equation in $abc$.

**(c)** Solve for $abc$ and use it to get the final answer.

> **Example** (Evan Chen, Fall 2015, 0♣)**.** Let $a$, $b$, $c$ be the distinct roots of the polynomial
>
> $$P(x) = x^3 - 10x^2 + x - 2015.$$
>
> The cubic polynomial $Q(x)$ is monic and has distinct roots $bc - a^2$, $ca - b^2$, $ab - c^2$. What is the sum of the coefficients of $Q$?

15OMOF12

**Walkthrough.** This can be done with brute force, by actually finding $Q$, but there is a trick to it.

**(a)** Show the answer is given by $Q(1) = (1 - bc + a^2)(1 - ca + b^2)(1 - ab + c^2)$.

**(b)** Show that $ab + bc + ca = 1$.

**(c)** Prove that $Q(1) = 2015000$.

> **Example** (USAMO 1975/3, 0♣). If $P(x)$ denotes a polynomial of degree $n$ such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \dots, n$, determine $P(n + 1)$.

75AMO3

**Walkthrough.** The main idea is to define $Q(x) = (x + 1)P(x) - x$.

**(a)** Compute $\deg Q$ (in terms of $n$).

**(b)** Determine the roots of $Q$.

**(c)** Use (a) and (b) to establish the factorization, of $Q$ up to a constant factor.

**(d)** Show that $Q(-1) = 1$ and use this to conclude that the leading coefficient of $Q$ is equal to

$$c = \frac{(-1)^{n+1}}{(n+1)!}.$$

**(e)** Compute $Q(n + 1)$.

**(f)** Prove that $P(n + 1) = \frac{n+1+(-1)^{n+1}}{n+2}$.

> **Example** (HMMT 2023 T2, 0♣). Prove there don't exist pairwise distinct complex numbers $a$, $b$, $c$, and $d$ such that
>
> $$a^3 - bcd = b^3 - cda = c^3 - dab = d^3 - abc.$$

23HMMTT2

**Walkthrough.** There is a brute-force approach along the lines of taking

$$a^3 - b^3 = bcd - cda$$

and factoring out the common $a - b$.

**(a)** Use this idea to show that $a^2 + b^2 = c^2 + d^2$.

**(b)** Similarly, show that $a^2 + c^2 = b^2 + d^2$, and $a^2 + d^2 = b^2 + c^2$.

However, I think the following Vieta-based approach is more conceptually nice, since it does not require any factoring and obviously generalizes to more variables.

**(c)** Let's assume $abcd \neq 0$. By scaling, show that we may in fact assume $abcd = 1$.

**(d)** Conclude that we may define the number $k$ by

$$k := a^3 - \frac{1}{a} = \cdots = d^3 - \frac{1}{d}.$$

**(e)** Let
$$P(X) = (X - a)(X - b)(X - c)(X - d).$$
Find the coefficients of $P$ in terms of $k$.

**(f)** Derive a contradiction by noticing $P(0) = -1$.

**(g)** Weed out the edge case $abcd = 0$ we didn't address earlier. This gives a complete solution to the problem.

## Practice problems

*Instructions*: Solve [40♣]. If you have time, solve [52♣].

> The Law speaks: you are cast out. You are un-dwarf. I AM A WITNESS!
>
> Angarthing in *The Hammer of Thursagan*, from *The Battle for Wesnoth*

**Problem 1** (Added by Eric Wang, 2♣)
Let $r$, $s$, and $t$ be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Compute $(r+s)^3 + (s+t)^3 + (t+r)^3$.

08AIME7

The sum of the roots is 0. The desired quantity is then

$$-t^3 - r^3 - s^3,$$

which is 753 by Newton sums.

**Problem 2** (USAMO 1973/4, 2♣)
Determine all triples $(x, y, z)$ of complex numbers satisfying

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

73AM04

Consider the cubic polynomial $(t-x)(t-y)(t-z)$. From Newton's sums and Vieta's, this cubic polynomial must equal $t^3 - 3t^2 + 3t - 1$. The only factorization of this is $(t-1)^3$, so the only solution must be $(x, y, z) = (1, 1, 1)$.

**Problem 3** (Canada 1996, added by Haozhe Yang, 2♣)
If $\alpha$, $\beta$, and $\gamma$ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$.

96CAN1

It is easy to show that $1-\alpha$, $1-\beta$, and $1-\gamma$ are the roots of the polynomial

$$x^3 - 3x^2 + 2x + 1$$

using Vieta's.

Then, we can easily calculate the desired expression using Vieta's as well. The answer is $-7$.

**Problem 4** (HMMT November 2016 Guts, added by Rohan Bodke, 2♣)
Let $r_1$, $r_2$, $r_3$, $r_4$ be the complex roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Calculate

$$\frac{r_1^2}{r_2^2 + r_3^2 + r_4^2} + \frac{r_2^2}{r_1^2 + r_3^2 + r_4^2} + \frac{r_3^2}{r_1^2 + r_2^2 + r_4^2} + \frac{r_4^2}{r_1^2 + r_2^2 + r_3^2}.$$

16HMNTGUTS27

By Newton's sums, the sum of the squares of the roots is 0. This means each term in the requested expression is $-1$, giving us a total answer of $-4$.

**Problem 5** (NIMO #8, 2♣)
Let $x$, $y$, $z$ be complex numbers satisfying

$$\begin{aligned} x^2 + 5y &= 10x \\ y^2 + 5z &= 10y \\ z^2 + 5x &= 10z. \end{aligned}$$

Find the sum of all possible values of $z$.

NIMO85

**Problem 6** (USAMO 1984/1, 2♣)
The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is $-32$. Determine $k$.

84AMO1

Notice that we can write the polynomial as

$$(x^2 + ax - 32)(x^2 + bx + 62)$$

for constants $a$ and $b$. Expanding this and matching coefficients, we get the system of equations

$$\begin{aligned} a + b &= -18 \\ 62a - 32b &= 200. \end{aligned}$$

We can solve this system to get $a = -4$, $b = -14$. We also know $k = 30 + ab$ from the earlier expansion, so $k = 86$.

**Required Problem 7** (3♣)
The cubic $x^3 - 7x^2 + 3x + 2$ has irrational roots $r > s > t$. There exists a unique set of rational numbers $A$, $B$, and $C$, such that the cubic $x^3 + Ax^2 + Bx + C$ has $r + s$ as a root. What is $A + B + C$?

JASONMAO

Consider the polynomial with roots $r + s$, $s + t$, and $r + t$. We will find its coefficients and show that it is the desired polynomial. Using Vieta's, we can see that

$$A = -2(r + s + t) = -14.$$

We can also see that

$$B = (r + s)(s + t) + (s + t)(r + t) + (r + s)(r + t).$$

Expanding and simplifying with Vieta's and Newton sums, we get $B = 52$.

The $C$ term is slightly more involved, but we can use a combination of Newton sums, Vieta's, and grouping of terms to get $C = -23$.

All these terms are rational, so overall, our answer is $A + B + C = -14 + 52 - 23 = 15$.

**Problem 8** (AIME II 2020, added by Benjamin Wang-Tie, 3♣)
Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of $x^2$ equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, compute $R(0)$.

20AIMEII11

**Problem 9** (AIME I 2019, added by Joshua Im, 2♣)
For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. Compute

$$\sum_{1 \le j < k \le 673} z_j z_k.$$

19AIME10

**Problem 10** (Austria 2016/6, added by Abdullahil Kafi, 3♣)
Let $a$, $b$, $c$ be three integers for which the sum

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer. Prove that each of the three numbers

$$\frac{ab}{c}, \quad \frac{ac}{b}, \quad \frac{bc}{a}$$

is an integer.

16AUT6

**Problem 11** (2♣)
Factor the polynomial

$$a(b - c)^3 + b(c - a)^3 + c(a - b)^3.$$

ZEAC3666

We notice that the polynomial vanishes whenever $a = b$, $a = c$, or $b = c$. So, the polynomial is divisible by $(a - b)(a - c)(b - c)$. We know the last factor must be a multiple of $a + b + c$. We can match the coefficient of $ab^3$ to get that the factored form is

$$(a - b)(a - c)(b - c)(-a - b - c).$$

**Problem 12** (3♣)
Let $a$, $b$, $c$ be real numbers. Prove that

$$a^3 + b^3 + c^3 = (a + b + c)^3 \quad \text{if and only if} \quad a^5 + b^5 + c^5 = (a + b + c)^5.$$

H1883820

**Problem 13** (2♣)
Let $a$, $b$, $c$, $d$ be distinct real numbers such that

$$a+b+c+d=0 \quad \text{and} \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0.$$

Prove that two of the four numbers have sum zero.

BMCQ54

We notice that quartic polynomial with roots $a, b, c, d$ is even, and the result follows.

**Problem 14** (AIME II 2003, added by Lincoln Liu, 3♣)
Consider the polynomials $P(x)=x^6-x^5-x^3-x^2-x$ and $Q(x)=x^4-x^3-x^2-1$. Given that $z_1$, $z_2$, $z_3$, and $z_4$ are the roots of $Q(x)=0$, find $P(z_1)+P(z_2)+P(z_3)+P(z_4)$.

03AIMEII9

Write

$$P(x)=(x^2+x)Q(x)+x^2-x+1.$$

Then,

$$\begin{aligned}
\sum_{i=1}^{4} P(z_i) &= \sum_{i=1}^{4}((x^2+x)Q(x)+x^2-x+1) \\
&= \sum_{i=1}^{4}(x^2-x+1) \\
&= \sum_{i=1}^{4} x^2 - \sum_{i=1}^{4} x + \sum_{i=1}^{4} 1 \\
&= 3-1+4 = \boxed{6}.
\end{aligned}$$

**Problem 15** (CMIMC 2018 A9, 9♣)
Given the polynomial identity

$$(x^2-3x+1)^{1009} = \sum_{k=0}^{2018} a_k x^k$$

calculate the remainder when $a_0^2+a_1^2+\cdots+a_{2018}^2$ is divided by 2017.

18CMIMCA9

**Required Problem 16** (Stanford Math Tournament 2011, 3♣)
Let $P(x)$ be a polynomial of degree 2011 such that $P(1)=0$, $P(2)=1$, $P(4)=2$, $\ldots$, and $P(2^{2011})=2011$. Find the coefficient of $x^1$ in $P$.

11SMTA7

We can notice that the polynomial $P(2x)-P(x)-1$ has roots $x=2^i$ for $0 \le i \le 2010$. Thus, we can write

$$P(2x)-P(x)-1 = c(x-2^0)(x-2^1)\cdots(x-2^{2010}).$$

Plugging in $x=0$, we can find $\frac{1}{c}=1+2+\cdots+2010$ (denote by $S$ this sum).

Now, let $a$ be the coefficient of the linear term in $P(x)$. Then, the linear term of $P(2x) - P(x) - 1$ is $2ax - ax = ax$. So, it suffices to find the linear coefficient of $c(x-2^0)(x-2^1)\cdots(x-2^{2010})$.

For this, we can use Vieta's. We end up with

$$a = 2^S + 2^{S-1} + \cdots + 2^{S-2010}.$$

We can simplify this to $a = 2 - \frac{1}{2^{2010}}$.

**Problem 17** (5♣)

Let $a$, $b$, $c$ be integers with $c \neq 0$. Suppose the cubic polynomial $x^3 + ax^2 + bx + c$ has roots $r \leq s \leq t$. Show that $\frac{r}{s} + \frac{s}{t} + \frac{t}{r}$ can be written as $u \pm \sqrt{v}$ for some rational numbers $u$ and $v$.

NOTVIETA

**Problem 18** (Mock ARML 2022, added by Shaheem Samsudeen, 2♣)

Let $a, b, c$ complex numbers with $ab + bc + ca = 61$ such that

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} = 5$$
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 32.$$

Find the value of $abc$.

H3233958

**Required Problem 19** (Stanford Math Tournament 2013, 3♣)

Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$, $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Compute

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}.$$

13SMTA9

Putting the three terms over a common denominator and factoring the numerator, we can find that the expression equals

$$a^2 + b^2 + c^2 + ab + bc + ca.$$

We can rewrite this as $(a+b+c)^2 - (ab+bc+ca)$.

Let $x = \sqrt{3}$, $y = \sqrt{5}$, $z = \sqrt{7}$, and $S = a + b + c = x + y + z$. Then, our desired expression is

$$S^2 - [(S-2x)(S-2y) + (S-2y)(S-2z) + (S-2z)(S-2x)].$$

We can simplify this to get the answer of

$$2S^2 - 4(xy + yz + zx) = 30.$$

**Problem 20** (Ritwin Narra, 5♣)

Fix an integer $n \neq 1$. Prove that if real numbers $a$, $b$, $c$, $d$ satisfy

$$a + b + c + d = a^n + b^n + c^n + d^n = 0$$

then two of $a$, $b$, $c$, $d$ sum to 0.

Z5394300

**Required Problem 21** (Black MOP 2012, 5♣)

Let $ABC$ be a triangle and let $h_A$, $h_B$, $h_C$ be the lengths of the altitudes from $A$, $B$, and $C$. Let $a = BC$, $b = CA$, $c = AB$. Suppose that

$$\sqrt{a+h_B} + \sqrt{b+h_C} + \sqrt{c+h_A} = \sqrt{a+h_C} + \sqrt{b+h_A} + \sqrt{c+h_B}.$$

Prove that triangle $ABC$ is isosceles.

12BLACKMOP

Let $\sqrt{a+h_B}$, $\sqrt{b+h_C}$, and $\sqrt{c+h_A}$ be the roots of a polynomial.

Then, we claim this polynomial also has roots $\sqrt{a+h_C}$, $\sqrt{b+h_A}$, and $\sqrt{c+h_B}$. This can be shown with Vieta's and Newton sums, along with the fact that

$$(a+h_B)(b+h_C)(c+h_A) = (a+h_C)(b+h_A)(c+h_B),$$

which can be shown by expanding and simplifying using the triangle area formula.

Thus, we have three cases:

1. $\sqrt{a+h_B} = \sqrt{a+h_C}$. Let $A$ be the area of the triangle. Then it is obvious that $b = c$.

2. $\sqrt{a+h_B} = \sqrt{b+h_A}$. We can derive that $a = b$ or $ab = -1$, the latter of which is impossible.

3. $\sqrt{a+h_B} = \sqrt{c+h_B}$. Obviously $a = c$.

In any case, the triangle is isosceles.

**Problem 22** (Added by Jason Lee, 5♣)

Consider all complex numbers $k$ for which there exist complex numbers $a$, $b$, $c$, $d$ and $e$ satisfying

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} &= 1 \\ \frac{b}{c} + \frac{c}{d} &= 2 \\ \frac{c}{d} + \frac{d}{e} &= 3 \\ \frac{d}{e} + \frac{e}{a} &= 4 \\ \frac{e}{a} + \frac{a}{b} &= k. \end{aligned}$$

Find the sum of all possible values of $k^4$.

H2954369

> **Remark.** For the previous problem by Jason Lee, avoid using a calculator — you can solve it with rather little arithmetic if you set it up correctly.

**Problem 23** (Prove the Newton sums in the reading, 5♣)
Suppose the complex-coefficient polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has complex roots $z_1, z_2, \ldots, z_n$. For each $d \geq 0$, define $p_d = z_1^d + z_2^d + \cdots + z_n^d$. Prove that for every integer $k \geq 1$ we have the identity

$$a_n p_k + a_{n-1}p_{k-1} + \cdots + a_{n-(k-1)}p_1 + k \cdot a_{n-k} = 0.$$

where we by convention let $a_i = 0$ if $i < 0$.

ZC01F573

**Required Problem 24** (Longlist 1985/19, 9♣)
Solve over $\mathbb{R}$ the system of simultaneous equations

$$\begin{aligned}
\sqrt{x} - \frac{1}{y} - 2w + 3z &= 1, \\
x + \frac{1}{y^2} - 4w^2 - 9z^2 &= 3, \\
x\sqrt{x} - \frac{1}{y^3} - 8w^3 + 27z^3 &= -5, \\
x^2 + \frac{1}{y^4} - 16w^4 - 81z^4 &= 15.
\end{aligned}$$

85LL19

After the obvious substitution we have

$$\begin{aligned}
a + b - c - d &= 1 \\
a^2 + b^2 - c^2 - d^2 &= 3 \\
a^3 + b^3 - c^3 - d^3 &= -5 \\
a^4 + b^4 - c^4 - d^4 &= 15.
\end{aligned}$$

Guessing small integer solutions, we find that $(a, b, c, d) = (1, -2, -1, -1)$ is a solution. We claim that it is the only solution (up to swapping $a$ and $b$).

Notice that we can write

$$a^n + b^n + (-1)^n + (-1)^n = c^n + d^n + (-2)^n + 1^n$$

for $n = 1, 2, 3, 4$. This means that the Newton sums of the polynomials with roots $\{a, b, -1, -1\}$ and $\{c, d, -2, 1\}$ are the same. This uniquely determines the polynomial (up to leading coefficient). Thus, the multisets of roots must be equal, and we are done.

Finally, our original substitution requires $a$ to be positive, so the only solution is

$$(x, y, w, z) = \left(1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}\right).$$

**Problem 25** ($\overline{\mathbb{Z}}$ is a ring, 9♣)

Suppose that $\alpha$ and $\beta$ are complex numbers and monic polynomials $P, Q \in \mathbb{Z}[x]$ satisfy $P(\alpha) = Q(\beta) = 0$.

(a) Show that there is monic polynomial $R \in \mathbb{Z}[x]$ such that $R(\alpha + \beta) = 0$.

(b) Show that there is a monic polynomial $S \in \mathbb{Z}[x]$ such that $S(\alpha\beta) = 0$.

Z40B5559

**Problem 26** (HMMT 2020, added by Guanjie Lu, 9♣)

Let $P(x) = x^{2020} + x + 2$. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let $\alpha$ satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

20HMMTA9