Poisson Approximation and the Chen-Stein Method

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Introduction

Poisson Law of Small Numbers

Let W $\sim$ Bin$(n, \lambda/n)$, $\quad \lambda > 0$, and let $Z \sim$ Poi(λ).
Then, as $n \to \infty$:

$$\mathbb{P}(W = k) \xrightarrow{d} e^{-\lambda}\frac{\lambda^k}{k!} = \mathbb{P}(Z = k), \quad k \in \mathbb{Z}^+.$$

In other words,

$d_{TV}(W, Z) \to 0, \quad$ where $\quad d_{TV}(W, Z) = \sup_{A \subseteq \mathbb{Z}^+} |W(A) - Z(A)|$

Introduction

Questions

- Relax assumption of *independence*?
- Relax assumption of *identically distributed*?
- How good is the Poisson approximation?

Chen-Stein operator

$$A_\lambda g(x) := \lambda g(x+1) - xg(x),$$

for every bounded function $g : \mathbb{Z}^+ \to \mathbb{R}$

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- To show that W is close to Z, we have to check

$$\mathbb{E}[A_\lambda g(W)] \to 0.$$

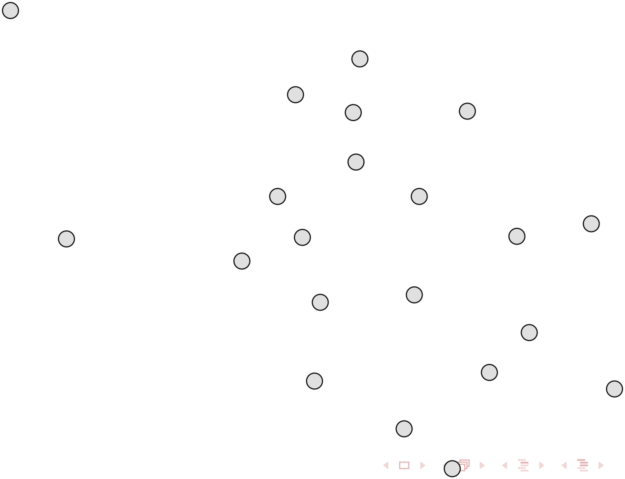
General setting

Let $X_\alpha, \alpha \in I$, with I a countable index set and:

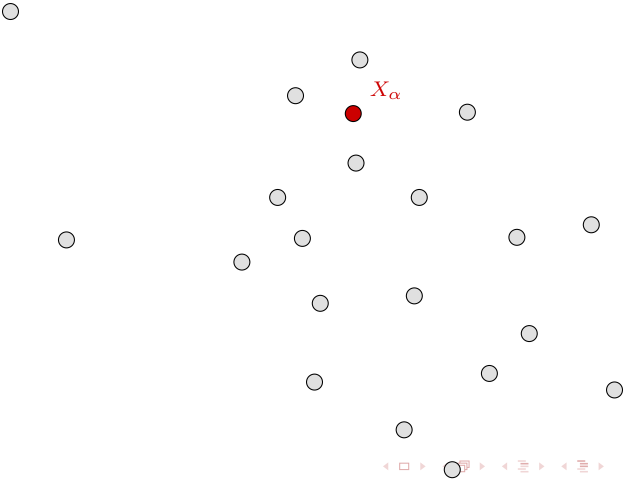
$$\mathbb{P}(X_\alpha = 1) = 1 - \mathbb{P}(X_\alpha = 0) = p_\alpha.$$

Define $W := \sum_{\alpha \in I} X_\alpha$, with $\lambda := \mathbb{E}[W]$,
and a neighbourhood B_α

Neighbourhood B_α



Neighbourhood B_α



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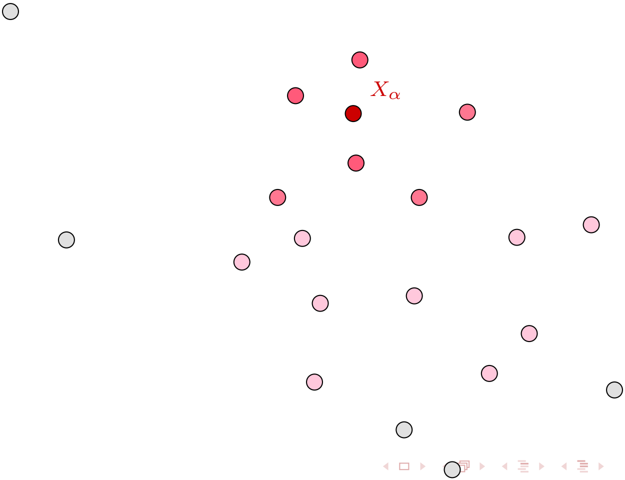

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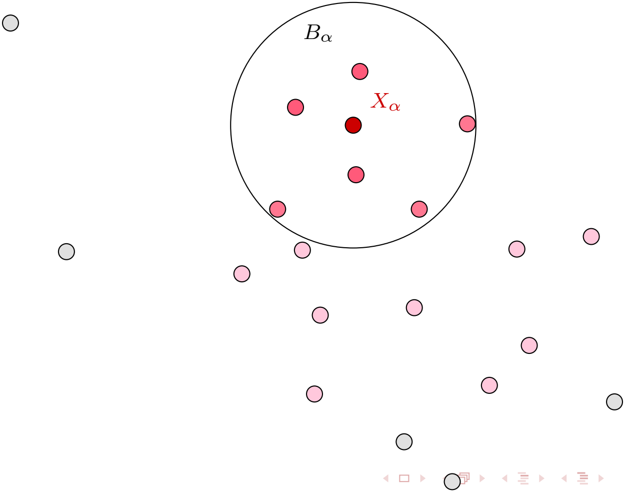

General setting

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta, \tag{1}$$

$$b_2 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha \setminus \{\alpha\}} \mathbb{E}[X_\alpha X_\beta], \tag{2}$$

$$b_3 = \sum_{\alpha \in I} \mathbb{E}\Big[\mathbb{E}[X_\alpha - p_\alpha | \sigma(X_\beta : \beta \notin B_\alpha)] \Big]. \tag{3}$$

Chen-Stein bound

Theorem

Let $W = \sum_\alpha X_\alpha$, *with* $\lambda = \mathbb{E}[W] < \infty$ *and let* $Z \sim Pois(\lambda)$. *Then:*

$$||\mathcal{L}(W) - \mathcal{L}(Z)||_{TV} \leq 2\left[(b_1 + b_2)\frac{1 - e^{-\lambda}}{\lambda} + b_3\left(1 \wedge \frac{1.4}{\sqrt{\lambda}}\right)\right],$$

and

$$|\mathbb{P}(W = 0) - e^{-\lambda}| \leq (b_1 + b_2 + b_3)\frac{1 - e^{-\lambda}}{\lambda}.$$


HAPPY BIRTHDAY

We have n people in the room and we are looking for a k-way coincidence.

Assume d days in the year, and a uniform distribution for birthdays throughout the year (i.e. the probability of being born on any given day is $\frac{1}{d}$).

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- Then W, the total number of coincidences, is given by $W = \sum_{\alpha \in I} X_\alpha$,
- And $\lambda = \mathbb{E}[W] = \sum_{\alpha \in I} d^{1-k} = \binom{n}{k} d^{1-k}$.

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- Approximate W with a Poisson random variable, Z, with mean λ.

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The approximation is always conservative when birthdays are uniform.

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- With this choice,

$$b_3 = \sum_{\alpha \in I} \mathbb{E}[|\mathbb{E}[X_\alpha - p_\alpha]|\sigma(X_\beta : \beta \notin B_\alpha)|] = 0.$$

$$
\begin{aligned}
b_1 &= \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta \\
&= \binom{n}{k} \left\{ \binom{n}{k} - \binom{n-k}{k} \right\} d^{2-2k}
\end{aligned}
$$

$$b_2 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha \setminus \{\alpha\}} \mathbb{E}[X_\alpha X_\beta]$$

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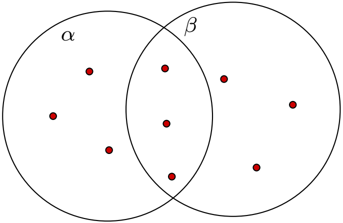
$$\mathbb{E}[X_\alpha X_\beta] = \mathbb{P}[X_\alpha = 1, X_\beta = 1]$$
$$= \mathbb{P}[\text{all people indexed by } \alpha \cup \beta \text{ share same bday}]$$



Bounds as n increases, for fixed d

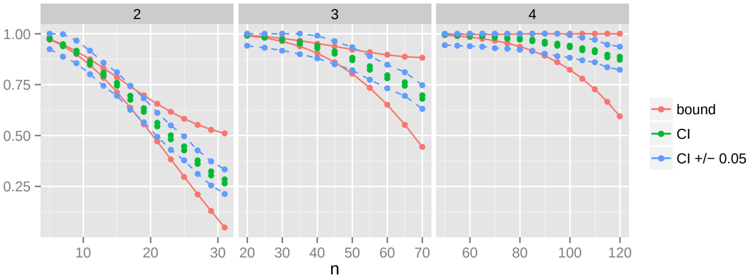
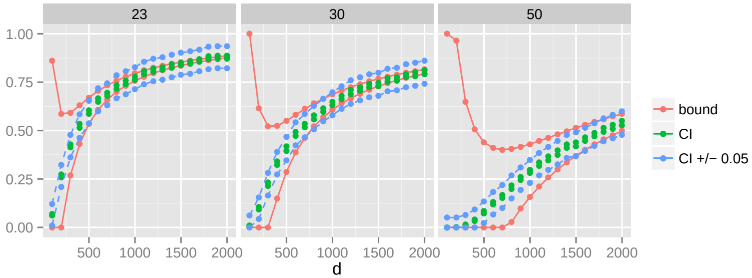


Figure: Simulations for $\mathbb{P}(W = 0)$, compared to the bounds given by the Chen-Stein method. The bounds are good when they (the red lines) are inside the blue lines, i.e. no more that 0.05 away from the simulated values. The bounds widen as n increases, for fixed $d = 365$, for each of $k = 2, 3, 4$.

Bounds as d increases, for fixed n



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- We fixed the ratio $\frac{n^k}{d^{1-k}}$ at 1.45 (the value it takes in the classic case).
- The order of the Chen-Stein bound here is the same as the order of b_2, which is

$$n^{k+1} d^{-k} \asymp n^{-1/(k-1)}.$$

Thus the Chen-Stein method yields that the total variation distance decays at a rate no slower than $O(n^{-1/(k-1)})$

Bounds as $n, d \to \infty$

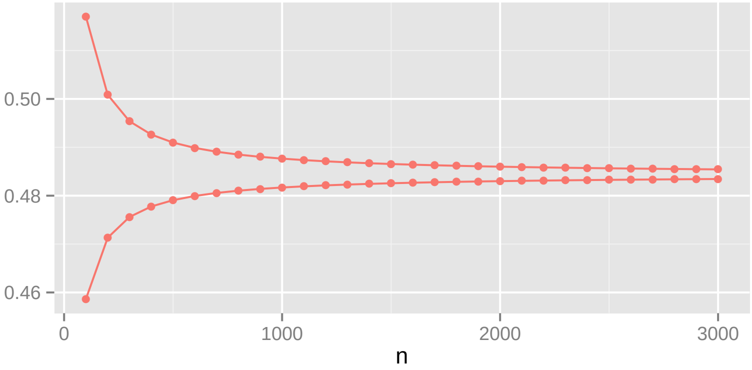

Figure: The Chen-Stein bounds on the distance between $\mathbb{P}(W = 0)$ and $\mathbb{P}(Z = 0)$ as n increases, keeping the ratio $\frac{n^k}{d^{1-k}}$ constant at 1.45

The Length of the Longest Head Run

Consider

- n independent coin tosses $C_i \sim \text{Ber}(p)$
- test length t

Goal: What's the probability of having a run of *at least* t heads in the sequence of tosses?

Runs could occur in **clumps**:

- [0, 1, 0, 1, 1, 1, 1, 0, 1]
- [0, 1, 0, 1, 1, 1, 1, 0, 1]

We count only the first sequence of t heads.

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Therefore

$$W = \sum_{\alpha \in I} X_\alpha \approx \mathrm{Poi}(\lambda) \quad \text{where } \lambda = \mathbb{E}(W).$$

$$\lambda = p^t[(1-p)(n-1)+1)].$$

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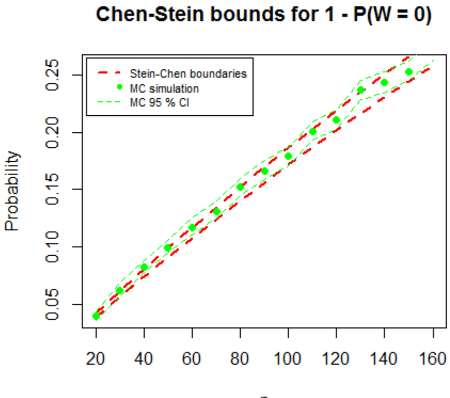
- $b_1 \; < \; \lambda^2(2t+1)/n + 2\lambda p^t$

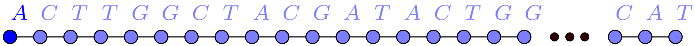
The Length of the Longest Head Run

Example: Consider a sequence $n = 110$ coin tosses, $p = 0.5$.

"What's the probability of obtaining a run of at least $t = 8$ heads?"



DNA example



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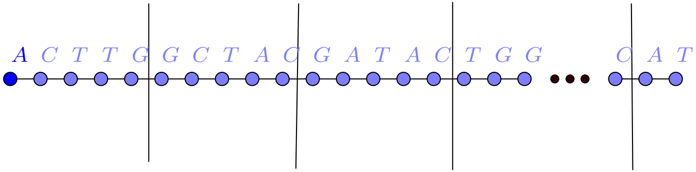

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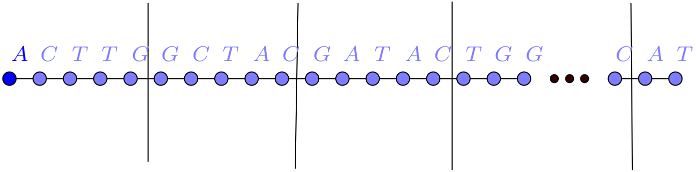
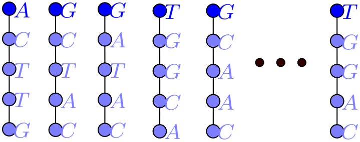

A C T T G G C T A C G A T A C T G G C A T


A C T T G
G C T A C
G A T A C
T G G C A
G C A A C
T G G A C

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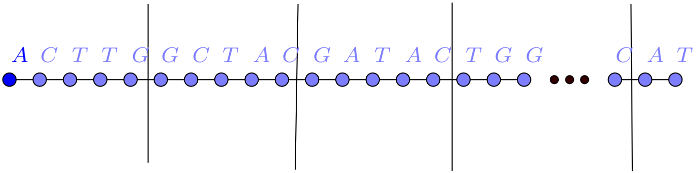
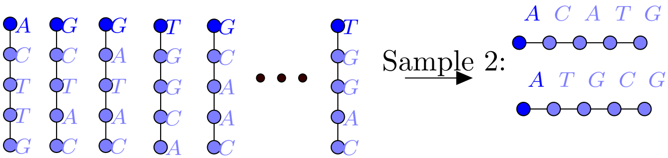
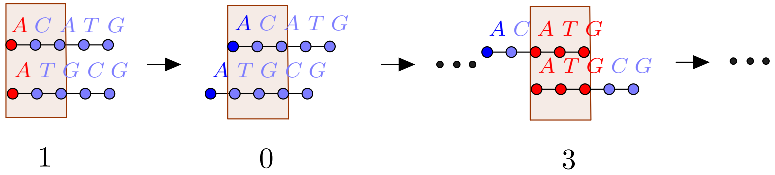

A C T T G G C T A C G A T A C T G G C A T


A C T T G
G C T A C
G A T A C
T G G C A
G C A A C
T G G A C
Sample 2:
A C A T G
A T G C G

DNA example



DNA example

We used a complete chloroplast genome of *Marchantia Polymorpha* (Liverwort), downloaded from GenBank. It consists of one sequence of 121,024 letters.

DNA example - algorithm

Data: Cut the sequence into 236 disjoint *stripes*, each consisting of 512 letters. Discard remaining letters.

w.size = 21; max.matches = vector[200];

for $i = 1 : 200$ **do**

  draw 2 *stripes* at random, store as str.A and str.B;

  current.max = 0;

  for *each possible placement of window of length w.size on str.A and str.B* **do**

    current.count = number of matches within window;

    if *current.count > current.max* **then**

      update current.max;

    end

  end

  max.matches[i] = current.max

end

DNA example

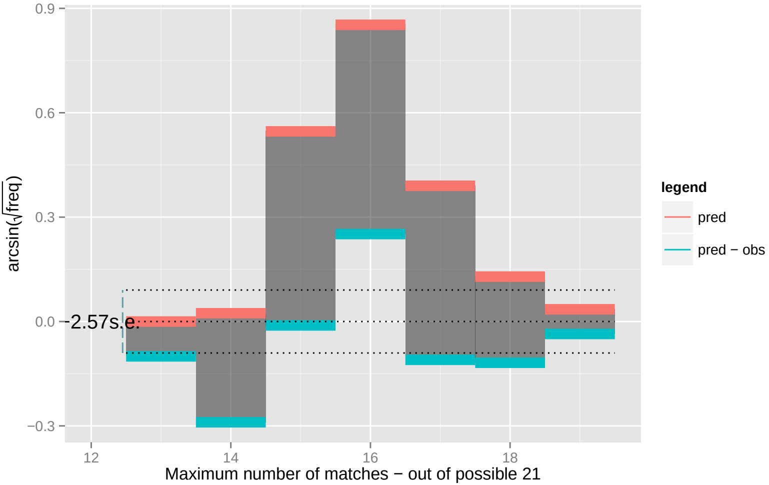
Suppose that $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are two *stripes*, where $A_i, B_i \in \{a, c, t, g\}$, chosen at random according to common distribution μ. Define:

$$M_n(t) = \max_{1 \le i,j \le n-t+1} \sum_{k=0}^{t-1} \mathbb{1}_{A_{i+k}=B_{j+k}}, \tag{4}$$

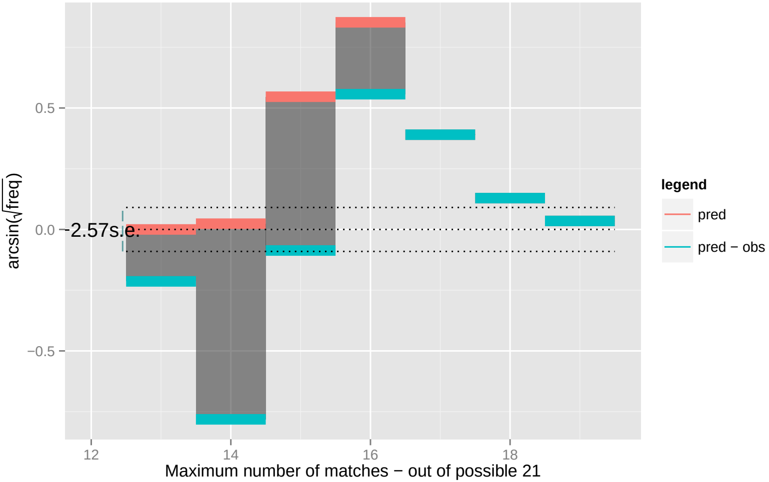
Then, under some regularity conditions:

$$\mathbb{P}[M_n(t) < s] - e^{-n(\frac{s}{t}-p)\mathbb{P}[Bin(t,p) \ge s]} \to 0. \tag{5}$$

DNA example


0.9
0.6
0.3
0.0
−0.3
arcsin(√freq)
−2.57s.e.
12
14
16
18
Maximum number of matches − out of possible 21
legend
pred
pred − obs

DNA example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