Spin-averated $\Lambda_c^{**+} \to \Lambda_c^+ \pi^+ \pi^-$ distribution

The notebook computes the **spin-averated** angular functions for the three body decays of the exited Λ_c^{**+} states under different hypotheses of its quantum numbers.

The spin factors are evaluated using conventions of the three-body decays introduced in the Dalitz plot decomposition paper.

The implementation in Julia comes from the ThreeBodyDecay.jl package. A dispatch on SymPy object is used to provide result in a nice symbolic form. The dispatch is defined in SymbolicThreeBodyDecay.jl package.

The spin sums are evaluated symbolically. The resulting decay intensity distribution is a bilinear form on isobars lineshapes.

$$I = |R^1|^2\, T_{1,1}(\theta,\zeta) + |R^2|^2\, T_{2,2}(\theta,\zeta) + 2\mathrm{Re}(R^{1*} R^2)\, T_{1,2}(\theta,\zeta)$$

where I is a differential decay rate (density of the events on the Dalitz plot), R^i is the parametrization of the lineshape of an intermediate resonance i, and θ, ζ are kimanatic angles that are unambiguesly expressed using the Mandelstam invariants.

- The LS couplings are used for parametrizing the decay vertices. The minimal orbital angular momentum is chosen is several value are possible.
- The matrix T gives only the angular factors, the break-up momentum for $p^L q^l$ is attributed to the lineshape function R, with L being the orbital angular momentum in the decay of Λ_c^{**+}, and l for the decay of an intermediate resonance.
- We incorporate **six** subchannel resonances: f_0, and ρ in $\pi\pi$, as well as $\Sigma_c(2455)$ and $\Sigma_c(2520)$ for both $\Lambda_c\pi$ channels
- Value of l is unambigues in all channels, $l = 0$ for $f_0 \to \pi\pi$, and $l = 1$ for all other resonances.
- The value of L in the decay of Λ_c^{**+} is indicated in the lineshape function:

$$R_{L,S}^{\text{resonance}}$$

How to related $R^{\Sigma(2455)++}$ and $R^{\Sigma(2455)0}$?

The isospin Clebsch in the Λ_c^{**+} decay is the same

$$\begin{aligned}
\langle \Sigma(2245)^{++}; \pi^- | \Lambda_c^{**+} \rangle = \langle 1, 1; 1, -1 | 0, 0 \rangle = \frac{1}{\sqrt{3}}\,, \\
\langle \Sigma(2245)^{0}; \pi^+ | \Lambda_c^{**+} \rangle = \langle 1, -1; 1, 1 | 0, 0 \rangle = \frac{1}{\sqrt{3}}\,,
\end{aligned}$$

There is a difference treatment of the Σ_c decays.

For $\Sigma_c^{++}(12)$ decaying to $\Lambda_c^+(1)\pi^+(2)$ in P-wave, the helicity coupling is related to the LS coupling with Clebshes which are the same

$$\begin{aligned}
H_{\nu;\nu,0}^{\Sigma_c^{++}\to\Lambda_c^+\pi^+} &= \langle 1, 0; 1/2, \nu | 1/2, \nu \rangle \, \langle 1/2, \nu; 0, 0 | 1/2, \nu \rangle \, H_{P-\text{wave}}^{\Sigma_c^{++}\to\Lambda_c^+\pi^+}\,, \\
H_{\nu;0,\nu}^{\Sigma_c^{0}\to\pi^-\Lambda_c^+} &= \langle 1, 0; 1/2, \nu | 1/2, \nu \rangle \, \langle 0, 0; 1/2, \nu | 1/2, \nu \rangle \, H_{P-\text{wave}}^{\Sigma_c^{0}\to\pi^-,\Lambda_c^+}
\end{aligned}$$

The symmetry of the LS coupling under permultation is the simplest,

$$H_{L-\text{wave}}^{\Sigma_c\to\Lambda_c\pi} = (-1)^L H_{L-\text{wave}}^{\Sigma_c\to\pi\Lambda_c}$$

The symmetry factor is derived in e.g Martin-Spearman book in Eq.(5.30) as $(-1)^{L+S-j_1-j_2}$. The relation gives the way to relate the lineshape functions of Σ_c^0 and Σ_c^{++}:

$$R^{\Sigma_c^0} = -R^{\Sigma_c^{++}}\,.$$

It holds for both $\Sigma_c(2455)$ and $\Sigma_c(2520)$.

```
begin
    cd(mktempdir())
    import Pkg
    Pkg.activate(".")
    Pkg.add([
        Pkg.PackageSpec(url="https://github.com/mmikhasenko/ThreeBodyDecay.jl"),
        Pkg.PackageSpec(url="https://github.com/mmikhasenko/SymbolicThreeBodyDecays.jl"),
        Pkg.PackageSpec("Plots"), Pkg.PackageSpec("Parameters"),
        Pkg.PackageSpec("JSON")
    ])
    #
    using JSON
    using Parameters
    using LinearAlgebra
    using ThreeBodyDecay
    using SymbolicThreeBodyDecays
    using SymbolicThreeBodyDecays.SymPy
    #
    using Plots
    import Plots.PlotMeasures: mm
end
```

```
theme(:wong2, frame=:box, grid=false, minorticks=true,
    guidefontvalign=:top, guidefonthalign=:right,
    xlim=(:auto, :auto), ylim=(:auto, :auto),
    lw=1.2, lab="", colorbar=false,
    bottom_margin=3mm, left_margin=3mm)
```

Integration with Symbolic Computation

```
begin
    @syms m1 m2 m3 m0
    @syms σ1 σ2 σ3
end;
```

`const ms =` (m1 = m_1, m2 = m_2, m3 = m_3, m0 = m_0)

```
const ms = ThreeBodyMasses(m1, m2, m3; m0)
```

`const σs =` (σ1 = σ_1, σ2 = σ_2, σ3 = $m_0^2 + m_1^2 + m_2^2 + m_3^2 - \sigma_1 - \sigma_2$)

```
const σs = Invariants(ms; σ1, σ2)
```

spinparity (generic function with 1 method)

```julia
function spinparity(p)
    pt = (p[2]..., p[1])
    jpv = str2jp.(pt)
    getproperty.(jpv, :j) .|> x2 |> ThreeBodyDecay.SpinTuple,
    getproperty.(jpv, :p) |> ThreeBodyDecay.ParityTuple
end
```

Setup

```julia
begin
    Base.@kwdef struct Setup{T}
        two_js
        decay_chains::Vector{T}
    end
    function Setup(chains::Vector{Tuple{String, String, Int64}}, ifstate)
        js, Ps = ifstate |> spinparity
        tbs = ThreeBodySystem(ms, js)
        #
        dcv = map(chains) do chain
            nameR, jpR, k = chain
            Rjp = str2jp(jpR)
            #
            d0 = DecayChainLS(k, identity; two_s=Rjp.j |> x2, parity=Rjp.p, Ps, tbs)
            two_L, two_S = d0.HRk.two_ls
            L = div(two_L, 2)
            symR = sympy.Symbol(nameR * "_{$(L), $(two_S)/2}", real=true)
            d = DecayChainLS(k, σ->symR; two_s=Rjp.j |> x2, parity=Rjp.p, Ps, tbs)
        end
        return Setup(tbs.two_js, dcv)
    end
end
```

unpolarized_intensity (generic function with 1 method)

```julia
function unpolarized_intensity(setup; refζs=(1, 1, 1, 1))
    #
    @unpack two_js, decay_chains = setup
    full_amplitude = sum(itr(two_js)) do two_λs
        A = sum(decay_chains) do dc
            amplitude(dc, σs |> StickySymTuple, two_λs .|> Sym; refζs)
        end
        abs2(A)
    end
    full_amplitude
end
```

intensity_matrix (generic function with 1 method)

```julia
function intensity_matrix(setup)
    I_symm = unpolarized_intensity(setup; refζs=(1, 1, 1, 1));
    lineshapes = map(d->d.Xlineshape(nothing), setup.decay_chains)
    H = sympy.hessian(I_symm, lineshapes)
    H_fully_simp = map(eachindex(Symmetric(H))) do i
        i[1]>i[2] ? Sym(0) :
        simplify(H[i].doit())
    end |> Symmetric;
    H_normalized = H_fully_simp.
            xreplace(sign_replacements |> Dict).
            xreplace(combine_replacements |> Dict)#.
            # xreplace(unsafe_uncombine_replacement[2:3] |> Dict);
    return H_normalized, lineshapes
end
```

Nice printing

latexalign (generic function with 1 method)

```julia
latexalign(H::Matrix) =
"""
```math
\\small
\\begin{align}
""" *
    prod(map(eachindex(Symmetric(H))) do i
        i[1]!=i[2] ? "" :
        "T_{$(i[1]),$(i[2])} &= " * sympy.latex(H[i[1],i[2]]) * "\\,,\\\\"
    end) *
    prod(map(eachindex(Symmetric(H))) do i
        i[1]<i[2] ?
        "T_{$(i[1]),$(i[2])} &= " * sympy.latex(H[i[1],i[2]]) * "\\,,\\\\" : ""
    end) *
    prod(map(eachindex(Symmetric(H))) do i
        i[1]>i[2] ? "T_{$(i[1]),$(i[2])} &= T_{$(i[2]),$(i[1])}" * "\\,,\\\\" : ""
    end) *
"""
\\end{align}
```
"""
```

latexintensity (generic function with 2 methods)

```
latexintensity(V::Vector, Isubscript="") =
"""
```math
\\small
\\begin{equation}
{\\Large I_{$(Isubscript)}}  =
\\begin{pmatrix}
""" *
    sympy.latex(V[1]) *
    prod(map(V[2:end]) do v
        " \\\\ " * sympy.latex(v)
    end) *
"""
\\end{pmatrix}^{\\Large\\dagger}
\\,\\, {\\Huge [T]} \\,\\,
\\begin{pmatrix}
""" *
    sympy.latex(V[1]) *
    prod(map(V[2:end]) do v
        " \\\\ " * sympy.latex(v)
    end) *
"""
\\end{pmatrix}
\\end{equation}
```
"""
```

printalign (generic function with 1 method)

```
printalign(H::Matrix) = Markdown.parse(latexalign(H))
```

Application

final_state = ("1/2+", "0-", "0-")

```
final_state = ("1/2+", "0-", "0-")
```

reaction = "3/2-" ⇒ ("1/2+", "0-", "0-")

```
reaction = "3/2-" => final_state
```

```
chains =
  [("R^{\\rho(770)0}", "1-", 1), ("R^{f_0}", "0+", 1), ("R^{\\Sigma(2455)0}", "1/2+", 2), ("
```

```julia
chains = [
    ("R^{\\rho(770)0}", "1-", 1), #pi+pi-
    ("R^{f_0}", "0+", 1), #pi+pi-
    #
    ("R^{\\Sigma(2455)0}", "1/2+", 2), #pi-Lc+
    ("R^{\\Sigma(2520)0}", "3/2+", 2), #pi-Lc+
    #
    ("R^{\\Sigma(2455)++}", "1/2+", 3), #Lc+pi+
    ("R^{\\Sigma(2520)++}", "3/2+", 3), #Lc+pi+
]
```

1/2- decay

1P states: 1/2-, 3/2-

```
s1h⁻ =
  Setup((two_h1 = 1, two_h2 = 0, two_h3 = 0, two_h0 = 1), [DecayChain{var"#9#11"{Sym}, Reco
```

```julia
s1h⁻ = Setup(chains, "1/2-" => final_state)
```

```julia
@time T1h⁻ = intensity_matrix(s1h⁻);
```

```
84.740753 seconds (326.71 k allocations: 16.982 MiB, 0.34% compilation tim
e)
```

$$I_{1/2-} = \begin{pmatrix} R^{\rho(770)0}_{0,1/2} \\ R^{f_0}_{1,1/2} \\ R^{\Sigma(2455)0}_{0,1/2} \\ R^{\Sigma(2520)0}_{2,3/2} \\ R^{\Sigma(2455)++}_{0,1/2} \\ R^{\Sigma(2520)++}_{2,3/2} \end{pmatrix}^{\dagger} [T] \begin{pmatrix} R^{\rho(770)0}_{0,1/2} \\ R^{f_0}_{1,1/2} \\ R^{\Sigma(2455)0}_{0,1/2} \\ R^{\Sigma(2520)0}_{2,3/2} \\ R^{\Sigma(2455)++}_{0,1/2} \\ R^{\Sigma(2520)++}_{2,3/2} \end{pmatrix}$$

```julia
latexintensity(T1h⁻[2], "1/2-") |> Markdown.parse
```

$$T_{1,1} = 2\,,$$
$$T_{2,2} = 2\,,$$
$$T_{3,3} = 2\,,$$
$$T_{4,4} = 4 - 3\sin^2\left(\theta_{31}\right),$$
$$T_{5,5} = 2\,,$$
$$T_{6,6} = 4 - 3\sin^2\left(\theta_{12}\right),$$
$$T_{1,2} = 2\cos\left(\theta_{23}\right),$$
$$T_{1,3} = 2\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right),$$
$$T_{2,3} = 2\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right),$$
$$T_{1,4} = -\frac{\sqrt{2}\left(\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right) + 3\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} + \frac{3\theta_{31}}{2}\right)\right)}{2},$$
$$T_{2,4} = -\frac{\sqrt{2}\left(\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right) + 3\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \frac{3\theta_{31}}{2}\right)\right)}{2},$$
$$T_{3,4} = \sqrt{2}\cdot\left(3\sin^2\left(\theta_{31}\right) - 2\right),$$
$$T_{1,5} = -2\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \theta_{23}\right),$$
$$T_{2,5} = -2\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right),$$
$$T_{3,5} = -2\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right),$$
$$T_{4,5} = \frac{\sqrt{2}\left(\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right) + 3\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right)\right)}{2},$$
$$T_{1,6} = \frac{\sqrt{2}\left(-3\sin\left(-\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} + \theta_{23}\right) + \sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \theta_{23}\right)\right)}{2},$$
$$T_{2,6} = \frac{\sqrt{2}\cdot\left(3\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} - \frac{3\theta_{12}}{2}\right) + \sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right)\right)}{2},$$
$$T_{3,6} = -\frac{\sqrt{2}\left(-3\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} - \frac{3\theta_{12}}{2} - \frac{\theta_{31}}{2}\right) - \sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)\right)}{2},$$
$$T_{4,6} = -\frac{3\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} - \frac{3\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{4} - \frac{9\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} - \frac{3\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right)}{4} - \frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \right.}{}$$
$$T_{5,6} = \sqrt{2}\cdot\left(3\sin^2\left(\theta_{12}\right) - 2\right),$$
$$T_{2,1} = T_{1,2}\,,$$
$$T_{3,1} = T_{1,3}\,,$$
$$T_{4,1} = T_{1,4}\,,$$
$$T_{5,1} = T_{1,5}\,,$$
$$T_{6,1} = T_{1,6}\,,$$
$$T_{3,2} = T_{2,3}\,,$$
$$T_{4,2} = T_{2,4}\,,$$

$T_{4,2} = T_{2,4}$,
$T_{5,2} = T_{2,5}$,
$T_{6,2} = T_{2,6}$,
$T_{4,3} = T_{3,4}$,
$T_{5,3} = T_{3,5}$,
$T_{6,3} = T_{3,6}$,
$T_{5,4} = T_{4,5}$,
$T_{6,4} = T_{4,6}$,
$T_{6,5} = T_{5,6}$,

```
printalign(T1h⁻[1])
```

```
s3h⁻ =
  Setup((two_h1 = 1, two_h2 = 0, two_h3 = 0, two_h0 = 3), [DecayChain{var"#9#11"{Sym}, Reco
```

```
s3h⁻ = Setup(chains, "3/2-" => final_state)
```

```
@time T3h⁻ = intensity_matrix(s3h⁻);
```

```
309.633427 seconds (493.14 k allocations: 15.683 MiB)
```

$$I_{3/2-} = \begin{pmatrix} R^{\rho(770)0}_{0,3/2} \\ R^{f_0}_{1,1/2} \\ R^{\Sigma(2455)0}_{2,1/2} \\ R^{\Sigma(2520)0}_{0,3/2} \\ R^{\Sigma(2455)++}_{2,1/2} \\ R^{\Sigma(2520)++}_{0,3/2} \end{pmatrix}^{\dagger} [T] \begin{pmatrix} R^{\rho(770)0}_{0,3/2} \\ R^{f_0}_{1,1/2} \\ R^{\Sigma(2455)0}_{2,1/2} \\ R^{\Sigma(2520)0}_{0,3/2} \\ R^{\Sigma(2455)++}_{2,1/2} \\ R^{\Sigma(2520)++}_{0,3/2} \end{pmatrix}$$

```
latexintensity(T3h⁻[2], "3/2-") |> Markdown.parse
```

$$T_{1,1} = 2\,,$$
$$T_{2,2} = 2\,,$$
$$T_{3,3} = 2\,,$$
$$T_{4,4} = 2\,,$$
$$T_{5,5} = 2\,,$$
$$T_{6,6} = 2\,,$$
$$T_{1,2} = 2\cos\left(\theta_{23}\right),$$
$$T_{1,3} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\cos\left(\frac{3\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \theta_{23} + \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{2,3} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\cos\left(\frac{3\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{1,4} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\cos\left(\frac{3\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{3\theta_{31}}{2}\right)}{2}\,,$$
$$T_{2,4} = \frac{3\cos\left(-\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \frac{3\theta_{31}}{2}\right)}{2} + \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{3,4} = 2 - 3\sin^2\left(\theta_{31}\right),$$
$$T_{1,5} = -\frac{3\sin\left(-\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \theta_{23}\right)}{2} - \frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \theta_{23}\right)}{2}\,,$$
$$T_{2,5} = -\frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right)}{2}\,,$$
$$T_{3,5} = -\frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{4,5} = -\frac{3\sin\left(-\frac{3\zeta^0_{1(2)}}{2} - \frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right)}{2} - \frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{1,6} = -\frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \theta_{23}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} - \theta_{23}\right)}{2}\,,$$
$$T_{2,6} = -\frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2}\right)}{2}\,,$$
$$T_{3,6} = -\frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{4,6} = -\frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{2} + \frac{3\sin\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} - \frac{3\theta_{31}}{2}\right)}{2}\,,$$
$$T_{5,6} = 2 - 3\sin^2\left(\theta_{12}\right),$$
$$T_{2,1} = T_{1,2}\,,$$
$$T_{3,1} = T_{1,3}\,,$$
$$T_{4,1} = T_{1,4}\,,$$
$$T_{5,1} = T_{1,5}\,,$$
$$T_{6,1} = T_{1,6}\,,$$

$$T_{3,2} = T_{2,3}\,,$$
$$T_{4,2} = T_{2,4}\,,$$
$$T_{5,2} = T_{2,5}\,,$$
$$T_{6,2} = T_{2,6}\,,$$
$$T_{4,3} = T_{3,4}\,,$$
$$T_{5,3} = T_{3,5}\,,$$
$$T_{6,3} = T_{3,6}\,,$$
$$T_{5,4} = T_{4,5}\,,$$
$$T_{6,4} = T_{4,6}\,,$$
$$T_{6,5} = T_{5,6}\,,$$

```
printalign(T3h⁻[1])
```

1D states: 3/2+ 5/2+

```
s3h⁺ =
  Setup((two_h1 = 1, two_h2 = 0, two_h3 = 0, two_h0 = 3), [DecayChain{var"#9#11"{Sym}, Reco
```

```
s3h⁺ = Setup(chains, "3/2+" => final_state)
```

```
@time T3h⁺ = intensity_matrix(s3h⁺);
```

```
352.904732 seconds (493.14 k allocations: 15.683 MiB)
```

$$I_{3/2+} = \begin{pmatrix} R^{\rho(770)0}_{1,1/2} \\ R^{f_0}_{2,1/2} \\ R^{\Sigma(2455)0}_{1,1/2} \\ R^{\Sigma(2520)0}_{1,3/2} \\ R^{\Sigma(2455)++}_{1,1/2} \\ R^{\Sigma(2520)++}_{1,3/2} \end{pmatrix}^{\dagger} \left[T\right] \begin{pmatrix} R^{\rho(770)0}_{1,1/2} \\ R^{f_0}_{2,1/2} \\ R^{\Sigma(2455)0}_{1,1/2} \\ R^{\Sigma(2520)0}_{1,3/2} \\ R^{\Sigma(2455)++}_{1,1/2} \\ R^{\Sigma(2520)++}_{1,3/2} \end{pmatrix}$$

```
latexintensity(T3h⁺[2], "3/2+") |> Markdown.parse
```

$$T_{1,1} = 2\,,$$
$$T_{2,2} = 2\,,$$
$$T_{3,3} = 2\,,$$
$$T_{4,4} = \frac{12\sin^2\left(\theta_{31}\right)}{5} + \frac{2}{5}\,,$$
$$T_{5,5} = 2\,,$$
$$T_{6,6} = \frac{12\sin^2\left(\theta_{12}\right)}{5} + \frac{2}{5}\,,$$
$$T_{1,2} = 2\cos\left(\theta_{23}\right),$$
$$T_{1,3} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} - \theta_{23} + \frac{\theta_{31}}{2}\right)}{2} + \frac{3\cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{2,3} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right)}{2} + \frac{3\cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{1,4} = \frac{\sqrt{5}\cdot\left(12\cos\left(-\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} + \frac{3\theta_{31}}{2}\right) - 8\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} - \theta_{23} + \frac{\theta_{31}}{2}\right) + 12\cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} \cdots\right.\right.}{40}$$
$$T_{2,4} = \frac{\sqrt{5}\cdot\left(3\cos\left(-\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \frac{3\theta_{31}}{2}\right) - 2\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right) + 3\cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right)\right)}{10}\,,$$
$$T_{3,4} = \frac{\sqrt{5}\cdot\left(3\cos\left(2\theta_{31}\right) + 1\right)}{10}\,,$$
$$T_{1,5} = \frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \theta_{23}\right)}{2} - \frac{3\sin\left(\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \theta_{23}\right)}{2}\,,$$
$$T_{2,5} = \frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right)}{2} - \frac{3\sin\left(\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right)}{2}\,,$$
$$T_{3,5} = \frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{2} - \frac{3\sin\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{2}\,,$$
$$T_{4,5} = \frac{\sqrt{5}\left(-3\sin\left(-\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right) - 2\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right) - 3\sin\left(\cdots\right.\right.}{10}$$
$$T_{1,6} = \frac{\sqrt{5}\left(-8\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \theta_{23}\right) + 12\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} + \theta_{23}\right) - 12\sin\left(\cdots\right.\right.}{40}$$
$$T_{2,6} = \frac{\sqrt{5}\left(-2\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right) + 3\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2}\right) - 3\sin\left(\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \cdots\right.\right.}{10}$$
$$T_{3,6} = \frac{\sqrt{5}\left(-2\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right) + 3\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} + \frac{\theta_{31}}{2}\right) - 3\sin\left(\frac{3\zeta^0_{1(\cdots}}{2}\cdots\right.\right.}{10}$$
$$T_{4,6} = \frac{3\sin\left(-\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right)}{5} + \frac{2\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{5} + \frac{9\sin\left(\frac{\zeta^0_{1(2)}}{2} + \cdots\right.}{\cdots}$$
$$T_{5,6} = \frac{\sqrt{5}\cdot\left(3\cos\left(2\theta_{12}\right) + 1\right)}{10}\,,$$
$$T_{2,1} = T_{1,2}\,,$$
$$T_{3,1} = T_{1,3}\,,$$

```
printalign(T3h⁺[1])
```

$T_{5,1} = T_{1,5}$,

```
s5h⁺ =
  Setup((two_h1 = 1, two_h2 = 0, two_h3 = 0, two_h0 = 5), [DecayChain{var"#9#11"{Sym}, Reco
```

```
s5h⁺ = Setup(chains, "5/2+" => final_state)
```

$T_{6,2} = T_{2,6}$,

```
@time T5h⁺ = intensity_matrix(s5h⁺);
```

```
702.026139 seconds (1.04 M allocations: 33.077 MiB)
```

$T_{6,4} = T_{4,6}$,

$$I_{5/2+} = \begin{pmatrix} R^{\rho(770)0}_{1,3/2} \\ R^{f_0}_{2,1/2} \\ R^{\Sigma(2455)0}_{3,1/2} \\ R^{\Sigma(2520)0}_{1,3/2} \\ R^{\Sigma(2455)++}_{3,1/2} \\ R^{\Sigma(2520)++}_{1,3/2} \end{pmatrix}^{\dagger} [T] \begin{pmatrix} R^{\rho(770)0}_{1,3/2} \\ R^{f_0}_{2,1/2} \\ R^{\Sigma(2455)0}_{3,1/2} \\ R^{\Sigma(2520)0}_{1,3/2} \\ R^{\Sigma(2455)++}_{3,1/2} \\ R^{\Sigma(2520)++}_{1,3/2} \end{pmatrix}$$

```
latexintensity(T5h⁺[2], "5/2+") |> Markdown.parse
```

$$T_{1,1} = \frac{12}{5} - \frac{3\sin^2(\theta_{23})}{5},$$
$$T_{2,2} = 2,$$
$$T_{3,3} = 2,$$
$$T_{4,4} = \frac{3\cos(2\theta_{31})}{10} + \frac{21}{10},$$
$$T_{5,5} = 2,$$
$$T_{6,6} = \frac{3\cos(2\theta_{12})}{10} + \frac{21}{10},$$
$$T_{1,2} = \frac{2\sqrt{30}\cos(\theta_{23})}{5},$$
$$T_{1,3} = \frac{\sqrt{30}\left(\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} - \theta_{23} + \frac{\theta_{31}}{2}\right) + \cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \theta_{23} + \frac{\theta_{31}}{2}\right) + \cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \frac{\theta_{31}}{2}\right.}{20}$$
$$T_{2,3} = \frac{\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right)}{2} + \frac{\cos\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\theta_{31}}{2}\right)}{4} + \frac{5\cos\left(\frac{5\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right)}{4},$$
$$T_{1,4} = \frac{3\cos\left(-\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \theta_{23} + \frac{3\theta_{31}}{2}\right)}{20} + \frac{3\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} - \theta_{23} + \frac{\theta_{31}}{2}\right)}{20} + \frac{3\cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \theta_{23} - \right.}{20}$$
$$T_{2,4} = \frac{\sqrt{30}\cdot\left(5\cos\left(-\frac{5\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \frac{3\theta_{31}}{2}\right) + \cos\left(-\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^1_{2(1)}}{2} + \frac{3\theta_{31}}{2}\right) + \cos\left(\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\theta_{31}}{2}\right) + \cos\right.}{20}$$
$$T_{3,4} = \frac{\sqrt{30}\cdot(6\cos(2\theta_{31}) + 2)}{20},$$
$$T_{1,5} = \sqrt{30}\left(\frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right)\cos(\theta_{23})}{10} - \frac{\sin\left(-\frac{5\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \theta_{23}\right)}{4} - \frac{\sin\left(\right.}{}\right.$$
$$T_{2,5} = \frac{\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right)}{2} - \frac{\sin\left(\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2}\right)}{4} + \frac{5\sin\left(\frac{5\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \right.}{4}$$
$$T_{3,5} = \frac{\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{2} - \frac{\sin\left(\frac{3\zeta^0_{1(2)}}{2} + \frac{3\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} - \frac{\theta_{31}}{2}\right)}{4} + \frac{5\sin\left(\frac{5\zeta^0_{1(2)}}{2} + \right.}{}$$
$$T_{4,5} = \frac{\sqrt{30}\left(-5\sin\left(-\frac{5\zeta^0_{1(2)}}{2} - \frac{5\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right) - \sin\left(-\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right) + \sin\right.}{20}$$
$$T_{1,6} = -\frac{3\sin\left(-\frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \theta_{23}\right)}{20} + \frac{3\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \theta_{23}\right)}{20} + \frac{3\sin\left(\frac{\zeta^0_{3(1)}}{2} + \right.}{}$$
$$T_{2,6} = \frac{\sqrt{30}\left(\sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2}\right) + \sin\left(\frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2}\right) - \sin\left(\frac{3\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \right.\right.}{20}$$
$$T_{3,6} = \frac{\sqrt{30}\left(\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right) + \sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} + \frac{3\theta_{12}}{2} + \frac{\theta_{31}}{2}\right) - \sin\left(\frac{3\zeta^0_{1(2)}}{2} + \right.\right.}{20}$$
$$T_{4,6} = -\frac{3\sin\left(-\frac{\zeta^0_{1(2)}}{2} - \frac{\zeta^0_{3(1)}}{2} + \frac{\zeta^1_{2(3)}}{2} + \frac{\theta_{12}}{2} + \frac{3\theta_{31}}{2}\right)}{20} + \frac{3\sin\left(\frac{\zeta^0_{1(2)}}{2} + \frac{\zeta^0_{3(1)}}{2} - \frac{\zeta^1_{2(3)}}{2} - \frac{\theta_{12}}{2} + \frac{\theta_{31}}{2}\right)}{20} + \frac{3\sin\left(\frac{\zeta^0_{1(2)}}{2}\right.}{}$$
$$\sqrt{30}\cdot(6\cos(2\theta_{12}) + 2)$$

```
printalign(T5h⁺[1])
```

$T_{2,1} = T_{1,2}$,

Save to file

$T_{6,1} = T_{1,6}$,

```
function wrap2dict(TV)
    T,V = TV
    d_sympy = Dict()
    d_cc = Dict()
    for i in 1:size(T,1), j in 1:size(T,2)
        d_sympy["T_$i,$j"] = sympy.srepr(T[i,j])
        d_cc["T_$i,$j"] = sympy.ccode(T[i,j])
    end
    Dict(
        "lineshapes" => sympy.srepr.(V),
        "intensity_latex" => latexintensity(V),
        "spin_averaged_crossings" => Dict(
            "latex" => latexalign(T),
            "sympy" => d_sympy,
            "ccode" => d_cc))
end;
```

```
function writejson(path, obj)
    open(path, "w") do io
        JSON.print(io, obj, 4)
    end
end;
```

```
computation_result =
  Dict("3/2+" ⇒ Dict("intensity_latex" ⇒ "```math\n\\small\n\\begin{equation}\n{\\Large I
```

```
computation_result = Dict("1/2-"=>wrap2dict(T1h⁻),
     "3/2-"=>wrap2dict(T3h⁻),
     "3/2+"=>wrap2dict(T3h⁺),
     "5/2+"=>wrap2dict(T5h⁺))
```

```
writejson(joinpath(@__DIR__, "LcXX2Lcpipi.json"), computation_result)
```

Simplifications

sign_replacements =

$$[\ \zeta^{0}_{2(1)} \Rightarrow -\zeta^{0}_{1(2)},\ \zeta^{0}_{3(2)} \Rightarrow -\zeta^{0}_{2(3)},\ \zeta^{0}_{1(3)} \Rightarrow -\zeta^{0}_{3(1)},\ \zeta^{1}_{3(1)} \Rightarrow -\zeta^{1}_{1(3)},\ \zeta^{1}_{1(2)} \Rightarrow -\zeta^{1}_{2(1)},\ \zeta^{1}_{3(2)} \Rightarrow -\zeta^{1}_{2(3)},\ \zeta^{2}_{1(2)} \Rightarrow -\zeta^{2}_{2(1)},\ \zeta^{2}_{1(3)} \Rightarrow -\zeta^{2}_{3(1)}$$

```julia
sign_replacements = let
    all_rotations = [(i,j,k) for (i,j,k)
        in Iterators.product(1:3,1:3,0:3) if i!=j]
    negative_rotations = filter(all_rotations) do (i,j,k)
        _wr = wr(i,j,k)
        !ispositive(_wr)
    end
    map(negative_rotations) do (i,j,k)
        symζ(wr(i,j,k)) => -symζ(wr(j,i,k))
    end
end
```

uncombine_replacement = $$[\ \zeta^{1}_{2(3)} \Rightarrow \zeta^{1}_{1(3)} + \zeta^{1}_{2(1)},\ \zeta^{2}_{3(1)} \Rightarrow \zeta^{2}_{2(1)} + \zeta^{2}_{3(2)},\ \zeta^{3}_{1(2)} \Rightarrow \zeta^{3}_{1(3)} + \zeta^{3}_{3(2)}\ ]$$

```julia
uncombine_replacement = [
    symζ(wr(2,3,1)) => symζ(wr(2,1,1))+symζ(wr(1,3,1)),
    symζ(wr(3,1,2)) => symζ(wr(3,2,2))+symζ(wr(2,1,2)),
    symζ(wr(1,2,3)) => symζ(wr(1,3,3))+symζ(wr(3,2,3))]
```

combine_replacements = $$[\ \zeta^{1}_{1(3)} \Rightarrow -\zeta^{1}_{2(1)} + \zeta^{1}_{2(3)},\ \zeta^{2}_{2(1)} \Rightarrow \zeta^{2}_{3(1)} - \zeta^{2}_{3(2)},\ \zeta^{3}_{3(2)} \Rightarrow \zeta^{3}_{1(2)} - \zeta^{3}_{1(3)}\ ]$$

```julia
combine_replacements = [
    symζ(wr(1,3,1)) => symζ(wr(2,3,1)) - symζ(wr(2,1,1)),
    symζ(wr(2,1,2)) => symζ(wr(3,1,2)) - symζ(wr(3,2,2)),
    symζ(wr(3,2,3)) => symζ(wr(1,2,3)) - symζ(wr(1,3,3))]
```

unsafe_uncombine_replacement =

$$[\ \zeta^{0}_{2(3)} \Rightarrow -\zeta^{0}_{1(2)} - \zeta^{0}_{3(1)} + 2\pi,\ \zeta^{0}_{3(1)} \Rightarrow -\zeta^{0}_{1(2)} - \zeta^{0}_{2(3)} + 2\pi,\ \zeta^{0}_{1(2)} \Rightarrow -\zeta^{0}_{2(3)} - \zeta^{0}_{3(1)} + 2\pi\ ]$$

```julia
unsafe_uncombine_replacement = [
    symζ(wr(2,3,0)) => 2PI-symζ(wr(3,1,0))-symζ(wr(1,2,0)),
    symζ(wr(3,1,0)) => 2PI-symζ(wr(1,2,0))-symζ(wr(2,3,0)),
    symζ(wr(1,2,0)) => 2PI-symζ(wr(2,3,0))-symζ(wr(3,1,0))
]
```