Tight Forests and the Chromatic Polynomial

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Abstract

Let G be a graph without three-cycles. A spanning forest F of G is said to be tight if, for each tree in F, all paths beginning with the smallest vertex of the tree avoid the patterns 231, 312, and 321. The generating function for tight forests of G is equal to the chromatic polynomial of G up to a sign change iff the total order on the vertices of G is a quasi-perfect order. This project strives to determine which graphs have such characteristics by first exploring the following concept: gluing two or more of such graphs to create a new quasi-perfect order graph. We proved that gluing along a single vertex produces our desired result if the vertex is the smallest of at least one graph for every connected pair of graphs. Additionally, we proved that if, for every pair of connected graphs, there are no three cycles and the two smallest vertices of at least one graph are adjacent, then gluing on that edge produces our desired result. Future research will consider other gluing cases.

Definitions

A path P: $a - c - b - v_1 - \cdots - v_m = d$ where $m \geq 1$, is a **Candidate Path** if $a < b < c$ and v_m is the only v_i smaller than c. [Hallam, Martin, Sagan 2016]

A total order on a graph's vertices is a **Quasi-Perfect Order** if, for all candidate paths in the graph, c and d are adjacent in the graph or, $d < b$ while c and d are adjacent in the graph. [Hallam, Martin, Sagan 2016]

Problem

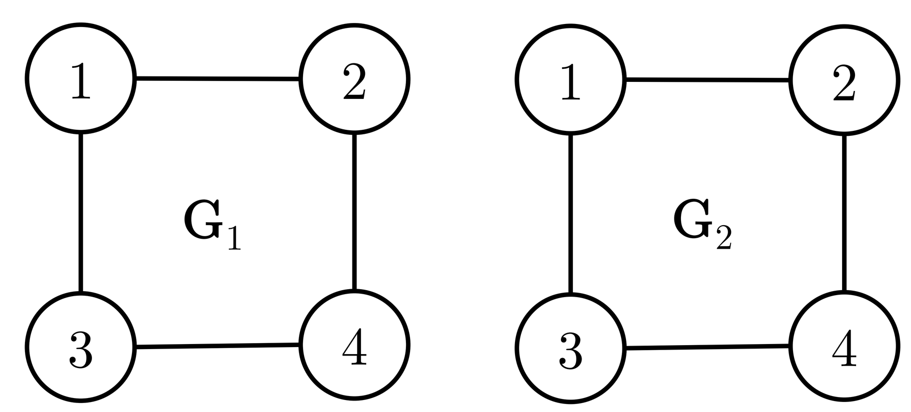
In combinatorics, while there are some characterizations, it is unknown (in general) which graphs have a quasi-perfect order and no three-cycles. To find such graphs, we began with the concept of gluing. Gluing allows us to reduce graphs to fundamental bits and study how to reconstruct them.

Research Findings

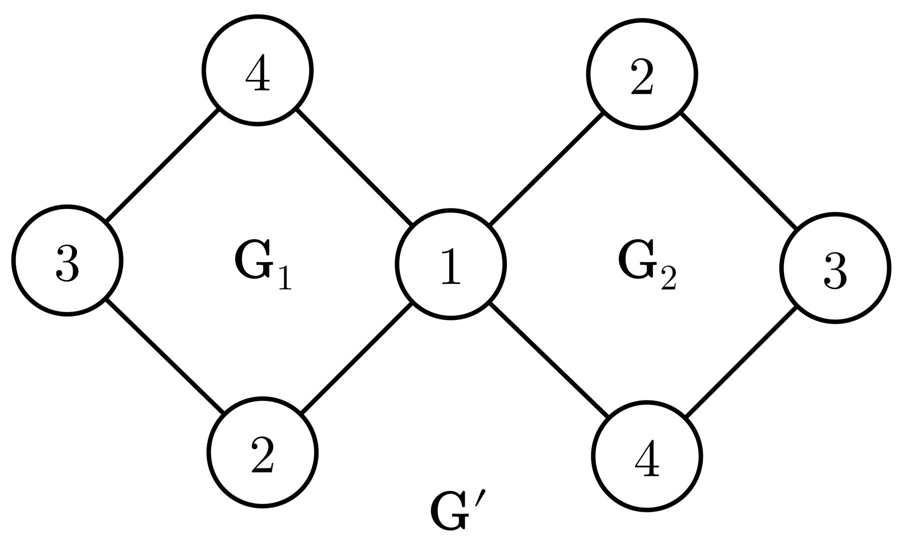
Theorem 1 (gluing on a vertex) If n quasi-perfect order graphs are glued together such that for all G_i and G_{i+1}, G_i and G_{i+1} intersect in the smallest vertex of G_{i+1}, then the resulting graph has a quasi-perfect order.

Theorem 2 (gluing on an edge) If n quasi-perfect order graphs without 3-cycles are glued together such that for all G_i and G_{i+1}, G_i and G_{i+1} intersect in an edge containing the smallest two vertices of G_{i+1}, then the resulting graph has a quasi-perfect order.

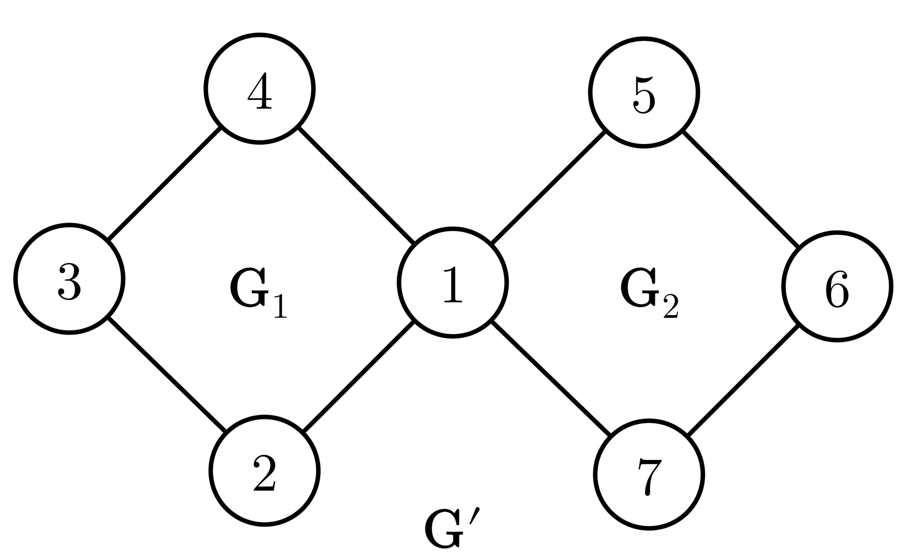
Example of Gluing on a Vertex



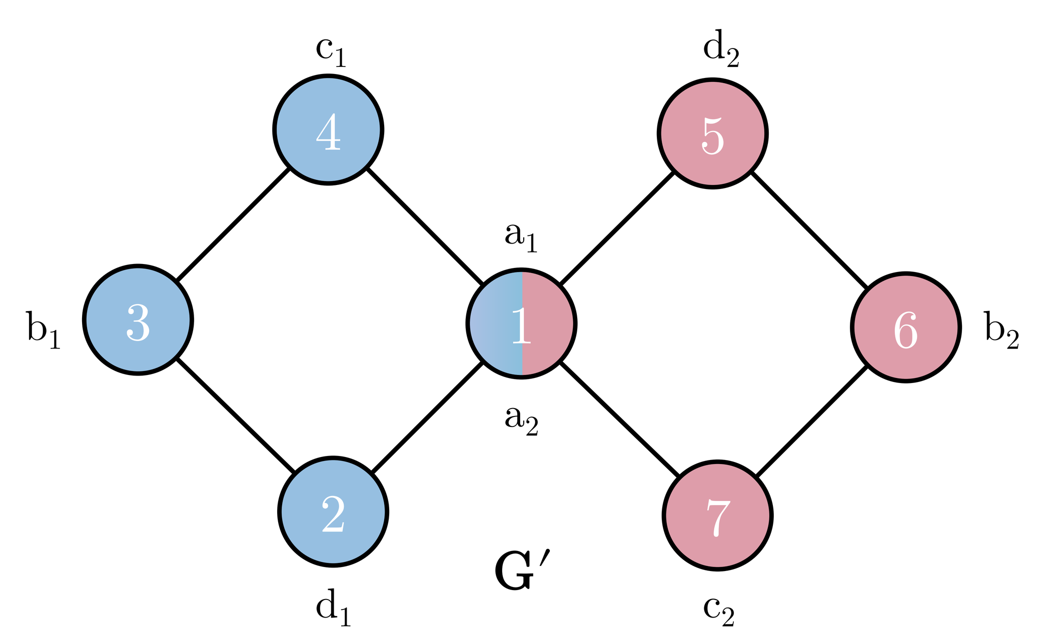
(i) Suppose we have two disjoint four-cycles. Four-cycles are one of the simplest examples of a graph that has a quasi-perfect order and no three-cycles.



(ii) Now, we glue the graphs at the smallest vertex of G_2 to form G'. The subgraphs, G_1 and G_2 still have their relative total orders. However, some vertices have the same labels.

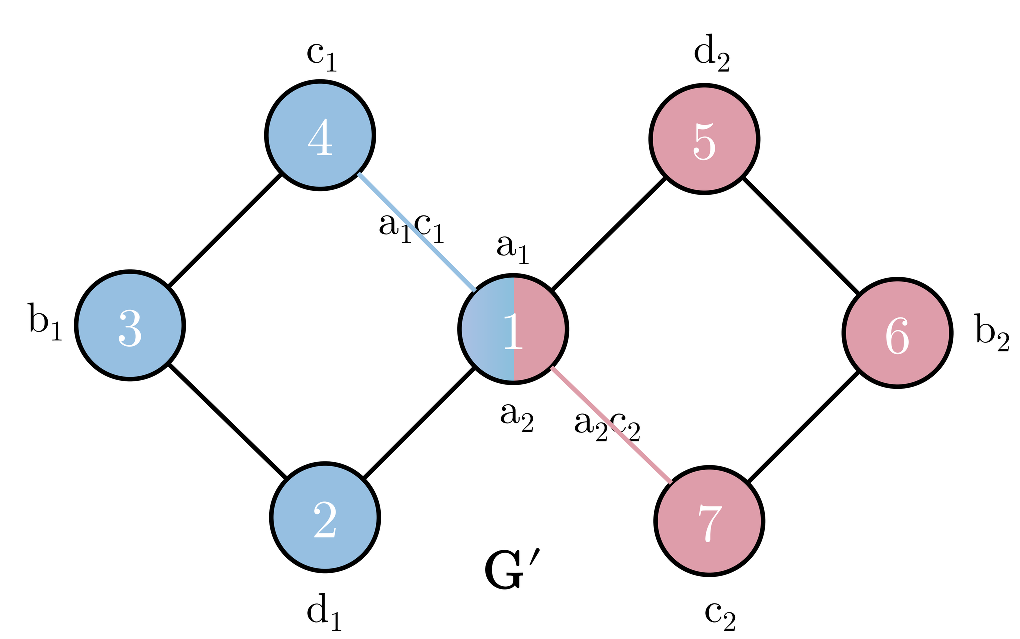


(iii) We can change the labels of the vertices in the path P: $2 - 3 - 4$ with $2 < 3 < 4$. By simply adding the value 3 to each vertex, we maintain their relative order.



(iv) To check if G' has a quasi-perfect order, we first find all candidate paths.

P_1: $1 - 4 - 3 - 2$ (blue)
P_2: $1 - 7 - 6 - 5$ (red)



(v) Finally, we test all candidate paths for required adjacencies.

In P_1; $a = 1$ is adjacent to $c = 2$
In P_2; $a = 1$ is adjacent to $c = 5$

Thus, G' has a quasi-perfect order!

Method

A Python program (using SageMath) was written in a Jupyter Notebook environment. The program automates checks for a quasi-perfect order in any given graph; it passed billions of tests. Moreover, as an aside, the program's algorithmic complexity has seen substantial improvements. Graphs constructed by gluing were passed to the program. The returned results were then studied for generalizable patterns. Finally, we mathematically proved two generalizable patterns (the research findings).

Significance/Conclusion

In general, this project is currently focusing on the "pure" mathematical aspects of graph theory. Graph theory is a subset of mathematics that is applied in many important technologies including Google Maps, social networks, DNA sequencing, etc. In terms of future research, this project will explore other ways to glue graphs while noting patterns that may reveal a common characteristic among quasi-perfect order graphs without three-cycles.

References

Hallam, Joshua & Martin, Jeremy & Sagan, Bruce. (2019). *Increasing spanning forests in graphs and simplicial complexes*. European Journal of Combinatorics. 76. 10.1016/j.ejc.2018.09.011.

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