

Problem Formalization

To state the result rigourously, we need to formally define all the objects. We begin by the game graph.

Definition 1 (Game graph). *Let $G = (V, E)$ be a directed graph with two absorbing vertices W and L . Every other vertex v has two outgoing edges $v \rightarrow v_+$ and $v \rightarrow v_-$.*

Furthermore, we denote a single vertex $s \in V \setminus \{W, L\}$ as starting node.

The game graph is the mathematical structure we will use to describe a tennis match. The vertexes are all the possible states of the match: each will correspond to a specific (punteggio in inglese). From a state v , if the first player wins, the game will transition to v_+ , else to v_- . The game being won by the first player corresponds to ending in W , lost in L .

Now we define the match itself as a random walk on the graph.

Definition 2 (Graph walk). *Given a game graph $G = (V, E)$, $s \in V$ and $p \in [0, 1]$. We define $\mathbf{X}^{(G,s,p)}$ to be a stochastic process on G starting from the starting the node $\mathbf{X}_0^{(G,s,p)} = s$, and with*

$$\mathbf{X}_{i+1}^{(G,s,p)} = \begin{cases} \left(\mathbf{X}_i^{(G,s,p)} \right)_+ & \text{with prob. } p, \\ \left(\mathbf{X}_i^{(G,s,p)} \right)_- & \text{with prob } 1 - p, \end{cases}$$

while $\mathbf{X}_i^{(G,s,p)} \notin \{W, L\}$, after that the random process ends. We call $\tau^{(G,s,p)} \in \mathbb{N} \cup \{\infty\}$ the number of steps needed to reach $\{W, L\}$. Finally, if $\tau^{(G,s,p)} < \infty$ we call the final state $O^{(G,s,p)} = \mathbf{X}_{\tau^{(G,s,p)}}^{(G,s,p)} \in \{W, L\}$.

The graph walk is the stochastic process corresponding to playing a match in the structure defined by G , and starting from punteggio 0 a 0, which corresponds to the node s . To continue with the parallels, $\tau^{(G,s,p)}$ is the number of points played during the match, and $\mathbf{X}_{\tau^{(G,s,p)}}^{(G,s,p)}$ is the outcome of the match.

Of course, in our analysis we are interested in those games which are balanced, meaning that assuming the first player is just as strong as the second one ($p = 1/2$), then the probability of the first player to win the match is also $1/2$, formally this means that:

$$\mathbb{P}(O_{(G,s,1/2)} = W) = \frac{1}{2}.$$

The last thing remained to defined is the fairness of the match. This is a quantity that indicates how much the stronger player actually benefits from the game structure.

Definition 3 (Game Fairness). *Given a game graph $G = (V, E)$, and a starting point s , we call the fairness of the match*

$$\mathcal{F}(G, s) := \frac{d}{dp} \mathbb{P}(O^{(G,s,p)} = W) \Big|_{p=\frac{1}{2}}.$$

While this definition could look esoteric there is a nice way to think about it. Assume the first player has an edge of ε , meaning that $p = 1/2 + \varepsilon$, then we will have that

$$\mathbb{P}(O_{(G,s,1/2+\varepsilon)} = W) = \frac{1}{2} + \varepsilon \mathcal{F}(G, s) + o(\varepsilon).$$

Thus, \mathcal{F} can be seen as the advantage multiplier factor for small ε .

Main Result

Theorem 1. *Given a game graph G and a starting position s , such that the game is balanced*

$$\mathbb{P}(O^{(G,s,1/2)} = W) = \frac{1}{2},$$

and the match has no hidden status

$$\mathbf{X}_i^{(G,s,p)} | \mathbf{X}_0^{(G,s,p)}, \dots, \mathbf{X}_{i-1}^{(G,s,p)} = \mathbf{X}_i^{(G,s,p)} | \mathbf{X}_{i-1}^{(G,s,p)},$$

then

$$\mathcal{F}(G, s) \leq \left(\mathbb{E}[\tau_{(G,s,1/2)}] \right)^2.$$

This result is tight, as we can see by the optimal game structure defined in the blog post. The two assumptions we do are also quite natural. The first one is that we are talking about a balanced match structure, where the first player has no inherent advantage to the second one. The second one is more subtle, and likely it does not hold in practice, basically we are assuming that the players will play each point independently of how the previous points in the match went. This assumption is however necessary, without it the result does not hold.

Proof

The proof relies on this key construction.

Definition 4 (Value function).

$$h_{(G)} : V \rightarrow [0, 1], \quad h_{(G)}(v) = \mathbb{P}(O^{(G,v,1/2)} = W).$$

Thus, $h_{(G)}(v)$ is the probability for the first player to win, given that the first player is just as strong as the second one ($p = 1/2$), and given that the current punteggio situation is v . Note that $h_{(G)}(v)$ is not a random variable, and it always assumes that $p = 1/2$.

This quantity is a good proxy for the state of the match, instead of a messy punteggio, which could be misleading, this indicates well how far (probabilistically) is a player from winning the match.

We now show that $h_{(G)}(v)$ has a key harmonic property, that later on will be crucial.

Lemma 1 ($h_{(G)}(v)$ is harmonic). *Given an edge $v \in G \setminus \{W, L\}$, let v_+ and v_- be the vertices reached by the outward edges from v , then*

$$h_{(G)}(v) = \frac{1}{2} (h_{(G)}(v_+) + h_{(G)}(v_-)).$$

Proof. The proof comes from rolling out one step of the random walk.

$$\begin{aligned} h_{(G)}(v) &= \mathbb{P}(O^{(G,v,1/2)} = W) \\ &\stackrel{(1)}{=} \mathbb{P}(O^{(G,v,1/2)} = W | \mathbf{X}_1^{(G,v,1/2)} = v_+) \mathbb{P}(\mathbf{X}_1^{(G,v,1/2)} = v_+) + \\ &\quad \mathbb{P}(O^{(G,v,1/2)} = W | \mathbf{X}_1^{(G,v,1/2)} = v_-) \mathbb{P}(\mathbf{X}_1^{(G,v,1/2)} = v_-) \\ &\stackrel{(2)}{=} h_{(G)}(v_+) \mathbb{P}(\mathbf{X}_1^{(G,v,1/2)} = v_+) + h_{(G)}(v_-) \mathbb{P}(\mathbf{X}_1^{(G,v,1/2)} = v_-) \\ &\stackrel{(3)}{=} \frac{1}{2} h_{(G)}(v_+) + \frac{1}{2} h_{(G)}(v_-). \end{aligned}$$

□

Define the random walk $X^{(p)} = (X_0^{(p)}, X_1^{(p)}, \dots)$ by

$$X_0^{(p)} = s,$$

$$X_{i+1}^{(p)} = \begin{cases} (X_i^{(p)})^+, & \text{if } b_i^{(p)} = 1, \\ (X_i^{(p)})^-, & \text{if } b_i^{(p)} = 0. \end{cases}$$

Let

$$\tau^{(p)} = \inf\{i \geq 0 : X_i^{(p)} \in \{W, L\}\}.$$

Theorem 2 (Fairness–Length Inequality). *Let*

$$\mathbb{P}_p(W) = \mathbb{P}_p(X_{\tau^{(p)}}^{(p)} = W), \quad \mathcal{F} = \frac{d}{dp} \mathbb{P}_p(W) \Big|_{p=1/2}.$$

Then

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}].$$

Proof. **Harmonic function.**

Define

$$h(v) = \mathbb{P}_{1/2}(X_{\tau^{(1/2)}}^{(p)} = W \mid X_0^{(p)} = v).$$

Then $h(W) = 1$, $h(L) = 0$, and for non-absorbing v ,

$$h(v) = \frac{h(v^+) + h(v^-)}{2}.$$

Under $p = \frac{1}{2}$, $h(X_0^{(p)}), \dots, h(X_{\tau^{(1/2)}}^{(p)})$ is a martingale.

Perturbation around 1/2.

By telescoping,

$$h(X_{\tau^{(p)}}^{(p)}) = h(X_0^{(p)}) + \sum_{i=1}^{\tau^{(p)}} (h(X_i^{(p)}) - h(X_{i-1}^{(p)})).$$

Taking expectation under \mathbb{P}_p ,

$$\mathbb{P}_p(W) = \frac{1}{2} + (p - \frac{1}{2}) \mathbb{E}_p \left[\sum_{i=1}^{\tau^{(p)}} \Delta(X_{i-1}^{(p)}) \right],$$

where

$$\Delta(v) := h(v^+) - h(v^-).$$

Thus

$$\mathcal{F} = \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta(X_{i-1}^{(p)}) \right].$$

Energy identity.

Under $p = \frac{1}{2}$,

$$h(X_i^{(p)}) - h(X_{i-1}^{(p)}) = \frac{1}{2} \Delta(X_{i-1}^{(p)}) (2b_{i-1}^{(p)} - 1).$$

Since the variables are independent and $\mathbb{E}_{1/2}[(2b_{i-1}^{(p)} - 1)^2] = 1$,

$$\text{Var}_{1/2} (h(X_{\tau^{(1/2)}}^{(p)})) = \frac{1}{4} \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right].$$

But this variance equals 1/4. Hence

$$\mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right] = 1.$$

Cauchy–Schwarz.

$$\mathcal{F} = \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta(X_{i-1}^{(p)}) \right].$$

By Cauchy–Schwarz,

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}] \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right].$$

Using the energy identity,

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}].$$

□