

Problem Formalization

To state the result rigourously, we need to formally define all the objects. We begin by the game graph.

Definition 1 (Game graph). *Let $G = (V, E)$ be a directed graph with two absorbing vertices W and L . Every other vertex v has two outgoing edges $v \rightarrow v_+$ and $v \rightarrow v_-$. Furthermore, we denote a single vertex $v \in V \setminus \{W, L\}$ as starting node.*

The game graph is the mathematical structure we will use to describe a tennis match. The vertexes are all the possible states of the match: each will correspond to a specific (punteggio in inglese). From a state v , if the first player wins, the game will transition to v_+ , else to v_- . The game being won by the first player corresponds to ending in W , lost in L .

Now we define the match itself as a random walk on the graph.

Definition 2 (Graph walk). *Given a game graph $G = (V, E)$ and $p \in [0, 1]$. We define X_0, \dots, X_τ , to be a stochastic process starting from $X_0 = s \in V$, and with*

$$X_i = \begin{cases} (X_i)_+ & \text{with prob. } p, \\ (X_i)_- & \text{with prob } 1 - p, \end{cases}$$

while $X_i \notin \{W, L\}$, after that the random process ends. We call $\tau \in \mathbb{N} \cup \{\infty\}$ the number of steps needed to reach $\{W, L\}$. We will use $\mathbb{P}_p(\cdot)$ and $\mathbb{E}_p[\cdot]$ to indicate the probability and the expected value with respect to p .

The graph walk is the stochastic process corresponding to playing a match in the structure defined by G . To continue with the parallel, τ is the number of points played during the match, and X_τ is the outcome of the match.

Of course, in our analysis we are interested in those games which are balanced, meaning that assuming the first player is just as strong as the second one ($p = 1/2$), then the probability of the first player to win the match is also $1/2$, formally this means that:

$$\mathbb{P}_{\frac{1}{2}}(X_\tau = W) = \frac{1}{2}.$$

The last thing remained to defined is the fairness of the match. This is a quantity that indicates how much the stronger player actually benefits from the game structure.

Definition 3 (Game Fairness). *Given a game graph $G = (V, E)$, we call the fairness of the match*

$$\mathcal{F}(G) := \left. \frac{d}{dp} \mathbb{P}_p(X_\tau = W) \right|_{p=\frac{1}{2}}.$$

While this definition could look esoteric there is a nice way to think about it. Assume the first player has an edge of ε , meaning that $p = 1/2 + \varepsilon$, then we will have that

$$\mathbb{P}_{1/2+\varepsilon}(X_\tau = W) = \frac{1}{2} + \varepsilon \mathcal{F}(G) + o(\varepsilon).$$

Thus, $\mathcal{F}(G)$ can be seen as the advantage multiplier factor for small ε .

Main Result

Theorem 1. *Given a game graph G , such that*

$$\mathbb{E}_{\frac{1}{2}}[\tau] < \infty \quad \text{and} \quad \mathbb{P}_{\frac{1}{2}}(X_\tau = W) = \frac{1}{2},$$

then

$$\mathcal{F}(G) \leq \left(\mathbb{E}_{\frac{1}{2}}[\tau] \right)^2.$$

For $p \in (0, 1)$, let

$$B^{(p)} = (b_0^{(p)}, b_1^{(p)}, \dots), \quad b_i^{(p)} \sim \text{Bernoulli}(p),$$

independent.

Define the random walk $X^{(p)} = (X_0^{(p)}, X_1^{(p)}, \dots)$ by

$$X_0^{(p)} = s,$$

$$X_{i+1}^{(p)} = \begin{cases} (X_i^{(p)})^+, & \text{if } b_i^{(p)} = 1, \\ (X_i^{(p)})^-, & \text{if } b_i^{(p)} = 0. \end{cases}$$

Let

$$\tau^{(p)} = \inf\{i \geq 0 : X_i^{(p)} \in \{W, L\}\}.$$

Theorem 2 (Fairness–Length Inequality). *Let*

$$\mathbb{P}_p(W) = \mathbb{P}_p(X_{\tau^{(p)}}^{(p)} = W), \quad \mathcal{F} = \left. \frac{d}{dp} \mathbb{P}_p(W) \right|_{p=1/2}.$$

Then

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}].$$

Proof. **Harmonic function.**

Define

$$h(v) = \mathbb{P}_{1/2}(X_{\tau^{(1/2)}}^{(p)} = W \mid X_0^{(p)} = v).$$

Then $h(W) = 1$, $h(L) = 0$, and for non-absorbing v ,

$$h(v) = \frac{h(v^+) + h(v^-)}{2}.$$

Under $p = \frac{1}{2}$, $h(X_0^{(p)}), \dots, h(X_{\tau^{(1/2)}}^{(p)})$ is a martingale.

Perturbation around 1/2.

By telescoping,

$$h(X_{\tau^{(p)}}^{(p)}) = h(X_0^{(p)}) + \sum_{i=1}^{\tau^{(p)}} (h(X_i^{(p)}) - h(X_{i-1}^{(p)})).$$

Taking expectation under \mathbb{P}_p ,

$$\mathbb{P}_p(W) = \frac{1}{2} + (p - \frac{1}{2}) \mathbb{E}_p \left[\sum_{i=1}^{\tau^{(p)}} \Delta(X_{i-1}^{(p)}) \right],$$

where

$$\Delta(v) := h(v^+) - h(v^-).$$

Thus

$$\mathcal{F} = \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta(X_{i-1}^{(p)}) \right].$$

Energy identity.

Under $p = \frac{1}{2}$,

$$h(X_i^{(p)}) - h(X_{i-1}^{(p)}) = \frac{1}{2} \Delta(X_{i-1}^{(p)}) (2b_{i-1}^{(p)} - 1).$$

Since the variables are independent and $\mathbb{E}_{1/2}[(2b_{i-1}^{(p)} - 1)^2] = 1$,

$$\text{Var}_{1/2} (h(X_{\tau^{(1/2)}}^{(p)})) = \frac{1}{4} \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right].$$

But this variance equals 1/4. Hence

$$\mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right] = 1.$$

Cauchy–Schwarz.

$$\mathcal{F} = \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta(X_{i-1}^{(p)}) \right].$$

By Cauchy–Schwarz,

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}] \mathbb{E}_{1/2} \left[\sum_{i=1}^{\tau^{(1/2)}} \Delta^2(X_{i-1}^{(p)}) \right].$$

Using the energy identity,

$$\mathcal{F}^2 \leq \mathbb{E}_{1/2}[\tau^{(1/2)}].$$

□