

# PSTAT 213B Section Notes, Winter 2020, Ming Min

## Section 1. (7.1.1, 7.1.4, 7.2.4, 7.2.1)

7.1.1 Let  $r > 1$ , define  $\|X\|_r = E[|X|^r]^{\frac{1}{r}}$ . Show that:

$$(a) \|c \cdot X\|_r = |c| \cdot \|X\|_r \text{ for } c \in \mathbb{R}.$$

$$(b) \|X + Y\|_r \leq \|X\|_r + \|Y\|_r.$$

$$(c) \|X\|_r = 0 \text{ iff } P(X=0) = 1.$$

$$\frac{1}{r} + \frac{1}{p} = 1$$

$$\frac{1}{p} = 1 - \frac{1}{r} = \frac{r-1}{r}$$

$$\text{Proof: (a)} \|c \cdot X\|_r = E[|c \cdot X|^r]^{\frac{1}{r}} = |c| \cdot \|X\|_r$$

$$(b) \|X + Y\|_r = E[|X+Y|^r]^{\frac{1}{r}}$$

$$\begin{aligned} E[|X+Y| \cdot |X+Y|^{r-1}] &\leq E[|X| \cdot |X+Y|^{r-1}] + E[|Y| \cdot |X+Y|^{r-1}] && \text{triangle} \\ &\leq E[|X|^r]^{\frac{1}{r}} \cdot E[|X+Y|^r]^{\frac{r-1}{r}} + E[|Y|^r]^{\frac{1}{r}} \cdot E[|X+Y|^r]^{\frac{r-1}{r}} && \text{holder.} \end{aligned}$$

$\Rightarrow$  desired result.

(c) " $\Rightarrow$ "

$$\text{if } \|X\|_r = E[|X|^r]^{\frac{1}{r}} = 0, \text{ then } E[|X|^r] = 0$$

$$\forall \epsilon \in \mathbb{Q}, \epsilon > 0. P(|X| > \epsilon) = P(|X|^r > \epsilon^r) \leq \frac{1}{\epsilon^r} E[|X|^r] = 0.$$

$$P(|X| > 0) = \lim_{\epsilon \rightarrow 0} P(|X| > \epsilon) = 0.$$

" $\Leftarrow$ " if  $P(X=0) = 1$ ,

$$\text{then } |X|^r = 0 \text{ a.s. } \Rightarrow E[|X|^r] = 0.$$

7.1.4 (Lévy metric) For two distribution functions  $F$  and  $G$ . Let

$$d(F, G) = \inf \{ \delta > 0 : F(x-\delta) - \delta \leq G(x) \leq F(x+\delta) + \delta \text{ for all } x \in \mathbb{R} \}$$

Show  $d$  is a metric on the space of distribution function.

Proof: (i)  $d(F, G) \geq 0$ . — by definition of  $d$ .

$$(ii) d(F, G) = 0 \Leftrightarrow F = G$$

$$(iii) d(F, G) = d(G, F)$$

$$(iv) d(F, G) + d(G, H) \geq d(F, H).$$

$$(v) d(F, G) = 0 \Rightarrow \forall \delta > 0, \left\{ \begin{array}{l} G(x) \leq F(x+\delta) + \delta \\ F(x) \leq G(x+\delta) + \delta \end{array} \right. \text{ (shift)}$$

let  $\delta \downarrow 0$ , we have  $G(x) \leq F(x)$ ,  $F(x) \leq G(x) \Rightarrow F = G$

if  $F = G$ , obvious.

$$(vi) \text{ denote } d(F, G) = \inf \{ \delta : \left\{ \begin{array}{l} G(x) \leq F(x+\delta) + \delta \\ F(x) \leq G(x+\delta) + \delta \end{array} \right. \}$$

$d(F, G)$  admits the same form.

(iv) Let  $\delta_n \downarrow d(F, G)$ ,  $\epsilon_n \downarrow d(G, H)$ . St.

$$\forall x, \quad G(x) \leq F(x + \delta_n) + \delta_n, \text{ and } F(x) \leq G(x + \delta_n) + \delta_n$$

$$\forall x, \quad G(x) \leq H(x + \epsilon_n) + \epsilon_n \text{ and } H(x) \leq G(x + \epsilon_n) + \epsilon_n.$$

$$\Rightarrow F(x) \leq G(x + \delta_n) + \delta_n \leq H(x + \delta_n + \epsilon_n) + (\delta_n + \epsilon_n)$$

$$\text{and } H(x) \leq F(x + \delta_n + \epsilon_n) + \delta_n + \epsilon_n$$

$$\Rightarrow d(F, H) \leq \lim_{n \rightarrow \infty} (\delta_n + \epsilon_n) = d(F, G) + d(G, H).$$

7.2.4.  $d(F_n, F) \rightarrow 0$  (denote  $d_n = d(F_n, F)$ ).

$\Leftrightarrow F_n(x) \rightarrow F$  at all  $x$  s.t.  $F(x)$  is continuous.

Proof.  $\Rightarrow d(F_n, F) \rightarrow 0$ . denote  $C^F = \{\text{continuous points of } F\}$ .

so  $\forall \epsilon > 0, \exists N$  s.t.  $\forall n \geq N$ . we have

$$F(x - \epsilon) - \epsilon \leq F_n(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x.$$

let  $n \rightarrow \infty$  and then  $\epsilon \rightarrow 0$

$$\lim_{n \rightarrow \infty} F_n(x) \leq F(x), \quad F(x) \leq \lim_{n \rightarrow \infty} F_n(x)$$

if  $F$  is continuous at  $x$ , then  $F(x) = \lim_{n \rightarrow \infty} F_n(x)$ .

" $\Leftarrow$ " If  $F_n(x) \rightarrow F(x)$  in  $C^F$ . let  $\epsilon > 0$ .

Let  $\epsilon > 0$ , find real numbers  $a = x_0 < x_1 < \dots < x_n = b$ , each being point of continuity of  $F$ .

with (i)  $F_m(a) < \epsilon$  for all  $m$ ,  $F(b) > -\epsilon$ .

(ii)  $|x_{i+1} - x_i| < \epsilon$  for  $1 \leq i \leq n$ .

$$\exists N \text{ s.t. } \forall n \geq N, \quad |F_n(x_i) - F(x_i)| < \epsilon$$

for all  $i$ , (since finitely many  $x_i$ 's).

Then for  $x \in [x_i, x_{i+1}]$ ,

$$F_n(x) \leq F_n(x_{i+1}) \leq F(x_{i+1}) + \epsilon \leq F(x + \epsilon) + \epsilon$$

$$F_n(x) \geq F_n(x_i) \geq F(x_i) - \epsilon \geq F(x - \epsilon) - \epsilon.$$

for  $x \in (-\infty, a]$ ,  $F_n(x) < \epsilon \leq F(x + \epsilon) + \epsilon$

and,  $F(x) \leq F(a) < \epsilon \Rightarrow F_n(x) > 0 \geq F(x - \epsilon) - \epsilon$ . (large enough  $n$ ).

for  $x \in (b, \infty)$ ,  $F_n(x) \leq 1 \leq F(x + \epsilon) + \epsilon$ .

$$F_n(x) \geq 1 - \epsilon \geq F(x - \epsilon) - \epsilon. \text{ (large enough } n\text{)}$$

This  $a$  can be found.

① find  $a'$  with  $F(a') < \frac{1}{2}\epsilon$  and  $a'$  continuity. so  $\exists M$  s.t.  $|F_n(a') - F(a')| \leq \frac{\epsilon}{2}$ ,  $\Rightarrow F_n(a') < \epsilon$  on  $n \geq M$ .

find  $a$  s.t.  $F_n(a) < \epsilon$  with  $a < a'$ , continu.

- 7.2.1. a)  $X_n \xrightarrow{r} X$  when  $r \geq 1$ , show  $E[X_n^r] \rightarrow E[X^r]$   
 b)  $X_n \xrightarrow{1} X$ . Show  $E[X_n] = E[X]$ . converse is not true  
 c)  $X_n \xrightarrow{2} X$  Show  $\text{Var}(X_n) \rightarrow \text{Var}(X)$

Proof: (a)  $X_n \xrightarrow{r} X \Rightarrow E[X_n - X]^r \rightarrow 0$ .

$$E[X_n^r]^k \leq E[(X_n - X)^r]^k + E[X^r]^k$$

$$\lim_{n \rightarrow \infty} E[X_n^r]^k \leq E[X^r]^k, \text{ and vice versa.}$$

(b)  $E[X_n - X] \rightarrow 0$ .

$$0 \leq |E[X_n] - E[X]| \leq E|X_n - X| \rightarrow 0.$$

(c)  $E[X_n - X]^2 \rightarrow 0 \Rightarrow E[X_n^2] \rightarrow E[X^2]$ .

$$E|X_n - X| \leq E[X_n - X]^2 \xrightarrow{k} 0 \Rightarrow E[X_n] \rightarrow E[X].$$

$$\text{so } \text{var}(X_n) = E[X_n^2] - E[X]^2 \rightarrow \text{var}(X).$$

## Section 2 (7.2.5(a), 7.2.7, 7.2.9, 7.3.1(a), 7.3.2)

7.2.5(a) Suppose  $X_n \xrightarrow{D} X$ ,  $Y_n \xrightarrow{P} c$ . Show  $X_n Y_n \xrightarrow{D} cX$ .  $\frac{X_n}{Y_n} \xrightarrow{D} \frac{X}{c}$  if  $c \neq 0$ .

Proof: let  $r \in C$ .  $P(X_n \leq r) \rightarrow P(X \leq r)$

$$\begin{aligned} P(X_n Y_n \leq r) &= P(X_n Y_n \leq r, |Y_n - c| \leq \varepsilon) + P(X_n Y_n \leq r, |Y_n - c| > \varepsilon) \\ &\leq P(X_n Y_n \leq r, |Y_n - c| \leq \varepsilon) + P(|Y_n - c| > \varepsilon) \end{aligned}$$

on  $\{|Y_n - c| \leq \varepsilon\}$ ,  $c - \varepsilon \leq Y_n \leq c + \varepsilon$ , let  $\varepsilon = \frac{1}{2}c$

let  $0 < \varepsilon < c$

if  $c > 0$ , then  $P(X_n Y_n \leq r, |Y_n - c| \leq \varepsilon)$

$$r > 0 \leq P(X_n \leq \frac{r}{c-\varepsilon})$$

$$\Rightarrow P(X_n Y_n \leq r) \leq P(X_n \leq \frac{r}{c-\varepsilon}) + P(|Y_n - c| > \varepsilon)$$

$$P(X_n Y_n \leq r) \geq P(X_n Y_n \leq r, |Y_n - c| < \varepsilon) \geq P(X_n \leq \frac{r}{c+\varepsilon})$$

$\Rightarrow$  let  $n \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ ,

$$\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} P(X_n \leq \frac{r}{c-\varepsilon}) \leq \lim_{n \rightarrow \infty} P(X_n Y_n \leq r) \leq \lim_{c \rightarrow 0} \lim_{n \rightarrow \infty} P(X_n \leq \frac{r}{c+\varepsilon})$$

as long as  $\frac{r}{c}$  is in  $C$ . (choose  $c$  s.t.  $[r, r+\varepsilon] \subseteq C$ )

$$\begin{aligned} P(|X_n - Y_n| > r) &= P(|X_n - Y_n| > r, |Y_n| > \varepsilon) + P(|X_n - Y_n| > r, |Y_n| \leq \varepsilon) \\ &\leq P(|Y_n| > \varepsilon) + P\left(|X_n| > \frac{r}{\varepsilon}\right) \rightarrow 0 \end{aligned}$$

7.2.7 Let  $\{X_n\}$  be a sequence of RV,  $\{c_n\}$  are Reals converging to c.

Show  $X_n \xrightarrow{P} X$  entails  $c_n X_n \xrightarrow{P} cX$  in (a.s.,  $L^r$ , P, D)

Proof: If  $X_n \xrightarrow{\text{a.s.}} X$  a.s.  $\omega_0 = \{w \in \Omega : \lim_{n \rightarrow \infty} X_n(w) = X(w)\}$ ,  $P(\omega_0) = 1$ .

so for  $\forall w \in \omega_0$ ,

$$\lim_{n \rightarrow \infty} c_n X_n(w) = c \cdot X(w) \Rightarrow c_n X_n \xrightarrow{\text{a.s.}} cX$$

If  $X_n \xrightarrow{L^r} X$ , then  $E[(X_n - X)^r] \rightarrow 0$ .

$$\begin{aligned} E[|c_n X_n - cX|^r] &\leq E[|c_n|^r |X_n - X|^r] + E[|(c_n - c)|^r |X|^r] \\ &\approx |c_n| \cdot E[|X_n - X|^r] + (|c_n - c| \cdot E[|X|^r])^r \rightarrow 0 \end{aligned}$$

If  $X_n \xrightarrow{P} X$ , then  $P(|X_n - X| > \varepsilon) \rightarrow 0$

$$\begin{aligned} P(|c_n X_n - cX| > \varepsilon) &\leq P(|c_n| \cdot |X_n - X| > \frac{\varepsilon}{2}) + P(|c_n - c| \cdot |X| > \frac{\varepsilon}{2}) \\ &\leq P(|X_n - X| > \frac{\varepsilon}{2|c_n|}) + P(|X| > \frac{\varepsilon}{2|c_n - c|}) \\ &\rightarrow 0 \end{aligned}$$

7.2.9 do it by yourself.

7.3.1 (a) Suppose  $X_n \xrightarrow{P} X$ . Show  $\{X_n\}$  is Cauchy convergent in P, i.e.  $\forall \varepsilon > 0$ ,  $P(|X_m - X_n| > \varepsilon) \rightarrow 0$  as  $m, n \rightarrow \infty$ . In what sense is the converse true?

Proof: If  $X_n \xrightarrow{P} X$ . then

$$\begin{aligned} P(|X_m - X_n| > \varepsilon) &\leq P(|X_m - X| > \frac{1}{2}\varepsilon) + P(|X_n - X| > \frac{1}{2}\varepsilon) \\ &\rightarrow 0 \end{aligned}$$

If Cauchy convergent in P.

$$\forall k, \exists n_k \text{ s.t. } P(|X_m - X_n| > \frac{1}{k^2}) < \frac{1}{k^2} \text{ for } \forall m, n \geq n_k$$

Define  $N_1 = n_1$ ,  $N_k = \max(N_{k-1} + 1, n_k)$

$$\text{So we have } \sum_{k=1}^{\infty} P(|X_{N_{k+1}} - X_{N_k}| > \frac{1}{k^2}) < \infty \Rightarrow$$

$A_k = \{|X_{N_{k+1}} - X_{N_k}| > \frac{1}{k^2}\}$  happens finitely often

so  $X_{N_1} + \sum_{k=1}^{\infty} (X_{N_{k+1}} - X_{N_k})$  converges, denote it by X.

$$X_{N_m} = X_{N_1} + \sum_{k=1}^{m-1} (X_{N_{k+1}} - X_{N_k}) \Rightarrow X - X_{N_m} = \sum_{k=m}^{\infty} (X_{N_{k+1}} - X_{N_k})$$

$\rightarrow X_{N_m} \rightarrow X$  a.s. as  $m \rightarrow \infty$ .

$$P(|X_n - X| > \epsilon) \leq P(|X_n - X_{N_m}| > \frac{\epsilon}{2}) + P(|X_{N_m} - X| > \frac{\epsilon}{2})$$

for  $\forall m$ , let  $m \rightarrow \infty$ ,  $n \rightarrow \infty$ .

7.3.2 Show that the prob. that infinitely many of events  $\{A_n : n \geq 1\}$  occur satisfies

$$P(A_n \text{ i.o.}) \geq (\limsup_{n \rightarrow \infty} P(A_n))$$

$$\text{Proof: } P(A_n \text{ i.o.}) = P(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) = \lim_{n \rightarrow \infty} P(\bigcup_{m=n}^{\infty} A_m)$$

$$P(\bigcup_{m=n}^{\infty} A_m) \geq P(A_m) \text{ for } \forall m \geq n$$

$$\Rightarrow P(\bigcup_{m=n}^{\infty} A_m) \geq \sup_{m \geq n} P(A_m)$$

$$\lim_{n \rightarrow \infty} P(\bigcup_{m=n}^{\infty} A_m) \geq \limsup_{n \rightarrow \infty} P(A_n)$$