Show that for a Binomial distribution

$\langle r \rangle = np$

$V(r) = np(1-p)$

Skewness $= \dfrac{1-2p}{\sqrt{np(1-p)}}$

and derive its kurtosis too.

i) $\langle r \rangle = np$

$$P(r) = \binom{n}{r} p^r (1-p)^{n-r}$$

n: trial num

p: success probability

r: success num

$$E(r) = \sum_{r=0}^{n} r \cdot P(r)$$

$$= \sum_{r=0}^{n} r \cdot \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{r!\,(n-r)!} p^r (1-p)^{n-r}$$

$$x! = x \cdot (x-1)!$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)!}{r(r-1)!\,(n-r)!}$$

$$= \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$\binom{n-1}{r-1} = \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!}$$

$$= \frac{(n-1)!}{(n-r)!\,(r-1)!}$$

$$r \cdot \binom{n}{r} = n \cdot \binom{n-1}{r-1}$$

$$E(r) = \sum_{r=1}^{n} n \cdot \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$= n \sum_{r=1}^{n} \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$= n \sum_{r=1}^{n} \binom{n-1}{r-1} p \cdot p^{r-1} (1-p)^{n-r}$$

$$= np \sum_{r=1}^{n} \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r}$$

$$n-1 = m$$

$$r-1 = j$$

$$= np \underbrace{\sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j}}_{\substack{\text{Sum of probability} \\ = 1}}$$

$$\therefore E(r) = np$$

ii) $V(r) = np(1-p)$

$$\text{Variance}(r) = E(r^2) - E(r)^2$$

$$E(r) = np$$

$$\text{Var}(r) = E(r^2) - (np)^2$$

$$E(r^2) = \sum_{r=0}^{n} r^2 \cdot \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \sum_{r=1}^{n} r^2 \binom{n}{r} p^r (1-p)^{n-r}$$

$$r^2\binom{n}{r} = r\cdot r\binom{n}{r} = r\cdot n\binom{n-1}{r-1}$$

$$E(r^2) = n\cdot\sum_{r=1}^{n} r\cdot\binom{n-1}{r-1}p^r(1-p)^{n-r}$$

$$= np\sum_{r=1}^{n} r\binom{n-1}{r-1}p^{r-1}(1-p)^{n-r}$$

$$= np\sum_{r=1}^{n} r\binom{n-1}{r-1}p^{r-1}(1-p)^{(n-1)-(r-1)}$$

$$n-1 = m$$

$$r-1 = j$$

$$\sum_{r=1}^{n} r\cdot\binom{n-1}{r-1}p^{r-1}(1-p)^{(n-1)-(r-1)}$$

$$= \sum_{j=0}^{m}(j+1)\cdot\binom{m}{j}p^j(1-p)^{m-j}$$

$$= \underbrace{j\cdot\sum_{j=0}^{m}\binom{m}{j}p^j(1-p)^{m-j}}_{mp} + \underbrace{\sum_{j=0}^{m}\binom{m}{j}p^j(1-p)^{m-j}}_{1}$$

$$= (n-1)p$$

$$\therefore\quad (n-1)p + 1$$

$$E(r^2) = np\{(n-1)p + 1\}$$

$$= n^2p^2 - np^2 + np$$

$$= (np)^2 + np(1-p)$$

$$Var(r) = E(r^2) - E(r)^2$$

$$= (np)^2 + np(1-p) - (np)^2$$

$$= np(1-p)$$

iii) Skewness

$$Skew(r) = \frac{E[(r-\mu)^3]}{\sigma^3}$$

$$= E\left(\left(\frac{r-\mu}{\sigma}\right)^3\right)$$

$$= \frac{E(r^3 - 3\mu r^2 + 3\mu^2 r - \mu^3)}{\sigma^3}$$

$$= \frac{E(r^3) - 3\mu E(r^2) + 3\mu^2 E(r) - \mu^3}{\sigma^3}$$

$$= \frac{E(r^3) - 3\mu(E(r^2) - \mu E(r)) - \mu^3}{\sigma^3}$$

$$= \frac{E(r^3) - 3\mu(E(r^2) - E(r)^2) - \mu^3}{\sigma^3}$$

$$= \frac{E(r^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

$$\mu = np$$

$$Var(r) = \sigma^2 = np(1-p)$$

$$E(r^3) = ?$$

$\vdots \quad \text{——} \quad ?$

$$\mathrm{Skew}(X) = \frac{E(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

$$= \frac{np + 3n^2p^2 - 3np^2 + n^3p^3 - 3n^2p^3 + 2np^3 - 3n^2p^2(1-p) - n^3p^3}{(np(1-p))^{3/2}}$$

$$= \frac{np - 3np^2 + 2np^3}{(np(1-p))^{3/2}}$$

$$= \frac{np(1-p)(1-2p)}{(np(1-p))^{3/2}}$$

$$= \frac{1-2p}{\sqrt{np(1-p)}}$$

iv) kurtosis $= \dfrac{1-6p(1-p)}{np(1-p)}$

$$\mathrm{kurt}(X) = E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right]$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{E(X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4)}{\sigma^4}$$

$$= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4}{\sigma^4}$$

$$= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^4 + \mu^4}{\sigma^4}$$

$$\mathrm{kurt}(X) = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^3}{\sigma^4}$$

$$= \frac{E(X^4) - 4np(np + 3n^2p^2 - 3np^2 + n^3p^3 - 3n^2p^3 + 2np^3) + 6n^2p^2(n^2p^2 + np(1-p)) - 3n^4p^4}{n^2p^2(1-p)^2}$$

$\Rightarrow E(X^4) = ?$

- Moment Generation function

$M_X(t) = (1 - p + pe^t)^n$

Using Moment generating function

$\Rightarrow E(X^4) = np + 3n^2p^2 - 3np^2 + n^3p^3 - 3n^2p^3$
$+ 2np^3 + 4n^2p^2 - 4np^2 + 5n^3p^3 - 15n^2p^3 + 10np^3$
$+ n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4$

$$\text{Kurt}(X) = \Big(np + 3n^2p^2 - 3np^2 + n^3p^3 - 3n^2p^3 + 2np^3 + 4n^2p^2 - 4np^2 + 5n^3p^3 - 15n^2p^3 + 10np^3 + n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4 - 4np(np + 3n^2p^2 - 3np^2 + n^3p^3 - 3n^2p^3 + 2np^3) + 6n^2p^2(n^2p^2 + np(1-p)) - 3n^4p^4 \Big)$$

$$\times \frac{1}{n^2p^2(1-p)^2}$$

$$\Rightarrow \frac{np - 7np^2 + 12np^3 - 6np^4}{n^2p^2(1-p)^2}$$

$$= \frac{np(1-p)(6p^2 - 6p + 1)}{n^2p^2(1-p)^2}$$

$$= \frac{1 + 6p(p-1)}{np(1-p)}$$

Show that for Poisson distribution

$\langle k \rangle = \lambda$, $Var(X) = \lambda$

$$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$E(X) = \lambda$$

$$E(X) = \sum x \cdot f_X(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=1}^{\infty} \frac{x}{x!} \lambda^x$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$\underline{z = x-1}$

$$= \lambda \cdot e^{-\lambda} \sum_{z=0}^{\infty} \frac{\lambda^z}{z!} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$= \lambda$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X)^2 = \sum x^2 p(x)$$

$$= \sum_{k=0} k^2 \frac{1}{k!} \lambda^k e^{-\lambda}$$

$$= \lambda e^{-\lambda} \left(\sum_{k=1} (k-1) \frac{1}{(k-1)!} \lambda^{k-1} \right.$$

$$\left. + \sum_{k=1} \frac{1}{(k-1)!} \lambda^{k-1} \right)$$

$$= \lambda \cdot e^{-\lambda} \left(\lambda \sum_{k=2} \frac{1}{(k-2)!} \lambda^{k-2} \right.$$

$$\left. + \sum_{k=1} \frac{1}{(k-1)!} \lambda^{k-1} \right)$$

$$= \lambda \cdot e^{-\lambda} \left(\lambda \sum_{i=0} \frac{1}{i!} \lambda^i + \sum_{j=0} \frac{1}{j!} \lambda^j \right)$$

$$= \lambda \cdot e^{-\lambda} \left(\lambda \cdot e^{\lambda} + e^{\lambda} \right)$$

$$= \lambda^2 + \lambda$$

$$\therefore Var(X) = E(X^2) - E(X)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$