전문통계분석법 HW3 Yang Mingyeong

1) Mean

X : random variable folling a normal distribution

$X \sim N(\mu, \sigma^2)$

$E(X) = \mu$

$E(X) = \int_X x \cdot f(x)\,dx \quad \cdots ①$

$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \cdots ②$

for normal distribution

$\Rightarrow E(X) = \int_X x \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$

$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} x \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$

$z = x - \mu$

$E(X) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty-\mu}^{\infty-\mu} (z+\mu) \cdot \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] d(z+\mu)$

$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{+\infty} (z+\mu) \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] dz$

$= \frac{1}{\sqrt{2\pi}\cdot\sigma}\left[\int_{-\infty}^{\infty} z \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot z^2\right) dz + \int_{-\infty}^{\infty} \mu \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot z^2\right) dz\right]$

The general antiderivatives are

$\int x \cdot \exp[-ax^2]\,dx = -\frac{1}{2a} \cdot \exp[-ax^2]$

$\int \exp[-ax^2]\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erf}[\sqrt{a}\,x]$

$\therefore \int_{-\infty}^{\infty} z \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right] dz$

$= -\frac{1}{2} \cdot 2\sigma^2 \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]$

$= -\sigma^2 \exp\left[\frac{1}{2\sigma^2} \cdot z^2\right]$

$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right] dz$

$= \frac{1}{2} \cdot \sqrt{\pi \cdot 2 \cdot \sigma^2} \cdot \mathrm{erf}\left(\frac{1}{\sqrt{2\sigma^2}} \cdot z\right)$

$= \sqrt{\frac{\pi}{2}} \cdot \sigma \cdot \mathrm{erf}\left(\frac{1}{\sqrt{2}\cdot\sigma} \cdot z\right)$

$E(X) = \frac{1}{\sqrt{2\pi}\sigma}\left(\left[-\sigma^2 \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right]_{-\infty}^{+\infty}\right.$

$\left. + \mu \cdot \left[\sqrt{\frac{\pi}{2}} \cdot \sigma \cdot \mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} \cdot z\right]\right]_{-\infty}^{+\infty}\right)$

$= \frac{1}{\sqrt{2\pi}\sigma}\left(\left[\lim_{z \to \infty}\left(-\sigma^2 \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right)\right.\right.$

$\left. - \lim_{z \to -\infty}\left(-\sigma^2 \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right)\right]$

$+ \mu\left[\lim_{z \to \infty}\left(\sqrt{\frac{\pi}{2}}\sigma\,\mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} \cdot z\right]\right)\right.$

$\left.\left. - \lim_{z \to -\infty}\left(\sqrt{\frac{\pi}{2}}\sigma\,\mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} \cdot z\right]\right)\right]\right)$

$= \frac{1}{\sqrt{2\pi}\sigma}\left([0-0] + \mu\left[\sqrt{\frac{\pi}{2}}\cdot\sigma - \left(-\sqrt{\frac{\pi}{2}}\cdot\sigma\right)\right]\right)$

$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \mu \cdot 2\sqrt{\frac{\pi}{2}} \cdot \sigma$

$= \mu$

Standard Deviation

$X \sim N(\mu, \sigma^2)$

$Var(X) = \sigma^2$

$$Var(X) = \int_X (x - E(X))^2 \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

$z = x - \mu$,

$$Var(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty-\mu}^{+\infty-\mu} z^2 \cdot \exp\left(-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right) d(z+\mu)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} z^2 \cdot \exp\left(-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right) dz$$

$z = \sqrt{2}\sigma x$

$$Var(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (\sqrt{2}\sigma x)^2 \cdot \exp\left[-\frac{1}{2}\left(\frac{\sqrt{2}\sigma x}{\sigma}\right)^2\right] d(\sqrt{2}\sigma x)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot 2\sigma^2 \cdot \sqrt{2}\sigma \int_{-\infty}^{\infty} x^2 \cdot \exp[-x^2]\, dx$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2} dx$$

the integrand is symmetric,

$$Var(X) = \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} x^2 \cdot e^{-x^2} dx$$

$z = x^2$, $x = \sqrt{z}$ $\quad dx = \frac{1}{2} \cdot z^{-1/2} dz$

$$Var(X) = \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z \cdot e^{-z} \cdot \frac{1}{2} \cdot z^{-\frac{1}{2}} dz$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z^{\frac{3}{2}-1} \cdot e^{-z} dz$$

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \cdot e^{-z} dz$$

$$\therefore Var(X) = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

$\Rightarrow$ Standard deviation $= \sqrt{Var(X)}$

$= \sigma$

Skewness

Normal distribution is a symmetric distribution and has a skewness of zero.

$$\gamma = E\left(\left(\frac{X-\mu}{\sigma}\right)^3\right)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\gamma = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^3 \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$u = x - \mu$

$$\gamma = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left(\frac{u}{\sigma}\right)^3 \cdot \exp\left[-\frac{u^2}{2\sigma^2}\right] du$$

$$(-u)^3 \cdot \exp\left(-\frac{(-u)^2}{2\sigma^2}\right) = -u^3 \cdot \exp\left(-\frac{u^2}{2\sigma^2}\right)$$

$\Rightarrow$ the integral is odd.

$$\therefore \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left(\frac{u}{\sigma}\right)^3 \cdot \exp\left(-\frac{u^2}{2\sigma^2}\right) du = 0$$

$\gamma = 0$

Kurtosis

$$\alpha = E\left(\left(\frac{X-\mu}{\sigma}\right)^4\right)$$

$$= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4}$$

$$= \frac{E(X^4) - 4\mu(3\mu\sigma^2 + \mu^3) + 6\mu^2(\sigma^2 + \mu^2) - 3\mu^4}{\sigma^4}$$

$E(X^4) = M_X^{(4)}(0)$ (Moment Generating Function)

$$M^{(4)}(t) = \left(3\sigma^4 + 6\sigma^2(\mu + \sigma^2 t)^2 + (\mu + \sigma^2 t)^4\right)\exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)$$

$$E(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

$$\alpha = \frac{(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4) - 4\mu(3\mu\sigma^2 + \mu^3) + 6\mu^2(\sigma^2 + \mu^2) - 3\mu^4}{\sigma^4}$$

$$= \frac{(1 - 4 + 6 - 3)\mu^4 + (6 - 12 + 6)\mu^2\sigma^2 + 3\sigma^4}{\sigma^4}$$

$$= \frac{3\sigma^4}{\sigma^4}$$

$$= 3$$