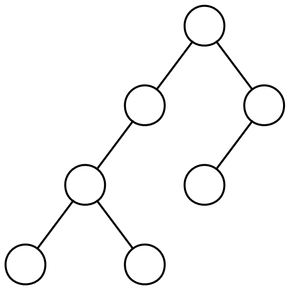
Algorithm Analysis Homework 5

April 10 (Due the next Tuesday class)

1. (a) Fill the nodes of the following binary tree with $1, 2, 3, 4, 5, 6$ and 7 for seven keys so that the tree becomes a binary search tree.



(b) Suppose that the probabilities that searches are performed for the seven keys are $p_1 = 0.05$, $p_2 = 0.1$, $p_3 = 0.3$, $p_4 = 0.1$, $p_5 = 0.15$, $p_6 = 0.2$, and $p_7 = 0.1$. What is the average cost (i.e., the number of key comparisons) for a search with this binary search tree?

2. (a) Write a computer program, given probabilities $(p_1, \ldots, p_n)$ of n keys $K_1 < \cdots < K_n$, that outputs

   (1) table of $\mathrm{Opt}(i, j)$, where

$$\begin{cases} \mathrm{Opt}(i,j) = 0, & i > j; \\ \mathrm{Opt}(i,j) = \min\limits_{i \le k \le j} \left(1 + \dfrac{p(i,k-1)}{p(i,j)} \mathrm{Opt}(i,k-1) + \dfrac{p(k+1,j)}{p(i,j)} \mathrm{Opt}(k+1,j) \right), & 1 \le i \le j \le n. \end{cases}$$

   and $p(i, j) = p_i + \cdots + p_j$.

   (2) table of k such that, for $i \le k \le j$,

$$1 + \frac{p(i,k-1)}{p(i,j)} \mathrm{Opt}(i,k-1) + \frac{p(k+1,j)}{p(i,j)} \mathrm{Opt}(k+1,j)$$

   is minimized,

   (3) (Optional) corresponding optimal binary search tree

   For example, for input $(p_1, \ldots, p_7) = (0.05, 0.1, 0.3, 0.1, 0.15, 0.2, 0.1)$, the output is as follows:

```
table of opt[i,j]

1.00 1.33 1.44 1.55 1.79 2.06 2.20
     1.00 1.25 1.40 1.69 2.00 2.16
          1.00 1.25 1.64 1.93 2.06
               1.00 1.40 1.67 1.82
                    1.00 1.43 1.56
                         1.00 1.33
                              1.00
```

```
table of k[i,j]

1  2  3  3  3  3  3
   2  3  3  3  3  3
      3  3  3  5  5
         4  5  5  6
            5  6  6
               6  6
                  7

Need to figure out how to print out BST :)

  3
 2   6
1   5 7
   4
```

(b) Run your program on the following inputs

(i) $(p_1, \ldots, p_4) = (0.4, 0.05, 0.25, 0.3)$

(ii) $(p_1, \ldots, p_7) = (0.1, 0.1, 0.2, 0.3, 0.1, 0.1, 0.1)$

(iii) $(p_1, \ldots, p_8) = (0.15, 0.2, 0.05, 0.3, 0.1, 0.1, 0.05, 0.05)$

(iv) $(p_1, \ldots, p_{10}) = (0.08, 0.12, 0.05, 0.15, 0.2, 0.05, 0.05, 0.1, 0.1, 0.1)$

and show the outputs.

3. **(Integer Partitions)** Given a positive integer n, an integer partition of n is a tuple $(n_1, \ldots, n_k)$ such that $n = n_1 + n_2 + \cdots + n_k$ and $n \geq n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. For example, there are 7 integer partitions of 5 as follows:

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

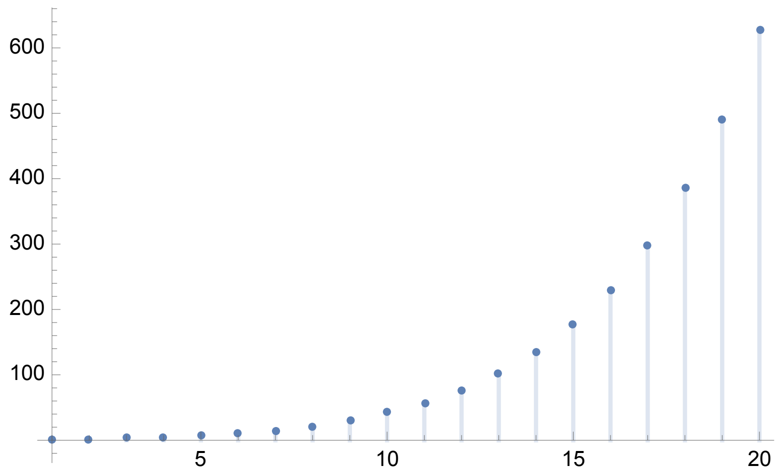
Also, for $n = 7$, we have

$$
\begin{aligned}
7 &= 7 \\
&= 6 + 1 \\
&= 5 + 2 \\
&= 5 + 1 + 1 \\
&= 4 + 3 \\
&= 4 + 2 + 1 \\
&= 4 + 1 + 1 + 1 \\
&= 3 + 3 + 1 \\
&= 3 + 2 + 2 \\
&= 3 + 2 + 1 + 1 \\
&\vdots \\
&= 1 + 1 + \cdots + 1
\end{aligned}
$$

How many integer partitions are there for $n = 7$? Let $p(n)$ be the number of integer partitions of n. For a small n's, we can tabulate $p(n)$ as follows:

n	$p(n)$
1	1
2	2
3	3
4	5
5	7
6	11
7	15

The function $p(n)$ increases very fast, and the following is a plot:



What is the exact value of $p(15)$? Or $p(100)$? To compute them, consider $p(n,k)$, the number of k-part integer partitions of n, that is, the number of k-tuples $(n_1, n_2, \ldots, n_k)$ such that

$n = n_1 + \cdots + n_k$ with $n \geq n_1 \geq n_2 \geq \cdots \geq n_k$. For example, for $n = 5$,

$$\begin{aligned} p(5,1) &= 1 \quad (5) \\ p(5,2) &= 2 \quad (4+1,\ 3+2) \\ p(5,3) &= 2 \quad (3+1+1,\ 2+2+1) \\ p(5,4) &= 1 \quad (2+1+1+1) \\ p(5,5) &= 1 \quad (1+1+1+1+1), \end{aligned}$$

and $p(5) = p(5,1) + p(5,2) + p(5,3) + p(5,4) + p(5,5)$. For $n = 7$, we have

$$\begin{aligned} p(7,1) &= 1 \\ p(7,2) &= 3 \\ p(7,3) &= 3 \\ &\vdots \end{aligned}$$

and $p(7) = p(7,1) + p(7,2) + \cdots + p(7,7)$. First, note that, clearly,

$$p(n) = p(n,1) + p(n,2) + \cdots + p(n,n). \tag{23}$$

And, as initial or boundary conditions, we have

$$p(n,n) = 1,\ p(n,1) = 1, \tag{24}$$

and, if $k > n$, then

$$p(n,k) = 0, \quad \text{if } k > n. \tag{25}$$

(a) Show that $p(n,k)$ satisfies the following equation, for $1 \leq k \leq n$:

$$p(n,k) = p(n-k,k) + p(n-1,k-1). \tag{26}$$

Hint: Consider 3-part partitions of 7. For $7 = n_1 + n_2 + n_3$, we have two cases: (1) $n_3 \geq 2$, and (2) $n_3 = 1$. The partitions of the case (1) have one-to-one correspondence to 3-part partitions of 4 because $7 = (n_1 - 1) + (n_2 - 1) + (n_3 - 1) + 3$ and $(n_1 - 1, n_2 - 1, n_3 - 1)$ is a partition of 4. The other case corresponds to the 2-part partitions of 6, because, in this case, $7 = n_1 + n_2 + 1$, and (n_1, n_2) is a partition of 6.

(b) What are the values of $p(15)$, $p(100)$, and $p(200)$?

(c) To answer the above question, did you use dynamic programming? If so, explain how you used dynamic programming.