$$X_1, \ldots, X_n \mid \mu, \lambda \overset{iid}{\sim} N(\mu, \lambda^{-1}),$$

where μ and λ are both <u>unknown</u>

$$\text{Normal Gamma} \left[\begin{array}{l} \mu \mid \lambda \sim N(m, (\underline{c\lambda})^{-1}) \\ \quad \lambda \sim \text{Gamma}(a, b) \end{array} \right.$$

Show $p(\mu, \lambda \mid x_1, \ldots, x_n)$ updated Normal Gamma

$$p(\mu, \lambda \mid x_1, \ldots, x_n) \propto p(x_1, \ldots, x_n \mid \mu, \lambda)\, p(\mu, \lambda)$$

$$\overset{ind}{=} \underbrace{p(x_1, \ldots, x_n \mid \mu, \lambda)}_{N(x_1, \ldots, x_n \mid \mu, \lambda)} \underbrace{p(\mu \mid \cancel{x_1, \ldots, x_n}\ \lambda)\, p(\lambda)}_{\text{NormalGamma}(\mu, \lambda \mid \underline{m, c}, \underline{a, b})}$$

① Calc the likelihood

② Calc the prior

① For a single x

$$N(x \mid \mu, \lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}} \exp\left\{-\frac{\lambda}{2}(x-\mu)^2\right\}$$

$$\underset{\mu, \lambda}{\propto} \underline{\lambda^{1/2}} \exp\left\{-\frac{\lambda}{2}\left[x^2 - 2x\mu + \mu^2\right]\right\}$$

(For n data pts $\rightarrow$ $\lambda^{n/2} \exp\left\{\frac{-\lambda}{2}\left[\sum x_i^2 - 2\sum x_i \mu + n\mu^2\right]\right\}$)

② For the joint prior

$$\text{NormalGamma}(\mu, \lambda \mid m, c, a, b)$$

$$\overset{\text{ind.}}{=} N(\mu \mid m, (c\lambda)^{-1})\ \mathrm{Gamma}(\lambda \mid a, b)$$

$$= \sqrt{\frac{c\lambda}{2\pi}} \exp\left\{-\frac{c\lambda}{2}(\mu - m)^2\right\} \times \boxed{\frac{b^a}{\Gamma(a)}} \lambda^{a-1} e^{-b\lambda}$$

$$\underset{\mu,\lambda}{\propto} \lambda^{1/2 + a - 1} \exp\left\{-\frac{c\lambda}{2}(\mu^2 - 2\mu m + m^2) - b\lambda\right\}$$

$$= \boxed{\lambda^{a - 1/2} \exp\left\{-\frac{\lambda}{2}\left(c\mu^2 - 2c\mu m + cm^2 + 2b\right)\right\}}$$

$$p(\mu, \lambda \mid x_{1:n}) \underset{\mu,\lambda}{\propto} \lambda^{n/2} \exp\left\{-\frac{\lambda}{2}\left[n\mu^2 - 2\sum_{i=1}^{n} x_i \mu + \sum_{i=1}^{n} x_i^2\right]\right\}$$

$$\times\ \lambda^{a - 1/2} \exp\left\{-\frac{\lambda}{2}\left[c\mu^2 - 2c\mu m + cm^2 + 2b\right]\right\}$$

$$\propto \lambda^{\overbrace{a + n/2}^{}\ - 1/2}_{A} \exp\left\{-\frac{\lambda}{2}\left[\mu^2 \overbrace{(c+n)}^{C} - 2\mu \overbrace{\left(\sum_i x_i + cm\right)}^{CM} + cm^2 + 2b + \sum_{i=1}^{n} x_i^2\right]\right\}$$

Let $A = a + \frac{n}{2}$, $C = c + n$, $CM = \sum x_i + cm$

$$cM^2 + 2b + \sum x_i^2 = CM^2 + 2B$$

Equ. (6) of NormalGamma.

$\rightarrow$ NormalGamma$(\mu, \lambda \mid M, C, A, B)$