Homework 5

STA 360: Assignment 5, Fall 2020

Due Friday September 18, 5 PM Standard Eastern Time

Hoff Exercise 3.12

3.12 a)

$Y \sim \text{binomial}(n,\theta) \rightarrow \binom{n}{y} \theta^y (1-\theta)^{n-y}$

$\log(p(y|\theta)) \rightarrow \log\binom{n}{y} + y\log(\theta) + (n-y)\log(1-\theta)$

$$\frac{\partial \log(p(y|\theta))}{\partial\theta} = y * \frac{1}{\theta} + (n-y) * \frac{1}{1-\theta} \times -1$$

$$= \frac{y}{\theta} - \frac{n-y}{1-\theta}$$

$$\frac{\partial^2 \log(p(y|\theta))}{\partial\theta} = -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}$$

$$I(\theta) = -E\left[\frac{-y}{\theta^2} - \frac{n-y}{(1-\theta)^2} \middle| \theta\right]$$

$E[y|\theta] = n\theta$ so $I(\theta)$ becomes

$$= -\frac{1}{\theta^2}E[y|\theta] - \frac{1}{(1-\theta)^2}E[n-y|\theta]$$

$$\frac{-n\theta}{\theta^2} - \frac{n(1-\theta)}{(1-\theta)^2} \xrightarrow{} \text{simplifies to } \frac{-n}{\theta} - \frac{n}{(1-\theta)} \rightarrow \frac{-n}{\theta(1-\theta)}$$

$$I(\theta) = -\frac{-n}{\theta(1-\theta)} = \frac{n}{\theta(1-\theta)} \quad P_J(\theta) = \sqrt{\frac{n}{\theta(1-\theta)}}$$

$$\sqrt{\frac{n}{\theta(1-\theta)}} = n^{1/2}\,\theta^{-1/2}\,(1-\theta)^{-1/2}$$

constant $\quad \theta^{-1/2+1-1}(1-\theta)^{-1/2+1-1} = \text{Beta}(-1/2+1, -1/2+1)$

$= \text{Beta}(1/2, 1/2)$

a.

b) $P(y|\psi) = \binom{n}{y} e^{\psi y}(1+e^{\psi})^{-n}$

$\log(p(y|\psi)) = \log\binom{n}{y} + \psi y - n\log(1+e^{\psi})$

$$\frac{\partial \log(p(y|\psi))}{\partial \psi} = y - \frac{ne^{\psi}}{(1+e^{\psi})}$$

$$\frac{\partial^2 \log(p(y|\psi))}{\partial \psi} = \frac{\partial}{\partial \psi}\left(\frac{-ne^{\psi}}{(1+e^{\psi})}\right) \xrightarrow{\text{product rule}} -n\left(\frac{e^{\psi}(1+e^{\psi}) - e^{\psi}e^{\psi}}{(1+e^{\psi})^2}\right)$$

simplify $= \dfrac{-ne^{\psi}}{(1+e^{\psi})^2}$

$$I(\psi) = -E\left[\frac{-ne^{\psi}}{(1+e^{\psi})^2}\Big|\psi\right] \xrightarrow[\text{on } \psi]{\text{condition}} \frac{ne^{\psi}}{(1+e^{\psi})^2}$$

$$P_J(\psi) = \sqrt{\frac{ne^{\psi}}{(1+e^{\psi})^2}} = \frac{n^{1/2}e^{1/2\psi}}{1+e^{\psi}} \underset{\psi}{\propto} = \boxed{\frac{e^{1/2\psi}}{1+e^{\psi}}}$$

b.

c.

$\theta \sim Beta(1/2, 1/2)$, $\psi = \log\left(\frac{\theta}{1-\theta}\right)$

$e^{\psi} = \frac{\theta}{1-\theta}$ Solve for $\theta \rightarrow \theta = \frac{e^{\psi}}{(1+e^{\psi})} = h(\psi)$

So

$$p_\theta\left(\frac{e^\psi}{(1+e^\psi)}\right) = \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}\left(\frac{e^\psi}{1+e^\psi}\right)^{-1/2}\left(1-\frac{e^\psi}{1+e^\psi}\right)^{-1/2}$$

$$\frac{\partial h(\psi)}{\partial \psi} = \frac{e^\psi(1+e^\psi)-e^{2\psi}}{(1+e^\psi)^2} \rightarrow \frac{e^\psi}{(1+e^\psi)^2}$$

Thus

$$p_\psi(\psi) = \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}\left(\frac{e^\psi}{1+e^\psi}\right)^{-1/2}\left(\frac{1}{1+e^\psi}\right)^{-1/2} * \frac{e^\psi}{(1+e^\psi)^2}$$

Simplify

$$\cancel{\frac{(1+e^\psi)^{1/2}}{e^{\psi/2}}}\ \cancel{\frac{(1+e^\psi)^{1/2}}{1}} \times \cancel{\frac{e^\psi}{(1+e^\psi)^2}} \rightarrow \frac{e^{1/2\psi}}{1+e^\psi}$$

$$\underbrace{\cancel{\frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}}}_{\text{Constants}}\frac{e^{1/2\psi}}{1+e^\psi} \underset{\psi}{\propto} = \frac{e^{1/2\psi}}{1+e^\psi}$$

So

$p_\psi(\psi) = \frac{e^{1/2\psi}}{1+e^\psi}$ Same as part b

c.

Lab Component

a. Task 4

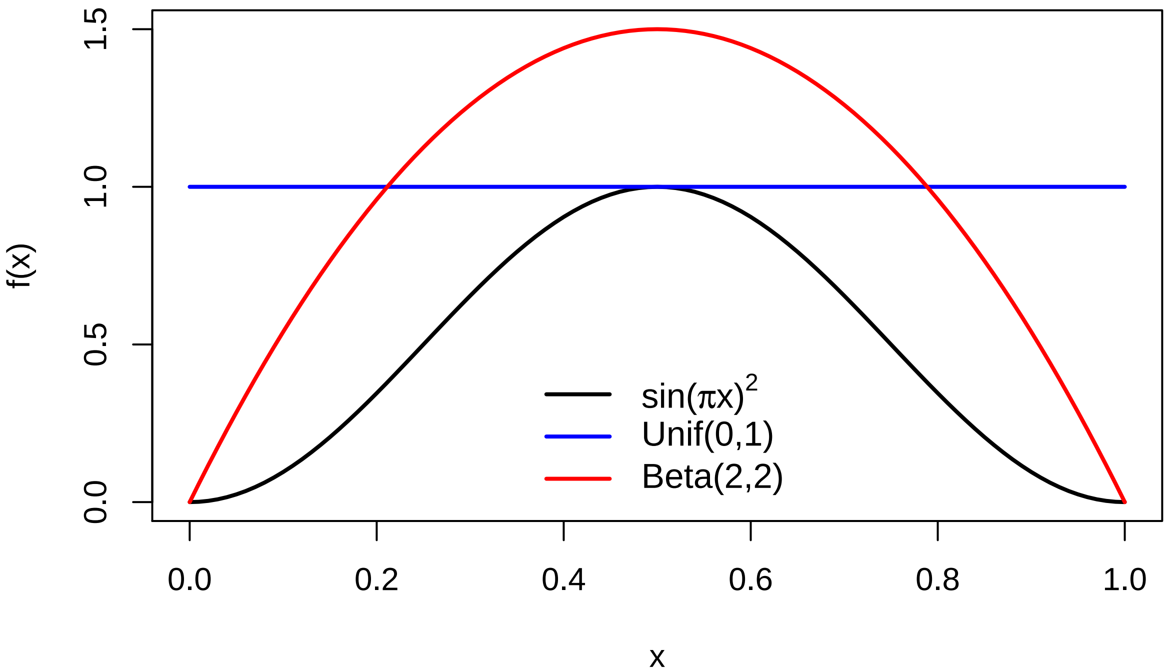
```r
#Code taken from Professor Steort's Lab 5 RMD file

# grid of points
x <- seq(0, 1, 10^-2)

fx <- function(x) sin(pi * x)^2

plot(fx, xlim = c(0,1), ylim = c(0,1.5), ylab = "f(x)", lwd = 2)
curve(dunif, add = TRUE, col = "blue", lwd = 2)
curve(dbeta(x,2,2), add = TRUE, col = "red", lwd = 2)
legend("bottom", legend = c(expression(paste("sin(",pi,"x)"^"2")),"Unif(0,1)",
```

```r
"Beta(2,2)"), col = c("black", "blue", "red"), lty = c(1,1,1), bty = "n", cex = 1.1, lwd = 2)
```

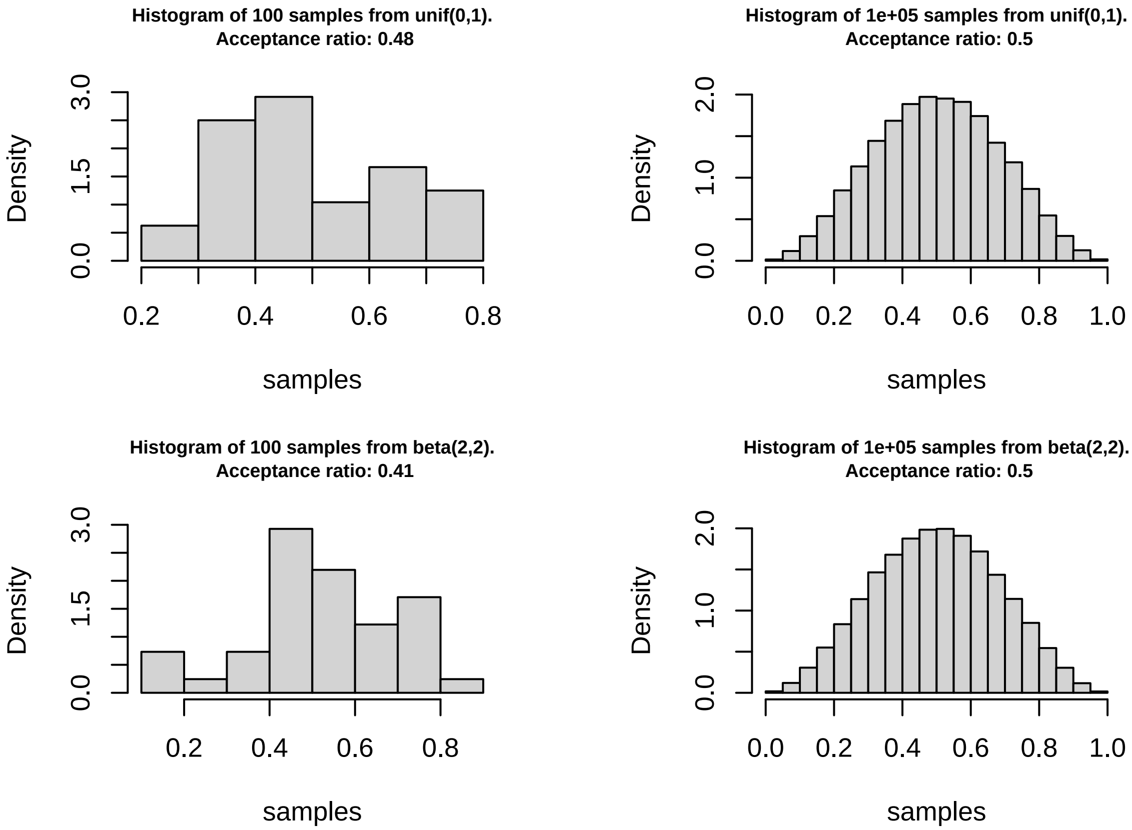



```r
sim_fun <- function(f, envelope = "unif", par1 = 0, par2 = 1, n = 10^2, plot = TRUE){
  r_envelope <- match.fun(paste0("r", envelope))
  d_envelope <- match.fun(paste0("d", envelope))
  proposal <- r_envelope(n, par1, par2)
  density_ratio <- f(proposal) / d_envelope(proposal, par1, par2)
  samples <- proposal[runif(n) < density_ratio]
  acceptance_ratio <- length(samples) / n
  if (plot) {
    hist(samples, probability = TRUE,
         main = paste0("Histogram of ",
                       n, " samples from ",
                       envelope, "(", par1, ",", par2,
                       ").\n Acceptance ratio: ",
                       round(acceptance_ratio,2)),
                       cex.main = 0.75)
  }
  list(x = samples, acceptance_ratio = acceptance_ratio)
}

par(mfrow = c(2,2), mar = rep(4, 4))

unif_1 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^2)
unif_2 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^5)

# Add in the Beta(2,2) Histograms
beta_1 <- sim_fun(fx, envelop = "beta", par1 = 2, par2 =2, n = 10^2)
beta_2 <- sim_fun(fx, envelop = "beta", par1 = 2, par2 =2, n = 10^5)
```




```r
par(mfrow = c(1,1))
```

From the histograms above, we can see that the more samples taken, the distribution of the acceptance ratios of our distributions look more normal with a mean value around 0.5. There is more variance with the 100 sample histograms; the average acceptance ratio of the samples vary more with the uniform having a slightly higher or lower ratio compared to the beta each time you run the code above. However, even with more variance ,the average acceptance ratio of the 100 samples still is close to the value of 0.5.