Homework 2

STA 360: Assignment 2, Fall 2020

Due Friday August 28, 5 PM Standard Eastern Time

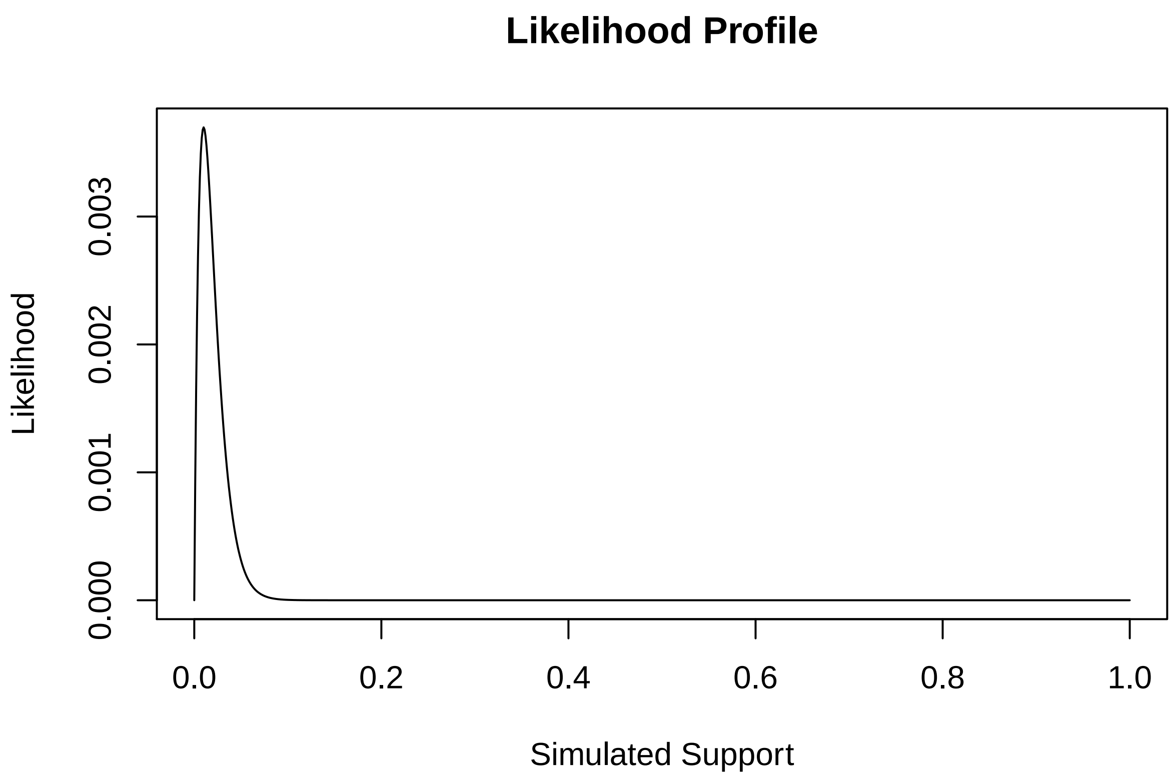
```
library(tidyverse) #load in tidyverse package
```

Lab Component

a. Task 3

```
set.seed(123)
### Bernoulli LH Function ###
# Input: obs.data, theta
# Output: bernoulli likelihood
obs.data <- rbinom(n = 100, size = 1, prob = 0.01)
bernLH <- function(obs.data, theta){
  N <- length(obs.data)
  x <- sum(obs.data)
  LH <- (theta ^x) *((1-theta)^{N-x})
  return (LH)
}

### Plot LH for a grid of theta values ###
# Create the grid #
theta.sim <- seq(from = 0, to = 1, length.out = 1000)
# Store the LH values
sim.LH <- bernLH(obs.data, theta = theta.sim)
# Create the Plot
plot(theta.sim, sim.LH, type = "l", main = "Likelihood Profile",
     xlab = "Simulated Support",
     ylab = "Likelihood")
```



b. Task 4

```
myBetaBernoulli <- function(obs.data, theta, a, b){
  N <- length(obs.data)
  x <- sum(obs.data)
  BeBern <- dbeta(theta, x + a, N-x+b)
  param1= x+a
  param2 = N-x+b
  print(param1)
  print(param2)
  return(BeBern)
}

non.informative.prior <- dbeta(theta.sim,1,1)
informative.prior <- dbeta(theta.sim,3,1)

posterior.non.informative <- myBetaBernoulli(obs.data, theta.sim, 1, 1)
```

```
## [1] 2
## [1] 100
```

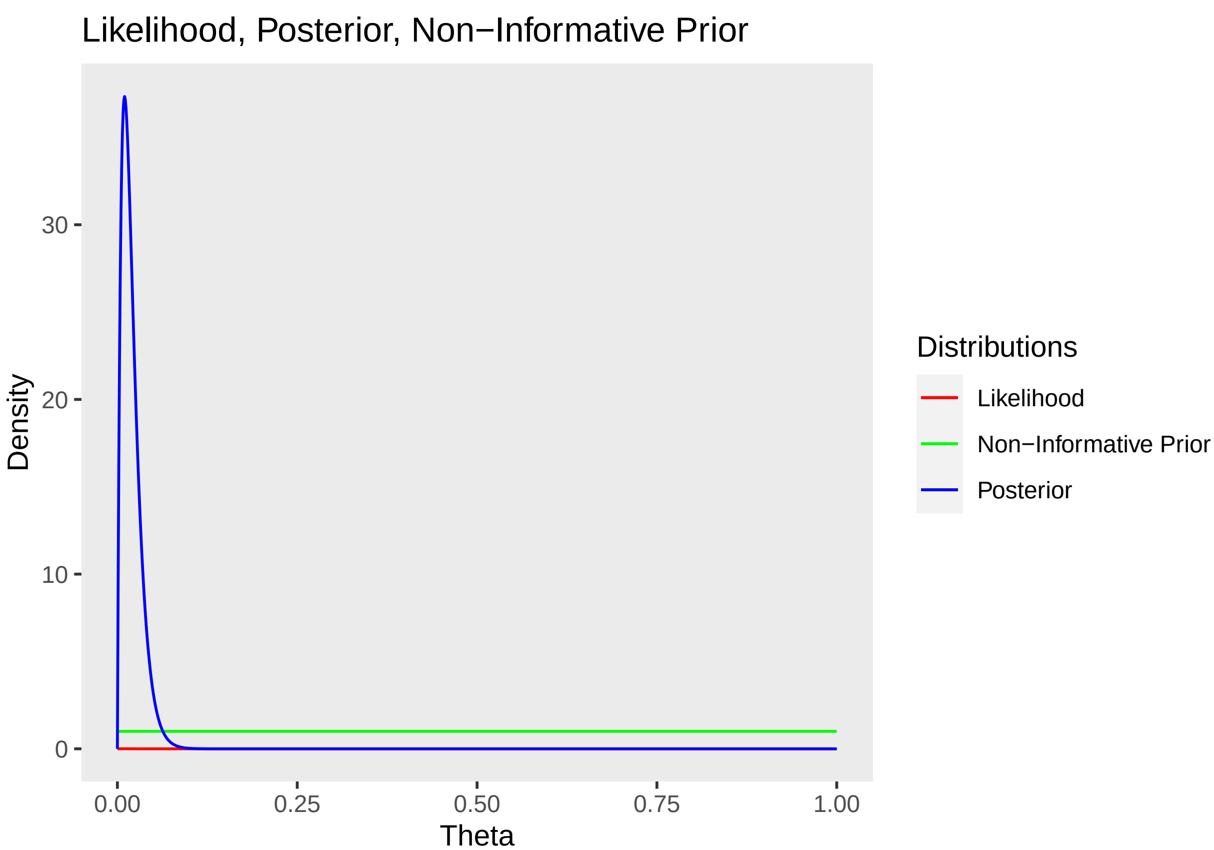
```
posterior.informative <- myBetaBernoulli(obs.data, theta.sim, 3, 1)
```

```
## [1] 4
## [1] 100
```

The parameters for the posterior with a non-informative prior are Beta(2, 100) and for the informative, the parameters are Beta(4, 100).

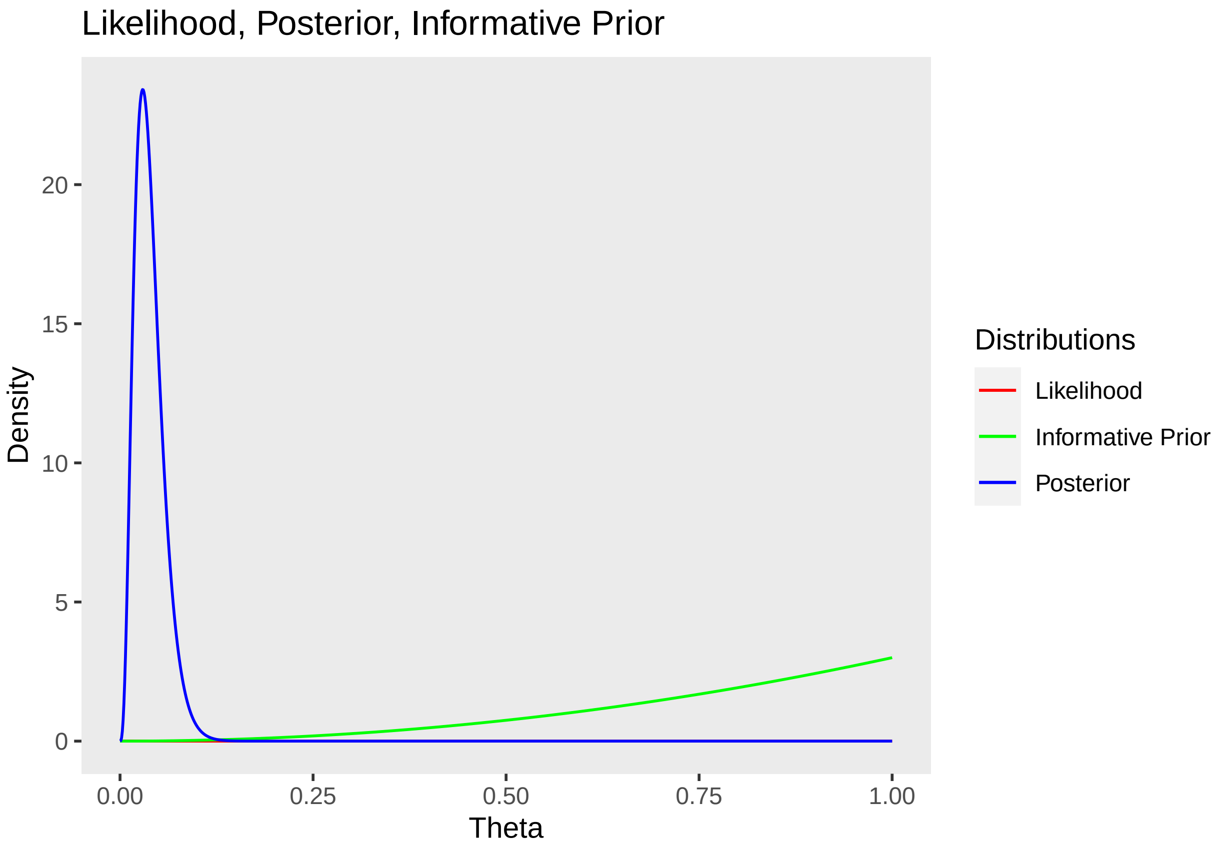
c. Task 5

```r
df<- data.frame(theta.sim, non.informative.prior, informative.prior, sim.LH)
ggplot(df, aes(theta.sim)) +
  geom_line(aes(y=sim.LH, color="Likelihood")) +
  geom_line(aes(y=non.informative.prior,
              color= "Non-Informative Prior")) +
  geom_line(aes(y=posterior.non.informative, color="Posterior"))+
  scale_color_manual(name = "Distributions",
     breaks = c("Likelihood","Non-Informative Prior",
              "Posterior"),
     values = c("red", "green","blue"))+
labs(title = "Likelihood, Posterior, Non-Informative Prior",
      y = "Density",
      x = "Theta" ) +
  theme(
    panel.grid.major = element_blank(),
    panel.grid.minor = element_blank(),
  )
```




```r
ggplot(df, aes(theta.sim)) +
  geom_line(aes(y=sim.LH, color="Likelihood")) +
  geom_line(aes(y=informative.prior,
              color= "Informative Prior")) +
  geom_line(aes(y=posterior.informative, color="Posterior"))+
  scale_color_manual(name = "Distributions",
     breaks = c("Likelihood","Informative Prior",
              "Posterior"),
     values = c("red", "green","blue"))+
```

```r
labs(title = "Likelihood, Posterior, Informative Prior",
       y = "Density",
       x = "Theta" ) +
  theme(
    panel.grid.major = element_blank(),
    panel.grid.minor = element_blank(),
  )
```



We can see that the informative prior is slightly more to the right than the non-informative prior. Having an informative prior shifted the posterior distribution so the value of theta it's centered about is greater than the non-informative prior posterior distribution. We can also see that the non-informative distribution is taller than the informative

The Exponential-Gamma Model

① $p(\theta|x) = \frac{p(\theta,x)}{p(x)} = \frac{p(x|\theta)p(\theta)}{p(x)} \propto p(x|\theta)\,p(\theta)$

In this case:

$p(x|\theta) = \text{Exp}(x|\theta) = \theta \exp(-\theta x)\ I(x>0)$

$p(\theta) = \text{Gamma}(\theta|a,b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta)\ I(\theta>0)$

so

② $p(x|\theta)p(\theta) = \theta e^{-\theta x} * \cancel{\frac{b^a}{\Gamma(a)}}\,\theta^{a-1} e^{-b\theta}\quad I(\theta>0)$

remove constant

③ $\theta e^{-\theta x} * \theta^{a-1} e^{-b\theta} \rightarrow \theta^{(a+1)-1} e^{-\theta(b+x)} \rightarrow \theta^{(a+1)-1}\exp(-\theta(b+x))$

group like terms

④ We can see that the posterior, $p(\theta|x)$, is a Gamma distribution with parameters $a+1$, $b+x$

$\text{Gamma}(\theta\,|\,a+1, b+x) \rightarrow \theta^{(a+1)-1}\exp(-\theta(b+x))$

Thus $p(\theta|x) = \text{Gamma}(\theta\,|\,a+1, b+x)$

a.

b. The posterior distribution is a proper density distribution function because it is an actual probability distribution, a Gamma distribution. Improper distributions are functions that do not integrate to 1. In this case, the Gamma distribution with parameters (a + 1, b + x) integrates to 1 with respect to theta.

c.

```
q2.obs.data <- c(20.9, 69.7, 3.6, 21.8, 21.4, 0.4, 6.7, 10.0)
q2.theta <- seq(from = 0.4, to = 69.7, length.out = 1000)
x <-sum(q2.obs.data)
alpha = 0.1
beta = 1.0
alpha2 = 0.1 + length(q2.obs.data)
beta2 = 1.0 + x
q2.prior <- dgamma(x = q2.theta, alpha , rate = 1.0/beta)
q2.posterior <-dgamma(x = q2.theta, alpha2, rate = 1.0/beta2)
df2<-data.frame(q2.theta, q2.prior, q2.posterior)
ggplot(df2, aes(x = q2.theta)) +
  geom_line(aes(y=q2.prior, color="Prior")) +
  geom_line(aes(y=q2.posterior,
                color= "Posterior")) +
  scale_color_manual(name = "Distributions",
     breaks = c("Prior",
```

```r
                "Posterior"),
     values = c("red", "green"))+
labs(title = "Posterior and Prior",
       y = "Density",
       x = "Theta" )
```