STA360 Practice Exam I Fall 2020

Instructions

- This exam is closed note and closed book.

- Write your name, NetID, and signature below.

- **Only what is on the exam will be graded (or written work submitted as one pdf file).**

- **Show all work and back up all your results for full credit.**

- **Only what's on the exam will be graded. It must be legible.**

- **You must label/assign pages when submitting to Gradescope to avoid losing points.**

- **The exam should be submitted via Gradescope. You will have 15 minutes to submit your exam to Gradescope after the exam as a single pdf file.**

Community Standard

To uphold the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;

- I will conduct myself honorably in all my endeavors; and

- I will act if the Standard is compromised.

I have adhered to the Duke Community Standard in completing this exam.

Name: _____

NetID: _____

Signature: _____

Score

(For TA use only — leave this section blank.)

1. _____/2.5

2. _____/10

Overall: _____/12.5

List of common distributions

$\text{Geometric}(x|\theta) = \theta(1-\theta)^x \, \mathbb{1}(x \in \{0, 1, 2, \ldots\})$ for $0 < \theta < 1$

$\text{Bernoulli}(x|\theta) = \theta^x(1-\theta)^{1-x} \, \mathbb{1}(x \in \{0, 1\})$ for $0 < \theta < 1$

$\text{Binomial}(x|n, \theta) = \binom{n}{x} \theta^x(1-\theta)^{n-x} \, \mathbb{1}(x \in \{0, 1, \ldots, n\})$ for $0 < \theta < 1$

$\text{Poisson}(x|\theta) = \dfrac{e^{-\theta}\theta^x}{x!} \, \mathbb{1}(x \in \{0, 1, 2, \ldots\})$ for $\theta > 0$

$\text{Exp}(x|\theta) = \theta e^{-\theta x} \, \mathbb{1}(x > 0)$ for $\theta > 0$

$\text{Uniform}(x|a, b) = \dfrac{1}{b-a} \, \mathbb{1}(a < x < b)$ for $a < b$

$\text{Gamma}(x|a, b = \text{rate}) = \dfrac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \, \mathbb{1}(x > 0)$ for $a, b > 0$,

$\text{Gamma}(x|a, b = \text{scale}) = \dfrac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b} \, \mathbb{1}(x > 0)$ for $a, b > 0$,

$\text{Pareto}(x|\alpha, c) = \dfrac{\alpha c^\alpha}{x^{\alpha+1}} \, \mathbb{1}(x > c)$ for $\alpha, c > 0$

$\text{Beta}(x|a, b) = \dfrac{1}{B(a, b)} x^{a-1}(1-x)^{b-1} \, \mathbb{1}(0 < x < 1)$ for $a, b > 0$

$\mathcal{N}(x|\mu, \sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{1}{2\sigma^2}(x-\mu)^2\right)$ for $\mu \in \mathbb{R}$, $\sigma^2 > 0$

$\mathcal{N}(x|\mu, \lambda^{-1}) = \sqrt{\dfrac{\lambda}{2\pi}} \exp\left(-\tfrac{1}{2}\lambda(x-\mu)^2\right)$ for $\mu \in \mathbb{R}$, $\lambda > 0$

1. (2.5 points) Assume the data is $x_{1:n} = (x_1, \ldots, x_n)$ and θ is the unknown parameter of interest. Let x_{n+1} be a new data point. **Circle the most correct answer (or correct answers).**

   (a) (2.5 points) Assume that you have the likelihood $p(x_{1:n} \mid \theta)$, the prior $p(\theta)$, and marginal distribution $p(x_{1:n})$ available to you. The posterior distribution $p(\theta \mid x_{1:n})$ can be derived (written) as the following:

   i.
   $$p(\theta \mid x_{1:n}) = \frac{p(x_{1:n} \mid \theta)p(\theta)}{p(x_{1:n})}$$

   ii.
   $$p(\theta \mid x_{1:n}) = \frac{p(x_{1:n}, \theta)}{p(x_{1:n})}$$

   iii.
   $$p(\theta \mid x_{1:n}) = \frac{p(\theta, x_{1:n})}{p(x_{1:n})}$$

   iv.
   $$p(\theta \mid x_{1:n}) \propto p(x_{1:n} \mid \theta)p(\theta)$$

2. (Normality and Bayes, 10 points) Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = x_i \beta + \epsilon_i, i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are fixed constants, $\epsilon_1, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$. Assume that σ^2 is known.

(a) (2 points) Find the maximum likelihood (ML) estimator of β, and denote it by $\hat{\beta}$.

(b) (1.5 points) Find the distribution of $\hat{\beta}$.

(c) (1.5 point) Find the distribution of an alternative estimator of β given by

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i}.$$

(d) (2.5 points) Find the posterior distribution of β under a normal prior with mean 0 and variance $\dfrac{\tau^2}{\sum_{i=1}^{n} x_i^2}$.

(e) (2.5 points) Show that the posterior expectation of β can be written as a function of $\hat{\beta}$.

Continued Workpage for Problem 2.

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