

Practice Exercise 5

Exercise

Let

$$p(y|\theta) = \theta^{-1}e^{-y/\theta}, \quad y > 0, \theta > 0.$$
$$p(\theta) = \theta^{-a}e^{-b/\theta}, \quad \theta > 0, a > 2, b > 0.$$

1. Find the posterior distribution of $\theta|y$.
2. Calculate the posterior mean and posterior variance.
3. Notice the prior is still proper when $1 < a \leq 2$. How would such a change affect the posterior mean and posterior variance?

Solution:

1. $\theta|y \sim \text{InverseGamma}(a, y + b)$.
2. $E(\theta|y) = \frac{y + b}{a - 1}$. $V(\theta|y) = \frac{(y + b)^2}{(a - 1)^2(a - 2)}$.
3. Suppose $1 < a \leq 2$. Then the posterior mean still exists, but the posterior variance doesn't (since we need $a > 2$).