$$Y_i = x_i\beta + \epsilon_i \qquad i=1,\dots,n \qquad \epsilon_i \overset{iid}{\sim} N(0,\sigma^2)$$

$$Y_i \sim N(x_i\beta, \sigma^2) \qquad i=1,\dots,n.$$

a. Find the MLE

$$p(y_i) = (2\pi\sigma^2)^{-1} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i\beta)^2\right\}$$

$$p(y \mid x, \beta) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i\beta)^2\right\}$$

$$\log p(y \mid x, \beta) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i\beta)^2$$

$$\frac{\partial \log p(y \mid x, \beta)}{\partial \beta} = \frac{\cancel{2}1}{\cancel{2}\sigma^2} \sum_{i=1}^{n} (y_i - x_i\beta)x_i \overset{set}{=} 0$$

$$\sum_{i=1}^{n} (y_i - x_i\beta)x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i = \beta \sum_i x_i^2$$

$$\Rightarrow \boxed{\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_i x_i^2}}$$

Soln is <u>unique</u> bc density is convex.

b. dist^{n} of $\hat{\beta}$ $\rightarrow$ normal

$$E\left[\frac{\sum_i x_i y_i}{\sum_i x_i^2}\right] = \frac{1}{\sum_i x_i^2} \sum_i x_i E[y_i] = \frac{\cancel{\sum_i x_i^2}\beta}{\cancel{\sum_i x_i^2}} = \beta$$

$$Var(\hat{\beta}) = \frac{\sum_i x_i^2}{(\sum_i x_i^2)^2} Var(Y_i) = \frac{\sum_i x_i^2}{(\sum_i x_i^2)^2} \sigma^2$$

$$= \frac{\sigma^2}{\sum_i x_i^2} . \quad \boxed{\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^n x_i^2}\right).}$$

c.) $\tilde{\beta} = \dfrac{\sum_i Y_i}{\sum_i x_i}$

$$E[\tilde{\beta}] = \frac{\sum_i E(Y_i)}{\sum_i x_i} = \frac{\sum_i x_i \beta}{\sum x_i} = \beta$$

$$\operatorname{Var}(\tilde{\beta}) = \frac{\sum_{i=1}^{n} \operatorname{var}(Y_i)}{(\sum_i x_i)^2} = \frac{\sum_{i=1}^{n} \sigma^2}{(\sum_i x_i)^2}$$

$$= \frac{n\sigma^2}{(\sum_i x_i)^2}$$

$$\tilde{\beta} \sim N\left(\beta, \frac{n\sigma^2}{(\sum x_i)^2}\right)$$