Homework 2

STA 360: Assignment 2, Fall 2020

Due Friday August 28, 5 PM Standard Eastern Time

```r
library(tidyverse) #load in tidyverse package
```

Lab Component

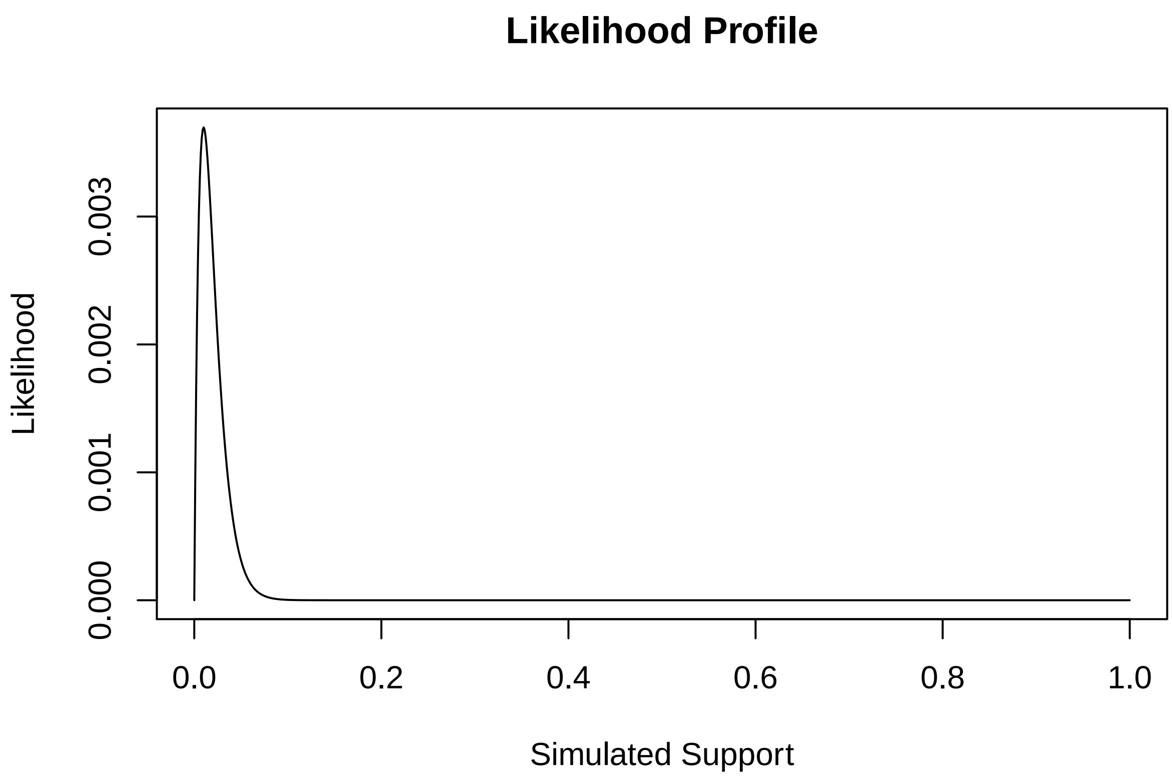
a. Task 3

Write a function that takes as its inputs that data you simulated (or any data of the same type) and a sequence of θ values of length 1000 and produces Likelihood values based on the Binomial Likelihood. Plot your sequence and its corresponding Likelihood function.

The likelihood function is given below. Since this is a probability and is only valid over the interval from $[0, 1]$ we generate a sequence over that interval of length 1000.

```r
set.seed(123)
### Bernoulli LH Function ###
# Input: obs.data, theta
# Output: bernoulli likelihood
obs.data <- rbinom(n = 100, size = 1, prob = 0.01)
bernLH <- function(obs.data, theta){
  N <- length(obs.data)
  x <- sum(obs.data)
  LH <- (theta ^x) *((1-theta)^{N-x})
  return (LH)
}

### Plot LH for a grid of theta values ###
# Create the grid #
theta.sim <- seq(from = 0, to = 1, length.out = 1000)
# Store the LH values
sim.LH <- bernLH(obs.data, theta = theta.sim)
# Create the Plot
plot(theta.sim, sim.LH, type = "l", main = "Likelihood Profile",
     xlab = "Simulated Support",
     ylab = "Likelihood")
```



b. Task 4 Write a function with, input: prior parameters a, b, and the observed data. output: parameters of the Beta posterior distribution of theta takes as its inputs prior parameters a and b for the Beta-Bernoulli model and the observed data, and produces the posterior parameters you need for the model

```
myBetaBernoulli <- function(obs.data, theta, a, b){
  N <- length(obs.data)
  x <- sum(obs.data)
  BeBern <- dbeta(theta, x + a, N-x+b)
  param1= x+a
  param2 = N-x+b
  print(param1)
  print(param2)
  return(BeBern)
}

non.informative.prior <- dbeta(theta.sim,1,1)
informative.prior <- dbeta(theta.sim,3,1)

posterior.non.informative <- myBetaBernoulli(obs.data, theta.sim, 1, 1)
```

```
## [1] 2
## [1] 100
```

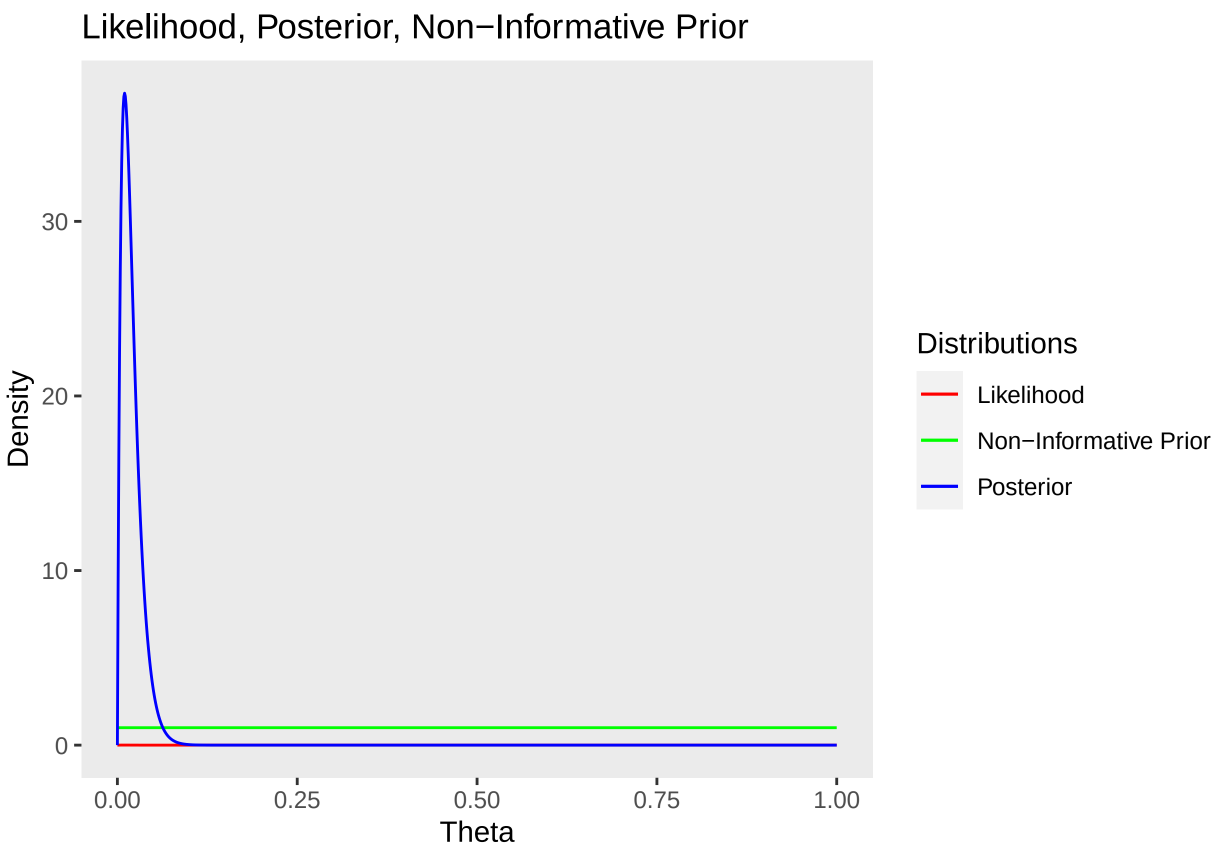
```
posterior.informative <- myBetaBernoulli(obs.data, theta.sim, 3, 1)
```

```
## [1] 4
## [1] 100
```

The parameters for the posterior with a non-informative prior are Beta(2, 100) and for the informative, the parameters are Beta(4, 100).

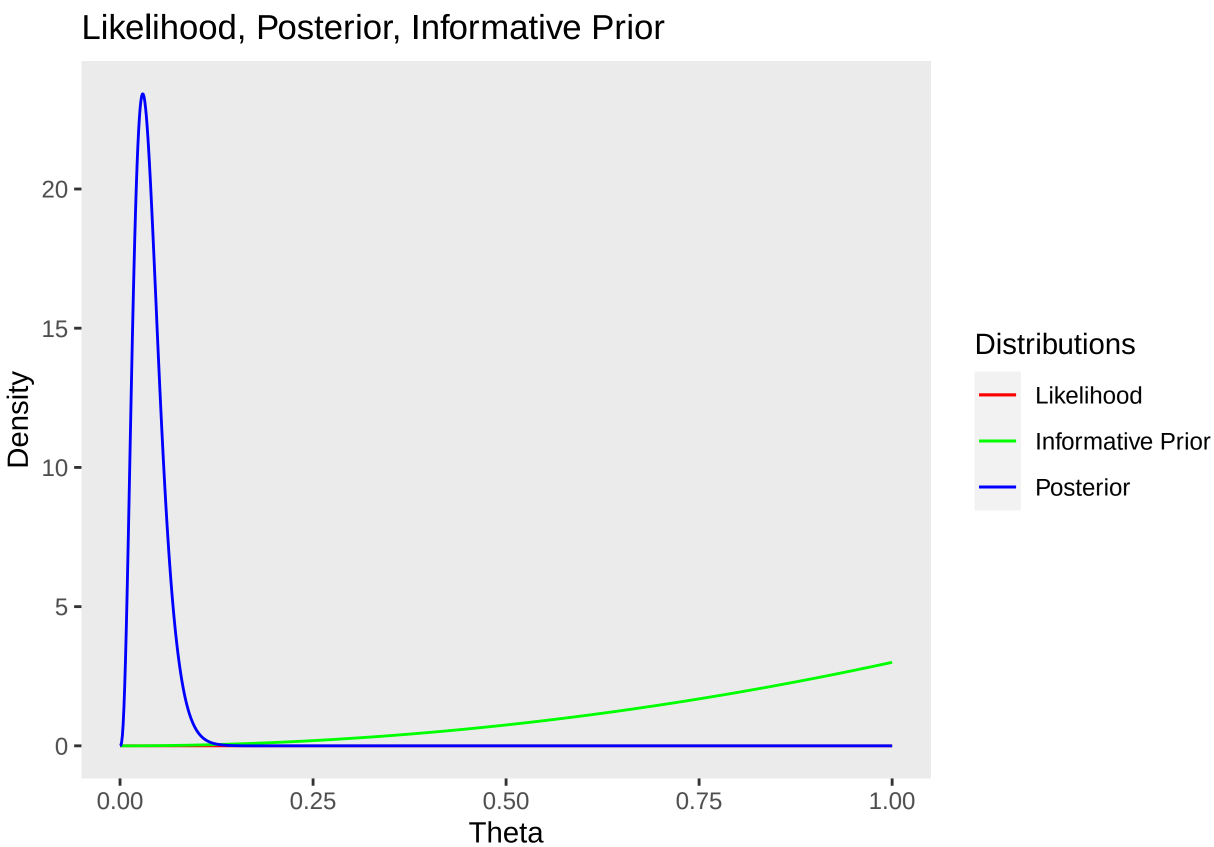
c. Task 5 Create two plots, one for the informative and one for the non-informative case to show the posterior distribution and superimpose the prior distributions on each along with the likelihood. What do you see? Remember to turn the y-axis ticks off since superimposing may make the scale non-sense.

```
df<- data.frame(theta.sim, non.informative.prior, informative.prior, sim.LH)
ggplot(df, aes(theta.sim)) +
  geom_line(aes(y=sim.LH, color="Likelihood")) +
  geom_line(aes(y=non.informative.prior,
                color= "Non-Informative Prior")) +
  geom_line(aes(y=posterior.non.informative, color="Posterior"))+
  scale_color_manual(name = "Distributions",
     breaks = c("Likelihood","Non-Informative Prior",
                "Posterior"),
     values = c("red", "green","blue"))+
labs(title = "Likelihood, Posterior, Non-Informative Prior",
       y = "Density",
       x = "Theta" ) +
  theme(
    panel.grid.major = element_blank(),
    panel.grid.minor = element_blank(),
  )
```




```
ggplot(df, aes(theta.sim)) +
  geom_line(aes(y=sim.LH, color="Likelihood")) +
  geom_line(aes(y=informative.prior,
                color= "Informative Prior")) +
  geom_line(aes(y=posterior.informative, color="Posterior"))+
  scale_color_manual(name = "Distributions",
     breaks = c("Likelihood","Informative Prior",
```

```
                 "Posterior"),
     values = c("red", "green","blue"))+
labs(title = "Likelihood, Posterior, Informative Prior",
       y = "Density",
       x = "Theta" ) +
  theme(
    panel.grid.major = element_blank(),
    panel.grid.minor = element_blank(),
  )
```



We can see that the informative prior is slightly more to the right than the non-informative prior. Having an informative prior shifted the posterior distribution so the value of theta it's centered about is greater than the non-informative prior posterior distribution. We can also see that the non-informative distribution is taller than the informative

The Exponential-Gamma Model

a. Derive the formula for the posterior density.

(1) $p(\theta|x) = \frac{p(\theta,x)}{p(x)} = \frac{p(x|\theta)p(\theta)}{p(x)} \propto p(x|\theta)\,p(\theta)$

In this case:

$p(x|\theta) = \text{Exp}(x|\theta) = \theta\exp(-\theta x)\ I(x>0)$

$p(\theta) = \text{Gamma}(\theta|a,b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta)\ I(\theta>0)$

so

(2) $p(x|\theta)p(\theta) = \theta e^{-\theta x} * \cancel{\frac{b^a}{\Gamma(a)}}\theta^{a-1}e^{-b\theta}\quad I(\theta>0)$

remove constant

(3) $\underline{\theta}\,\underline{\underline{e^{-\theta x}}} * \underline{\theta^{a-1}}\,\underline{\underline{e^{-b\theta}}} \rightarrow \theta^{(a+1)-1}e^{-\theta(b+x)} \rightarrow \theta^{(a+1)-1}\exp(-\theta(b+x))$

group like terms

(4) We can see that the posterior, $p(\theta|x)$, is a Gamma distribution with parameters $a+1$, $b+x$

$\text{Gamma}(\theta\,|\,a+1, b+x) \rightarrow \theta^{(a+1)-1}\exp(-\theta(b+x))$

Thus $p(\theta|x) = \text{Gamma}(\theta\,|\,a+1, b+x)$