5. $p(y|\theta) = \theta^{-1} e^{-y/\theta}$, $y > 0$, $\theta > 0$

$p(\theta) = \theta^{-a} e^{-b/\theta}$, $\theta > 0$, $a > 2$, $b > 0$.

1.) $p(\theta|y) \propto \theta^{-1} e^{-y/\theta} \theta^{-a} e^{-b/\theta}$

$\propto \theta^{-a-1} e^{-(y+b)/\theta}$

$\Rightarrow \theta | y \sim \text{Inverse Gamma}(a, y+b)$

$a > 2$; $\theta >$

$p(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}$ is an Inv. Gamma (α, β) w/ shape, scale parameters.

2.) $E[\theta | y] = \frac{y+b}{a-1}$ (defn of mean of Inv. Gamma)

you can altern. show this by

$$E[\theta \mid y] = \int \theta \; p(\theta \mid y) \, d\theta$$ Let $\beta = y + b$

$$= \int \theta \; IG(\theta \mid a, y+b) \, d\theta$$

$$= \int \theta \frac{\beta^{a}}{\Gamma(a)} \theta^{-a-1} e^{-(y+b)/\theta} \, d\theta$$

$$= \frac{\beta^{a}}{\Gamma(a)} \int \theta^{-a+1-1} e^{-(y+b)/\theta} \, d\theta$$

$$= \frac{\Gamma(a-1)}{\beta^{a-1}} \times \frac{\beta^{a}}{\Gamma(a)} \underbrace{\int \theta^{-(a-1)-1} e^{-(y+b)/\theta} \times \frac{\beta^{a-1}}{\Gamma(a-1)} \, d\theta}_{IG(a-1,\, y+b)}$$

$$= \frac{\Gamma(a-1)\,\beta^{a}}{\beta^{a-1}\,\Gamma(a)} = \frac{\Gamma(a-1)}{\Gamma(a)} \cdot \beta^{a-a+1}$$

$$= \frac{y+b}{a-1},$$ by defn of the gamma funct and subs

Fact:

$\Gamma(x+1) = x\Gamma(x)$

$\Rightarrow \Gamma(a) = (a-1)\Gamma(a-1)$

of

$\beta = y + \delta$.

To find the $Var(\theta | y)$, either use the fact that $Var(\theta|y) = \dfrac{(y+\delta)^2}{(a-1)^2(a-1)}$

for an Inverse Gamma or

derive

$E[\theta^2 | y]$ and use this

to derive the $var(\theta | y)$ using:

$$V(\theta|y) = E[\theta^2|y] - \{E[\theta|y]\}^2.$$

[I'm leaving $E[\theta^2|y]$

as an exercise to work

through on your own].

3.] If $1 < a \leq 2$, then
the posterior mean exists.
Unfortunately since
$$\text{var}(\theta|y) = \frac{(y+b)^2}{(a-1)^2(a-2)}$$
the $\text{var}(\theta|y)$ doesn't exist
since $a > 2$ by defn in
the problem statement.