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Q1

(i) By the Well Ordering Property every nonempty subset of the set of positive integers has a least value.
(ii) $(k \in \mathbb{Z}^+ \land n \in \mathbb{Z}^+) \to n^k \in \mathbb{Z}^+$
Proof: We assume that there is an integer smaller than 1 in the set of positive integers. We choose $n \in \mathbb{Z}^+$ such that $n < 1$. If we multiply both sides by n we get this inequality $n^2 < n$. However since $n^2 \in \mathbb{Z}^+$ by (ii) then this is a contradiction since we assumed n is the smallest integer but now n^2 is smaller than n. So our assumption is false.
Therefore by (i) and the proof by contradiction, 1 is the smallest integer in the set of positive integers.

Q2

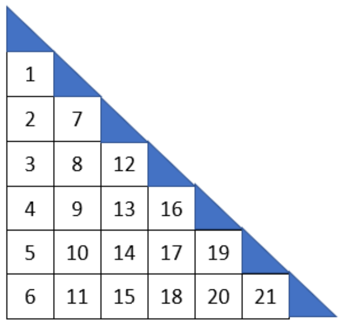
(1) First we prove that S(m,1) by using induction.
Base case: $x_1 = 1$ which we find that there is$f(1,1) = \frac{1!}{1!0!} = 1$ solutions which is correct because there can be only one solution thus the base case holds.
Inductive case: Assume that S(m,1) is true. That is for $x_1 + x_2 + ... + x_m = 1$ the number of solutions is equal to f(m,1) $= \frac{m!}{1!(m-1)!} = m$.
We have to show that S(m+1,1) holds.
(i) We know for $x_1 + x_2 + ... + x_m = 1$ there is one solution and (ii) for one variable $x_{m+1} = 1$ there is one solution (base case).
Thus, if we add x_{m+1} to $x_1 + x_2 + ... + x_m$ we get $x_1 + x_2 + ... + x_m + x_{m+1}$ which means there is $m + 1$ solutions by (i) and (ii). Since $f(m + 1, 1) = \frac{m+1!}{1!(m)!} = m + 1$ then it is equal to the conclusion in (iii). So we have proven the inductive case.
$\therefore$ S(m,1) is proven by induction.

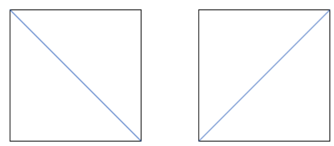
(2) We must show S(1,n) is true by using induction.
Base case: S(1,1) denotes that the number of solutions for $x_1 = 1$ is equal to $f(1, 1) = \frac{1!}{1!0!} = 1$ which is again correct because there can be only one solution thus the base case holds.
Inductive case: Assume S(1,n) is true. We must show S(1,n+1) is true.
$S(1, n)$ means that there are $\binom{n}{n} = 1$ solutions.
Since for $S(1, n + 1)$ there are $\binom{n+1}{n+1} = 1$ solutions as well, then $S(1, n)$ is proved by induction.

(3) Proof for $S(m, n)$:
Basis step: $S(1, n + 1)$ and $S(m + 1, 1)$ holds from (1) and (2).
Inductive Hypothesis: Assume $S(m, n + 1)$ and $S(m + 1, n)$ are true. We know for $S(m, n + 1)$ the combination formula is $\binom{n+m}{n+1}$ and for $S(m + 1, n)$ it is $\binom{n+m}{n}$. From Pascals Identity (Chapter 6.4) we know that:
$\binom{n+m}{n+1} + \binom{n+m}{n} = \frac{(n+m)!}{(n+1)!(m-1)!} + \frac{(n+m)!}{(n)!(m)!} = (n + m)!\left(\frac{1}{(n+1)!(m-1)!} + \frac{1}{n!m!}\right) = (n + m)!\left(\frac{m+n+1}{(n+1)!m!}\right) = \frac{(m+n+1)!}{(n+1)!m!}$
Since $\frac{(m+n+1)!}{(n+1)!m!} = \binom{n+m+1}{n+1}$ is the combination formula equivalent to the statement $f(m + 1, n + 1)$ then $S(m + 1, n + 1)$ holds.
$\therefore$ $S(m, n)$ is proved by induction.

Q3

a. We can draw 21 squares in the given grid. Each square can be split into 4 different triangles in different orientations. So we have $4 \times 21 = 84$ triangles. We also can draw 7 triangles in the empty remaining spaces.
So we can have $84 + 7 = 91$ triangles drawn on the grid.





b. By the product rule the total number of functions $= 4^6$
To find the number of onto functions we subtract the number of functions that are not onto.
We start from leaving one element out from the co-domain and find the number of functions with 3 elements in co-domain. When we do this then we need to add back the case where 2 elements of the co-domain is left out because of the Inclusion-Exclusion Principle (Chapter 6.1). When we that add back then the number of not-onto functions that have 3 elements left out of the co-domain is again added so we subtract that case again as well. So we have:

$$4^6 - \binom{4}{1} \times 3^6 + \binom{4}{2} \times 2^6 - \binom{4}{3} \times 1^6 + \binom{4}{4} \times 0^6 = 1560$$

1560 onto functions.

Q4

a. Let a_n be the recurrence relation. We know if we have a string of length n then we have 2 cases:
(i) A string of length $n-1$ which is valid (meaning it contains 2 consecutive elements). Then we have 3 choices to add to the string to make it length n. Therefore we have $3.a_{n-1}$.
(ii) A string of length $n-1$ which is not valid. The number of such strings are $3^{n-1} - a_{n-1}$.
If we add both cases using the sum rule:
$a_n = 3.a_{n-1} + (3^{n-1} - a_{n-1}) = 2.a_{n-1} + 3^{n-1}$

b. Initial conditions are:
$a_1 = 0$
$a_2 = 3$

c. $a_n - 2.a_{n-1} = 3^{n-1}$
Characteristic equation: $r - 2 = 0 \rightarrow r = 2$
Homogeneous solution: $a_n^H = A.2^n$
Particular solution: $a_n^P = B.3^n$
Plugging particular solution into the equation:
$3B.3^{n-1} - 2.B.3^{n-1} = 3^{n-1}$
$\Rightarrow B.3^{n-1} = 3^{n-1} \Rightarrow B = 1$ Using initial condition to find A:
$a_n = A.2^n + 3^n$ and $a_1 = 0 \Rightarrow 2A + 3 = 0 \Rightarrow A = -\frac{3}{2}$
$a_n = -\frac{3}{2}.2^n + 3^n$