19 April 2023

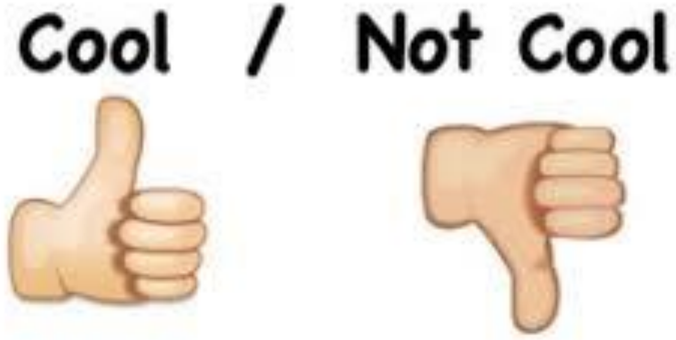
UNIVERSITY OF GLOUCESTERSHIRE

CT4031
Maths for Data Science

Week 10 – Introduction to Mathematical Proof

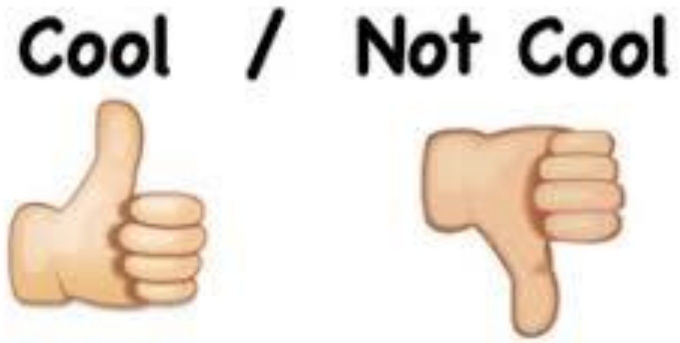
Introduction to Mathematical Proof

How this sounds to you?



Introduction to Mathematical Proof

Any better?



"***The search for a mathematical proof is the search for a knowledge which is more absolute than the knowledge accumulated by any other discipline***." Simon Singh

Introduction to Mathematical Proof

- Let's understand some concepts.
- What is a proof?

Introduction to Mathematical Proof

Let's understand some concepts.

What is a proof?

- A proof is a sequence of logical statements, one implying another, which explains why a given statement is true.
- Previously established theorems may be used to deduce the new ones; one may also refer to axioms, which are the starting points, “rules” accepted by everyone.

Introduction to Mathematical Proof

Let's understand some concepts.
What is a mathematical proof?

Introduction to Mathematical Proof

Let's understand some concepts.

What is a mathematical proof?

- Mathematical proof is absolute, which means that once a theorem is proved, it is proved forever.
- Until proven though, the statement is never accepted as a true one.
- Writing proofs is the essence of mathematics studies.
- At University every word will be defined, notations clearly presented, and each theorem proved.

Introduction to Mathematical Proof

Let's understand some concepts.

What is a mathematical proof?

- Mathematical proof is absolute, which means that once a theorem is proved, it is proved forever.
- Until proven though, the statement is never accepted as a true one.
- Writing proofs is the essence of mathematics studies.
- At University every word will be defined, notations clearly presented, and each theorem proved (***hopefully.....***).

Introduction to Mathematical Proof

What are the main types of mathematical proof?
Quick research task!

Introduction to Mathematical Proof

- Direct proof
- Proof by induction
- Proof by contrapositive
- Proof by contradiction

Introduction to Mathematical Proof

- Direct proof
- Say you want to prove that P = Q
- A direct proof is to use P to show that Q is true.

Direct proof

- you start with the given hypothesis or premise and use logical reasoning to arrive at the conclusion.

- In other words, you directly prove that the conclusion must be true given the premises.

Direct Proof: Example

Premise: If a and b are both even integers, then a + b is an even integer.
Conclusion: If a = 4 and b = 6, then a + b is an even integer.

Proof:
a = 4 and b = 6 (even integers)
a + b= 4 + 6 = 10(even integer)

Therefore, we can conclude that if a = 4 and b = 6, then a + b is an even integer, as stated in the conclusion.

Direct Proof: Example-2

Premise: If a number is divisible by 6, it is also divisible by 2 and 3..

Conclusion: 132 is divisible by 6.

Proof:

We know that 132 is an even integer.

Find the sum of the digits of 132=1+3+2 = 6 (divisible by 2 and 3)

Since 132 is divisible by both 2 and 3, we can conclude that it is divisible by 6, as stated in the premise.

Introduction to Mathematical Proof

- Proof by induction
- Say you want to prove that P = Q
- A proof that P = Q holds in a sequence of numbers N, in all N+1, N+2, N+3, … known cases.

Proof by induction

- a technique used in mathematics to prove that a statement is true for all positive integers (or for some infinite set of positive integers).

- Provide proof in two main steps:

  - **Base case:** Show that the statement is true for the smallest possible value of the integer (often n=1).

  - **Inductive step**: Assume that the statement is true for some arbitrary value of n and show that it must also be true for n+1.

Proof by induction Example

Statement: For all positive integers n, 1 + 2 + 3 + … + n = n(n+1)/2.

Proof:

Base case:

- When n=1, the left-hand side (LHS) of the equation is 1, and
- The right-hand side (RHS) is 1(1+1)/2 = 1. Therefore, the statement is true for the base case.

Inductive step:

- We want to show that the statement is also true for n+1. That is, we want to show that 1 + 2 + 3 + … + n + (n+1) = (n+1)(n+2)/2.
- add (n+1) to both sides of the assumed statement:
  - 1 + 2 + 3 + … + n + (n+1) = n(n+1)/2 + (n+1)
  - 1 + 2 + 3 + … + n + (n+1) = (n+1)/2 * (n + 2)
  - Simplifying the expression on the right-hand side, we get:
  - 1 + 2 + 3 + … + n + (n+1) = (n+1)(n+2)/2

- Therefore, we have shown that if the statement is true for n, then it must also be true for n+1. By the principle of mathematical induction, we can conclude that the statement is true for all positive integers n.

Introduction to Mathematical Proof

- Proof by contrapositive
- Say you want to prove that P = Q
- We know that P = Q also means that ¬P = ¬Q.
- So, if you can prove ¬P = ¬Q, then you can prove P = Q.

Proof by contrapositive

- a method of proof in logic and mathematics that shows that a statement is true by proving its contrapositive.
- What is Contrapositive?
- How it is formed?
  - Negate both the hypothesis and the conclusion and reverse their order.

Proof by contrapositive..

- "If it is raining, then the ground is wet"
- Contrapositive statement: "If the ground is not wet, then it is not raining.“

- To use proof by contrapositive, we assume the negation of the conclusion and show that it implies the negation of the hypothesis. If this is true, then the original statement must also be true.

Proof by contrapositive: Example-1

- **Statement**: "If n is an even integer, then n^2 is an even integer."

- **Contrapositive**: "If n^2 is an odd integer, then n is an odd integer.“

- **Proof:**
  - Let’s assume that n^2 is an odd integer.
  - This means that n must be an odd integer
  - since the square of an even integer is always even. Therefore, the contrapositive statement is true, and so is the original statement.

Proof by contrapositive: Example-2

- **Statement**: "If a number is even, then it is divisible by 2."

- **Contrapositive**: it is not divisible by 2", which means it is not an integer or it is an odd integer.
- **Proof:**
  - a number is even if it can be written as 2k for some integer k.
  - If the number is not divisible by 2, then it cannot be written as 2k, so it is not even.
  - Therefore, we have proven the contrapositive statement, which means that the original conditional statement is also true.

Introduction to Mathematical Proof

- Proof by contradiction
- Say you want to prove that P = Q
- We show that P = Q by assuming the OPPOSITE, that P = ¬ Q and that both are TRUE
- Through the proof we should reach a conclusion that P = ¬ Q is IMPOSSIBLE, or it is an ABSURD.
- Hence, a CONTRADICTION.

Proof by contradiction: Example

- **Statement:** There are infinitely many prime numbers.
- **Assumption:** There are only finitely many prime numbers (use the assumption to arrive at contradiction)

- Proof: Consider a number called N, which is the product of all the finite prime numbers.
- Let's say that the prime numbers in question are p1, p2, p3, ..., pn.
- Then, N = p1 * p2 * p3 * ... * pn.
- Now, consider the number M = N + 1.
- Since M>N, so, either M is prime or it is not prime.
- If M is prime, then it must be a new prime number that was not included in our original list of finitely many prime numbers.

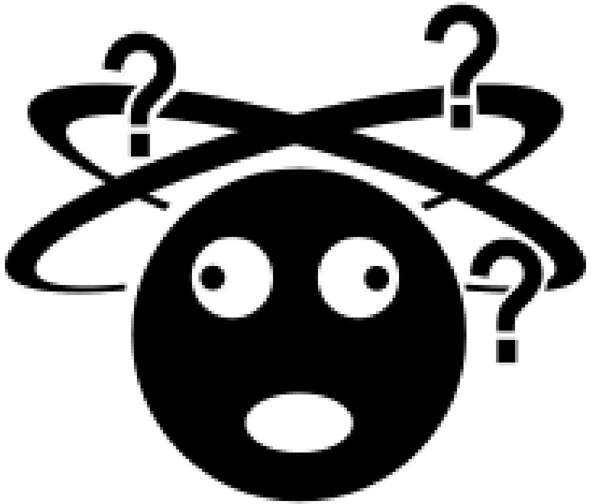
Proof by contradiction: Example..

- This contradicts our assumption that there are only finitely many prime numbers.
- On the other hand, if M is not prime, then it must have at least one prime factor.
- However, none of the prime numbers p1, p2, p3, ..., pn are factors of M.
- So, this prime factor must be a new prime number that was not included in our original list of finitely many prime numbers.
- Again, this contradicts our assumption that there are only finitely many prime numbers.

- Since both possibilities lead to a contradiction, we have proven that our assumption ("There are only finitely many prime numbers") must be false. Therefore, the original statement ("There are infinitely many prime numbers") must be true.

Introduction to Mathematical Proof

Are you confused now?



Introduction to Mathematical Proof

Let's try an example

Introduction to Mathematical Proof

Let's try an example

Say we want to prove the following statement:

"The sum of any consecutive numbers is odd"

Introduction to Mathematical Proof

- Let's try an example
- Say we want to prove the following statement:

"The sum of any consecutive numbers is odd"

- Let's use all four proof methods and try that.

Introduction to Mathematical Proof

- ***"The sum of any consecutive numbers is odd"***

- First, some definitions
- **Definition 1**: An integer number n is even if and only if there is an integer k such that $n = 2k$
- **Definition 2**: An integer number n is odd if and only if there is an integer k such that $n = 2k+1$
- **Definition 3**: Two integers a and b are consecutive if and only if $b = a + 1$.

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Direct proof
- Assume *a* and *b* are consecutives
- $b = a + 1$
- So …. $(a + b) = a + (a + 1)$

$$= 2a + 1$$

- $2a + 1 = 2k + 1$
- By definition 2, we have proved.

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by induction
- Show that P(x) is true, and that P(x+1) is true as well.
- So, it will be true to any basis case (x) and its consecutives (x+1)

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by induction
- Let the function P(x) be P(x) = x + (x +1)
- Let's assume that P(x) = odd is true.
- Let's test....
    - x=1 / P(1) = 1 + 2 = 3 (odd)
    - x=2 / P(1) = 2 + 3 = 5 (odd)
    - x=3 / P(1) = 3 + 4 = 7 (odd)
- Proved the basis case.

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by induction
- Let's calculate $P(x+1)$
- Knowing that $P(x) = x + (x +1)$
- Then, $P(x+1) = x+1 + (x+1 + 1)$

$$= x + 1 + (x + 2)$$

$$=2x + 1 + 2$$

- Knowing that $2x+1$ is ODD, adding 2 to an odd number gives an ODD number.
- Proved the inductive case.

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by contraposition
- If a + b is NOT ODD, then a and b are NOT consecutive integers
- Assume that a + b is NOT ODD
- Then they would be EVEN, so, a + b = 2k
- However, that DOES NOT hold, otherwise a = b

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by contraposition
- In this case, a + b = 2k + 1 and it is ODD
- Also, 2k + 1 => k + k + 1
- So, they are consecutive.
- Hence, both claims are WRONG: If a + b is NOT ODD, then a and b are NOT consecutive integers
- Proved that ¬P = ¬Q

Introduction to Mathematical Proof

"The sum of any consecutive numbers is odd"

- Proof by contradiction
- Assume *a* and *b* are consecutives
- Assume that a + b is NOT odd
- So, there is a k, such that (a + b = 2k + 1).
- But a + b = a + a + 1 or 2a + 1
- In this case, we are saying that 2a + 1 ≠ 2k + 1
- This is an ABSURD, then the OPPOSITE is true.

Introduction to Mathematical Proof

Now..... I am going to read your mind!!!

Introduction to Mathematical Proof

Now..... I am going to read your mind!!!
Are you ready???

Introduction to Mathematical Proof

Now..... I am going to read your mind!!!
Are you ready???

You might be thinking now.....

Introduction to Mathematical Proof

Now..... I am going to read your mind!!!
Are you ready???

You might be thinking now.....

Why do I need to learn that?!!!!

Introduction to Mathematical Proof

Now….. I am going to read your mind!!!
Are you ready???

You might be thinking now…..

Why do I need to learn that?

I will show you how this can be applied in real computing problems, using my favourite mathematical proof!

Introduction to Mathematical Proof

My favourite is……

Introduction to Mathematical Proof

My favourite is……

Introduction to Mathematical Proof

My favourite is......

reductio ad absurdum

Introduction to Mathematical Proof

My favourite is……

Aka – proof by contradiction
Or reduction to an absurd

Introduction to Mathematical Proof

Let's use proof by contradiction to prove that this code is correct.

```python
def my_function(n):
  output = []
  for i in range (0,n,2):
    output.append(i + (i + 1))

  return output
```

Introduction to Mathematical Proof

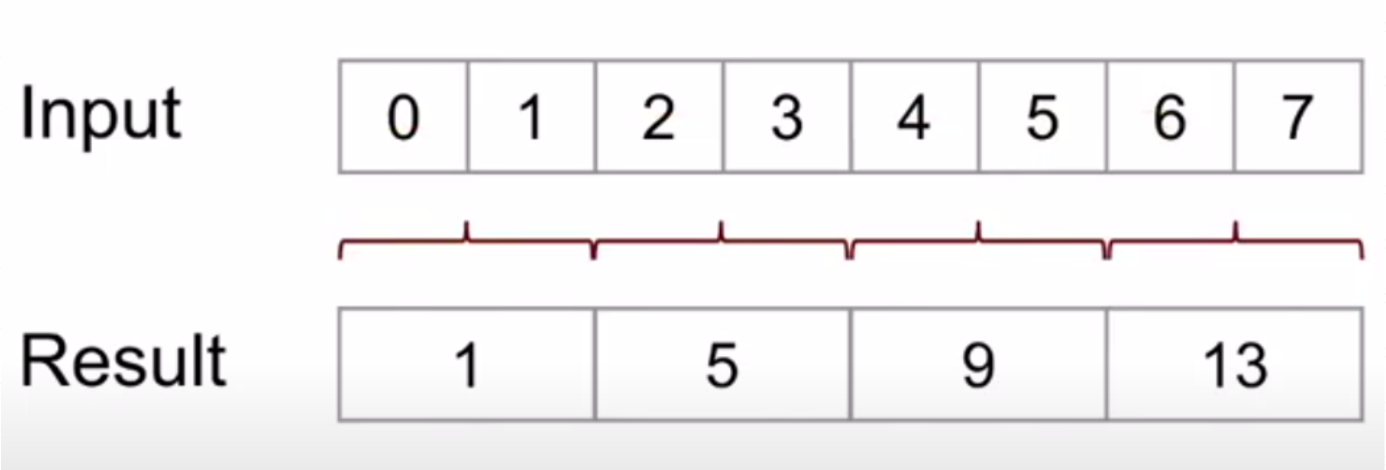
Let's use proof by contradiction to prove that this code is correct.
What does this code do? Let's look at it.

```
def my_function(n):
  output = []
  for i in range (0,n,2):
    output.append(i + (i + 1))

  return output
```

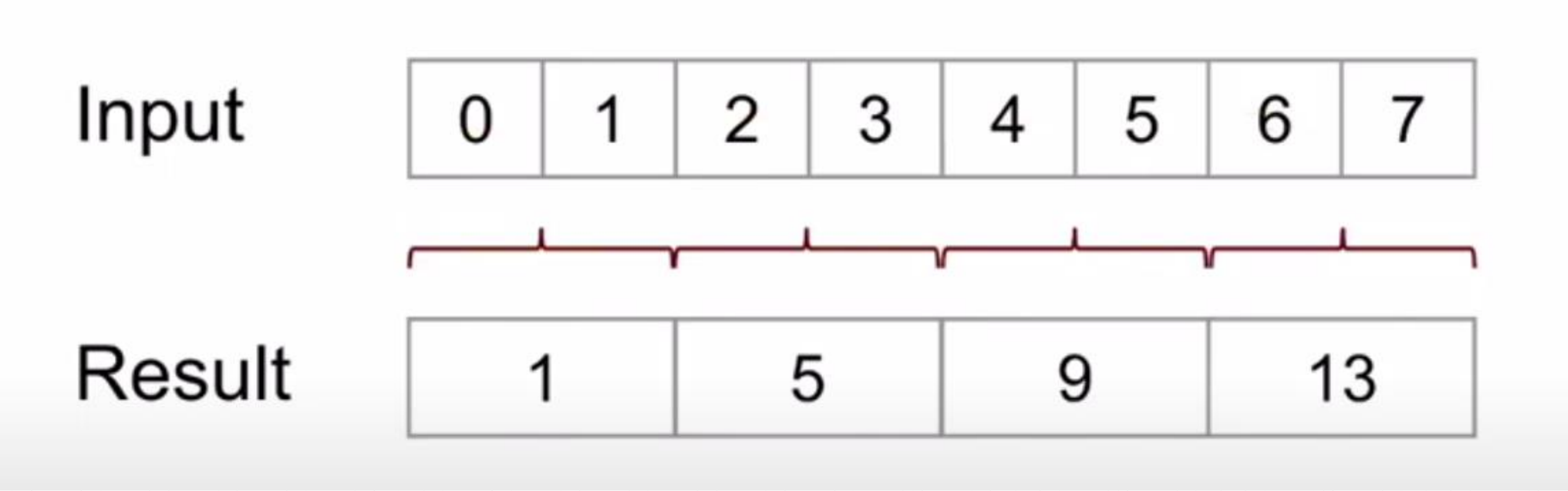
Introduction to Mathematical Proof

- Let's use proof by contradiction to prove that this code is correct.
- What does this code do? Let's look at it.
- When designed it was planned to receive an **integer** and return a list of **integers.**
- If n = 8



Introduction to Mathematical Proof

- Prove that for all ***n*** received, this function always returns a list where ALL integers are ODD numbers.



Introduction to Mathematical Proof

- Prove that for all ***n*** received, this function always returns a list where ALL integers are ODD numbers.
- Let's use proof by contradiction

Introduction to Mathematical Proof

- Prove that for all ***n*** received, this function always returns a list where ALL integers are ODD numbers.
- Let's use proof by contradiction
- Let's say that THERE IS at least ONE element in the RESULT that is EVEN
- So, we assume that there is an integer **j** and that at least one **output[i] = 2j**

Introduction to Mathematical Proof

- Prove that for all ***n*** received, this function always returns a list where ALL integers are ODD numbers.
- Let's use proof by contradiction
- Let's say that THERE IS at least ONE element in the RESULT that is EVEN
- So, we assume that there is an integer **j** and that at least one **output[i] = 2j**
- Based on the code:
    - we know that any **output[i] = i + (i+1)**

Introduction to Mathematical Proof

- So, we assume that at least one **output[i] = 2j**
- We know that **output[i] = i + (i+1)**
- Therefore...

$$2j = i + (i+1) \Rightarrow$$
$$2j = 2i + 1 \Rightarrow$$
$$2(j - I) = 1 \Rightarrow$$
$$j - i = ½$$

Introduction to Mathematical Proof

- So, we assume that at least one **output[i] = 2j**
- We know that **output[i] = i + (i+1)**
- Therefore...

$$2j = i + (i+1) \Rightarrow$$
$$2j = 2i + 1 \Rightarrow$$
$$2(j - I) = 1 \Rightarrow$$
$$j - i = ½$$

- However, what is an ABSURD here?

Introduction to Mathematical Proof

- **j - i = ½ ?!!**
- Look at the code

```
def my_function(n):
    output = []
    for i in range (0,n,2):
        output.append(i + (i + 1))

    return output
```

Introduction to Mathematical Proof

- We know from assumption that **j** is an integer.
- We know from the code that **i** is expected to be an integer.
- Therefore, **j - i = ½** is an **ABSURD**
- A substraction of two integers **CAN'T** end in a fraction.

Introduction to Mathematical Proof

So... we just used....

reductio ad absurdum

Introduction to Mathematical Proof

- Did you like that?
- You can get more here.
- Finally, any ideas of the applications for such a formal method in programming?

Post-sessional work

- What is Data Mining?
- Discuss three different Data Mining techniques.

Quick practical

Python example

Assignment

We are getting close to the assignment deadline.
Questions?

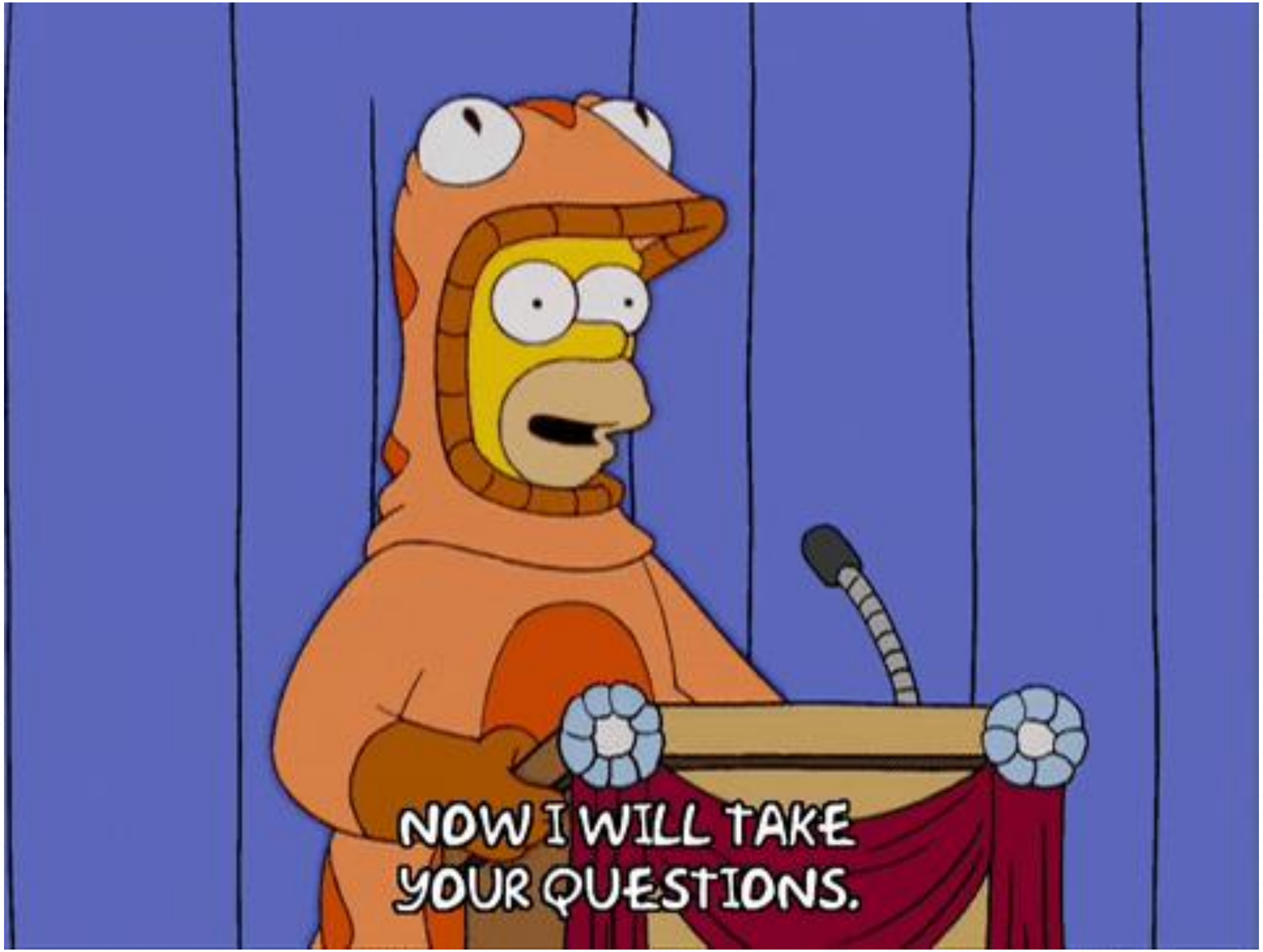
References

Four Basic Proof Techniques Used in Mathematics, Available at: (https://www.youtube.com/watch?v=V5tUc-J124s) (Accessed: 16th July 2021).

Proof by Contradiction in Algorithms Available at: (https://www.youtube.com/watch?v=aLvIpdIexmg) (Accessed: 16th July 2021).

Next Session !

- Introduction to Data Mining


NOW I WILL TAKE
YOUR QUESTIONS.