Cointegration lecture 1

Gray Calhoun

November 4th, 2014, version 0.4.0

Simple example of cointegration

- Start with k-dimensional VAR(1)

$$y_t = a_0 + Ay_{t-1} + e_t$$

  - Stationarity implies that eigenvalues of A are all less than 1.
  - Suppose that some are equal to 1 and some less than 1
  - Last class, we (essentially) worked with the case where all of the eigenvalues equaled 1
- Let $\Pi = A - I$ and rewrite the VAR as

$$\Delta y_t = a_0 + \Pi y_{t-1} + e_t$$

  - Unit eigenvalues of A become zero eigenvalues of Π, so Π will not have full rank
- Suppose Π has rank r, then we can write

$$\Pi = \alpha\beta'$$

  where α and β both are $k \times r$ with full rank.
- Then $\Delta y_t = a_0 + \alpha\beta' y_{t-1} + e_t$

Simple example of cointegration

- Intuitively, $\Delta y_t \sim I(0)$ since $y_t \sim I(1)$
- Then we must have $\beta' y_t \sim I(0)$ as well
- r is the "cointegrating rank" of the system
  - $r = 0$ implies "no cointegration"
  - $r = k-1$ implies that there is a single unit root process driving all of the series
- Formally:

$$
\begin{aligned}
y_t &= a_0 + A y_{t-1} + e_t \\
&= y_0 + t a_0 + \sum_{s=1}^{t} A^{t-s} e_t \\
&= y_0 + t a_0 + \sum_{s=1}^{t} \Gamma \Lambda^{t-s} \Gamma' e_s \\
y_{t-1} &= y_0 + (t-1) a_0 + \sum_{s=1}^{t-1} \Gamma \Lambda^{t-s-1} \Gamma' e_s
\end{aligned}
$$

Simple example of cointegration

$$
\begin{aligned}
y_t - y_{t-1} &= a_0 + e_t + \sum_{s=1}^{t-1} \Gamma(\Lambda^{t-s} - \Lambda^{t-s-1})\Gamma' e_s \\
&\equiv a_0 + e_t + \sum_{s=1}^{t-1} \Psi_{t-s} e_s
\end{aligned}
$$

with

$$
\begin{aligned}
\Lambda^{t-s} - \Lambda^{t-s-1} &= (\underbrace{1, \ldots, 1}_{r}, \lambda_1^{t-s}, \ldots, \lambda_{k-r}^{t-s}) \\
&\quad - (\underbrace{1, \ldots, 1}_{r}, \lambda_1^{t-s-1}, \ldots, \lambda_{k-r}^{t-s-1}) \\
&= (\underbrace{0, \ldots, 0}_{r}, \lambda_1^{t-s}(1-\lambda_1^{-1}), \ldots, \lambda_{k-r}^{t-s}(1-\lambda_{k-r}^{-1}))
\end{aligned}
$$

- Since $\sum_j \|\Psi_j\|$ is finite, $\Delta y_t \sim I(0)$.

Simple example of cointegration

- For $\beta' y_{t-1}$, we have

$$\begin{aligned} \alpha_0 + \alpha\beta' y_{t-1} + e_t &= \Delta y_t \\ &= a_0 + e_t + \sum_{s=1}^{t-1} \Psi_{t-s} e_s \\ \alpha'\alpha\beta' y_{t-1} &= \sum_{s=1}^{t-1} \alpha' \Psi_{t-s} e_s \\ \beta' y_{t-1} &= \sum_{s=1}^{t-1} (\alpha'\alpha)^{-1} \alpha' \Psi_{t-s} e_s \end{aligned}$$

- α and β are not unique

$$\Pi = \alpha\beta' = (\alpha R)(\beta R'^{-1})' = \tilde{\alpha}\tilde{\beta}'$$

- They are unique up to rotations, so they define the same dynamics

Granger-Representation Theorem

- Let y_t be a k-dimensional VAR(p):

$$y_t = a_0 + \sum_{i=1}^{p} A_i y_{t-1} + e_t.$$

- Let $\Pi = I - A_1 - \cdots A_p$ and let $r = \text{rank}(\Pi)$.
- Let $C_j = -\sum_{i=j+1}^{p} A_i$
- There are r stationary linear combinations of the variables in y_t, and we can write the VAR as a VECM

$$\Delta y_t = a_0 + \alpha\beta' y_{t-1} + \sum_{i=1}^{p-1} C_j \Delta y_{t-i} + e_t$$

- α and β are both $k \times r$ and $\beta' y_t$ is stationary

Interpretation

- Unit-root components represent stochastic trends
- Cointegrating vectors can represent long-run equilibria
- The same sort of behavior can hold for *any* persistent process (irregular breaking patterns, etc.)
    - i.e., a single process exhibiting instability, and a second process that is the first plus a stationary error component
    - Typically called *comovement* or *cobreaking* in that case
- We can imagine a "structural" VECM, just like SVAR:

$$C_0 \Delta y_t = a_0 + \alpha\beta' y_{t-1} + \sum_{i=1}^{p-1} C_j \Delta y_{t-i} + e_t$$

- Same identification issues as with VARs: we can't estimate all of the elements of C_0 without imposing external economic theory
- Same approaches as with VARs can apply here

Estimating the VECM

- If β is known, OLS is consistent and asymptotically normal
  - "Known" can mean "known under the null hypothesis," so this isn't as crazy as it might appear
- If β is not known we need to estimate it
- Typically we will need to also estimate r
- Identifying specific cointegrating vectors can be tricky as well

Park-Phillips triangular representation

If $\beta' y_t$ is stationary, so is (assuming $\beta_{11}, \beta_{12}, \ldots, \beta_{1r} \neq 0$)

$$\begin{pmatrix} 1 & \beta_{21}/\beta_{11} & \beta_{31}/\beta_{11} & \cdots & \beta_{k1}/\beta_{11} \\ 1 & \beta_{22}/\beta_{12} & \beta_{32}/\beta_{12} & \cdots & \beta_{k2}/\beta_{12} \\ \vdots & & & & \\ 1 & \beta_{2r}/\beta_{1r} & \beta_{3r}/\beta_{1r} & \cdots & \beta_{kr}/\beta_{1r} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & \beta_{21}/\beta_{11} & \beta_{31}/\beta_{11} & \cdots & \beta_{k1}/\beta_{11} \\ 0 & \beta_{22}/\beta_{12} - \beta_{21}/\beta_{11} & \beta_{32}/\beta_{12} - \beta_{31}/\beta_{11} & \cdots & \beta_{k2}/\beta_{12} - \beta_{k1}/\beta_{11} \\ \vdots & & & & \\ 0 & \beta_{2r}/\beta_{1r} - \beta_{21}/\beta_{11} & \beta_{3r}/\beta_{1r} - \beta_{31}/\beta_{11} & \cdots & \beta_{kr}/\beta_{1r} - \beta_{k1}/\beta_{11} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix}$$

and (dot dot dot)

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & b_{r+1,1} & \cdots & b_{k1} \\ 0 & 1 & \ldots & 0 & b_{r+1,2} & \cdots & b_{k2} \\ \vdots & & & & & & \\ 0 & 0 & \ldots & 1 & b_{r+1,r} & \cdots & b_{kr} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix} = [I\ B'] y_t$$

Park-Phillips triangular representation

- Partition y_t' as (y_{1t}', y_{2t}').
- Now define

$$\begin{aligned}\mu_1 &= \mathrm{E}([I\ B']y_t)\\ u_{1t} &= [I\ B']y_t - \mu_1\end{aligned}$$

- We can also define

$$\begin{aligned}\mu_2 &= \mathrm{E}\,\Delta y_{2t}\\ u_{2t} &= \Delta y_{2t} - \mu_2\end{aligned}$$

- Then

$$\begin{aligned}y_{1t} &= \mu_1 - B' y_{2t} + u_{1t}\\ \Delta y_{2t} &= \mu_2 + u_{2t}\end{aligned}$$

- Note that $u_t = (u_{1t}', u_{2t}')'$ will be an MA process.

Engle-Granger approach to estimating cointegrating vector

- We can estimate

$$y_{1t} = \mu_1 - B' y_{2t} + u_{1t}$$

with OLS: let $D_T = (T^{1/2}, T)$

$$D_T^{-1} \sum_{t=1}^{T} \begin{pmatrix} 1 & y_{2t}' \\ y_{2t} & y_{2t} y_{2t}' \end{pmatrix} D_T^{-1} \Rightarrow \begin{pmatrix} 1 & \int_0^1 W_2'(s) ds \Sigma_2' \\ \Sigma_2 \int_0^1 W_2(s) ds & \Sigma_2 \int_0^1 W_2(s) W_2(s)' ds \Sigma_2' \end{pmatrix}$$

and

$$D_T^{-1} \sum_{t=1}^{T} \begin{pmatrix} u_{1t} \\ y_{2t} u_{1t} \end{pmatrix} \Rightarrow \begin{pmatrix} \omega_1 W_1(1) \\ \Sigma_2 \int_0^1 W_2(s) dW_1(s) \sigma_1 + \omega_1/2 \end{pmatrix}$$

- $\hat{\mu}_1 = \mu_1 + O_p(T^{-1/2})$
- $\hat{B} = B + O_p(T^{-1})$

Engle-Granger approach

- If you know:
  - r
  - Which elements of y_t are certain to be in the cointegrating relationships

  then you can estimate the cointegrating vector with OLS
- This estimator is superconsistent (you can treat it as known in future inference steps)
  1. Estimate cointegrating relationships with OLS:

     $$y_{1t} = \mu_1 - B' y_{2t} + u_{1t}$$

  2. Plug in $\hat{B}$ and estimate VECM with OLS

     $$\Delta y_t = a_0 + \alpha [I\ \hat{B}'] y_{t-1} + \sum_{i=1}^{p-1} \Delta y_{t-i} + e_t$$

  3. We can ignore estimation error in $\hat{B}$ in this second equation.

Engle-Granger approach:

- This also leads to a test for cointegration:
- When we estimate

$$y_{1t} = \mu_1 - B' y_{2t} + u_{1t}$$

we get superconsistency only if u_{1t} is I(0), which requires cointegration to hold

- Otherwise, $y_{1t} + B' y_{2t}$ has a unit root, so u_{1t} has a unit root.
- We can test whether $\hat{u}_{1t}$ has a unit root by doing an ADF-type test and regressing $\hat{u}_{1t}$ on $\hat{u}_{1,t-1}$:

$$\hat{\rho} = \frac{\sum_{t=2}^{T} \hat{u}_{1,t-1} \hat{u}_{1,t}}{\sum_{t=2}^{T} \hat{u}_{1,t-1}^2}$$

- $\hat{\rho}$ has a *nonstandard* nonstandard distribution, so you can't use the ADF tables. (see Hamiltion Proposition 19.4)

Why not just work with the differences?

- Δy_t is stationary, so can't we just invoke Wold representation theorem:

$$\Delta y_t = C(L)e_t$$

- Use Beveridge-Nelson decomposition ($C_j^* = -\sum_{s=j+1}^{\infty} C_s$)

$$\Delta y_t = C(1)e_t + C^*(L)(e_t - e_{t-1})$$

so

$$y_t = y_0 + \sum_{t=0}^{t} \Delta y_t = y_0 + C(1)w_t + C^*(L)e_t$$

where $w_t = \sum_{s=0}^{t} e_t$ (a unit root process)

Why not just work with the differences?

- Cointegration implies that $\beta' y_t$ is I(0), so

$$\beta' y_0 + \beta' C(1) w_t + \beta' C^*(L) e_t$$

  must be I(0) as well, which only happens if the w_t term is a.s. zero, so we need

$$\beta' C(1) = 0$$

  as a consequence of cointegration.
- Remember that for an MA(∞) to be invertible, we need the solutions to $\det(C(z)) = 0$ to all be outside the unit circle, which we just ruled out.
- Δy_t **does not have a VAR representation**
- $t^{-1/2} y_t$ has limiting variance of $C(1)\Sigma C(1)'$,
- $\operatorname{avar}(T^{-1/2} \sum_{t=1}^{T} \Delta y_t)$ has the same asymptotic variance, which doesn't have full rank.

License and copying