Cointegration lecture 2

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Problem with our last results

- We used a normalization:

$$y_{1t} = B' y_{2t} + u_t$$

  and estimate this with OLS
- Requires us to know the number of cointegrating relationships
  - i.e. the dimension of y_{1t}
- Normalization fails if some of the terms in y_{1t} don't really belong in the cointegrating relationship
  - Note that it's okay if some of the terms in y_{2t} don't belong in the relationship, because then we're just estimating zero.
- Useful reading: Hamilton Chapter 20, Watson (1994) *Vector Autoregressions and Cointegration*

An intermediate approach (Johansen)

- Suppose we know there are r cointegrating relationships, but don't have a normalization:

$$\Delta y_t = a_0 + \Pi y_{t-1} + \sum_{i=1}^{p-1} A_i \Delta y_{t-i} + e_t$$

  where $\Pi = \alpha\beta'$ and α and β are both $k \times r$

- We can estimate this equation by MLE (called reduced rank regression) assuming normality
- Asymptotic distribution will not depend on normality

An intermediate approach (Johansen)

- Given any values of $\hat{\alpha}$ and $\hat{\beta}$, MLE of the remaining parameters is easy

$$\Delta y_t - \hat{\Pi} y_{t-1} = a_0 + \sum_{i=1}^{p-1} A_i \Delta y_{t-i} + e_t$$

so the MLE of a_0 and A_i come from OLS

- Let $\hat{u}_t$ and $\hat{v}_t$ be the residuals from the regressions

$$\Delta y_t = a_0 + \sum_{i=1}^{p-1} A_i \Delta y_{t-i} + u_t$$

$$y_{t-1} = a_0 + \sum_{i=1}^{p-1} A_i \Delta y_{t-i} + v_t$$

so

$$\hat{e}_t = \hat{u}_t - \hat{\Pi} \hat{v}_t$$

An intermediate approach (Johansen)

- Given $\hat{\beta}$ and the other MLEs, we know that the MLE of $\hat{\alpha}$ comes from regressing Δu_t on $\hat{\beta}' v_t$, so

$$\hat{\alpha}' = \left(\hat{\beta}' \sum_{t=p+1}^{n} \hat{v}_t \hat{v}_t' \hat{\beta}\right)^{-1} \hat{\beta}' \sum_{t=p}^{n} \hat{v}_t \hat{u}_t$$

- Under Normality, assuming MLEs for the other parameters, the log-likelihood of β is equal to (plus a constant)

$$\begin{aligned}
&-(n/2) \log \det \hat{\Sigma} \\
&\qquad = (-n/2) \log \det \sum_{t=p}^{n} (\hat{u}_t - \hat{\alpha}\hat{\beta}' \hat{v}_t)(\hat{u}_t - \hat{\alpha}\hat{\beta}' \hat{v}_t)' \\
&\qquad = (-n/2) \log \det (\hat{S}_{uu} - \hat{S}_{uv} \hat{\beta} (\hat{\beta}' \hat{S}_{vv} \hat{\beta})^{-1} \hat{\beta} \hat{S}_{uv}')
\end{aligned}$$

with $\hat{S}_{uu} = \sum \hat{u}\hat{u}'$, etc.

An intermediate approach (Johansen)

- Then $\hat{\beta}$ minimizes

$$\det(\hat{S}_{uu} - \hat{S}_{uv}\hat{\beta}(\hat{\beta}'\hat{S}_{vv}\hat{\beta})^{-1}\hat{\beta}\hat{S}_{uv}')$$

- One can show (we won't) that we can proceed by:
  1. Finding the eigenvalues of $\hat{S}_{vv}^{-1}\hat{S}_{uv}'\hat{S}_{uu}^{-1}\hat{S}_{uv}$, order them $\hat{\lambda}_1 > \cdots > \hat{\lambda}_k$
  2. Let $\hat{b}_1, \ldots, \hat{b}_r$ be the eigenvectors associated with the first h eigenvalues of that matrix
  3. The cointegrating vectors can be written as linear combinations of the $\hat{b}_i$.
  4. $\hat{\beta} = [\hat{b}_1, \ldots, \hat{b}_r]$
- Log likelihood becomes

$$-(kn/2)\log 2\pi - (kn/2) - (n/2)\log\det\hat{S}_{uu} - (n/2)\sum_{i=1}^{r}\log(1-\hat{\lambda}_i)$$

Moving from estimation to testing

- This gives us an estimate of $\hat{\beta}$ given a particular value of r
- May want to conduct inference on β
- May want to conduct inference on α and A
- May want to conduct inference on r.

Conducting inference on the number of cointegrating relationships

- Inference on r is easy: we have a formula for the maximum of the log-likelihood function under the null that the number of cointegrating relationships is r.
- We can do an LM test easily
- Can do an LRT against the alternative that there are more than r cointegrating relationships:

$$2(L_A - L_0) = -n \sum_{i=r+1}^{k-1} \log(1 - \hat{\lambda}_i)$$

(Johansen's, 1988, "trace test")

- Or against the alternative that there are $r+1$ relationships:

$$2(L_A - L_0) = -n \log(1 - \hat{\lambda}_{r+1})$$

(Johansen's "maximum eigenvalue test")

- Asymptotic distribution of these is not chi-square. See Hamilton 20.3 for distributional results.

Deciding on the number of cointegrating relationships

1. Test $r = 0$ against $r > 0$.
2. If that test rejects, test $r = 1$ against $r > 1$
3. continue until you fail to reject

This approach will cause problems if we want to do inference on the parameters afterwards, though.

Quick summary of results for testing the other VECM parameters

- Just like before $\hat{\beta}$ is superconsistent

$$T(\hat{\beta} - \beta) = O_p(1)$$

- Nonstandard terms cancel out: LRT for hypotheses about β typically is chi-square
- Since $\hat{\beta}$ is superconsistent, other VECM parameters are asymptotically normal and well-behaved

Other issues with cointegration

- Pretesting issues in cointegrating rank
  - Typically use “most conservative” critical value
  - There are ways to formalize
- local-to-unity issues with stochastic trends
- Bayesian analysis is not especially reassuring

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