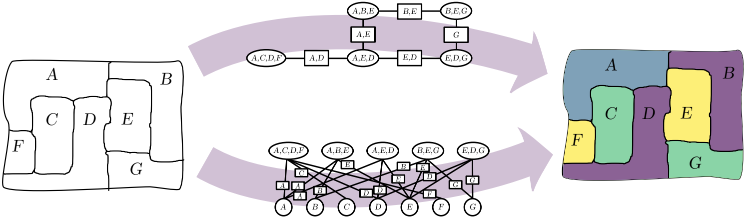
Graph Coloring: Comparing Cluster Graphs to Factor Graphs



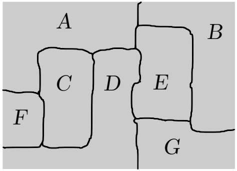
Simon Streicher and Johan du Preez
Stellenbosch University

Graph Coloring

Graph Coloring

Practical example is a **four coloring map problem**:

You only need four colors to color-in a map with no neighboring countries having the same color.



Noted by Francis Guthrie in 1852
Theorem proven by Appel and Haken in 1976

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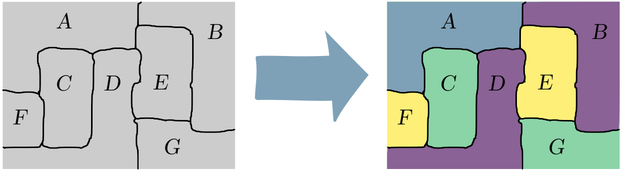

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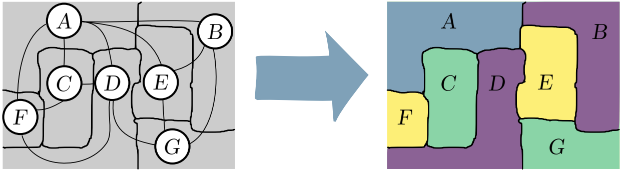

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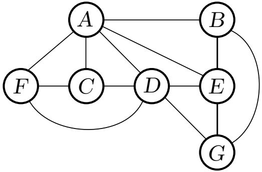
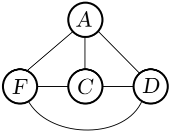
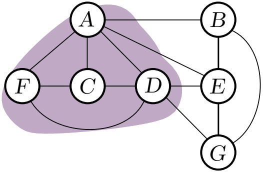


Undirected graph

Graph Coloring





Maximal cliques

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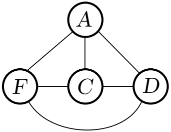

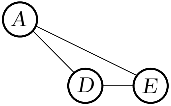

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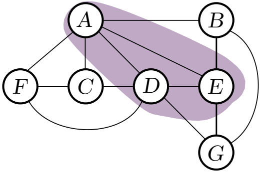

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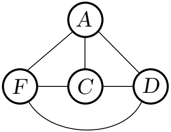

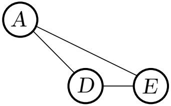

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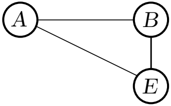

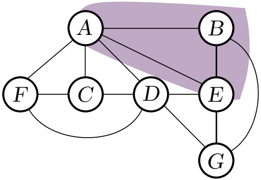

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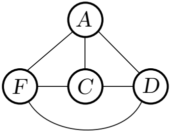

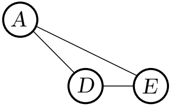

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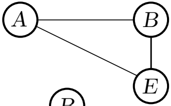

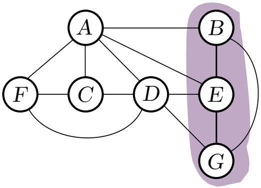

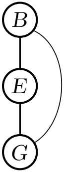

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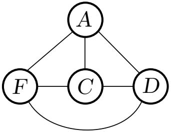

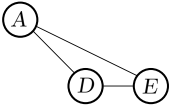

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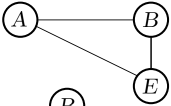

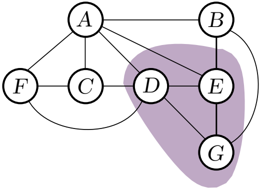

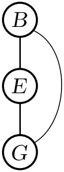

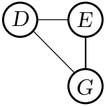

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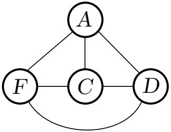

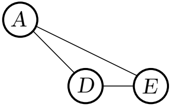

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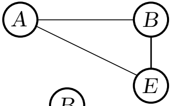

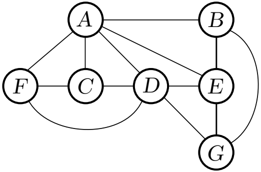

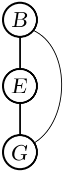

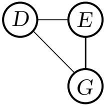

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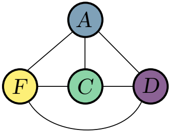

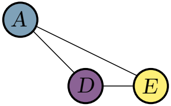

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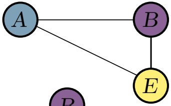

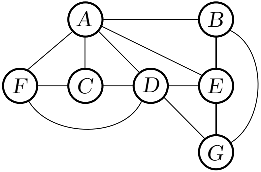

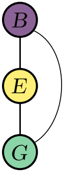

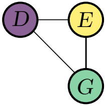

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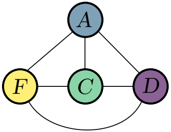

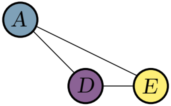

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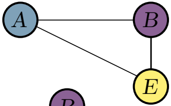

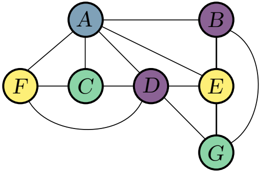

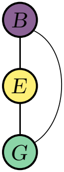

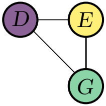

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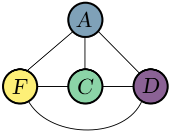

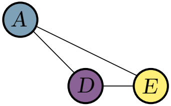

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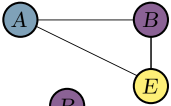

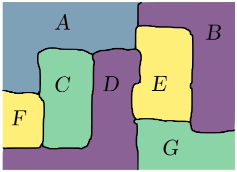

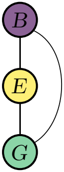

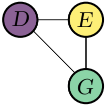

Graph Coloring

Sudoku is also a graph coloring problem

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4x4 Sudoku example:

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

Graph Coloring

Maximal cliques:

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

rows

A	B	C	D

E	F	G	H

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rows

A	B	C	D

E	F	G	H

I	J	K	L

M	N	O	P

blocks

A	B
E	F

C	D
G	H

I	J
M	N

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I	J	K	L

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blocks

A	B
E	F

C	D
G	H

I	J
M	N

K	L
O	P

columns

A
E
I
M

B
F
J
N

C
G
K
O

D
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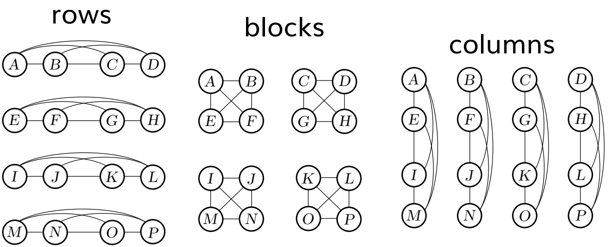

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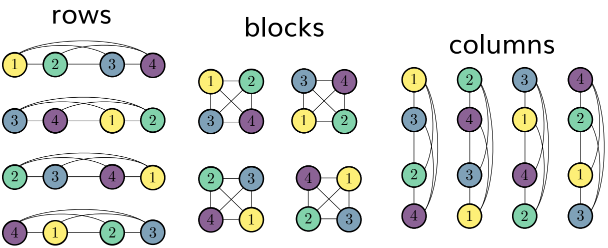

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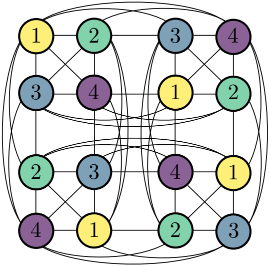
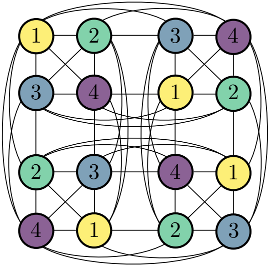


Graph Coloring

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3



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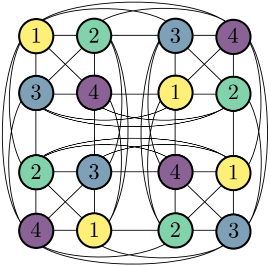

But how do we take all constraints into account ?

Probabilistic Graphical Models

PROBABILISTIC GRAPHICAL MODELS

In a general sense, a PGM of a system

- clusters information into local sections, and
- let the sections communicate about their combined outcome

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In a probabilistic sense, these "local sections" are
- prior distributions,
- marginal distributions, and/or
- conditional distributions;

together, a compact representation of a larger space

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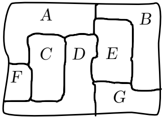

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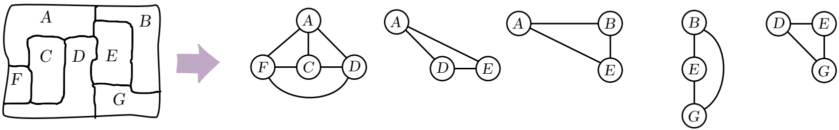

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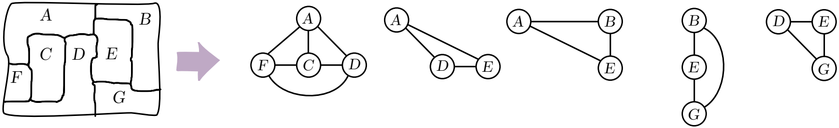

$$P(A,B,C,D,E,F,G) = \mathrm{f}_1(A,C,D,F) \cdot \mathrm{f}_2(A,D,E) \cdot \mathrm{f}_3(A,B,E) \cdot \mathrm{f}_4(B,E,G) \cdot \mathrm{f}_5(D,E,G)$$

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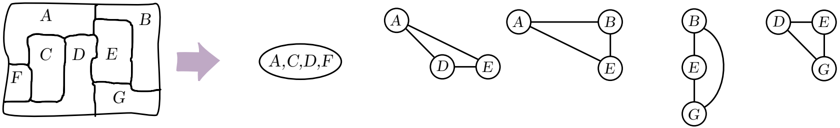

$$P(A,B,C,D,E,F,G) = \mathrm{f}_1(A,C,D,F) \cdot \mathrm{f}_2(A,D,E) \cdot \mathrm{f}_3(A,B,E) \cdot \mathrm{f}_4(B,E,G) \cdot \mathrm{f}_5(D,E,G)$$

1	2	3	4	1
1	2	4	3	1
1	3	2	4	1
	⋮			
4	3	2	1	1
elsewhere				0

$P(A,B,C,D)$
non normalized

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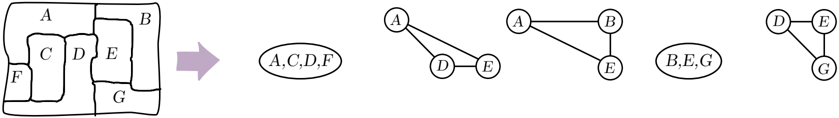

$$P(A,B,C,D,E,F,G) = f_1(A,C,D,F)\cdot f_2(A,D,E)\cdot f_3(A,B,E)\cdot f_4(B,E,G)\cdot f_5(D,E,G)$$

1	2	3	4	1
1	2	4	3	1
1	3	2	4	1
	⋮			
4	3	2	1	1
elsewhere				0

$P(A,B,C,D)$
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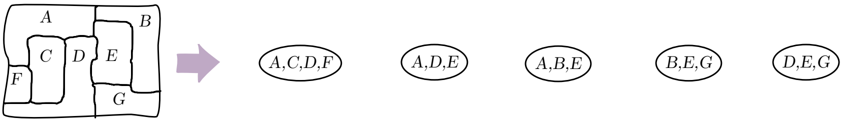

$$P(A,B,C,D,E,F,G) \;=\; \mathrm{f}_1(A,C,D,F) \cdot \mathrm{f}_2(A,D,E) \cdot \mathrm{f}_3(A,B,E) \cdot \mathrm{f}_4(B,E,G) \cdot \mathrm{f}_5(D,E,G)$$

PROBABILISTIC GRAPHICAL MODELS

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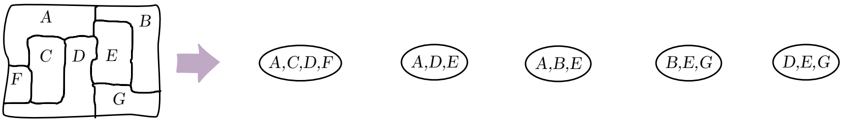

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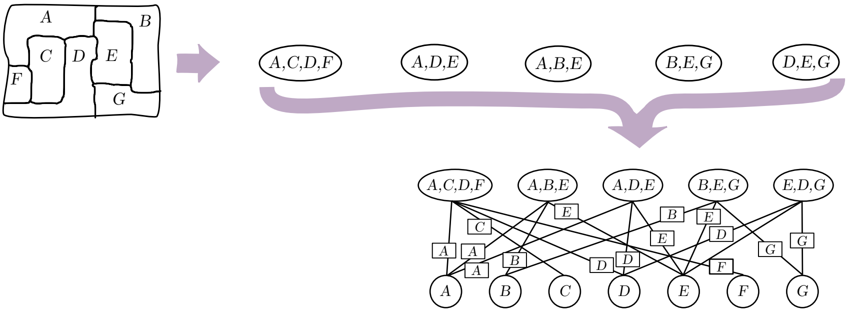

Factor/Bethé graph

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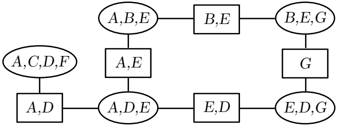
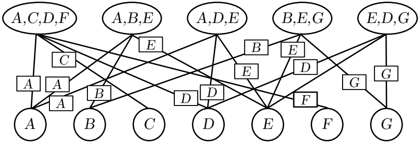


Cluster graph



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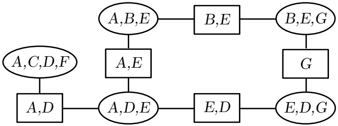

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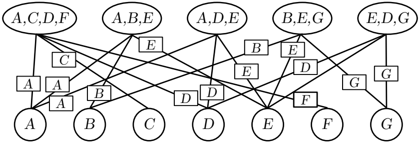

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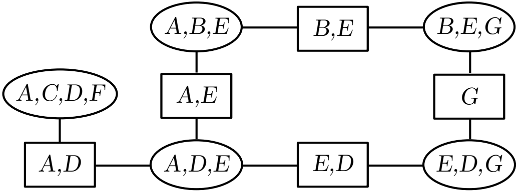

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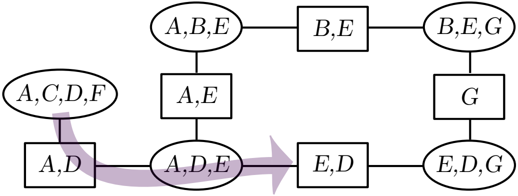

Cluster graph

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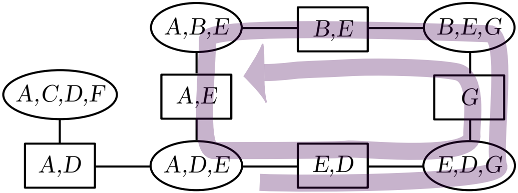

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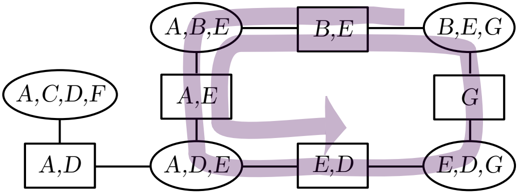

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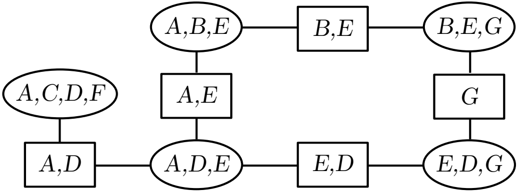

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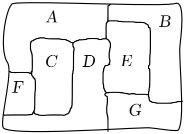

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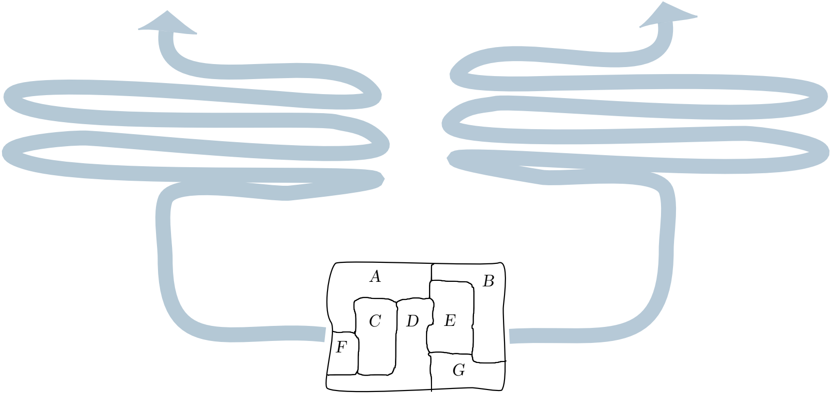

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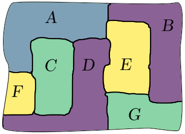
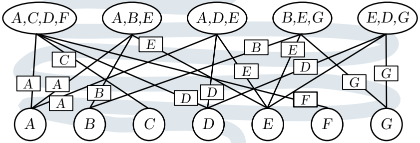
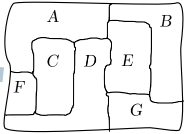
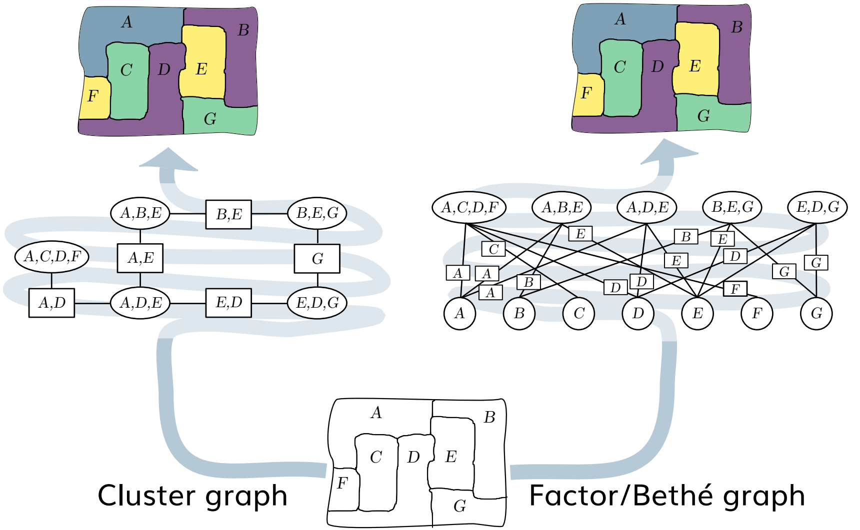






Factor/Bethé graph

PROBABILISTIC GRAPHICAL MODELS



CLUSTER GRAPHS

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We found that

- Graph structure influence convergence speed and accuracy
- Factor graphs are predominant in PGM literature
- Cluster graphs outperform factor graphs

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Why is cluster graphs the underdog?

- Multiple solutions for the same clusters
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- Graph structure influence convergence speed and accuracy
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Why is cluster graphs the underdog?

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- Absence of a generic construction procedure

We propose the LTRIP procedure as a solution

CLUSTER GRAPHS

Factor/Bethé graph:

variables	A B C D E F G
clusters	A,B,G A,B,C,D B,E,F B,C,D,E,F B,C,G

CLUSTER GRAPHS

Factor/Bethé graph:

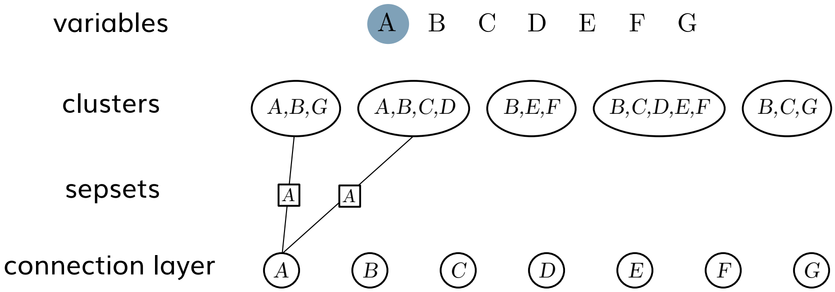
variables A B C D E F G

clusters A,B,G A,B,C,D B,E,F B,C,D,E,F B,C,G

connection layer A B C D E F G

CLUSTER GRAPHS

Factor/Bethé graph:



*running intersection property

CLUSTER GRAPHS

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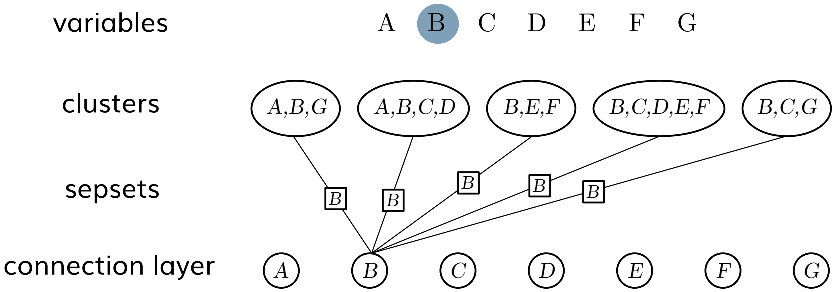

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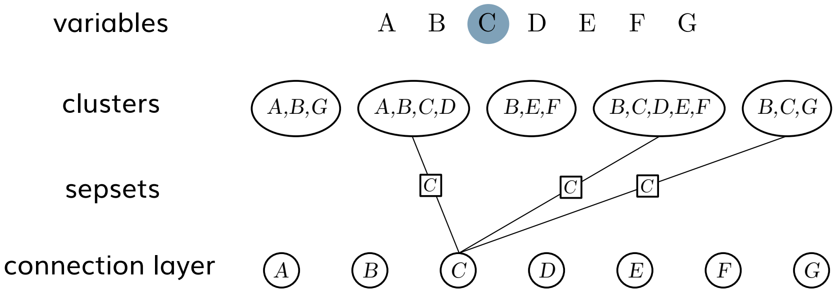

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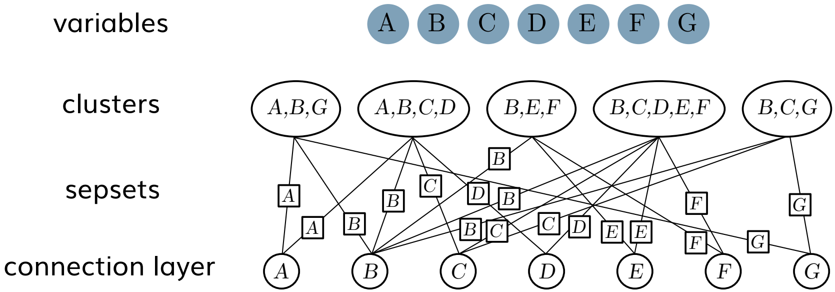

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CLUSTER GRAPHS

Cluster graph:

variables A B C D E F G

clusters A,B,G A,B,C,D B,E,F B,C,D,E,F B,C,G

connection layer A B C D E F G

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variables A B C D E F G

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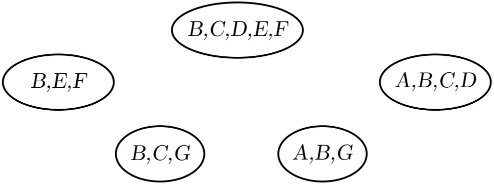
CLUSTER GRAPHS

LTRIP procedure:

variables $A \quad B \quad C \quad D \quad E \quad F \quad G$

clusters



multivar. sepsets

*running intersection property

CLUSTER GRAPHS

LTRIP procedure:

variables

clusters

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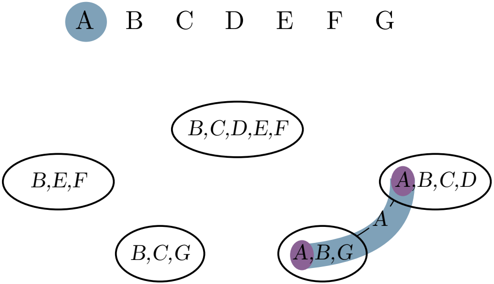

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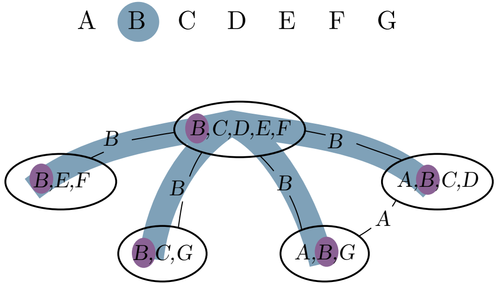

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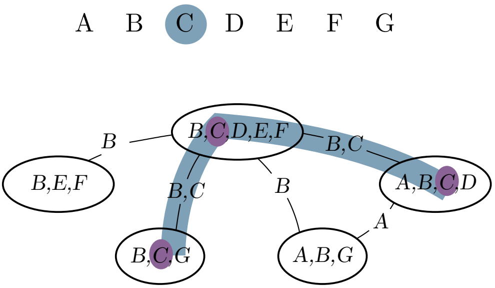

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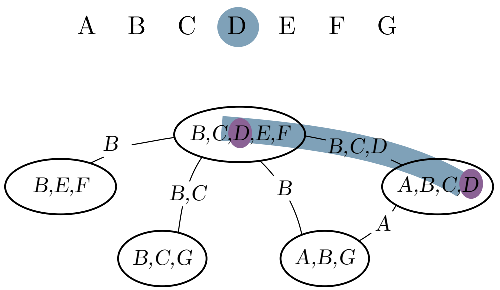

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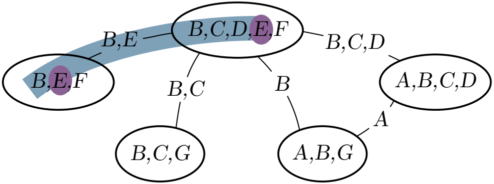

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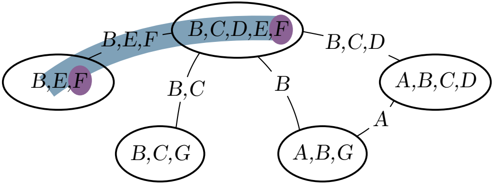

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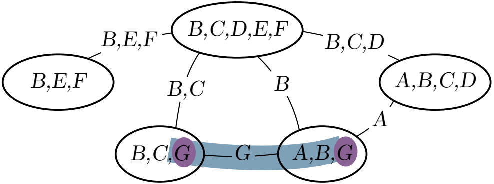

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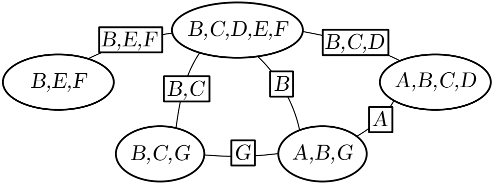

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CLUSTER GRAPHS

Sudoku cluster graph:

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

CLUSTER GRAPHS

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A	B	C	D
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E	F	G	H

I	J	K	L

M	N	O	P

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E	F

C	D
G	H

I	J
M	N

K	L
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A
E
I
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B
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J
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C
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CLUSTER GRAPHS

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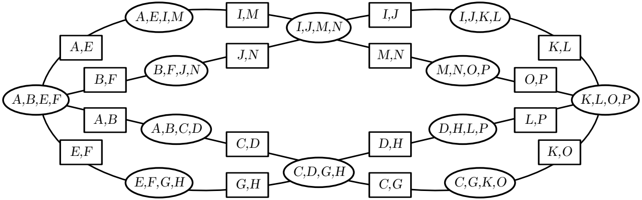
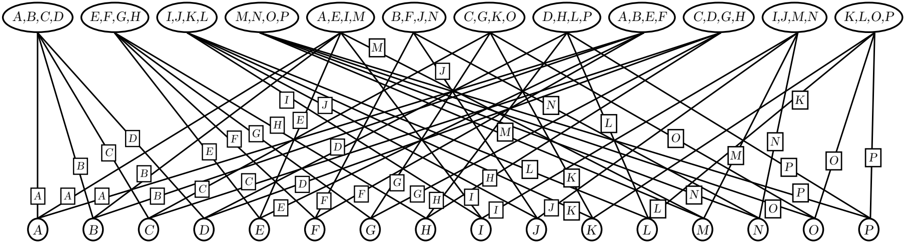


CLUSTER GRAPHS

Sudoku factor/Bethé graph:



RESULTS

RESULTS

A comparison of cluster graphs vs. factor graphs on PGMs build from Sudoku puzzles

Datasets used:

- Project Euler @ projecteuler.net/problem=96
- Sterten's 95 hardest Sudokus @ magictour.free.fr/top95

Tested with different cluster sizes by splitting-up clusters

RESULTS

9x9 Sudoku puzzle

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
$\ldots$								

RESULTS

A	B	C	D	E	F	G	H	I

J	K	L	M	N	O	P	Q	R

$\vdots$

9x9 Sudoku puzzle

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
$\dots$								

RESULTS

Split each 9 variable clique

A	B	C	D	E	F	G	H	I

into cliques of 3 variables

C	B	A	D	E	B	D	E	F
D	E	A	C	B	D	F	E	G
F	G	A	F	E	C	H	E	I
H	I	A	H	C	G	F	I	H
H	I	B	D	I	C	F	G	H
F	B	G	D	G	H	H	I	G

RESULTS

Split each 9 variable clique

A	B	C	D	E	F	G	H	I

into cliques of 5 variables

E	D	B	A	C
F	I	G	A	H
F	B	I	G	H
E	D	F	B	C
D	I	G	H	C
E	D	F	G	H
E	F	I	G	H

RESULTS

Split each 9 variable clique

A	B	C	D	E	F	G	H	I

into cliques of 7 variables

E	D	F	B	G	A	C

E	F	B	I	G	A	H

E	D	F	B	G	H	C

E	D	F	I	G	H	C

RESULTS

Split each 9 variable clique

A	B	C	D	E	F	G	H	I

into cliques of 9 variables

A	B	C	D	E	F	G	H	I

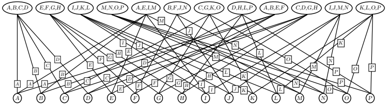
RESULTS

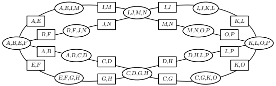
Split each 9 variable clique into cliques of 9 variables

A	B	C	D	E	F	G	H	I

A	B	C	D	E	F	G	H	I

Build these clusters into both a factor graph and a cluster graph





RESULTS

Split each 9 variable clique

A	B	C	D	E	F	G	H	I

into cliques of 9 variables

A	B	C	D	E	F	G	H	I

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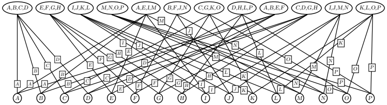

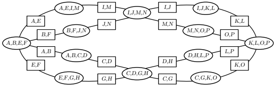

Run message passing on both graphs

(note, for a solved Suduko all clusters is reduced to a single entry)

A	B	C	D	E	F	G	H	I	
1	3	2	6	4	5	7	9	8	1
elsewhere									0

$P(A,B,C,D,E,F,G,H,I)$

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

RESULTS

Project Euler: solution count

Size 3

	factor graph ✗	factor graph ✓	
cluster graph ✗	36	0	36
cluster graph ✓	1	13	14
	37	13	

cluster graph slower

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

Size 5

cluster graph \ factor graph	✗	✓	
✗	25	0	25
✓	2	23	25
	27	23	

cluster graph slower

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

Size 5

cluster graph \ factor graph	✗	✓	
✗	25	0	25
✓	2	23	25
	27	23	

cluster graph slower

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

cluster graph slower

Size 5

cluster graph \ factor graph	✗	✓	
✗	25	0	25
✓	2	23	25
	27	23	

cluster graph slower

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

cluster graph slower

Size 5

cluster graph \ factor graph	✗	✓	
✗	25	0	25
✓	2	23	25
	27	23	

cluster graph slower

Size 7

cluster graph \ factor graph	✗	✓	
✗	5	0	5
✓	4	41	45
	9	41	

cluster graph faster

Size 9

cluster graph \ factor graph	✗	✓	
✗	0	0	0
✓	4	46	50
	4	46	

cluster graph faster

RESULTS

Project Euler: solution count

Size 3

cluster graph \ factor graph	✗	✓	
✗	36	0	36
✓	1	13	14
	37	13	

cluster graph slower

Size 5

cluster graph \ factor graph	✗	✓	
✗	25	0	25
✓	2	23	25
	27	23	

cluster graph slower

Size 7

cluster graph \ factor graph	✗	✓	
✗	5	0	5
✓	4	41	45
	9	41	

cluster graph faster

Size 9

cluster graph \ factor graph	✗	✓	
✗	0	0	0
✓	4	46	50
	4	46	

cluster graph faster

Cluster graphs more successful than factor graphs
Naive solver to test graph structures - can improve!

CONCLUSION

- Main contribution is LTRIP for constructing cluster graphs
- These cluster graphs show great promise over factor graphs
- We hope LTRIP will enhance the popularity of cluster graphs

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FUTURE WORK

- Investigate more advance techniques for graph coloring PGMs
- Mutual information approach for LTRIP's max spanning trees
- Investigate cluster graphs on wider set of problems