Information Theory SV2

Question sources: *Matthew Ireland's SV worksheet, 23/24 Examples Sheet*

Question 1: Explain why a terminating character is included in Arithmetic coding. If it were not included, what information would you need to decode a message? If it were included, what effect does the probability assigned to it have on the code?

Question 2: Consider a (7, 4) Hamming code which maps $k = 4$ information bits to a length $n = 7$ codeword. Assume 0 and 1 are equiprobable in the input data.

Use the convention used by the rest of the world, not the one presented in lectures. The transmitted codeword is `[b1, b2, b3, b4, b5, b6, b7]` where:

$$b_3, b_5, b_6, b_7 \text{ are the source bits,}$$

$$b_4 = b_5 \oplus b_6 \oplus b_7, \quad b_2 = b_3 \oplus b_6 \oplus b_7, \quad b_1 = b_3 \oplus b_5 \oplus b_7.$$

(a) Suppose that a codeword is transmitted over a binary symmetric channel (BSC), and the received codeword is `r = [1, 1, 0, 1, 0, 1, 1]`. Decode the received sequence to a codeword.

(b) Calculate the probability of block error p_B of the (7, 4) Hamming code as a function of the bit error p and show that to leading order it goes as $21p^2$.

(c) If the (7, 4) Hamming code can correct any one-bit error, might there be a (14, 8) code that can correct any two errors?

Question 3: Consider using the repetition code R_5 to encode binary input symbols for transmission through a binary symmetric channel with $f = 0.3$. Assuming $p_0 < 0.5$, find the maximum value of p_0 for which the optimal decoder's rule is not simply "pick the majority vote".

Question 4: A binary symmetric channel receives as input a bit whose values $\{0, 1\}$ have probabilities $\{p, 1 - p\}$, but in either case, a transmission error can occur with probability ε, which flips the bit. The surface plot describes the mutual information of this channel as a function $M(\varepsilon, p)$ of these probabilities:

(a) At the point marked A, the error probability is $\varepsilon = 0$. Why then is the channel mutual information minimal in this case: $M(\varepsilon, p) = 0$?

(b) At the point marked B, an error always occurs ($\varepsilon = 1$). Why then is the channel mutual information maximal in this case: $M(\varepsilon, p) = 1$?

(c) At the point marked C, the input bit values are equiprobable ($p = 0.5$), so the symbol source has maximal entropy. Why then is the channel mutual information in this case $M(\varepsilon, p) = 0$?

Mackay ex 6.4, p.119 Prove that any uniquely decodable code from $\{0,1\}^+$ to $\{0,1\}^+$ necessarily makes some strings longer if it makes some strings shorter.

Mackay ex 6.5, p.120 Encode the string `000000000000100000000000` using the basic Lempel–Ziv algorithm.