A CLASS OF REGULARIZED ALGORITHMS WITH NEW STABILIZER FOR UNSTABLE EQUILIBRIUM PROBLEMS *

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Abstract. In this paper, we study methods for solving equilibrium programming problems with new stabilizer, provided that the input data in the problem is known inaccurately. We concentrate on two aspects: one is general stabilization method; the other is specific (extragradient) algorithm. The problems considered in this paper are generally unstable to perturbations of the input data, and therefore the application of standard approaches to solving such problems in most cases will not lead to an acceptable accuracy of the answer. In this regard, special methods called regularization methods are usually used to solve such problems. For the former aspect, we propose the stabilization method with the use of a new version of the stabilizer, which has the property of strong skew symmetry. We also prove the convergence of method. For the latter, we give a specific regularized method of numerical implementation of the proposed general approach. The convergence of specific method is proved.

Key words. Equilibrium programming; ill-posed problem; new stabilizer; stabilization method; regularized extragradient method

1. Introduction.

Many important problems of operations research, computational mathematics, mathematical economics, related to the search for a compromise and the reconciliation of partially (or completely) opposing interests of the parties to the conflict, are reduced to the study of mathematical models, which is the field of mathematics called equilibrium programming. Many classical problems can be written in the form of equilibrium programming, such as optimal control, multi-criteria decision-making under uncertainty, inverse optimization problems, variational inequalities, saddle Lagrange problems, Nash equilibrium search problems, etc. [**?**].

The main problem of equilibrium programming is usually formulated as follows: let there be some function $\Phi(v, w)$ given on the Cartesian product $\mathbf{W} \times \mathbf{W}$, where $\mathbf{W} = \{w \in \mathbf{W_0} \subseteq \mathbb{R}^n : g_1(w) \leqslant 0, ..., g_m(w) \leqslant 0\}$ - a given set from the space $\mathbb{R}^n$, $\mathbf{W_0}$ is a set from $\mathbb{E}^n$ (usually has a simple form, perhaps $\mathbf{W_0} = \mathbb{E}^n$). We need to find a point v_* from $\mathbf{W}$ that satisfies the inequality

$$\Phi(v_*, v_*) \leqslant \Phi(v_*, w), \quad \forall w \in \mathbf{W}. \tag{1.1}$$

Such a point is called the **equilibrium point** of the (1.1) problem. If the function $\Phi(v, w)$ does not depend on the variable v, then the problem (1.1) turns into a regular mathematical programming problem.

By now, the problem of the existence and uniqueness of the equilibrium point of the unperturbed problem [**?**] has been studied quite well. Also in [**?**][**?**], a result on the convexity and closure of the set of solutions is presented. Various specific methods for finding equilibrium points are discussed, for example, in [**?**]-[**?**]. Such methods are usually iterative in nature. However, they were studied under significant restrictions on $\Phi(v, w)$ and the set of $\mathbf{W}$, which is rarely not easily applied in practice. Special attention is paid to continuous methods that generate entire families of [**?**] methods.

In practice, attention is also drawn to another feature of the problem, namely instability. This problem, generally speaking, is **unstable** to perturbations of the function $\Phi(v, w)$ and the set $\mathbf{W}$, as evidenced by a simple example:

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Let's assume that the following conditions are met for the perturbation of the function and its gradient:

$$\lim_{\delta\to 0}|\Phi^{\delta}(v,w)-\Phi(v,w)|=0;\quad \lim_{\delta\to 0}\|\nabla^{\delta}_{w}\Phi(v,w)-\nabla_{w}\Phi(v,w)\|=0 \tag{1.2}$$

Let $\Phi(v,w)=vw$, for an approximate function we take

$$\Phi^{\delta}(v,w)=vw-\delta w^2, \mathbf{W}=\{w\in\mathbb{E}^1:|w|\leqslant 1\}.$$

Then $\nabla_w\Phi^{\delta}(v,w)=v-2\delta w$. Obviously, the condition (1.2) is satisfied, and the set of solutions $\mathbf{W}_*$ consists of a single point $v_*=0$, but the approximate problem

$$v^{\delta}_{*}\in\mathbf{W},\quad \Phi^{\delta}(v^{\delta}_{*},v^{\delta}_{*})\leqslant\Phi^{\delta}(v^{\delta}_{*},w),\quad \forall w\in\mathbf{W} \tag{1.3}$$

has no solution for any arbitrarily small $\delta>0$.

It is clear that a simple approximation of the problem does not give a positive result, and to solve it, we need to use methods that implement the idea of **regularization**, which put forward by academician A.N. Tikhonov. As expected, there are regularized variants of methods for solving the problem.In various works, methods for regularization of unstable equilibrium problems [**?**]-[**?**] are proposed, in which a stabilizer of the form of the scalar product $\langle v,w\rangle$ is usually used. Accordingly, we can try other types of stabilizer, for example, $-\|v-w\|^2$. It should be noted that a new approach is required to solve this unstable problem.

One of the important concepts in the theory of equilibrium programming is **strong skew symmetry**. If satisfied

$$\Phi(w,w)-\Phi(w,v)-\Phi(v,w)+\Phi(v,v)\geqslant\mu\|v-w\|^2,\quad \forall w,v\in\mathbf{W_0},$$

where $\mu>0$ is some constant, then we say that the function $\Phi(v,w)$ is strongly skew-symmetric.

It is easy to check that $\langle v,w\rangle$ and $-\|v-w\|^2$ are strong skew-symmetric:

$$\begin{aligned}&\langle v,w\rangle:\Phi(w,w)-\Phi(w,v)-\Phi(v,w)+\Phi(v,v)=1\cdot\|v-w\|^2,\forall v,w\in\mathbb{R}^n,\\ &-\|v-w\|^2:\Phi(w,w)-\Phi(w,v)-\Phi(v,w)+\Phi(v,v)=2\cdot\|v-w\|^2,\forall v,w\in\mathbb{R}^n.\end{aligned}$$

The goal and the main contribution of the paper is to develop stabilization method using a new norm square stabilizer for solving equilibrium programming problems with inaccurate data, as well as the certain regularized method using the new stabilizer. In this paper, we don't consider all variants of constructive methods, but mainly concentrates on extragradient method.

The paper uses the theory and methods of optimization, the theory of equilibrium programming, the theory and methods of solving ill-posed problems and numerical methods for solving various practical problems. The methods developed in this paper can be applied to solving problems of equilibrium programming, the function and set of which are unstable to the errors of setting the initial data. The author expresses gratitude to his scientific supervisor B.A. Budak for the valuable comments made during the discussion of this work.

2. A stabilization method for unstable equilibrium problems. The first question which is discussed in this paper is: what will be obtained in the stabilization method if we use a stabilizer of the form $-\|v-w\|^2$. We consider a stabilization method that uses the ideas of the penalty method.

Notation	Description
$\Phi(v,w)$	a real function of two variables from $\mathbb{R}^n$ defined on $\mathbf{W}\times\mathbf{W}$
$g(w)$	A constraint function that specifies a set $\mathbf{W}$
v_*	equilibrium point or normal solution of the problem
$\Omega(v,w)$	stabilizer of two variables
∇_w	the gradient of the variable w
Pr	operation of projection
$\langle\cdot,\cdot\rangle$	inner product of two elements in $\mathbb{R}^n$
$\|\cdot\|$	norm in $\mathbb{R}^n$

Table 1
Notations and abbreviations.

A natural question arises: what is the reason for using such a new stabilizer? It turns out that it has a good property called *strong skew symmetry*. This property is similar to the property of strong convexity in optimization theory. As is known, in the idea of Tikhonov regularization, the term $\|u\|^2$, which is strongly convex, is added to the minimized functional. Accordingly, we add a strongly skew-symmetric term $-\|v-w\|^2$ to the function $\Phi(v,w)$.

2.1. Some preliminary considerations. Let's return to the problem statement: find the point v_* from the conditions

$$\begin{gathered} v_*\in\mathbf{W}=\{w\in\mathbf{W_0}: g_i(w)\leqslant 0, i=1,\dots,m; g_i(w)=0, i=m+1,\dots,s\},\\ \Phi(v_*,v_*)\leqslant\Phi(v_*,w),\quad \forall w\in\mathbf{W}, \end{gathered}\tag{2.1}$$

where $\mathbf{W_0}$ is a given set from the Euclidean space $\mathbb{E}^n$. The functions $\Phi(v,w), g_i(w)$ are defined on the set $\mathbf{W_0}$. The points v_* that satisfy the conditions (2.1) are called equilibrium points. The set of equilibrium points will be denoted by $\mathbf{W}_*$. It is assumed that $\mathbf{W}_*\neq\varnothing$.

Definition 2.1. *We will call the function*

$$P(w)=\sum_{i=1}^{s}[g_i^+(w)]^p,\quad w\in\mathbf{W_0}, p>0\tag{2.2}$$

***the simplest penalty function**, where*

$$g_i^+=\max\{0;g_i\}, i=1,...,m; g_i^+=|g_i|, i=m+1,...,s.$$

Let the approximations $\Phi_\delta(v,w), P_\delta(w)$ for the functions $\Phi(v,w), P(w)$ be such that

$$\begin{aligned} &\big|\Phi_\delta(v,w)-\Phi(v,w)\big|\leqslant\delta\big(1+\|v\|^2+\|w\|^2\big),\quad v,w\in\mathbf{W_0},\delta>0,\\ &\big|P_\delta(w)-P(w)\big|\leqslant\delta\big(1+\|w\|^2\big),\quad w\in\mathbf{W_0},\delta>0. \end{aligned}\tag{2.3}$$

Taking into account the constraints of the set and perturbations, we introduce the Tikhonov function using a stabilizer of the form $\Omega(v,w)=-\|v-w\|^2$:

$$t_\delta(v,w)=\Phi_\delta(v,w)+AP_\delta(w)-\alpha\|v-w\|^2,\quad v,w\in\mathbf{W_0},\alpha>0,A>0.\tag{2.4}$$

Note that this stabilizer is a stabilizer that depends on two variables, which the same as $\langle v, w\rangle$,and it has the property of strong skew symmetry. We will search for the point v_δ that satisfies the conditions

$$v_\delta \in \mathbf{W_0}, \quad t_\delta(v_\delta, v_\delta) \leqslant t_\delta(v_\delta, w) + \varepsilon, \quad \forall w \in \mathbf{W_0}, \varepsilon > 0, \tag{2.5}$$

i.e. to search for the equilibrium point of the Tikhonov function. Suppose that for some point $v_* \in \mathbf{W}_*$ there are constants $v > 0, c_i \geqslant 0$ such that

$$\Phi(v_*, v_*) \leqslant \Phi(v_*, w) + \sum_{i=1}^{s} c_i [g_i^+(w)]^v, \quad \forall w \in \mathbf{W_0} \tag{2.6}$$

We have already seen this condition in the previous research of regularization methods. [?] We formulate the first auxiliary proposition. [?], page. 5

Proposition 2.2. *If the condition (2.6), $p \geqslant v$, is met for the point $v_* \in \mathbf{W}_*$, then the inequalities*

$$\begin{aligned}
&\sum_{i=1}^{s} c_i [g_i^+(w)]^v \leqslant AP(w) + BA^{-v/(p-v)}, \quad \forall w \in \mathbf{W_0}, A > 0, p > v,\\
&\sum_{i=1}^{s} c_i [g_i^+(w)]^v \leqslant AP(w), \quad \forall w \in \mathbf{W_0}, A > |c|_\infty = \max_{1 \leqslant i \leqslant s} c_i, p = v,
\end{aligned} \tag{2.7}$$

are valid, where $B = (p - v)v^{v/(p-v)} p^{-v/(p-v)} |c|^{v/(p-v)}, |c| = (\sum_{i=1}^{s} |c_i|^{p/(p-v)})^{(p-v)/p}$ when $p > v$.

Next, we give and prove a proposition about a sufficient condition for the non-emptiness of the set $\mathbf{W}_{*\delta}$, i.e. the set of equilibrium points of the Tikhonov function.

Proposition 2.3. *If*

$$\begin{aligned}
&BA^{-v/(p-v)} + \delta\|v_*\|^2(3 + A) + 2(\delta + A\delta) \leqslant 1/2\varepsilon(\delta),\\
&\alpha\|v_* - w\|^2 + \|w\|^2(\delta + A\delta) \leqslant 1/2\varepsilon(\delta), \quad \forall w \in \mathbf{W_0}, \delta > 0,
\end{aligned} \tag{2.8}$$

where the point $v_* \in \mathbf{W}_*$ meets the condition (2.6), $p \geqslant v$, then in (2.5) it is possible to accept $v_\delta = v_*$, that is, under these assumptions, the set of $\mathbf{W}_{*\delta}$ points v_δ satisfying the condition (2.5) when choosing $\Phi_\delta(v, w), P_\delta(w)$ from (2.3), nonempty.

Remark 2.4. The second condition of the sentence will be met, for example, if a constraint condition is imposed on the set $\mathbf{W_0}$, i.e. $\|w\| \leqslant \text{const}, \forall w \in \mathbf{W_0}$.

Note that the further discussion does not depend on the method of searching for points $v_\delta \in \mathbf{W}_{*\delta}$, only the fact of the existence of such points will be important for us.

Definition 2.5. *If satisfied*

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geqslant 0, \quad \forall w, v \in \mathbf{W_0}, \tag{2.9}$$

*then we say that the function $\Phi(v, w)$ is **skew-symmetric**.*

Remark 2.6. The properties of functions belonging to the skew-symmetric class can be viewed, for example, in [?]. Note that from (2.9) for $v = v_*$ and the condition (2.6) we derive that

$$\Phi(w, w) - \Phi(w, v_*) \geqslant \Phi(v_*, w) - \Phi(v_*, v_*) \geqslant -\sum_{i=1}^{s} c_i [g_i^+(w)]^v, \quad \forall w \in \mathbf{W_0}. \tag{2.10}$$

Another proposition that is used during the proof of the method: [?], page. 99, lemma 11.

PROPOSITION 2.7. *Let $z, b, d \geqslant 0, p > 1$ such, that*

$$0 \leqslant z^p \leqslant bz + d. \tag{2.11}$$

Then

$$0 \leqslant z \leqslant (b^q + qd)^{1/p}, \tag{2.12}$$

where q is defined by $\frac{1}{p} + \frac{1}{q} = 1$.

2.2. Description of the stabilization method using a new stabilizer. We turn to the very description of the stabilization method. The main theorem describing the convergence of the stabilization method is formulated and proved as follows:

THEOREM 2.8. *Let the following conditions be met*
1) *$\mathbf{W_0}$ is a closed bounded set, the functions $g_i(w), i = 1, ..., m, |g_i(w)|, i = m + 1, \dots, s, \Phi(w, w)$ are semi-continuous from below on $\mathbf{W_0}$; the function $\Phi(v, w)$ is semi-continuous from above on v on $\mathbf{W_0}$ for any fixed $w \in \mathbf{W_0}$; the set of $\mathbf{W}_*$ solutions to the problem (2.1) is nonempty; for some point $v_* \in \mathbf{W}_*$, the inequality (2.6) is fulfilled; the function $\Phi(v, w)$ is skew-symmetric on $\mathbf{W_0}$, i.e. the inequality (2.9) is satisfied;*
2) *$\Omega(v, w) = -\alpha_k \|v - w\|^2$ is the stabilizer of the problem (2.1), $P(w)$ is the penalty function defined by the formula (2.2) for $p \geqslant v$;*
3) *approximations $\Phi_\delta(v, w), P_\delta(w)$ of the functions $\Phi(v, w), P(w)$ satisfy the conditions (2.3);*
4) *the parameters $\alpha = \alpha(\delta) > 0, A = A(\delta) > 0, \varepsilon = \varepsilon(\delta) > 0, \delta > 0$, satisfy the conditions (2.8) and, in addition,*

$$\begin{gathered}\lim_{\delta\to 0} \alpha(\delta) = 0, \quad \lim_{\delta\to 0} A(\delta) = 0, \quad \lim_{\delta\to 0} \varepsilon(\delta) = 0, \quad \lim_{\delta\to 0} \delta A(\delta) = 0,\\ \sup_{\delta>0} \frac{3\delta + \delta A(\delta)}{\alpha(\delta)} < +\infty, \quad \sup_{\delta>0} \frac{\varepsilon(\delta)}{\alpha(\delta)} < +\infty.\end{gathered} \tag{2.13}$$

Then the set $\mathbf{W}_{*\delta} \neq \varnothing$ for all $\delta > 0$, and

$$\lim_{\delta\to 0} \rho(v_\delta, \mathbf{W}_*) = 0, \quad \lim_{\delta\to 0} \rho(\Phi(v_\delta, v_\delta), \Phi_*) = 0, \tag{2.14}$$

where Φ_* is the set of values of the functions $\Phi(v, v)$, when v runs through the set $\mathbf{W}_*$, and the convergence in (2.14) is uniform with respect to the choice of $\Phi_\delta(v, w), P_\delta(w)$ from (2.3) and the point v_δ from $\mathbf{W}_{*\delta}$.

So, the stabilization method using the stabilizer $-\alpha_k\|v - w\|^2$ is formally described.

3. Regularized extragradient method. In the last section, we considered the stabilization method, but the specific method of finding the equilibrium points of the Tikhonov function was not investigiated. In this section, we will explore one of the specific methods - *extragradient method.*

Consider the following iterative process

$$\begin{gathered}u_k = \mathrm{Pr}_{\mathbf{W_0}}(v_k - \beta_k[\nabla_w^k \Phi(v_k, v_k) + A_k \nabla_w^k P(v_k)]),\\ v_{k+1} = \mathrm{Pr}_{\mathbf{W_0}}(v_k - \beta_k[\nabla_w^k \Phi(v_k, u_k) + A_k \nabla_w^k P(u_k) - 2\alpha_k(u_k - v_k)]), \quad k = 0, 1, ...\end{gathered} \tag{3.1}$$

In general, it is similar to the gradient method of the predictive type, which was mentioned in some works[**?**]. Here instead of the original function, we take its approximation, and add the penalty term and the derivative of the stabilizer. We introduce the Tikhonov function

$$\begin{aligned} T_k(v,w) &= \Phi(v,w) + A_k P(w) - \alpha_k \|v-w\|^2, \quad v,w \in \mathbf{W_0}, \\ & A_k > 0, \alpha_k > 0, \quad k = 0,1,... \end{aligned} \tag{3.2}$$

It is assumed that the functions $\Phi(v,w), g_i(w), i = 1,...,s$ are differentiable in $\mathbf{W_0}$ and the gradient of the function (3.2) by w exists and is equal to

$$\begin{aligned} &\nabla_w T_k(v,w) = \nabla_w \Phi(v,w) + A_k \nabla_w P(w) - 2\alpha(w-v), \\ &v,w \in \mathbf{W_0}, A_k > 0, \alpha_k > 0, k = 0,1,... \end{aligned} \tag{3.3}$$

Instead of the exact values $\nabla_w \Phi(v,w), \nabla_w P(w)$, we know the sequences of approximations $\{\nabla_w^k \Phi(v,w)\}, \{\nabla_w^k P(w)\}$ such that

$$\begin{aligned} &\|\nabla_w^k \Phi(v,w) - \nabla_w \Phi(v,w)\| \leqslant \delta_k(1+\|v\|+\|w\|), \quad \forall v,w \in \mathbf{W_0}, \\ &\|\nabla_w^k P(w) - \nabla_w P(w)\| \leqslant \delta_k(1+\|w\|), \quad \forall w \in \mathbf{W_0}, \delta_k > 0, k = 0,1,... \end{aligned} \tag{3.4}$$

3.1. Auxiliary theorems. The following theorem was proposed on the convexity and closure of the set of solutions $\mathbf{W}_*$. It also establishes the uniqueness of the normal solution. [[**?**] page. 3.]

THEOREM 3.1. *Let the following conditions be met*
1) $\mathbf{W}_0$ *is a convex closed set of* $\mathbb{E}^n$*,*
2) *the function* $\Phi(v,w)$ *is continuous over the set of variables* $(v,w) \in \mathbf{W_0} \times \mathbf{W_0}$*, convex over the variable* w *by* $\mathbf{W_0}$ *for every fixed* $v \in \mathbf{W_0}$*, satisfies the condition of skew symmetry* (2.9)*.*
3) *The functions* $g_i(w)$ *for* $i = 1,...,m$ *are continuous, convex on* $\mathbf{W_0}$*; the function* $g_i(w)$ *for* $i = m+1,...,s$ *are affine, i.e. representable as* $g_i(w) = \langle a_i, w\rangle - b_i, a_i \in \mathbb{E}^n, b_i \in \mathbb{E}^1$*.*
4) *Let the set of* $\mathbf{W}_*$ *solutions to the problem* (2.1) *be nonempty.*
Then $\mathbf{W}_*$ *is convex, closed, and the problem* (2.1) *has a unique solution* v_* *that has the minimum norm among all solutions and is called the normal solution of the problem* (2.1)*.*

Next, we will study the behavior of sequence $\{z_k\}$ - equilibrium points of the function (3.2), defined by the condition

$$z_k \in \mathbf{W_0}, \quad T_k(z_k,z_k) \leqslant T_k(z_k,w) \quad \forall w \in \mathbf{W_0}, \quad k = 0,1,... \tag{3.5}$$

Similar results are given in the following theorem, its proof uses the facts of the previous theorem.

THEOREM 3.2. *Let the conditions of the* 3.1 *theorem be satisfied, and*
1) *the functions* $\Phi(v,w), g_i(w), i = 1,...,m$*, have continuous gradients* $\nabla_w \Phi(v,w)$*,* $\nabla_w g_i(w)$ *on* $\mathbf{W_0}$*;*
2) *the Tikhonov function* (3.2) *is convex by* w *for every* $k = 0,1,...$*;*
3) *there are constants* $v > 0, c_i \geqslant 0, i = 1,...,s$*, such that*

$$\Phi(v_*,v_*) \leqslant \Phi(v_*,w) + \sum_{i=1}^{s} c_i (g_i^+(w))^v, \quad \forall w \in \mathbf{W_0}, \tag{3.6}$$

where v_* is the normal solution of the problem (2.1);
***4)** the parameter p from the penalty function (2.2) satisfies the conditions $p \geqslant v, p > 1$; the sequences $\{\alpha_k\}, \{A_k\}$ are such that*

$$\begin{aligned} &\alpha_k > 0, \quad A_k > 0, \quad k = 0, 1, ..., \\ &\lim_{k\to\infty} \alpha_k = 0, \quad \lim_{k\to\infty} A_k = +\infty, \quad \lim_{k\to\infty} \alpha_k A_k^{v/(p-v)} = +\infty. \end{aligned} \tag{3.7}$$

(for $p = v$, the last condition is not necessary).

Then the points z_k satisfying the conditions (3.5) exist, are uniquely determined for every $k = 0, 1...$ and are such that

$$\|z_k\| \leqslant R_k \leqslant \sup_{k\geqslant 0} R_k = R, \quad k = 0, 1, ...,$$

$$\begin{aligned} R_k &= \left(\frac{B}{\alpha_k A_k^{v/(p-v)}}\right)^{1/2} + \|v_*\|, \quad B = (p-v)v^{v/(p-v)}p^{-p/(p-v)}|c|^{p/(p-v)}, \\ |c| &= \left(\sum_{i=1}^{s} c_i^{p/(p-v)}\right)^{(p-v)/p} \text{ when } p > v, \quad R_k = \|v_*\| \text{ when } p = v, \end{aligned} \tag{3.8}$$

$$\lim_{k\to\infty} A_k P(z_k) = 0, \tag{3.9}$$

$$\lim_{k\to\infty} \|z_k - v_*\| = 0, \tag{3.10}$$

$$\begin{aligned} \|z_k - z_m\| &\leqslant \frac{|A_m - A_k| R_1}{2\alpha_k}, \quad \forall k, m = 0, 1, ... \\ R_1 &= \max_{\|w\|\leqslant R} \|\nabla_w P(w)\|. \end{aligned} \tag{3.11}$$

Remark 3.3. The classes of problems for which the (3.6) condition is met are given in [**?**].

3.2. Description of a regularized extragradient method using a new stabilizer. Let's start investigating the convergence of the iterative method (3.1). The following theorem is satisfied:

THEOREM 3.4. *Let all the conditions of the theorems 3.1,3.2 be satisfied, and*
***1)** let the gradients $\nabla_w \Phi(v, w), \nabla_w P(w)$ satisfy the Lipschisch condition:*

$$\begin{aligned} &\max\{\|\nabla_w \Phi(v, v) - \nabla_w \Phi(w, w)\|, \|\nabla_w P(v) - \nabla_w P(w)\|\} \leqslant L\|v - w\|, \\ &\forall v, w \in \mathbf{W_0}, L = \text{const} > 0. \end{aligned} \tag{3.12}$$

Let the modified Lipschitz condition of the form also be fulfilled:

$$\|\nabla_w \Phi(v, w) - \nabla_w \Phi(w, w)\| \leqslant L\|v - w\|, \quad \forall v, w \in \mathbf{W_0}, L = \text{const} > 0. \tag{3.13}$$

***2)** Instead of the exact values of the gradients $\nabla_w \Phi(v, w), \nabla_w P(w)$, we know sequence of approximations $\{\nabla_w^k \Phi(v, w)\}, \{\nabla_w^k P(w)\}$ satisfying the conditions (3.4).*
***3)** The parameters $\{\alpha_k\}, \{\beta_k\}, \{\delta_k\}, \{A_k\}$ of the (3.1) method are such that*

$$\begin{aligned} &\alpha_k > 0, A_{k+1} \geqslant A_k > 0, \beta_k > 0, \delta_k > 0, \lim_{k\to\infty} \beta_k = 0, \lim_{k\to\infty} \delta_k = 0, \\ &\sup_{k\geqslant 0} \beta_k(1 + A_k) < \frac{1}{L}, \lim_{k\to\infty} \frac{\delta_k + \delta_k A_k}{\alpha_k} = 0, \lim_{k\to\infty} \frac{A_{k+1} - A_k}{\alpha_k^2 \beta_k} = 0. \end{aligned} \tag{3.14}$$

Then the sequence $\{v_k\}$ *generated by the method* (3.1) *for any choice of the initial approximation* $v_0 \in \mathbf{W_0}$ *converges to the normal solution* v_* *tasks* (2.1)*, i.e.*

$$\lim_{k\to\infty} \|v_k - v_*\| = 0 \tag{3.15}$$

moreover, the convergence in (3.15) *is uniform with respect to the choice* $\{\nabla_w^k \Phi(v,w)\}$, $\{\nabla_w^k P(w)\}$ *from* (3.4).

Remark 3.5. As sequences of parameters $\{\alpha_k\}$,$\{\beta_k\}$,$\{\delta_k\}$,$\{A_k\}$ that meet the conditions (3.14), we can, for example, take

$$\alpha_k = (k+1)^{-\alpha}, A_k = (k+1)^A, \delta_k = (k+1)^{-\delta}, \beta_k = \frac{1}{2L(1+A_k)}$$

where α, A, δ are positive numbers.