Exe 1. Week 10

1.a) $\nabla f(x+\varepsilon p) = \nabla f(x) + \varepsilon \nabla^2 f(x) p + O(\varepsilon^2)$

$$\Rightarrow \nabla^2 f(x) p \approx \frac{\nabla f(x+\varepsilon p) - \nabla f(x)}{\varepsilon}$$

1.b) $\nabla f(x-\varepsilon p) = \nabla f(x) - \varepsilon \nabla^2 f(x) p + O(\varepsilon^2)$

$$\Rightarrow \nabla^2 f(x) p \approx \frac{\nabla f(x+\varepsilon p) - \nabla f(x-\varepsilon p)}{2\varepsilon}$$

2) $f(x+\varepsilon e_i) = f(x) + \varepsilon \nabla f(x)^T e_i + \frac{\varepsilon^2}{2} e_i^T \nabla^2 f(x) e_i + O(\varepsilon^3)$

$f(x+\varepsilon e_j) = f(x) + \varepsilon \nabla f(x)^T e_j + \frac{\varepsilon^2}{2} e_j^T \nabla^2 f(x) e_j + O(\varepsilon^3)$

$f(x+\varepsilon(e_i+e_j)) = f(x) + \varepsilon \nabla f(x)^T (e_i+e_j)$

$+ \frac{\varepsilon^2}{2} (e_i+e_j)^T \nabla^2 f(x) (e_i+e_j) + O(\varepsilon^3)$

$= f(x) + \varepsilon \nabla f(x)^T (e_i+e_j)$

$+ \frac{\varepsilon^2}{2} [e_i^T \nabla^2 f(x) e_i + e_j^T \nabla^2 f(x) e_j + 2 e_i^T \nabla^2 f(x) e_j] + O(\varepsilon^3)$

$$\Rightarrow f(x+\varepsilon(e_i+e_j)) - f(x+\varepsilon e_i) - f(x+\varepsilon e_j)$$

$$= -f(x) + \varepsilon^2 e_i^T \nabla^2 f(x) e_j + O(\varepsilon^3)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{f(x+\varepsilon(e_i+e_j)) - f(x+\varepsilon e_i) - f(x+\varepsilon e_j) + f(x)}{\varepsilon^2}$$

$$+ O(\varepsilon)$$