Pearploy Chaichaorensin 6381278

Task 3 : Quick Sort Recurrence

(ii) Let $g(n) = \frac{f(n)}{n+1}$. Can you write what you have in terms of the function g ?

From (3) $$n \cdot f(n) = 2n + (n+1)f(n-1)$$

$$f(n) = \frac{2n + (n+1)f(n-1)}{n}$$

Then $$g(n) = \frac{f(n)}{n+1} = \frac{2n + (n+1)f(n-1)}{n} \cdot \frac{1}{n+1}$$

$$= \frac{2n}{n(n+1)} + \frac{(n+1)f(n-1)}{n(n+1)}$$

$$g(n) = \frac{2}{(n+1)} + \frac{f(n-1)}{n} \qquad \text{———(4)}$$

Since $g(n) = \frac{f(n)}{n+1}$ $then$ $g(n-1) = \frac{f(n-1)}{n}$, let's plug it in equation (4)

So $$g(n) = \frac{2}{(n+1)} + g(n-1)$$

(iii) Your task in this step is to find a closed form for g

Since, $g(n) = \frac{f(n)}{n+1}$ and $f(0) = 0$, then $g(0) = \frac{0}{1} = 0$

$$g(n) = \frac{2}{(n+1)} + g(n-1)$$

$$g(n) = \frac{2}{(n+1)} + \frac{2}{(n)} + g(n-2)$$

$$g(n) = \frac{2}{(n+1)} + \frac{2}{(n)} + \frac{2}{(n-1)} + g(n-3)$$

$$g(n) = \frac{2}{(n+1)} + \frac{2}{(n)} + \frac{2}{(n-1)} + \frac{2}{(n-2)} + g(n-4)$$

$$g(n) = \frac{2}{(n+1)} + \frac{2}{(n)} + \frac{2}{(n-1)} + \frac{2}{(n-2)} + \ldots + \frac{2}{3} + \frac{2}{2} + 2$$

$$g(n) = 2\,[\frac{1}{(n+1)} + \frac{1}{(n)} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \ldots + \frac{1}{3} + \frac{1}{2} + 1] \qquad \text{———(5)}$$

We now see the pattern of the n-th Harmonic number $H_n = \frac{1}{(n)} + \frac{1}{(n-1)} + \frac{1}{(n-2)} +...+ \frac{1}{2} + 1$

Then $H_{n+1} = \frac{1}{(n+1)} + \frac{1}{(n)} + \frac{1}{(n-1)} + \frac{1}{(n-2)} +...+ \frac{1}{2} + 1$, substitute H_{n+1} in (5)

$g(n) = 2\,[H_{n+1}]$

In terms of an expression involving Harmonic numbers, $g(n) = 2\,[H_{n+1}]$

(iv) Let's find a closed form for $f(n)$ and conclude that $f(n)$ is $O(\,n\,ln(n))$

Since $g(n) = \frac{f(n)}{n+1}$ then $f(n) = (n + 1) \cdot g(n)$

$$f(n) = (n + 1) \cdot g(n) \qquad \text{———(6)}$$

From(iii), we know that $g(n) = 2\,[H_{n+1}]$, let substitute $g(n)$ in (6)

So, $$f(n) = (n + 1)(2\,H_{n+1})$$

In terms of Big-O calculation the constant can be eliminated and now we have.

Running time of $f(n) = O(\,n) \cdot O(H_{n+1})$

According from the fact that $H_n \leq 1 + ln(n)$, then the worst-case running time H_n is $O(ln(n))$.

And the worst-case running time H_{n+1} is $O(ln(n + 1))$.

Running time of $f(n) = O(\,n) \cdot O(ln(n + 1))$

To conclude, the running time of $f(n)$ is $O(\,n\,ln(n))$