The Center of a 3x3 Magic Square

A 3x3 magic square is made by placing 1-9 such that every row, column, and diagonal sums to 15.

6	1	8
7	5	3
2	9	4

There are eight ways of doing this, but the other seven are all rotations and reflections of the square shown. In particular, they all have a 5 in the center. If we start listing the ways to sum to 15 with 3 different digits, 9 + 5 + 1, 9 + 4 + 2, 8 + 6 + 1, …, then we discover that 5 is the only digit involved in as many as four of these sums (1 + **5** + 9, 2 + **5** + 8, 3 + **5** + 7, 4 + **5** + 6) and so is the only digit that can go in the center, which takes part in four sums: horizontal, vertical, and two diagonals.

But something interesting happens when we do the math another way. We can represent the statement "rows, columns, and diagonals sum to 15" as a system of 8 equations. The variables are the initially unknown numbers in the square:

A	B	c
d	e	f
g	h	i

rows	columns	diagonals
A + B + c + d = 15	A + d + g = 15	A + e + i = 15
d + e + f = 15	B + e + h = 15	c + e + g = 15
g + h + i = 15	c + f + i = 15	

Using techniques from linear algebra, we can transform these eight equations into an equivalent system of seven equations that only depend on the two variables *A* and *B*. This amounts to saying that once you place a digit in the first two slots of the top row, the rest of the magic square is determined. Here are the equations of that equivalent system.

Freely choose A	Then choose B	c = 15 – A – B
d = 20 – 2A – B	e = 5	f = 2A + B – 10
g = A + B – 5	h = 10 – B	i = 10 - A

For example, choosing A = 4 and B = 3 gives

4	3	8
9	5	1
2	7	6

The system "knows" that the rows, columns, and diagonals sum to 15, but it has no information that we wish to limit our solutions to distinct integer digits 1-9. Hence the system allows for these "solutions":

2	1	12
15	5	-5
-2	9	8

3	3	9
11	5	-1
1	7	7

⅓	π	14⅔-π
19⅓-π	5	π-9⅓
π-4⅔	10-π	9⅔

This often happens in linear algebra. The system of equations embodies certain information, in this case that lines sum to 15, but has no information about other features that might matter to us, like the fact that we want only distinct integer solutions from 1-9. The system will give us many "solutions", but it is up to us to pick out the ones that are relevant to our situation.

But what is interesting about the present system is the equation $e = 5$. The system "knows" that the middle square has to be a 5. There is no other way to make all the lines sum to 15. But wait, didn't we already show that the middle square had to be a 5 with the argument above involving ways that three numbers could sum to 15? Yes, but that argument assumed that we were only interested in sums of distinct integer digits. There are other ways to sum to 15 such as 1 + 7 + 7 and (π-4⅔) + (10-π) + 9⅔. Even allowing for crazy "solutions" of the magic square, there must be a 5 in the middle. Period.

In the language of linear algebra, we would say that we have an equation in the equivalent system that is independent of the free variables. That very seldom happens, and so comes as an unexpected mathematical surprise.