Problem Set 11
7.2: Trigonometric Integrals

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	
Members:	KEY

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\begin{aligned} \sin A \cos B &= \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right], \\ \sin A \sin B &= \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right], \\ \cos A \cos B &= \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right], \end{aligned}$$

Problem 1

Find the integral $\int \sin^3 x \cos^2 x dx$.

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1-\cos^2 x)\cos^2 x \sin x \, dx$$

$$\boxed{u = \cos x \to du = -\sin x \, dx}$$

$$= -\int (1-u^2)u^2 \, du = -\int (u^2 - u^4) \, du$$

$$= -\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Problem 2

Find the integral $\int \sin^2 x \cos^4 x dx$.

$$= \int \frac{1}{2}(1-\cos(2x)) \cdot \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \frac{1}{8} \int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x)) dx$$

$$= \frac{1}{8} \int (1+2\cos(2x)+\cos^2(2x)-\cos(2x)-2\cos^2(2x)-\cos^3(2x)) dx$$

$$= \frac{1}{8} \int (-\cos^3(2x)-\cos^2(2x)+\cos(2x)+1) dx$$

$$I = \frac{1}{8}\Big[\underbrace{\int [\cos(2x)+1] dx}_{I_1} - \underbrace{\int \cos^2(2x) dx}_{I_2} - \underbrace{\int \cos^3(2x) dx}_{I_3}\Big]$$

$$I_1 = \int [\cos(2x)+1] dx = \frac{1}{2}\sin(2x) + x + C_1$$

$$I_2 = \int \frac{1}{2}[1+\cos(4x)] dx = \frac{1}{2}x + \frac{1}{8}\sin(4x) + C_2$$

$$I_3 = \int \cos^2(2x)\cos(2x)\,dx = \int \left(1-\sin^2(2x)\right)\cos 2x\,dx$$

$$\boxed{u = \sin(2x) \qquad du = 2\cos(2x)\,dx}$$

$$= \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\left[u - \frac{1}{3}u^3\right] = \frac{1}{2}u - \frac{1}{6}u^3 + C_3$$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C_3$$

$$I = \frac{1}{8}\left(I_1 - I_2 - I_3\right)$$

$$= \frac{1}{8}\left[\cancel{\frac{1}{2}\sin(2x)} + x - \frac{1}{2}x - \frac{1}{8}\sin(4x) - \cancel{\frac{1}{2}\sin(2x)} + \frac{1}{6}\sin^3(2x)\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\right] + C$$

$$= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C$$

$$\boxed{\text{Note: } C = \frac{1}{8}\left(C_1 - C_2 - C_3\right)}$$

Problem 3

Find the integral $\int \csc^6(2x)\cot^3(2x)dx$.

$$= \int \csc^4(2x)\cot^3(2x) \quad \csc^2(2x)\,dx$$

$$= \int (1+\cot^2(2x))^2 \cot^3(2x) \quad \csc^2(2x)\,dx$$

$$\boxed{u = \cot(2x) \rightarrow du = -2\csc^2(2x)\,dx}$$

$$= \frac{-1}{2}\int (1+u^2)^2 u^3\, du = -\frac{1}{2}\int (1+2u^2+u^4)u^3\, du$$

$$= -\frac{1}{2}\int (u^3+2u^5+u^7)\, du$$

$$= -\frac{1}{2}\left[\frac{1}{4}u^4+\frac{1}{3}u^6+\frac{1}{8}u^8\right]+C$$

$$= -\frac{1}{2}\left[\frac{1}{4}\cot^4(2x)+\frac{1}{3}\cot^6(2x)+\frac{1}{8}\cot^8(2x)\right]+C$$

Problem 4

Find the integral $\int \sec^4 x\,(1-\tan^2 x)\,dx$.

$$= \int \sec^4 x\,(1-\sec^2 x+1)\,dx$$

$$= \int (2\sec^4 x - \sec^6 x)\,dx = 2\int \sec^4 x\,dx - \int \sec^6 x\,dx$$

$$= 2\int \sec^2 x \cdot \sec^2 x\,dx - \int \sec^4 x \;\sec^2 x\,dx$$

$$= 2\int (1+\tan^2 x)\sec^2 x\,dx - \int (1+\tan^2 x)^2 \sec^2 x\,dx$$

$$\boxed{u = \tan x \qquad du = \sec^2 x\,dx}$$

$$= 2\int (1+u^2)\,du - \int (1+u^2)^2\,du$$

$$= 2\left[u + \frac{1}{3}u^3\right] - \int (1+2u^2+u^4)\,du$$

$$= 2u + \frac{2}{3}u^3 - \left[u + \frac{2}{3}u^3 + \frac{1}{5}u^5\right]$$

$$= \frac{-1}{5}u^5 + u + C = \frac{-1}{5}\tan^5 x + \tan x + C$$

Problem 5

Using the formula:

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$

Find the integral $\int_0^{\frac{\pi}{6}} \sin(5x)\cos(3x)dx$.

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left[\sin(2x)+\sin(8x)\right]dx$$

$$= \frac{1}{2}\left[\frac{-1}{2}\cos(2x) - \frac{1}{8}\cos(8x)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\left(\frac{-1}{4}+\frac{1}{16}\right)-\left(\frac{-1}{2}-\frac{1}{8}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-3}{16}+\frac{10}{16}\right]$$

$$= \frac{7}{32}$$

Problem (DO NOT SUBMIT)

Find the integral $\int e^x \sec^4(e^x)\tan^3(e^x)dx.= I$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$$

$$I = \int \sec^4(u)\tan^3(u)\,du$$

$$= \int \sec^2(u)\tan^3(u)\,\sec^2(u)\,du$$

$$= \int (1+\tan^2(u))\tan^3(u)\,\sec^2(u)\,du$$

$$\boxed{w = \tan(u) \rightarrow dw = \sec^2(u)\,du}$$

$$= \int (1+w^2)w^3\,dw = \int (w^3+w^5)\,dw$$

$$= \frac{1}{4}w^4 + \frac{1}{6}w^6 + C$$

$$\boxed{w = \tan(u)}$$

$$= \frac{1}{4}\tan^4(u) + \frac{1}{6}\tan^6(u) + C$$

$$\boxed{u = e^x}$$

$$= \frac{1}{4}\tan^4(e^x) + \frac{1}{6}\tan^6(e^x) + C$$