Problem Set 9
6.5: Average Value of a Function

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	
Members:	

Solution key
many thanks to
Majed Bamardouf and
his team

Problem 1

Find the average value of the function $f(x) = x^2\sqrt{x+1}$ over $[-1,0]$.

$$f_{ave} = \frac{1}{0-(-1)} \int_{-1}^{0} \left[x^2\sqrt{x+1}\right] dx$$

$$\boxed{\begin{array}{l} x^2 = (u-1)^2 \\ u = x+1 \\ du = dx \\ x=-1 \to u=0 \\ x=0 \to u=1 \end{array}}$$

$$= \int_0^1 \left[(u-1)^2 \cdot \sqrt{u}\right] du$$

$$= \int_0^1 \left[(u^2-2u+1)u^{\frac{1}{2}}\right] du = \int_0^1 \left(u^{\frac{5}{2}} - 2u^{3/2} + u^{\frac{1}{2}}\right) du$$

$$= \left[\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{7} - \frac{4}{5} + \frac{2}{3} = \frac{16}{105}$$

Problem 2

Find the average value of the function $f(x) = \cos^2(\pi x)$ over $[-1, 1]$.

$$f_{ave} = \frac{1}{1-(-1)} \int_{-1}^{1} \cos^2(\pi x)\, dx$$

$$= \frac{1}{2} \cdot 2 \int_{0}^{1} \left(\cos^2(\pi x)\right) dx$$

$$= \int_{0}^{1} \left[\frac{1+\cos(2\pi x)}{2} \right] dx = \int_{0}^{1} \frac{1}{2}\, dx + \int_{0}^{1} \frac{\cos(2\pi x)}{2} dx$$

$$= \int_{0}^{1} \frac{1}{2}\, dx + \int_{0}^{2\pi} \frac{\cos(u)}{4\pi}\, du$$

$$\boxed{\begin{array}{l} u = 2\pi x \\ du = 2\pi\, dx \\ dx = \frac{du}{2\pi} \\ x = 0 \to u = 0 \\ x = 1 \to u = 2\pi \end{array}}$$

$$= \frac{1}{2} + \frac{1}{4\pi} \Big[\sin(u) \Big]_{0}^{2\pi} = \frac{1}{2}$$

Problem 3

Find the number $a < 0$ such that the average value of $f(x) = 3x^2 - 2x + 2$ on the interval $[a, 0]$ is equal to 8.

$$f_{ave} = 8 = \frac{1}{0-a} \int_a^0 \left[3x^2 - 2x + 2\right] dx$$

$$-8a = \left[x^3 - x^2 + 2x\right]_a^0$$

$$-8a = -\left(a^3 - a^2 + 2a\right)$$

$$8a = a^3 - a^2 + 2a$$

$$a^3 - a^2 - 6a = 0$$

$$a\left(a^2 - a - 6\right) = 0$$

$$a(a-3)(a+2) = 0$$

$$a = 0 \quad \text{or} \quad a = 3 \quad \text{or} \quad \underline{a = -2} < 0$$

$$\therefore \boxed{a = -2}$$

Problem (Revision)

If f is an EVEN continuous function and $\int_0^4 f(x)dx = 5$. Find $\int_{-2}^{2} [xf(x^2) + f(2x)]\, dx$.

$$\int_{-2}^{2}[xf(x^2)+f(2x)]dx = \int_{-2}^{2}\underbrace{(xf(x^2))}_{\text{odd}}dx + \int_{-2}^{2}\underbrace{(f(2x))}_{\text{even}}dx$$

$$= 0 + 2\int_0^2 f(2x)\,dx$$

$$= 2\int_0^4 \frac{f(u)}{2}\,du = \int_0^4 f(u)\,du = 5$$

$$\boxed{\begin{aligned} &u = 2x \\ &du = 2\,dx \\ &dx = \frac{du}{2} \\ &x = 2 \to u = 4 \\ &x = 0 \to u = 0 \end{aligned}}$$