Problem Set 18
11.1: Sequences

Please indicate the members who are present. Also indicate the group coordinator.

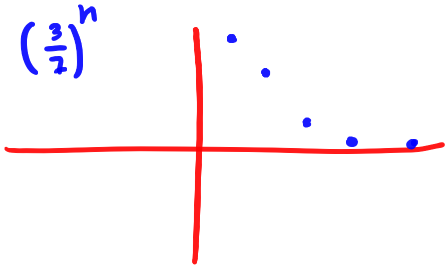
Group Number:	
Members:	KEY

Problem 1

Determine whether the sequence converges or diverges. If it converges, find the limit.

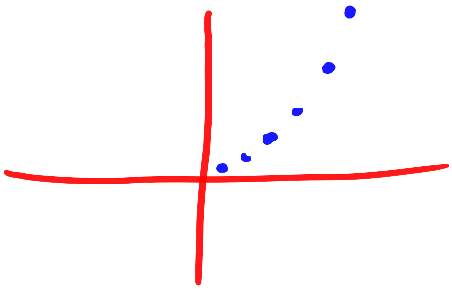
$$a_n = 3^n 7^{-n}, \qquad b_n = 3^{-n} 7^n$$

$\left(\frac{3}{7}\right)^n$



$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{3}{7}\right)^n = 0$$

Converges to 0



$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{7}{3}\right)^n = \infty$$

diverges

Problem 2

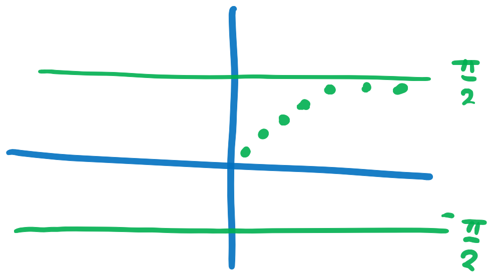
Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \tan\left(\frac{3n\pi}{5+12n}\right), \qquad b_n = \arctan(3n)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan\left(\frac{3n\pi}{5+12n}\right) = \tan\left(\lim_{n\to\infty} \frac{3n\pi}{5+12n}\right)$$

$$\tan\left(\frac{3\pi}{12}\right) = 1$$

Converges to 1



$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \tan^{-1}(3n) = \frac{\pi}{2}$$

Converges to $\frac{\pi}{2}$

Problem 3

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{\frac{(-1)^n n}{n^2+2}\right\}, \qquad \left\{\frac{(-1)^n n^2}{n^2+2}\right\}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n n}{n^2+2}\right| = \lim_{n\to\infty}\left|\frac{n}{n^2+2}\right| = 0$$

$$\therefore \lim_{n\to\infty}\left(\frac{(-1)^n n}{n^2+2}\right) = 0$$

converges to 0

$$\lim_{n\to\infty}\left|\frac{(-1)^n n^2}{n^2+2}\right| = \lim_{n\to\infty}\left|\frac{n^2}{n^2+2}\right| = 1$$

$$\therefore \lim_{n\to\infty}\left(\frac{(-1)^n n^2}{n^2+2}\right)$$

DNE

diverges

Problem 4

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{ \left(1 + \frac{2}{n}\right)^n \right\}_{n=1}^{\infty}$$

let $f(x) = \left(1 + \frac{2}{x}\right)^x$

$$L = \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x \quad (1^\infty \text{ form})$$

$$\ln L = \lim_{x \to \infty} \left(x \ln\left(1 + \frac{2}{x}\right)\right)$$

$$= \lim_{x \to \infty} \left(\frac{\ln\left(1 + \frac{2}{x}\right)}{x^{-1}} \right) = \lim_{x \to \infty} \left(\frac{\frac{\cancel{-}2/\cancel{x^2}}{1 + \frac{2}{x}}}{\cancel{-}1/\cancel{x^2}} \right)$$

$$= \lim_{x \to \infty} \left(\frac{2}{1 + \frac{2}{x}} \right) = 2$$

$$\ln L = \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x \Rightarrow L = e^2$$

We can say that $\left(1 + \frac{2}{n}\right)^n$ Converges to e^2

Problem 5

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{2+\frac{\sin n}{n}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}\left(2+\frac{\sin n}{n}\right) = 2 + \lim_{n\to\infty}\frac{\sin n}{n} = 2$$

by Squeeze theorem

$$-1 \leq \sin n \leq 1$$

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$n\to\infty = 0$ $\qquad$ $n\to\infty = 0$

$$\downarrow$$

$n\to\infty = 0$

Converges to 2

Problem 6

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{n!}{2^n}$$

$$a_n = \frac{n!}{2^n} = \frac{n(n-1)(n-2)\dots 3\ 2\ 1}{2\cdot 2\cdot 2\ \dots 2\cdot 2\cdot 2}$$

$$= \frac{n}{2}\cdot \underbrace{\frac{(n-1)(n-2)\dots 3\cdot 2\cdot 1}{2\cdot 2\dots 2\cdot 2\cdot 2}}_{>1}$$

by comparing:

$\frac{n}{2}$ is divergent $\left(\lim_{n\to\infty}\frac{n}{2}=\infty\right)$

Therefore $\frac{n!}{2^n} = \frac{n}{2}\cdot\frac{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{2\cdot 2\cdot 2\cdot 2\cdot 2}$ is also divergent

Problem 7

Find the limit of the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$

$$a_n = \sqrt{2} \quad , \quad a_{n+1} = \sqrt{2\, a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n}$$

$$L = \sqrt{2L}$$

$$L^2 = 2L$$

$$L^2 - 2L = 0$$

$$L(L-2)$$

$$L \neq 0 \quad \text{or} \quad L = 2$$

$$a_n \geq \sqrt{2}$$

& the sequence is increasing

Problem 8

The sequence

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Why is the sequence convergent? Find the limit.

Since the sequence is increasing and there is an upper limit ($a_n < 3$) then the sequence is convergent.

$$\lim_{n\to\infty} a_{n+1} = 3 - \frac{1}{\lim_{n\to\infty} a_n}$$

$$L = 3 - \frac{1}{L}$$

$$L^2 - 3L + 1 = 0$$

$$L = \frac{3 \pm \sqrt{9-4}}{2} = \cancel{\frac{3-\sqrt{5}}{2}} \text{ or } \frac{3+\sqrt{5}}{2}$$

Rejected because the sequence is increasing & a_n starts from 1

the sequence converges to $\frac{3+\sqrt{5}}{2}$