Problem Set 7-Wednesday
6.2: Volumes (Method of Cross-Sections)

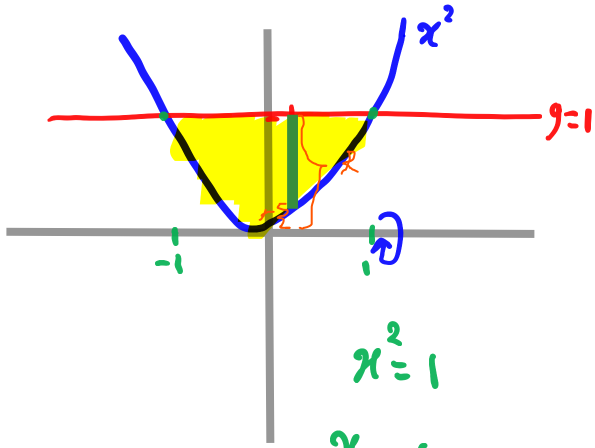
Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	Solution Key By group 1, section 35
Members:	

many thanks to
Majed Bamardouf and his team

Problem 1

Find the volume of the solid generated by rotating the region bounded by the curves $y = x^2$ and $y = 1$ about the x-axis.



$x^2 = 1$

$x = \pm 1$

$R = 1 - 0 = 1$

$r = x^2 - 0 = x^2$

$$V = \int_{-1}^{1} \left[(1)^2\pi - (x^2)^2\pi\right] dx$$

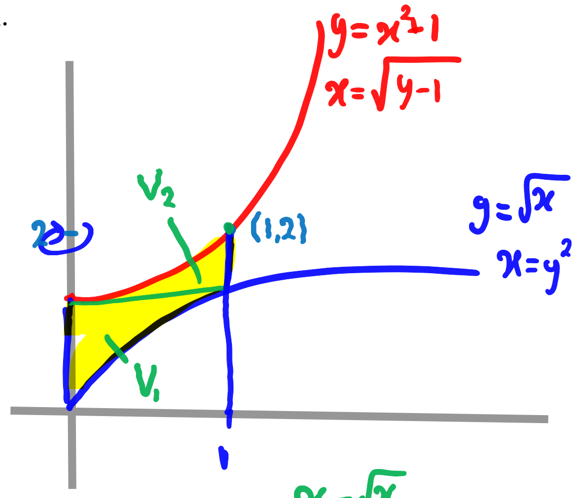
$$= \pi \int_{-1}^{1} (1 - x^4)\, dx$$

$$= \pi \left[x - \frac{1}{5}x^5\right]_{-1}^{1} = \pi\left[\frac{4}{5} + \frac{4}{5}\right]$$

$$= \frac{8\pi}{5}$$

Problem 2

The region bounded by the curves $y = \sqrt{x}$ and $y = x^2 + 1$ between $x = 0$ and $x = 1$ is revolved about the y-axis. Find the volume of the solid generated.



$x=\sqrt{x}$
$\therefore x=1, 0$

V_1 (disk): $R = y^2$

$$V_1 = \int_0^1 \pi \cdot (y^2)^2 \, dy = \pi \int_0^1 y^4 \, dy$$

$$= \pi \left[\frac{1}{5} y^5\right]_0^1 = \frac{\pi}{5}$$

V_2 (Washer):

$R = 1$, $r = \sqrt{y-1}$

$$V_2 = \int_1^2 \pi \left(\sqrt{y-1}\right)^2 dy = \pi \int_1^2 (y-1)\, dy$$

$$V_2 = \pi \left[\frac{y^2}{2} - y\right]_1^2 = \pi \left[(2-2) - \left(\frac{1}{2} - 1\right)\right]$$

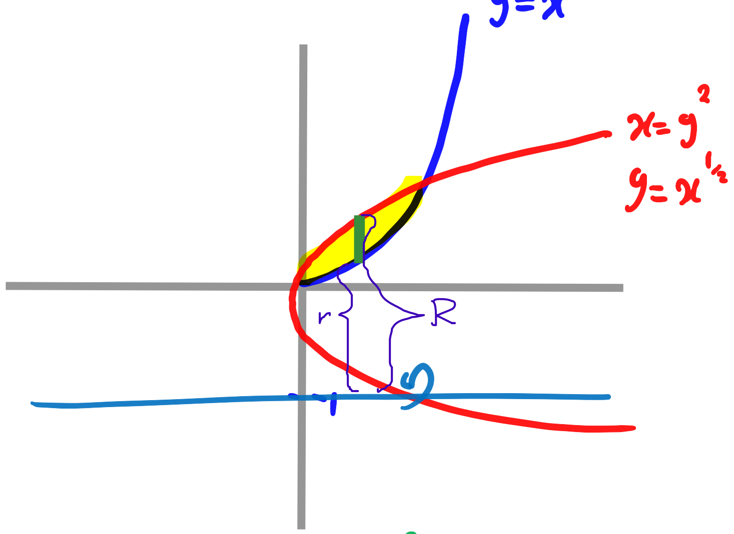
$$= \frac{\pi}{2}$$

$\therefore V = V_1 + V_2 = \frac{\pi}{5} + \frac{\pi}{2}$

$$V = \frac{7\pi}{10}$$

Problem 3

Find the volume of the solid generated by rotating the region bounded by the curves $y = x^3$ and $x = y^2$ about the line $y = -1$.



$$R = \sqrt{x} + 1$$

$$r = x^3 + 1$$

$$x^3 = \sqrt{x}$$

$$x^6 = x$$

$$x = 1$$

$$V = \pi \int_0^1 \left[\left(\sqrt{x}+1\right)^2 - \left(x^3+1\right)^2 \right] dx$$

$$= \pi \int_0^1 \left[\left(x + 2\sqrt{x} + \cancel{1}\right) - \left(x^6 + 2x^3 + \cancel{1}\right) \right] dx$$

$$= \pi \int_0^1 \left(-x^6 - 2x^3 + 2\sqrt{x} + x\right) dx$$

$$= \pi \left[-\frac{1}{7}x^7 - \frac{1}{2}x^4 + \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^1 = \pi \left[-\frac{1}{7} + \frac{4}{3} \right]$$

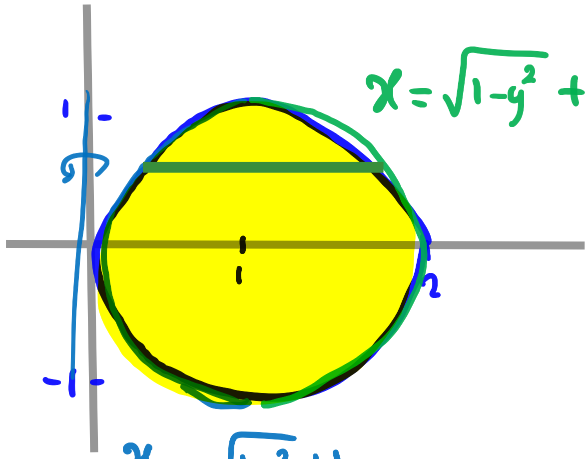
$$= \frac{25\pi}{21}$$

Problem 4

The area enclosed by the circle $y^2 + (x-1)^2 = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

$$R = \sqrt{1-y^2} + 1$$

$$r = -\sqrt{1-y^2} + 1$$



$$V = \pi \int_{-1}^{1} \left[\left(\sqrt{1-y^2}+1\right)^2 - \left(-\sqrt{1-y^2}+1\right)^2 \right] dy$$

$$= \pi \int_{-1}^{1} \left[\left(\cancel{1-y^2} + 2\sqrt{1-y^2} \cancel{+1}\right) - \left(\cancel{1-y^2} - 2\sqrt{1-y^2} \cancel{+1}\right) \right] dy$$

$$= \pi \int_{-1}^{1} \left(4\sqrt{1-y^2}\right) dy = 4\pi \int_{-1}^{1} \left(\overline{\sqrt{1-y^2}}^{\text{Semicircle } r=1}\right) dy$$

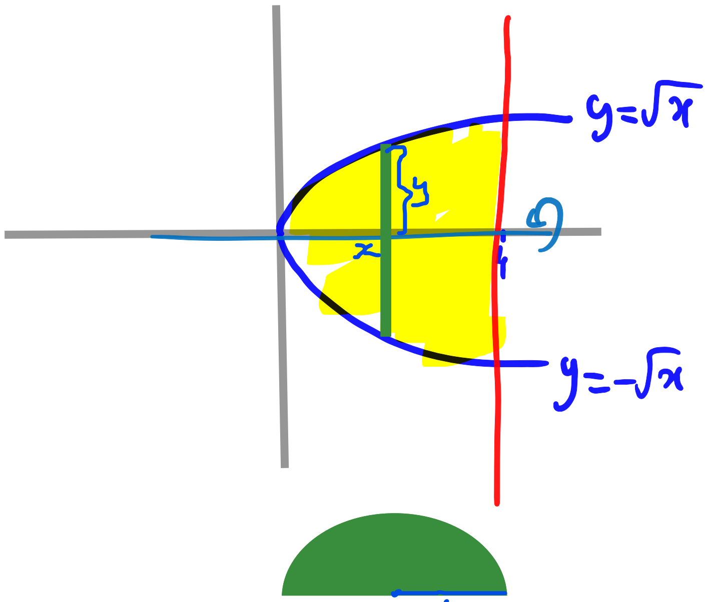
$$= 4\pi \cdot \left(\frac{1}{2}\pi \cdot 1^2\right) = 2\pi^2$$

Problem 5

The base of a solid is bounded by the curves $x = y^2$ and $x = 4$. The cross sections of the solid, perpendicular to the x-axis, are semicircles. Find the volume of the solid.

$$S = \sqrt{x} + \sqrt{x} = 2\sqrt{x}$$

$$\therefore r = \frac{1}{2} \cdot 2\sqrt{x} = \sqrt{x}$$



$$A(x) = \frac{1}{2} y^2 = \frac{1}{2}(\sqrt{x})^2 = \frac{1}{2}x$$

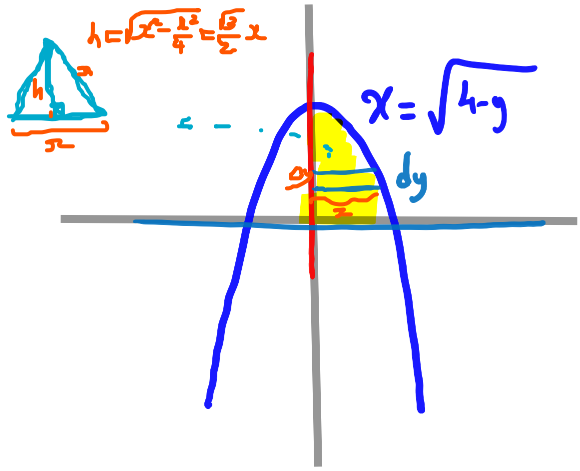
$$V = \int_0^4 \frac{1}{2}\pi(\sqrt{x})^2\, dx$$

$$= \frac{1}{2}\pi \int_0^4 x\, dx$$

$$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

Problem 6

A solid has a base lying in the first quadrant and bounded by the curves $y = 4 - x^2$, $x = 0$ and $y = 0$. If the cross sections of the solid perpendicular to the y-axis are equilateral triangles with the base running from the y-axis to the curve. Find the volume of the solid.



$$S = \sqrt{4-y}$$

$$A(y) = \frac{\sqrt{3}}{4} S^2$$

$$V = \int_0^4 \frac{\sqrt{3}}{4} \cdot (4-y)\, dy$$

$$= \frac{\sqrt{3}}{4}\left[4y - \frac{y^2}{2}\right]_0^4 = \frac{\sqrt{3}}{4}\left[16-8\right]$$

$$= 2\sqrt{3}$$