Problem Set 19
11.2: Series

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	
Members:	KEY

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \text{ converges if } |r| < 1 \text{ and its sum is } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

if $|r| \geq 1$, the geometric series is divergent.

Divergence Test If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Problem 1

The sequence $s_n = \dfrac{n^2-8}{5n^2+3}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. Does this series converge or diverge? If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{n^2-8}{5n^2+3} = \lim_{n\to\infty} \frac{1-\frac{8}{n^2}}{5+\frac{3}{n^2}} = \frac{1}{5}$$

$\therefore$ Conv to $\frac{1}{5}$

$$\therefore \text{Sum} = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \frac{1}{5}$$

Problem 2

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$a_n = \frac{1}{3^{2n-1}}, \quad b_n = \frac{2}{3^{2n}}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3^{2n-1}} = \sum_{n=1}^{\infty} \frac{1}{\frac{9^n}{3}} = \sum_{n=1}^{\infty} \frac{3}{9^n}$$

$$= \sum_{n=1}^{\infty} 3\left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{9}\right)^{n-1}$$

geometric series

$$\boxed{a = \frac{1}{3}}, \quad \boxed{r = \frac{1}{9}}$$

Since $|r| = \frac{1}{9} < 1$, it's convergent

with sum $= \dfrac{a}{1-r} = \dfrac{1/3}{1-1/9} = \dfrac{3}{8}$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{3^{2n}} = \sum_{n=1}^{\infty} 2\left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty} \frac{2}{9}\left(\frac{1}{9}\right)^{n-1}$$

geometric series

$$\boxed{b = \frac{2}{9}}, \quad \boxed{r = \frac{1}{9}}$$

Since $|r| = \frac{1}{9} < 1$, it's convergent

with sum $= \dfrac{b}{1-r} = \dfrac{2/9}{1-1/9} = \dfrac{2}{8}$

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

it's convergent

Problem 3

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$$

$$\lim_{n\to\infty} \frac{2+n}{1-2n} = \frac{\frac{2}{n}+1}{\frac{1}{n}-2} = \frac{-1}{2}$$

it's divergent ; using divergence test

$\left(\lim_{n\to\infty} a_n \neq 0\right)$

Problem 4

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$$

using divergence test:

$$\lim_{n\to\infty} \frac{1}{n^3-n} = 0 \text{ ; it may be conv or div.}$$

Partial fraction decomposition

$$= \frac{1}{n(n-1)(n+1)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+1}$$

$$1 = A(n-1)(n+1) + B\cdot n(n+1) + C\cdot n(n-1)$$

$$n=0 \rightarrow \boxed{A=-1} \quad \Big| \quad n=1 \rightarrow \boxed{B=\tfrac{1}{2}} \quad \Big| \quad n=-1 \rightarrow \boxed{C=\tfrac{1}{2}}$$

$$S_n \sum_{n=2}^{n} \left(\frac{1}{2(n+1)} + \frac{1}{2(n-1)} - \frac{1}{n} \right) =$$

$$\left(\cancel{\tfrac{1}{6}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{8}} + \tfrac{1}{4} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{10}} + \cancel{\tfrac{1}{6}} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{12}} + \cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{5}}\right)$$

$$+ \cancel{\cdot}\cancel{\cdot}\cancel{\cdot}\cancel{\cdot} + \left(\cancel{\tfrac{1}{2n}} + \cancel{\tfrac{1}{2(n-2)}} - \cancel{\tfrac{1}{n-1}}\right) + \left(\tfrac{1}{2(n+1)} + \cancel{\tfrac{1}{2(n-1)}} - \cancel{\tfrac{1}{n}}\right)$$

$$S_n = \frac{1}{4} - \frac{1}{2(n+1)}$$

$$\text{Sum} = \lim_{n\to\infty} S_n = \frac{1}{4} \qquad \text{Convergent}$$

Problem 5

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$$

using the divergence test:

$\lim_{n\to\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right) = 0$; It may be Conv or div.

in fact, it is a telescoping series, So it is convergent.

$$S_n = \sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(e^{\frac{1}{i}} - e^{\frac{1}{i+1}}\right)$$

$$= \left(e - \cancel{e^{\frac{1}{2}}}\right) + \left(\cancel{e^{\frac{1}{2}}} - \cancel{e^{\frac{1}{3}}}\right) + \left(\cancel{e^{\frac{1}{3}}} - \cancel{e^{\frac{1}{4}}}\right) \cancel{+} \cdots \cancel{+} \left(\cancel{e^{\frac{1}{n-1}}} - \cancel{e^{\frac{1}{n}}}\right) + \left(\cancel{e^{\frac{1}{n}}} - e^{\frac{1}{n+1}}\right)$$

$$S_n = e - e^{\frac{1}{n+1}}$$

$$\text{Sum} = \lim_{n\to\infty}\left(e - e^{\frac{1}{n+1}}\right) = e - 1$$

Problem 6

Express the number as a ratio of integers $0.\bar{8} = 0.8888....$

$$0.888\ldots = \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \ldots$$

$\div$ $r = 1/10$ $\qquad$ $\div$ $r = 1/10$

$$a = \frac{8}{10}, \quad r = \frac{1}{10}$$

it is a convergent series with sum $= \dfrac{a}{1-r}$

$$\text{Sum} = \frac{8/10}{1 - 1/10} = \frac{8}{9}$$

$$\boxed{0.\bar{8} = \frac{8}{9}}$$

Problem 7

If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+2}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

$$a_n = S_n - S_{n-1} = \frac{n-1}{n+2} - \frac{n-2}{n+1}$$

$$= \frac{n^2-1 - n^2+4}{(n+2)(n+1)} = \frac{3}{(n+2)(n+1)}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n+2} = 1$$