Problem Set 13
7.4: Integration of Rational Functions By Partial Fractions

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	
Members:	KEY

Problem 1

Write the form of the partial fraction decomposition of

$$\frac{4-2x}{x^2(x^2+4)^3(x^2-4)^2(x^2+x+1)^2}.$$

$$= \frac{4-2x}{x^2(x^2+4)^3(x-2)^2(x+2)^2(x^2+x+1)^2} = \frac{-2\cancel{(x-2)}}{x^2(x^2+4)^3(x-2)^{\cancel{2}}(x+2)^2(x^2+x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+4)} + \frac{Ex+F}{(x^2+4)^2} + \frac{Gx+H}{(x^2+4)^3} + \frac{I}{x-2}$$

$$+ \frac{J}{(x+2)} + \frac{K}{(x+2)^2} + \frac{Lx+M}{x^2+x+1} + \frac{Nx+P}{(x^2+x+1)^2}$$

Problem 2

Find the integral $\int \frac{3x^3+4x^2+2}{x^4+x^2}dx. = I$

Partial fraction decomposition

$$\frac{3x^3+4x^2+2}{x^4+x^2} = \frac{3x^3+4x^2+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$3x^3+4x^2+2 = A\cdot x\cdot(x^2+1) + B(x^2+1) + (Cx+D)\,x^2$$

When $x=0$: $\therefore \boxed{B=2}$

$$3x^3+4x^2+2 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$Ax=0$

$\therefore \boxed{A=0}$

$3x^3 = \cancel{Ax^3} + Cx^3$

$\therefore \boxed{C=3}$

$4x^2 = Bx^2 + Dx^2$

$4x^2 = 2x^2 + Dx^2$

$\therefore \boxed{D=2}$

$$\therefore \frac{3x^3+4x^2+2}{x^4+x^2} = \frac{2}{x^2} + \frac{3x+2}{x^2+1}$$ = the integrand

$$\therefore I = \underbrace{\int \frac{2}{x^2}\,dx}_{I_1} + \underbrace{\int \frac{3x}{x^2+1}\,dx}_{I_2} + \underbrace{\int \frac{2}{x^2+1}\,dx}_{I_3}$$

$$I_1 = \frac{-2}{x} + C_1$$

$$I_2 = \int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1) + C_2$$

$$I_3 = \int \frac{2}{x^2+1}\,dx = 2\tan^{-1}x + C_3$$

$$\therefore I = \frac{-2}{x} + \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}x + C$$

$$\boxed{C = C_1 + C_2 + C_3}$$

Problem 3

Find the integral $\int \frac{2x^3+13}{x^3-x^2+4x-4}dx$.

$$\begin{array}{r|l} & 2 \\ x^3-x^2+4x-4 & 2x^3+13 \\ & -\;(2x^3-2x^2+8x-8) \\ \hline & 2x^2-8x+21 \end{array}$$

$$\therefore\ I = \underbrace{\int 2\,dx}_{I_2} + \underbrace{\int \frac{2x^2-8x+21}{x^3-x^2+4x-4}}_{I_1}$$

$\boxed{I_1}$

Partial fraction decomposition

$$\frac{2x^2-8x+21}{x^2(x-1)+4(x-1)}$$

$$= \frac{2x^2-8x+21}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$$

$$2x^2-8x+21 = (Ax+B)(x-1) + C(x^2+4)$$

$x=1:$ $\quad 15 = 5C$

$$\boxed{C=3}$$

$$2x^2-8x+21 = Ax^2-Ax+Bx-B+Cx^2+4C$$

$$2x^2 = Ax^2+Cx^2$$

$$2x^2 = Ax^2+3x^2$$

$$\boxed{A=-1}$$

$$21=-B+4C$$

$$21=-B+12$$

$$\boxed{B=-9}$$

$$I_1 = \int \frac{-x-9}{x^2+4}\,dx + \int \frac{3}{x-1}\,dx$$

$$= \int \frac{-x}{x^2+4}\,dx + \int \frac{-9}{x^2+4}\,dx + \int \frac{3}{x-1}\,dx$$

$$= \frac{-1}{2}\ln(x^2+4) - \frac{9}{2}\tan^{-1}\left(\frac{x}{2}\right) + 3\ln|x-1| + C_1$$

$$I = 2x - \frac{1}{2}\ln(x^2+4) - \frac{9}{2}\tan^{-1}\left(\frac{x}{2}\right) + 3\ln|x-1| + C$$

Problem 4

Find the integral $\int \frac{\cos x}{(2-\cos^2 x)\sin x} dx$.

$$I = \int \frac{\cos x}{(2-1+\sin^2 x)\sin x} dx = \int \frac{\cos x}{(1+\sin^2 x)\sin x} dx$$

$$\boxed{\text{let } u = \sin x, \quad du = \cos x \, dx}$$

$$I = \int \frac{1}{(1+u^2)u} du$$

Partial fraction decomposition

$$\frac{1}{(1+u^2)u} = \frac{Au+B}{1+u^2} + \frac{C}{u}$$

$$1 = u(Au+B) + C(1+u^2)$$

$x=0: \to \boxed{C=1}$

Plug $c=1$:

$$1 = Au^2 + B + 1 + u^2$$

$\boxed{B=0}$ $\boxed{A=-1}$

$$\therefore I = \underbrace{\int \frac{-u}{1+u^2} du}_{I_1} + \underbrace{\int \frac{1}{u} du}_{I_2}$$

$$I_1 = -\frac{1}{2}\ln(1+u^2)+c_1 \ , \quad I_2 = \ln|u| + c_2$$

$$\therefore I = -\frac{1}{2}\ln(1+u^2) + \ln|u| + C \qquad \boxed{C = c_1 + c_2}$$

$$\therefore I = -\frac{1}{2}\ln(1+\sin^2 x) + \ln|\sin x| + C$$

Problem 5

Find the integral $\int \frac{1}{\cos x + \sin x} dx$.

let $t = \tan\left(\frac{x}{2}\right)$, Where $-\pi < x < \pi$, then:

- $dx = \frac{2dt}{1+t^2}$
- $\sin x = \frac{2t}{1+t^2}$
- $\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{1}{\left(\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{1-t^2+2t} dt = -\int \frac{2}{t^2-2t-1} dt$$

Partial fraction decomposition

$$\frac{2}{(t^2-2t+1)-1-1} = \frac{2}{(t-1)^2-2}$$

$$= \frac{2}{(t-1-\sqrt{2})(t-1+\sqrt{2})} = \frac{A}{t-1-\sqrt{2}} + \frac{B}{t-1+\sqrt{2}}$$

$$2 = A(t-1+\sqrt{2}) + B(t-1-\sqrt{2})$$

$$2 = At - A + A\sqrt{2} + Bt - B - B\sqrt{2}$$

$At + Bt = 0$

$\boxed{A = -B}$

$2 = -A + A\sqrt{2} - B - B\sqrt{2}$

$2 = -A + A\sqrt{2} + A + A\sqrt{2}$

$\boxed{A = \frac{1}{\sqrt{2}}}$ $\boxed{B = \frac{-1}{\sqrt{2}}}$

$$I = -\int \frac{\frac{1}{\sqrt{2}}}{t-1-\sqrt{2}}\,dt - \int \frac{-\frac{1}{\sqrt{2}}}{t-1+\sqrt{2}}\,dt$$

$$= -\frac{1}{\sqrt{2}} \ln|t-1-\sqrt{2}| + \frac{1}{\sqrt{2}} \ln|t-1+\sqrt{2}| + C$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}}\right| + C$$