Problem Set 12
7.3: Trigonometric Substitution

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	
Members:	KEY

For , $\sqrt{a^2 - x^2}$, use $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

For , $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

For , $\sqrt{x^2 - a^2}$, use $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.

Problem 1

Find the integral $\int \frac{x^2}{\sqrt{4-x^2}}dx. = I$

- let $x = 2\sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- $dx = 2\cos\theta \, d\theta$
- $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\cos\theta$

$$I = \int \frac{4\sin^2\theta}{2\cos\theta} \cdot 2\cos\theta \, d\theta$$

$$= 4\int \sin^2\theta \, d\theta$$

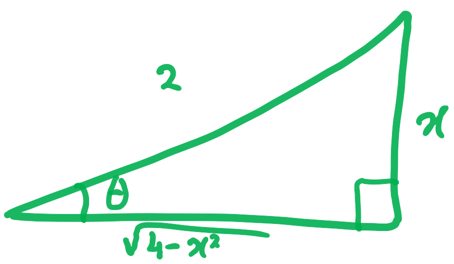
$$= 4\int \left[\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= 2\int \left[1-\cos 2\theta\right] d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right] = 2\left[\theta - \frac{1}{2}\cdot 2\sin\theta\cos\theta\right]$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}\right] + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C$$



$\sin\theta = \frac{x}{2}$

Problem 2

Evaluate the integral $\int_5^{5\sqrt{3}} \frac{1}{x^2\sqrt{x^2+25}}dx$.

- let $x = 5\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- $dx = 5\sec^2\theta \, d\theta$
- $\sqrt{x^2+25} = \sqrt{25\tan^2\theta + 25} = 5\sqrt{\tan^2\theta + 1} = 5\sqrt{\sec^2\theta} = 5\sec\theta$

- When $x = 5\sqrt{3} \rightarrow 5\sqrt{3} = 5\tan\theta \Rightarrow \boxed{\theta = \frac{\pi}{3}}$
- When $x = 5 \longrightarrow 5 = 5\tan\theta \rightarrow \boxed{\theta = \frac{\pi}{4}}$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{5\sec^2\theta}{25\tan^2\theta \cdot 5\sec\theta} d\theta$$

$$= \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta} d\theta = \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} d\theta$$

$$= \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta} d\theta = \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\theta\csc\theta \, d\theta = \frac{1}{25}\Big[-\csc\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{25}\left[-\frac{2\sqrt{3}}{3} + \sqrt{2}\right] = \frac{-2\sqrt{3}}{75} + \frac{\sqrt{2}}{25}$$

Problem 3

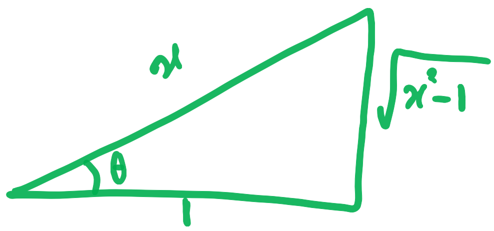
Find the integral $\int \frac{\sqrt{x^2-1}}{x}dx, \quad x > 1.$

$$\boxed{\begin{array}{l} \bullet\ x = \sec\theta, \text{ where } 0 \leq \theta < \frac{\pi}{2} \\ \bullet\ dx = \sec\theta \tan\theta \, d\theta \\ \bullet\ \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta \end{array}}$$

$$I = \int \frac{\tan\theta}{\sec\theta} \sec\theta \tan\theta \, d\theta = \int \tan^2\theta \, d\theta$$

$$= \int [\sec^2\theta - 1] \, d\theta$$

$$= \tan\theta - \theta + C$$



$\sec\theta = x$

$$= \sqrt{x^2-1} + \sec^{-1}x + C$$

Problem 4

Find the integral $\int \frac{x}{\sqrt{x^2-6x+13}}dx$.

$$x^2-6x+13 = (x^2-6x)+13 = (x^2-6x+9)+13-9 = (x-3)^2+4$$

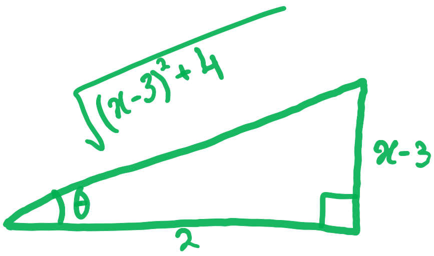
- $x-3 = 2\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- $dx = 2\sec^2\theta \, d\theta$
- $\sqrt{(x-3)^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2\sec\theta$

$$I = \int \frac{2\tan\theta+3}{2\sec\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= \int \left[2\tan\theta\sec\theta + 3\sec\theta\right] d\theta$$

$$= 2\sec\theta + 3\ln\left|\sec\theta+\tan\theta\right| + C$$

$$= 2\,\frac{\sqrt{(x-3)^2+4}}{2} + 3\ln\left|\frac{\sqrt{(x-3)^2+4}}{2} + \frac{x-3}{2}\right| + C$$



$\tan\theta = \frac{x-3}{2}$

$$= 2\,\frac{\sqrt{(x-3)^2+4}}{2} + 3\ln\left|\sqrt{(x-3)^2+4} + x-3\right| \underbrace{-3\ln(2)+C}$$

$$= 2\,\frac{\sqrt{(x-3)^2+4}}{2} + 3\ln\left|\sqrt{(x-3)^2+4} + x-3\right| + C_1$$

$\boxed{\text{Where } C_1 = -3\ln 2 + C}$

Problem 5

Find the integral $\int (5+4x-x^2)^{3/2}dx$.

$$5+4x-x^2 = -(x^2-4x)+5$$
$$= -(x^2-4x+4)+5+4$$
$$= 9-(x-2)^2$$

$$\boxed{\begin{array}{l} \bullet\ x-2 = 3\sin\theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \bullet\ dx = 3\cos\theta\, d\theta \\ \bullet\ \sqrt{9-(x-2)^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta \end{array}}$$

$$I = \int (3\cos\theta)^3 \cdot 3\cos\theta\, d\theta = 81\int \cos^4\theta\, d\theta$$

$$= 81\int \left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta$$

$$= \frac{81}{4}\int \left(1+2\cos 2\theta+\cos^2 2\theta\right) d\theta$$

$$= \frac{81}{4}\int \left(1+2\cos 2\theta+\frac{1}{2}+\frac{\cos 4\theta}{2}\right) d\theta$$

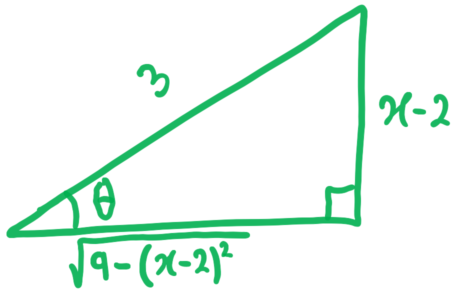
$$= \frac{81}{4}\int \left(\frac{3}{2}+2\cos 2\theta+\frac{\cos 4\theta}{2}\right) d\theta$$

$$= \frac{81}{4}\left[\frac{3}{2}\theta+2\cdot\frac{1}{2}\sin 2\theta+\frac{1}{2}\cdot\frac{1}{4}\sin 4\theta\right]+C$$

$$= \frac{81}{4}\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right] + C$$

$$= \frac{81}{4}\left[\frac{3}{2}\theta + 2\sin\theta\cos\theta + \frac{1}{8}\,2\cdot\sin(2\theta)\cdot\cos(2\theta)\right] + C$$

$$= \frac{81}{4}\left[\frac{3}{2}\theta + 2\sin\theta\cos\theta + \frac{1}{4}\cdot 2\sin\theta\cdot(1-2\sin^2\theta)\right] + C$$



$$\sin\theta = \frac{x-2}{3}$$

$$I = \frac{81}{4}\left[\frac{3}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{2(x-2)}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3} + \frac{1}{2}\cdot\frac{x-2}{3}\left(1-\frac{2(x-2)^2}{9}\right)\right] + C$$