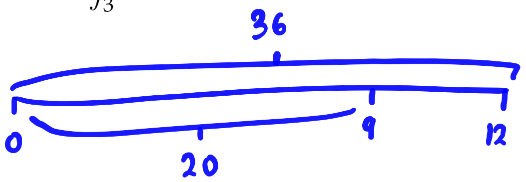
Problem Set 16
E1: Revision

Please indicate the members who are present. Also indicate the group coordinator.

Group Number:	KEY
Members:	

Problem 1

If $\int_{12}^{0} f(x)dx = -36$ and $\int_{0}^{9} f(x)dx = 20$. Find $\int_{3}^{4} f(3x)dx$



$$\boxed{\begin{array}{ll} u = 3x & x=4 \to u=12 \\ du = 3dx & x=3 \to u=9 \end{array}}$$

$$3\int_{9}^{12} f(u)\,du = 3\left[\int_{0}^{12} f(x)\,dx - \int_{0}^{9} f(x)\,dx\right]$$

$$= 3\left[36-20\right]$$

$$= 3 \cdot 16 = 48$$

Problem 2

Evaluate $\int_{0}^{1} (x^{10} + 10^x)dx$.

$$= \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_0^1$$

$$= \frac{1}{11} + \frac{10}{\ln 10} - \frac{1}{\ln 10}$$

$$= \frac{1}{11} + \frac{9}{\ln 10}$$

Problem 3

Find the limit, if exists,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{31i}{n}\right)^{-4/5}$$

.

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{31}{2}\cdot\frac{2}{n}i\right)^{-4/5}$$

$$\Delta x = \frac{2}{n}$$

$$x_i = \frac{2}{n}i = a + i\Delta x$$

$$a = 0 \quad , \therefore b = 2$$

$$I = \int_0^2 \left(1+\frac{31}{2}x\right)^{-\frac{4}{5}} dx$$

$$u = 1+\frac{31}{2}x, \quad du = \frac{31}{2}dx$$

$$x = 2 \to u = 32$$

$$x = 0 \to u = 1$$

$$I = \frac{2}{31}\int_1^{33} u^{-\frac{4}{5}}\, du$$

$$= \frac{2}{31}\left[5\ u^{\frac{1}{5}}\right]_1^{33}$$

$$= \frac{2}{31}\cdot 5\left[32^{1/5} - 1\right]$$

$$= \frac{10}{31}$$

Problem 4

Find the average value of $f(x) = \dfrac{x}{(x^2+1)^3}$ from 1 to 3.

$$f_{avg} = \frac{1}{3-1}\int_1^3 \frac{x}{(x^2+1)^3}dx$$

$$u = x^2+1, \quad du = 2x\, dx$$

$$x=3 \to u = 10 \quad , x = 1 \to u = 2$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_2^{10}\frac{1}{u^3}du$$

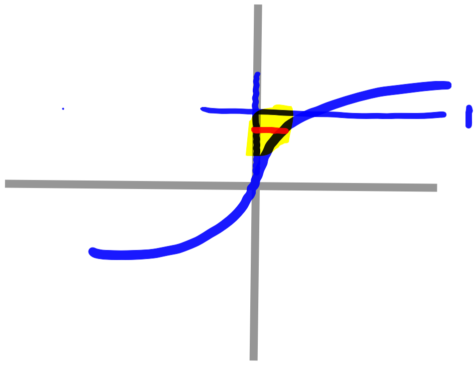
$$= \frac{1}{4}\left[\frac{1}{-2u^2}\right]_2^{10}$$

$$= \frac{1}{4}\left[\frac{-1}{200}+\frac{1}{8}\right]$$

$$= \frac{3}{100}$$

Problem 5

The base of a solid S is bounded by $x = y^3, y = 1$ and the y-axis. If parallel cross-sections perpendicular to y-axis are equilateral triangles. Find the volume of the solid.



thickness: Δy

$A(x) = \frac{\sqrt{3}}{4} s^2$

$= \frac{\sqrt{3}}{4} (y^3)^2$

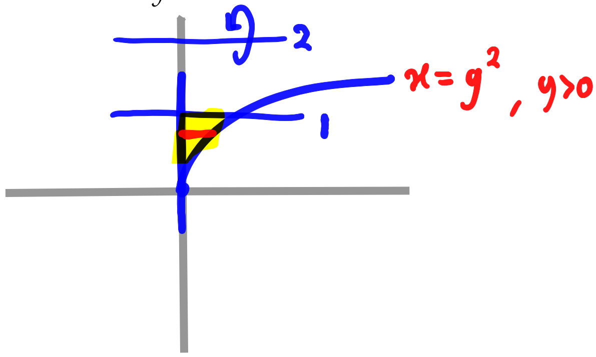
$V = \int_0^1 \frac{\sqrt{3}}{4} (y^6)\, dy$

$= \frac{\sqrt{3}}{4} \left[\frac{1}{7} y^7\right]_0^1$

$= \frac{\sqrt{3}}{28}$

Problem 6

Using the method of cylindrical shells, find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}, x = 0$ and $y = 1$ about the line $y = 2$.



thickness: Δy

$h = y^2$, $r = 2 - y$

$V = 2\pi \int_0^1 (2y^2 - y^3)\, dy$

$= 2\pi \left[\frac{2}{3} y^3 - \frac{1}{4} y^4\right]_0^1$

$= 2\pi \left[\frac{5}{12}\right] = \frac{5\pi}{6}$

Problem 7

Evaluate $\int \frac{dx}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}$
(DO IT IN TWO DIFFERENT METHODS).

$\boxed{1}$

$$I = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \csc^2 x\,dx - \int \cot x \csc x\,dx$$

$$= -\cot x + \csc x + C$$

$\boxed{2}$ let $t = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$

$$dx = \frac{2\,dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2+1-t^2}\,dt$$

$$= \int 1\,dt = t + C = \tan\left(\frac{x}{2}\right) + C$$

Problem 8

Find $\int \tan^3 x \sec^5 x\,dx$.

$$\int \tan^2 x \sec^4 x \quad \tan x \sec x\,dx$$

$$= \int (\sec^2 x - 1)\sec^4 x \quad \tan x \sec x\,dx$$

$$\boxed{u = \sec x, \quad du = \tan x \sec x\,dx}$$

$$= \int (u^2-1)u^4\,du$$

$$= \int (u^6 - u^4)\,du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

Problem 9

Evaluate $\int_0^{\pi/2} \cos(3x)\cos(2x)dx$.

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos(x) + \cos(5x)\right] dx$$

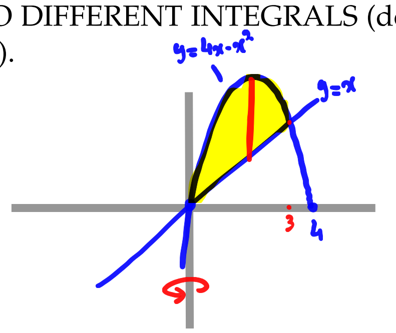
$$= \frac{1}{2}\left[\sin(x) + \frac{1}{5}\sin(5x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[1 + \frac{1}{5}\right] = \frac{1}{2}\cdot\frac{6}{5}$$

$$= \frac{3}{5}$$

Problem 10

Describe the volume of the solid generated by rotating the region bounded by the curves $y = 4x - x^2$ and $y = x$ about the $y-$axis by TWO DIFFERENT INTEGRALS (do not evaluate).



Cylinderical Shell

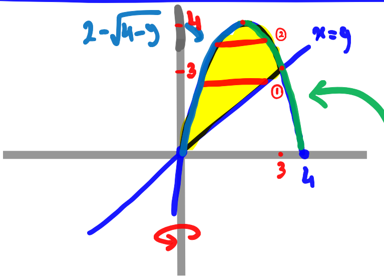
thickness : Δx

$h = 4x - x^2 - x = 3x - x^2$

$r = x$

$$V = 2\pi \int_0^3 x\left(3x - x^2\right)dx$$

$$= 2\pi \int_0^3 \left(3x^2 - x^3\right) dx$$



$y = 4x - x^2$

$y = -(x^2 - 4x + 4) + 4$

$y = 4 - (x-2)^2$

$x = 2 + \sqrt{4-y}$

Washer ① : $\begin{pmatrix} y=0 \text{ to} \\ y=3 \end{pmatrix}$

thickness: Δy

$R = y$

$r = 2 - \sqrt{4-y}$

$$V = \pi \int_0^3 y^2 - \left(2 - \sqrt{4-y}\right)^2 dy$$

Washer ② : $\begin{pmatrix} y=3 \text{ to} \\ y=4 \end{pmatrix}$

thickness: Δy

$R = 2 + \sqrt{4-y}$

$r = 2 - \sqrt{4-y}$

$$V = \pi \int_3^4 \left(2+\sqrt{4-y}\right)^2 - \left(2 - \sqrt{4-y}\right)^2 dy$$

$$\therefore V = \pi \int_0^3 y^2 - \left(2-\sqrt{4-y}\right)^2 dy + \pi \int_3^4 \left(2+\sqrt{4-y}\right)^2 - \left(2-\sqrt{4-y}\right)^2 dy$$

Problem 11

Evaluate $\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta$.

$$= \int_0^{\frac{\pi}{2}} \left[4 - 4\sin\theta + \sin^2\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[4 - 4\sin\theta + \frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right] d\theta$$

$$= \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{4} - 4$$

Problem 12

Find $\int \frac{dx}{\sqrt{x^2+9}}$ (using trigonometric substitution).

$x = 3\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3\sec^2\theta \, d\theta$

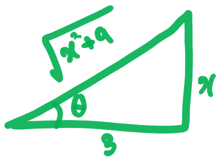
$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\tan^2\theta + 1}$

$= 3\sqrt{\sec^2\theta} = 3\sec\theta$

$$I = \int \frac{3\sec^2\theta}{3\sec\theta} d\theta$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$



$\tan\theta = \frac{x}{3}$

Problem 13

Evaluate $\int_0^{\pi} (3x+2)\cos\left(\frac{x}{2}\right)\,dx$.

d		I
$3x+2$	$+$	$\cos\left(\frac{x}{2}\right)$
3	$-$	$2\sin\left(\frac{x}{2}\right)$
0		$-4\cos\left(\frac{x}{2}\right)$

$$I = \left[2(3x+2)\sin\left(\frac{x}{2}\right) + 12\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$= 6\pi + 4 - 12 = 6\pi - 8$$

Problem 14

Evaluate $\int_{\ln(\pi/4)}^{\ln(\pi/2)} e^x \tan^{-1}(e^x)\,dx$.

let $w = e^x$

$dw = e^x\,dx$

$x = \ln\left(\frac{\pi}{2}\right) \to w = \frac{\pi}{2}$

$x = \ln\left(\frac{\pi}{4}\right) \to w = \frac{\pi}{4}$

$$I = \int_{\pi/4}^{\pi/2} \tan^{-1}(w)\,dw$$

$u = \tan^{-1} w \qquad dv = dw$

$du = \frac{1}{1+w^2}\,dw \qquad v = w$

$$I = \left[w\tan^{-1} w - \int \frac{w}{1+w^2}\,dw\right]_{\pi/4}^{\pi/2}$$

$$= \left[w\tan^{-1} w - \frac{1}{2}\ln(1+w^2)\right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{2}\tan^{-1}\left(\frac{\pi}{2}\right) - \frac{1}{2}\ln\left(1+\frac{\pi^2}{4}\right) - \frac{\pi}{4}\tan^{-1}\left(\frac{\pi}{4}\right) + \frac{1}{2}\ln\left(1+\frac{\pi^2}{16}\right)$$