# Problem Set 20
# 11.3: The Integral Test

*Please indicate the members who are present. Also indicate the group coordinator.*

| Group Number: | |
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| Members: | KEY |
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The $p-$ series
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges if $p > 1$ and diverges if $p \leq 1$.

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Suppose $f(k) = a_k$, where $f$ is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$, then
$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

## Problem 1

Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

it's P-Series $P=\sqrt{2} > 1$

$\therefore$ It's convergent

## Problem 2

Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$

$$= \sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{n^2} + \frac{4}{n^2} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

P-Series $P = \frac{3}{2} > 1$

convergent

P-Series $P = 2 > 1$

convergent

$$\therefore \sum_{n=1}^{\infty} \left( \frac{\sqrt{n}+4}{n^2} \right)$$

is convergent.

## Problem 3

Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$

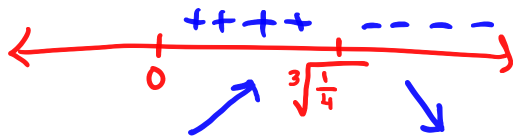
$$f(x) = \frac{\sqrt{x}}{1+x^{3/2}}, \quad x \in \mathbb{R}$$

- $f(x)$ is continues.
- $f(x)$ is $+$.
- $f(x)$ is decresing on $[1, \infty)$.

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(1+x^{3/2}) - \frac{3}{2}\sqrt{x}\cdot\sqrt{x}}{(1+x^{3/2})^2}$$

$$= \frac{\frac{1+x^{3/2}}{2\sqrt{x}} - \frac{3}{2}x}{(1+x^{3/2})^2}$$

$$= \frac{1+x^{3/2}-3x^{3/2}}{2\sqrt{x}\,(1+x^{3/2})^2} = \frac{1-2x^{3/2}}{2\sqrt{x}\,(1+x^{3/2})^2}$$




$$\int_1^{\infty} \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \lim_{t\to\infty} \int_1^t \frac{\sqrt{x}}{1+x^{3/2}}\,dx$$

$u = 1+x^{3/2} \to du = \frac{3}{2}\sqrt{x}\,dx$

$$\lim_{t\to\infty} \frac{2}{3}\int \frac{1}{u}\,du = \frac{2}{3}\ln u$$

$$= \lim_{t\to\infty} \frac{2}{3}\left[\ln(1+x^{3/2})\right]_1^t = \lim_{t\to\infty} \frac{2}{3}\left[\ln(1+t^{3/2}) - \ln 2\right] = \infty$$

the Series is divergent by the integral test

## Problem 4

Determine whether the series is convergent or divergent $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

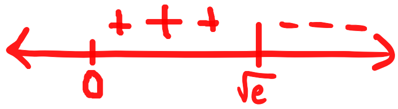
$$f(x) = \frac{\ln x}{x^2}, \quad x \in (0, \infty)$$

- $f(x)$ is continues
- $f(x)$ is $+$
- $f(x)$ is decreasing $[2, \infty)$

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - 2x \ln x}{x^4}$$

$$= \frac{x - 2x \ln x}{x^4} = \frac{x - 2x \ln x}{x^4}$$

$$= \frac{1 - \ln x}{x^3}$$




$$\int_2^{\infty} x^{-2} \ln x$$

$$= \lim_{t \to \infty} \int_2^t x^{-2} \ln x \, dx$$

$u = \ln x \qquad dv = x^{-2}\, dx$

$du = \frac{1}{x}\, dx \qquad v = \frac{-1}{x}$

$$= \lim_{t \to \infty} \left[ \frac{-\ln x}{x} \Big|_2^t + \int_2^t x^{-2}\, dx \right] = \lim_{t \to \infty} \left[ \frac{-\ln t}{t} + \frac{\ln 2}{2} + \left[ \frac{-1}{x} \right]_2^t \right]$$

$$= \lim_{t \to \infty} \left[ \frac{-\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{t} + \frac{1}{2} \right] = \lim_{t \to \infty} \left[ \frac{-1}{t} \right] + \lim_{t \to \infty} \left[ \frac{\ln 2}{2} - \frac{1}{t} + \frac{1}{2} \right]$$

$$= \frac{\ln 2 + 1}{2}$$

## Problem 5

Explain why the Integral Test can't be used to determine whether the series is convergent $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$.

$$f(x) = \frac{\cos \pi x}{\sqrt{x}}, \quad x \in (0, \infty)$$

f(x) is not always positive; since cosπx is in the range [-1,1]

f(x) is oscillating because of cosπx (increasing and decreasing periodically)

## Problem 6

Find the values of $p$ for which the series is convergent.

$$\sum_{n=1}^{\infty} n(1+n^2)^p$$

$$= \int_1^{\infty} x(1+x^2)^p\,dx = \lim_{t\to\infty} \int_1^{t} x(1+x^2)^p\,dx$$

let $u = 1+x^2$ $\quad x = t \to u = 1+t^2$

$du = 2x\,dx$ $\quad x = 1 \to u = 2$

$$= \lim_{t\to\infty} \frac{1}{2}\int_2^{1+t^2} u^p\,du = \lim_{t\to\infty} \frac{1}{2}\left[\frac{u^{p+1}}{p+1}\right]_2^{1+t^2}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\frac{(1+t^2)^{p+1}}{p+1} - \frac{2^{p+1}}{p+1}\right] \qquad p+1 \neq 0$$

the limit exist only when $p+1 < 0$

$p < -1$

$\therefore$ the Series is convergent

when $\boxed{p < -1}$

## Problem 7

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ find the sum of $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

$$\therefore \sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{3} \frac{1}{n^2}$$

$$= \frac{\pi^2}{6} - \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2}\right)$$

$$= \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9}\right)$$

$$= \frac{\pi^2}{6} - \frac{49}{36}$$

## Problem 8

How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?

$$R_n \leq \int_n^{\infty} \frac{1}{x(\ln x)^2}\,dx$$

$$\int_n^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_n^{t} \frac{1}{x(\ln x)^2}\,dx$$

let $u = \ln x$, $du = \frac{1}{x}\,dx$

$x = t \rightarrow \ln t$

$x = n \rightarrow \ln n$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2}\,du = \lim_{t\to\infty} \left[\frac{-1}{u}\right]_{\ln n}^{\ln t}$$

$$= \lim_{t\to\infty} \left[\frac{-1}{\ln t} + \frac{1}{\ln n}\right] = \frac{1}{\ln n}$$

$$\frac{1}{\ln n} \leq 0,01$$

$$\ln n \geq 100$$

$$\boxed{n \geq e^{100}}$$