MATH101

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Syllabus

2.1: A Preview of Calculus

Objectives

> “
>
> - Understand what calculus is and how it compares with precalculus.
> - Understand that the tangent line problem is basic to calculus.
> - Understand that the area problem is also basic to calculus.

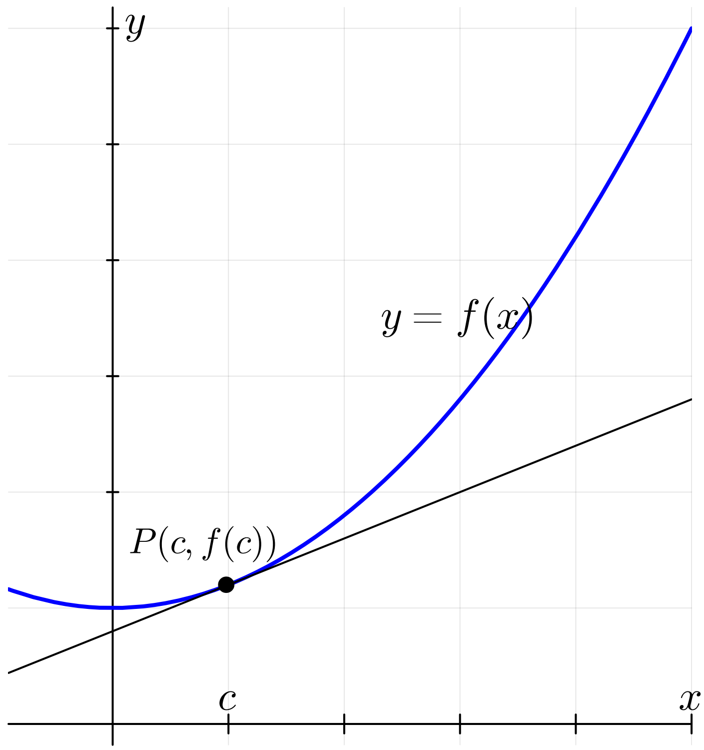
intro.

What is Calculus?

Precalculus Matematics $\Rightarrow$ **Limit process** $\Rightarrow$ ***Calculus***

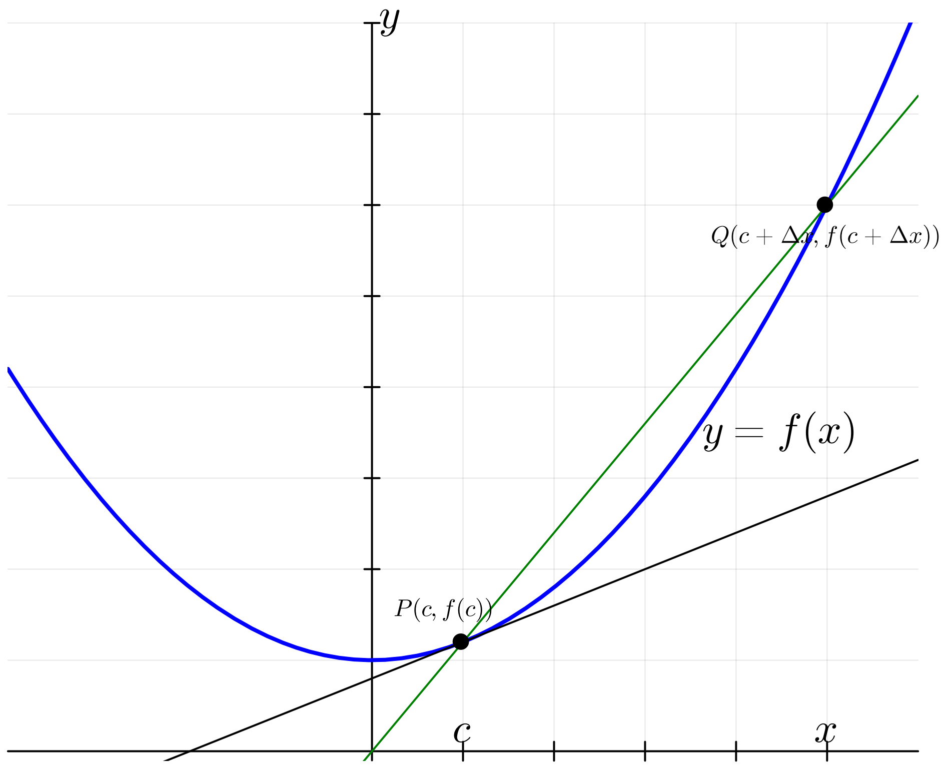
The Tangent Line Problem

What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx

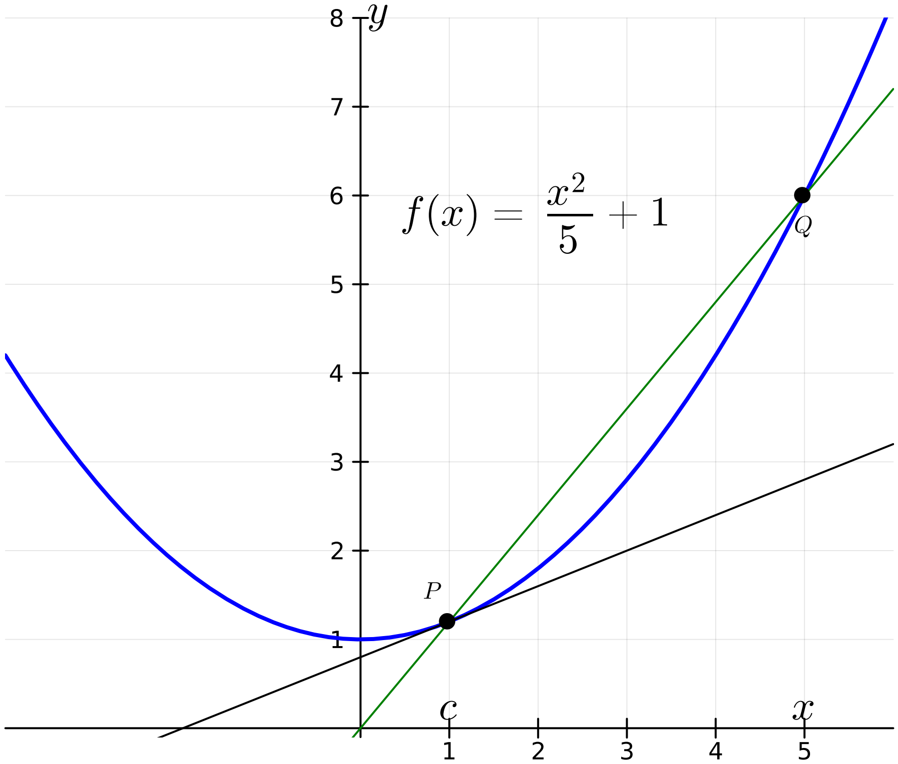
Find the equation of the secant line



$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Δx

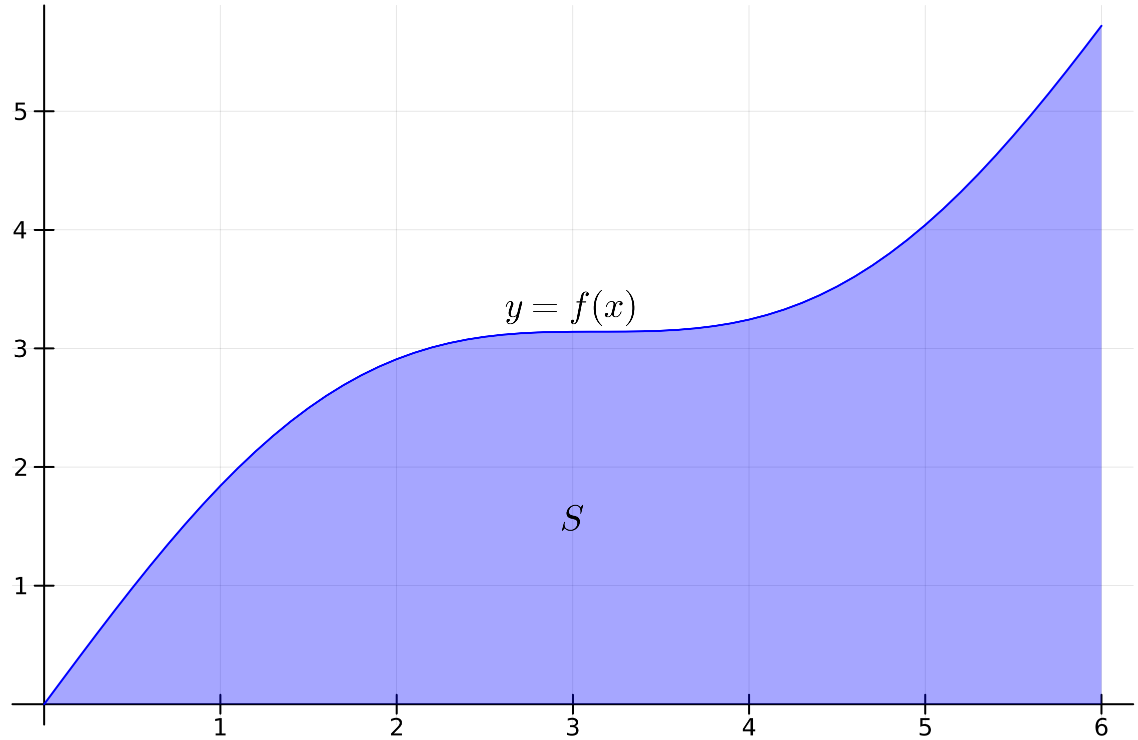
Example: **Find the equation of the secant line**



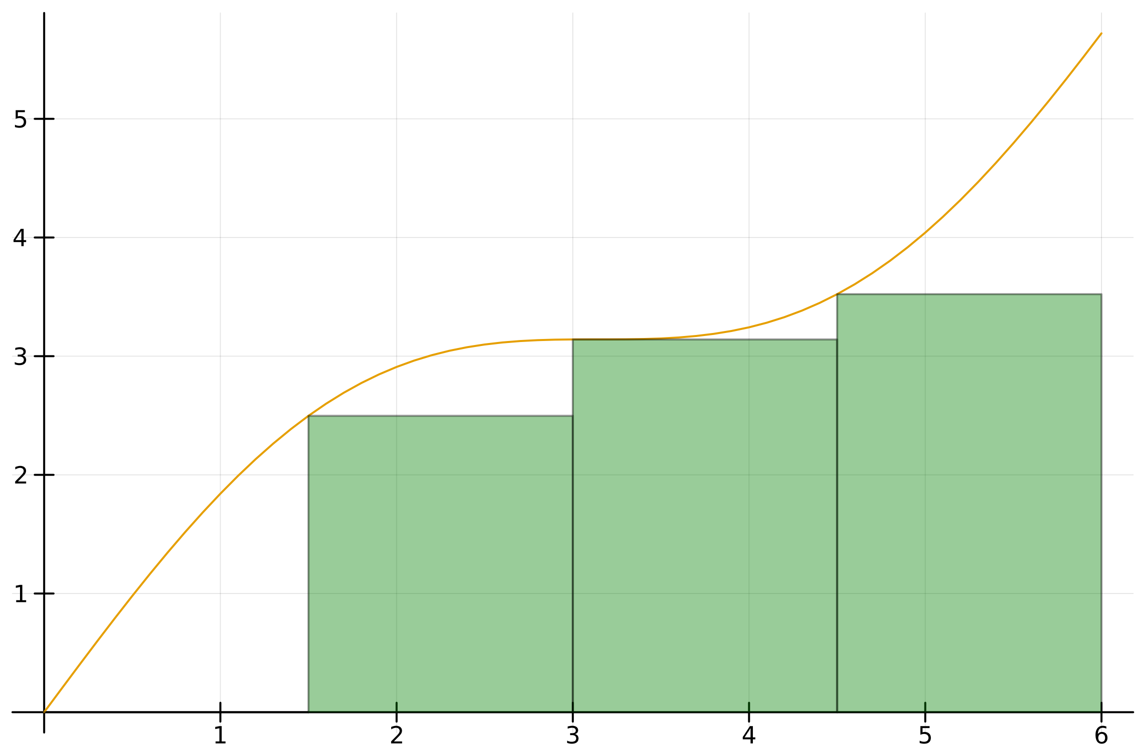
$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = 4 a = 0 b = 6 method = Left



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

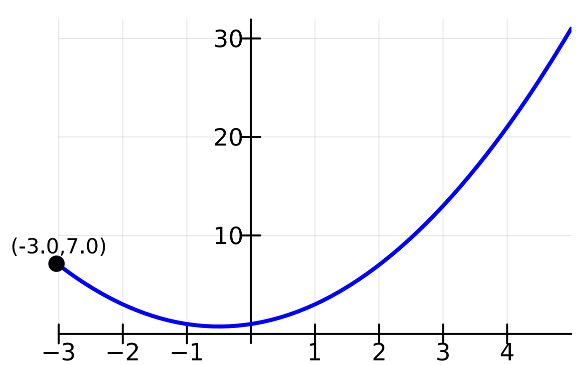
> “
>
> - Estimate a limit using a **numerical** or **graphical approach**.
> - Learn different ways that a limit can fail to exist.
> - <s>Study and use a formal definition of limit</s>.

An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$ 0.0 x **approaches** 1 from Left



x approacheds 1 (from left)	$f(x)$ approaches
-3.0	7.0

Remark

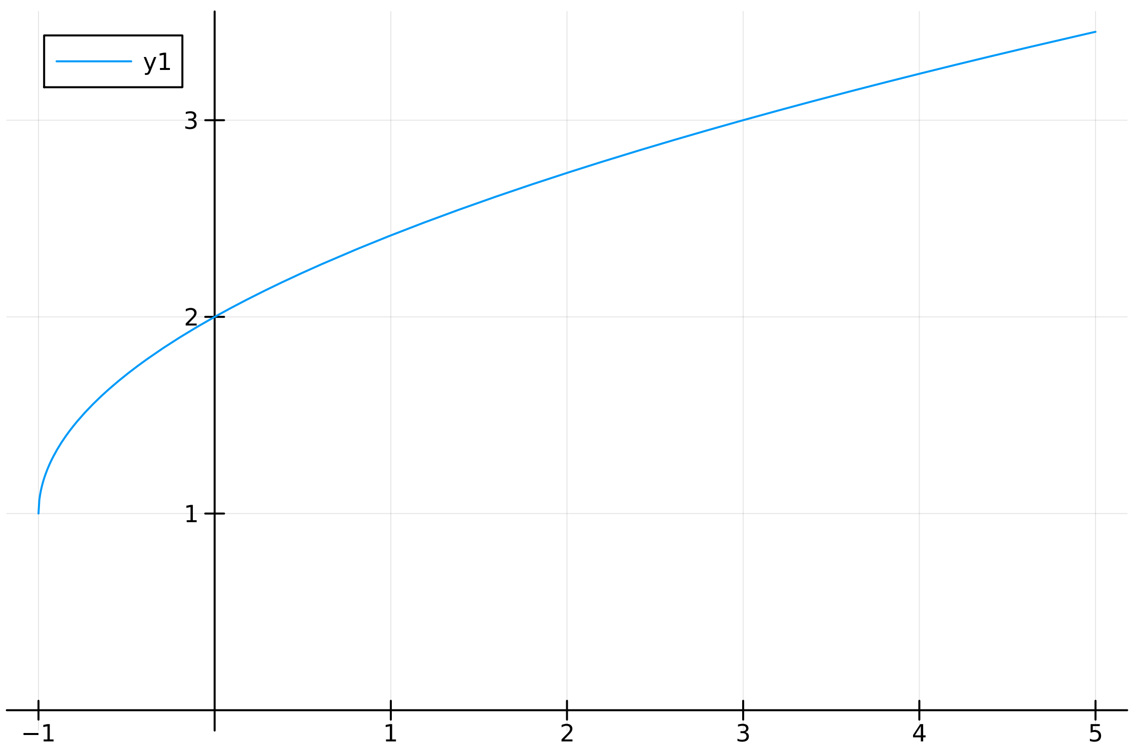
$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1: Estimating a Limit Numerically

Evaluate the function $f(x) = \dfrac{x}{\sqrt{x+1} - 1}$ at several x-values near 0 and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph




```
1.9999995001202078
```

```
begin
    whatever(x)=x/(sqrt(x+1)-1)
    whatever(-0.000001)
end
```

Example 2: **Finding a Limit**

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark **Problem solving**

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: `MATLAB`, `python`, `Julia`)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist

Example 3: **Different Right and Left Behavior**

Show that the limit $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

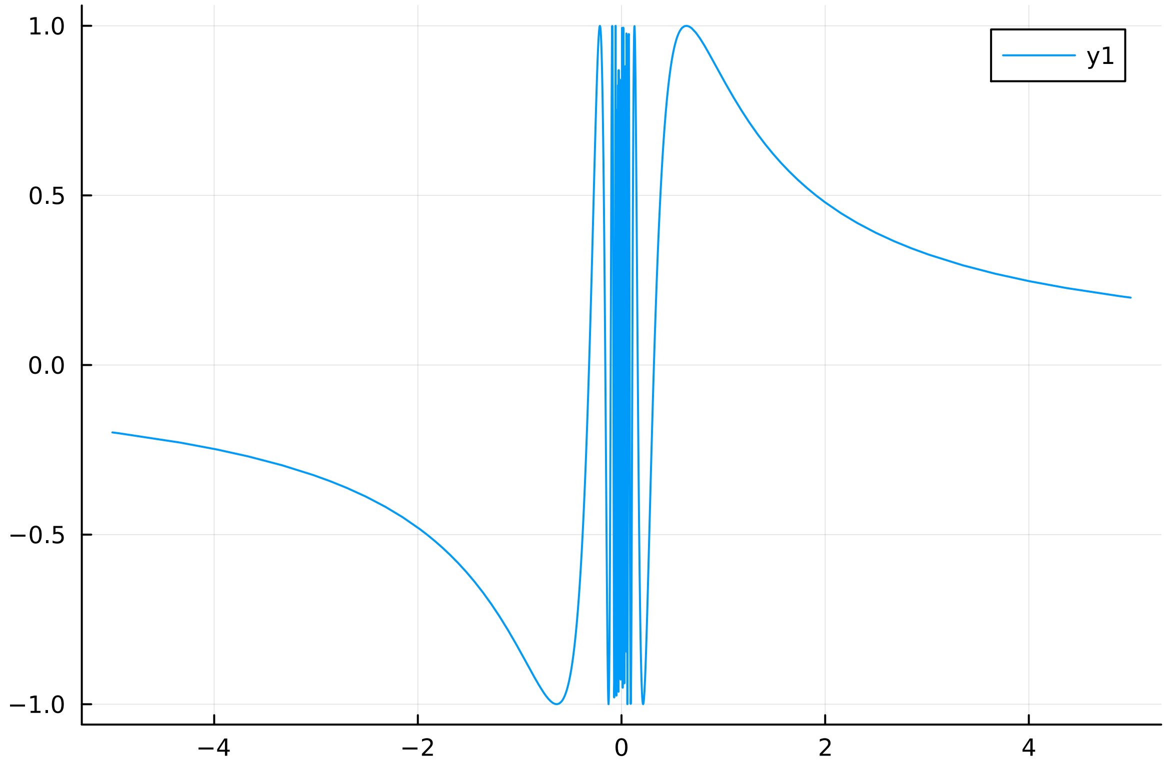
Example 4: **Unbounded Behavior**

Discuss the existence of the limit $\lim\limits_{x \to 0} \dfrac{1}{x^2}$

Example 5: **Oscillating Behavior**

Discuss the existence of the limit $\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

```
9.999999999999998e9
```




```
plot(x->sin(1/x))
```

A Formal Definition of Limit (Redaing Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \to c} f(x) = L$$

implies two statements—the limit exists and the limit is L.

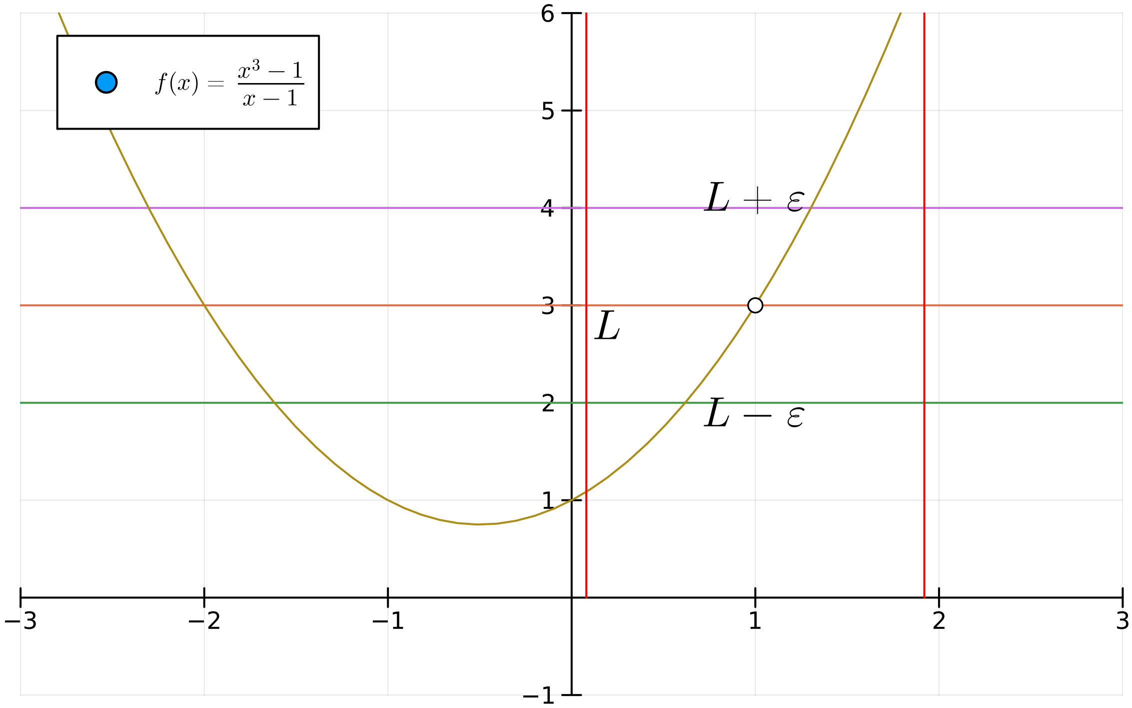
Example:

Prove that

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

ϵ = 1.0 δ = 0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

> “
>
> - Evaluate a limit using properties of limits.
> - Develop and use a strategy for finding limits.
> - Evaluate a limit using the dividing out technique.
> - Evaluate a limit using the rationalizing technique.
> - Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem **Some Bacic Limits**

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Theorem **Properties of Limits**

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L, \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. **Scalar multiple** $\lim_{x \to c} \left[bf(x)\right] = bL$
2. **Sum or difference** $\lim_{x \to c} \left[f(x) \pm g(x)\right] = L \pm K$
3. **Product** $\lim_{x \to c} \left[f(x)g(x)\right] = LK$
4. **Quotient** $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}, \quad K \neq 0$
5. **Power** $\lim_{x \to c} \left[f(x)\right]^n = L^n$

Example 2: The Limit of a Polynomial

Find $\lim_{x \to 2} (4x^2 + 3)$.

Theorem **Limits of Polynomial and Rational Functions**

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3: The Limit of a Rational Function

Find

$$\lim_{x\to 1} \frac{x^2+x+2}{x+1}.$$

Theorem **The Limit of a Function Involving a Radical**

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem **The Limit of a Composite Function**

If f and g are functions such that $\lim\limits_{x\to c} g(x) = L$ and $\lim\limits_{x\to c} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Theorem **Limits of Transcendental Functions**

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \to c} \sin(x) = \sin(c)$
2. $\lim_{x \to c} \cos(x) = \cos(c)$
3. $\lim_{x \to c} \tan(x) = \tan(c)$
4. $\lim_{x \to c} \cot(x) = \cot(c)$
5. $\lim_{x \to c} \sec(x) = \sec(c)$
6. $\lim_{x \to c} \csc(x) = \csc(c)$
7. $\lim_{x \to c} a^x = a^c, \quad a > 0$
8. $\lim_{x \to c} \ln(x) = \ln(c)$

A Strategy for Finding Limits

Theorem **Functions That Agree at All but One Point**

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Remarks **A Strategy for Finding Limits**

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c.

Dividing Out Technique

Example 7: **Dividing Out Technique**

Find the limit $\displaystyle\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8: **Rationalizing Technique**

Find the limit $\displaystyle\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$.

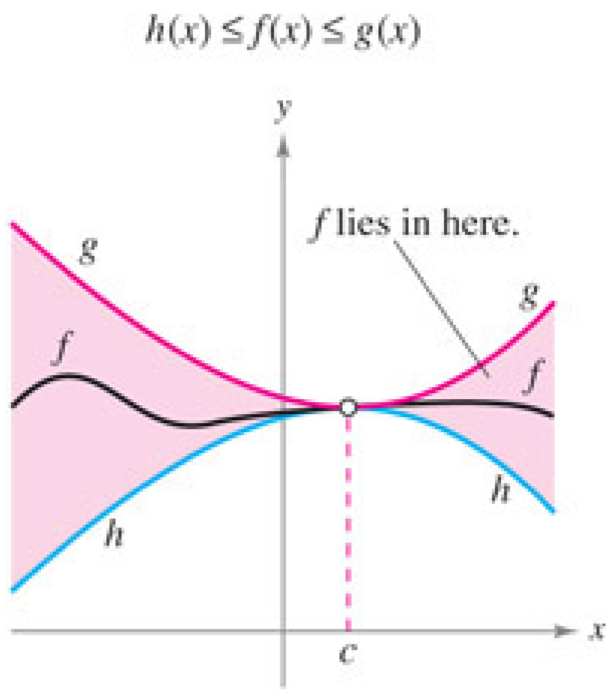
The Squeeze Theorem

Theorem **The Squeeze Theorem**

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ **exists** and equal to L.



Theorem **Three Special Limits**

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$
3. $\lim_{x \to 0} (1 + x)^{1/x} = e.$

Example 9: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$.

Example 10: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$.

Exercises

2.5: Infinite Limits

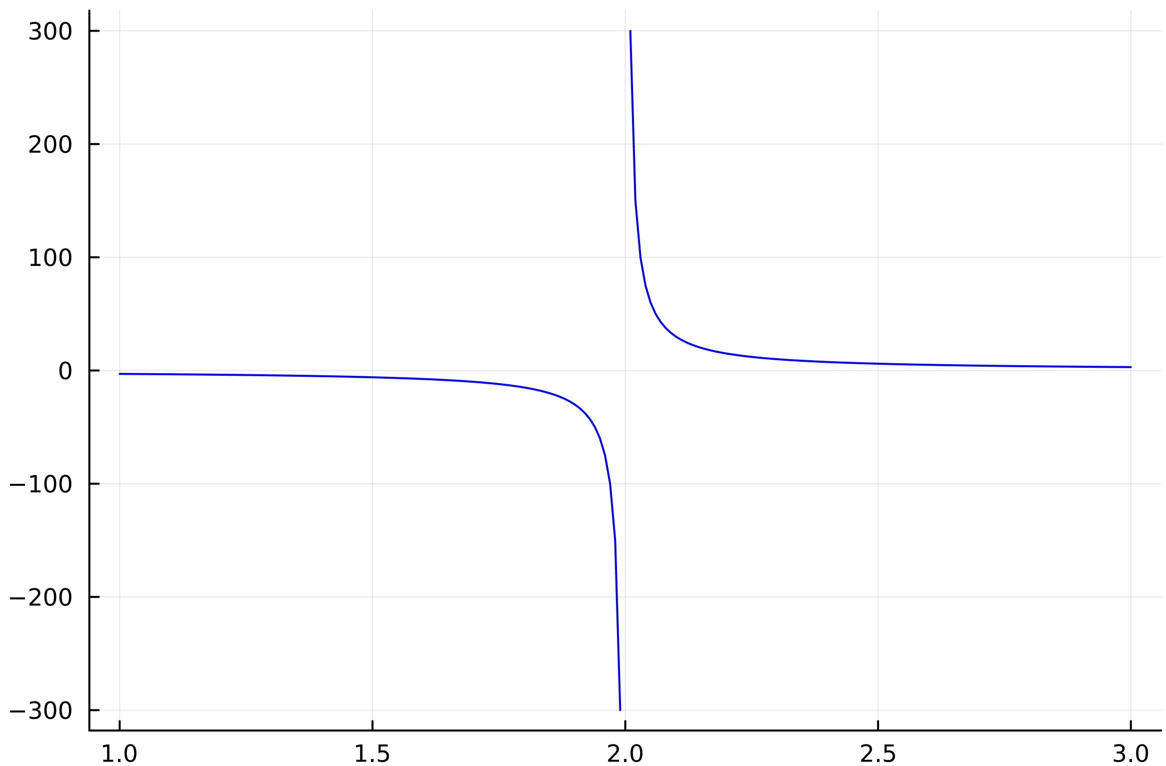
Objectives

> “
>
> - Determine infinite limits from the left and from the right.
> - Find and sketch the vertical asymptotes of the graph of a function.

Example: **Infinite Limit**

Consider

$$f(x) = \frac{3}{x-2}$$




```
plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-
2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c.

Theorem **Vertical Asymptotes**

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

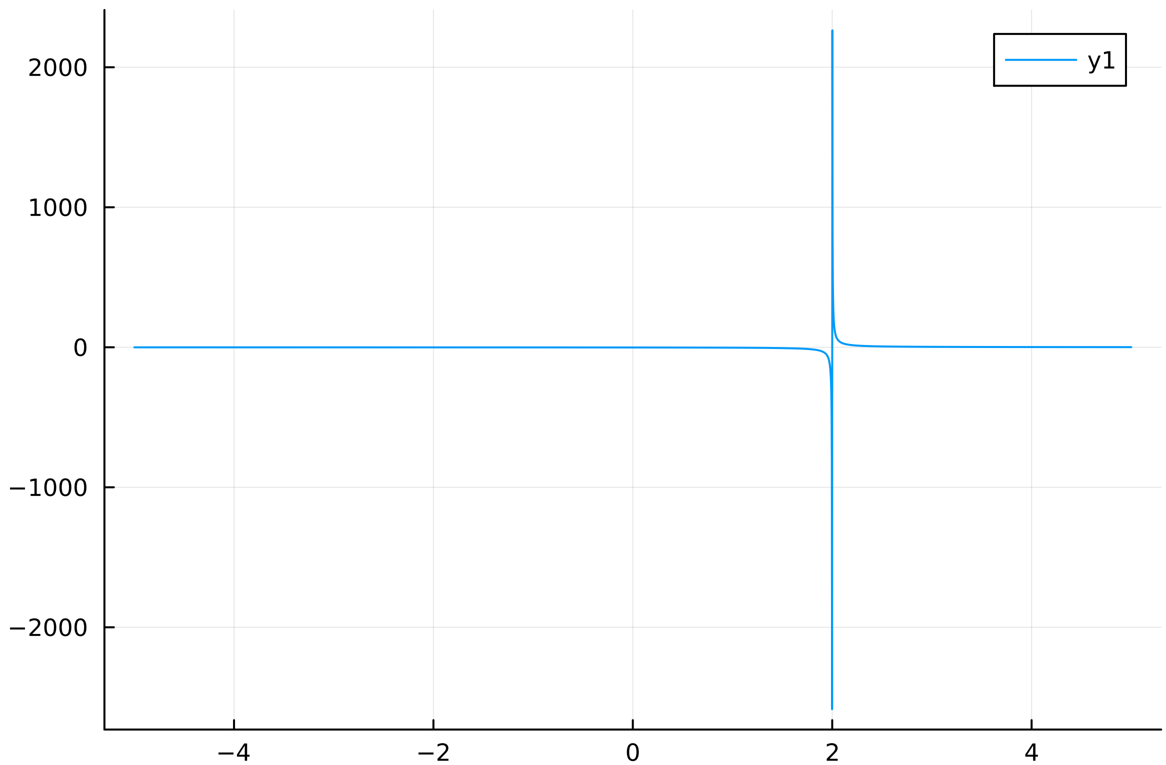
has a vertical asymptote at c.

Example 2: Finding Vertical Asymptotes

1. $h(x) = \dfrac{1}{2(x+1)}$.

2. $h(x) = \dfrac{x^2+1}{x^2-1}$.

3. $h(x) = \cot x = \dfrac{\cos x}{\sin x}$.




```
plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- desmos.com
- geogebra.org

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^3 - 3x}{x - 1} \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{x^3 - 3x}{x - 1}$$

Theorem **Properties of Infinite Limits**

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L$$

1. **Sum or difference**: $\lim\limits_{x \to c} \left[f(x) \pm g(x) \right] = \infty$
2. **Product**:

$$\begin{aligned} \lim_{x \to c} \left[f(x) g(x) \right] &= \infty, \quad L > 0 \\ \lim_{x \to c} \left[f(x) g(x) \right] &= -\infty, \quad L < 0 \end{aligned}$$

3. Quotient: $\lim\limits_{x \to c} \left[\dfrac{g(x)}{f(x)} \right] = 0$

Remark

2. is **not true** if $\lim\limits_{x \to c} g(x) = 0$

Exercises

4.5: Limits at Infinity

Objectives

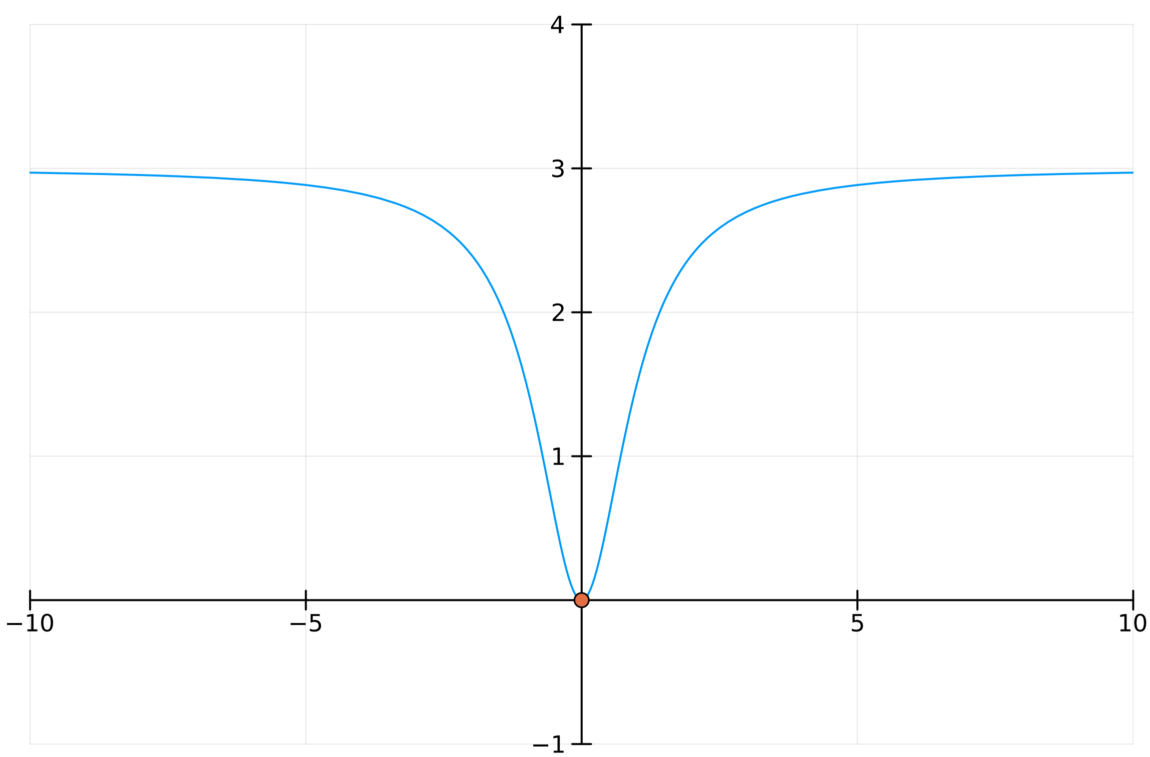
> - Determine (finite) limits at infinity.
> - Determine the horizontal asymptotes, if any, of the graph of a function.
> - Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2+1}$$

$x =$ 0

$f(x) =$ 0.0



we write

$$\lim_{x\to\infty} \frac{3x^2}{x^2+1} = 3, \quad \lim_{x\to-\infty} \frac{3x^2}{x^2+1} = 3$$

Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L$$

Remarks

- Limits at infinity have many of the same properties of limits discussed in Section 2.3.
- For example, if $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ both exist, then
  - $\lim\limits_{x\to\infty} [f(x)+g(x)] = \lim\limits_{x\to\infty} f(x) + \lim\limits_{x\to\infty} g(x)$
  - $\lim\limits_{x\to\infty} [f(x)g(x)] = \left[\lim\limits_{x\to\infty} f(x)\right]\left[\lim\limits_{x\to\infty} g(x)\right]$
- Similar properties hold for limits at $-\infty$.

Theorem **Limits at Infinity**

1. If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim\limits_{x\to-\infty} e^x = 0$ and $\lim\limits_{x\to\infty} e^{-x} = 0$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

$$h(x) = \frac{p(x)}{q(x)}$$

1. $\deg p < \deg q$, then the limit is 0.
2. $\deg p = \deg q$, then the **limit** of the rational function **is** the **ratio** of the **leading coefficients**.
3. $\deg p > \deg q$, then the **limit** of the rational function **does not exist**.

Examples

```
# begin
#   xx=symbols("xx",real=true)
#   limit(xx*sin(1/xx),xx,0)
# end
```

Infinite Limits at Infinity

Remark

Determining whether a function has an infinite limit at infinity is useful in analyzing the **“end behavior”** of its graph. You will see examples of this in Section 4.6 on curve sketching.

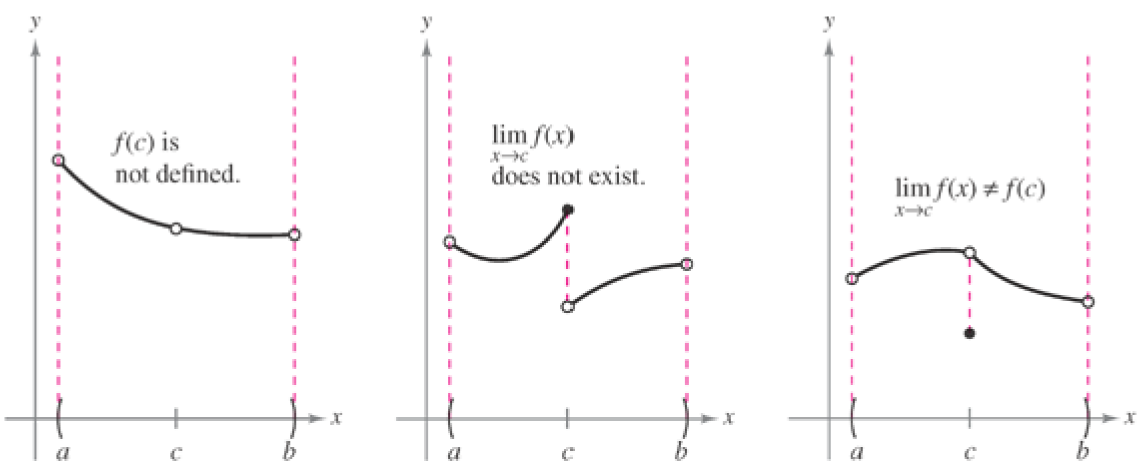
2.4: Continuity and One-Sided Limits

Objectives

> “
>
> - Determine continuity at a point and continuity on an open interval.
> - Determine one-sided limits and continuity on a closed interval.
> - Use properties of continuity.
> - Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

The graph of f is no contnious at $x = c$



In Figure __above__, it appears that continuity at ``x=c`` can be __destroyed__ by any one of __three conditions__.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

- A function f is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval.
- A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Remarks

- If a function f is defined on an open interval I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c.
- Discontinuities fall into two categories:
  - **removable**: A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$.
  - **nonremovable**: there is no way to define $f(c)$ so as to make the function continuous at $x = c$.

Example 1:

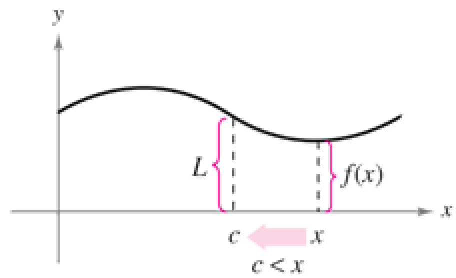
Discuss the continuity of each function

$$\begin{aligned}
&\text{a.} \quad f(x) = \frac{1}{x} \\
&\text{b.} \quad g(x) = \frac{x^2 - 1}{x - 1} \\
&\text{c.} \quad h(x) = \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases} \\
&\text{d.} \quad y = \sin x
\end{aligned}$$

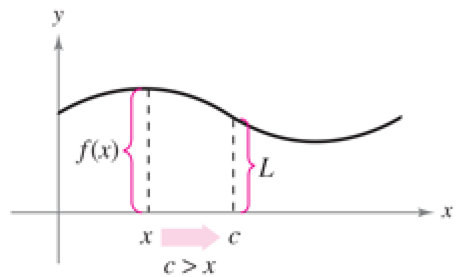
Examples

One-Sided Limits and Continuity on a Closed Interval

(a) Limit from right $\lim_{x \to c^+} f(x) = L$



(a) Limit as x approaches c from the right.



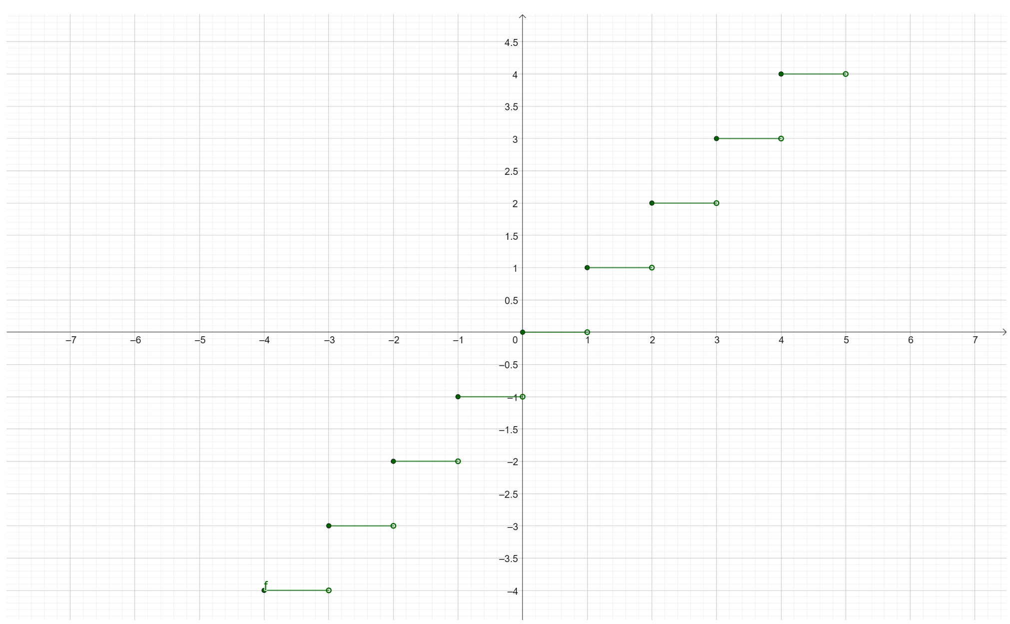
(b) Limit as x approaches c from the left.

(b) Limit from left $\lim_{x \to c^-} f(x) = L$

STEP FUNCTIONS

(**greatest integer function**)

$$[\![x]\!] = \textbf{greatest integer } n \textbf{ such that } n \leq x.$$



Theorem **The Existence of a Limit**

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is if and only if

$$\lim_{x \to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b).$$

Example 4: **Continuity on a Closed Interval**

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}$$

Properties of Continuity

Theorem **Properties of Continuity**

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. **Scalar multiple:** bf
2. **Sum or difference:** $f \pm g$
3. **Product:** fg
4. **Quotient:** $\dfrac{f}{g}, \quad g(c) \neq 0$,

Remarks

1. **Polynomials** are continuous at every point in their domains.
2. **Rational functions** are continuous at every point in their domains.
3. **Radical functions** are continuous at every point in their domains.
4. **Trigonometric functions** are continuous at every point in their domains.
5. **Exponential and logarithmic functions** are continuous at every point in their domains.

Theorem **Continuity of a Composite Function**

If g is continuous at c and f is continuous at $g(c)$ then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Remark

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

provided f and g satisfy the conditions of the theorem.

Example 7: **Testing for Continuity**

Describe the interval(s) on which each function is continuous.

$$
\begin{array}{ll}
\text{a.} & f(x) = \tan x \\
\text{b.} & g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \\
\text{c.} & h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}
\end{array}
$$

Exercises

The Intermediate Value Theorem

Theorem **Intermediate Value Theorem**

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Example 8: **An Application of the Intermediate Value Theorem**

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

3.1: The Derivative and the Tangent Line Problem

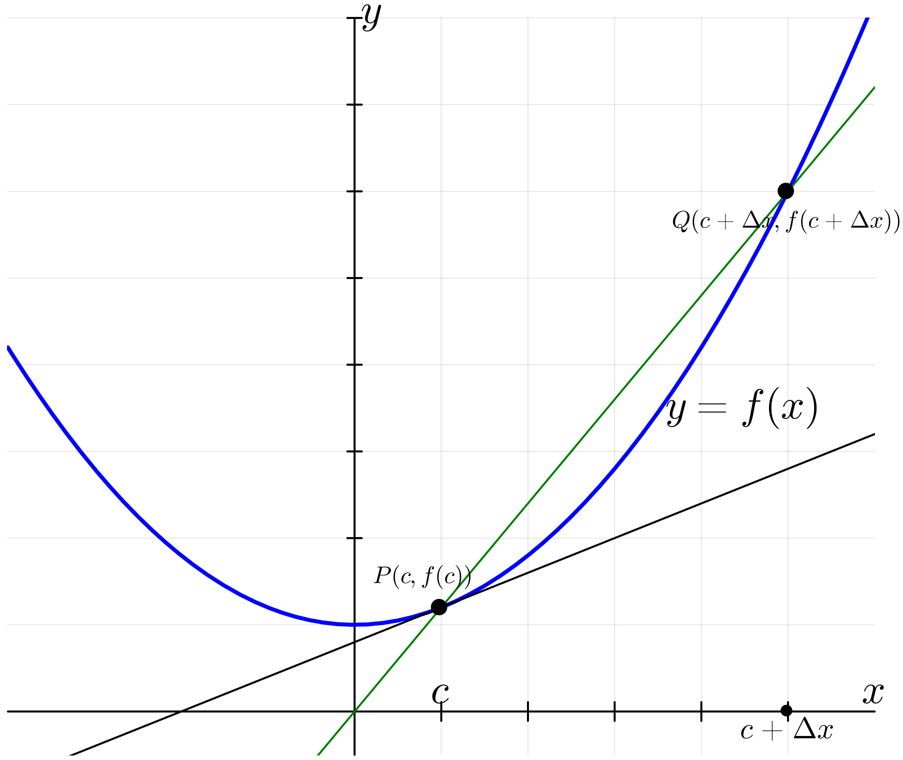
Objectives

> “
>
> - Find the slope of the tangent line to a curve at a point.
> - Use the limit definition to find the derivative of a function.
> - Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Δx 4.0

Find the equation of the secant line



Slope of secant line

$$\mathrm{m}_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Definition of Tangent Line with Slope

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Remark

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$**.

Example 1: The Slope of the Graph of a Linear Function

Find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$.

Example 2: Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$.

Remarks

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line.
- For vertical tangent lines, you can use the **following definition**. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the **vertical line** $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f.

The Derivative of a Function

Definition **Derivative of a Function**

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Remarks

- The notation $f'(x)$ is read as "f prime of x."
- $f'(x)$ is a **function** that gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.
- The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.
- The process of finding the derivative of a function is called **differentiation**.
- A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval** (a, b) when it is differentiable at every point in the interval.

Notation

$$y = f(x)$$

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}[f(x)]$
- $D_x[y]$

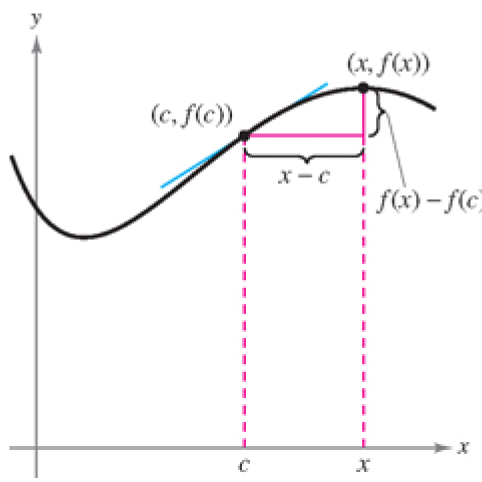
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Examples 3,4,5: **Finding the Derivative by the Limit Process**

Find the derivative of

- $f(x) = x^3 + 2x$
- $f(x) = \sqrt{x}$
- $y = \dfrac{2}{t}$ with respect to t.

Differentiability and Continuity



Alternative form of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Remarks

derivative from the left

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

derivative from the right

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Example:

$$f(x) = [[x]]$$

Example 6: **A Graph with a Sharp Turn**

$$f(x) = |x - 2|$$

Example 7: **A Graph with a Vertical Tangent Line**

$$f(x) = x^{\frac{1}{3}}$$

Theorem **Differentiability Implies Continuity**

If f is differentiable at $x = c$, then f is continuous at $x = c$.

remarks

The relationship between continuity and differentiability is summarized below.

- If a function f is differentiable at $x = c$, then it is continuous at $x = c$. So, **differentiability** implies ($\Rightarrow$) **continuity**.
- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, **continuity does not imply differentiability**.

Exercises

3.2: Basic Differentiation Rules and Rates of Change

Objectives

> “
>
> - Find the derivative of a function using the Constant Rule.
> - Find the derivative of a function using the Power Rule.
> - Find the derivative of a function using the Constant Multiple Rule.
> - Find the derivative of a function using the Sum and Difference Rules.
> - Find the derivatives of the sine function and of the cosine function.
> - Find the derivatives of exponential functions.
> - Use derivatives to find rates of change.

The Constant Rule

Theorem **The Constant Rule**

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

The Power Rule

Theorem **The Power Rule**

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at 0, n must be a number such that x^{n-1} is defined on an interval containing 0.

The Constant Multiple Rule

Theorem **The Constant Multiple Rule**

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

The Sum and Difference Rules

Theorem **The Sum and Difference Rules**

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x) \qquad \text{Difference Rule}$$

Derivatives of the Sine and Cosine Functions

Theorem **Derivatives of the Sine and Cosine Functions**

$$\frac{d}{dx}[\sin(x)] = \cos x \quad , \qquad \frac{d}{dx}[\cos(x)] = -\sin x$$

Derivatives of Exponential Functions

Theorem **Derivative of the Natural Exponential Function**

$$\frac{d}{dx}[e^x] = e^x$$

$$2\sin\left(t\right)$$

```
begin
    t = symbols("t", real=true)
    g(t)=-2cos(t)-5
    plot(x->g(x))
    diff(g(t),t)
end
```

Rates of Change

- The derivative can be used to determine the **rate of change** of one variable with respect to another.
- Applications involving rates of change, sometimes referred to as **instantaneous rates of change**, occur in a wide variety of fields.
- A common use for rate of change is to describe **the motion of an object moving in a straight line.** (+ direction and -direction)
- The function s that gives **the position (relative to the origin)** of an object as a **function of time** t is called a **position function**. If, over a period of time Δt, the object changes its position by the amountthen, by the familiar formula

$$\Delta s = s(t + \Delta t) - s(t)$$

- then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}.$$

-the average velocity is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} \qquad \text{Average Velocity}.$$

- In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the velocity of the object at time t is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \qquad \text{Velocity function}.$$

Example:

If a ball is thrown into the air with a velocity of $4m/s$, its height (in meters (m)) t seconds later is given by

$$y = 4t - 4.9t^2.$$

1. Find the average velocity for the time period from $t = 1$ to $t = 3$.
2. Find the instantaneous rate of change at $t = 2$.

Example 11: **Using the Derivative to Find Velocity**

At time $t = 0$, a diver jumps from a platform diving board that is 9.8 meters above the water. The initial velocity of the diver is 4.9 meters per second. When does the diver hit the water? What is the diver's velocity at impact?

3.3: Product and Quotient Rules and Higher-Order Derivatives

Objectives

> “
>
> - Find the derivative of a function using the Product Rule.
> - Find the derivative of a function using the Quotient Rule.
> - Find the derivative of a trigonometric function.
> - Find a higher-order derivative of a function.

The Product Rule

Theorem **The Product Rule**

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the **first** function **times** the **derivative of the second**, **plus** the **second** function times the **derivative of the first**.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Example:

Find the derivative of $f(x) = xe^x$.

The Quotient Rule

Theorem **The Quotient Rule**

The quotient of two differentiable functions f and g is itself differentiable at all values of for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the **denominator times** the **derivative of the numerator minus** the **numerator times the derivative of the denominator**, all **divided by** the **square of the denominator**.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0.$$

Example:

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Derivatives of Trigonometric Functions

Theorem **Derivatives of Trigonometric Functions**

$$\begin{array}{rcl|rcl} \frac{d}{dx}(\tan x) & = & \sec^2 x & \frac{d}{dx}(\cot x) & = & -\csc^2 x \\ \frac{d}{dx}(\sec x) & = & \sec x \tan x & \frac{d}{dx}(\csc x) & = & -\csc x \cot x \end{array}$$

Example:

Differentiate

$$y = \frac{1 - \cos x}{\sin x}$$

Higher-Order Derivatives

Remarks

Rates of changes

$$\begin{aligned} & & & s(t) & & \text{Position function} \\ & & v(t) &= s'(t) & & \text{Velocity function} \\ a(t) &= & v'(t) &= s''(t) & & \text{Acceleration function} \end{aligned}$$

Higher Derivatives

$$\begin{array}{llllll}
\textbf{First derivative:} & y', & f'(x), & \frac{dy}{dx}, & \frac{d}{dx}[f(x)], & D_x[y] \\
\textbf{Second derivative:} & y'', & f''(x), & \frac{d^2y}{dx^2}, & \frac{d^2}{dx^2}[f(x)], & D_x^2[y] \\
\textbf{Third derivative:} & y''', & f'''(x), & \frac{d^3y}{dx^3}, & \frac{d^3}{dx^3}[f(x)], & D_x^3[y] \\
\textbf{Fourth derivative:} & y^{(4)}, & f^{(4)}(x), & \frac{d^4y}{dx^4}, & \frac{d^4}{dx^4}[f(x)], & D_x^4[y] \\
\vdots & & & & & \\
\textbf{nth derivative:} & y^{(n)}, & f^{(n)}(x), & \frac{d^ny}{dx^n}, & \frac{d^n}{dx^n}[f(x)], & D_x^n[y]
\end{array}$$

Exercises

3.4: The Chain Rule

Objectives

> “
>
> - Find the derivative of a composite function using the Chain Rule.
> - Find the derivative of a function using the General Power Rule.
> - Simplify the derivative of a function using algebra.
> - Find the derivative of a transcendental function using the Chain Rule.
> - Find the derivative of a function involving the natural logarithmic function.
> - Define and differentiate exponential functions that have bases other than .

The Chain Rule

Theorem **The Chain Rule**

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f((x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Remark

■: Outer function ■: Inner function

$$\text{If} \quad y = f(g(x)) = f(u), \text{ then}$$

$$y' = \frac{dy}{dx} = f'(g(x))g'(x)$$

or, equivalently

$$y' = \frac{dy}{dx} = f'(u)\frac{du}{dx}$$

Example: Using the Chain Rule

Find $\dfrac{dy}{dx}$ for

$$y = \left(x^2 + 1\right)^3.$$

Examples

The General Power Rule

Theorem **The General Power Rule**

If $y = [u(x)]^n$, where u is a differentiable function of x and is a rational n number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{dy}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u(x)]^n = n[u]^{n-1}u'$$

Example:

Find the derivative of $y = (3x - 2x^2)^3$.

Simplifying Derivatives

Example:

Find the derivative of

1. $f(x) = x^2\sqrt{1 - x^2}$.
2. $f(x) = \dfrac{x}{\sqrt[3]{x^2 + 4}}$.

Transcendental Functions and the Chain Rule

$$
\begin{aligned}
\frac{d}{dx}[\sin u] &= (\cos u)u' \quad , & \frac{d}{dx}[\cos u] &= -(\sin u)u' \\
\frac{d}{dx}[\tan u] &= (\sec^2 u)u' \quad , & \frac{d}{dx}[\cot u] &= -(\csc^2 u)u' \\
\frac{d}{dx}[\sec u] &= (\sec u \tan u)u' \quad , & \frac{d}{dx}[\csc u] &= -(\csc u \cot u)u' \\
\frac{d}{dx}[e^u] &= (e^u)u'
\end{aligned}
$$

Examples

The Derivative of the Natural Logarithmic Function

Theorem **Derivative of the Natural Logarithmic Function**

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, x > 0$

Theorem **Derivative Involving Absolute Value**

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln |u|] = \frac{u'}{u}$$

Examples:

Find y' for

1. $y = \ln \sqrt{x+1}$
2. $y = \left(\frac{3x-1}{x^2+3}\right)^2$
3. $y = \ln \left[\frac{x(x^2+1)^2}{\sqrt{2x^3+1}}\right]$

Bases Other than e

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base** a is denoted by a^x and is defined by

$$a^x = e^{x \ln a}$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number ($a \neq 1$) and x is any **positive** real number, then the **logarithmic function to the base** a is denoted by $\log_a x$ and is defined by

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Theorem **Derivatives for Bases Other than e**

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a^x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a^u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

Examples

3.5: Implicit Differentiation

objectives

> “
>
> - Distinguish between functions written in implicit form and explicit form.
> - Use implicit differentiation to find the derivative of a function.
> - Find derivatives of functions using logarithmic differentiation.

Example: Implicit Differentiation

Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$.

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$.

Example: Finding the Slope of a Graph Implicitly

Determine the slope of the graph of

$$3(x^2+y^2)^2 = 100xy$$

at the point $(3,1)$.

Example: Determining a Differentiable Function

Find $\frac{dy}{dx}$ implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x.

Example: Finding the Second Derivative Implicitly

Given $x^2+y^2=25$, find $\frac{d^2y}{dx^2}$.

Definition Normal Line

The normal line at a point is the line **perpendicular** to the tangent line at the point.

Example (exercise 63): Normal Lines

Find the equations for the tangent line and normal line to the circle

$$x^2+y^2=25$$

at the points $(4,3)$ and $(-3,4)$.

Logarithmic Differentiation

Example: **Logarithmic Differentiation**

Find the derivative of

1. $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}, \qquad x \neq 2.$
2. $y = x^{2x}, \qquad x > 0.$
3. $y = x^{\pi}.$

Definition **Orthogonal Trajectories**

Two graphs (curves) are **orthogonal** if at their point(s) of intersection, their **tangent lines** are **perpendicular** to each other.

Excercise 81:

Are the following curves **orthogonal**?

$$2x^2 + y^2 = 6, \qquad y^2 = 4x$$

Examples

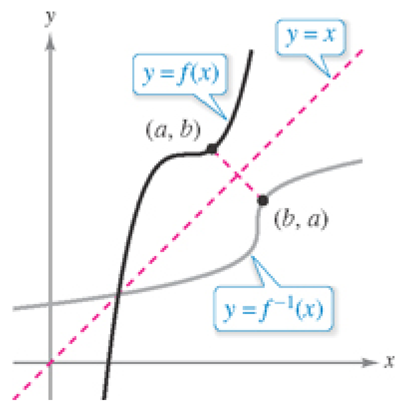
3.6: Derivatives of Inverse Functions

Objectives

> “
>
> - Find the derivative of an inverse function.
> - Differentiate an inverse trigonometric function.

Derivative of an Inverse Function

The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.



Theorem **Continuity and Differentiability of Inverse Functions**

Let f be a function whose domain is an interval I . If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Theorem **The Derivative of an Inverse Function**

Let f be a function that is differentiable on an interval I . If f has an inverse function g, then g differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

Example: **Graphs of Inverse Functions Have Reciprocal Slopes**

Let $f(x) = x^2, \quad x \geq 0$. Find

1. $f^{-1}(x)$
2. Find the slopes of the graphs of f and f^{-1} at the points $(2, 4)$ and $(4, 2)$ respectively

Derivatives of Inverse Trigonometric Functions

Theorem **Derivatives of Inverse Trigonometric Functions**

Let u be a differentiable function of x.

$$\begin{aligned}
\frac{d}{dx}\left[\sin^{-1} u\right] &= \frac{u'}{\sqrt{1-u^2}} \quad , & \frac{d}{dx}\left[\cos^{-1} u\right] &= \frac{-u'}{\sqrt{1-u^2}} \\
\frac{d}{dx}\left[\tan^{-1} u\right] &= \frac{u'}{1+u^2} \quad , & \frac{d}{dx}\left[\cot^{-1} u\right] &= \frac{-u'}{1+u^2} \\
\frac{d}{dx}\left[\sec^{-1} u\right] &= \frac{u'}{|u|\sqrt{u^2-1}} \quad , & \frac{d}{dx}\left[\csc^{-1} u\right] &= \frac{-u'}{|u|\sqrt{u^2-1}}
\end{aligned}$$

Examples

3.7: Related Rates

Objectives

> “
>
> - Find a related rate.
> - Use related rates to solve real-life problems.

Finding Related Rates

Example 1: Two Rates That Are Related

The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find $\frac{dy}{dt}$ when $x = 1$, given that $\frac{dx}{dt} = 2$ when $x = 1$.

Problem Solving with Related Rates

In **Example 1**

- **Equation:** $y = x^2 + 3$.
- **Given:** $\dfrac{dx}{dt} = 2$ when $x = 1$.
- **Find:** $\dfrac{dy}{dt}$ when $x = 1$.

Guidelines for Solving Related-Rate Problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation with respect to time t.
4. After completing **Step 3**, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Example 2: **Ripples in a Pond**

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in

Russ Bishop/Alamy Stock Photo

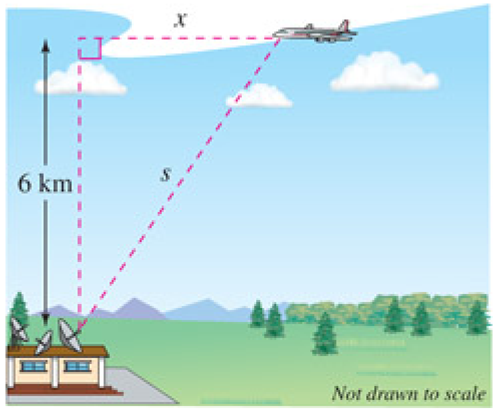
The radius r of the outer ripple is increasing at a constant rate of 0.5 meter per second. When the radius is 2 meters, at what rate is the total area A of the disturbed water changing?

Example 3: **An Inflating Balloon**

Air is being pumped into a spherical balloon at a rate of 1.5 cubic meters per minute. Find the rate of change of the radius when the radius is 2 meters

Example 4: The Speed of an Airplane Tracked by Radar

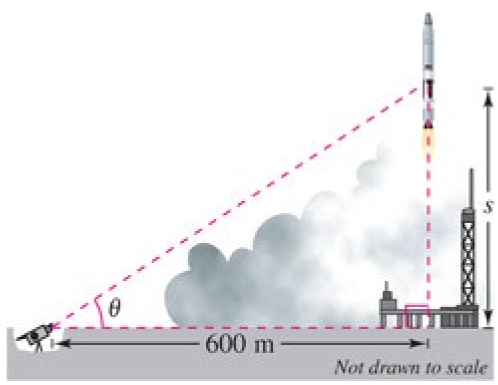
An airplane is flying on a flight path that will take it directly over a radar tracking station shown



The distance s is decreasing at a rate of 400 kilometers per hour when $s = 10$ kilometers. What is the speed of the plane?

Example 5: A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in



at 10 seconds after lift-off.

Excercise 17:

At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic meters per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 4 meters high? (Hint: The formula for the volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$.)

3.8: Newton's Method

Objectives

> “
>
> - Approximate a zero of a function using Newton's Method.

Newton's Method

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate , use these steps.

1. Make an initial estimate x_1 that is close to c . (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. When $|x_{n+1} - x_n|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration**.

Example 1: Using Newton's Method

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

```
1.4142156862745099
```

```
begin
    f(x)=x^2-2
    df(x)=2x
    x1=1
    x2=x1-f(x1)/df(x1)
    x3=x2-f(x2)/df(x2)
    x4=x3-f(x3)/df(x3)
end
```

Example 2: Using Newton's Method

Use Newton's Method to approximate the zero(s) of

$$f(x) = e^x + x$$

Continue the iterations until two successive approximations differ by less than 0.0001 .

Example 3: An Example in Which Newton's Method Fails

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

Revision

4.1: Extrema on an Interval

> “
>
> - Understand the definition of extrema of a function on an interval.
> - Understand the definition of relative extrema of a function on an open interval.
> - Find extrema on a closed interval.

Extrema of a Function

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I.

- The **minimum** and **maximum** of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval.
- The **minimum** and **maximum** of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema can occur at interior points or endpoints of an interval.
- **Extrema** that occur at the endpoints are called **endpoint extrema**.

Theorem **The Extreme Value Theorem**

If f is **continuous** on a **closed interval** $[a, b]$, then f has both a **minimum** and a **maximum** on the interval.

Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a **maximum**, then $f(c)$ is called a **relative maximum of f**, or you can say that f has a **relative maximum at $(c, f(c))$**.
2. If there is an open interval containing c on which $f(c)$ is a **minimum**, then $f(c)$ is called a **relative minimum of f**, or you can say that f has a **relative minimum at $(c, f(c))$**.

- The **plural** of relative maximum is **relative maxima**, and
- the **plural** of relative minimum is **relative minima**.
- **Relative maximum** and **relative minimum** are sometimes called **local maximum** and **local minimum**, respectively.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

Theorem **Relative Extrema Occur Only at Critical Numbers**

If f has a relative minimum or relative maximum at c, then c is a critical number of f.

Finding Extrema on a Closed Interval

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.

1. Find the **critical numbers** of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$..
4. The least of these values is the minimum. The greatest is the maximum.

Example 2: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 3x^4 - 4x^3$$

on the interval $[-1, 2]$.

Example 3: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 2x - 3x^{2/3}$$

on the interval $[-1, 3]$.

Example 4: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 2\sin x - \cos 2x$$

on the interval $[0, 2\pi]$.

Revision

4.2: Rolle's Theorem and the Mean Value Theorem

Objectives

> “
>
> - Understand and use Rolle's Theorem.
> - Understand and use the Mean Value Theorem.

Rolle's Theorem

Theorem **Rolle's Theorem**

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Example 1: **Illustrating Rolle's Theorem**

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

Example 2: **Illustrating Rolle's Theorem**

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

The Mean Value Theorem

Theorem **The Mean Value Theorem**

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example:

Consider the graph of the function $f(x) = -x^2 + 5$.

1. Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.
2. Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line.
3. Find the equation of the tangent line through c.

Revision

4.3: Increasing and Decreasing Functions and the First Derivative Test

Objectives

> “
>
> - Determine intervals on which a function is increasing or decreasing.
> - Apply the First Derivative Test to find relative extrema of a function.

Increasing and Decreasing Functions

Definitions of Increasing and Decreasing Functions

- A function \(f\) is **increasing** on an interval when, for any two numbers \(x_1\) and \(x_2\) in the interval, \(x_1 implies \(f(x_1).
- A function \(f\) is **descreasing** on an interval when, for any two numbers \(x_1\) and \(x_2\) in the interval, \(x_1 implies $f(x_1) > f(x_2)$.

Theorem **Test for Increasing and Decreasing Functions**

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is **increasing** on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is **decreasing** on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is **constant** on $[a, b]$.

Example 1: **Intervals on Which Is Increasing or Decreasing**

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

The First Derivative Test

Theorem **The First Derivative Test**

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum at** c.
2. If $f'(x)$ changes from positive to negative at c, then f has a **relative maximum at** c.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.

Example 2: Applying the First Derivative Test

Find the relative extrema of

$$f(x) = \frac{1}{2}x - \sin x$$

in the interval $(0, 2\pi)$.

Example 3: Applying the First Derivative Test

Find the relative extrema of

$$f(x) = \left(x^2 - 4\right)^{2/3}.$$

Example 4: Applying the First Derivative Test

Find the relative extrema of

$$f(x) = \frac{x^4 + 1}{x^2}.$$

4.4: Concavity and the Second Derivative Test

Objectives

> “
>
> - Determine intervals on which a function is concave upward or concave downward.
> - Find any points of inflection of the graph of a function.
> - Apply the Second Derivative Test to find relative extrema of a function.

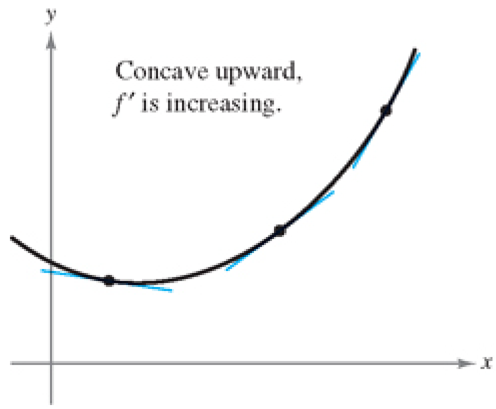
Concavity

Introduction

Definition of Concavity

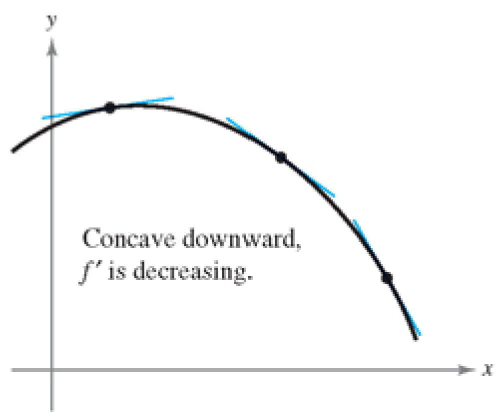
Let f be differentiable on an open interval I. The graph of f is

- ⬆ *concave upward* on I when f' **is increasing on the interval**.



(a) The graph of f lies above its tangent lines.

- ⬇ *concave downward* on I when f' **is decreasing on the interval**.



(b) The graph of f lies below its tangent lines.

Theorem **Test for Concavity**

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is **concave upward** on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is **concave downward** on I.

Example 2: **Determining Concavity**

Determine the open intervals on which the graph of

$$f(x) = \frac{x^2+1}{x^2-4}$$

is concave upward or concave downward.

Points of Inflection

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at the point $(c, f(c))$, then this point is **a point of inflection** of the graph of f when the concavity of changes from upward to downward (or downward to upward) at the point.

Theorem **Points of Inflection**

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Example 3: **Finding Points of Inflection**

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3.$$

The Second Derivative Test

Theorem **Second Derivative Test**

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a **relative minimum** at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a **relative maximum** at $(c, f(c))$.
3. If $f''(x) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the **First Derivative Test**.

Example 4: Using the Second Derivative Test

Find the relative extrema of

$$f(x) = -3x^5 + 5x^3.$$

Exercises

5.6: Indeterminate Forms and L'Hôpital's Rule

Objectives

> “
>
> - Recognize limits that produce indeterminate forms.
> - Apply L'Hôpital's Rule to evaluate a limit.

Indeterminate Forms

The forms

$$\frac{0}{0}, \frac{\infty}{\infty}$$

are called **indeterminate** because

- they do not guarantee that a limit exists,
- nor do they indicate what the limit is, if one does exist.

Remark There are other indeterminate forms such

- $0 \cdot \infty$
- 1^{∞}
- ∞^{0}
- 0^{0}
- $\infty - \infty$

L'Hôpital's Rule

Theorem **The Extended Mean Value Theorem**

If f and g are differentiable on an open interval (a,b) and continuous on $[a,b]$ such that $g'(x)\neq 0$ for any x in (a,b), then there exists a point c in (a,b) such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Theorem **L'Hôpital's Rule**

Let f and g be functions that are differentiable on an open interval (a,b) containing c, except possibly at c itself. Assume that $g'(x)\neq 0$ for all x in (a,b), except possibly at c itself. If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\frac{0}{0}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces any one of the indeterminate forms $\frac{\infty}{\infty}$, $\frac{(-\infty)}{\infty}$, $\frac{\infty}{(-\infty)}$, or $\frac{(-\infty)}{(-\infty)}$.

Remarks

L'Hôpital's Rule can also be applied to **one-sided limits**. For instance, if the limit of $f(x)/g(x)$ as x approaches c from the right produces the indeterminate form $0/0$, then

$$\lim_{x\to c^+}\frac{f(x)}{g(x)}=\lim_{x\to c^+}\frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

Example 1: Indeterminate Form 0/0

Evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

Example 2: Indeterminate Form ∞/∞

Evaluate

$$\lim_{x \to \infty} \frac{\ln x}{x}.$$

Example 3: Applying L'Hôpital's Rule More than Once

Evaluate

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}}.$$

Example 4: Indeterminate Form $0 \cdot \infty$

Evaluate

$$\lim_{x \to \infty} e^{-x}\sqrt{x}.$$

Example 5: Indeterminate Form 1^∞

Evaluate

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

Example 6: Indeterminate Form 0^0

Evaluate

$$\lim_{x \to 0^+} (\sin x)^x.$$

Example 7: Indeterminate Form ∞-∞

Evaluate

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

4.6: A Summary of Curve Sketching

Objectives

> - Analyze and sketch the graph of a function.

Analyzing the Graph of a Function

Usefull concepts in analyzing the graph of a function.

- Info from f:
  - x-intercepts and y-intercepts **(Section 1.1)**
  - Symmetry **(Section 1.1)** and Periodicity
  - Domain and range **(Section 1.3)**
  - Continuity **(Section 2.4)**
  - Vertical asymptotes **(Section 2.5)**
  - Horizontal asymptotes **(Section 4.5)**
  - Infinite limits at infinity **(Section 4.5)**
- Info from f':
  - Differentiability **(Section 3.1)**
  - Relative extrema **(Section 4.1)**
  - Increasing and decreasing functions **(Section 4.3)**
- Info from f'':
  - Concavity **(Section 4.4)**
  - Points of inflection **(Section 4.4)**

Example 1: Sketching the Graph of a Rational Function

Analyze and sketch the graph of

$$f(x) = \frac{2(x^2-9)}{x^2-4}.$$

- f info
  - Damain: all real numners except ± 2.
  - Inctercepts:
    - $x-$intercepts: $(-3,0)$ and $(3,0)$.
    - $y-$untercepts: $(0,\frac{9}{2})$.
  - Symmetry: The graph of f is symmetric with respect to $y-$axis.
  - Asymptotes:
    - **V.A**: $x=\pm 2$
    - **H.A**: $y=2$
- f' info: $f'(x) = \dfrac{20x}{(x^2-4)^2}$
  - Critical numbers: $x=0$.
  - Increasing/Decreasing:
    - f is decreasing on $(-\infty,-2)\cup(-2,0)$
    - f is increasing on $(0,2)\cup(2,\infty)$
  - Local extrema;
    - f has a local minimum at $(0,\frac{9}{2})$.
- f'' info: $f''(x) = \dfrac{-20(3x^2+4)}{(x^2-4)^3}$
  - Inflection points: none
  - Concavity:
    - **Concave upwords**: $(-2,2)$.
    - **Concave downwords**: $(-\infty,-2)\cup(2,\infty)$.

```
1 Enter cell code...
```

Remark

The graph of a rational function (having no common factors and whose denominator is of degree or greater) has a **slant asymptote** when the **degree of the numerator** *exceeds* the **degree of the denominator by exactly** .

Example 2: Sketching the Graph of a Rational Function

Analyze and sketch the graph of

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}.$$

Example 3: Sketching the Graph of a Logistic Function

Analyze and sketch the graph of

$$f(x) = \frac{1}{1 + e^{-x}}.$$

Example 4: Sketching the Graph of a Radical Function

Analyze and sketch the graph of

$$f(x) = 2x^{5/3} - 5x^{4/3}.$$

Example 5: Sketching the Graph of a Polynomial Function

Analyze and sketch the graph of

$$f(x) = x^4 - 12x^3 + 48x^2 - 64x.$$

Example 6: Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of

$$f(x) = \frac{\cos x}{1 + \sin x}.$$

Example 7: Sketching the Graph of a Inverse Trigonometric Function

Analyze and sketch the graph of

$$f(x) = (\arctan x)^2.$$

Example 8: Sketching the Graph of a Logorithmic Function

Analyze and sketch the graph of

$$f(x) = \ln(x^2 + 2x + 3).$$

4.7: Optimization Problems

Objectives

> - Solve applied minimum and maximum problems.

Applied Minimum and Maximum Problems

Guidelines for Solving Applied Minimum and Maximum Problems

1. If possible, make a **sketch** and identify all
   - **given** quantities and
   - quantities to be **determined**.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented on the Formulas from Geometry formula card.)
3. Reduce the primary equation to one having a **single independent variable**. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 4.1, 4.2, 4.3, and 4.4.

Example 1: Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square centimeters. What dimensions will produce a box with maximum volume?

```
3.0
```

```
(108-36)/24
```

Example 2: Finding Minimum Distance

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Example 3: Finding Minimum Area

A rectangular page is to contain 216 square centimeters of print. The margins at the top and bottom of the page are to be 3 centimeters, and the margins on the left and right are to be 2 centimeters. What should the dimensions of the page be so that the least amount of paper is used?

Example 4: Finding Minimum Length

Two posts, one 12 meters high and the other 28 meters high, stand 30 meters apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

Example 5: An Endpoint Maximum

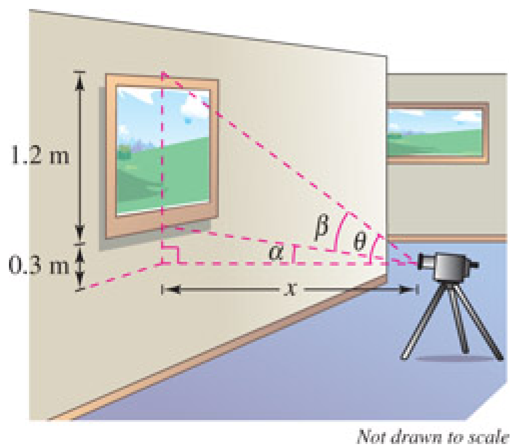
Four meters of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

```
(1.0, 0.560099, 1.27324)
```

```
begin
    A(r) = π*r^2 + (1-(π/2)*r)^2
    A(0),A(2/(4+π)),A(2/π)
end
```

Example 6: Maximizing an Angle

A photographer is taking a picture of a 12-meter painting hung in an art gallery. The camera lens is 0.3 meter below the lower edge of the painting, as shown below.



How far should the camera be from the painting to maximize the angle subtended by the camera lens?

Example 7: Finding a Maximum Revenue

The demand function for a product is modeled by

$$p = 56e^{-0.000012x} \qquad \text{Demand function}$$

where p is the price per unit (in dollars) and x is the number of units. What price will yield a maximum revenue?

4.8: Differentials

Objectives

> “
>
> - Understand the concept of a tangent line approximation.
> - Compare the value of the differential, dy, with the actual change in y, Δy.
> - Estimate a propagated error using a differential.
> - Find the differential of a function using differentiation formulas.

Tangent Line Approximations

Consider a function f that is differentiable at c. The equation of the tangent line at the point $(c, f(c))$ is

$$y = f(c) + f'(c)(x - c).$$

and is called

- **the tangent line approximation** (or **linear approximation**) of f at c.

Example 1: **Using a Tangent Line Approximation**

Find the tangent line approximation of $f(x) = 1 + \sin x$ at the point $(0, 1)$. Then use a table to compare the x-values of the linear function with those of on an open interval containing $x = 0$.

```
(0.0410757, 6)
```

```
begin
    h(x) = 1 + sin(x)
    LL(x) = 1 + x
    c=5
    h(c), LL(c)
end
```

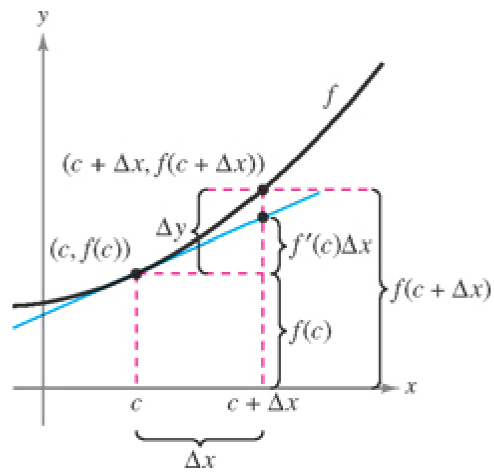
```
Enter cell code...
```

Differentials

Let

- Δx: the change in x (i.e. $\Delta x = x - c$)
- Δy: the change in y So,

$$\begin{aligned} \Delta y &= f(c+\Delta x) - f(c) && \textcolor{red}{\text{Actual change in } y} \\ &\approx f'(c)\Delta x && \textcolor{red}{\text{Approximate change in } y} \end{aligned}$$



Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of** x (denoted by dx) is any nonzero real number. The **differential of** y (denoted by dy) is

$$dy = f'(x)dx$$

remarks

In many types of applications, we use

$$\Delta y \approx dy \quad \text{or} \quad \Delta y \approx f'(x)dx$$

Example 2: **Comparing Δy and dy**

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$.

Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

```
0.020100000000000007
```

```
(1.01)^2-1
```

Error Propagation

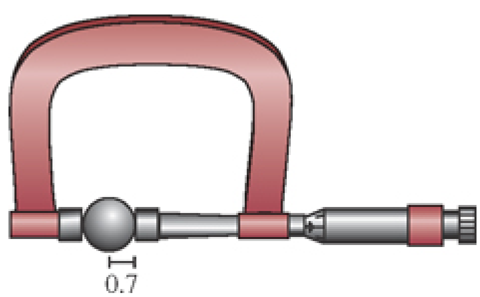
$$f(x + \Delta x) - f(x) = \Delta y$$

- Δx: Measurement error
- $f(x + \Delta x)$: Exact value
- $f(x)$: Measured value
- Δy: Propagated error

Example 3: **Estimation of Error**

The measured radius of a ball bearing is 0.7 centimeter, as shown in the figure. The measurement is correct to within 0.01 centimeter.

1. Estimate the propagated error in the volume of the ball bearing.
2. Find the relative error.
3. Find percent error



Calculating Differentials

- Each of the differentiation rules that you studied in Chapter 3 can be written in **differential form**.

Differential Formulas Let u and v be differentiable functions of x. Then

$$du = u'dx, \quad dv = v'dx$$

- **Constant multiple**: $d[cu] = cdu$
- **Sum or difference**: $d[u \pm v] = du \pm dv$
- **Product**: $d[uv] = udv + vdu$
- **Quotient**: $d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

Example 7: **Approximating Function Values**

Use differentials to approximate $\sqrt{16.5}$.

5.1: Antiderivatives and Indefinite Integration

Objectives

> - Write the general solution of a differential equation and use indefinite integral notation for antiderivatives.
> - Use basic integration rules to find antiderivatives.
> - Find a particular solution of a differential equation.

Antiderivatives

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when

$$F'(x) = f(x) \quad \text{for all} \quad x \quad \text{in} \quad I.$$

Theorem **Representation of Antiderivatives**

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form

$$G(x) = F(x) + C \quad \text{for all } x \text{ in } I,$$

where C is a constant.

Remarks

- The constant C is called the **constant of integration**.
- The **family of functions** represented by G is the **general antiderivative** of f and
- $G(x) = F(x) + C$ is **the general solution** of the differential equation

$$G'(x) = f(x)$$

- A **differential equation** in x and y is an equation that involves x, y, and derivatives of y.

Example 1: **Solving a Differential Equation**

Find the general solution of the differential equation

$$\frac{dy}{dx} = 2.$$

Remarks

When solving $\dfrac{dy}{dx} = f(x)$, we conveniently write

$$dy = f(x)dx \tag{*}$$

- **antidifferentiation (or indefinite integration)**: is the operation of finding all solutions of Equation $(*)$ and is denoted by an integral sign $\int$.
- The general solution is denoted by

$$y = \int f(x)dx = F(x) + C$$

- Here
  - $f(x)$ is called **Integrand**
  - dx means x the **Variable of Integration**
  - $F(x)$ is **An antiderivative of $f(x)$**
  - C is **Constant of Integration**
- In this course, the notation $\int f(x)dx = F(x) + C$ means that F is an antiderivative of f on an interval I.

Basic Integration Rules

$$\int F'(x)dx = F(x) + C \quad \text{Integration is the "inverse" of differentiation}$$

if $\int f(x)dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \quad \text{Differentiation is the "inverse" of integration}$$

Basic Integration Rules

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}[a^x] = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$

Example

Initial Conditions and Particular Solutions

- In many applications of integration, you are given enough information to determine a **particular solution**.
- To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition**.

Exasmples

Example 9: **Solving a Vertical Motion Problem**

A ball is thrown upward with an initial velocity of 19.6 meters per second from an initial height of 24.5 meters. [Assume the acceleration is $a(t) = -9.8$ meters per second pere second.]

1. Find the position function giving the height s as a function of the time t.
2. When does the ball hit the ground?

5.9: Hyperbolic Functions

Objectives

> “
>
> - Develop properties of hyperbolic functions.
> - Differentiate and integrate hyperbolic functions.
> - Develop properties of inverse hyperbolic functions.
> - Differentiate and integrate functions involving inverse hyperbolic functions.

Hyperbolic Functions

Definitions of the Hyperbolic Functions

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} & \qquad \operatorname{csch} x &= \frac{1}{\sinh x}, \quad x \neq 0 \\ \cosh x &= \frac{e^x + e^{-x}}{2} & \qquad \operatorname{sech} x &= \frac{1}{\cosh x}, \\ \tanh x &= \frac{\sinh x}{\cosh x} & \qquad \coth x &= \frac{\cosh x}{\sinh x}, \quad x \neq 0 \end{aligned}$$

Hyperbolic Identities

HYPERBOLIC IDENTITIES

$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x + \operatorname{sech}^2 x = 1$$
$$\coth^2 x - \operatorname{csch}^2 x = 1$$
$$\sinh^2 x = \frac{-1 + \cosh 2x}{2}$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$
$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Differentiation and Integration of Hyperbolic Functions

Theorem **Derivatives and Integrals of Hyperbolic Functions**

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\, du = \sinh u + C$$
$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\, du = \cosh u + C$$
$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$
$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$
$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$
$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example 2: Finding Relative Extrema

Find the relative extrema of

$$f(x) = (x-1)\cosh x \quad - \quad \sinh x.$$

""

""

```
rotate_xy (generic function with 2 methods)
```

```
begin
    using CommonMark, ImageIO, FileIO, ImageShow
    using PlutoUI
    using Plots, PlotThemes, LaTeXStrings, Random
    using PGFPlotsX
    using SymPy
    using HypertextLiteral: @htl, @htl_str
    using ImageTransformations
    using Colors
end
```