

MATH101

2.1: A Preview of Calculus

What is Calculus?

The Tangent Line Problem

The Area Problem

2.2: Finding Limits Graphically and Numerically

An Introduction to Limits

Limits That Fail to Exist

A Formal Definition of Limit (Redaing Only)

2.3: Evaluating Limits Analytically

Properties of Limits

A Strategy for Finding Limits

Dividing Out Technique

Rationalizing Technique

The Squeeze Theorem

2.5: Infinite Limits

Vertical Asymptotes

4.5: Limits at Infinity

Horizontal Asymptotes

Infinite Limits at Infinity

2.4: Continuity and One-Sided Limits

Continuity at a Point and on an Open Interval

One-Sided Limits and Continuity on a Closed Interval

Properties of Continuity

The Intermediate Value Theorem

3.1: The Derivative and the Tangent Line Problem

The Tangent Line Problem

The Derivative of a Function

Differentiability and Continuity

3.2: Basic Differentiation Rules and Rates of Change

The Constant Rule

The Power Rule

The Constant Multiple Rule

The Sum and Difference Rules

Derivatives of the Sine and Cosine Functions

Derivatives of Exponential Functions

Rates of Change

3.3: Product and Quotient Rules and Higher-Order Derivatives

The Product Rule

The Quotient Rule

Derivatives of Trigonometric Functions

Higher-Order Derivatives

3.4: The Chain Rule

The Chain Rule

The General Power Rule

Simplifying Derivatives

Transcendental Functions and the Chain Rule

The Derivative of the Natural Logarithmic Function

Bases Other than e

3.5: Implicit Differentiation

Logarithmic Differentiation

3.6: Derivatives of Inverse Functions

Derivative of an Inverse Function

Derivatives of Inverse Trigonometric Functions

3.7: Related Rates

Finding Related Rates

Problem Solving with Related Rates

3.8: Newton's Method

Newton's Method

4.1: Extrema on an Interval

Extrema of a Function

Relative Extrema and Critical Numbers

Finding Extrema on a Closed Interval

Syllabus



2.1: A Preview of Calculus

Objectives

“

- Understand what calculus is and how it compares with precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

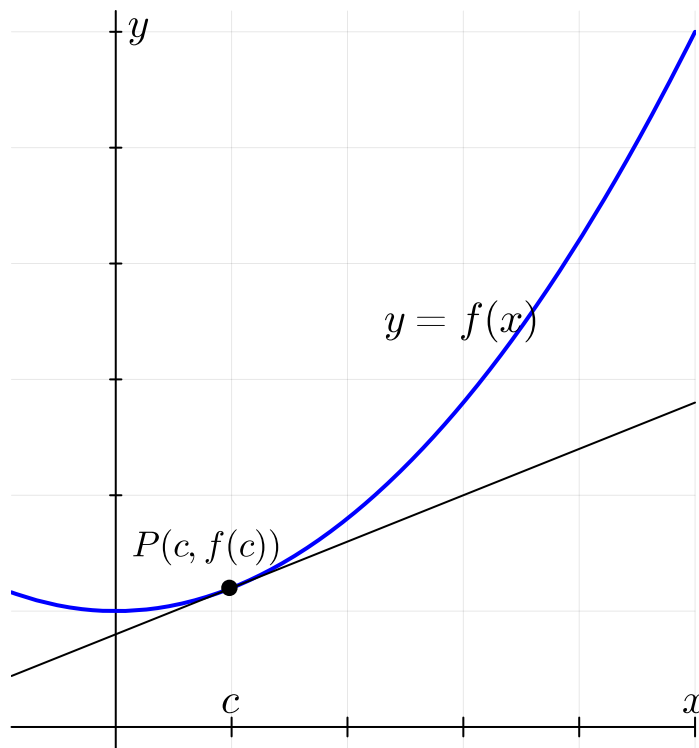
intro.

What is Calculus?

Precalculus Mathematics \Rightarrow Limit process \Rightarrow *Calculus*

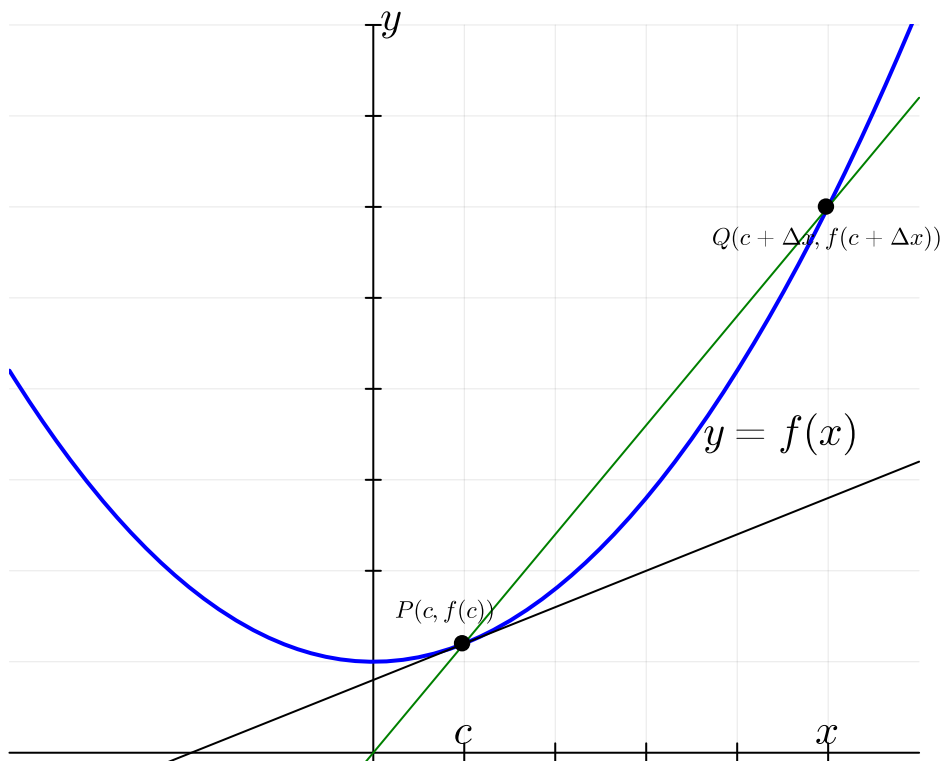
The Tangent Line Problem

What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx 

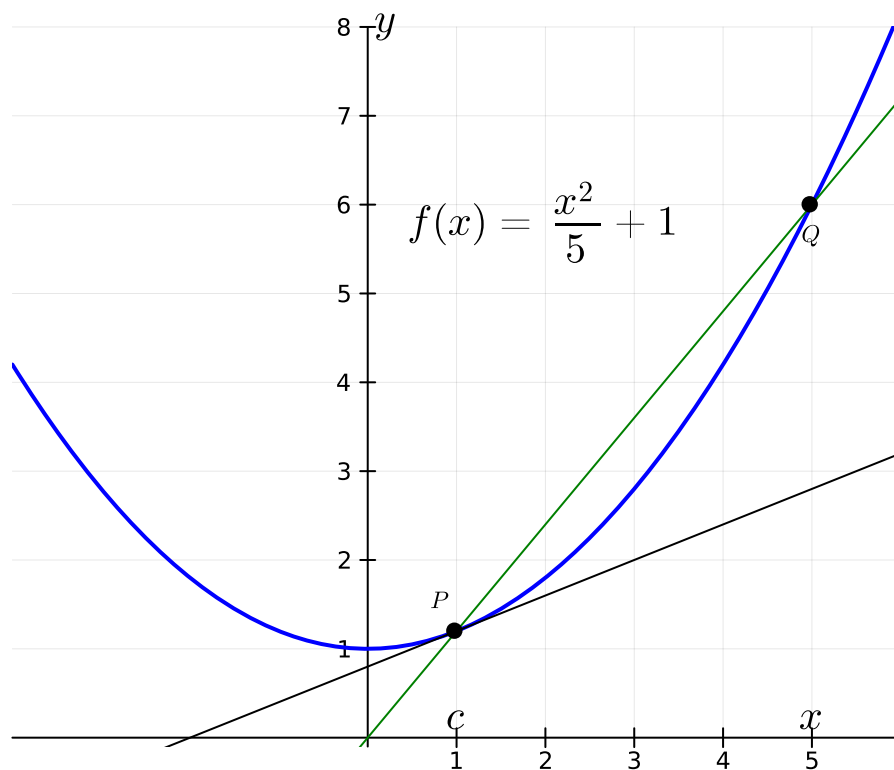
Find the equation of the secant line



$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Δx 

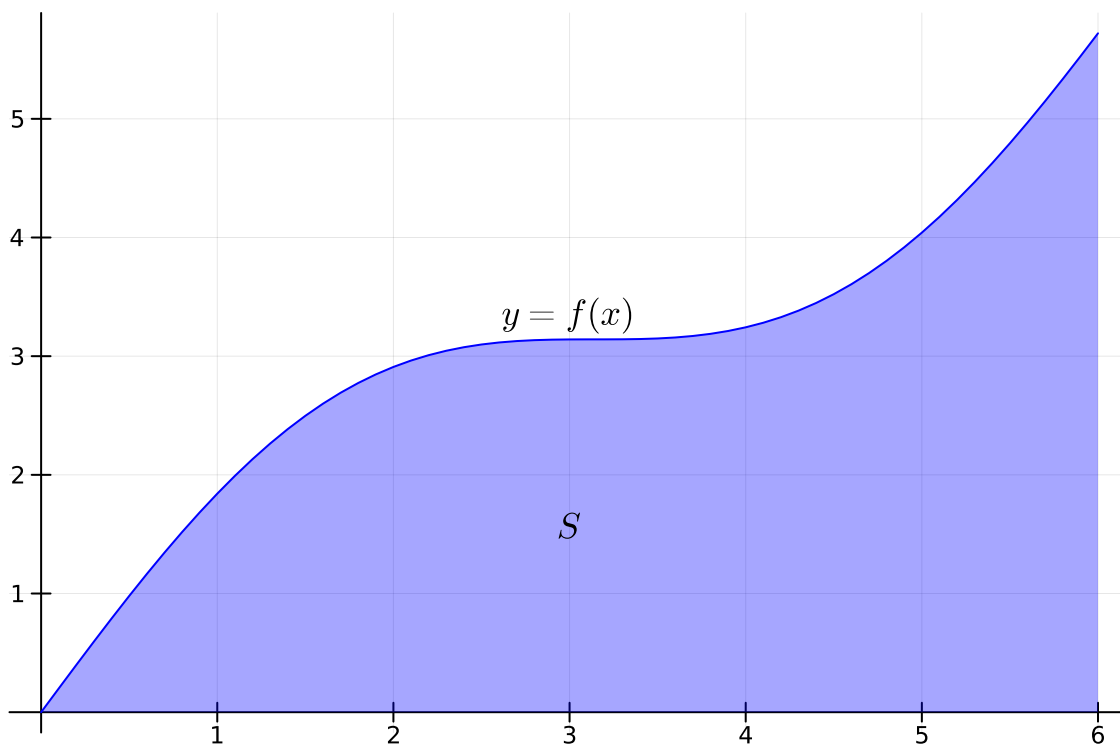
Example: Find the equation of the secant line



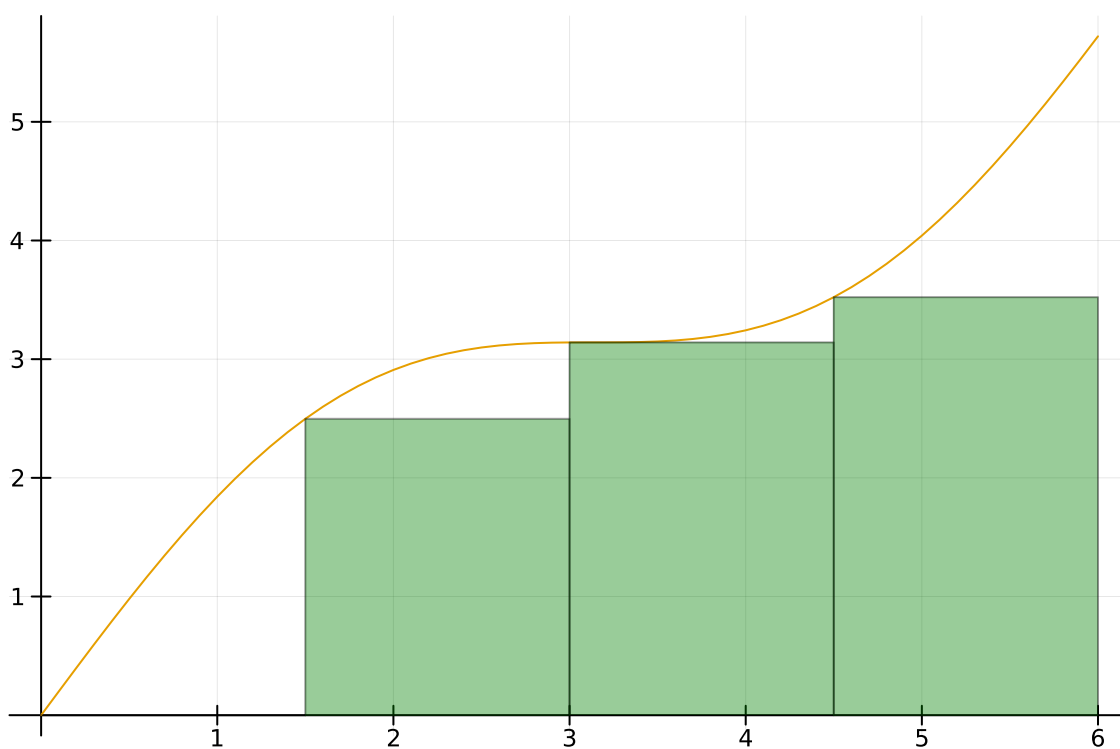
$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = a = b = method =



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

“

- Estimate a limit using a **numerical** or **graphical approach**.
- Learn different ways that a limit can fail to exist.
- <s>Study and use a formal definition of limit</s>.

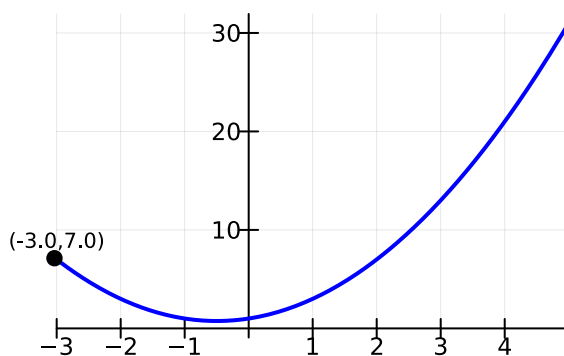
An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$  0.0

x approaches 1 from Left ▼



x approacheds 1 (from left)	$f(x)$ approaches
-3.0	7.0

Remark

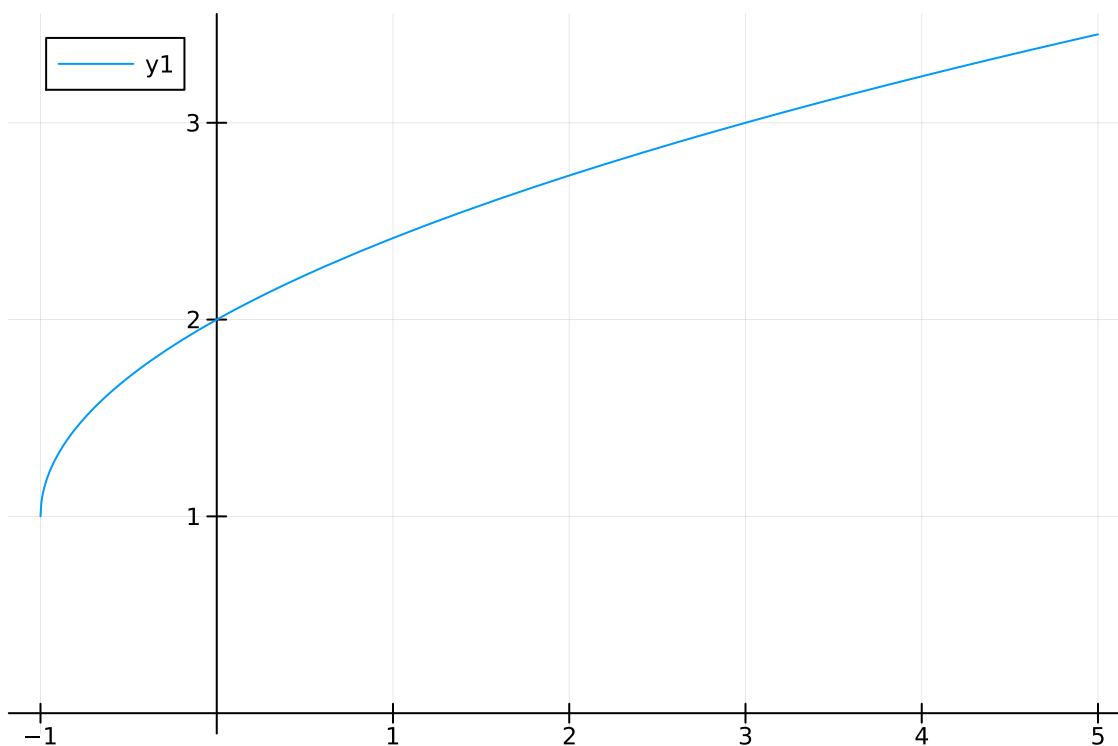
$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1: Estimating a Limit Numerically

Evaluate the function $f(x) = \frac{x}{\sqrt{x+1} - 1}$ at several x -values near 0 and use the results to estimate the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph



1.9999995001202078

```
1 begin
2   whatever(x)=x/(sqrt(x+1)-1)
3   whatever(-0.000001)
4 end
```

Example 2: Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark Problem solving

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: MATLAB, python, Julia)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist

Example 3: Different Right and Left Behavior

Show that the limit $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

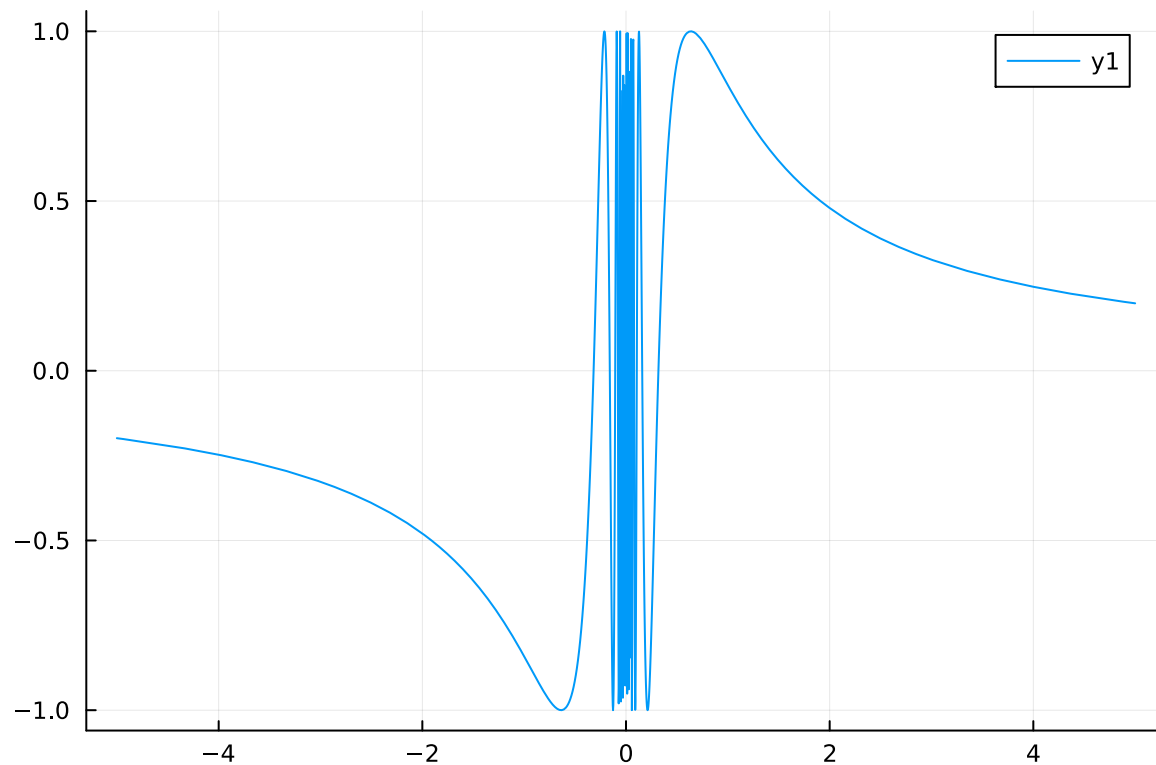
Example 4: Unbounded Behavior

Discuss the existence of the limit $\lim_{x \rightarrow 0} \frac{1}{x^2}$

Example 5: Oscillating Behavior

Discuss the existence of the limit $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

9.999999999999998e9



```
1 plot(x->sin(1/x))
```

A Formal Definition of Limit (Redaig Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \rightarrow c} f(x) = L$$

implies two statements—the limit exists and the limit is L .

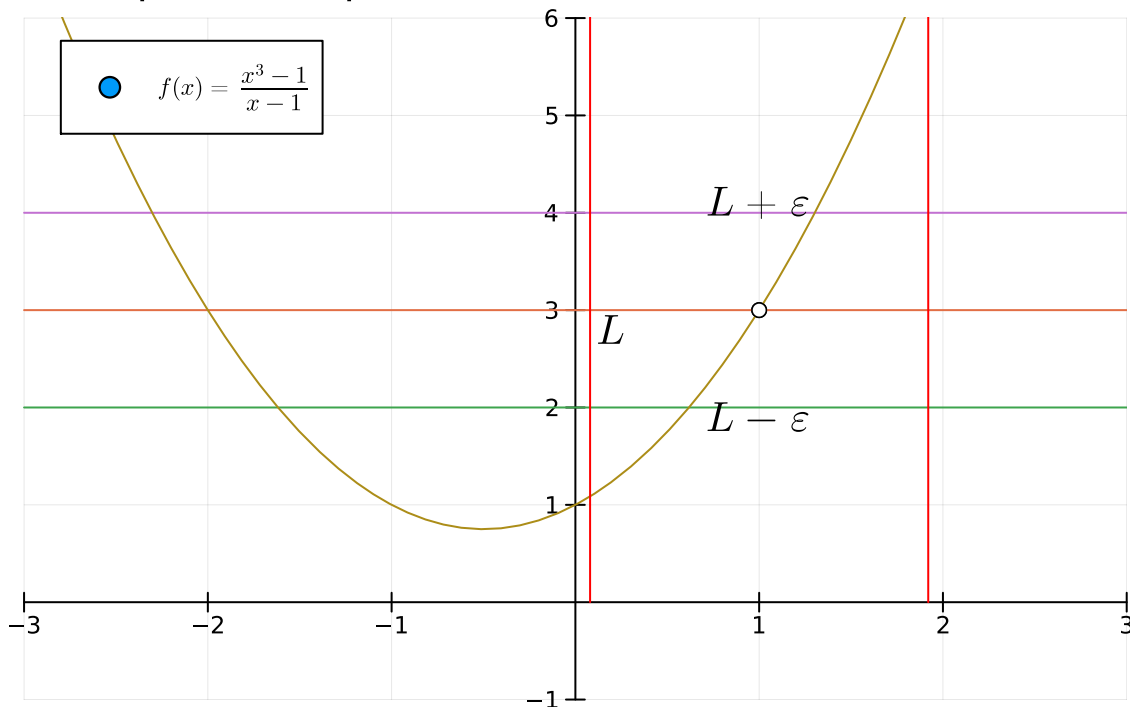
Example:

Prove that

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

$\epsilon =$  1.0 $\delta =$  0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

“

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$

Theorem Properties of Limits

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \rightarrow c} f(x) = L, \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. **Scalar multiple** $\lim_{x \rightarrow c} [bf(x)] = bL$
2. **Sum or difference** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. **Product** $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. **Quotient** $\lim_{x \rightarrow c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}, \quad K \neq 0$
5. **Power** $\lim_{x \rightarrow c} [f(x)]^n = L^n$

Example 2: The Limit of a Polynomial

Find $\lim_{x \rightarrow 2} (4x^2 + 3)$.

Theorem Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3: The Limit of a Rational Function

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x + 2}{x + 1}.$$

Theorem The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem The Limit of a Composite Function

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow c} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

Theorem Limits of Transcendental Functions

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \rightarrow c} \sin(x) = \sin(c)$
2. $\lim_{x \rightarrow c} \cos(x) = \cos(c)$
3. $\lim_{x \rightarrow c} \tan(x) = \tan(c)$
4. $\lim_{x \rightarrow c} \cot(x) = \cot(c)$
5. $\lim_{x \rightarrow c} \sec(x) = \sec(c)$
6. $\lim_{x \rightarrow c} \csc(x) = \csc(c)$
7. $\lim_{x \rightarrow c} a^x = a^c, \quad a > 0$
8. $\lim_{x \rightarrow c} \ln(x) = \ln(c)$



A Strategy for Finding Limits

Theorem Functions That Agree at All but One Point

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x).$$

Remarks A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c .

Dividing Out Technique

Example 7: Dividing Out Technique

Find the limit $\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8: Rationalizing Technique

Find the limit $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$.

The Squeeze Theorem

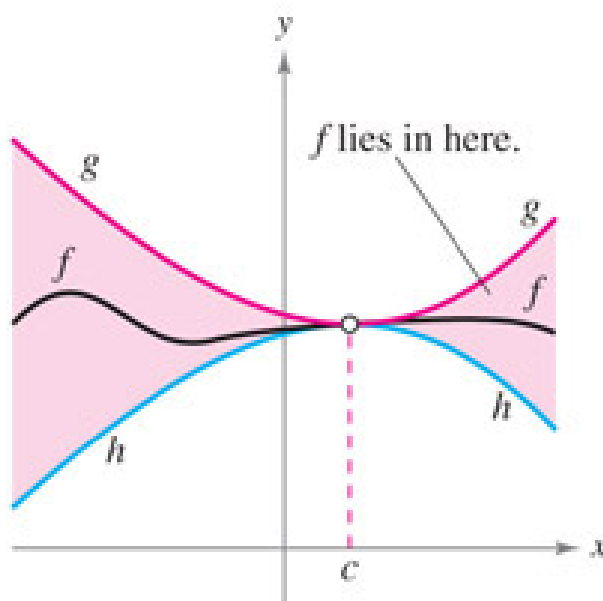
Theorem The Squeeze Theorem

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then $\lim_{x \rightarrow c} f(x)$ exists and equal to L .

$$h(x) \leq f(x) \leq g(x)$$



Theorem Three Special Limits

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$
3. $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e.$

Example 9: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\tan x}{x}$.

Example 10: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$.

Exercises



2.5: Infinite Limits

Objectives

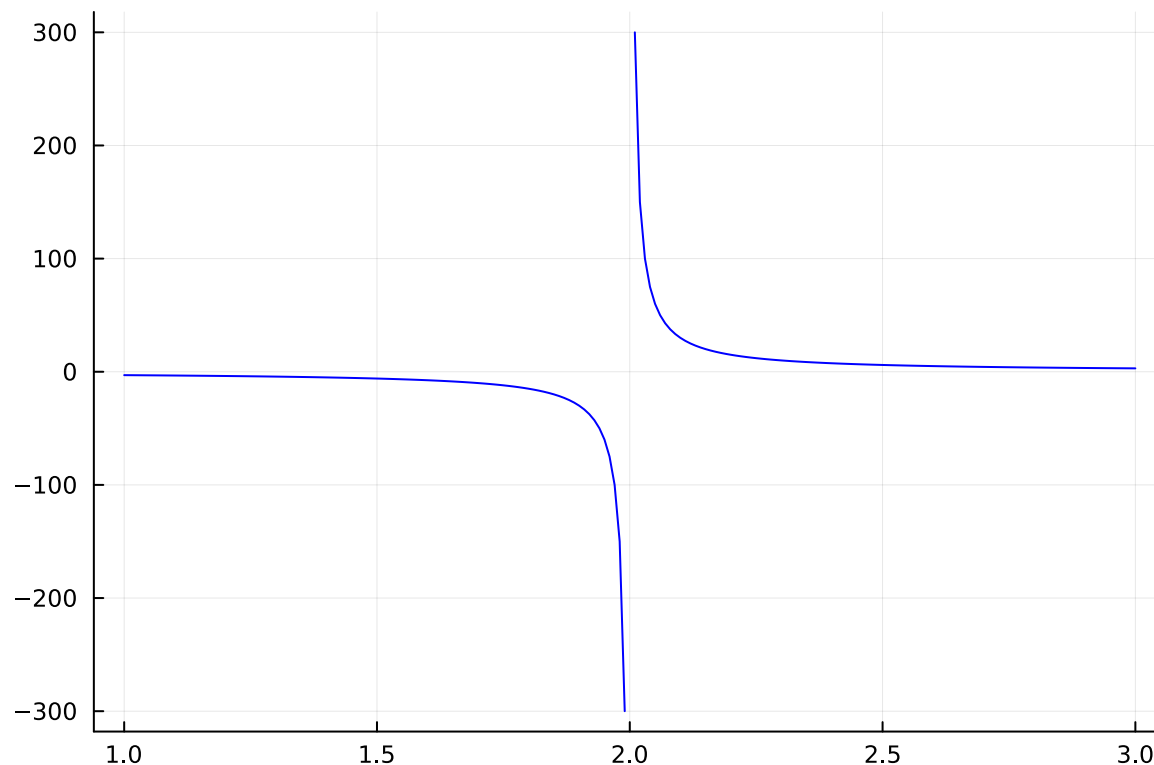
“

- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Example: Infinite Limit

Consider

$$f(x) = \frac{3}{x-2}$$



```
1 plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c .

Theorem Vertical Asymptotes

Let f and g be continuous on an open interval containing c . If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

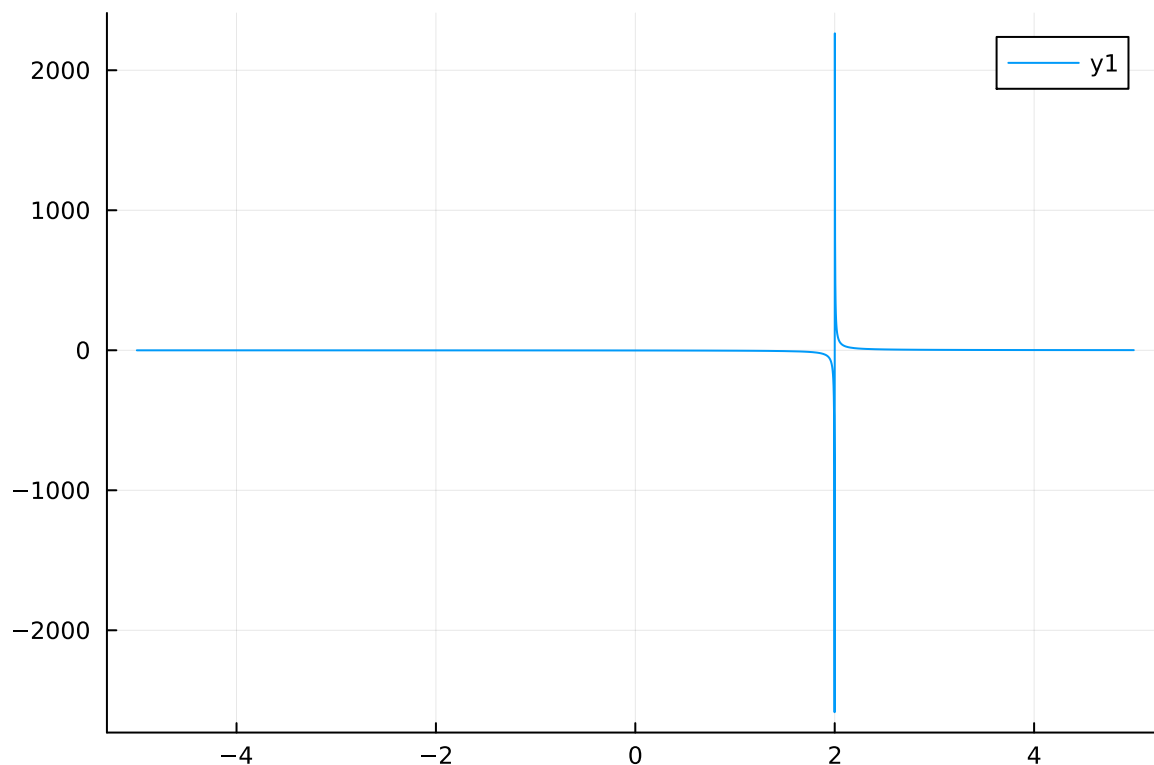
has a vertical asymptote at c .

Example 2: Finding Vertical Asymptotes

1. $h(x) = \frac{1}{2(x+1)}.$

2. $h(x) = \frac{x^2 + 1}{x^2 - 1}.$

3. $h(x) = \cot x = \frac{\cos x}{\sin x}.$



```
1 plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- desmos.com
- geogebra.org

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x \rightarrow 1^-} \frac{x^3 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^3 - 3x}{x - 1}$$

Theorem Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L$$

1. **Sum or difference:** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$

2. **Product:**

$$\lim_{x \rightarrow c} [f(x)g(x)] = \infty, \quad L > 0$$

$$\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, \quad L < 0$$

3. **Quotient:** $\lim_{x \rightarrow c} \left[\frac{g(x)}{f(x)} \right] = 0$

Remark

2. is **not true** if $\lim_{x \rightarrow c} g(x) = 0$

Exercises



4.5: Limits at Infinity

Objectives

“

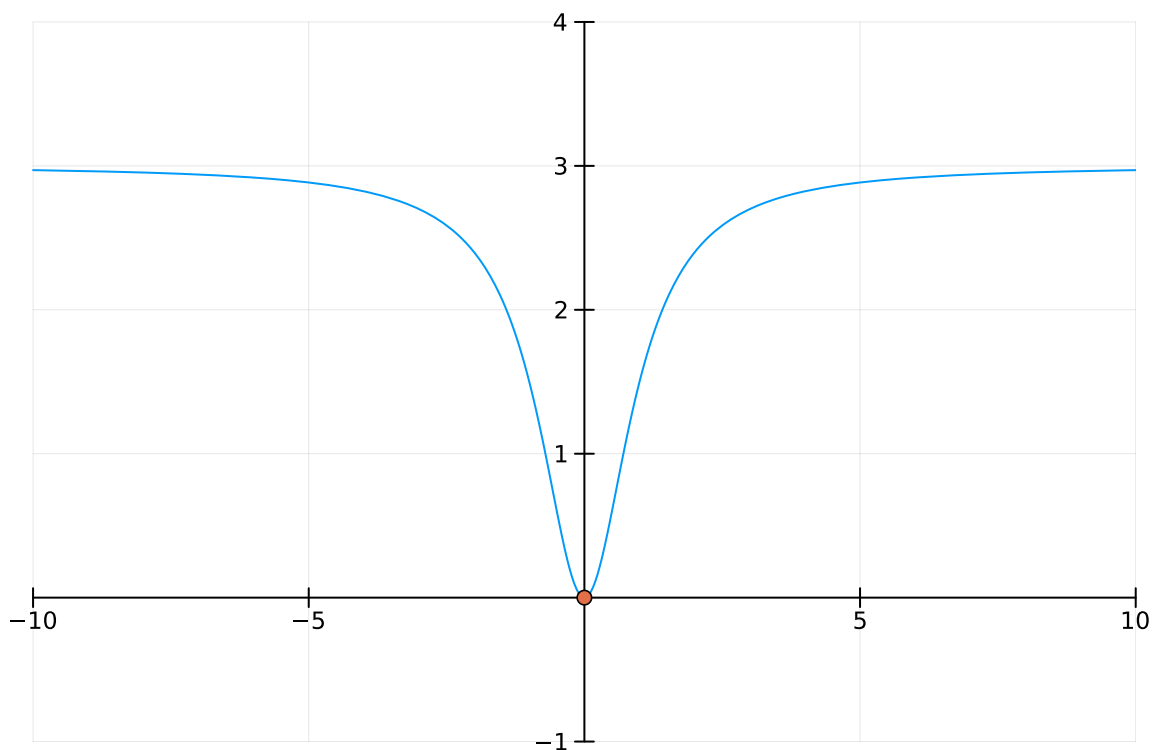
- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2 + 1}$$

$x =$

$f(x) = 0.0$



we write

$$\lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + 1} = 3, \quad \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2 + 1} = 3$$

Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = L$$

Remarks

- Limits at infinity have many of the same properties of limits discussed in Section 2.3.
- For example, if $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} g(x)$ both exist, then
 - $\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
 - $\lim_{x \rightarrow \infty} [f(x)g(x)] = \left[\lim_{x \rightarrow \infty} f(x) \right] \left[\lim_{x \rightarrow \infty} g(x) \right]$
- Similar properties hold for limits at $-\infty$.

Theorem Limits at Infinity

1. If r is a positive rational number and c is any real number, then

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim_{x \rightarrow -\infty} e^x = 0$ and $\lim_{x \rightarrow \infty} e^{-x} = 0$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

$$h(x) = \frac{p(x)}{q(x)}$$

1. $\deg p < \deg q$, then the limit is 0.
2. $\deg p = \deg q$, then the **limit** of the rational function is the **ratio** of the **leading coefficients**.
3. $\deg p > \deg q$, then the **limit** of the rational function **does not exist**.

Examples



```
1 # begin
2 #   xx=symbols("xx",real=true)
3 #   limit(xx*sin(1/xx),xx,0)
4 # end
```

Infinite Limits at Infinity

Remark

Determining whether a function has an infinite limit at infinity is useful in analyzing the “**end behavior**” of its graph. You will see examples of this in Section 4.6 on curve sketching.

2.4: Continuity and One-Sided Limits

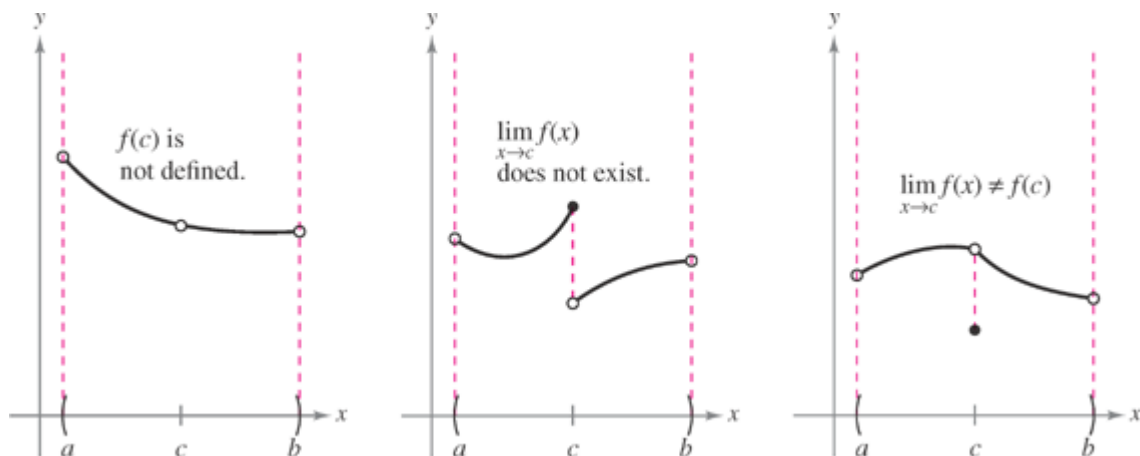
Objectives

“

- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

The graph of f is not continuous at $x = c$



In Figure __above__, it appears that continuity at $x=c$ can be __destroyed__ by any one of __three conditions__.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \rightarrow c} f(x)$ exists.
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Continuity on an Open Interval

- A function f is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval.
- A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Remarks

- If a function f is defined on an open interval I (except possibly at c), and f is not continuous at c , then f is said to have a **discontinuity** at c .
- Discontinuities fall into two categories:
 - **removable**: A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$.
 - **nonremovable**: there is no way to define $f(c)$ so as to make the function continuous at $x = c$.

Example 1:

Discuss the continuity of each function

a. $f(x) = \frac{1}{x}$

b. $g(x) = \frac{x^2 - 1}{x - 1}$

c. $h(x) = \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases}$

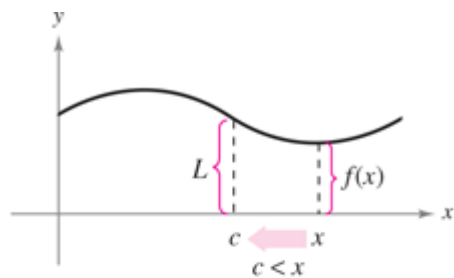
d. $y = \sin x$

Examples

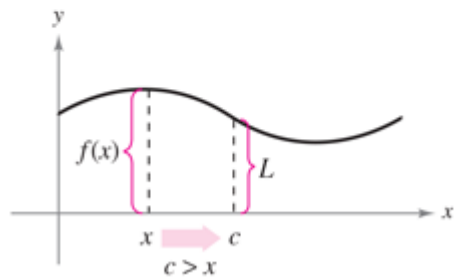


One-Sided Limits and Continuity on a Closed Interval

(a) Limit from right $\lim_{x \rightarrow c^+} f(x) = L$



(a) Limit as x approaches c from the right.



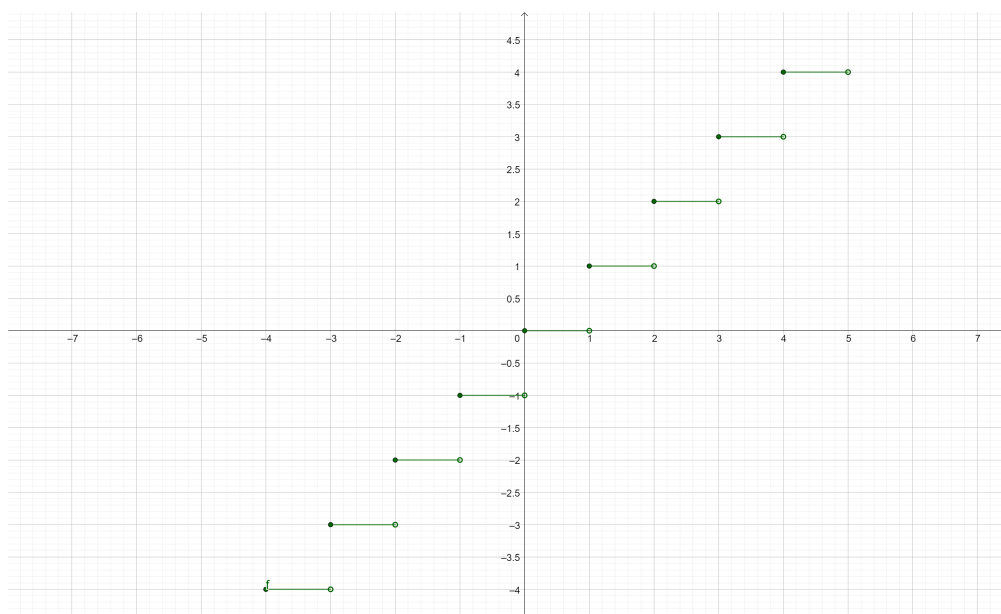
(b) Limit as x approaches c from the left.

(b) Limit from left $\lim_{x \rightarrow c^-} f(x) = L$

STEP FUNCTIONS

(greatest integer function)

$[x] = \text{greatest integer } n \text{ such that } n \leq x.$



Theorem The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is if and only if

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and

$$\lim_{x \rightarrow b^-} f(x) = f(b).$$

Example 4: Continuity on a Closed Interval

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}$$

Properties of Continuity

Theorem Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c .

1. **Scalar multiple:** bf
2. **Sum or difference:** $f \pm g$
3. **Product:** fg
4. **Quotient:** $\frac{f}{g}$, $g(c) \neq 0$,

Remarks

1. **Polynomials** are continuous at every point in their domains.
2. **Rational functions** are continuous at every point in their domains.
3. **Radical functions** are continuous at every point in their domains.
4. **Trigonometric functions** are continuous at every point in their domains.
5. **Exponential and logarithmic functions** are continuous at every point in their domains.

Theorem Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$ then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c .

Remark

$$\lim_{x \rightarrow c} f(g(x)) = f(g(c))$$

provided f and g satisfy the conditions of the theorem.

Example 7: Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$

b. $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Exercises



The Intermediate Value Theorem

Theorem Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Example 8: An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

3.1: The Derivative and the Tangent Line Problem

Objectives

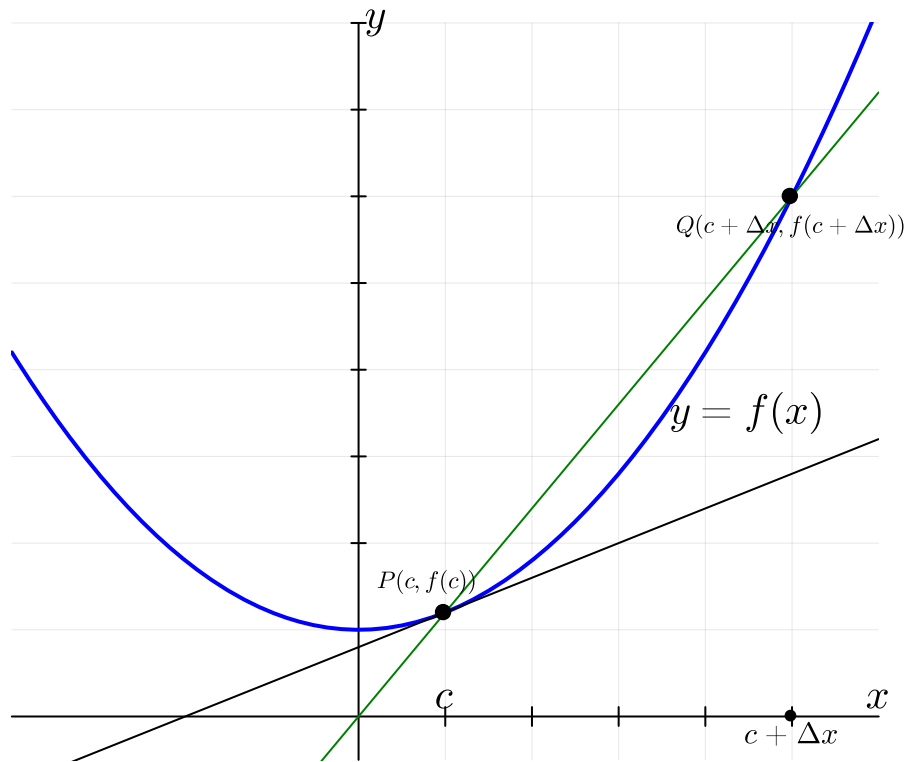
“

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Δx  4.0

Find the equation of the secant line



Slope of secant line

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Definition of Tangent Line with Slope

If f is defined on an open interval containing c , and if the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Remark

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$** .

Example 1: The Slope of the Graph of a Linear Function

Find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$.

Example 2: Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$.

Remarks

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line.
- For vertical tangent lines, you can use the **following definition**. If f is continuous at c and

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the **vertical line** $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f .

The Derivative of a Function

Definition Derivative of a Function

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x .

Remarks

- The notation $f'(x)$ is read as “ f prime of x .”
- $f'(x)$ is a **function** that gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.
- The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.
- The process of finding the derivative of a function is called **differentiation**.
- A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval** (a, b) when it is differentiable at every point in the interval.

Notation

$$y = f(x)$$

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}[f(x)]$
- $D_x[y]$

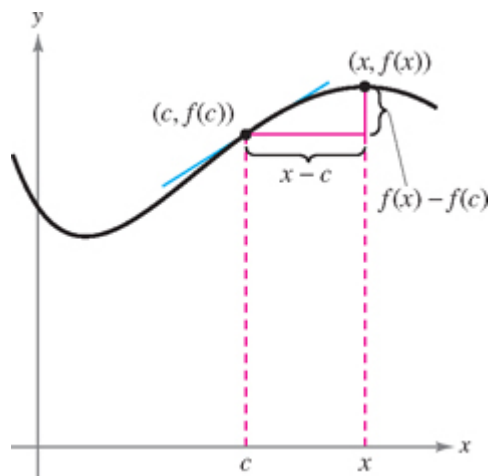
$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Examples 3,4,5: Finding the Derivative by the Limit Process

Find the derivative of

- $f(x) = x^3 + 2x$
- $f(x) = \sqrt{x}$
- $y = \frac{2}{t}$ with respect to t .

Differentiability and Continuity



Alternative form of derivative

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

Remarks

derivative from the left

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$$

derivative from the right

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

Example:

$$f(x) = \llbracket x \rrbracket$$

Example 6: A Graph with a Sharp Turn

$$f(x) = |x - 2|$$

Example 7: A Graph with a Vertical Tangent Line

$$f(x) = x^{\frac{1}{3}}$$

Theorem Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

remarks

The relationship between continuity and differentiability is summarized below.

- If a function f is differentiable at $x = c$, then it is continuous at $x = c$. So, **differentiability** implies (\Rightarrow) **continuity**.
- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, **continuity does not imply differentiability**.

Exercises



3.2: Basic Differentiation Rules and Rates of Change

Objectives

“

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Find the derivatives of exponential functions.
- Use derivatives to find rates of change.

The Constant Rule

Theorem The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

The Power Rule

Theorem The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at 0 , n must be a number such that x^{n-1} is defined on an interval containing 0 .

The Constant Multiple Rule

Theorem The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

The Sum and Difference Rules

Theorem The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g .

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{Difference Rule}$$

Derivatives of the Sine and Cosine Functions

Theorem Derivatives of the Sine and Cosine Functions

$$\frac{d}{dx}[\sin(x)] = \cos x, \quad \frac{d}{dx}[\cos(x)] = -\sin x$$

Derivatives of Exponential Functions

Theorem Derivative of the Natural Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$



$$2 \sin(t)$$

```
1 begin
2     t = symbols("t", real=true)
3     g(t)=-2cos(t)-5
4     plot(x->g(x))
5     diff(g(t),t)
6 end
```

Rates of Change

- The derivative can be used to determine the **rate of change** of one variable with respect to another.
- Applications involving rates of change, sometimes referred to as **instantaneous rates of change**, occur in a wide variety of fields.
- A common use for rate of change is to describe **the motion of an object moving in a straight line**. (+ direction and -direction)
- The function **s** that gives **the position (relative to the origin)** of an object as a **function of time t** is called a **position function**. If, over a period of time Δt , the object changes its position by the amount Δs , then, by the familiar formula

$$\Delta s = s(t + \Delta t) - s(t)$$

- then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}.$$

-the average velocity is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} \quad \text{Average Velocity.}$$

- In general, if **$s = s(t)$** is the position function for an object moving along a straight line, then the velocity of the object at time **t** is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \quad \text{Velocity function.}$$

Example:

If a ball is thrown into the air with a velocity of **4m/s** , its height (in meters (m)) **t** seconds later is given by

$$y = 4t - 4.9t^2.$$

1. Find the average velocity for the time period from **$t = 1$** to **$t = 3$** .
2. Find the instantaneous rate of change at **$t = 2$** .

Example 11: Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is **9.8** meters above the water. The initial velocity of the diver is **4.9** meters per second. When does the diver hit the water? What is the diver's velocity at impact?

3.3: Product and Quotient Rules and Higher-Order Derivatives

Objectives

“

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

Theorem The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the **first** function **times** the **derivative of the second**, **plus** the **second** function times the **derivative of the first**.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Example:

Find the derivative of $f(x) = xe^x$.

The Quotient Rule

Theorem The Quotient Rule

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the **denominator times the derivative of the numerator minus the numerator times the derivative of the denominator**, all **divided by the square of the denominator**.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$$

Example:

Find an equation of the tangent line to the graph of $f(x) = \frac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Derivatives of Trigonometric Functions

Theorem Derivatives of Trigonometric Functions

$$\begin{array}{lcl} \frac{d}{dx}(\tan x) & = & \sec^2 x \\ \frac{d}{dx}(\sec x) & = & \sec x \tan x \end{array} \quad \left| \quad \begin{array}{lcl} \frac{d}{dx}(\cot x) & = & -\csc^2 x \\ \frac{d}{dx}(\csc x) & = & -\csc x \cot x \end{array} \right.$$

Example:

Differentiate

$$y = \frac{1 - \cos x}{\sin x}$$

Higher-Order Derivatives

Remarks

Rates of changes

$$\begin{array}{rcll} & & s(t) & \text{Position function} \\ v(t) & = & s'(t) & \text{Velocity function} \\ a(t) & = & v'(t) = s''(t) & \text{Acceleration function} \end{array}$$

Higher Derivatives

$$\text{First derivative: } y', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad D_x[y]$$

$$\text{Second derivative: } y'', \quad f''(x), \quad \frac{d^2y}{dx^2}, \quad \frac{d^2}{dx^2}[f(x)], \quad D_x^2[y]$$

$$\text{Third derivative: } y''', \quad f'''(x), \quad \frac{d^3y}{dx^3}, \quad \frac{d^3}{dx^3}[f(x)], \quad D_x^3[y]$$

$$\text{Fourth derivative: } y^{(4)}, \quad f^{(4)}(x), \quad \frac{d^4y}{dx^4}, \quad \frac{d^4}{dx^4}[f(x)], \quad D_x^4[y]$$

⋮

$$\text{nth derivative: } y^{(n)}, \quad f^{(n)}(x), \quad \frac{d^ny}{dx^n}, \quad \frac{d^n}{dx^n}[f(x)], \quad D_x^n[y]$$

Exercises



3.4: The Chain Rule

Objectives

“

- Find the derivative of a composite function using the Chain Rule.
- Find the derivative of a function using the General Power Rule.
- Simplify the derivative of a function using algebra.
- Find the derivative of a transcendental function using the Chain Rule.
- Find the derivative of a function involving the natural logarithmic function.
- Define and differentiate exponential functions that have bases other than .

The Chain Rule

Theorem The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Remark

■: Outer function ■: Inner function

If $y = f(g(x)) = f(u)$, then

$$y' = \frac{dy}{dx} = f'(g(x))g'(x)$$

or, equivalently

$$y' = \frac{dy}{dx} = f'(u) \frac{du}{dx}$$

Example: Using the Chain Rule

Find $\frac{dy}{dx}$ for

$$y = (x^2 + 1)^3.$$

Examples



The General Power Rule

Theorem The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \frac{dy}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u(x)]^n = n[u]^{n-1}u'$$

Example:

Find the derivative of $y = (3x - 2x^2)^3$.

Simplifying Derivatives

Example:

Find the derivative of

1. $f(x) = x^2\sqrt{1-x^2}$.
2. $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$.

Transcendental Functions and the Chain Rule

$$\begin{aligned}\frac{d}{dx}[\sin u] &= (\cos u)u' & , & & \frac{d}{dx}[\cos u] &= -(\sin u)u' \\ \frac{d}{dx}[\tan u] &= (\sec^2 u)u' & , & & \frac{d}{dx}[\cot u] &= -(\csc^2 u)u' \\ \frac{d}{dx}[\sec u] &= (\sec u \tan u)u' & , & & \frac{d}{dx}[\csc u] &= -(\csc u \cot u)u' \\ \frac{d}{dx}[e^u] &= (e^u)u'\end{aligned}$$

Examples



The Derivative of the Natural Logarithmic Function

Theorem Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x .

$$\begin{aligned} 1. \frac{d}{dx} [\ln x] &= \frac{1}{x}, x > 0 \\ 2. \frac{d}{dx} [\ln u] &= \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, x > 0 \end{aligned}$$

Theorem Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx} [\ln |u|] = \frac{u'}{u}$$

Examples:

Find y' for

$$\begin{aligned} 1. y &= \ln \sqrt{x+1} \\ 2. y &= \left(\frac{3x-1}{x^2+3} \right)^2 \\ 3. y &= \ln \left[\frac{x(x^2+1)^2}{\sqrt{2x^3+1}} \right] \end{aligned}$$

Bases Other than e

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{x \ln a}$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number ($a \neq 1$) and x is any **positive** real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined by

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Theorem Derivatives for Bases Other than e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x .

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$



3.5: Implicit Differentiation

objectives

“

- Distinguish between functions written in implicit form and explicit form.
- Use implicit differentiation to find the derivative of a function.
- Find derivatives of functions using logarithmic differentiation.

Example: Implicit Differentiation

Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$.

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x .
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$.

Example: Finding the Slope of a Graph Implicitly

Determine the slope of the graph of

$$3(x^2 + y^2)^2 = 100xy$$

at the point $(3, 1)$.

Example: Determining a Differentiable Function

Find $\frac{dy}{dx}$ implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x .

Example: Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$.

Definition Normal Line

The normal line at a point is the line **perpendicular** to the tangent line at the point.

Example (exercise 63): Normal Lines

Find the equations for the tangent line and normal line to the circle

$$x^2 + y^2 = 25$$

at the points $(4, 3)$ and $(-3, 4)$.

Logarithmic Differentiation

Example: Logarithmic Differentiation

Find the derivative of

1. $y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$
2. $y = x^{2x}, \quad x > 0.$
3. $y = x^\pi.$

Definition Orthogonal Trajectories

Two graphs (curves) are **orthogonal** if at their point(s) of intersection, their **tangent lines** are **perpendicular** to each other.

Excercise 81:

Are the following curves **orthogonal**?

$$2x^2 + y^2 = 6, \quad y^2 = 4x$$



3.6: Derivatives of Inverse Functions

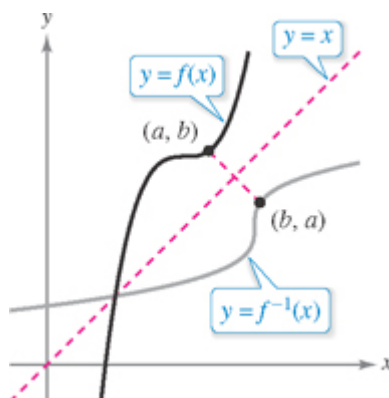
Objectives

“

- Find the derivative of an inverse function.
- Differentiate an inverse trigonometric function.

Derivative of an Inverse Function

The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.



Theorem Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I . If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Theorem The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I . If f has an inverse function g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

Example: Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$, $x \geq 0$. Find

1. $f^{-1}(x)$
2. Find the slopes of the graphs of f and f^{-1} at the points $(2, 4)$ and $(4, 2)$ respectively

Derivatives of Inverse Trigonometric Functions

Theorem Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x .

$$\begin{aligned}\frac{d}{dx} [\sin^{-1} u] &= \frac{u'}{\sqrt{1-u^2}} & , & & \frac{d}{dx} [\cos^{-1} u] &= \frac{-u'}{\sqrt{1-u^2}} \\ \frac{d}{dx} [\tan^{-1} u] &= \frac{u'}{1+u^2} & , & & \frac{d}{dx} [\cot^{-1} u] &= \frac{-u'}{1+u^2} \\ \frac{d}{dx} [\sec^{-1} u] &= \frac{u'}{|u|\sqrt{u^2-1}} & , & & \frac{d}{dx} [\csc^{-1} u] &= \frac{-u'}{|u|\sqrt{u^2-1}}\end{aligned}$$

Examples



3.7: Related Rates

Objectives

“

- Find a related rate.
- Use related rates to solve real-life problems.

Finding Related Rates

Example 1: Two Rates That Are Related

The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find $\frac{dy}{dt}$ when $x = 1$, given that $\frac{dx}{dt} = 2$ when $x = 1$.

Problem Solving with Related Rates

In **Example 1**

- **Equation:** $y = x^2 + 3$.
- **Given:** $\frac{dx}{dt} = 2$ when $x = 1$.
- **Find:** $\frac{dy}{dt}$ when $x = 1$.

Guidelines for Solving Related-Rate Problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation with respect to time t .
4. After completing **Step 3**, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Example 2: Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in



Russ Bishop/Alamy Stock Photo

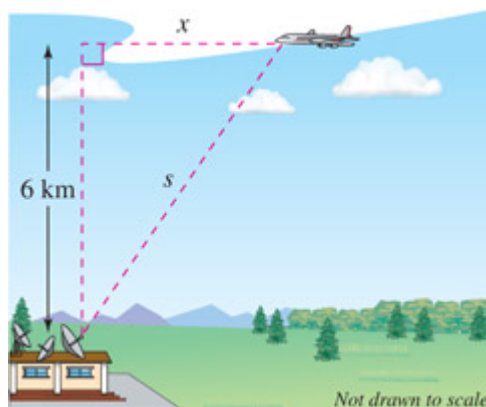
The radius r of the outer ripple is increasing at a constant rate of **0.5** meter per second. When the radius is **2** meters, at what rate is the total area A of the disturbed water changing?

Example 3: An Inflating Balloon

Air is being pumped into a spherical balloon at a rate of **1.5** cubic meters per minute. Find the rate of change of the radius when the radius is **2** meters

Example 4: The Speed of an Airplane Tracked by Radar

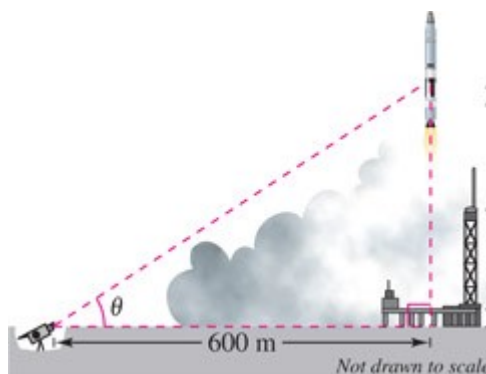
An airplane is flying on a flight path that will take it directly over a radar tracking station shown



The distance s is decreasing at a rate of 400 kilometers per hour when $s = 10$ kilometers. What is the speed of the plane?

Example 5: A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in



at 10 seconds after lift-off.

Excercise 17:

At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic meters per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 4 meters high? (Hint: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

3.8: Newton's Method

Objectives

“

- Approximate a zero of a function using Newton's Method.

Newton's Method

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c . Then, to approximate , use these steps.

1. Make an initial estimate x_1 that is close to c . (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. When $|x_{n+1} - x_n|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration**.

Example 1: Using Newton's Method

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

1.4142156862745099

```
1 begin
2   f(x)=x^2-2
3   df(x)=2x
4   x1=1
5   x2=x1-f(x1)/df(x1)
6   x3=x2-f(x2)/df(x2)
7   x4=x3-f(x3)/df(x3)
8 end
```

1 Enter cell code...

Example 2: Using Newton's Method

Use Newton's Method to approximate the zero(s) of

$$f(x) = e^x + x$$

Continue the iterations until two successive approximations differ by less than **0.0001**.

Example 3: An Example in Which Newton's Method Fails

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

4.1: Extrema on an Interval

“

- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

Definition of Extrema

Let f be defined on an interval I containing c .

1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I .
 2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I .
- The **minimum** and **maximum** of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval.
 - The **minimum** and **maximum** of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema can occur at interior points or endpoints of an interval.
 - **Extrema** that occur at the endpoints are called **endpoint extrema**.

Theorem The Extreme Value Theorem

If f is **continuous** on a **closed interval** $[a, b]$, then f has both a **minimum** and a **maximum** on the interval.

Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a **maximum**, then $f(c)$ is called a **relative maximum of f** , or you can say that f has a **relative maximum at $(c, f(c))$** .
 2. If there is an open interval containing c on which $f(c)$ is a **minimum**, then $f(c)$ is called a **relative minimum of f** , or you can say that f has a **relative minimum at $(c, f(c))$** .
- The **plural** of relative maximum is **relative maxima**, and
 - the **plural** of relative minimum is **relative minima**.
 - **Relative maximum** and **relative minimum** are sometimes called **local maximum** and **local minimum**, respectively.

Definition of a Critical Number

Let f be defined at c . If $f'(c) = 0$ or if f is not differentiable at c , then c is a **critical number** of f .

Theorem Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at c , then c is a critical number of f .

Finding Extrema on a Closed Interval

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.

1. Find the **critical numbers** of f in (a, b) .
2. Evaluate f at each critical number in (a, b) .
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Example 2: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 3x^4 - 4x^3$$

on the interval $[-1, 2]$.

Example 3: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 2x - 3x^{2/3}$$

on the interval $[-1, 3]$.

Example 4: Finding Extrema on a Closed Interval

Find the extrema of

$$f(x) = 2 \sin x - \cos 2x$$

on the interval $[0, 2\pi]$.

rotate_xy (generic function with 2 methods)

"""

"""

```
1 begin
2   using CommonMark, ImageIO, FileIO, ImageShow
3   using PlutoUI
4   using Plots, PlotThemes, LaTeXStrings, Random
5   using PGFPlotsX
6   using SymPy
7   using HypertextLiteral: @html, @html_str
8   using ImageTransformations
9   using Colors
10 end
```