

MATH101

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Syllabus



2.1: A Preview of Calculus

Objectives

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- Understand what calculus is and how it compares with precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

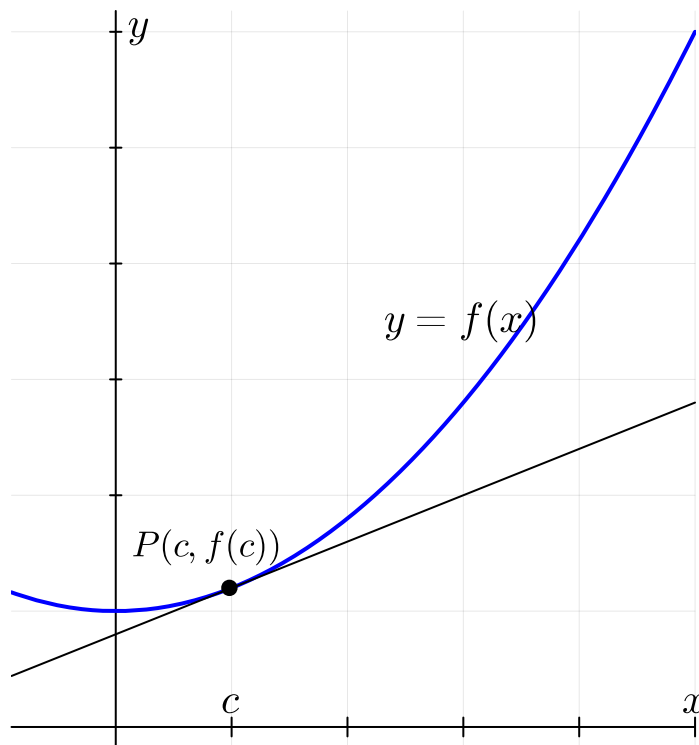
intro.

What is Calculus?

Precalculus Mathematics \Rightarrow Limit process \Rightarrow *Calculus*

The Tangent Line Problem

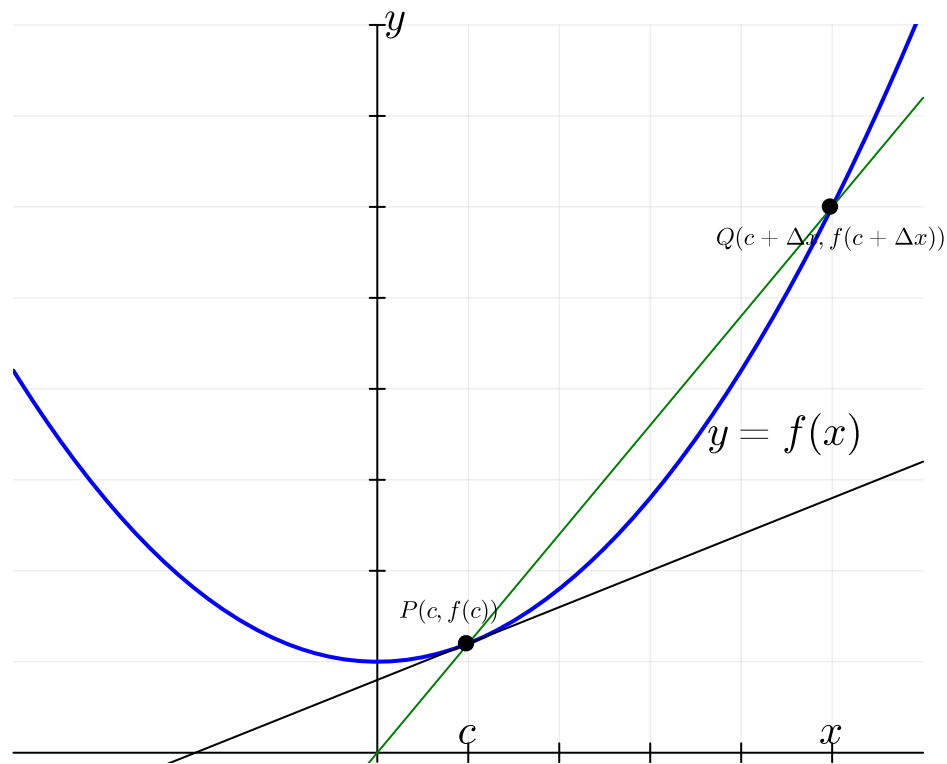
What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx



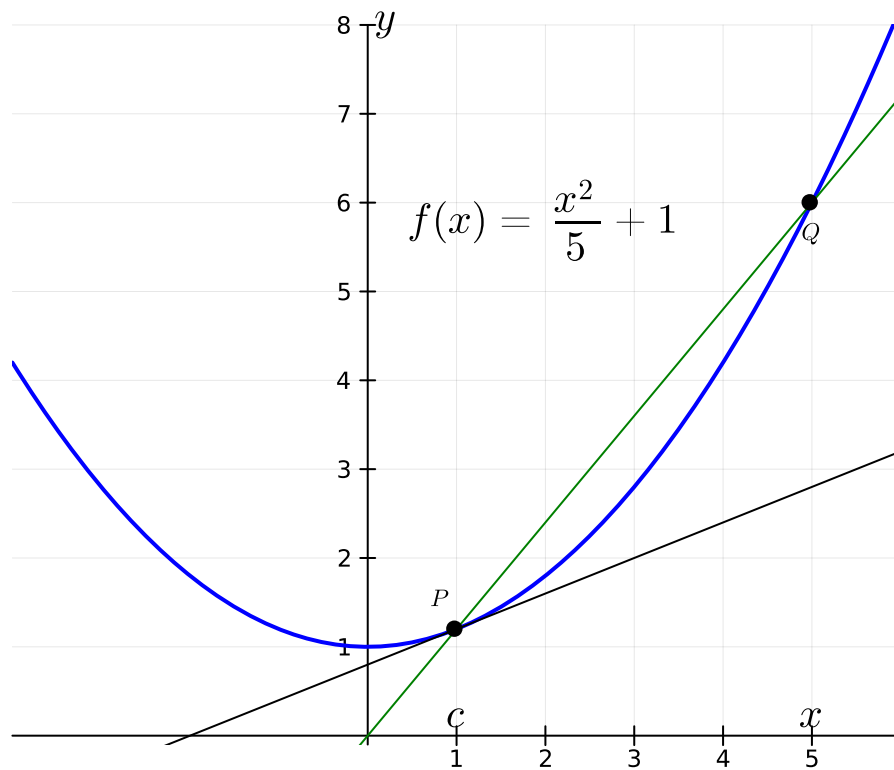
Find the equation of the secant line



$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Δx 

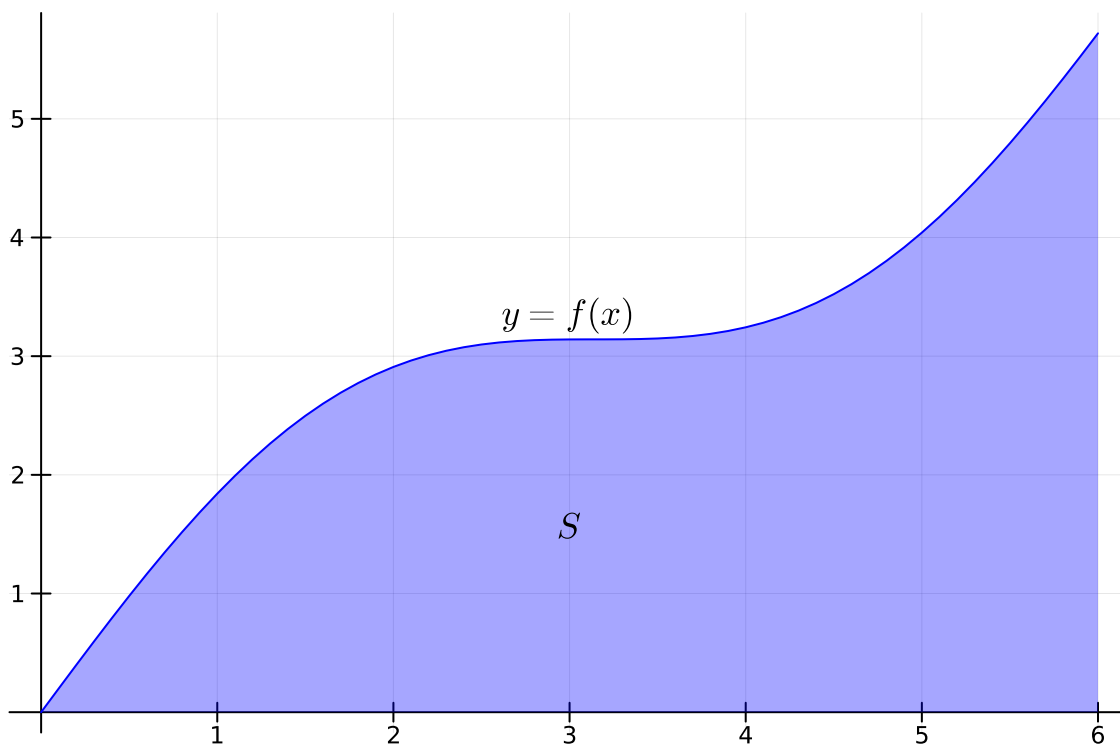
Example: **Find the equation of the secant line**



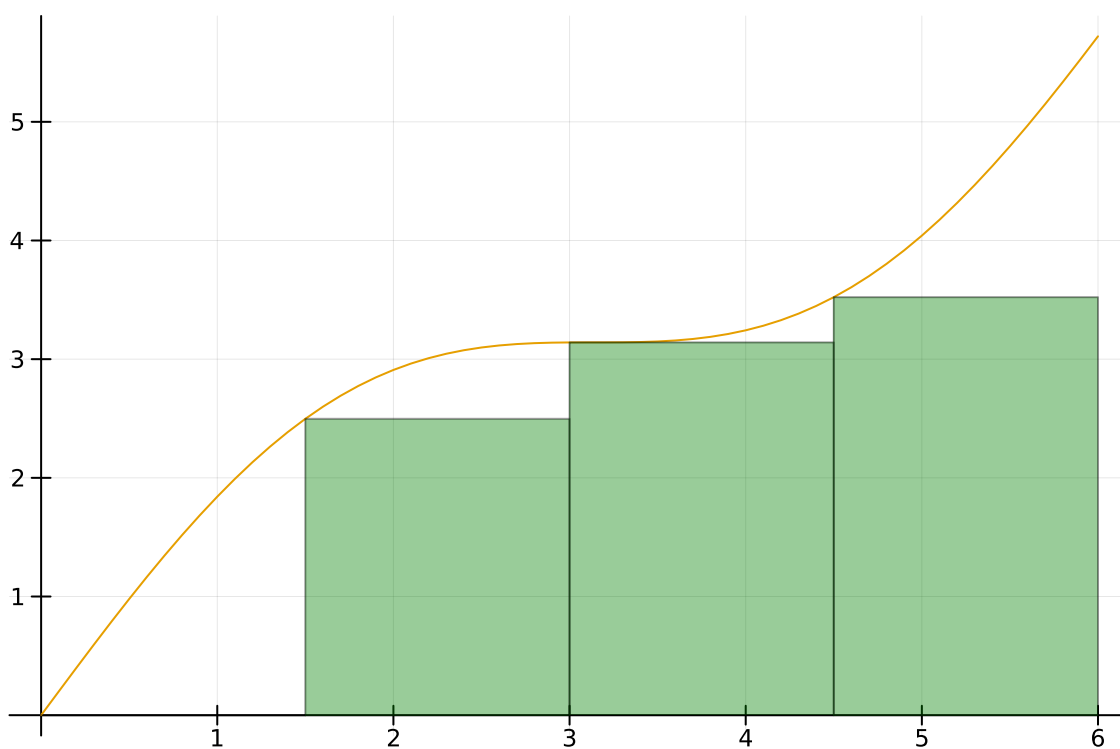
$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = a = b = method =



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

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- Estimate a limit using a **numerical** or **graphical approach**.
- Learn different ways that a limit can fail to exist.
- <s>Study and use a formal definition of limit</s>.

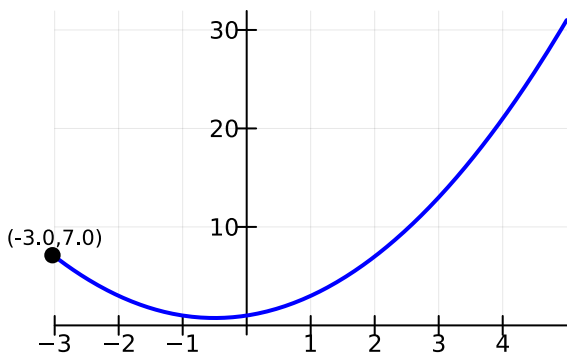
An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$  0.0

x approaches 1 from Left ▾



x approacheds 1 (from left)	$f(x)$ approaches
-3.0	7.0

Remark

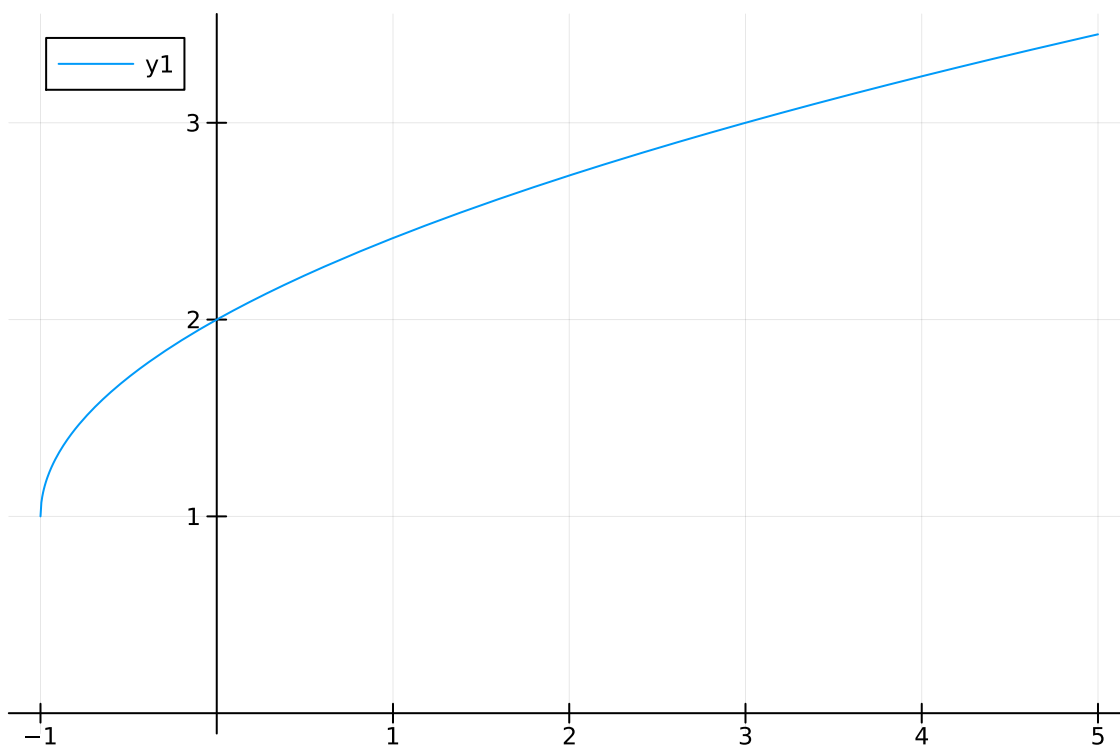
$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1: Estimating a Limit Numerically

Evaluate the function $f(x) = \frac{x}{\sqrt{x+1} - 1}$ at several x -values near 0 and use the results to estimate the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph



1.9999995001202078

```
1 begin
2   whatever(x)=x/(sqrt(x+1)-1)
3   whatever(-0.000001)
4 end
```

Example 2: Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark Problem solving

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: MATLAB, python, Julia)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist

Example 3: Different Right and Left Behavior

Show that the limit $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

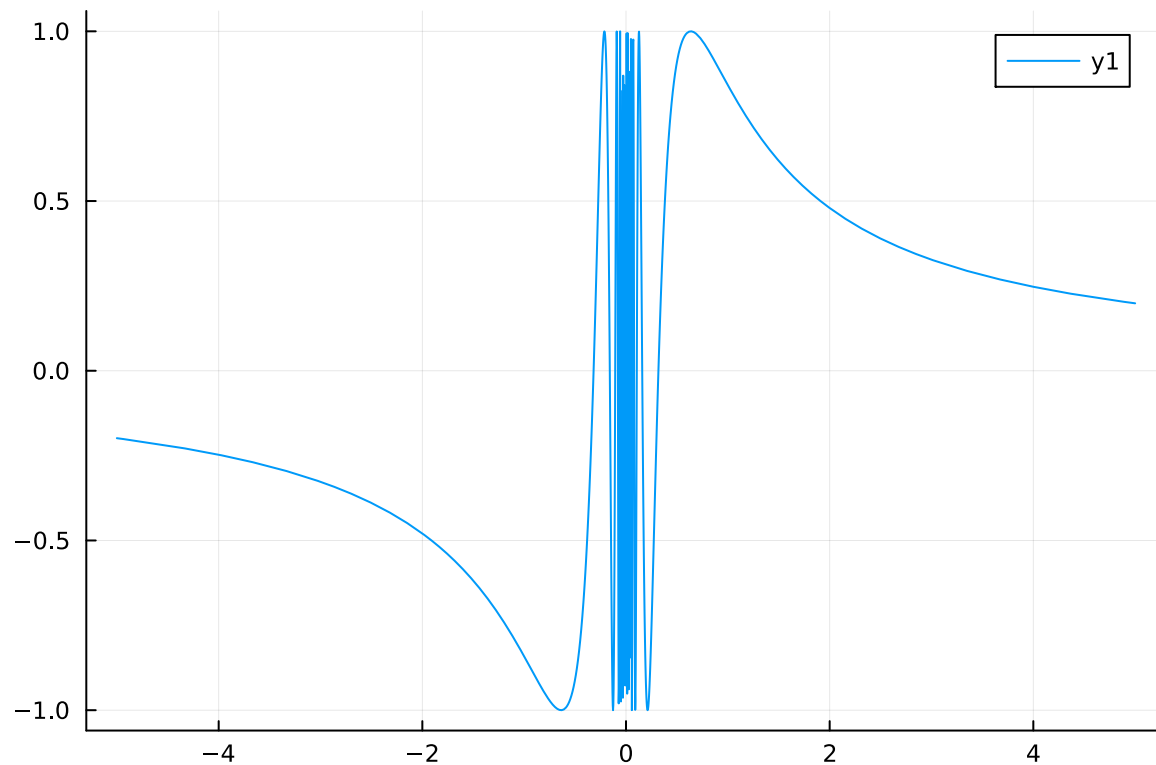
Example 4: Unbounded Behavior

Discuss the existence of the limit $\lim_{x \rightarrow 0} \frac{1}{x^2}$

Example 5: Oscillating Behavior

Discuss the existence of the limit $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

9.999999999999998e9



```
1 plot(x->sin(1/x))
```

A Formal Definition of Limit (Redaig Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \rightarrow c} f(x) = L$$

implies two statements—the limit exists and the limit is L .

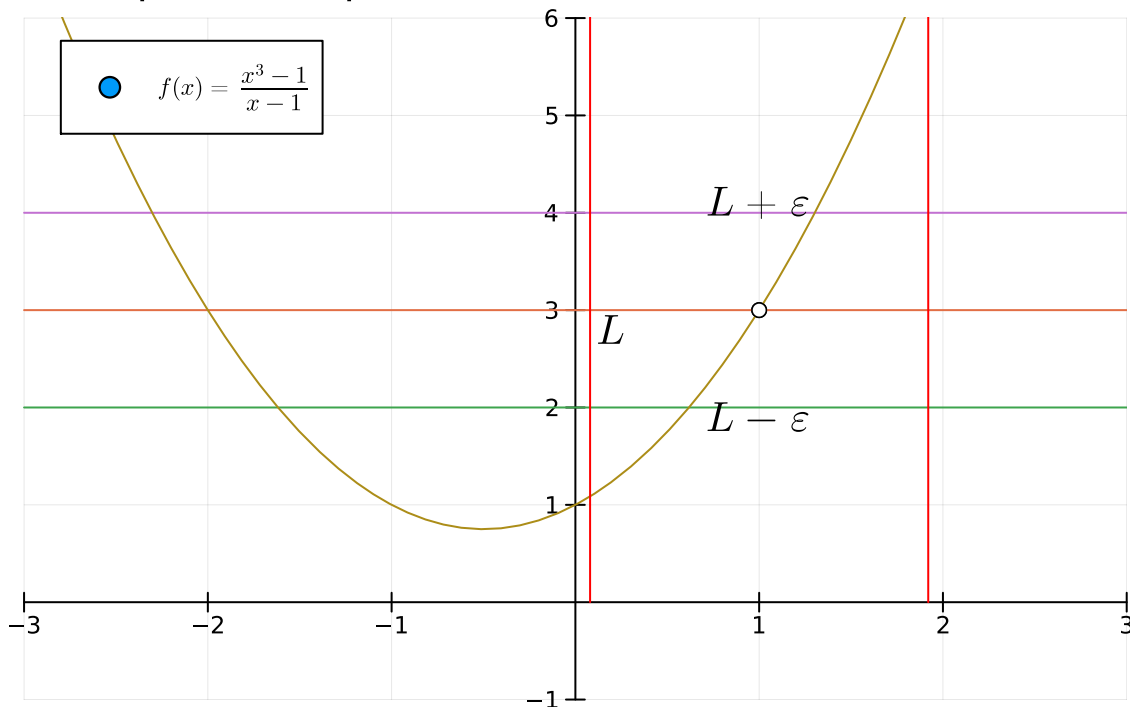
Example:

Prove that

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

$\epsilon =$  1.0 $\delta =$  0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

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- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$

Theorem Properties of Limits

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \rightarrow c} f(x) = L, \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. **Scalar multiple** $\lim_{x \rightarrow c} [bf(x)] = bL$
2. **Sum or difference** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. **Product** $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. **Quotient** $\lim_{x \rightarrow c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}, \quad K \neq 0$
5. **Power** $\lim_{x \rightarrow c} [f(x)]^n = L^n$

Example 2: The Limit of a Polynomial

Find $\lim_{x \rightarrow 2} (4x^2 + 3)$.

Theorem Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3: The Limit of a Rational Function

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x + 2}{x + 1}.$$

Theorem The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem The Limit of a Composite Function

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow c} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

Theorem Limits of Transcendental Functions

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \rightarrow c} \sin(x) = \sin(c)$
2. $\lim_{x \rightarrow c} \cos(x) = \cos(c)$
3. $\lim_{x \rightarrow c} \tan(x) = \tan(c)$
4. $\lim_{x \rightarrow c} \cot(x) = \cot(c)$
5. $\lim_{x \rightarrow c} \sec(x) = \sec(c)$
6. $\lim_{x \rightarrow c} \csc(x) = \csc(c)$
7. $\lim_{x \rightarrow c} a^x = a^c, \quad a > 0$
8. $\lim_{x \rightarrow c} \ln(x) = \ln(c)$



A Strategy for Finding Limits

Theorem Functions That Agree at All but One Point

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x).$$

Remarks A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c .

Dividing Out Technique

Example 7: Dividing Out Technique

Find the limit $\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8: Rationalizing Technique

Find the limit $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$.

The Squeeze Theorem

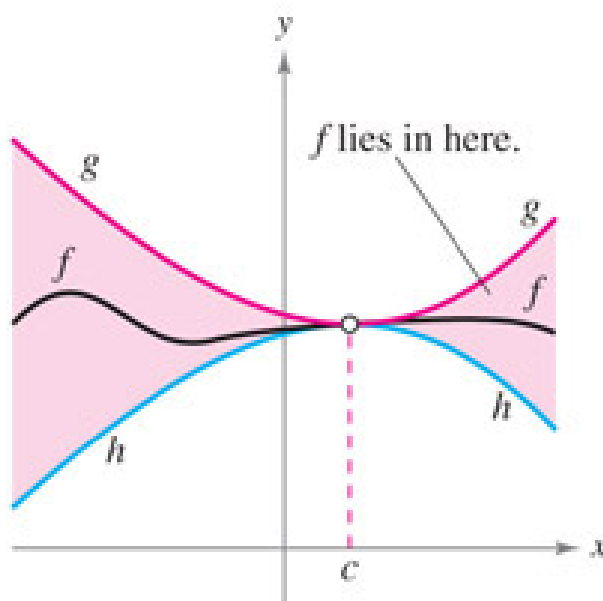
Theorem The Squeeze Theorem

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then $\lim_{x \rightarrow c} f(x)$ exists and equal to L .

$$h(x) \leq f(x) \leq g(x)$$



Theorem Three Special Limits

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$
3. $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e.$

Example 9: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\tan x}{x}$.

Example 10: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$.

Exercises



2.5: Infinite Limits

Objectives

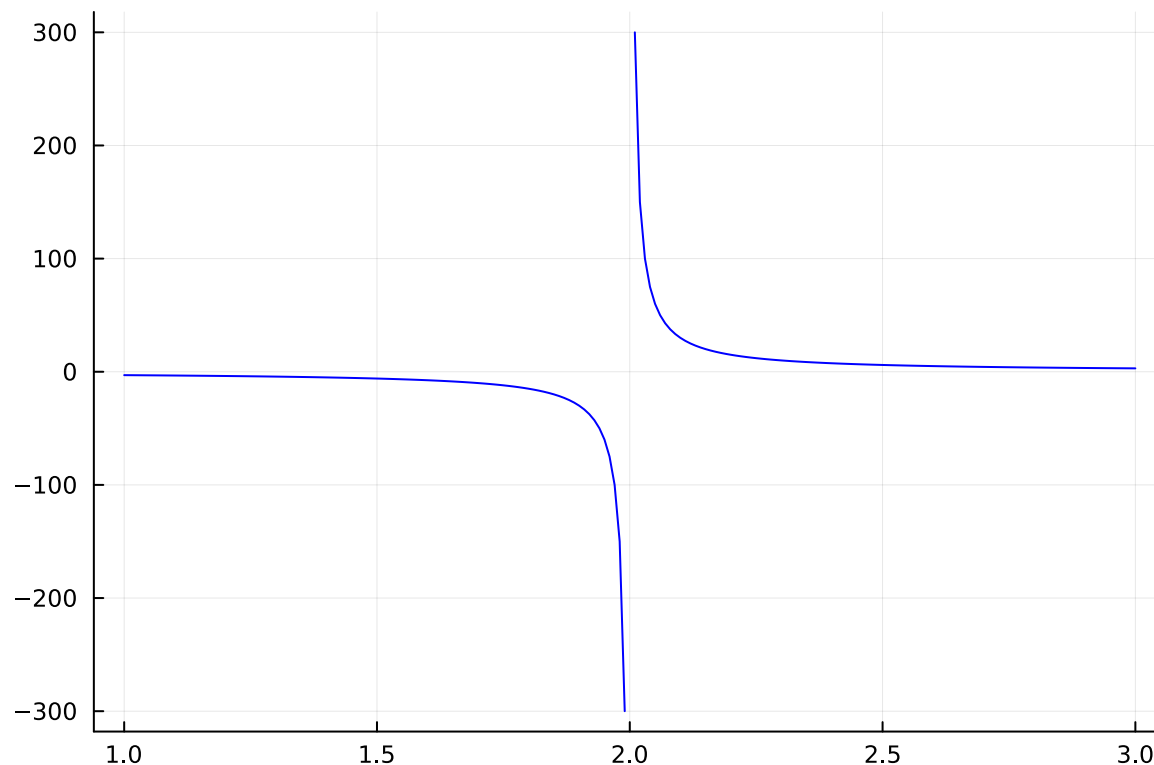
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- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Example: Infinite Limit

Consider

$$f(x) = \frac{3}{x-2}$$



```
1 plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c .

Theorem Vertical Asymptotes

Let f and g be continuous on an open interval containing c . If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

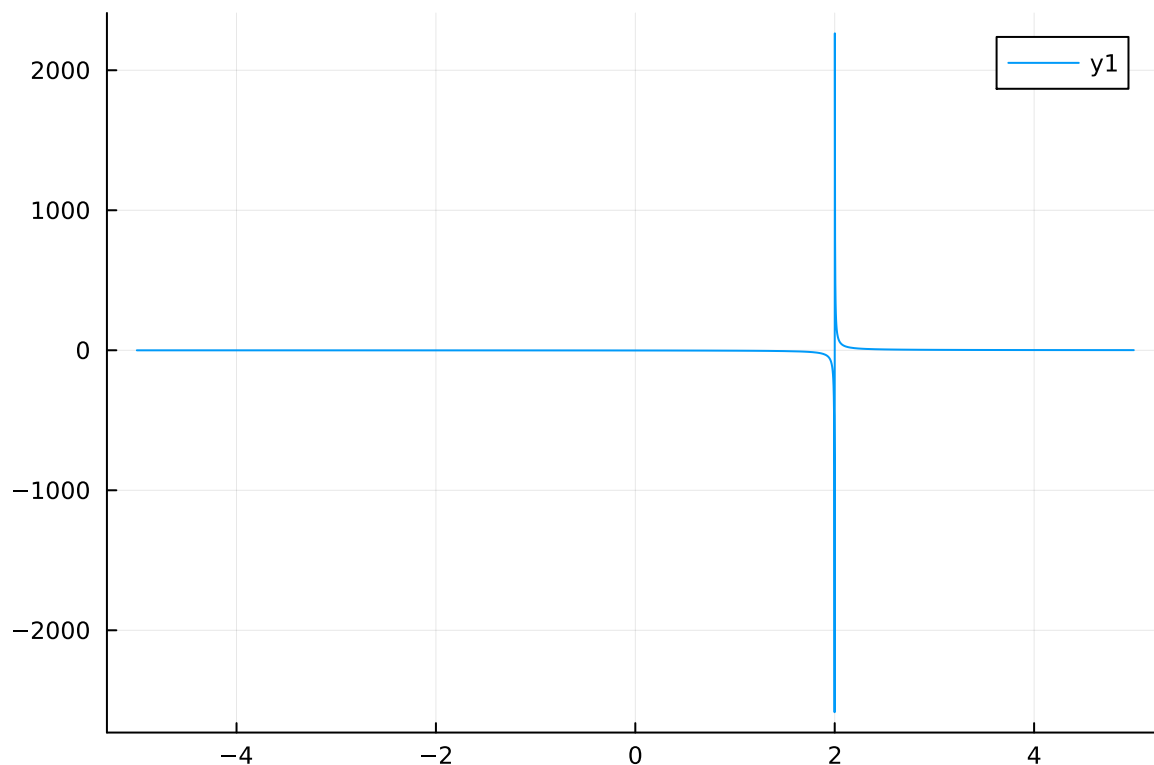
has a vertical asymptote at c .

Example 2: Finding Vertical Asymptotes

1. $h(x) = \frac{1}{2(x+1)}.$

2. $h(x) = \frac{x^2 + 1}{x^2 - 1}.$

3. $h(x) = \cot x = \frac{\cos x}{\sin x}.$



```
1 plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- desmos.com
- geogebra.org

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x \rightarrow 1^-} \frac{x^3 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^3 - 3x}{x - 1}$$

Theorem Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L$$

1. **Sum or difference:** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$

2. **Product:**

$$\lim_{x \rightarrow c} [f(x)g(x)] = \infty, \quad L > 0$$

$$\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, \quad L < 0$$

3. **Quotient:** $\lim_{x \rightarrow c} \left[\frac{g(x)}{f(x)} \right] = 0$

Remark

2. is **not true** if $\lim_{x \rightarrow c} g(x) = 0$

Exercises



4.5: Limits at Infinity

Objectives

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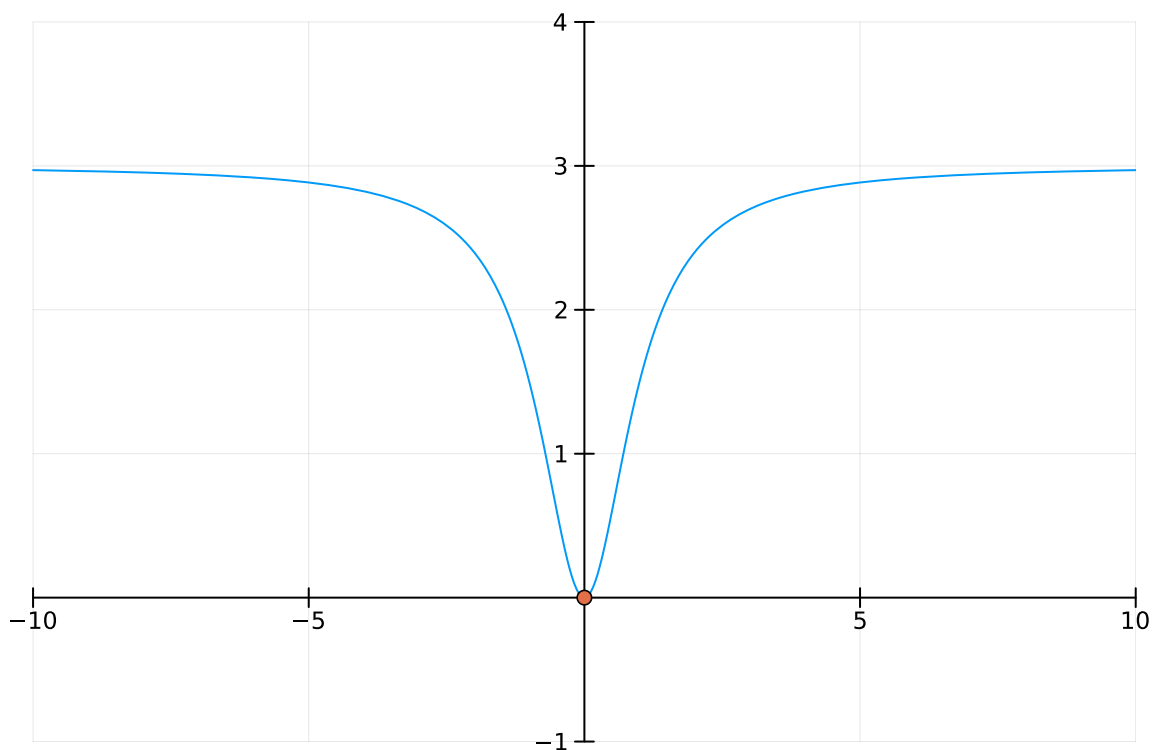
- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2 + 1}$$

$x =$

$f(x) = 0.0$



we write

$$\lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + 1} = 3, \quad \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2 + 1} = 3$$

Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = L$$

Remarks

- Limits at infinity have many of the same properties of limits discussed in Section 2.3.
- For example, if $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} g(x)$ both exist, then
 - $\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
 - $\lim_{x \rightarrow \infty} [f(x)g(x)] = \left[\lim_{x \rightarrow \infty} f(x) \right] \left[\lim_{x \rightarrow \infty} g(x) \right]$
- Similar properties hold for limits at $-\infty$.

Theorem Limits at Infinity

1. If r is a positive rational number and c is any real number, then

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim_{x \rightarrow -\infty} e^x = 0$ and $\lim_{x \rightarrow \infty} e^{-x} = 0$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

$$h(x) = \frac{p(x)}{q(x)}$$

1. $\deg p < \deg q$, then the limit is 0.
2. $\deg p = \deg q$, then the **limit** of the rational function is the **ratio** of the **leading coefficients**.
3. $\deg p > \deg q$, then the **limit** of the rational function **does not exist**.

Examples



```
1 # begin
2 #   xx=symbols("xx",real=true)
3 #   limit(xx*sin(1/xx),xx,0)
4 # end
```

Infinite Limits at Infinity

Remark

Determining whether a function has an infinite limit at infinity is useful in analyzing the “**end behavior**” of its graph. You will see examples of this in Section 4.6 on curve sketching.

2.4: Continuity and One-Sided Limits

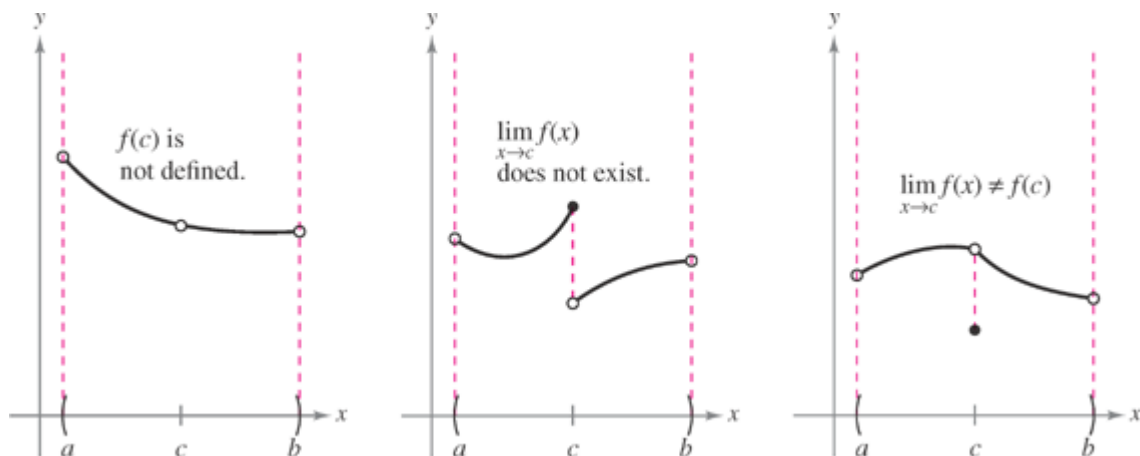
Objectives

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- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

The graph of f is not continuous at $x = c$



In Figure __above__, it appears that continuity at $x=c$ can be __destroyed__ by any one of __three conditions__.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \rightarrow c} f(x)$ exists.
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Continuity on an Open Interval

- A function f is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval.
- A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Remarks

- If a function f is defined on an open interval I (except possibly at c), and f is not continuous at c , then f is said to have a **discontinuity** at c .
- Discontinuities fall into two categories:
 - **removable**: A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$.
 - **nonremovable**: there is no way to define $f(c)$ so as to make the function continuous at $x = c$.

Example 1:

Discuss the continuity of each function

a. $f(x) = \frac{1}{x}$

b. $g(x) = \frac{x^2 - 1}{x - 1}$

c. $h(x) = \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases}$

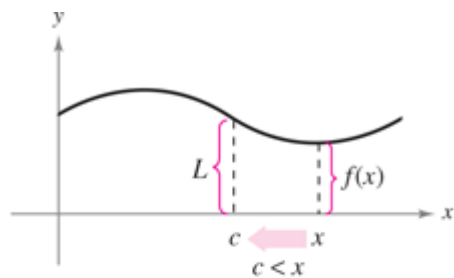
d. $y = \sin x$

Examples

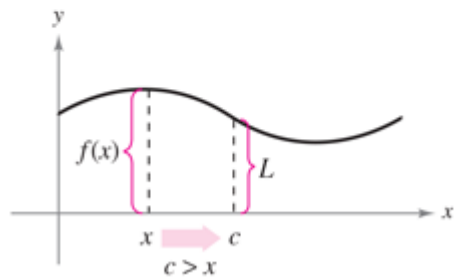


One-Sided Limits and Continuity on a Closed Interval

(a) Limit from right $\lim_{x \rightarrow c^+} f(x) = L$



(a) Limit as x approaches c from the right.



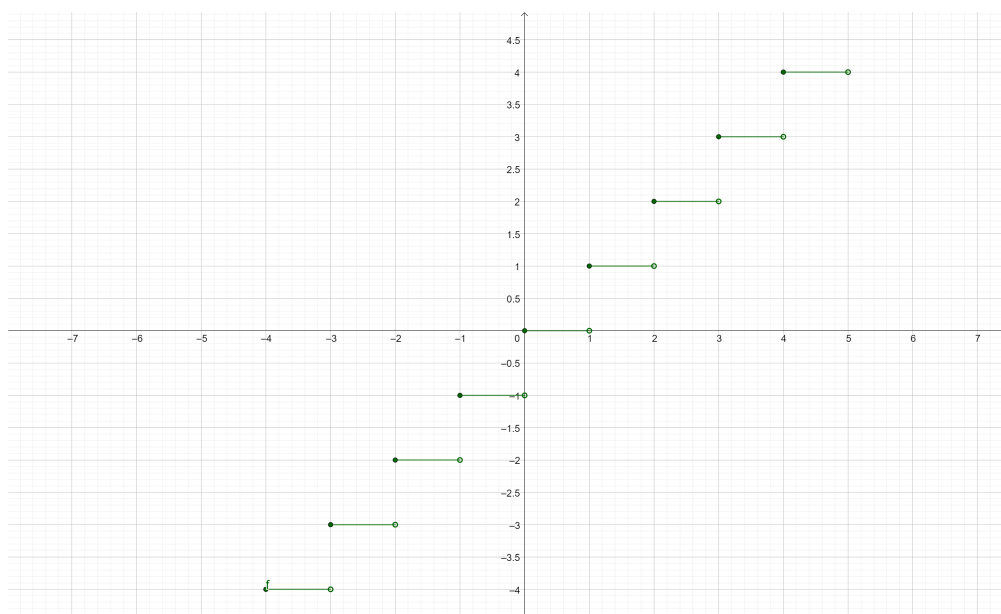
(b) Limit as x approaches c from the left.

(b) Limit from left $\lim_{x \rightarrow c^-} f(x) = L$

STEP FUNCTIONS

(greatest integer function)

$[x] = \text{greatest integer } n \text{ such that } n \leq x.$



Theorem The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is if and only if

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and

$$\lim_{x \rightarrow b^-} f(x) = f(b).$$

Example 4: Continuity on a Closed Interval

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}$$

Properties of Continuity

Theorem Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c .

1. **Scalar multiple:** bf
2. **Sum or difference:** $f \pm g$
3. **Product:** fg
4. **Quotient:** $\frac{f}{g}$, $g(c) \neq 0$,

Remarks

1. **Polynomials** are continuous at every point in their domains.
2. **Rational functions** are continuous at every point in their domains.
3. **Radical functions** are continuous at every point in their domains.
4. **Trigonometric functions** are continuous at every point in their domains.
5. **Exponential and logarithmic functions** are continuous at every point in their domains.

Theorem Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$ then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c .

Remark

$$\lim_{x \rightarrow c} f(g(x)) = f(g(c))$$

provided f and g satisfy the conditions of the theorem.

Example 7: Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$

b. $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Exercises



The Intermediate Value Theorem

Theorem Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Example 8: An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

3.1: The Derivative and the Tangent Line Problem

Objectives

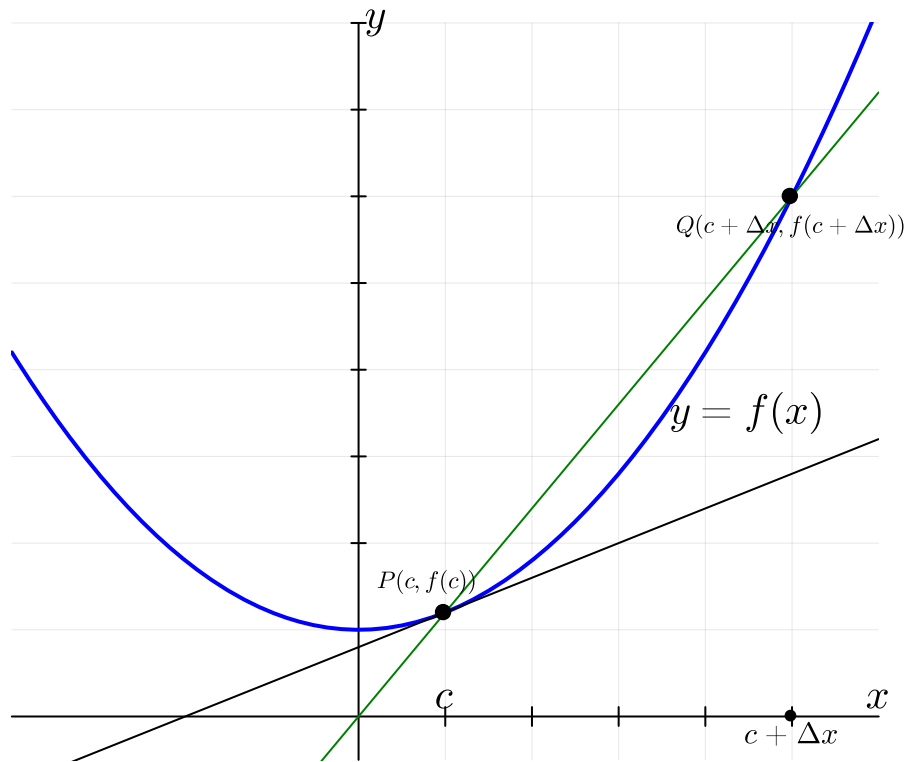
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- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Δx  4.0

Find the equation of the secant line



Slope of secant line

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Definition of Tangent Line with Slope

If f is defined on an open interval containing c , and if the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Remark

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$** .

Example 1: The Slope of the Graph of a Linear Function

Find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$.

Example 2: Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$.

Remarks

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line.
- For vertical tangent lines, you can use the **following definition**. If f is continuous at c and

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the **vertical line** $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f .

The Derivative of a Function

Definition Derivative of a Function

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x .

Remarks

- The notation $f'(x)$ is read as “ f prime of x .”
- $f'(x)$ is a **function** that gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.
- The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.
- The process of finding the derivative of a function is called **differentiation**.
- A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval** (a, b) when it is differentiable at every point in the interval.

Notation

$$y = f(x)$$

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}[f(x)]$
- $D_x[y]$

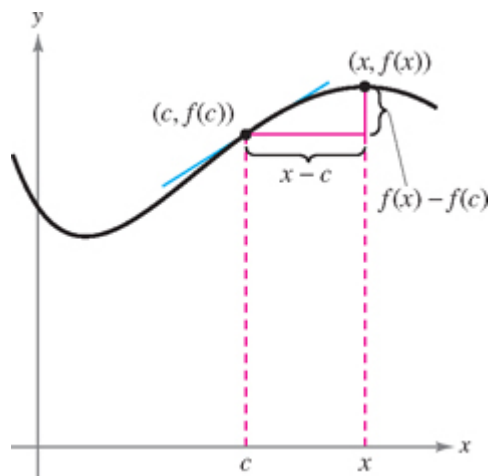
$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Examples 3,4,5: Finding the Derivative by the Limit Process

Find the derivative of

- $f(x) = x^3 + 2x$
- $f(x) = \sqrt{x}$
- $y = \frac{2}{t}$ with respect to t .

Differentiability and Continuity



Alternative form of derivative

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

Remarks

derivative from the left

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$$

derivative from the right

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

Example:

$$f(x) = \lfloor x \rfloor$$

Example 6: A Graph with a Sharp Turn

$$f(x) = |x - 2|$$

Example 7: A Graph with a Vertical Tangent Line

$$f(x) = x^{\frac{1}{3}}$$

Theorem Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

remarks

The relationship between continuity and differentiability is summarized below.

- If a function f is differentiable at $x = c$, then it is continuous at $x = c$. So, **differentiability** implies (\Rightarrow) **continuity**.
- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, **continuity does not imply differentiability**.

Exercises



3.2: Basic Differentiation Rules and Rates of Change

Objectives

“

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Find the derivatives of exponential functions.
- Use derivatives to find rates of change.

The Constant Rule

Theorem **The Constant Rule**

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

The Power Rule

Theorem The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at 0 , n must be a number such that x^{n-1} is defined on an interval containing 0 .

The Constant Multiple Rule

Theorem The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

The Sum and Difference Rules

Theorem The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g .

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{Difference Rule}$$

Derivatives of the Sine and Cosine Functions

Theorem Derivatives of the Sine and Cosine Functions

$$\frac{d}{dx}[\sin(x)] = \cos x, \quad \frac{d}{dx}[\cos(x)] = -\sin x$$

Derivatives of Exponential Functions

Theorem Derivative of the Natural Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$



$$2 \sin(t)$$

```
1 begin
2     t = symbols("t", real=true)
3     g(t)=-2cos(t)-5
4     plot(x->g(x))
5     diff(g(t),t)
6 end
```

Rates of Change

- The derivative can be used to determine the **rate of change** of one variable with respect to another.
- Applications involving rates of change, sometimes referred to as **instantaneous rates of change**, occur in a wide variety of fields.
- A common use for rate of change is to describe **the motion of an object moving in a straight line**. (+ direction and -direction)
- The function s that gives **the position (relative to the origin)** of an object as a **function of time t** is called a **position function**. If, over a period of time Δt , the object changes its position by the amount Δs , then, by the familiar formula

$$\Delta s = s(t + \Delta t) - s(t)$$

- then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}.$$

-the average velocity is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} \quad \text{Average Velocity.}$$

- In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the velocity of the object at time t is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \quad \text{Velocity function.}$$

Example:

If a ball is thrown into the air with a velocity of 4 m/s , its height (in meters (m)) t seconds later is given by

$$y = 4t - 4.9t^2.$$

1. Find the average velocity for the time period from $t = 1$ to $t = 3$.
2. Find the instantaneous rate of change at $t = 2$.

Example 11: Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is **9.8** meters above the water. The initial velocity of the diver is **4.9** meters per second. When does the diver hit the water? What is the diver's velocity at impact?

3.3: Product and Quotient Rules and Higher-Order Derivatives

Objectives

“

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

Theorem The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the **first** function **times** the **derivative of the second**, **plus** the **second** function times the **derivative of the first**.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Example:

Find the derivative of $f(x) = xe^x$.

The Quotient Rule

Theorem The Quotient Rule

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the **denominator times the derivative of the numerator minus the numerator times the derivative of the denominator**, all **divided by the square of the denominator**.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$$

Example:

Find an equation of the tangent line to the graph of $f(x) = \frac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Derivatives of Trigonometric Functions

Theorem Derivatives of Trigonometric Functions

$$\begin{array}{lcl} \frac{d}{dx}(\tan x) & = & \sec^2 x \\ \frac{d}{dx}(\sec x) & = & \sec x \tan x \end{array} \quad \left| \quad \begin{array}{lcl} \frac{d}{dx}(\cot x) & = & -\csc^2 x \\ \frac{d}{dx}(\csc x) & = & -\csc x \cot x \end{array} \right.$$

Example:

Differentiate

$$y = \frac{1 - \cos x}{\sin x}$$

Higher-Order Derivatives

Remarks

Rates of changes

$$\begin{array}{rcll} & & s(t) & \text{Position function} \\ v(t) & = & s'(t) & \text{Velocity function} \\ a(t) & = & v'(t) = s''(t) & \text{Acceleration function} \end{array}$$

Higher Derivatives

$$\text{First derivative: } y', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad D_x[y]$$

$$\text{Second derivative: } y'', \quad f''(x), \quad \frac{d^2y}{dx^2}, \quad \frac{d^2}{dx^2}[f(x)], \quad D_x^2[y]$$

$$\text{Third derivative: } y''', \quad f'''(x), \quad \frac{d^3y}{dx^3}, \quad \frac{d^3}{dx^3}[f(x)], \quad D_x^3[y]$$

$$\text{Fourth derivative: } y^{(4)}, \quad f^{(4)}(x), \quad \frac{d^4y}{dx^4}, \quad \frac{d^4}{dx^4}[f(x)], \quad D_x^4[y]$$

⋮

$$\text{nth derivative: } y^{(n)}, \quad f^{(n)}(x), \quad \frac{d^ny}{dx^n}, \quad \frac{d^n}{dx^n}[f(x)], \quad D_x^n[y]$$

Exercises



"""

"""

```
1 begin
2     using CommonMark, ImageIO, FileIO, ImageShow
3     using PlutoUI
4     using Plots, PlotThemes, LaTeXStrings, Random
5     using PGFPlotsX
6     using SymPy
7     using HypertextLiteral: @html, @html_str
8     using ImageTransformations
9     using Colors
10 end
```