MATH101

2.1: A Preview of Calculus

Objectives

> “
>
> - Understand what calculus is and how it compares with precalculus.
> - Understand that the tangent line problem is basic to calculus.
> - Understand that the area problem is also basic to calculus.

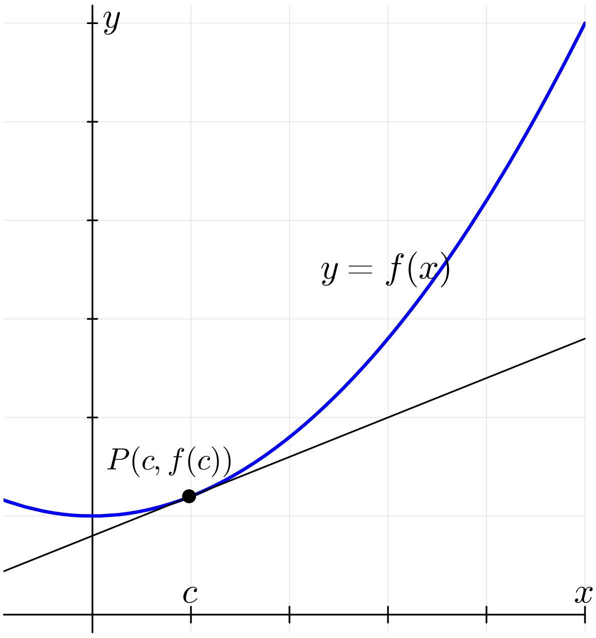
intro.

What is Calculus?

Precalculus Matematics $\Rightarrow$ Limit process $\Rightarrow$ *Calculus*

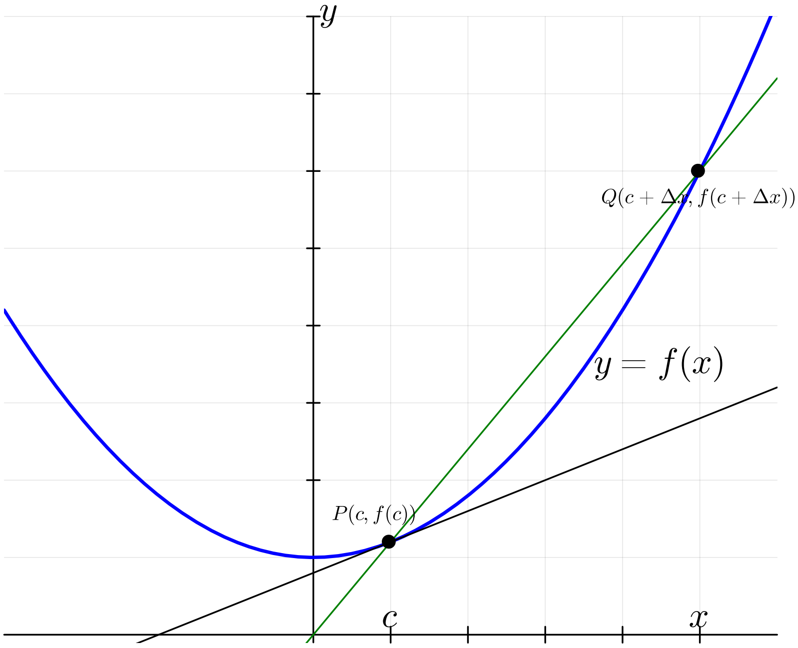
The Tangent Line Problem

What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx

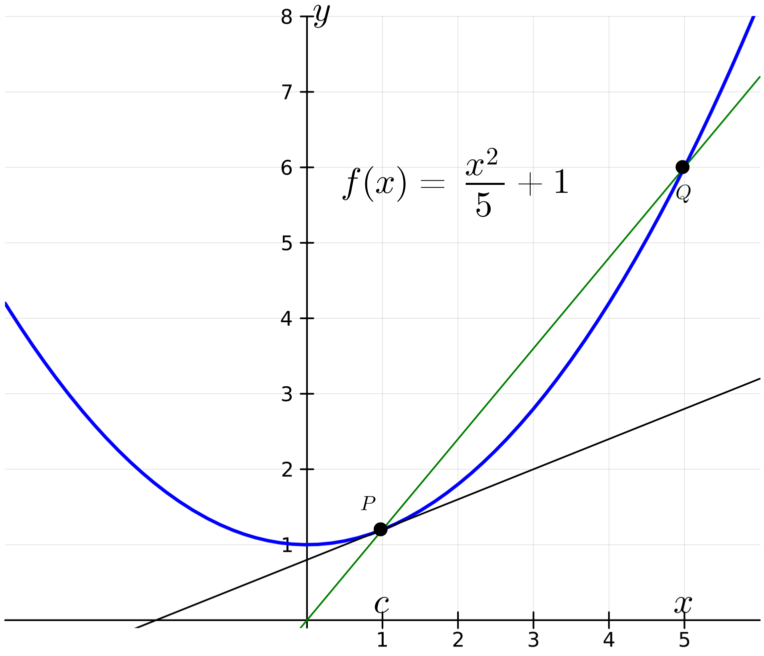
Find the equation of the secant line



$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Δx

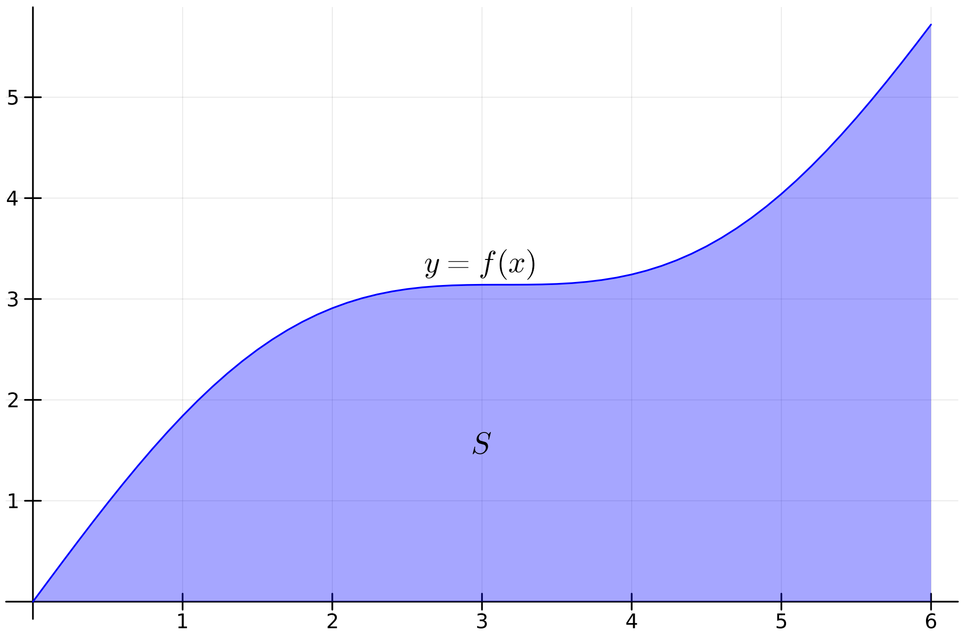
Example: **Find the equation of the secant line**



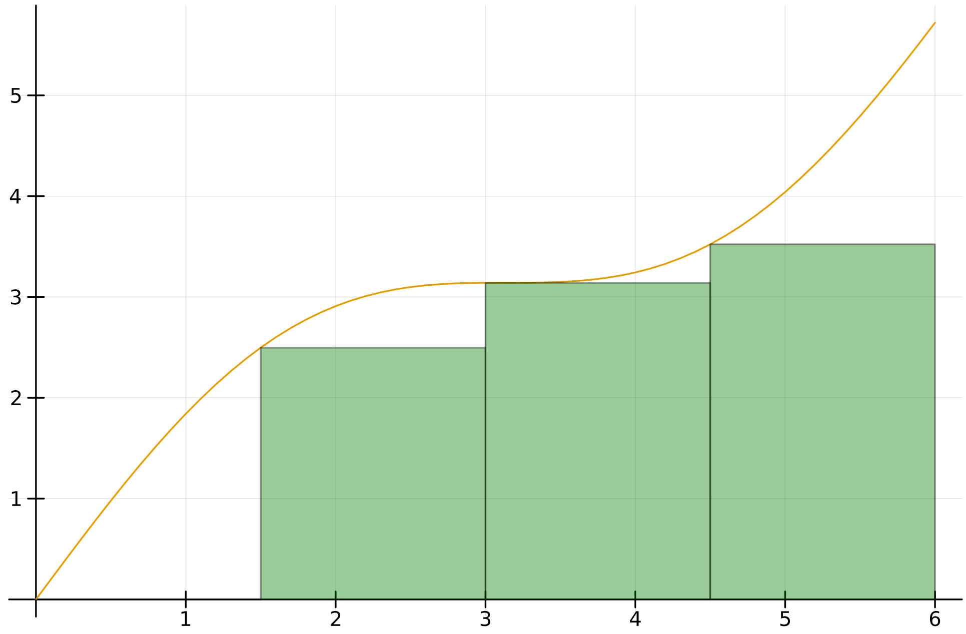
$$\mathrm{m}_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = 4 a = 0 b = 6 method = Left



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

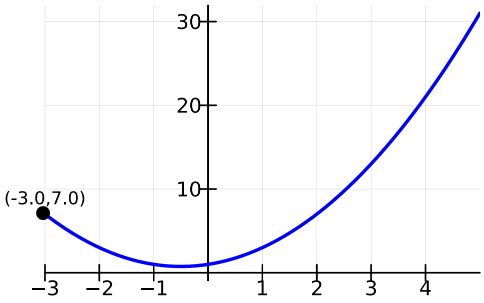
> “
>
> - Estimate a limit using a **numerical** or **graphical approach**.
> - Learn different ways that a limit can fail to exist.
> - <s>Study and use a formal definition of limit</s>.

An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$ 0.0 x **approaches** 1 from Left



x approacheds 1 (from left)	$f(x)$ approaches
-3.0	7.0

Remark

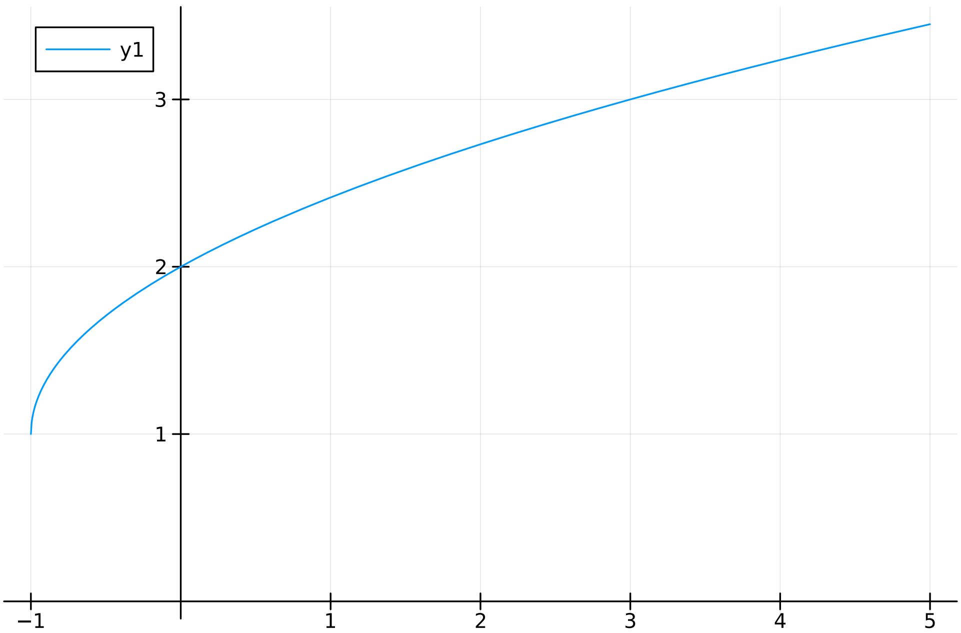
$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1: **Estimating a Limit Numerically**

Evaluate the function $f(x) = \dfrac{x}{\sqrt{x+1} - 1}$ at several x-values near 0 and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph




```
1.9999995001202078
```

```
begin
    whatever(x)=x/(sqrt(x+1)-1)
    whatever(-0.000001)
end
```

Example 2: **Finding a Limit**

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark **Problem solving**

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: `MATLAB`, `python`, `Julia`)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist

Example 3: **Different Right and Left Behavior**

Show that the limit $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

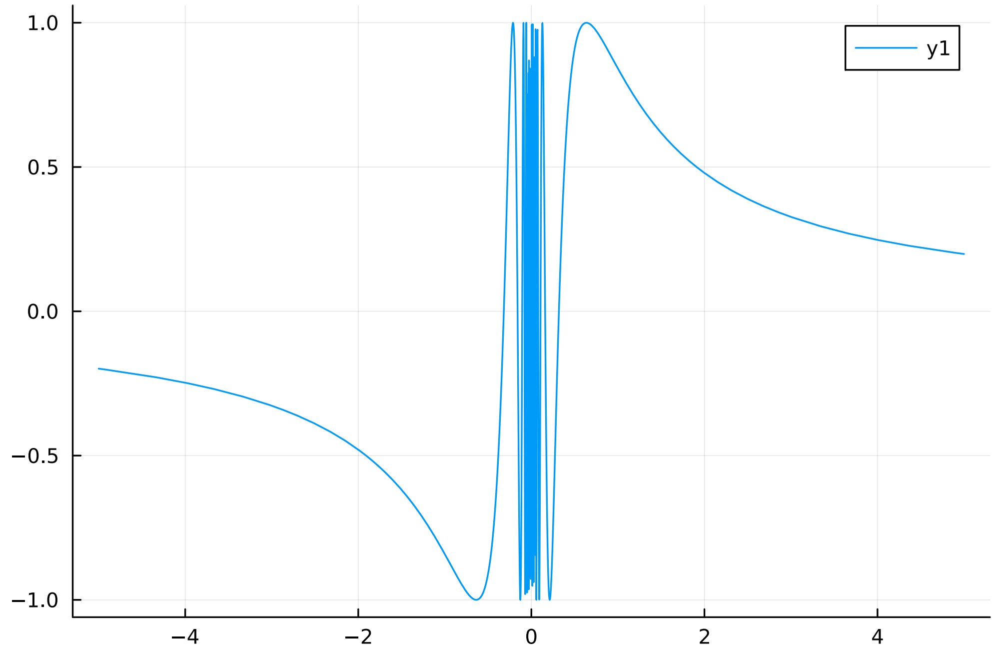
Example 4: **Unbounded Behavior**

Discuss the existence of the limit $\lim_{x\to 0} \frac{1}{x^2}$

Example 5: **Oscillating Behavior**

Discuss the existence of the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

```
9.999999999999998e9
```




```
plot(x->sin(1/x))
```

A Formal Definition of Limit (Redaing Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \to c} f(x) = L$$

implies two statements—the limit exists and the limit is L.

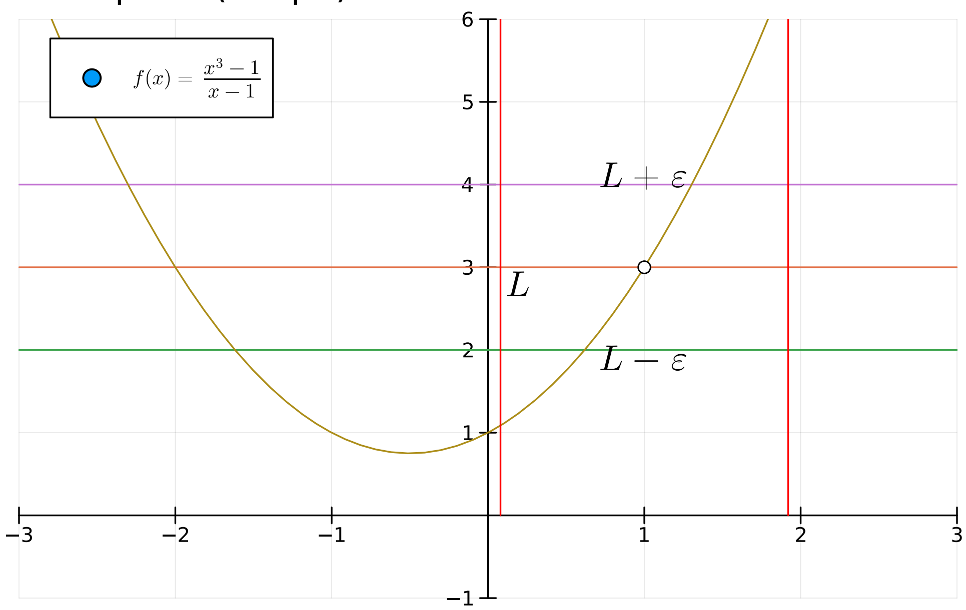
Example:

Prove that

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

ϵ = 1.0 δ = 0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

> - Evaluate a limit using properties of limits.
> - Develop and use a strategy for finding limits.
> - Evaluate a limit using the dividing out technique.
> - Evaluate a limit using the rationalizing technique.
> - Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem **Some Bacic Limits**

Let b and c be real numbers, and let n be a positive integer.

1. $\lim\limits_{x \to c} b = b$
2. $\lim\limits_{x \to c} x = c$
3. $\lim\limits_{x \to c} x^n = c^n$

Theorem **Properties of Limits**

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L, \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. **Scalar multiple** $\lim\limits_{x \to c} \left[b f(x)\right] = bL$
2. **Sum or difference** $\lim\limits_{x \to c} \left[f(x) \pm g(x)\right] = L \pm K$
3. **Product** $\lim\limits_{x \to c} \left[f(x) g(x)\right] = LK$
4. **Quotient** $\lim\limits_{x \to c} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{K}, \quad K \neq 0$
5. **Power** $\lim\limits_{x \to c} \left[f(x)\right]^n = L^n$

Example 2: **The Limit of a Polynomial**

Find $\lim\limits_{x \to 2} (4x^2 + 3)$.

Theorem **Limits of Polynomial and Rational Functions**

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3: **The Limit of a Rational Function**

Find

$$\lim_{x \to 1} \frac{x^2 + x + 2}{x + 1}.$$

Theorem **The Limit of a Function Involving a Radical**

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem **The Limit of a Composite Function**

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Theorem **Limits of Transcendental Functions**

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \to c} \sin(x) = \sin(c)$
2. $\lim_{x \to c} \cos(x) = \cos(c)$
3. $\lim_{x \to c} \tan(x) = \tan(c)$
4. $\lim_{x \to c} \cot(x) = \cot(c)$
5. $\lim_{x \to c} \sec(x) = \sec(c)$
6. $\lim_{x \to c} \csc(x) = \csc(c)$
7. $\lim_{x \to c} a^x = a^c, \quad a > 0$
8. $\lim_{x \to c} \ln(x) = \ln(c)$

A Strategy for Finding Limits

Theorem **Functions That Agree at All but One Point**

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Remarks **A Strategy for Finding Limits**

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c.

Dividing Out Technique

Example 7: **Dividing Out Technique**

Find the limit $\lim\limits_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8: **Rationalizing Technique**

Find the limit $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$.

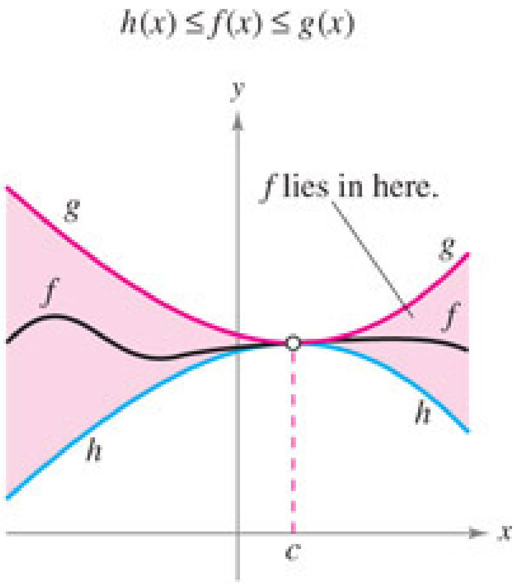
The Squeeze Theorem

Theorem **The Squeeze Theorem**

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ **exists** and equal to L.



Theorem **Three Special Limits**

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0.$
3. $\lim_{x \to 0} (1+x)^{1/x} = e.$

Example 9: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$.

Example 10: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$.

Exercises

2.5: Infinite Limits

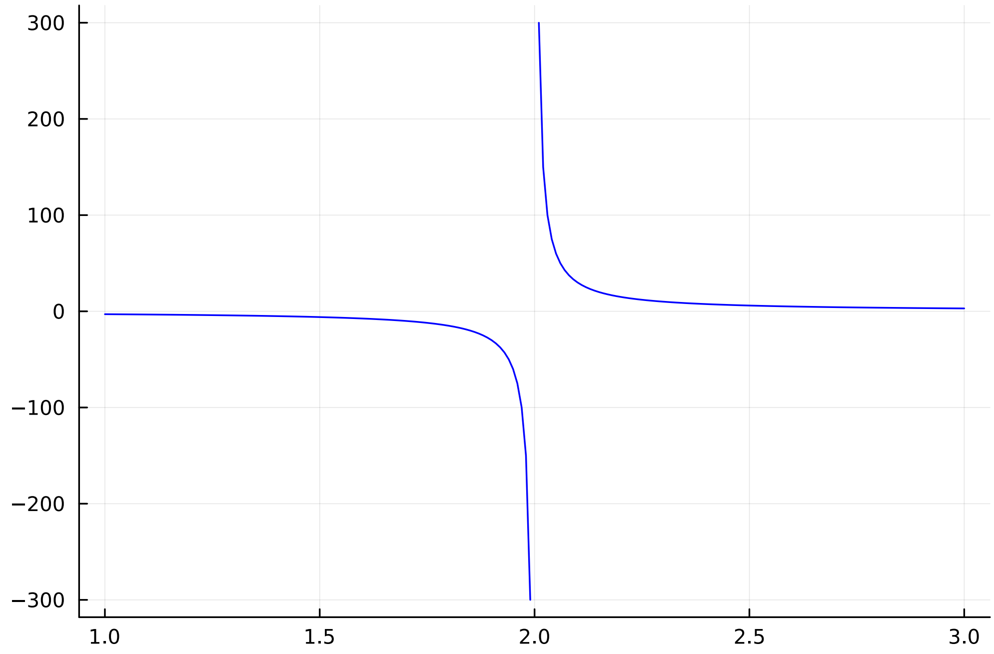
Objectives

> “
>
> - Determine infinite limits from the left and from the right.
> - Find and sketch the vertical asymptotes of the graph of a function.

Example: **Infinite Limit**

Consider

$$f(x) = \frac{3}{x-2}$$




```
plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c.

Theorem **Vertical Asymptotes**

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

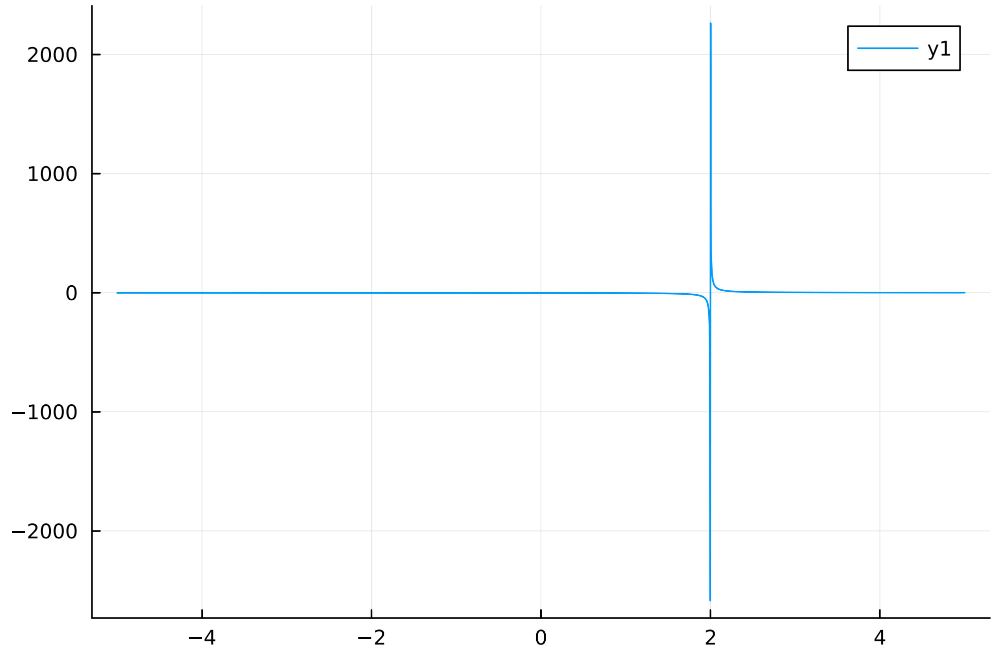
has a vertical asymptote at c.

Example 2: **Finding Vertical Asymptotes**

1. $h(x) = \dfrac{1}{2(x+1)}$.

2. $h(x) = \dfrac{x^2+1}{x^2-1}$.

3. $h(x) = \cot x = \dfrac{\cos x}{\sin x}$.




```
plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- desmos.com
- geogebra.org

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2+2x-8}{x^2-4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x\to 1^-} \frac{x^3-3x}{x-1} \quad \text{and} \quad \lim_{x\to 1^+} \frac{x^3-3x}{x-1}$$

Theorem **Properties of Infinite Limits**

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L$$

1. **Sum or difference**: $\lim\limits_{x \to c} \left[f(x) \pm g(x)\right] = \infty$
2. **Product**:

$$\begin{aligned} \lim_{x \to c} \left[f(x)g(x)\right] &= \infty, \quad L > 0 \\ \lim_{x \to c} \left[f(x)g(x)\right] &= -\infty, \quad L < 0 \end{aligned}$$

3. Quotient: $\lim\limits_{x \to c} \left[\dfrac{g(x)}{f(x)}\right] = 0$

Remark

2. is **not true** if $\lim\limits_{x \to c} g(x) = 0$

Exercises

4.5: Limits at Infinity

Objectives

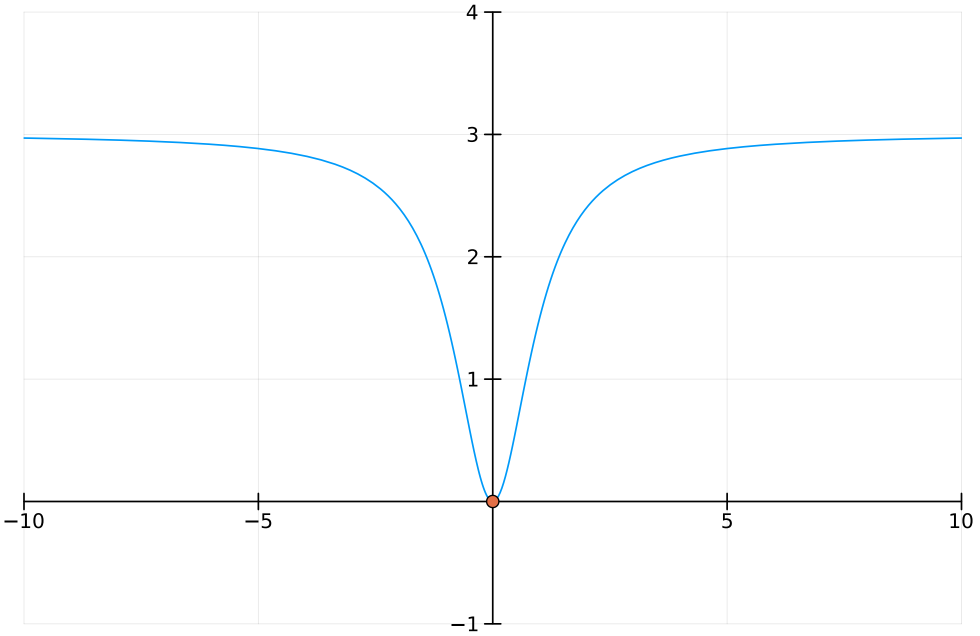
> “
>
> - Determine (finite) limits at infinity.
> - Determine the horizontal asymptotes, if any, of the graph of a function.
> - Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2+1}$$

$x =$ 0

$f(x) =$ 0.0



we write

$$\lim_{x\to\infty} \frac{3x^2}{x^2+1} = 3, \quad \lim_{x\to-\infty} \frac{3x^2}{x^2+1} = 3$$

Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L$$

Remarks

- Limits at infinity have many of the same properties of limits discussed in Section 2.3.
- For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then
  - $\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$
  - $\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right]$
- Similar properties hold for limits at $-\infty$.

Theorem

Limits at Infinity

1. If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^{-x} = 0$

Guidelines for Finding Limits at ±∞ of Rational Functions

$$h(x) = \frac{p(x)}{q(x)}$$

1. $\deg p < \deg q$, then the limit is 0.
2. $\deg p = \deg q$, then the **limit** of the rational function **is** the **ratio** of the **leading coefficients**.
3. $\deg p > \deg q$, then the **limit** of the rational function **does not exist**.

Examples

```
# begin
#   xx=symbols("xx",real=true)
#   limit(xx*sin(1/xx),xx,0)
# end
```

Infinite Limits at Infinity

Remark

Determining whether a function has an infinite limit at infinity is useful in analyzing the **"end behavior"** of its graph. You will see examples of this in Section 4.6 on curve sketching.

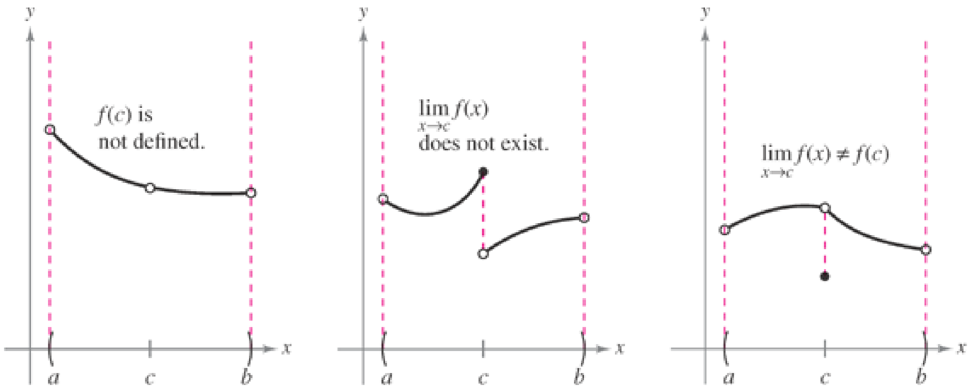
2.4: Continuity and One-Sided Limits

Objectives

> “
>
> - Determine continuity at a point and continuity on an open interval.
> - Determine one-sided limits and continuity on a closed interval.
> - Use properties of continuity.
> - Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

The graph of f is no contnious at $x = c$



In Figure __above__, it appears that continuity at ``x=c`` can be __destroyed__ by any one of __three conditions__.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

- A function f is **continuous on an open interval** (a, b) when the function is continuous at each point in the interval.
- A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Remarks

- If a function f is defined on an open interval I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c.
- Discontinuities fall into two categories:
  - **removable**: A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$.
  - **nonremovable**: there is no way to define $f(c)$ so as to make the function continuous at $x = c$.

Example 1:

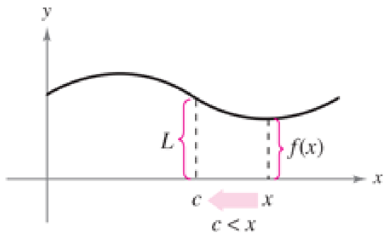
Discuss the continuity of each function

$$
\begin{aligned}
&\text{a.} \quad f(x) = \frac{1}{x} \\
&\text{b.} \quad g(x) = \frac{x^2 - 1}{x - 1} \\
&\text{c.} \quad h(x) = \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases} \\
&\text{d.} \quad y = \sin x
\end{aligned}
$$

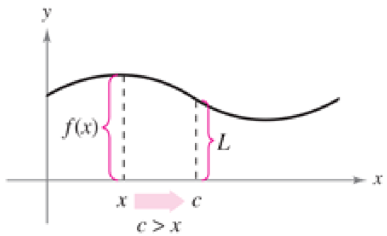
Examples

One-Sided Limits and Continuity on a Closed Interval

(a) Limit from right $\lim_{x \to c^+} f(x) = L$



(a) Limit as x approaches c from the right.



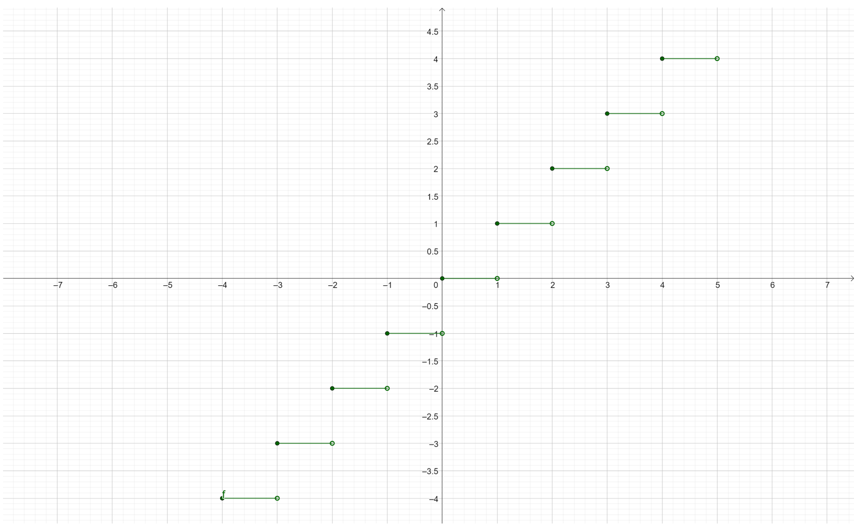
(b) Limit as x approaches c from the left.

(b) Limit from left $\lim_{x \to c^-} f(x) = L$

STEP FUNCTIONS

(**greatest integer function**)

$$[\![x]\!] = \textbf{ greatest integer } n \textbf{ such that } n \leq x.$$



Theorem **The Existence of a Limit**

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is if and only if

$$\lim_{x\to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a,b]$ when f is continuous on the open interval (a,b) and

$$\lim_{x\to a^+} f(x) = f(a)$$

and

$$\lim_{x\to b^-} f(x) = f(b).$$

Example 4: **Continuity on a Closed Interval**

Discuss the continuity of

$$f(x) = \sqrt{1-x^2}$$

Properties of Continuity

Theorem **Properties of Continuity**

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. **Scalar multiple:** bf
2. **Sum or difference:** $f \pm g$
3. **Product:** fg
4. **Quotient:** $\dfrac{f}{g}$, $g(c) \neq 0$,

Remarks

1. **Polynomials** are continuous at every point in their domains.
2. **Rational functions** are continuous at every point in their domains.
3. **Radical functions** are continuous at every point in their domains.
4. **Trigonometric functions** are continuous at every point in their domains.
5. **Exponential and logarithmic functions** are continuous at every point in their domains.

Theorem **Continuity of a Composite Function**

If g is continuous at c and f is continuous at $g(c)$ then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Remark

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

provided f and g satisfy the conditions of the theorem.

Example 7: **Testing for Continuity**

Describe the interval(s) on which each function is continuous.

$$\begin{aligned} &\text{a.} \quad f(x) = \tan x \\ &\text{b.} \quad g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \\ &\text{c.} \quad h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{aligned}$$

Exercises

The Intermediate Value Theorem

Theorem **Intermediate Value Theorem**

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Example 8: **An Application of the Intermediate Value Theorem**

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

3.1: The Derivative and the Tangent Line Problem

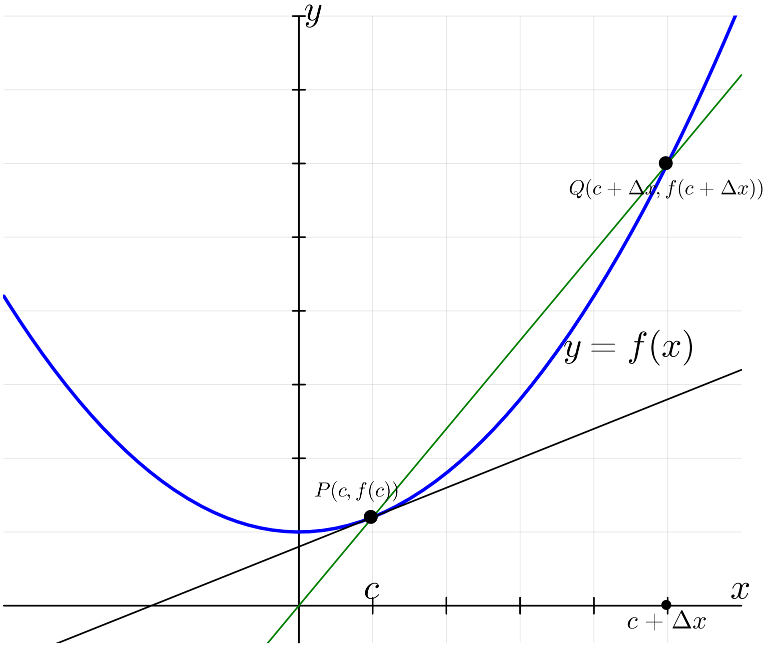
Objectives

> “
>
> - Find the slope of the tangent line to a curve at a point.
> - Use the limit definition to find the derivative of a function.
> - Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Δx 4.0

Find the equation of the secant line



Slope of secant line

$$m_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Definition of Tangent Line with Slope

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Remark

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$**.

Example 1: **The Slope of the Graph of a Linear Function**

Find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$.

Example 2: **Tangent Lines to the Graph of a Nonlinear Function**

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$.

Remarks

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line.
- For vertical tangent lines, you can use the **following definition**. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = -\infty$$

then the **vertical line** $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f.

The Derivative of a Function

Definition **Derivative of a Function**

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Remarks

- The notation $f'(x)$ is read as "f prime of x."
- $f'(x)$ is a **function** that gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.
- The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.
- The process of finding the derivative of a function is called **differentiation**.
- A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval** (a, b) when it is differentiable at every point in the interval.

Notation

$$y = f(x)$$

- $f'(x)$
- $\dfrac{dy}{dx}$
- y'
- $\dfrac{d}{dx}[f(x)]$
- $D_x[y]$

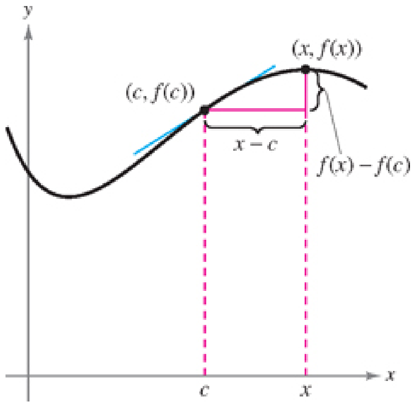
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Examples 3,4,5: **Finding the Derivative by the Limit Process**

Find the derivative of

- $f(x) = x^3 + 2x$
- $f(x) = \sqrt{x}$
- $y = \dfrac{2}{t}$ with respect to t.

Differentiability and Continuity



Alternative form of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Remarks

derivative from the left

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

derivative from the right

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Example:

$$f(x) = [[x]]$$

Example 6: A Graph with a Sharp Turn

$$f(x) = |x - 2|$$

Example 7: A Graph with a Vertical Tangent Line

$$f(x) = x^{\frac{1}{3}}$$

Theorem **Differentiability Implies Continuity**

If f is differentiable at $x = c$, then f is continuous at $x = c$.

remarks

The relationship between continuity and differentiability is summarized below.

- If a function f is differentiable at $x = c$, then it is continuous at $x = c$. So, **differentiability** implies ($\Rightarrow$) **continuity**.
- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, **continuity does not imply differentiability**.

Exercises

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```julia
begin
    using CommonMark, ImageIO, FileIO, ImageShow
    using PlutoUI
    using Plots, PlotThemes, LaTeXStrings, Random
    using PGFPlotsX
    using SymPy
    using HypertextLiteral: @htl, @htl_str
    using ImageTransformations
    using Colors
end
```