

☰ MATH101

2.1: A Preview of Calculus

What is Calculus?
The Tangent Line Problem
The Area Problem

2.2: Finding Limits Graphically and Numerically

An Introduction to Limits
Limits That Fail to Exist
A Formal Definition of Limit (Redaing Only)

2.3: Evaluating Limits Analytically

Properties of Limits
A Strategy for Finding Limits
Dividing Out Technique
Rationalizing Technique
The Squeeze Theorem

2.5: Infinite Limits

Vertical Asymptotes

4.5: Limits at Infinity

Horizontal Asymptotes

Syllabus



2.1: A Preview of Calculus

Objectives

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- Understand what calculus is and how it compares with precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

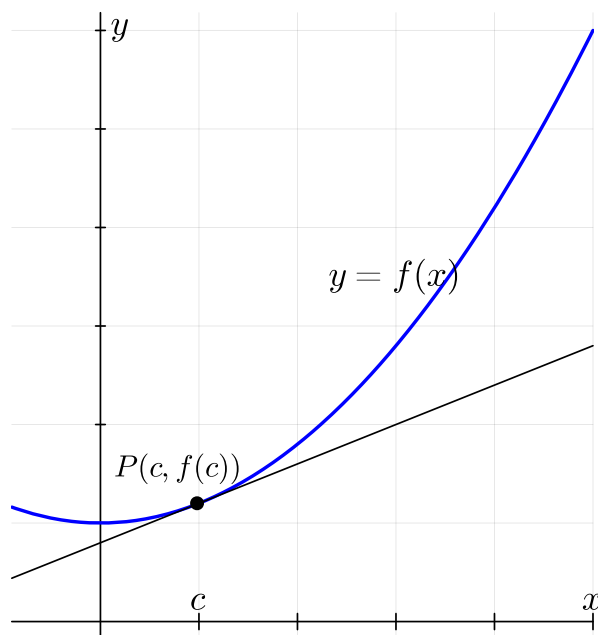
intro.

What is Calculus?

Precalculus Mathematics \Rightarrow Limit process \Rightarrow Calculus

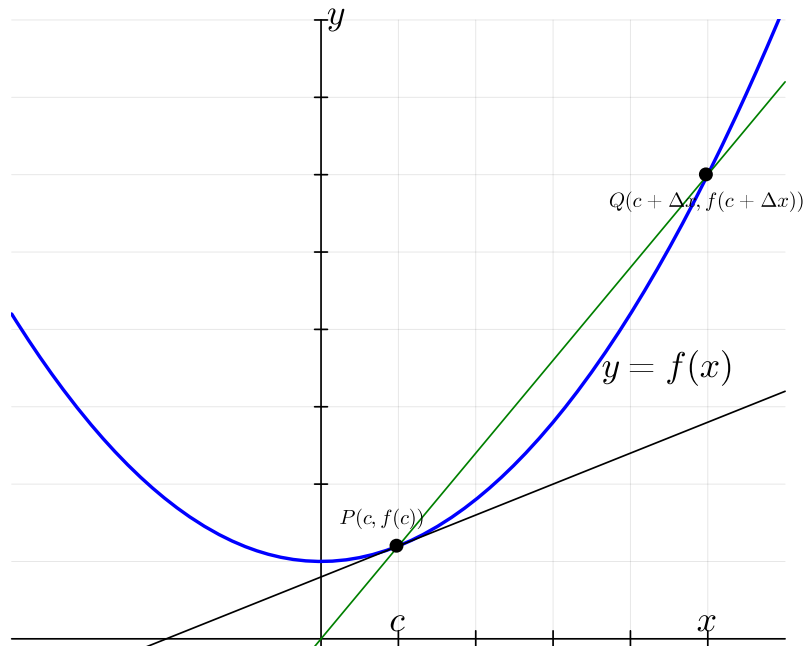
The Tangent Line Problem

What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx 

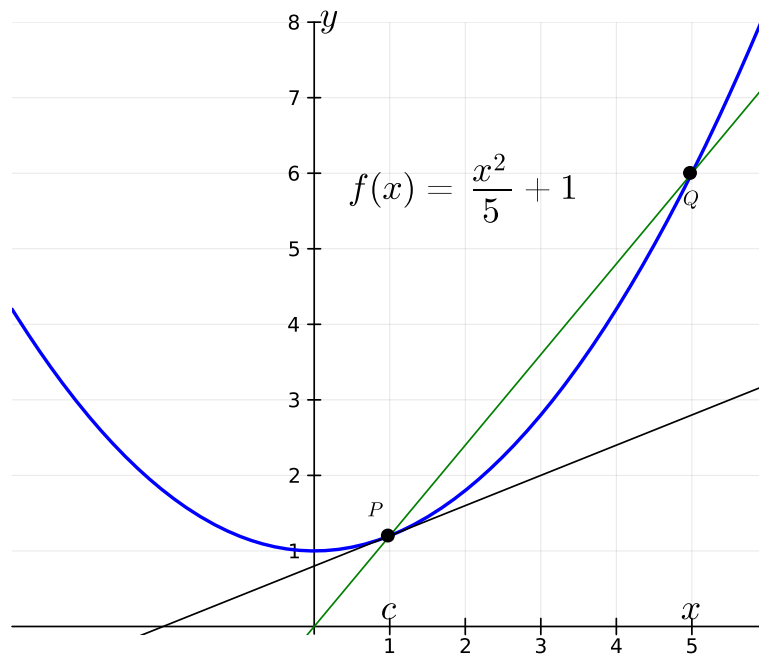
Find the equation of the secant line



$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Δx 

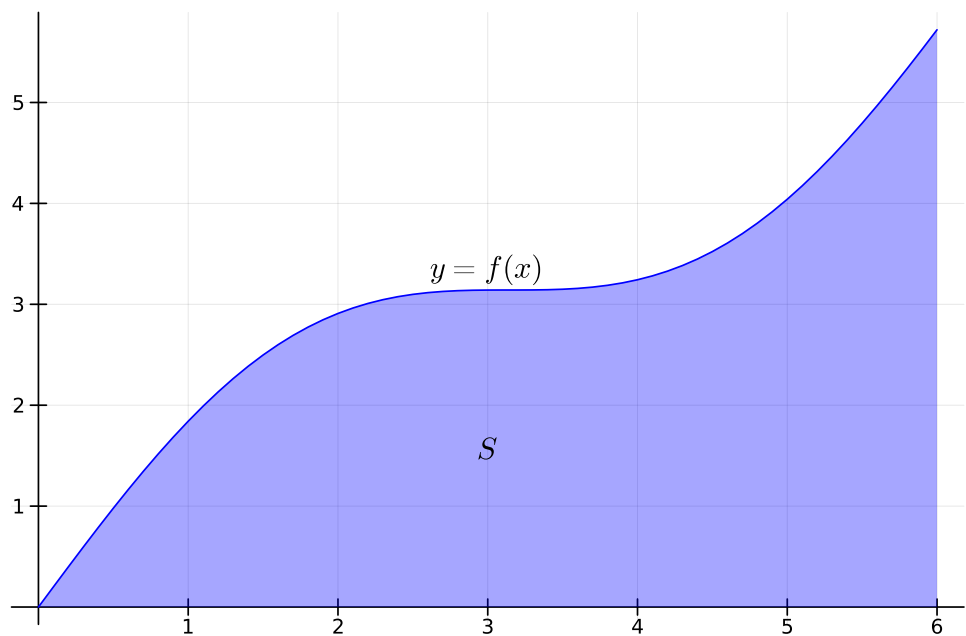
Example: Find the equation of the secant line



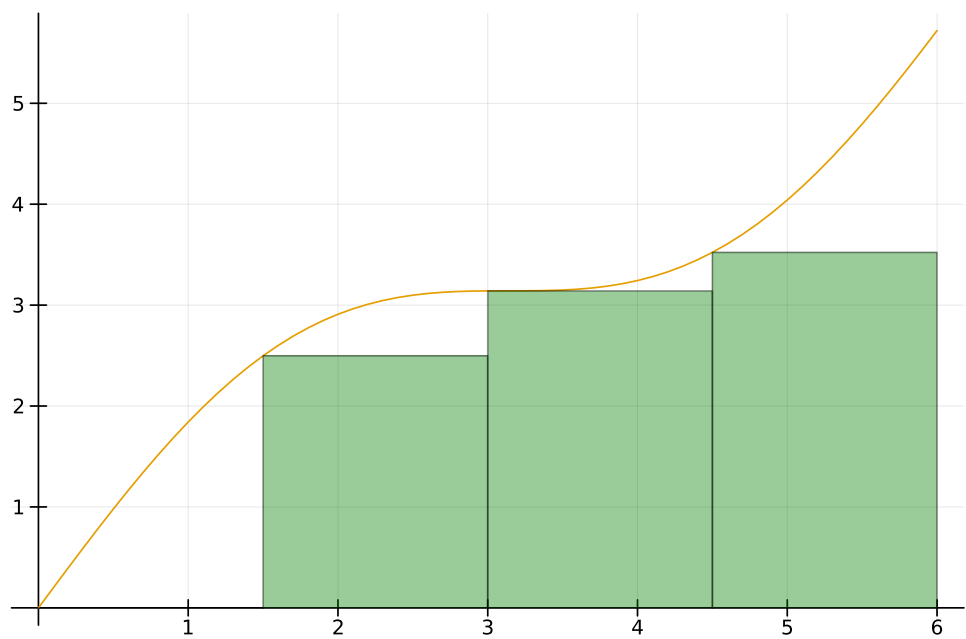
$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = a = b = method =



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

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- Estimate a limit using a **numerical** or **graphical approach**.
- Learn different ways that a limit can fail to exist.
- Study and use a formal definition of limit.

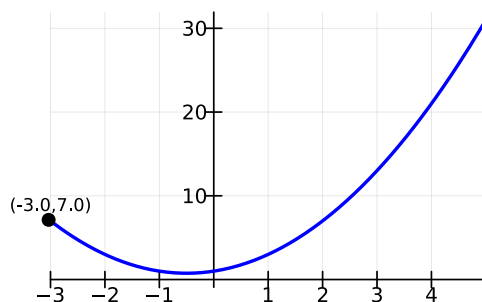
An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$

x approaches 1 from



x approaches 1 (from left)	$f(x)$ approaches
------------------------------	-------------------

-3.0	7.0
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Remark

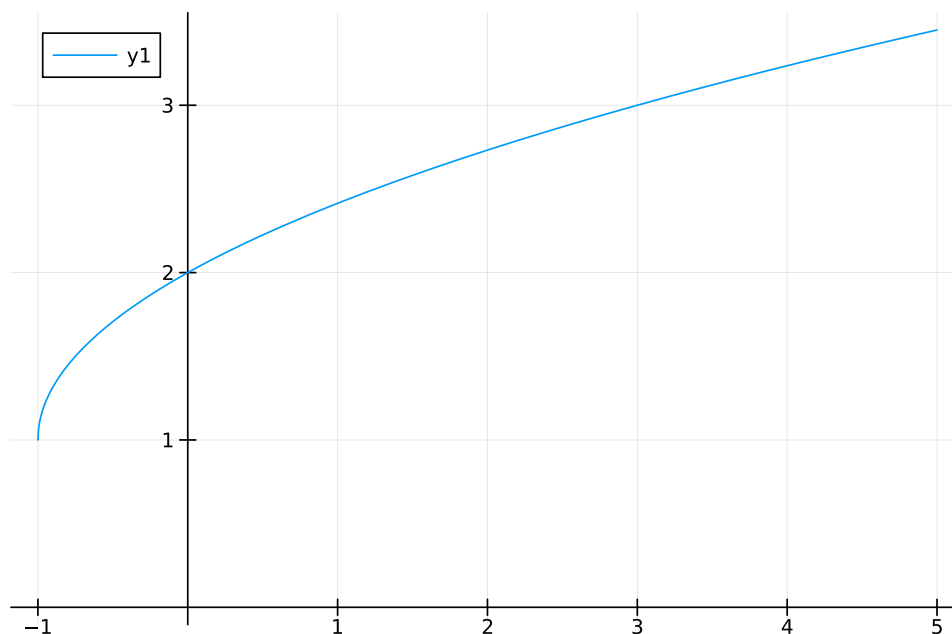
$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1:**Estimating a Limit Numerically**

Evaluate the function $f(x) = \frac{x}{\sqrt{x+1} - 1}$ at several x -values near **0** and use the results to estimate the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph



1.9999995001202078

```
1 begin
2     whatever(x)=x/(sqrt(x+1)-1)
3     whatever(-0.000001)
4 end
```

Example 2:**Finding a Limit**

Find the limit of $f(x)$ as x approaches **2**, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark**Problem solving**

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: MATLAB, python, Julia)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist



Example 3:**Different Right and Left Behavior**

Show that the limit $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

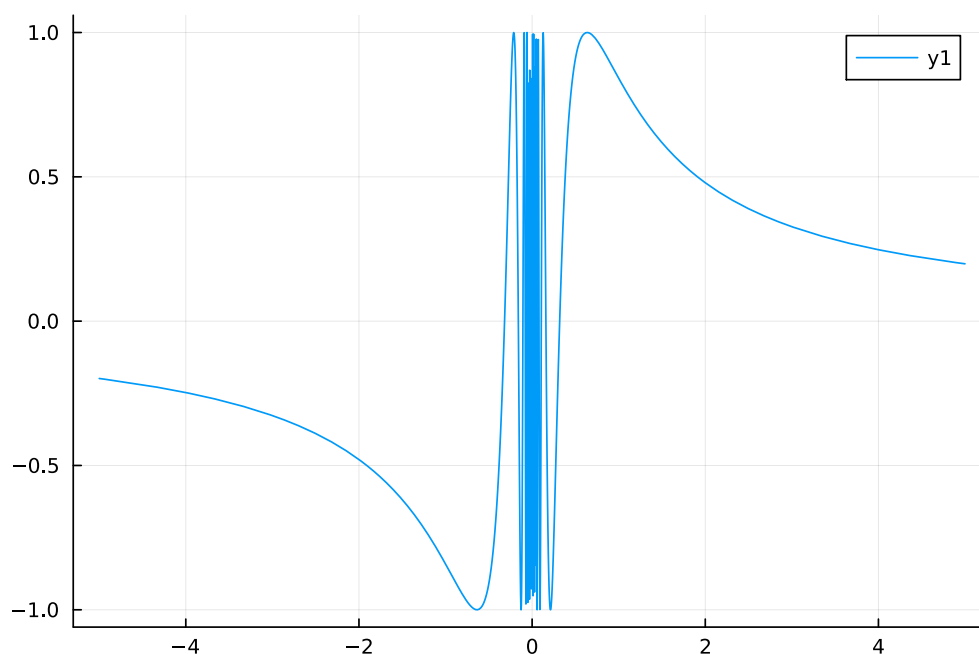
Example 4:**Unbounded Behavior**

Discuss the existence of the limit $\lim_{x \rightarrow 0} \frac{1}{x^2}$

Example 5:**Oscillating Behavior**

Discuss the existence of the limit $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

9.999999999999998e9



```
1 plot(x->sin(1/x))
```

```
1 begin
2   x=1
3   (x,sin(1 / x))
4 end
```

A Formal Definition of Limit (Redaing Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \rightarrow c} f(x) = L$$

implies two statements—the limit exists and the limit is L .

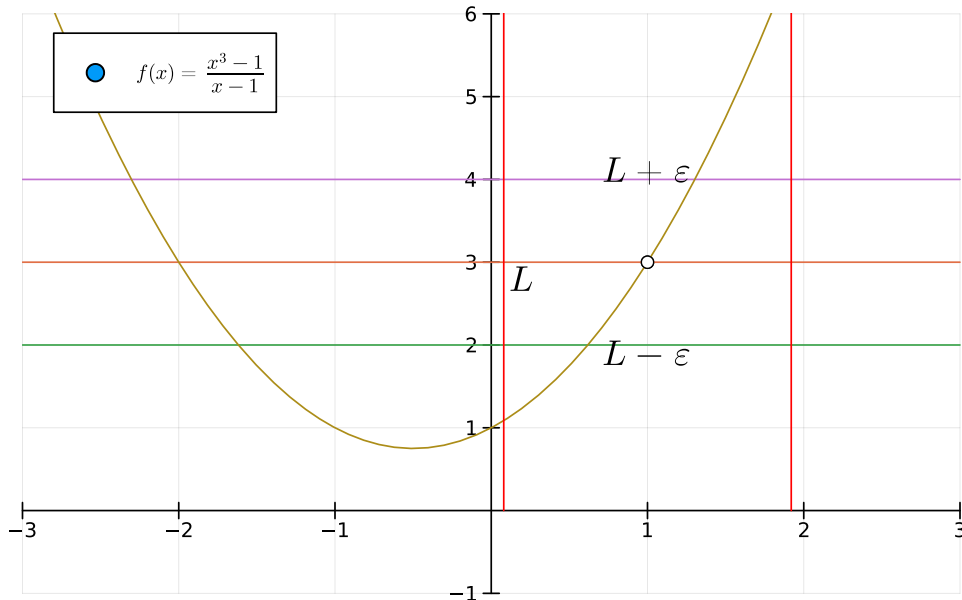
Example:

Prove that

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

$\epsilon =$  1.0 $\delta =$  0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

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- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem

Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$

Theorem**Properties of Limits**

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \rightarrow c} f(x) = L, \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. **Scalar multiple** $\lim_{x \rightarrow c} [bf(x)] = bL$
2. **Sum or difference** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. **Product** $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. **Quotient** $\lim_{x \rightarrow c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}, \quad K \neq 0$
5. **Power** $\lim_{x \rightarrow c} [f(x)]^n = L^n$

Example 2:**The Limit of a Polynomial**

Find $\lim_{x \rightarrow 2} (4x^2 + 3)$.

Theorem**Limits of Polynomial and Rational Functions**

If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3:**The Limit of a Rational Function**

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x + 2}{x + 1}.$$

Theorem**The Limit of a Function Involving a Radical**

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem**The Limit of a Composite Function**

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow c} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

Theorem**Limits of Transcendental Functions**

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \rightarrow c} \sin(x) = \sin(c)$
2. $\lim_{x \rightarrow c} \cos(x) = \cos(c)$
3. $\lim_{x \rightarrow c} \tan(x) = \tan(c)$
4. $\lim_{x \rightarrow c} \cot(x) = \cot(c)$
5. $\lim_{x \rightarrow c} \sec(x) = \sec(c)$
6. $\lim_{x \rightarrow c} \csc(x) = \csc(c)$
7. $\lim_{x \rightarrow c} a^x = a^c, \quad a > 0$
8. $\lim_{x \rightarrow c} \ln(x) = \ln(c)$



A Strategy for Finding Limits

Theorem

Functions That Agree at All but One Point

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x).$$

Remarks

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c .

Dividing Out Technique

Example 7:

Dividing Out Technique

Find the limit $\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8:

Rationalizing Technique

Find the limit $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$.

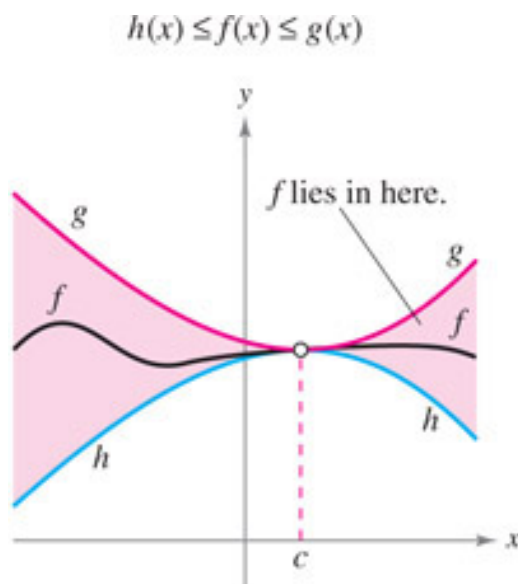
The Squeeze Theorem

Theorem The Squeeze Theorem

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then $\lim_{x \rightarrow c} f(x)$ exists and equal to L .



Theorem Three Special Limits

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$
3. $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e.$

Example 9: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\tan x}{x}.$

Example 10: A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}.$



2.5: Infinite Limits

Objectives

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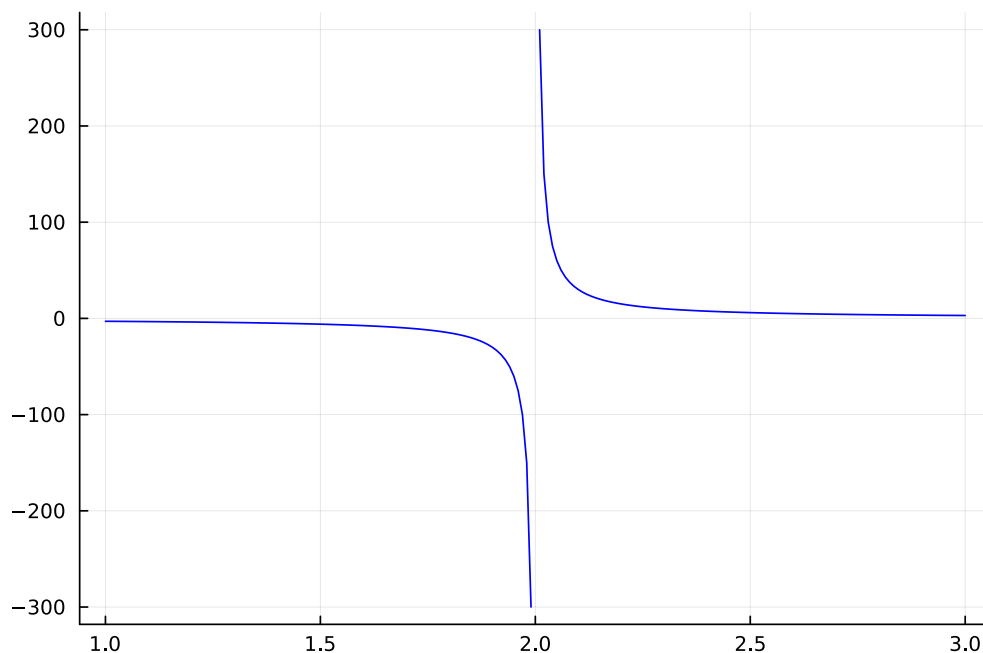
- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Example:

Infinite Limit

Consider

$$f(x) = \frac{3}{x - 2}$$



```
1 plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c .

Theorem Vertical Asymptotes

Let f and g be continuous on an open interval containing c . If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

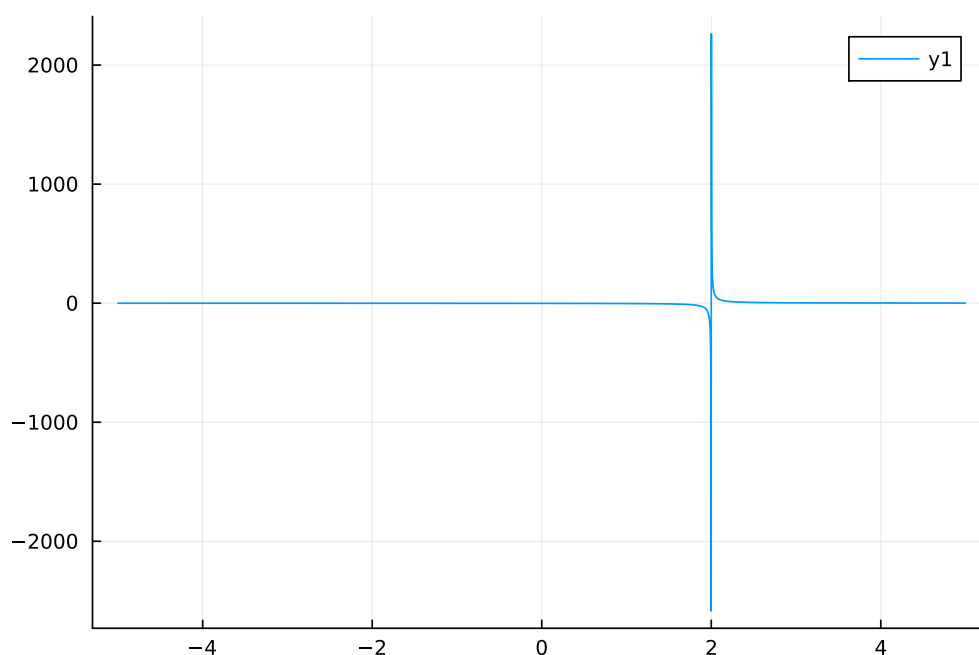
has a vertical asymptote at c .

Example 2:**Finding Vertical Asymptotes**

1. $h(x) = \frac{1}{2(x+1)}.$

$$2. h(x) = \frac{x^2 + 1}{x^2 - 1}.$$

$$3. h(x) = \cot x = \frac{\cos x}{\sin x}.$$



```
1 plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- [desmos.com](https://www.desmos.com)
- [geogebra.org](https://www.geogebra.org)

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x \rightarrow 1^-} \frac{x^3 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^3 - 3x}{x - 1}$$

Theorem**Properties of Infinite Limits**

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L$$

1. **Sum or difference:** $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$

2. **Product:**

$$\lim_{x \rightarrow c} [f(x)g(x)] = \infty, \quad L > 0$$

$$\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, \quad L < 0$$

3. **Quotient:** $\lim_{x \rightarrow c} \left[\frac{g(x)}{f(x)} \right] = 0$

Remark

2. is **not true** if $\lim_{x \rightarrow c} g(x) = 0$

Exercises

4.5: Limits at Infinity

Objectives

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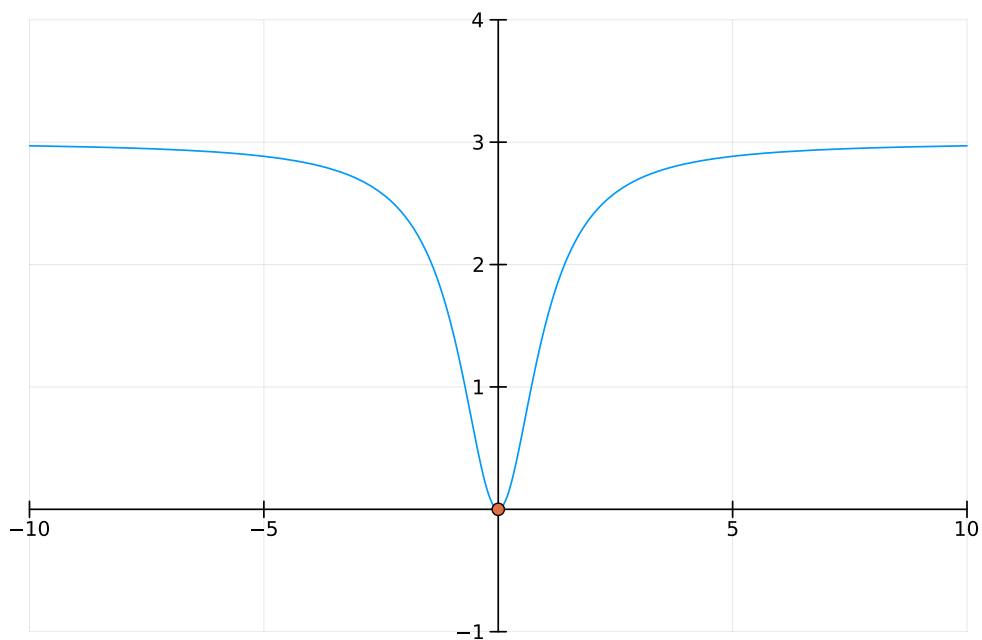
- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2 + 1}$$

$$x = 0$$

$$f(x) = 0.0$$



we write

$$\lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + 1} = 3, \quad \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2 + 1} = 3$$

Horizontal Asymptotes

1 Enter cell code...

```
1 # begin
2 #   xx=symbols("xx",real=true)
3 #   limit(xx*sin(1/xx),xx,0)
4 # end
```

""

""

```
1 begin
2   using CommonMark, ImageIO, FileIO, ImageShow
3   using PlutoUI
4   using Plots, PlotThemes, LaTeXStrings, Random
5   using PGFPlotsX
6   using SymPy
7   using HypertextLiteral: @html, @html_str
8   using ImageTransformations
9   using Colors
10 end
```