MATH101

2.1: A Preview of Calculus

Objectives

> “
>
> - Understand what calculus is and how it compares with precalculus.
> - Understand that the tangent line problem is basic to calculus.
> - Understand that the area problem is also basic to calculus.

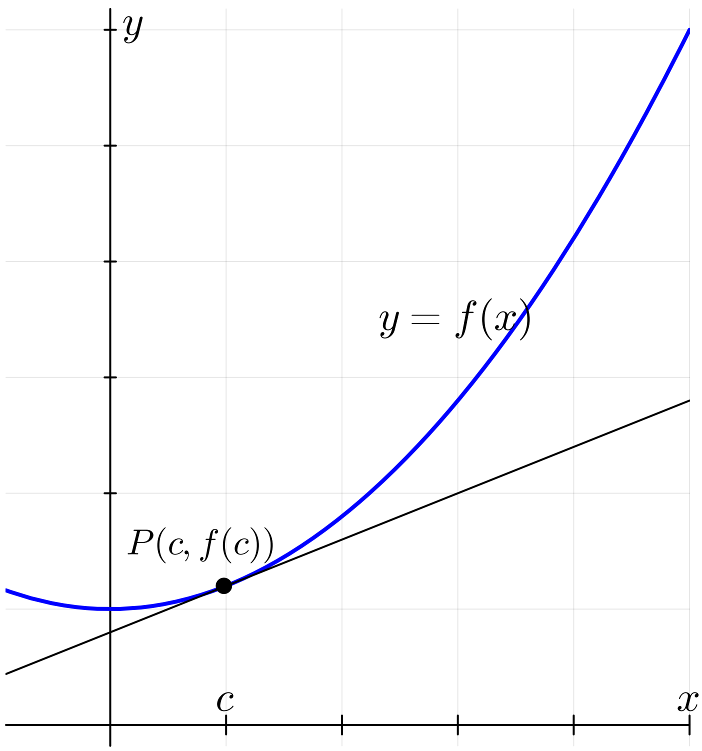
intro.

What is Calculus?

Precalculus Matematics $\Rightarrow$ Limit process $\Rightarrow$ *Calculus*

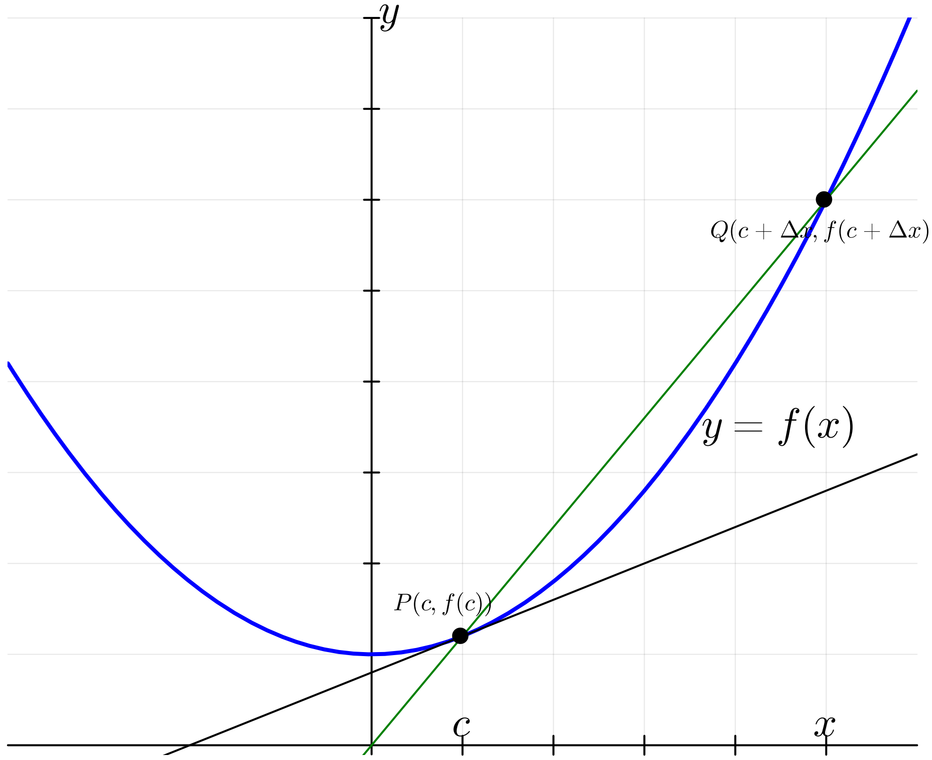
The Tangent Line Problem

What is the slope of the line (called *tangent line*) passing through the point $P(c, f(c))$?



Δx

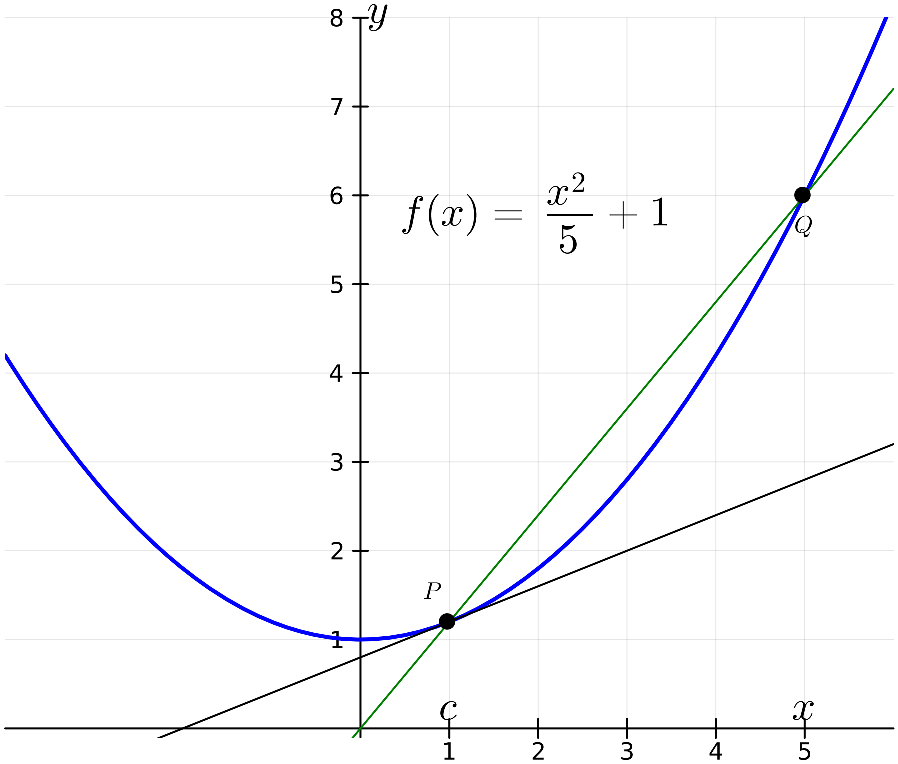
Find the equation of the secant line



$$\mathbf{m}_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Δx

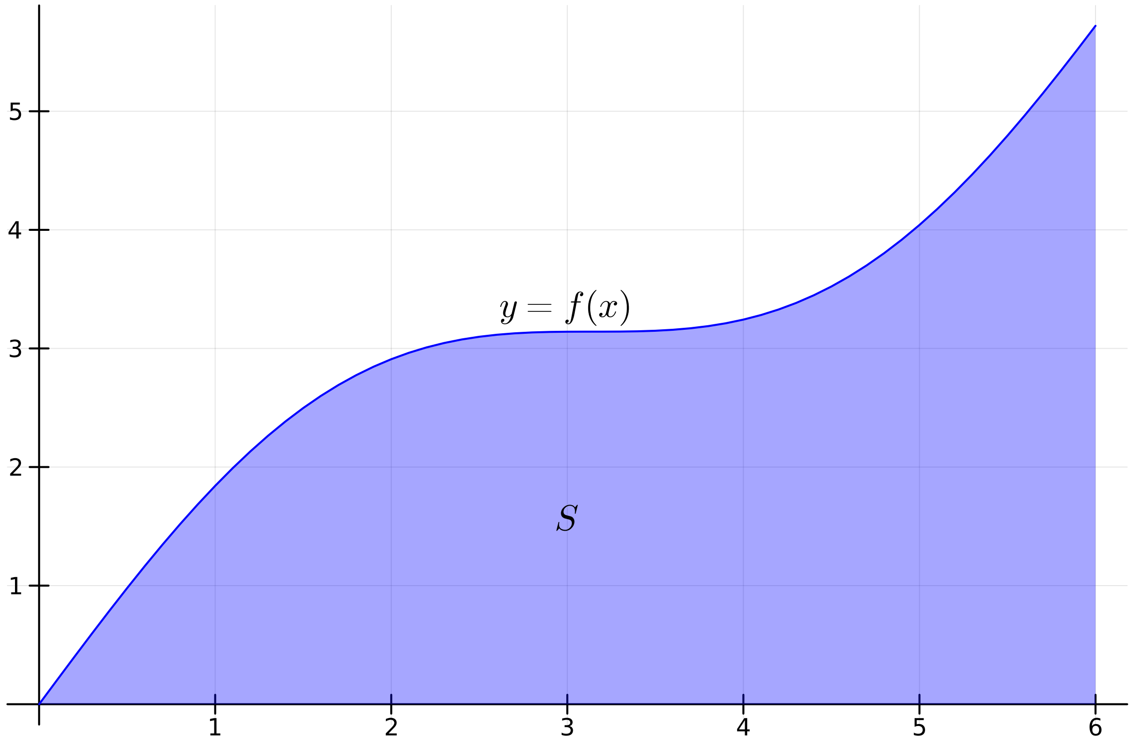
Example: **Find the equation of the secant line**



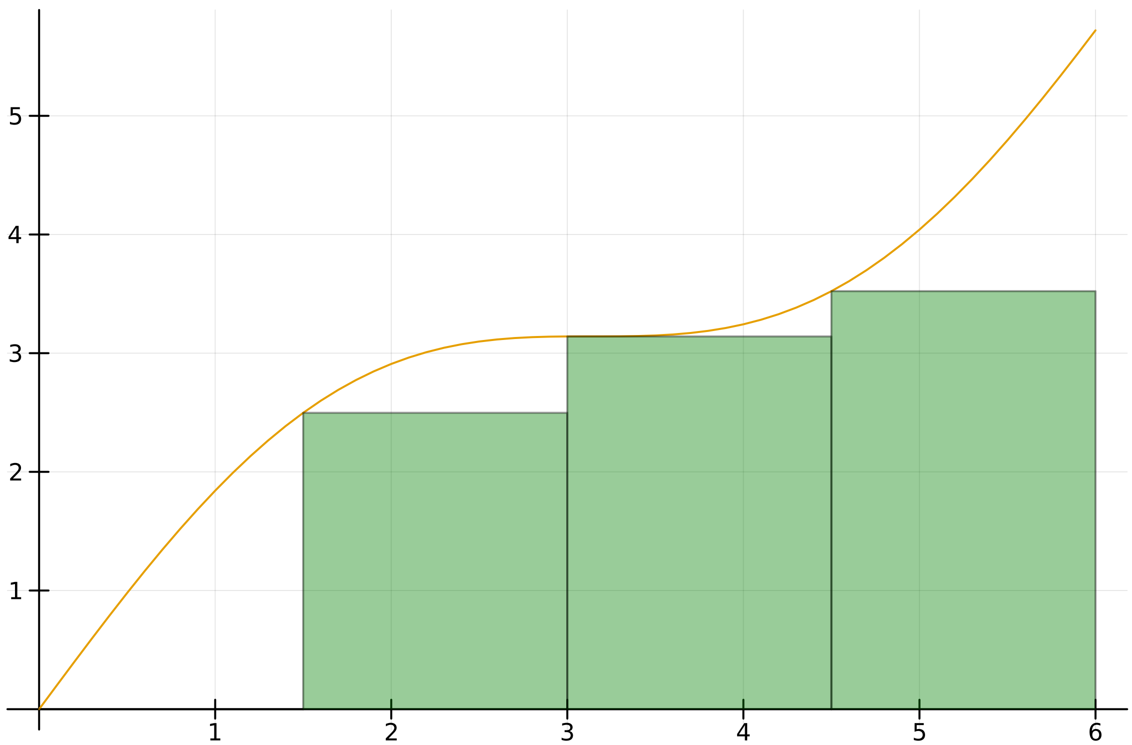
$$\mathrm{m}_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x} = 1.2$$

The Area Problem

Find the area under the curve?



n = 4 a = 0 b = 6 method = Left



outro.

2.2: Finding Limits Graphically and Numerically

Objectives

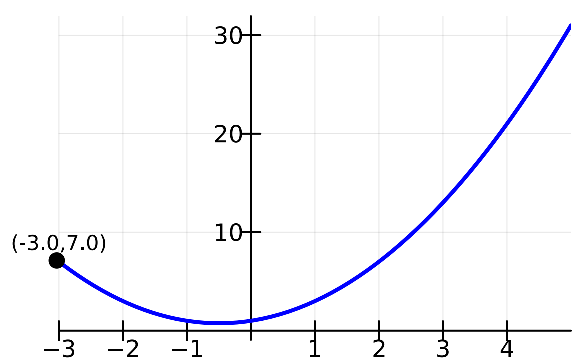
> “
>
> - Estimate a limit using a **numerical** or **graphical approach**.
> - Learn different ways that a limit can fail to exist.
> - <s>Study and use a formal definition of limit</s>.

An Introduction to Limits

Consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

$\Delta x =$ 0.0 x **approaches** 1 from Left



x approacheds 1 (from left)	**$f(x)$ approaches**
-3.0	7.0

Remark

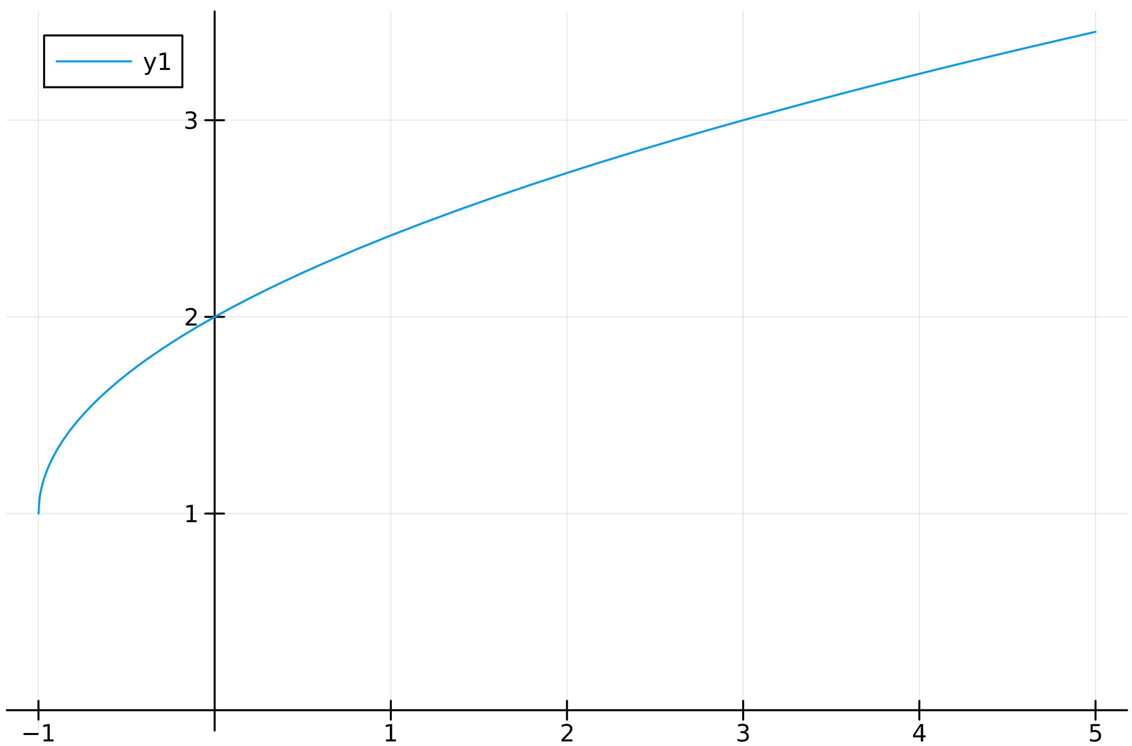
$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

Example 1: Estimating a Limit Numerically

Evaluate the function $f(x) = \dfrac{x}{\sqrt{x+1} - 1}$ at several x-values near 0 and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Graph




```
1.9999995001202078
```

```julia
begin
    whatever(x)=x/(sqrt(x+1)-1)
    whatever(-0.000001)
end
```

Example 2: **Finding a Limit**

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2, \\ 0, & x = 2 \end{cases}$$

Remark **Problem solving**

1. Numerical values (using table of values)
2. Graphical (drawing a graph by hand or by technology: `MATLAB`, `python`, `Julia`)
3. Analytical (using algebra or of course **calculus**)

Limits That Fail to Exist

Example 3: **Different Right and Left Behavior**

Show that the limit $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

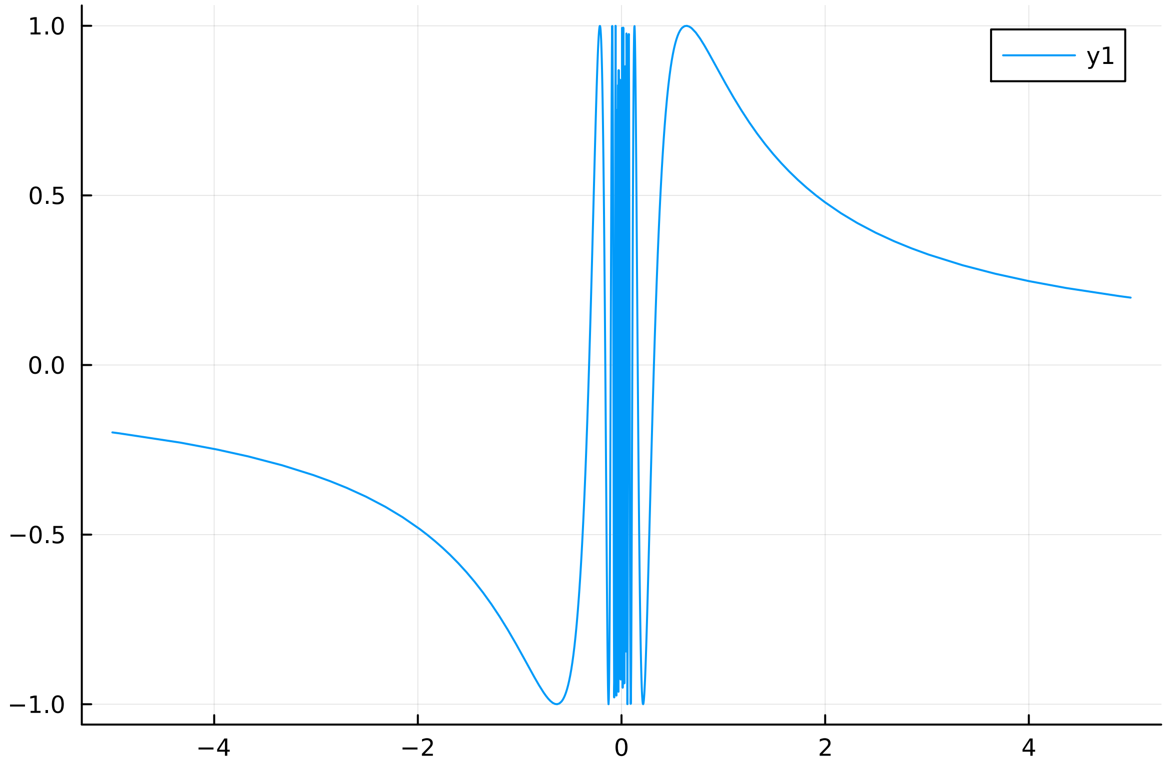
Example 4: **Unbounded Behavior**

Discuss the existence of the limit $\lim\limits_{x \to 0} \dfrac{1}{x^2}$

Example 5: **Oscillating Behavior**

Discuss the existence of the limit $\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

```
9.999999999999998e9
```




```
plot(x->sin(1/x))
```

A Formal Definition of Limit (Redaing Only)

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \epsilon$$

Remark

Throughout this text, the expression

$$\lim_{x \to c} f(x) = L$$

implies two statements—the limit exists and the limit is L.

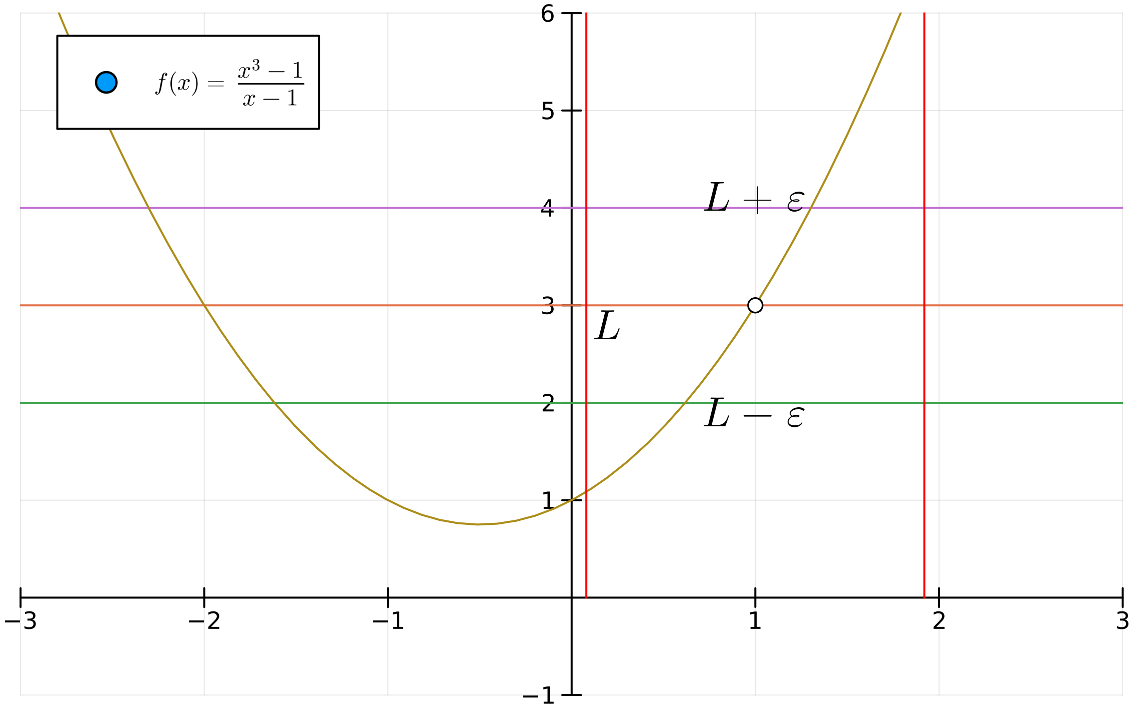
Example:

Prove that

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

ϵ = 1.0 δ = 0.92

Example 1 (Graph)



2.3: Evaluating Limits Analytically

Objectives

> - Evaluate a limit using properties of limits.
> - Develop and use a strategy for finding limits.
> - Evaluate a limit using the dividing out technique.
> - Evaluate a limit using the rationalizing technique.
> - Evaluate a limit using the Squeeze Theorem.

Properties of Limits

Theorem **Some Bacic Limits**

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Theorem **Properties of Limits**

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L, \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. **Scalar multiple** $\lim_{x \to c} \left[bf(x)\right] = bL$
2. **Sum or difference** $\lim_{x \to c} \left[f(x) \pm g(x)\right] = L \pm K$
3. **Product** $\lim_{x \to c} \left[f(x)g(x)\right] = LK$
4. **Quotient** $\lim_{x \to c} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{K}, \quad K \neq 0$
5. **Power** $\lim_{x \to c} \left[f(x)\right]^n = L^n$

Example 2: **The Limit of a Polynomial**

Find $\lim_{x \to 2} (4x^2 + 3)$.

Theorem **Limits of Polynomial and Rational Functions**

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3: The Limit of a Rational Function

Find

$$\lim_{x\to 1} \frac{x^2+x+2}{x+1}.$$

Theorem **The Limit of a Function Involving a Radical**

Let n be a positive integer. The limit below is valid for all c when n is **odd**, and is valid for $c > 0$ when n is **even**.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem **The Limit of a Composite Function**

If f and g are functions such that $\lim\limits_{x\to c} g(x) = L$ and $\lim\limits_{x\to c} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Theorem **Limits of Transcendental Functions**

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \to c} \sin(x) = \sin(c)$
2. $\lim_{x \to c} \cos(x) = \cos(c)$
3. $\lim_{x \to c} \tan(x) = \tan(c)$
4. $\lim_{x \to c} \cot(x) = \cot(c)$
5. $\lim_{x \to c} \sec(x) = \sec(c)$
6. $\lim_{x \to c} \csc(x) = \csc(c)$
7. $\lim_{x \to c} a^x = a^c, \quad a > 0$
8. $\lim_{x \to c} \ln(x) = \ln(c)$

A Strategy for Finding Limits

Theorem **Functions That Agree at All but One Point**

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Remarks **A Strategy for Finding Limits**

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function $g(x)$ that agrees with f for all other x than c.

Dividing Out Technique

Example 7: **Dividing Out Technique**

Find the limit $\displaystyle \lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalizing Technique

Recall

- **rationalizing** the numerator (denominator) means **multiplying** the numerator and denominator by **the conjugate** of the numerator (denominator)

Example 8: **Rationalizing Technique**

Find the limit $\displaystyle \lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$.

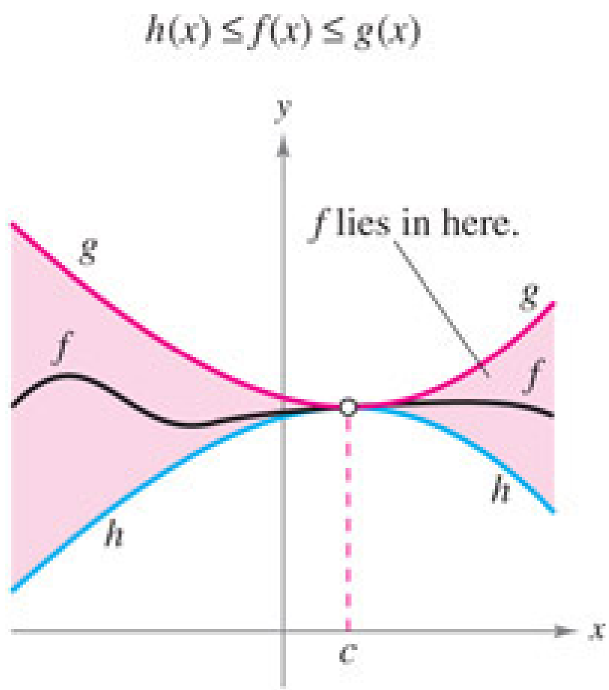
The Squeeze Theorem

Theorem **The Squeeze Theorem**

if $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ **exists** and equal to L.



Theorem **Three Special Limits**

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1.$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$
3. $\lim_{x \to 0} (1 + x)^{1/x} = e.$

Example 9: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

Example 10: **A Limit Involving a Trigonometric Function**

Find the limit: $\lim\limits_{x\to 0}\dfrac{\sin 4x}{x}$.

Exercises

2.5: Infinite Limits

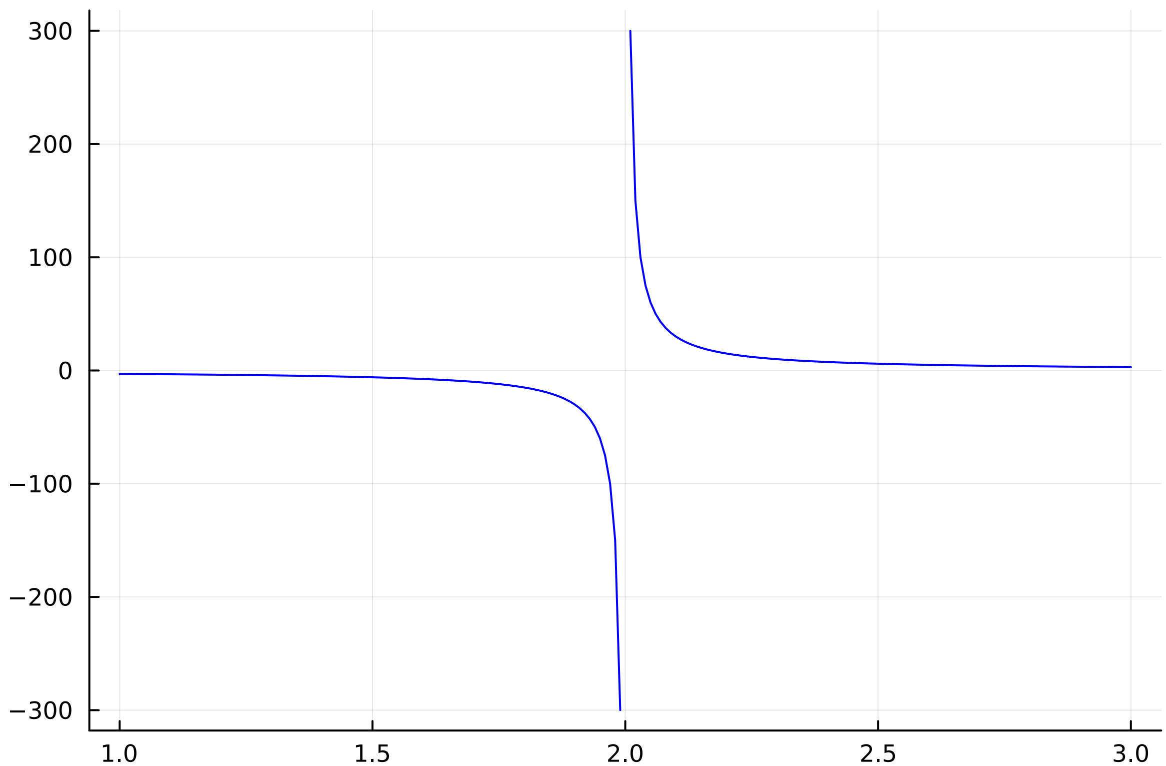
Objectives

> - Determine infinite limits from the left and from the right.
> - Find and sketch the vertical asymptotes of the graph of a function.

Example: **Infinite Limit**

Consider

$$f(x) = \frac{3}{x-2}$$




```
plot(1:0.01:1.99,x->3/(x-2),label=nothing,c=:blue);plot!(2.01:0.01:3,x->3/(x-2),label=nothing,c=:blue)
```

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of .

Remark

If the graph of a function f has a vertical asymptote at $x = c$, then f is not continuous at c.

Theorem **Vertical Asymptotes**

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at c.

Example 2: Finding Vertical Asymptotes

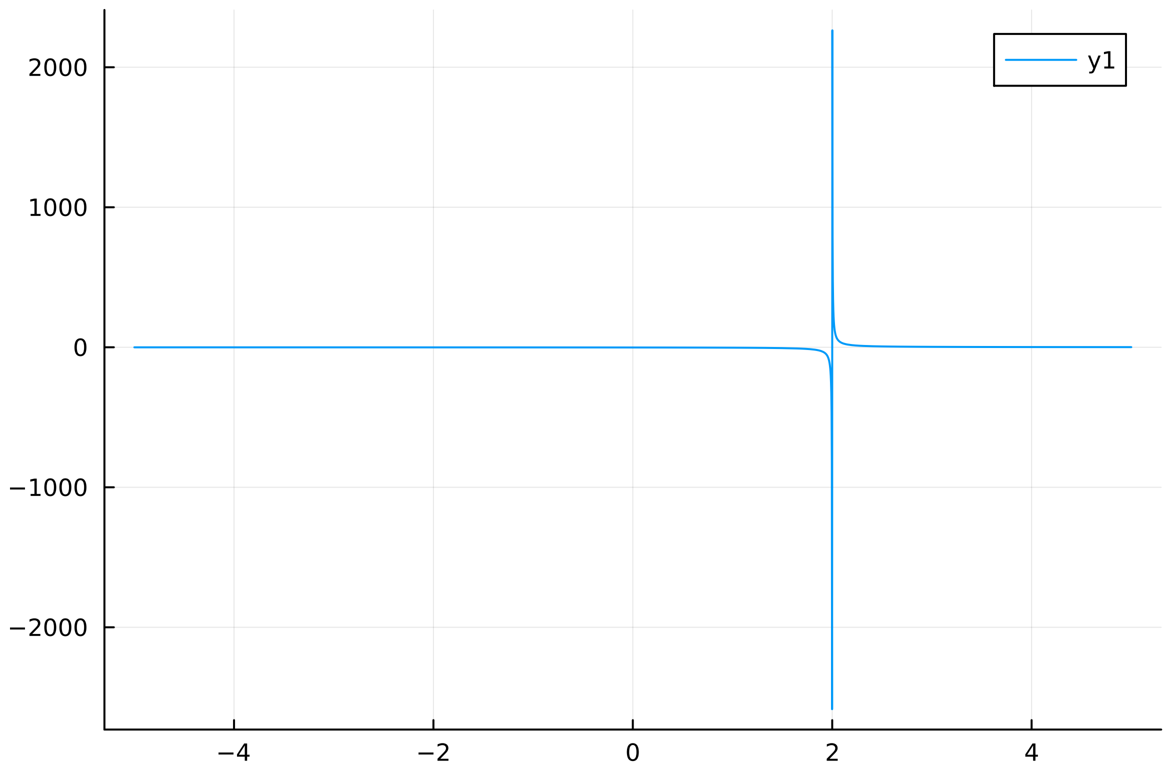
1. $h(x) = \dfrac{1}{2(x+1)}$.

---

2. $h(x) = \dfrac{x^2+1}{x^2-1}$.

---

3. $h(x) = \cot x = \dfrac{\cos x}{\sin x}$.




```
plot(x->3/(x-2))
```

Remark

There are good online graphing tools that you use

- desmos.com
- geogebra.org

Example 3: A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4: Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^3 - 3x}{x - 1} \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{x^3 - 3x}{x - 1}$$

Theorem **Properties of Infinite Limits**

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L$$

1. **Sum or difference**: $\lim\limits_{x \to c} \left[f(x) \pm g(x)\right] = \infty$
2. **Product**:

$$\begin{aligned} \lim_{x \to c} \left[f(x) g(x)\right] &= \infty, \quad L > 0 \\ \lim_{x \to c} \left[f(x) g(x)\right] &= -\infty, \quad L < 0 \end{aligned}$$

3. Quotient: $\lim\limits_{x \to c} \left[\dfrac{g(x)}{f(x)}\right] = 0$

Remark

2. is **not true** if $\lim\limits_{x \to c} g(x) = 0$

Exercises

4.5: Limits at Infinity

Objectives

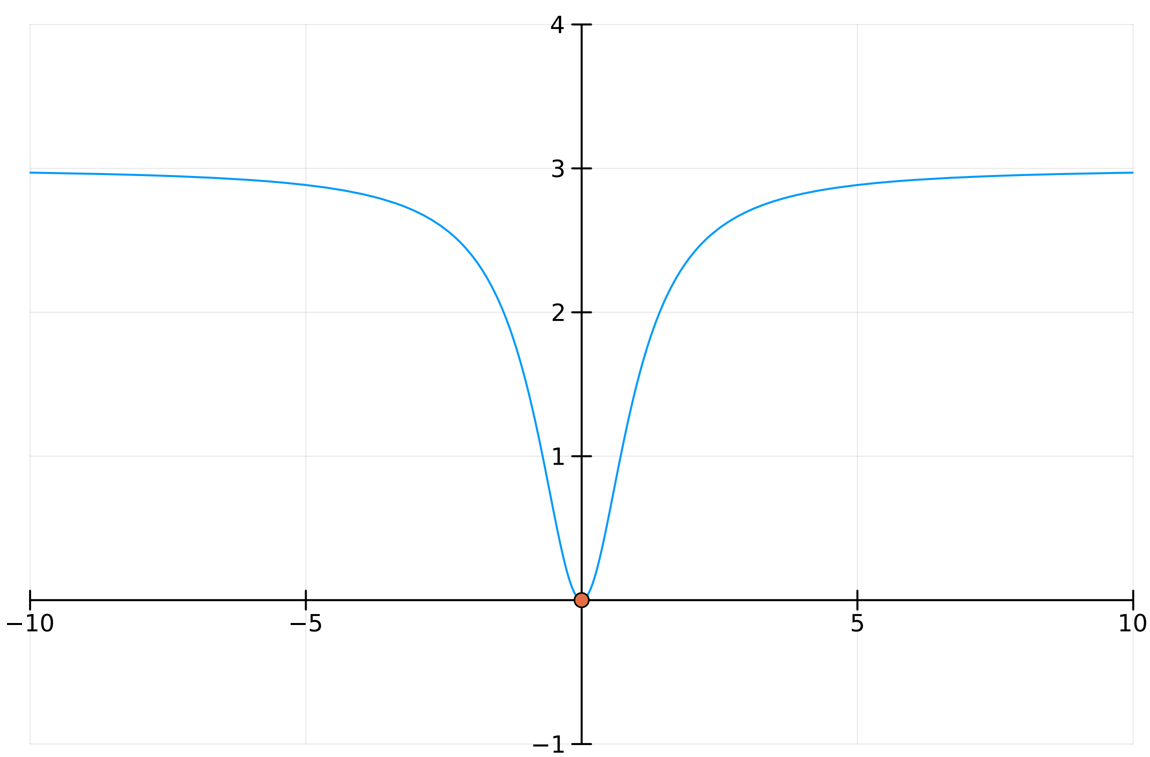
> “
>
> - Determine (finite) limits at infinity.
> - Determine the horizontal asymptotes, if any, of the graph of a function.
> - Determine infinite limits at infinity.

Consider

$$f(x) = \frac{3x^2}{x^2+1}$$

$x =$ 0

$f(x) =$ 0.0



we write

$$\lim_{x\to\infty} \frac{3x^2}{x^2+1} = 3, \quad \lim_{x\to-\infty} \frac{3x^2}{x^2+1} = 3$$

Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L$$

Remarks

- Limits at infinity have many of the same properties of limits discussed in Section 2.3.
- For example, if $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ both exist, then
  - $\lim\limits_{x\to\infty} [f(x)+g(x)] = \lim\limits_{x\to\infty} f(x) + \lim\limits_{x\to\infty} g(x)$
  - $\lim\limits_{x\to\infty} [f(x)g(x)] = \left[\lim\limits_{x\to\infty} f(x)\right]\left[\lim\limits_{x\to\infty} g(x)\right]$
- Similar properties hold for limits at $-\infty$.

Theorem **Limits at Infinity**

1. If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim\limits_{x\to-\infty} e^x = 0$ and $\lim\limits_{x\to\infty} e^{-x} = 0$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

$$h(x) = \frac{p(x)}{q(x)}$$

1. $\deg p < \deg q$, then the limit is 0.
2. $\deg p = \deg q$, then the **limit** of the rational function **is** the **ratio** of the **leading coefficients**.
3. $\deg p > \deg q$, then the **limit** of the rational function **does not exist**.

Examples

```
# begin
#   xx=symbols("xx",real=true)
#   limit(xx*sin(1/xx),xx,0)
# end
```

Infinite Limits at Infinity

> **Remark**
>
> Determining whether a function has an infinite limit at infinity is useful in analyzing the **“end behavior”** of its graph. You will see examples of this in Section 4.6 on curve sketching.

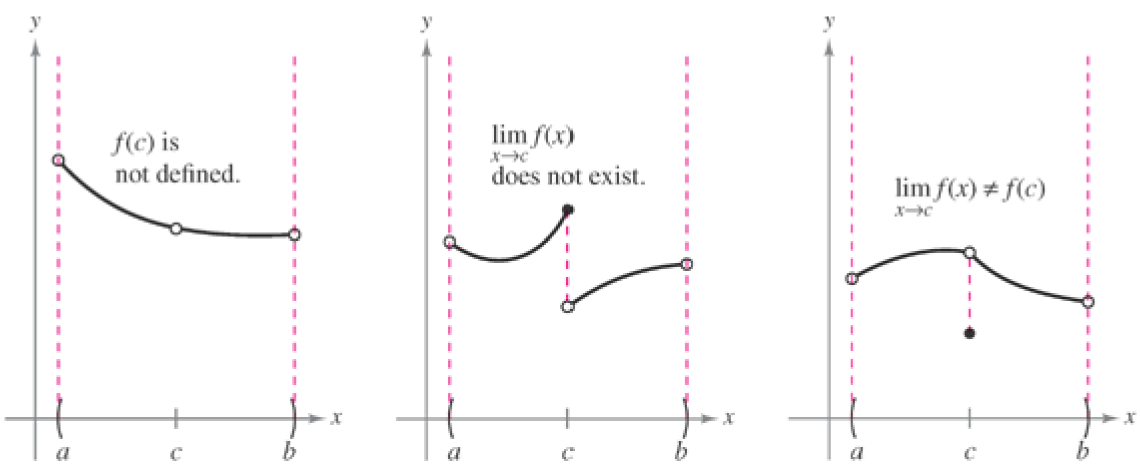
2.4: Continuity and One-Sided Limits

Objectives

> “
>
> - Determine continuity at a point and continuity on an open interval.
> - Determine one-sided limits and continuity on a closed interval.
> - Use properties of continuity.
> - Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

The graph of f is no contnious at $x = c$



In Figure __above__, it appears that continuity at ``x=c`` can be __destroyed__ by any one of __three conditions__.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

Definition of Continuity

Continuity at a Point

A function f is **continuous at** c when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

- A function f is **continuous on an open interval** (a, b) when the function is continuous at each point in the interval.
- A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Remarks

- If a function f is defined on an open interval I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c.
- Discontinuities fall into two categories:
    - **removable**: A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$.
    - **nonremovable**: there is no way to define $f(c)$ so as to make the function continuous at $x = c$.

Example 1:

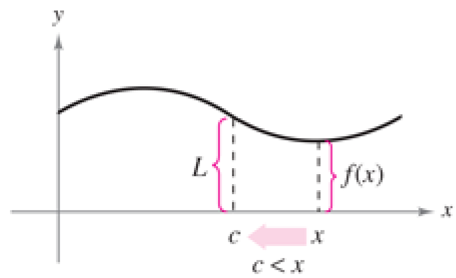
Discuss the continuity of each function

$$
\begin{aligned}
&\text{a.} \quad f(x) = \frac{1}{x} \\
&\text{b.} \quad g(x) = \frac{x^2 - 1}{x - 1} \\
&\text{c.} \quad h(x) = \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases} \\
&\text{d.} \quad y = \sin x
\end{aligned}
$$

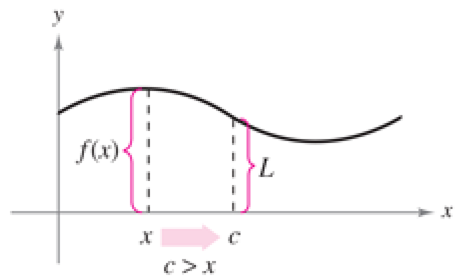
Examples

One-Sided Limits and Continuity on a Closed Interval

(a) Limit from right $\lim_{x \to c^+} f(x) = L$



(a) Limit as x approaches c from the right.



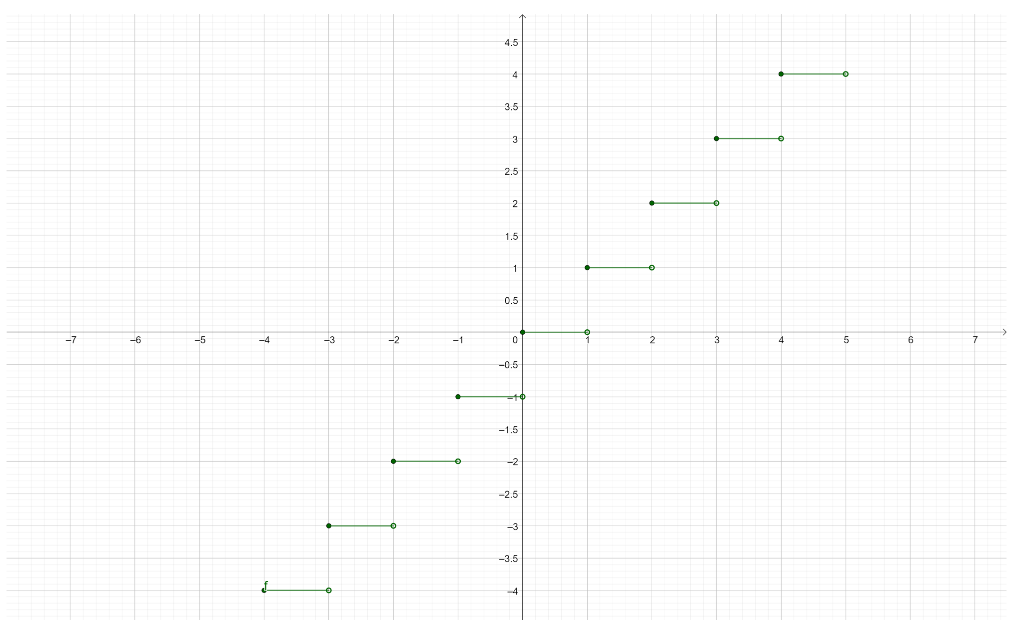
(b) Limit as x approaches c from the left.

(b) Limit from left $\lim_{x \to c^-} f(x) = L$

STEP FUNCTIONS

(**greatest integer function**)

$$[\![x]\!] = \textbf{greatest integer } n \textbf{ such that } n \leq x.$$



Theorem **The Existence of a Limit**

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is if and only if

$$\lim_{x\to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a)$$

and

$$\lim_{x\to b^-} f(x) = f(b).$$

Example 4: **Continuity on a Closed Interval**

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}$$

Properties of Continuity

Theorem **Properties of Continuity**

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. **Scalar multiple:** bf
2. **Sum or difference:** $f \pm g$
3. **Product:** fg
4. **Quotient:** $\dfrac{f}{g}, \quad g(c) \neq 0$,

Remarks

1. **Polynomials** are continuous at every point in their domains.
2. **Rational functions** are continuous at every point in their domains.
3. **Radical functions** are continuous at every point in their domains.
4. **Trigonometric functions** are continuous at every point in their domains.
5. **Exponential and logarithmic functions** are continuous at every point in their domains.

Theorem **Continuity of a Composite Function**

If g is continuous at c and f is continuous at $g(c)$ then the **composite function** given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Remark

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

provided f and g satisfy the conditions of the theorem.

Example 7: **Testing for Continuity**

Describe the interval(s) on which each function is continuous.

$$\begin{array}{ll} \text{a.} & f(x) = \tan x \\ \text{b.} & g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \\ \text{c.} & h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{array}$$

Exercises

The Intermediate Value Theorem

Theorem **Intermediate Value Theorem**

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Example 8: **An Application of the Intermediate Value Theorem**

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

3.1: The Derivative and the Tangent Line Problem

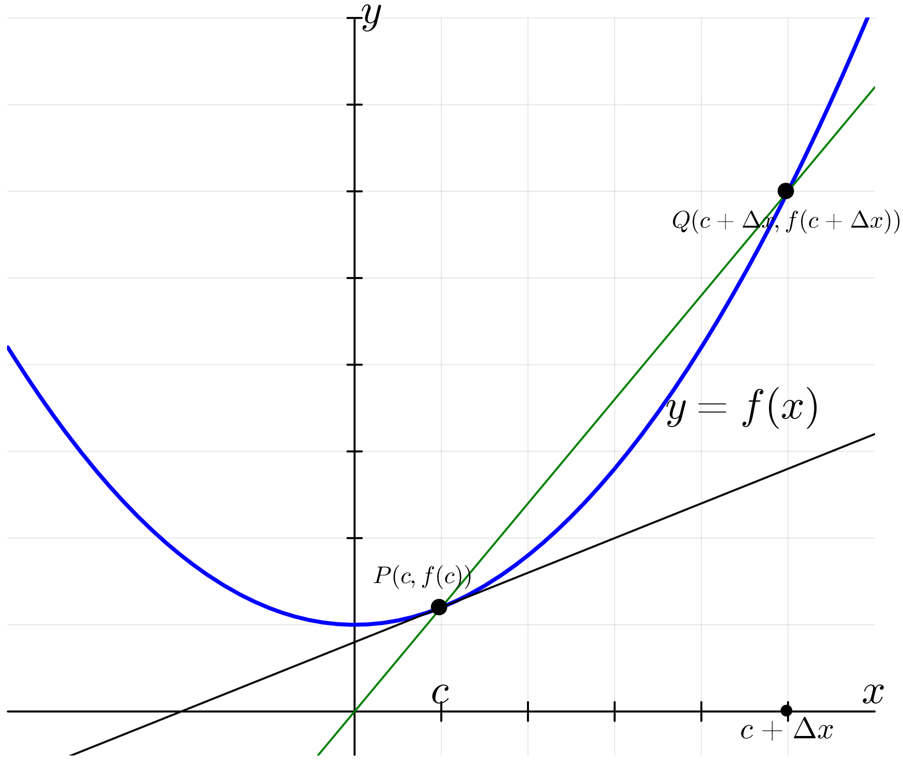
Objectives

> “
>
> - Find the slope of the tangent line to a curve at a point.
> - Use the limit definition to find the derivative of a function.
> - Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Δx 4.0

Find the equation of the secant line



Slope of secant line

$$\mathrm{m}_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Definition of Tangent Line with Slope

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Remark

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$**.

Example 1: The Slope of the Graph of a Linear Function

Find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$.

Example 2: Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$.

Remarks

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line.
- For vertical tangent lines, you can use the **following definition**. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the **vertical line** $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f.

The Derivative of a Function

Definition **Derivative of a Function**

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Remarks

- The notation $f'(x)$ is read as "f prime of x."
- $f'(x)$ is a **function** that gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.
- The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.
- The process of finding the derivative of a function is called **differentiation**.
- A function is **differentiable** at x when its derivative exists at x and is **differentiable on an open interval** (a, b) when it is differentiable at every point in the interval.

Notation

$$y = f(x)$$

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}[f(x)]$
- $D_x[y]$

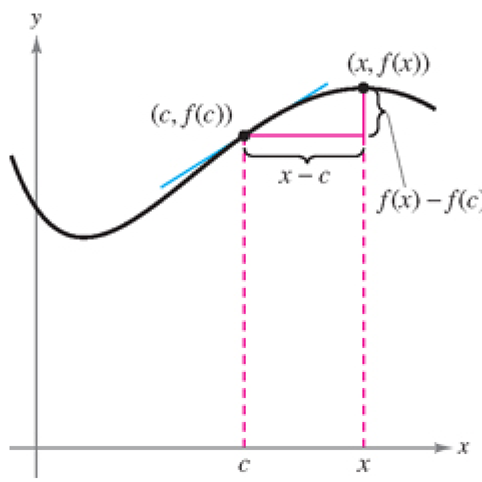
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Examples 3,4,5: **Finding the Derivative by the Limit Process**

Find the derivative of

- $f(x) = x^3 + 2x$
- $f(x) = \sqrt{x}$
- $y = \dfrac{2}{t}$ with respect to t.

Differentiability and Continuity



Alternative form of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Remarks

derivative from the left

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

derivative from the right

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Example:

$$f(x) = [[x]]$$

Example 6: A Graph with a Sharp Turn

$$f(x) = |x - 2|$$

Example 7: A Graph with a Vertical Tangent Line

$$f(x) = x^{\frac{1}{3}}$$

Theorem **Differentiability Implies Continuity**

If f is differentiable at $x = c$, then f is continuous at $x = c$.

remarks

The relationship between continuity and differentiability is summarized below.

- If a function f is differentiable at $x = c$, then it is continuous at $x = c$. So, **differentiability** implies ($\Rightarrow$) **continuity**.
- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, **continuity does not imply differentiability**.

Exercises

3.2: Basic Differentiation Rules and Rates of Change

Objectives

> “
>
> - Find the derivative of a function using the Constant Rule.
> - Find the derivative of a function using the Power Rule.
> - Find the derivative of a function using the Constant Multiple Rule.
> - Find the derivative of a function using the Sum and Difference Rules.
> - Find the derivatives of the sine function and of the cosine function.
> - Find the derivatives of exponential functions.
> - Use derivatives to find rates of change.

The Constant Rule

Theorem **The Constant Rule**

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

The Power Rule

Theorem **The Power Rule**

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at 0, n must be a number such that x^{n-1} is defined on an interval containing 0.

The Constant Multiple Rule

Theorem **The Constant Multiple Rule**

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

The Sum and Difference Rules

Theorem **The Sum and Difference Rules**

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= f'(x)+g'(x) && \text{Sum Rule}\\ \frac{d}{dx}[f(x)-g(x)] &= f'(x)-g'(x) && \text{Difference Rule}\end{aligned}$$

Derivatives of the Sine and Cosine Functions

Theorem **Derivatives of the Sine and Cosine Functions**

$$\frac{d}{dx}[\sin(x)] = \cos x \quad , \quad \frac{d}{dx}[\cos(x)] = -\sin x$$

Derivatives of Exponential Functions

Theorem **Derivative of the Natural Exponential Function**

$$\frac{d}{dx}[e^x] = e^x$$

Rates of Change

- The derivative can be used to determine the **rate of change** of one variable with respect to another.
- Applications involving rates of change, sometimes referred to as **instantaneous rates of change**, occur in a wide variety of fields.
- A common use for rate of change is to describe **the motion of an object moving in a straight line.** (+ direction and -direction)
- The function s that gives **the position (relative to the origin)** of an object as a **function of time t** is called a **position function**. If, over a period of time Δt, the object changes its position by the amountthen, by the familiar formula

$$\Delta s = s(t + \Delta t) - s(t)$$

- then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}.$$

-the average velocity is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} \qquad \text{Average Velocity}.$$

- In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the velocity of the object at time t is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \qquad \text{Velocity function}.$$

Example:

If a ball is thrown into the air with a velocity of $4m/s$, its height (in meters (m)) t seconds later is given by

$$y = 4t - 4.9t^2.$$

1. Find the average velocity for the time period from $t = 1$ to $t = 3$.
2. Find the instantaneous rate of change at $t = 2$.

Example 11: **Using the Derivative to Find Velocity**

At time $t = 0$, a diver jumps from a platform diving board that is 9.8 meters above the water. The initial velocity of the diver is 4.9 meters per second. When does the diver hit the water? What is the diver's velocity at impact?

3.3: Product and Quotient Rules and Higher-Order Derivatives

Objectives

> “
>
> - Find the derivative of a function using the Product Rule.
> - Find the derivative of a function using the Quotient Rule.
> - Find the derivative of a trigonometric function.
> - Find a higher-order derivative of a function.

The Product Rule

Theorem **The Product Rule**

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the **first** function **times** the **derivative of the second**, **plus** the **second** function times the **derivative of the first**.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Example:

Find the derivative of $f(x) = xe^x$.

The Quotient Rule

Theorem **The Quotient Rule**

The quotient of two differentiable functions f and g is itself differentiable at all values of for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the **denominator times** the **derivative of the numerator minus** the **numerator times the derivative of the denominator**, all **divided by** the **square of the denominator**.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0.$$

Example:

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x+5}$ at $(-1, 1)$.

Derivatives of Trigonometric Functions

Theorem **Derivatives of Trigonometric Functions**

$$\begin{array}{rcl|rcl} \frac{d}{dx}(\tan x) & = & \sec^2 x & \frac{d}{dx}(\cot x) & = & -\csc^2 x \\ \frac{d}{dx}(\sec x) & = & \sec x \tan x & \frac{d}{dx}(\csc x) & = & -\csc x \cot x \end{array}$$

Example:

Differentiate

$$y = \frac{1 - \cos x}{\sin x}$$

Higher-Order Derivatives

Remarks

Rates of changes

$$
\begin{array}{rcrcll}
 & & & & s(t) & \quad \text{Position function} \\
 & & v(t) & = & s'(t) & \quad \text{Velocity function} \\
a(t) & = & v'(t) & = & s''(t) & \quad \text{Acceleration function}
\end{array}
$$

Higher Derivatives

$$
\begin{array}{llllll}
\textbf{First derivative:} & y', & f'(x), & \frac{dy}{dx}, & \frac{d}{dx}[f(x)], & D_x[y] \\[2ex]
\textbf{Second derivative:} & y'', & f''(x), & \frac{d^2y}{dx^2}, & \frac{d^2}{dx^2}[f(x)], & D_x^2[y] \\[2ex]
\textbf{Third derivative:} & y''', & f'''(x), & \frac{d^3y}{dx^3}, & \frac{d^3}{dx^3}[f(x)], & D_x^3[y] \\[2ex]
\textbf{Fourth derivative:} & y^{(4)}, & f^{(4)}(x), & \frac{d^4y}{dx^4}, & \frac{d^4}{dx^4}[f(x)], & D_x^4[y] \\[2ex]
\vdots & & & & & \\[2ex]
\textbf{nth derivative:} & y^{(n)}, & f^{(n)}(x), & \frac{d^ny}{dx^n}, & \frac{d^n}{dx^n}[f(x)], & D_x^n[y]
\end{array}
$$

Exercises

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```
begin
    using CommonMark, ImageIO, FileIO, ImageShow
    using PlutoUI
    using Plots, PlotThemes, LaTeXStrings, Random
    using PGFPlotsX
    using SymPy
    using HypertextLiteral: @htl, @htl_str
    using ImageTransformations
    using Colors
end
```