MATH102

Syllabus

5.2: Area

Objectives

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>
> 1. Use sigma notation to write and evaluate a sum.
> 2. Understand the concept of area.
> 3. Approximate the area of a plane region.
> 4. Find the area of a plane region using limits.

Sigma Notation

Sigma Notation

The sum of n terms $a_1, a_2, \cdots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

where i is the **index of summation**, a_i is the th ***i*th term** of the sum, and the upper and lower bounds of summation are n and 1.

Summation Properties

$$\begin{aligned}\sum_{i=1}^{n} ka_i &= k\sum_{i=1}^{n} a_i \\ \sum_{i=1}^{n}(a_i \pm b_i) &= \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i\end{aligned}$$

Theorem Summation Formulas

$$(1) \quad \sum_{i=1}^{n} c = cn, \quad c \text{ is a constant}$$

$$(2) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(3) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(4) \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1: Evaluating a Sum

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n}$ for $n = 10, 100, 1000$ and $10,000$.

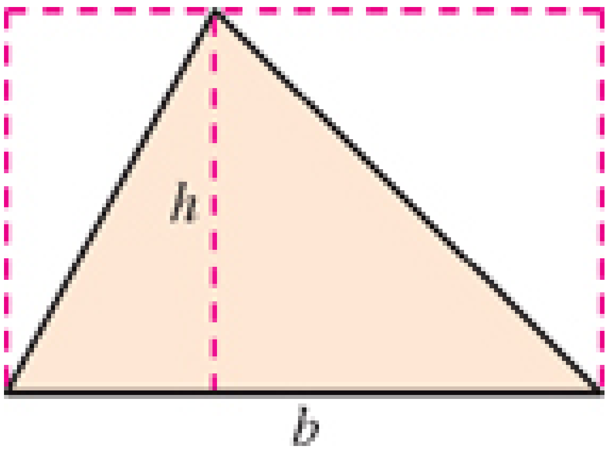
Area

In **Euclidean geometry**, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is

$$A = bh$$

it is actually more proper to say that this is the *definition* of the **area of a rectangle**.

For a triangle $A = \frac{1}{2}bh$



The Area of a Plane Region

Example

Use **five** rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = 5 - x^2$$

and the x-axis between $x = 0$ and $x = 2$.

f (generic function with 1 method)

```
f(x) = 5 - x^2
```

n = 5 a = 0 b = 2 method = Left

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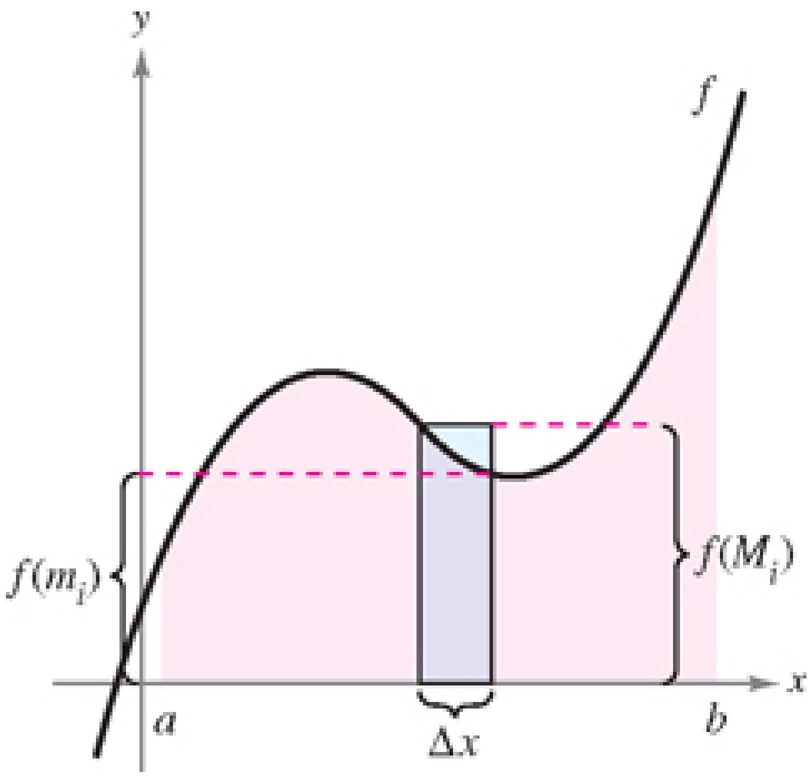
Finding Area by the Limit Definition

Find the area of a plane region bounded above by the graph of a nonnegative, **continuous** function

$$y = f(x)$$

The region is bounded below by the x- axis and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$



- To approximate the area of the region, begin by subdividing the interval into subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

- The endpoints of the intervals are

$$\overbrace{a+0(\Delta x)}^{a=x_0} < \overbrace{a+1(\Delta x)}^{a=x_1} < \overbrace{a+2(\Delta x)}^{a=x_2} < \cdots < \overbrace{a+n(\Delta x)}^{a=x_n}.$$

- Let

$$\begin{aligned} f(m_i) &= \text{Minimum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \\ f(M_i) &= \text{Maximum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \end{aligned}$$

- Define an **inscribed rectangle** lying inside the i^{th} subregion
- Define an **circumscribed rectangle** lying outside the i^{th} subregion

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \leq f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

- The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of the circumscribed rectangles is called an **upper sum**.

$$\begin{aligned} \text{Lower sum} &= s(n) &= \sum_{i=1}^{n} f(m_i)\Delta x \quad \textcolor{red}{\text{Area of inscribed rectangle}} \\ \text{Upper sum} &= S(n) &= \sum_{i=1}^{n} f(M_i)\Delta x \quad \textcolor{red}{\text{Area of circumscribed rectangle}} \end{aligned}$$

- The actual area of the region lies between these two sums.

$$s(n) \leq (\text{Area of region}) \leq S(n).$$

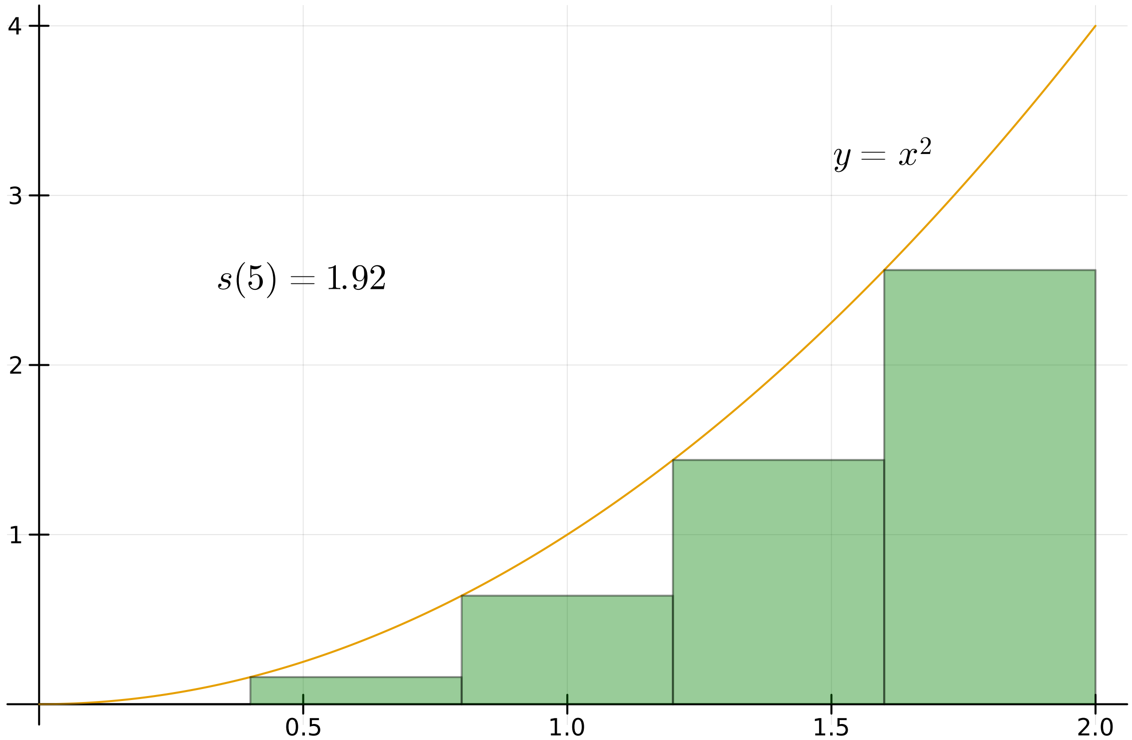
Example 4: Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the $x-$axis between $x = 0$ and $x = 2$.

n = 5 a = 0 b = 2 method = Left

f4 (generic function with 1 method)

```
f4(x) = x^2
```



Theorem **Limits of the Lower and Upper Sums**

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

where

$$\Delta x = \frac{b-a}{n}$$

and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the ith subinterval.

Definition **Area of a Region in the Plane**

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $y = b$ is

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

where

$$x_{i-1} \leq c_i \leq x_i \quad \text{and} \quad \Delta x = \frac{b-a}{n}.$$

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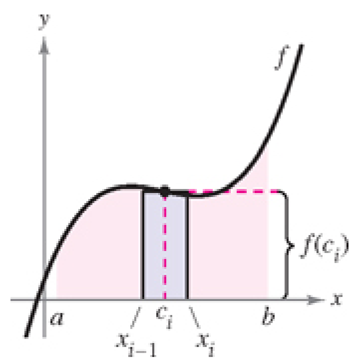


Example 5: Finding Area by the Limit Definition

Find the area of the region bounded by the graph of $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Example 7: A Region Bounded by the *y*-axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \leq y \leq 1$.))

Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

Example 8: **Approximating Area with the Midpoint Rule**

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = \sin x$ and the x-axis for $0 \leq x \leq \pi$.

```
2.0523443059540623
```

```
begin
    f8(x)=sin(x)
    Δx28 = π/4
    A = Δx28*(f8(π/8)+f8(3π/8)+f8(5π/8)+f8(7π/8))
end
```

5.3: Riemann Sums and Definite Integrals

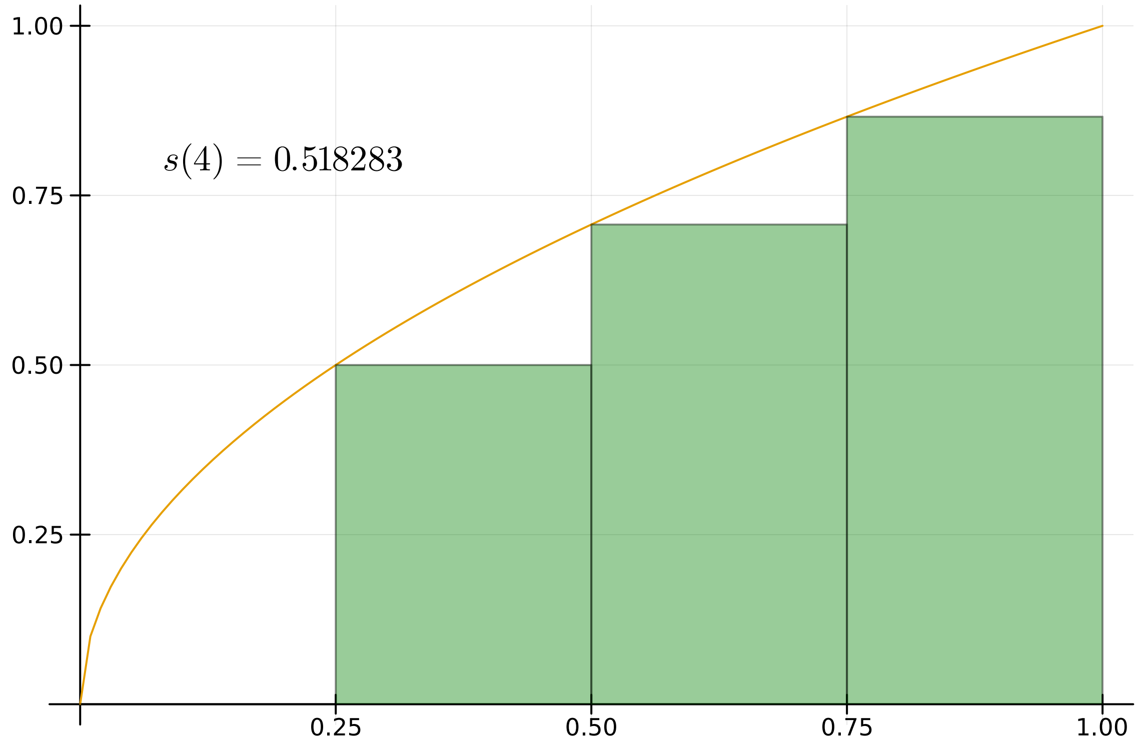
> **Objectives**
>
> 1. Understand the definition of a Riemann sum.
> 2. Evaluate a definite integral using limits and geometric formulas.
> 3. Evaluate a definite integral using properties of definite integrals.

Riemann Sums

```
g (generic function with 1 method)
```

```
g(x) = √x
```

n = 4 a = 0 b = 1 method = Left



Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the th subinterval

$$[x_{i-1}, x_i] \quad \textit{i}\textbf{th subinterval}$$

If c_i is any point in the th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

Remark

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$.

- If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n} \quad \textcolor{red}{\text{Regular partition}}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \quad \textcolor{red}{\text{General partition}}$$

- Note that

$$\|\Delta\| \to 0 \quad \text{implies that} \quad n \to \infty.$$

Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists, then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Theorem **Continuity Implies Integrability**

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is,

$$\int_a^b f(x)dx \quad \text{exists.}$$

Theorem **The Definite Integral as the Area of a Region**

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)dx$$

Example 3: Areas of Common Geometric Figures

Evaluate each integral using a geometric formula.

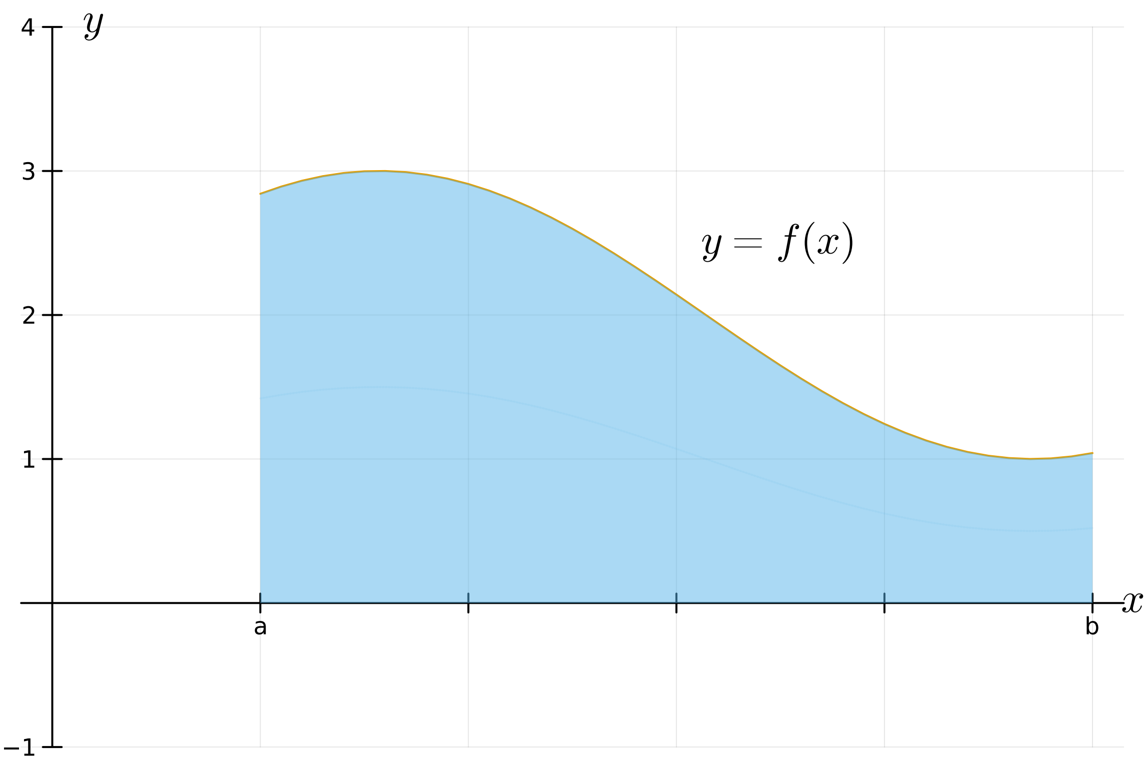
- $\int_1^3 4dx$
- $\int_0^3 (x+2)dx$
- $\int_{-2}^2 \sqrt{4-x^2}dx$

Remark The definite integral is a **number**

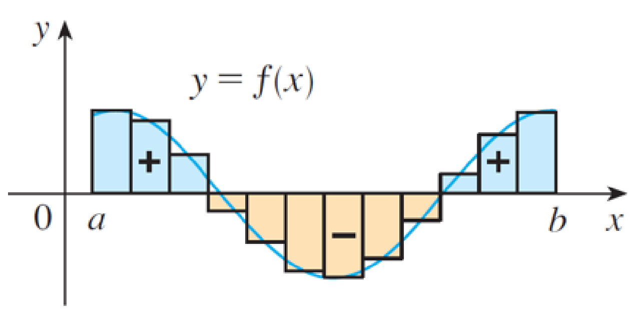
- It does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

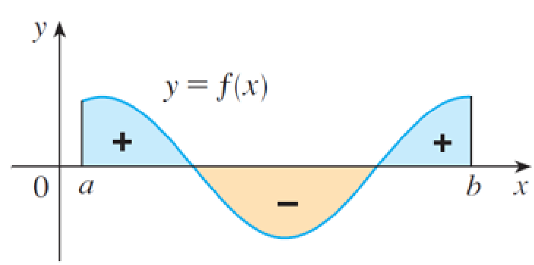
$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(w)dw = \int_a^b f(\text{😀})d\text{😀}$$

- If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from a to b.



- $\int_a^b f(x)dx$ is the net area





Properties of Definite Integrals

Definitions **Two Special Definite Integrals**

- If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$.
- If f is integrable on $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Theorem **Additive Interval Property**

If f is integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem **Properties of Definite Integrals**

- If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and
  1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
  2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

Theorem **Preservation of Inequality**

- If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)dx.$$

- If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

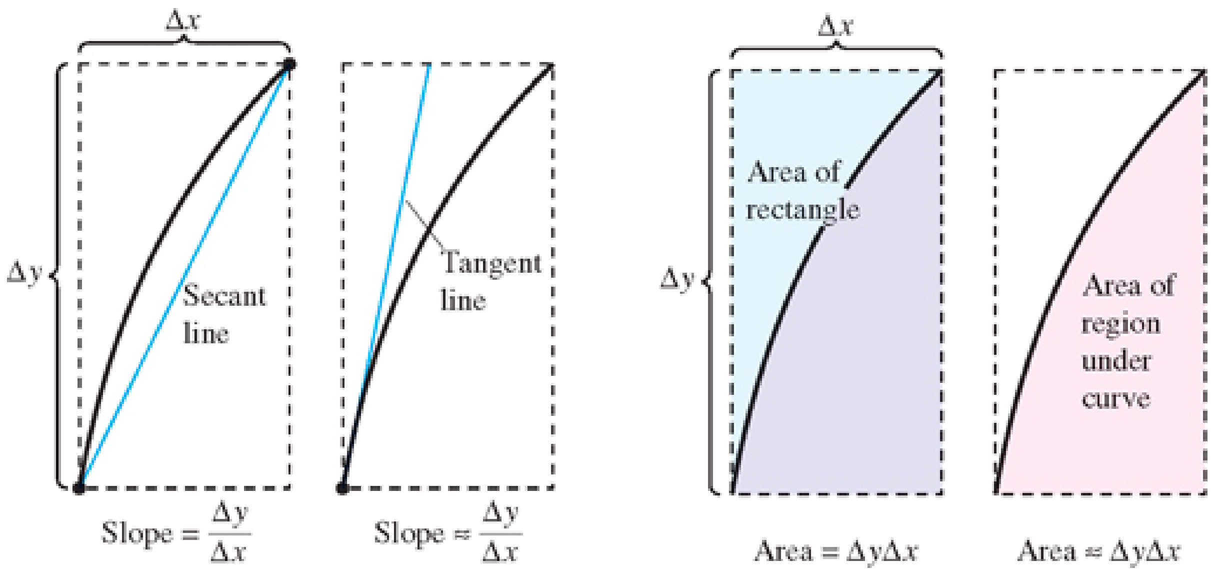
Examples:

5.4: The Fundamental Theorem of Calculus

> **Objectives**
>
> 1. Evaluate a definite integral using the Fundamental Theorem of Calculus.
> 2. Understand and use the Mean Value Theorem for Integrals.
> 3. Find the average value of a function over a closed interval.
> 4. Understand and use the Second Fundamental Theorem of Calculus.
> 5. Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus

Antidifferentiation and Definite Integration



(a) Differentiation (b) Definite integration

- $\int_a^b f(x)dx$
  - definite integral
  - number
- $\int f(x)dx$
  - indefinite integral
  - function

Theorem **The Fundamental Theorem of Calculus**

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark

We use the notation

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{or} \quad \int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example 1: Evaluating a Definite Integral

Evaluate each definite integral.

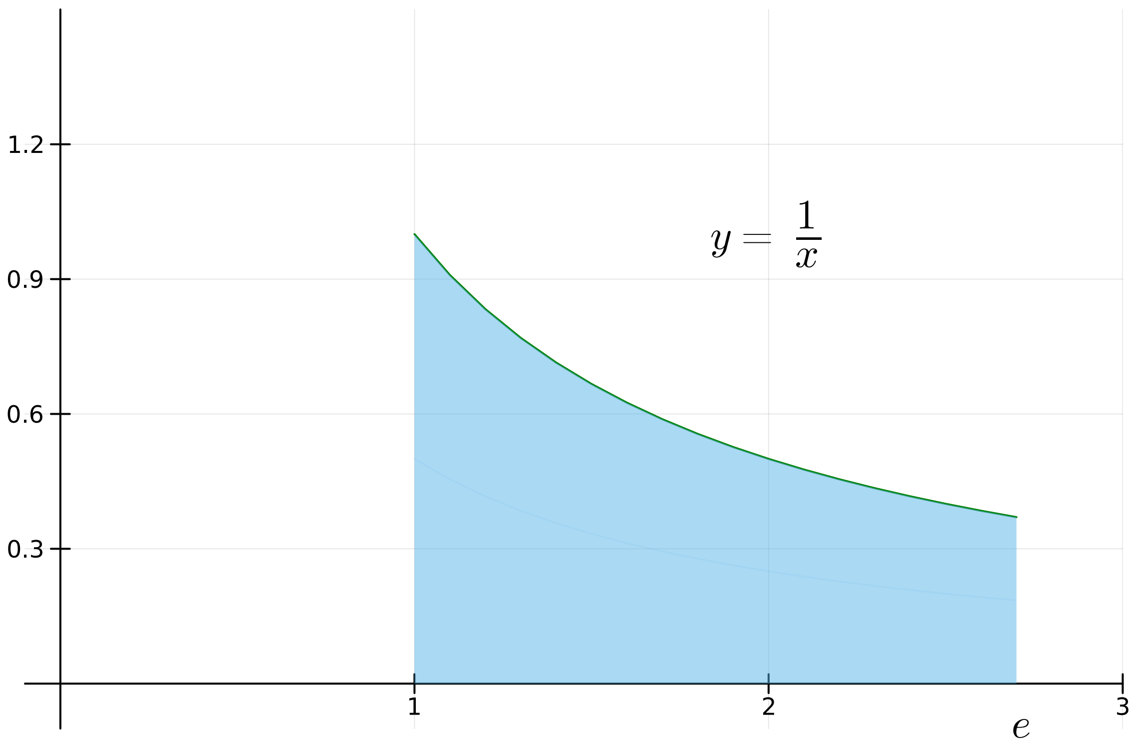
- $\int_1^2 (x^2 - 3)dx$
- $\int_1^4 3\sqrt{x}dx$
- $\int_0^{\pi/4} \sec^2 x dx$
- $\int_0^2 \left|2x - 1\right| dx$

Example 3: **Using the Fundamental Theorem to Find Area**

Find the area of the region bounded by the graph of

$$y = \frac{1}{x}$$

the x-axis, and the vertical lines $x = 1$ and $x = e$.

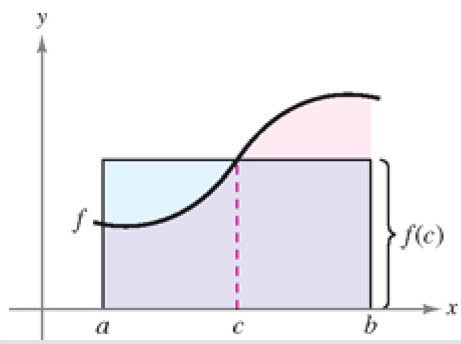


The Mean Value Theorem for Integrals

Theorem **The Mean Value Theorem for Integrals**

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$



Average Value of a Function

Definition **the Average Value of a Function on an Interval**

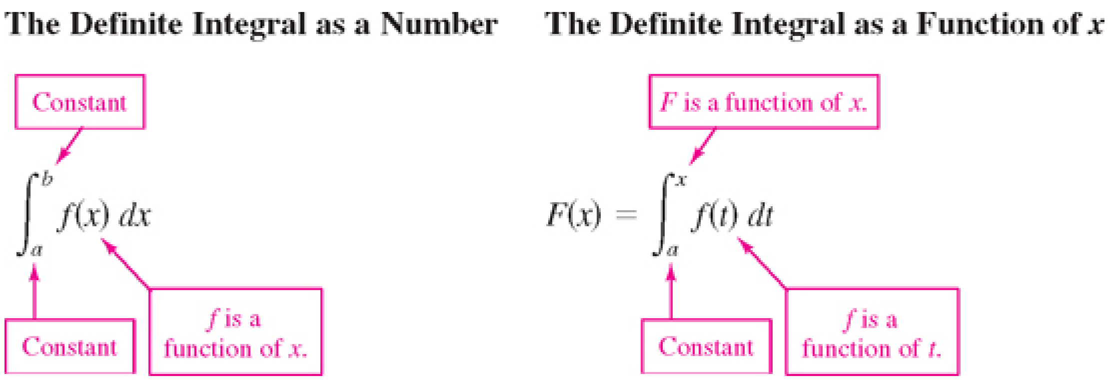
If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\textbf{Avergae value} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 4: **Finding the Average Value of a Function**

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

The Second Fundamental Theorem of Calculus

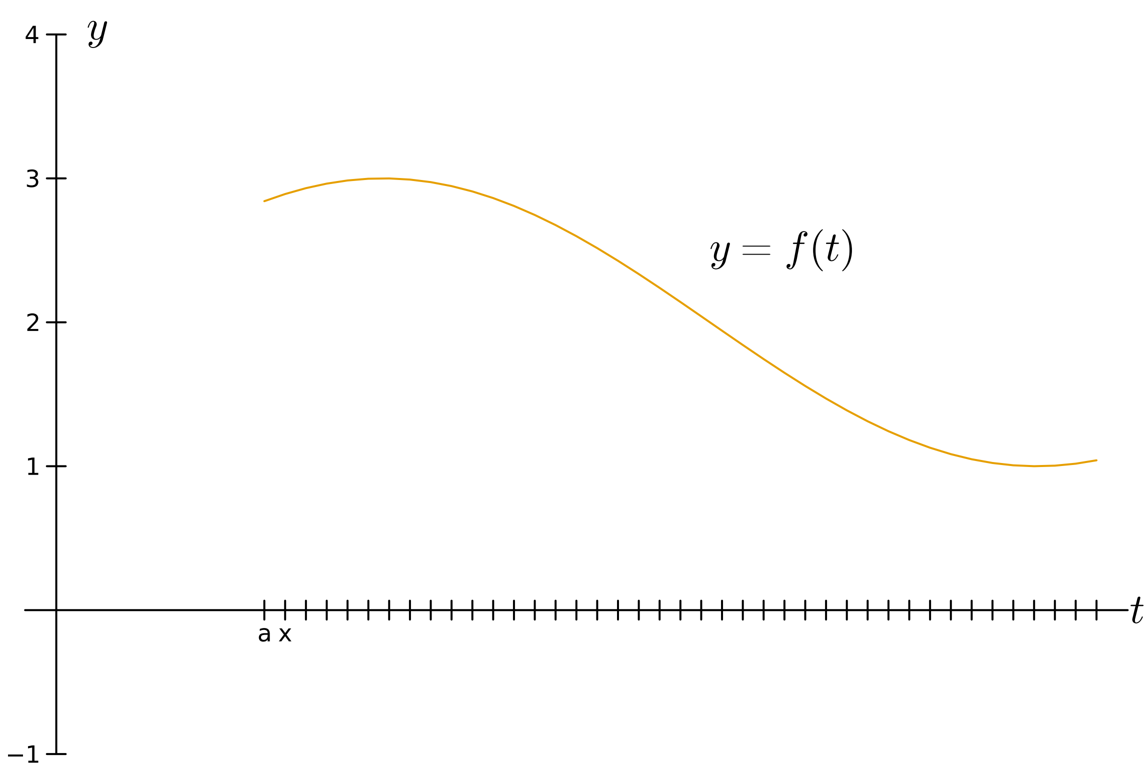


Consider the following function

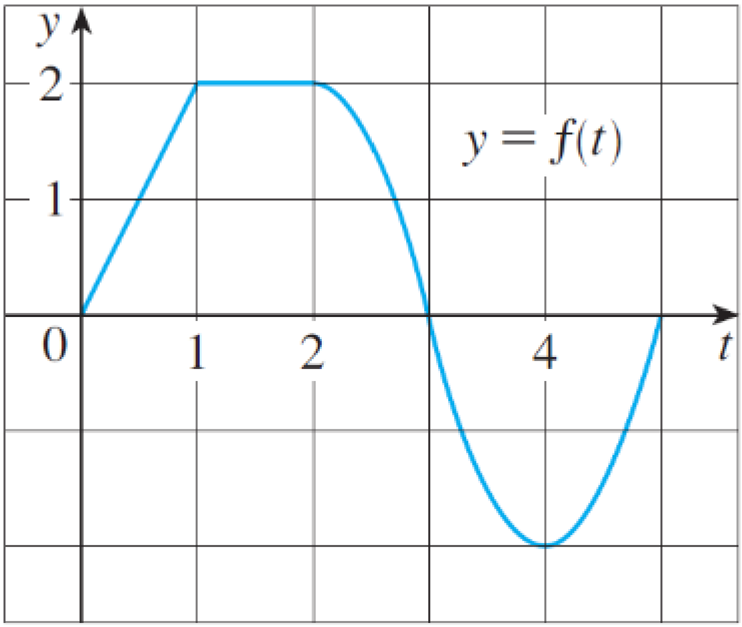
$$F(x) = \int_a^x f(t)dt$$

where f is a continuous function on the interval $[a, b]$ and $x \in [a, b]$.

x =



Example If $g(x) = \int_0^x f(t)dt$



Find $g(2)$

Theorem **The Second Fundamental Theorem of Calculus**

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\right] = f(x).$$

Remarks

- $\frac{d}{dx}\left(\int_a^x f(u)du\right) = f(x)$
- $g(x)$ is an **antiderivative** of f

Examples

Find the derivative of

(1) $g_1(x) = \int_0^x \sqrt{1+t}dt$.

(2) $g_2(x) = \int_x^0 \sqrt{1+t}dt$.

(3) $g_3(x) = \int_0^{x^2} \sqrt{1+t}dt$.

(4) $g_4(x) = \int_{\sin(x)}^{\cos(x)} \sqrt{1+t}dt$.

💣 BE CAREFUL:

Evaluate $\int_{-3}^{6} \frac{1}{x} dx$

Net Change Theorem

Question: If $y = F(x)$, then what does $F'(x)$ represents?

Theorem **The Net Change Theorem**

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)dx = F(b) - F(a) \qquad \text{Net change of } F(x)$$

```
begin
    using FileIO, ImageIO, ImageShow, ImageTransformations
    using SymPy
    using PlutoUI
    using CommonMark
    using Plots, PlotThemes, LaTeXStrings
    using HypertextLiteral: @htl, @htl_str
    using Colors
    using Random
end
```