MATH102

5.2: Area

Objectives

> 1. Use sigma notation to write and evaluate a sum.
> 2. Understand the concept of area.
> 3. Approximate the area of a plane region.
> 4. Find the area of a plane region using limits.

Sigma Notation

Sigma Notation

The sum of n terms $a_1, a_2, \cdots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

where i is the **index of summation**, a_i is the th **ith term** of the sum, and the upper and lower bounds of summation are n and 1.

Summation Properties

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Theorem

Summation Formulas

$$(1) \quad \sum_{i=1}^{n} c = cn, \quad c \text{ is a constant}$$

$$(2) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(3) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(4) \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1: Evaluating a Sum

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n}$ for $n = 10, 100, 1000$ and $10,000$.

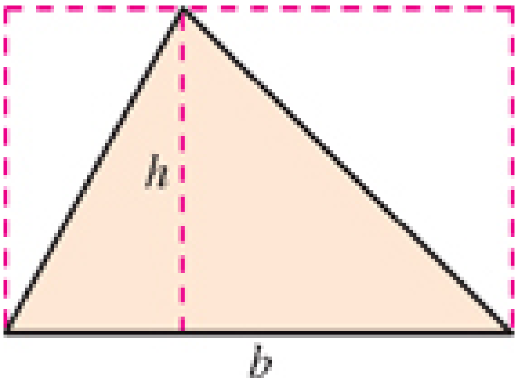
Area

In **Euclidean geometry**, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is

$$A = bh$$

it is actually more proper to say that this is the *definition* of the **area of a rectangle**.

For a triangle $A = \frac{1}{2}bh$



The Area of a Plane Region

Example

Use **five** rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = 5 - x^2$$

and the x-axis between $x = 0$ and $x = 2$.

```
f (generic function with 1 method)
```

```
f(x) = 5 - x^2
```

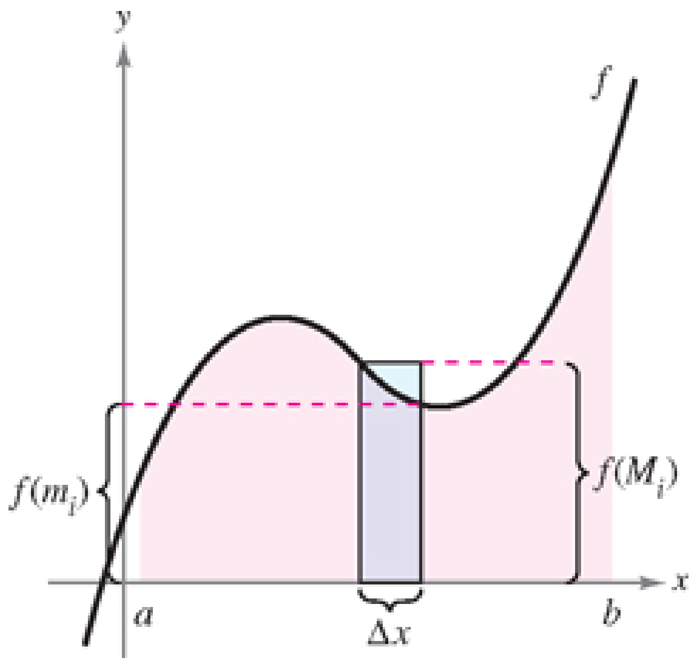
n = 5 a = 0 b = 2 method = Left

Finding Area by the Limit Definition

Find the area of a plane region bounded above by the graph of a nonnegative, **continuous** function

$$y = f(x)$$

The region is bounded below by the x- axis and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$



- To approximate the area of the region, begin by subdividing the interval into subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

- The endpoints of the intervals are

$$\overbrace{a+0(\Delta x)}^{a=x_0} < \overbrace{a+1(\Delta x)}^{a=x_1} < \overbrace{a+2(\Delta x)}^{a=x_2} < \cdots < \overbrace{a+n(\Delta x)}^{a=x_n}.$$

- Let

$$\begin{aligned} f(m_i) &= \text{Minimum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \\ f(M_i) &= \text{Maximum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \end{aligned}$$

- Define an **inscribed rectangle** lying inside the i^{th} subregion
- Define an **circumscribed rectangle** lying outside the i^{th} subregion

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \leq f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

- The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of the circumscribed rectangles is called an **upper sum**.

$$\begin{aligned} \text{Lower sum} &= s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \quad \text{Area of inscribed rectangle} \\ \text{Upper sum} &= S(n) = \sum_{i=1}^{n} f(M_i)\Delta x \quad \text{Area of circumscribed rectangle} \end{aligned}$$

- The actual area of the region lies between these two sums.

$$s(n) \leq (\text{Area of region}) \leq S(n).$$

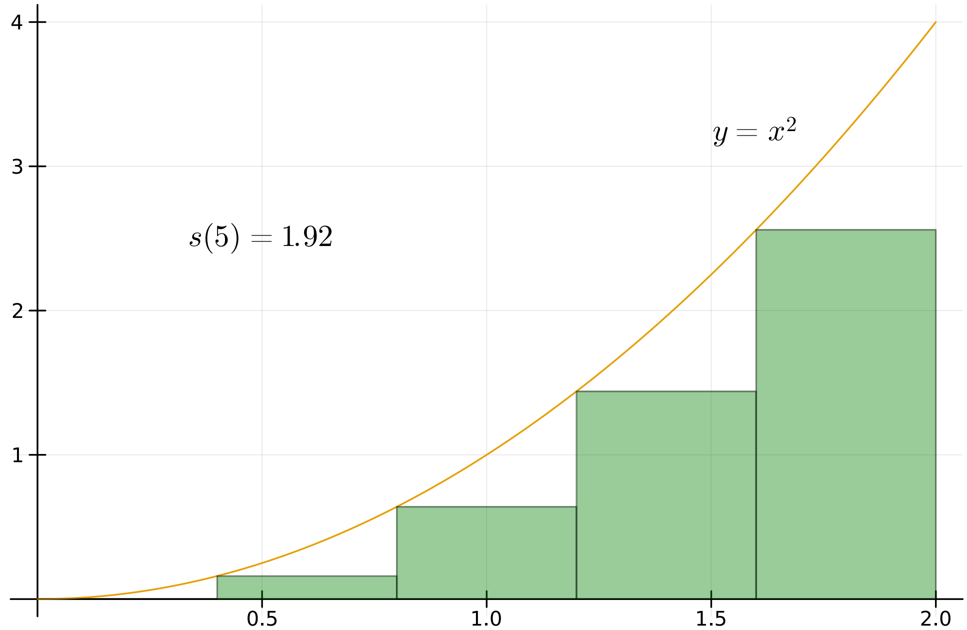
Example 4: **Finding Upper and Lower Sums for a Region**

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the $x-$axis between $x = 0$ and $x = 2$.

n = 5 a = 0 b = 2 method = Left

```
f4 (generic function with 1 method)
```

```
f4(x) = x^2
```



Theorem **Limits of the Lower and Upper Sums**

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

where

$$\Delta x = \frac{b-a}{n}$$

and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the ith subinterval.

Definition **Area of a Region in the Plane**

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $y = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where

$$x_{i-1} \le c_i \le x_i \quad \text{and} \quad \Delta x = \frac{b-a}{n}.$$

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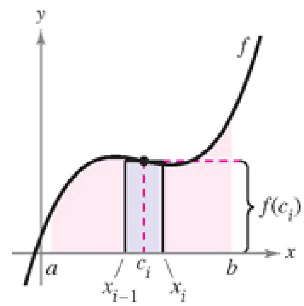


Example 5: **Finding Area by the Limit Definition**

Find the area of the region bounded by the graph of $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Example 7: **A Region Bounded by the *y*-axis**

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$.))

Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x.$$

Example 8: **Approximating Area with the Midpoint Rule**

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = \sin x$ and the x-axis for $0 \le x \le \pi$.

2.0523443059540623

```
begin
    f8(x)=sin(x)
    Δx28 = π/4
    A = Δx28*(f8(π/8)+f8(3π/8)+f8(5π/8)+f8(7π/8))
end
```

5.3: Riemann Sums and Definite Integrals

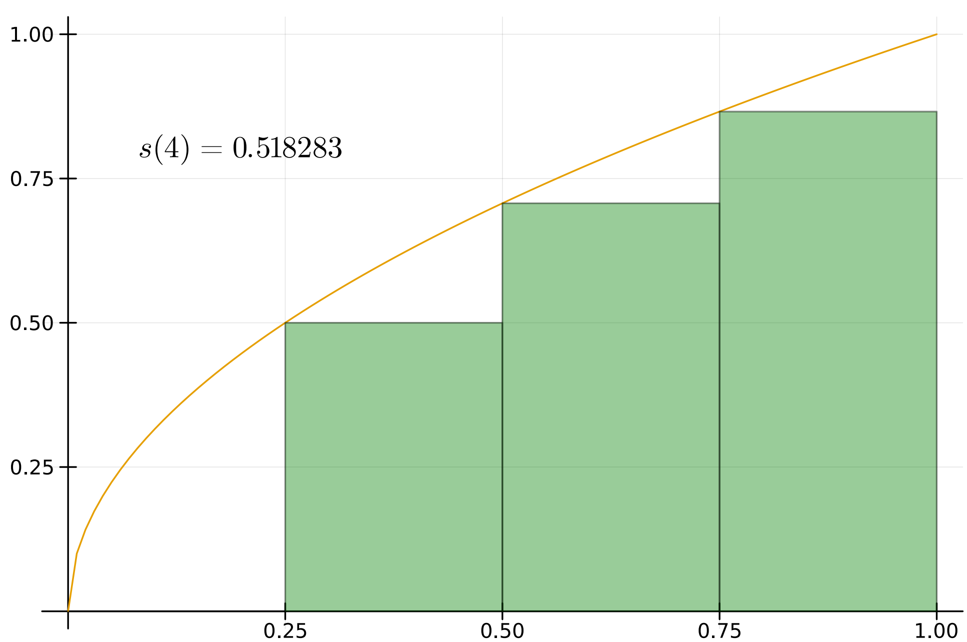
> **Objectives**
>
> 1. Understand the definition of a Riemann sum.
> 2. Evaluate a definite integral using limits and geometric formulas.
> 3. Evaluate a definite integral using properties of definite integrals.

Riemann Sums

g (generic function with 1 method)

```
g(x) = √x
```

n = 4 a = 0 b = 1 method = Left



Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the th subinterval

$$[x_{i-1}, x_i] \quad \textit{i}\text{th subinterval}$$

If c_i is any point in the th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

Remark

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$.

- If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n} \quad \text{Regular partition}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \le n \quad \text{General partition}$$

- Note that

$$\|\Delta\| \to 0 \quad \text{implies that} \quad n \to \infty.$$

Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists, then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Theorem — Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is,

$$\int_a^b f(x) dx \quad \text{exists.}$$

Theorem — The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x) dx$$

Example 3: Areas of Common Geometric Figures

Evaluate each integral using a geometric formula.

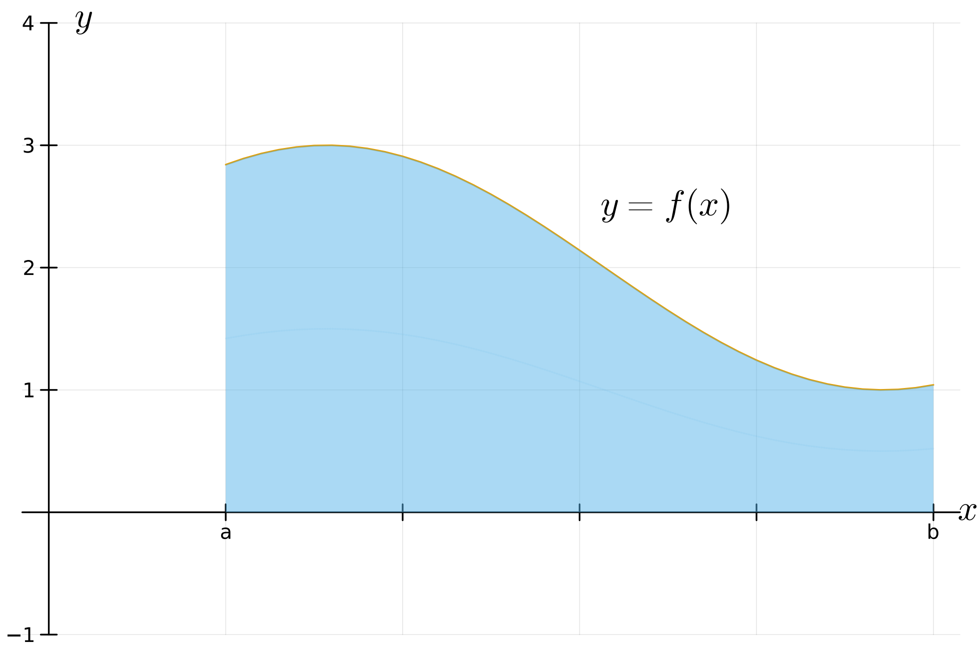
- $\int_1^3 4dx$
- $\int_0^3 (x+2)dx$
- $\int_{-2}^2 \sqrt{4-x^2}dx$

Remark The definite integral is a **number**

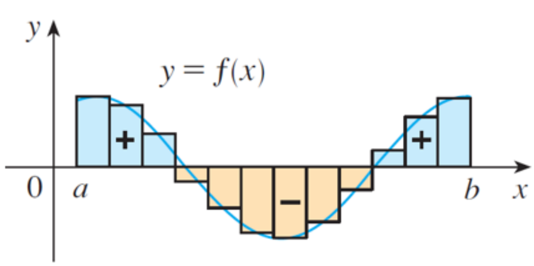
- It does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

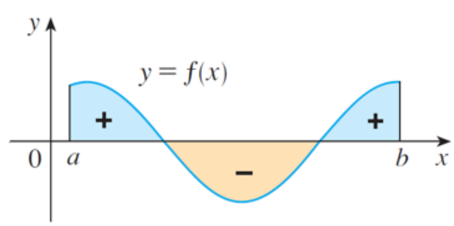
$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(w)dw = \int_a^b f(😀)d😀$$

- If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from a to b.



- $\int_a^b f(x)dx$ is the net area





Properties of Definite Integrals

Definitions **Two Special Definite Integrals**

- If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$.
- If f is integrable on $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Theorem **Additive Interval Property**

If f is integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem **Properties of Definite Integrals**

- If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and
  1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
  2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

Theorem **Preservation of Inequality**

- If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)dx.$$

- If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

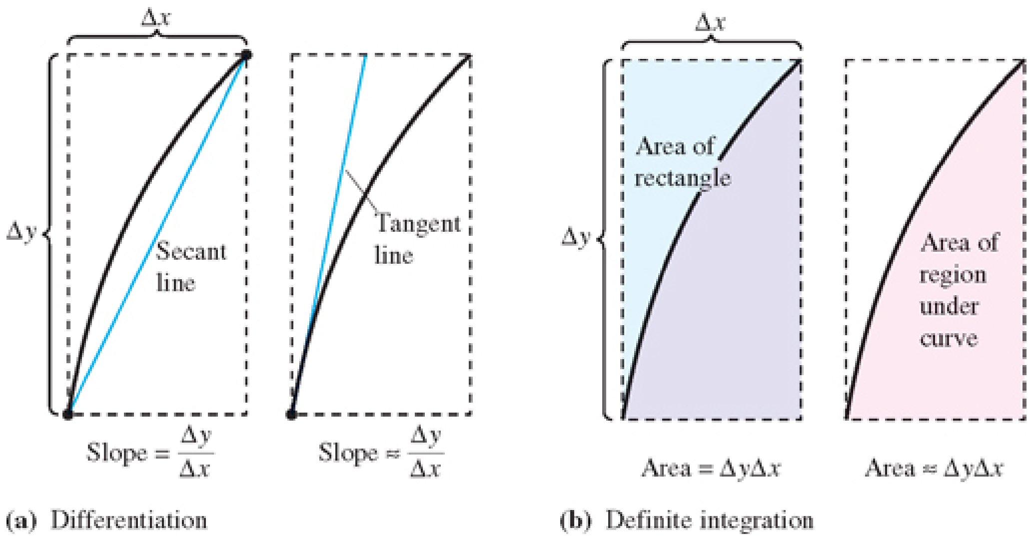
Examples:

5.4: The Fundamental Theorem of Calculus

Objectives

1. Evaluate a definite integral using the Fundamental Theorem of Calculus.
2. Understand and use the Mean Value Theorem for Integrals.
3. Find the average value of a function over a closed interval.
4. Understand and use the Second Fundamental Theorem of Calculus.
5. Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus



- ✏ $\int_a^b f(x)dx$
  - definite integral
  - number
- ✏ $\int f(x)dx$
  - indefinite integral
  - function

Theorem **The Fundamental Theorem of Calculus**

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark

We use the notation

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{or} \quad \int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example 1: **Evaluating a Definite Integral**

Evaluate each definite integral.

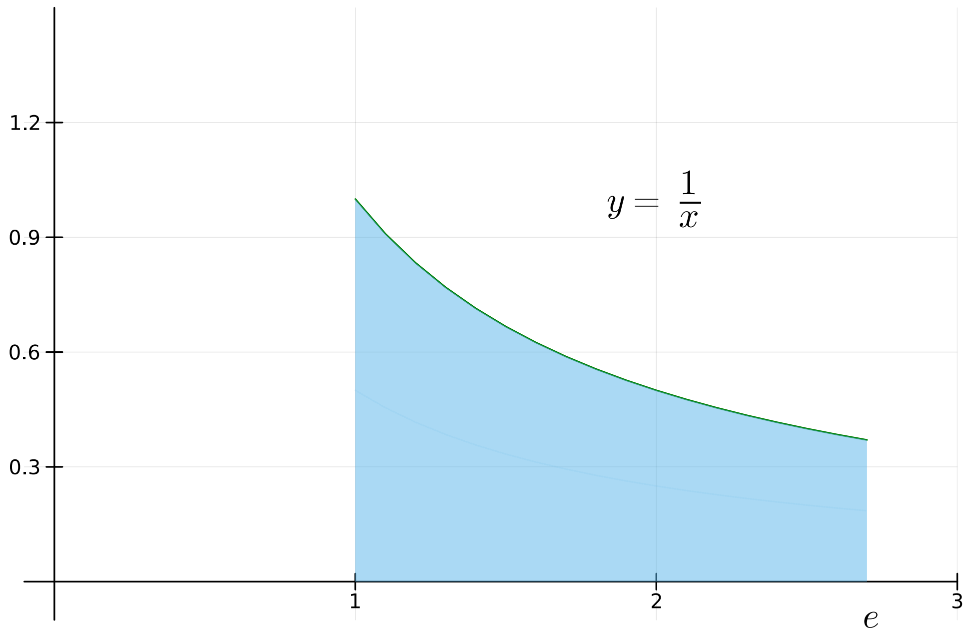
- $\int_{1}^{2} (x^2 - 3)dx$
- $\int_{1}^{4} 3\sqrt{x}dx$
- $\int_{0}^{\pi/4} \sec^2 x dx$
- $\int_{0}^{2} \left|2x - 1\right| dx$

Example 3: **Using the Fundamental Theorem to Find Area**

Find the area of the region bounded by the graph of

$$y = \frac{1}{x}$$

the x-axis, and the vertical lines $x = 1$ and $x = e$.

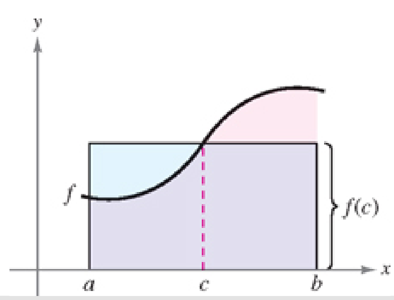


The Mean Value Theorem for Integrals

Theorem **The Mean Value Theorem for Integrals**

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$



Average Value of a Function

Definition the Average Value of a Function on an Interval

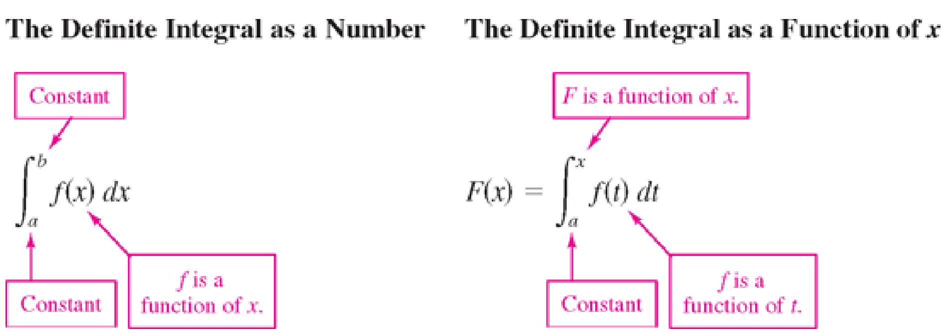
If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\textbf{Avergae value} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 4: Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

The Second Fundamental Theorem of Calculus

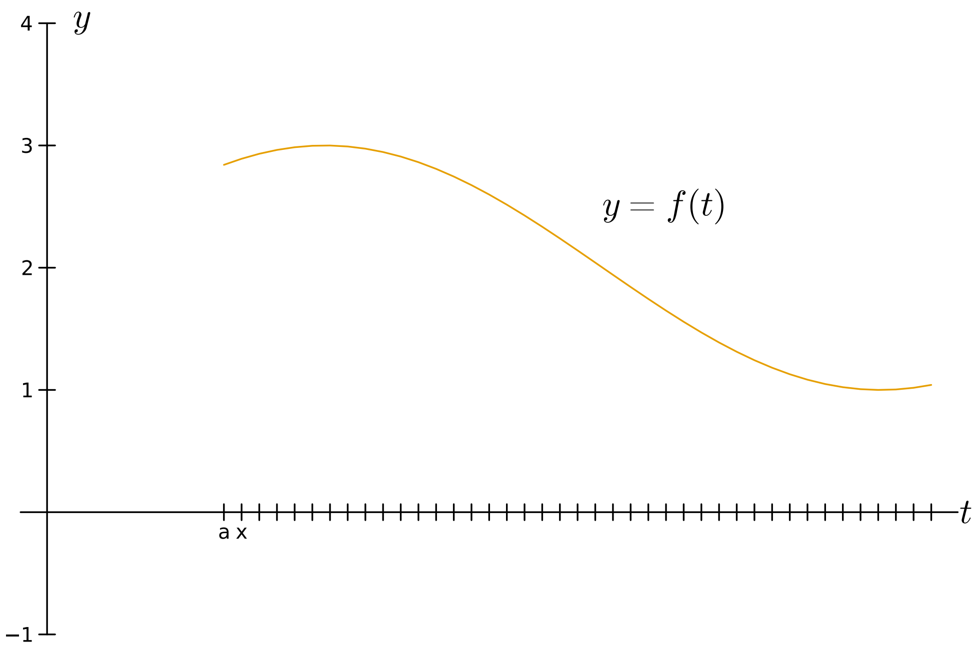


Consider the following function

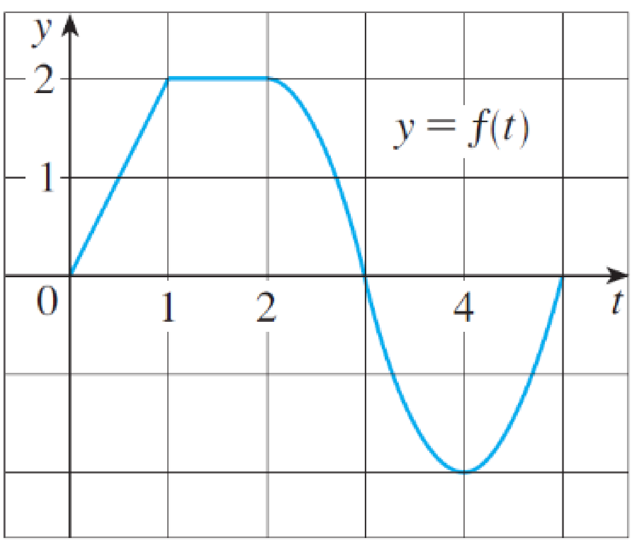
$$F(x) = \int_a^x f(t)dt$$

where f is a continuous function on the interval $[a, b]$ and $x \in [a, b]$.

x =



Example If $g(x) = \int_0^x f(t)dt$



Find $g(2)$

Theorem **The Second Fundamental Theorem of Calculus**

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\right] = f(x).$$

Remarks

- $\frac{d}{dx}\left(\int_a^x f(u)du\right) = f(x)$
- $g(x)$ is an **antiderivative** of f

Examples

Find the derivative of

(1) $g_1(x) = \int_0^x \sqrt{1+t}dt$.

(2) $g_2(x) = \int_x^0 \sqrt{1+t}dt$.

(3) $g_3(x) = \int_0^{x^2} \sqrt{1+t}dt$.

(4) $g_4(x) = \int_{\sin(x)}^{\cos(x)} \sqrt{1+t}dt$.

BE CAREFUL:

Evaluate $\int_{-3}^{6} \frac{1}{x}dx$

Net Change Theorem

Question: If $y = F(x)$, then what does $F'(x)$ represents?

Theorem **The Net Change Theorem**

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)dx = F(b) - F(a) \qquad \text{Net change of } F(x)$$

- There are many applications, we will focus on one

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

- **Remarks**

$$\begin{aligned} \text{displacement} &= \int_{t_1}^{t_2} v(t)dt \\ \text{total distance traveled} &= \int_{t_1}^{t_2} |v(t)|dt \end{aligned}$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1) \quad \text{is the change in velocity from time to time .}$$

Example 10: **Solving a Particle Motion Problem**

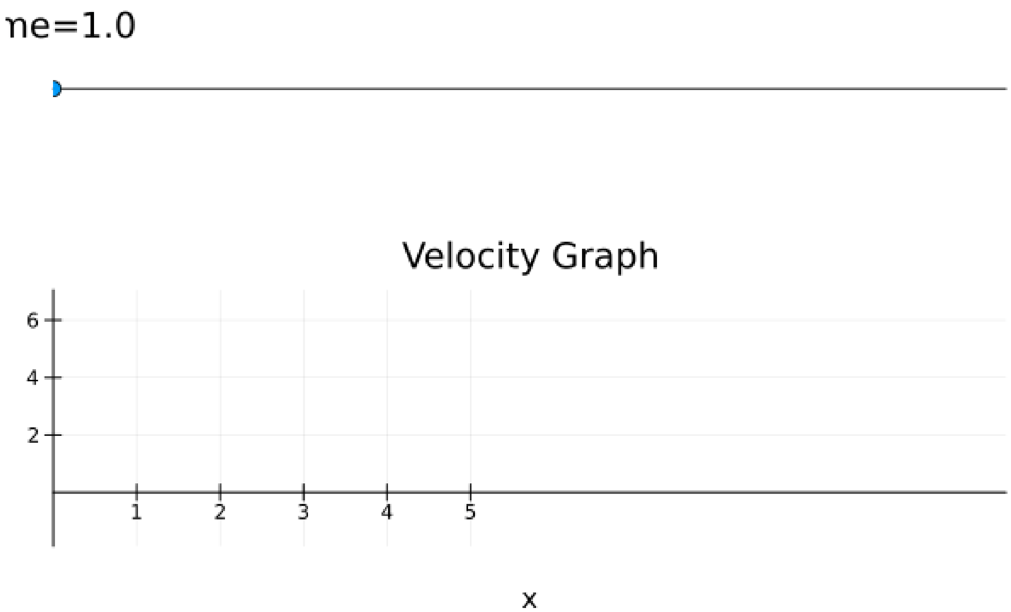
A particle is moving along aline. Its velocity function (in m/s^2) is given by

$$v(t) = t^3 - 10t^2 + 29t - 20,$$

a. What is the **displacement** of the particle on the time interval 1≤ t≤ 5?
b. What is the **total distance** traveled by the particle on the time interval 1≤ t≤ 5?

```
v (generic function with 1 method)
```

```
v(t) = t^3 - 10 * t^2 + 29 * t - 20
```




```
ⓘ Saved animation to /home/code/src/example_fps15.gif
```

5.5: The Substitution Rule

> **Objectives**
>
> 1. Use pattern recognition to find an indefinite integral.
> 2. Use a change of variables to find an indefinite integral.
> 3. Use the General Power Rule for Integration to find an indefinite integral.
> 4. Use a change of variables to evaluate a definite integral.
> 5. Evaluate a definite integral involving an even or odd function.

	solve
$\int 2x\sqrt{1+x^2}\;dx$	$\int \sqrt{u}\;du$

Pattern Recognition

Theorem **Antidifferentiation of a Composite Function**

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C.$$

$$\int \overbrace{f}^{\text{Outside function}}(\underbrace{g(x)}_{\text{Inside function}})\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

Substitution Rule says: It is permissible to operate with dx and du after integral signs as if they were differentials.

Example Find

(i) $\quad \int (x^2+1)^2(2x)dx$

(ii) $\quad \int 5e^{5x}dx$

(iii) $\quad \int \frac{x}{\sqrt{1-4x^2}}dx$

(iv) $\quad \int \sqrt{1+x^2}\;\; x^5 dx$

(v) $\quad \int \tan x dx$

Change of Variables for Indefinite Integrals

Example: Find

$$
\begin{aligned}
&(i) \qquad \int \sqrt{2x-1}dx \\
&(ii) \qquad \int x\sqrt{2x-1}dx \\
&(iii) \qquad \int \sin^2 3x \cos 3x dx
\end{aligned}
$$

The General Power Rule for Integration

Theorem **The General Power Rule for Integration**

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example: Find

$$(i) \quad \int 3(3x-1)^4 dx$$

$$(ii) \quad \int (e^x+1)(e^x+x)dx$$

$$(iii) \quad \int 3x^2\sqrt{x^3-2}\, dx$$

$$(iv) \quad \int \frac{-4x}{(1-2x^2)^2}\, dx$$

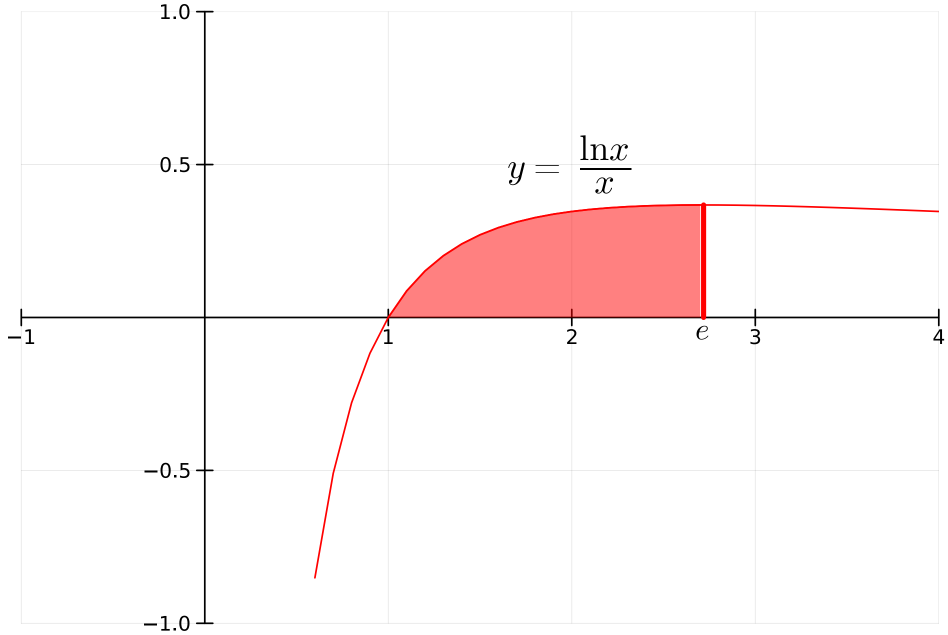
$$(v) \quad \int \cos^2 x \sin x \, dx$$

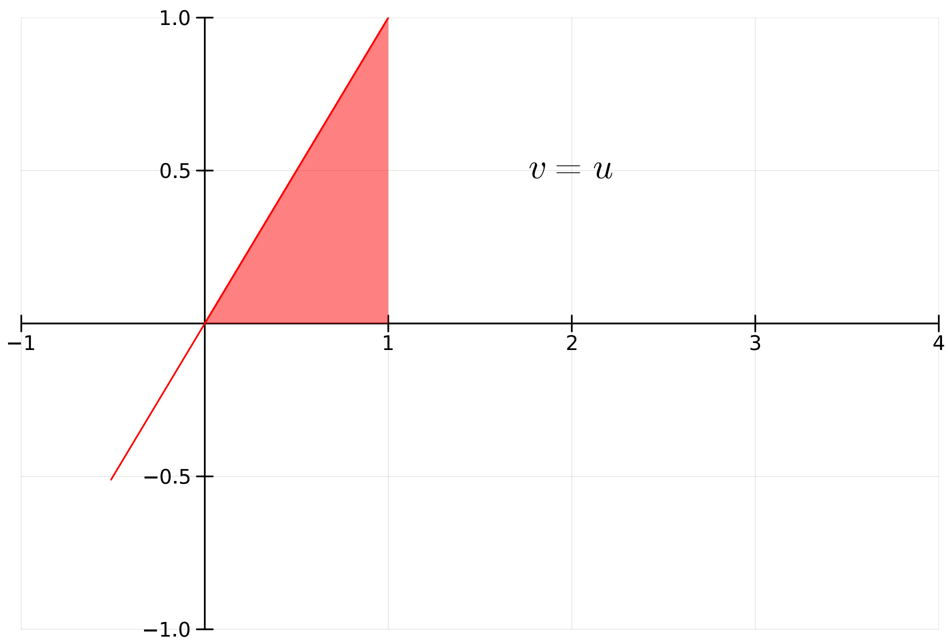
Change of Variables for Definite Integrals

Substitution: Definite Integrals

Example: Evaluate

$$\int_1^e \frac{\ln x}{x} dx$$





Example: Evaluate

$$(i) \quad \int_1^2 \frac{dx}{(3-5x)^2}$$

$$(iii) \quad \int_0^1 x(x^2+1)^3\, dx$$

$$(iv) \quad \int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$$

Integration of Even and Odd Functions

Theorem **Integration of Even and Odd Functions**

Let f be integrable on $[-a, a]$.

- If f is **even** $[f(-x) = f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

- If f is **odd** $[f(-x) = -f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

Example Find

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx$$

5.7: The Natural Logarithmic Function: Integration

Objectives

1. Use the Log Rule for Integration to integrate a rational function.
2. Integrate trigonometric functions.

Log Rule for Integration

Theorem **Log Rule for Integration**

Let u be a differentiable function of x.

$$\begin{array}{lrcl} \text{(i)} & \displaystyle\int \frac{1}{x}dx & = & \ln|x| + C \\ \text{(ii)} & \displaystyle\int \frac{1}{u}du & = & \ln|u| + C \end{array}$$

Remark

$$\int \frac{u'}{u}dx = \ln|u| + C$$

Example 1: **Using the Log Rule for Integration**

$$\int \frac{2}{x} dx$$

Example 3: **Finding Area with the Log Rule**

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.

Example 5: **Using Long Division Before Integrating**

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx$$

Examples Find

$$\text{(i)} \qquad \int \frac{1}{4x-1}dx$$

$$\text{(ii)} \qquad \int \frac{3x^2+1}{x^3+x}dx$$

$$\text{(iii)} \qquad \int \frac{\sec^2 x}{\tan x}dx$$

$$\text{(iv)} \qquad \int \frac{x^2+x+1}{x^2+1}dx$$

$$\text{(v)} \qquad \int \frac{2x}{(x+1)^2}dx$$

Example 7: **Solve the differential equation**

Solve

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

Integrals of Trigonometric Functions

Example 8: **Using a Trigonometric Identity**

$$\int \tan x dx$$

Example 9: **Derivation of the Secant Formula**

$$\int \sec x dx$$

5.8:Inverse Trigonometric Functions: Integration

Objectives

1. Integrate functions whose antiderivatives involve inverse trigonometric functions.
2. Use the method of completing the square to integrate a function.
3. Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

Theorem Integrals Involving Inverse Trigonometric Functions

Let u be a differential function of x, and let $a > 0$.

$$\begin{aligned} 1.\quad & \int \frac{du}{\sqrt{a^2-u^2}} &&= \arcsin\tfrac{u}{a}+C \\ 2.\quad & \int \frac{du}{a^2+u^2} &&= \tfrac{1}{a}\arctan\tfrac{u}{a}+C \\ 3.\quad & \int \frac{du}{u\sqrt{u^2-a^2}} &&= \tfrac{1}{a}\operatorname{arcsec}\tfrac{|u|}{a}+C \end{aligned}$$

Examples Find

- $$\int \frac{dx}{\sqrt{4-x^2}},$$
- $$\int \frac{dx}{2+9x^2},$$
- $$\int \frac{dx}{x\sqrt{4x^2-9}},$$
- $$\int \frac{dx}{\sqrt{e^{2x}-1}},$$
- $$\int \frac{x+2}{\sqrt{4-x^2}}dx.$$

Completing the Square

Example 5: **Completing the Square**

Find

$$\int \frac{dx}{x^2-4x+7}.$$

Example 6: **Completing the Square**

Find the area of the region bounded by the graph of

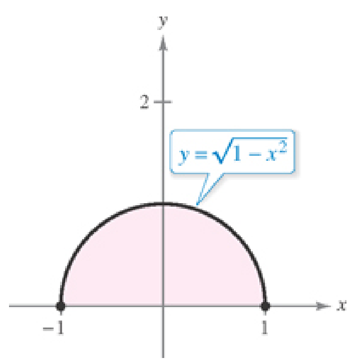
$$f(x) = \frac{1}{\sqrt{3x-x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

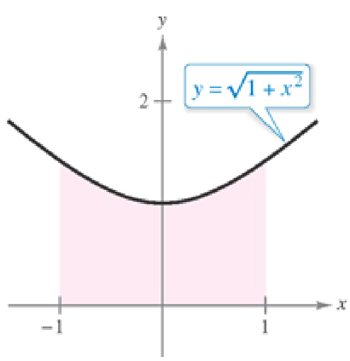
5.9: Hyperbolic Functions

> **Objectives**
>
> 1. Develop properties of hyperbolic functions (MATH101).
> 2. Differentiate (MATH101) and integrate hyperbolic functions.
> 3. Develop properties of inverse hyperbolic functions (Reading only).
> 4. Differentiate and integrate functions involving inverse hyperbolic functions. (Reading only).

Circle: $x^2 + y^2 = 1$



Hyperbola: $-x^2 + y^2 = 1$

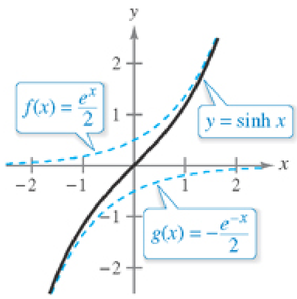


Definitions of the Hyperbolic Functions

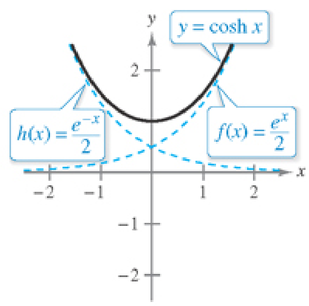
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x},\ x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

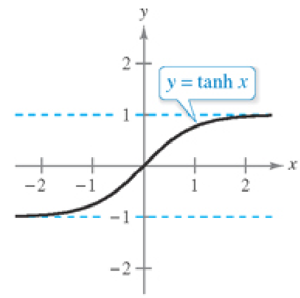
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x},\ x \neq 0$$



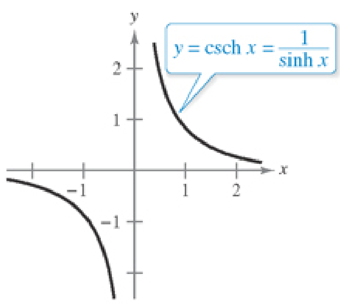
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



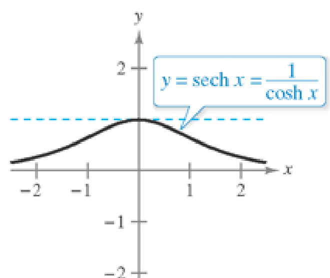
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



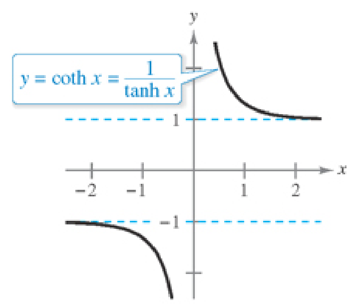
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1, \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1, \qquad \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh$$

$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}, \qquad \cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sin 2x = 2 \sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Theorem Differentiation and Integration of Hyperbolic Functions

Theorem Let u be a differentiable function of x.

$$\begin{aligned}
\frac{d}{dx}(\sinh u) &= (\cosh u)u', & \int \cosh u du &= \sinh u + C \\
\frac{d}{dx}(\cosh u) &= (\sinh u)u', & \int \sinh u du &= \cosh u + C \\
\frac{d}{dx}(\tanh u) &= (\operatorname{sech}^2 u)u', & \int \operatorname{sech}^2 u du &= \tanh u + C \\
\frac{d}{dx}(\coth u) &= -(\operatorname{csch}^2 u)u', & \int \operatorname{csch}^2 u du &= -\coth u + C \\
\frac{d}{dx}(\operatorname{sech} u) &= -(\operatorname{sech} u \tanh u)u', & \int \operatorname{sech} u \tanh u du &= -\operatorname{sech} u + C \\
\frac{d}{dx}(\operatorname{csch} u) &= -(\operatorname{csch} u \coth u)u', & \int \operatorname{csch} u \coth u du &= -\operatorname{csch} u + C
\end{aligned}$$

Example 4: Integrating a Hyperbolic Function

Find

$$\int \cosh 2x \sinh^2 2x dx$$

7.1: Area of a Region Between Two Curves

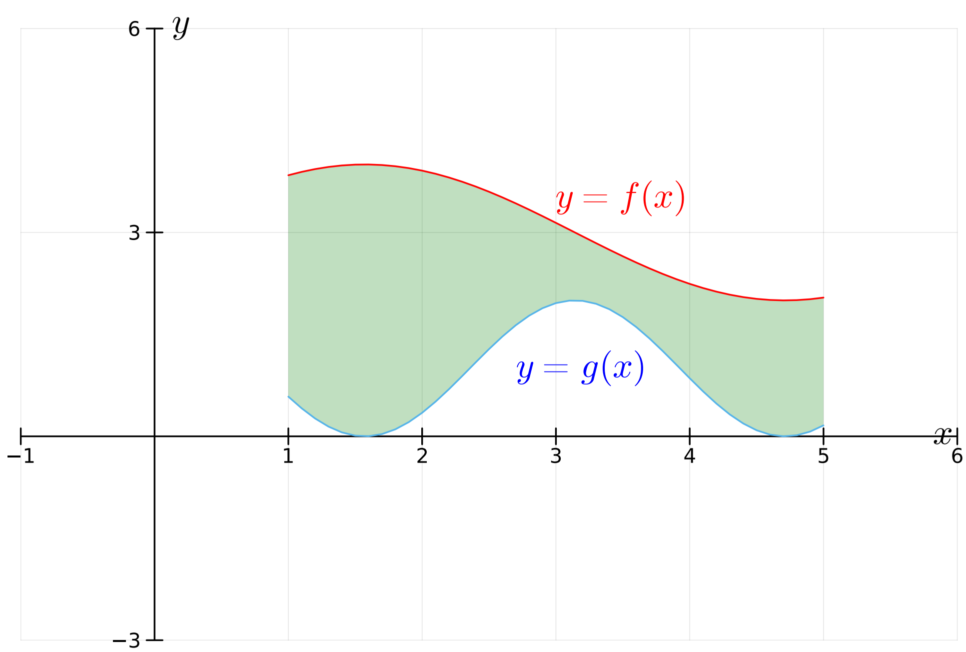
Objectives

> “
>
> 1. Find the area of a region between two curves using integration.
> 2. Find the area of a region between intersecting curves using integration.
> 3. Describe integration as an accumulation process.

......

Area of a Region Between Two Curves

How can we find the area between the two curves?



$$Area = \int_a^b [f(x) - g(x)]dx$$

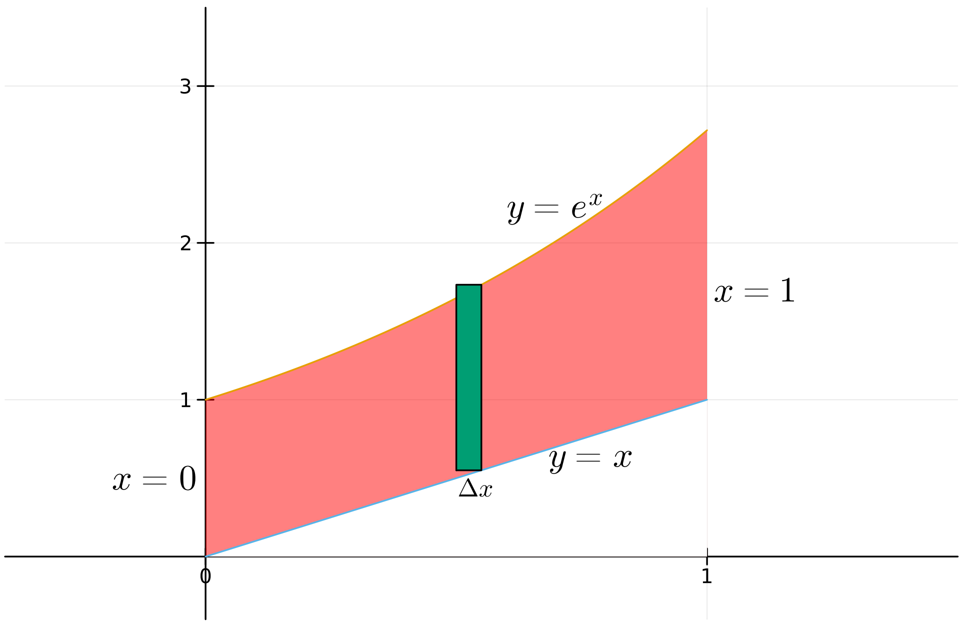
Remark

- Area = $y_{top} - y_{bottom}$.

Example 1: **Finding the Area of a Region Between Two Curves**

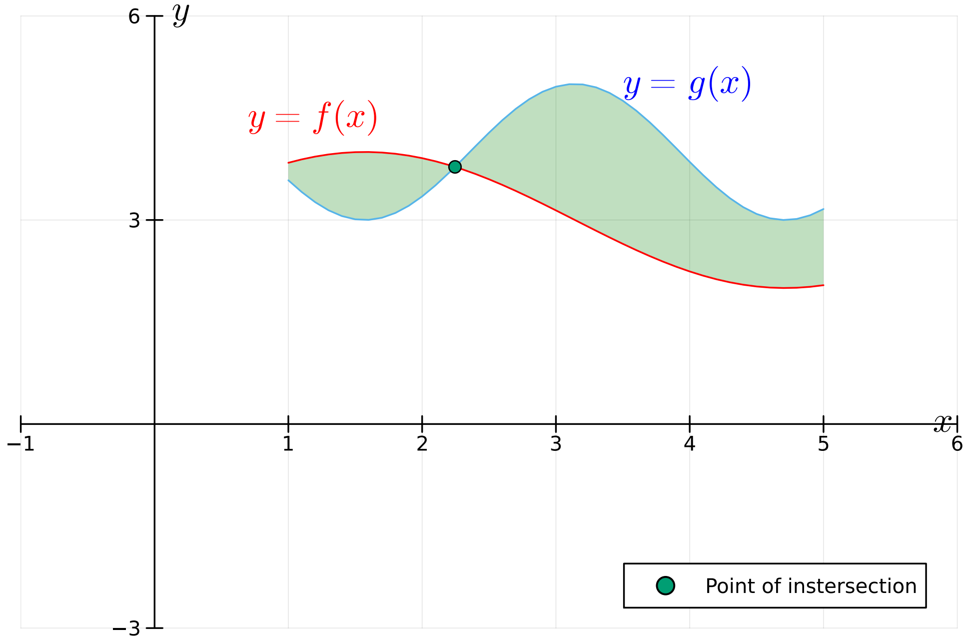
Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, bounded on the sides by $x = 0$ and $x = 1$.

Solution



Area of a Region Between Intersecting Curves

In geberal,



$$Area = \int_a^b |f(x) - g(x)| dx$$

Example 2: **A Region Lying Between Two Intersecting Graphs**

Find the area of the region enclosed by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution in class

Example 3: **A Region Lying Between Two Intersecting Graphs**

Find the area of the region bounded by the curves

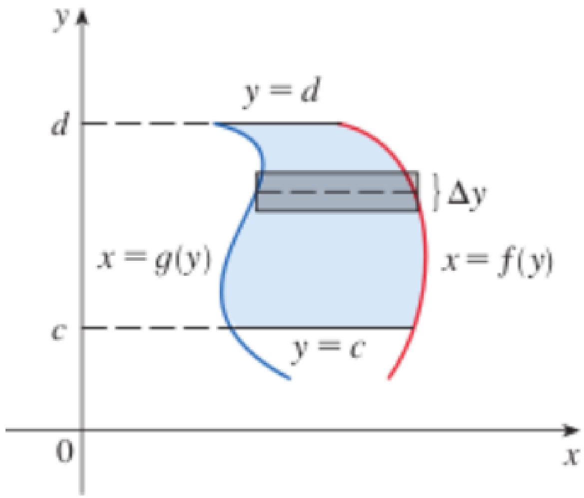
$$y = \cos(x), \quad y = \sin(x), \quad x = 0, \quad x = \frac{\pi}{2}$$

Example 4: **Curves That Intersect at More than Two Points**

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x, \qquad g(x) = -x^2 + 2x.$$

Integrating with Respect to y



Example 5: **Horizontal Representative Rectangles**

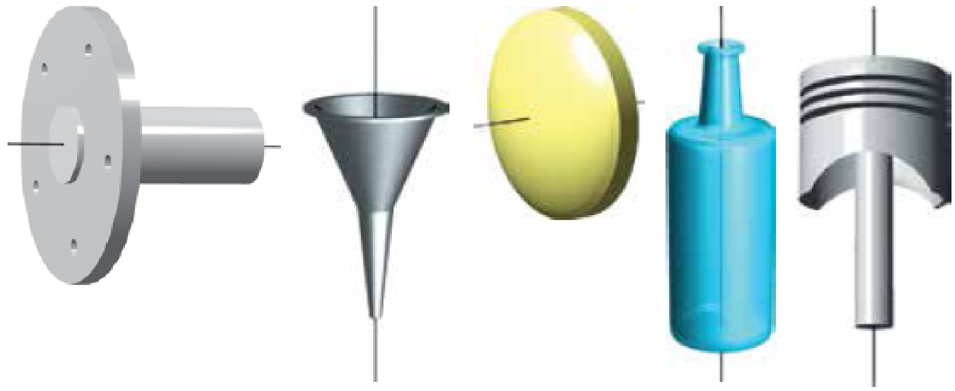
Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

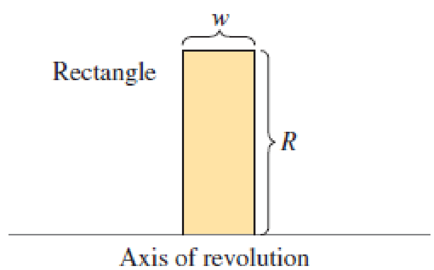
7.2: Volume: The Disk Method

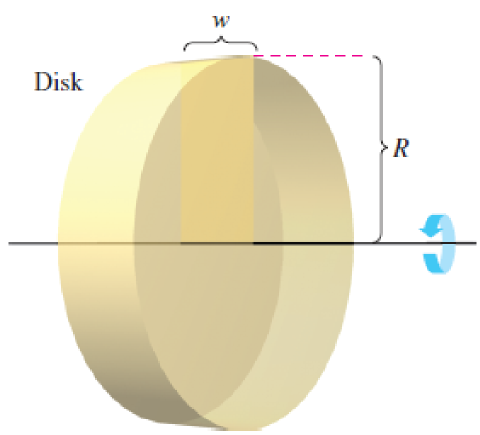
> **Objectives**
>
> - Find the volume of a solid of revolution using the disk method.
> - Find the volume of a solid of revolution using the washer method.
> - Find the volume of a solid with known cross sections.

The Disk Method

Solids of Revolution



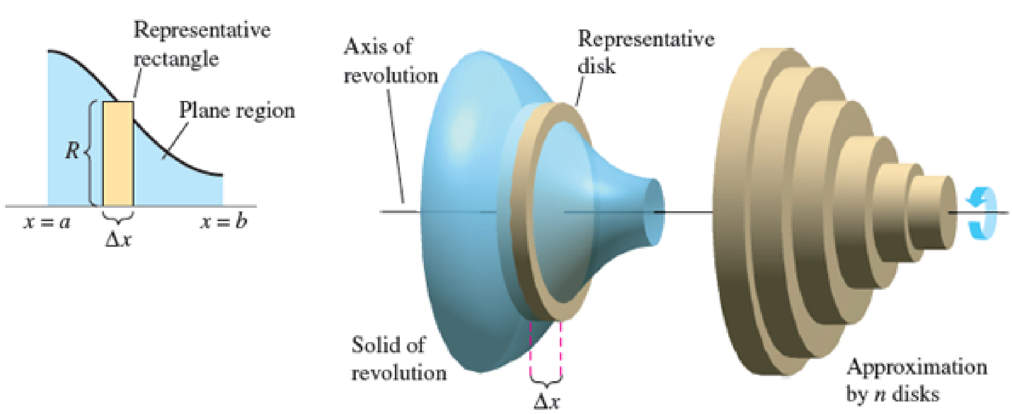




Volume of a disk

$$V = \pi R^2 w$$

Disk Method



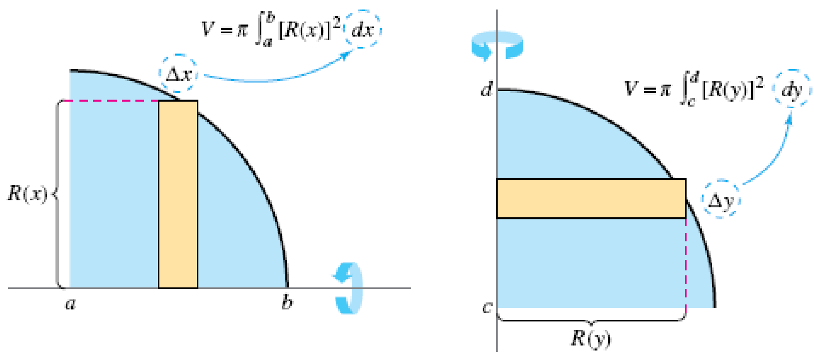
$$
\begin{aligned}
\text{Volume of solid} &\approx \sum_{i=1}^{n} \pi \left[R(x_i)\right]^2 \Delta x \\
&= \pi \sum_{i=1}^{n} \left[R(x_i)\right]^2 \Delta x
\end{aligned}
$$

Taking the limit $\|\Delta\| \to 0 (n \to \infty)$, we get

$$
\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} \left[R(x_i)\right]^2 \Delta x = \pi \int_a^b \left[R(x)\right]^2 dx.
$$

Disk Method

To find the volume of a solid of revolution with the disk method, use one of the formulas below



Horizontal axis of revolution

Vertical axis of revolution

Example 1: **Using the Disk Method**

Find the volume of the solid formed by revolving the region bounded by the graph of

$$
f(x) = \sqrt{\sin x}
$$

and the x-axis ($0 \leq x \leq \pi$) about the x-axis

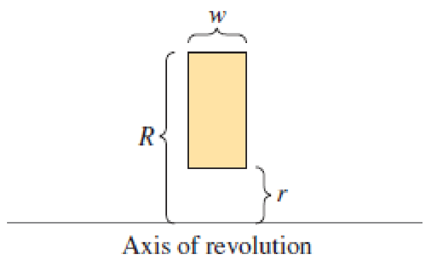
Example 2: **Using a Line That Is Not a Coordinate Axis**

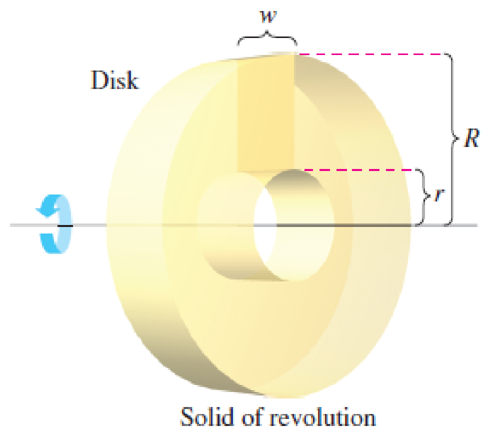
Find the volume of the solid formed by revolving the region bounded by the graphs of

$$
f(x) = 2 - x^2
$$

and $g(x) = 1$ about the line $y = 1$.

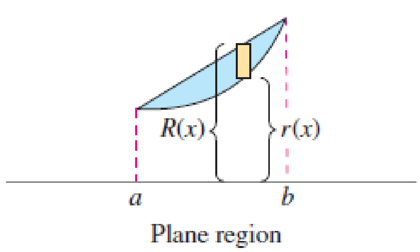
The Washer Method

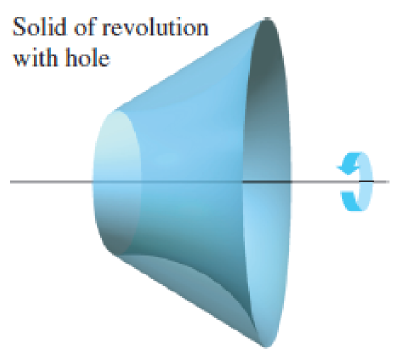




$$\text{Volume of washer} = \pi(R^2 - r^2)w$$

Washer Method





$$V = \pi \int_a^b [(R[x])^2 - (r[x])^2) dx$$

Example 3: **Using the Washer Method**

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = \sqrt{x} \qquad \text{and} \qquad y = x^2$$

about the x-axis.

Example 4: **Integrating with Respect to `y`: Two-Integral Case**

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

about the y-axis

Solids with Known Cross Sections

Example 1 | Example 2

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$V = \int_a^b A(x)dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$V = \int_c^d A(y)dy$$

Example 6: **Triangular Cross Sections**

The base of a solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \quad g(x) = -1 + \frac{x}{2} \quad \text{and} \quad x = 0.$$

The cross sections perpendicular to the x-axis are equilateral triangles.

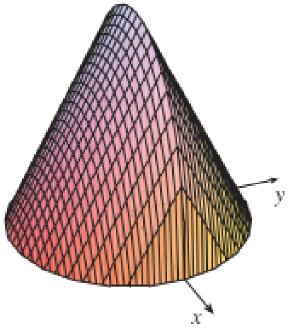
Exercise Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Exercise The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $y = 2$.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $x = -1$.

Exercise Figure below shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



7.3: Volume: The Shell Method

> **Objectives**
>
> 1. Find the volume of a solid of revolution using the shell method.
> 2. Compare the uses of the disk method and the shell method.

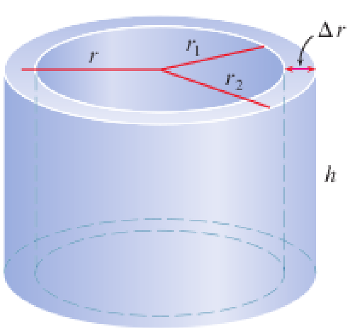
Problem Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y—axis.

Step 1: ☐ Step 2: ☐ Step 3: ☐

""

The Shell Method

A shell is a hallow circular cylinder

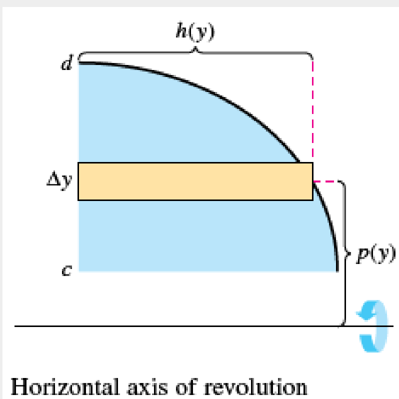


$$V = 2\pi r h \Delta r = [\text{circumference}][\text{height}][\text{thickness}]$$

Cylindrical Shells Illustration

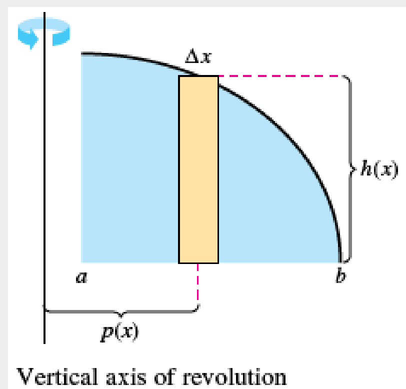
Horizontal Axis of Revolution

$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)dy$$



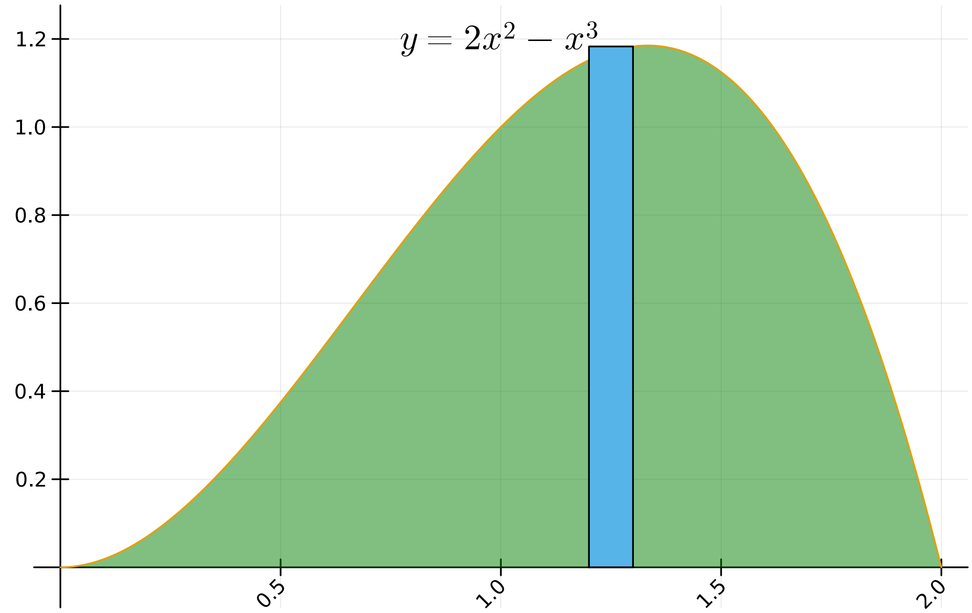
Vertical Axis of Revolution

$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)dx$$



Example: Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the $y-$axis.

Solution:



Example : Find the volume of the solid obtained by rotating about the $y-$axis the region between $y = x$ and $y = x^2$.

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Example 4: **Shell Method Preferable**

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1.$$

about the y-axis.

Example 5: **Shell Method Necessary**

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$.

7.4: Arc Length and Surfaces of Revolution

> **Objectives**
>
> 1. Find the arc length of a smooth curve.
> 2. Find the area of a surface of revolution.

Arc Length

Definition **Arc Length**

Let the function $y = f(x)$ represents a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 2: **Finding Arc Length**

Find the arc length of the graph of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$.

Example 3: **Finding Arc Length**

Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0, 8]$.

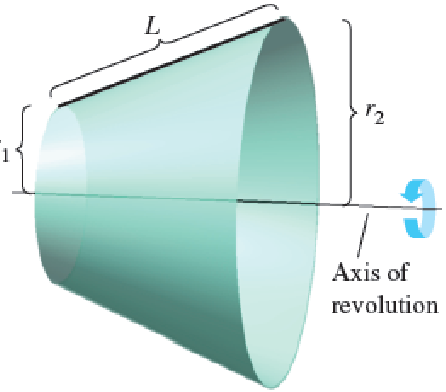
Example 4: **Finding Arc Length**

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$.

Area of a Surface of Revolution

Definition **Surface of Revolution**

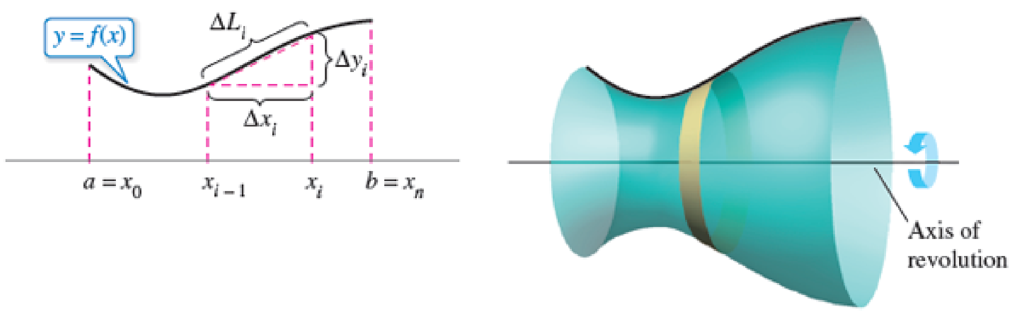
When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.



Surface Area of *frustum*

$$S = 2\pi r L, \quad \text{where} \quad r = \frac{r_1 + r_2}{2}$$

Consider a function f that has a continuous derivative on the interval $[a, b]$. The graph of f is revolved about the x-axis

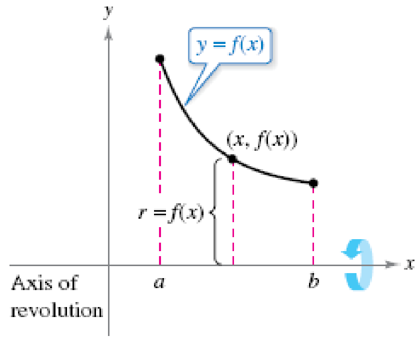


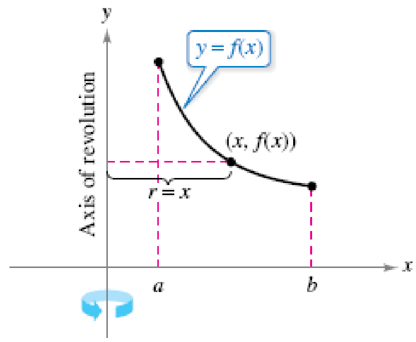
Surface Area Formula

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2} dx.$$

Definition **Area of a Surface of Revolution**

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$.





The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx, \quad y \text{ is a function of x .}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution.

If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_a^b r(y)\sqrt{1 + [g'(y)]^2}dy, \quad x \text{ is a function of y .}$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Remark

The formulas can be written as

$$S = 2\pi \int_a^b r(x)ds, \quad y \text{ is a function of x .}$$

and

$$S = 2\pi \int_c^d r(y)ds, \quad x \text{ is a function of y .}$$

where

$$ds = \sqrt{1 + \left[f'(x)\right]^2}dx \quad \text{and} \quad ds = \sqrt{1 + \left[g'(y)\right]^2}dy \quad \text{respectively.}$$

Example 6: **The Area of a Surface of Revolution**

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis.

Example 7: **The Area of a Surface of Revolution**

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis.

8.1: Basic Integration Rules

> **Objectives**
>
> 1. Review procedures for fitting an integrand to one of the basic integration rules.

Review of Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, n \neq -1$

5. $\int \dfrac{du}{u} = \ln|u| + C$

6. $\int e^u du = e^u + C$

7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right) a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \arctan \dfrac{u}{a} + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{|u|}{a} + C$

Example 3:

Find $\displaystyle\int \frac{x^2}{\sqrt{16-x^6}}dx$.

Example 4: **A Disguised Form of the Log Rule**

Find $\displaystyle\int \frac{dx}{1+e^x}$.

8.2: Integration by Parts

> **Objectives**
>
> 1. Find an antiderivative using integration by parts.

The integration rule that corresponds to the Product Rule for differentiation is called ***integration by parts***

Indefinite Integrals

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Theorem **Integration by Parts**

If u and v are functions of x and have continuous derivatives, then

$$\int udv = uv - \int vdu.$$

Example 1: **Integration by Parts**

Find $\int xe^x dx$.

Example 2: **Integration by Parts**

Find $\int x^2 \ln x dx$.

Example 3: **An Integrand with a Single Term**

Evaluate $\int_0^1 \arcsin x dx$.

Example 4: **Repeated Use of Integration by Parts**

Find $\int x^2 \sin x dx$.

Example 5: **Integration by Parts**

Find $\int \sec^3 x dx$.s

Example 7: **Using the Tabular Method**

Find $\int x^2 \sin 4x dx$.

8.3: Trigonometric Integrals

Objectives

1. Solve trigonometric integrals involving powers of sine and cosine.
2. Solve trigonometric integrals involving powers of secant and tangent.
3. Solve trigonometric integrals involving sine-cosine products.

RECALL

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\begin{aligned} \sin mx \sin nx &= \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x], \\ \sin mx \cos nx &= \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)], \\ \cos mx \cos nx &= \tfrac{1}{2}[\cos(m-n) + \cos(m+n)], \end{aligned}$$

$$\int \tan x dx = \ln|\sec x| + C, \quad \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x dx = -\ln|\csc x| + C, \quad \int \csc x dx = \ln|\csc x - \cot x| + C$$

Integrals of Powers of Sine and Cosine

$$\int \sin^m x \cos^n x dx$$

- m is odd, write as $\int \sin^{m-1} x \cos^n x \sin x dx$. *Example*: $\int \sin^5 x \cos^2 x dx$
- n is odd, write as $\int \sin^m x \cos^{n-1} \cos x dx$. *Example* $\int \sin^5 x \cos^3 x dx$
- m and n are even, use formulae (*Example* $\int \cos^2 x dx$ and $\int \sin^4 x dx$)

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}.$$

Example 1: **Power of Sine Is Odd and Positive**

Find $\displaystyle\int \sin^3 x \cos^4 x dx$.

Example 2: **Power of Cosine Is Odd and Positive**

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$.

Example 3: **Power of Cosine Is Even and Nonnegative**

Find $\int \cos^4 x dx$.

Integrals of Powers of Secant and Tangent

$$\int \tan^m x \sec^n x dx$$

- n is even, write as $\int \tan^m x \sec^{n-2} \sec^2 x dx$. *Example* $\int \tan^6 x \sec^4 x dx$
- m is odd, write as $\int \tan^{m-1} x \sec^{n-1} \tan x \sec x dx$. *Example* $\int \tan^5 x \sec^7 x dx$.

Example 4: **Power of Tangent Is Odd and Positive**

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$.

Example 5: **Power of Secant Is Even and Positive**

Find $\int \sec^4 3x \tan^3 3x dx$.

Example 6: **Power of Tangent Is Even**

Evaluate $\int_0^{\pi/4} \tan^4 x dx$.

Example 7: **Converting to Sines and Cosines**

Find $\int \frac{\sec x}{\sqrt{\tan^2 x}} dx$.

Integrals Involving Sine-Cosine Products

Example 8: Using a Product-to-Sum Formula

Find $\int \sin 5x \cos 4x dx$.

```
begin
    using FileIO, ImageIO, ImageShow, ImageTransformations
    using SymPy
    using PlutoUI
    using CommonMark
    using Plots, PlotThemes, LaTeXStrings
    using HypertextLiteral: @htl, @htl_str
    using Colors
    using Random
end
```