

MATH102

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5.2: Area

Objectives

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- 1 Use sigma notation to write and evaluate a sum.
- 2 Understand the concept of area.
- 3 Approximate the area of a plane region.
- 4 Find the area of a plane region using limits.

Sigma Notation

Sigma Notation

The sum of n terms a_1, a_2, \dots, a_n is written as

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$$

where i is the **index of summation**, a_i is the i th term of the sum, and the upper and lower bounds of summation are n and 1 .

Summation Properties

$$\sum_{i=1}^n ka_i = k \sum_{i=1}^n a_i$$

$$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

Theorem Summation Formulas

$$(1) \quad \sum_{i=1}^n c = cn, \quad c \text{ is a constant}$$

$$(2) \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$(3) \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(4) \quad \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Example 1: Evaluating a Sum

Evaluate $\sum_{i=1}^n \frac{i+1}{n}$ for $n = 10, 100, 1000$ and $10,000$.

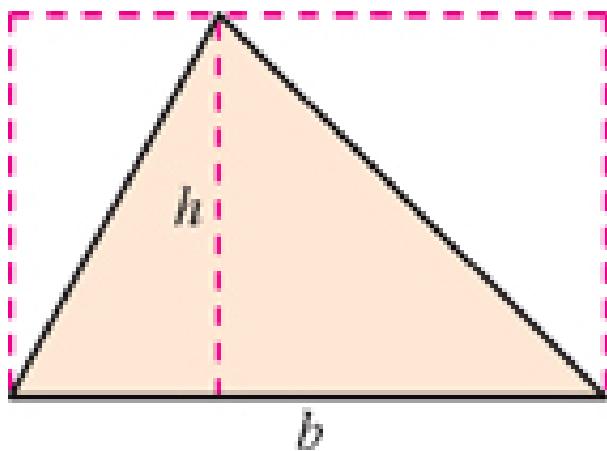
Area

In **Euclidean geometry**, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is

$$A = bh$$

it is actually more proper to say that this is the *definition* of the **area of a rectangle**.

For a triangle $A = \frac{1}{2}bh$



The Area of a Plane Region

Example

Use **five** rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = 5 - x^2$$

and the x -axis between $x = 0$ and $x = 2$.

f (generic function with 1 method)

```
1 f(x) = 5 - x^2
```

n = a = b = method =

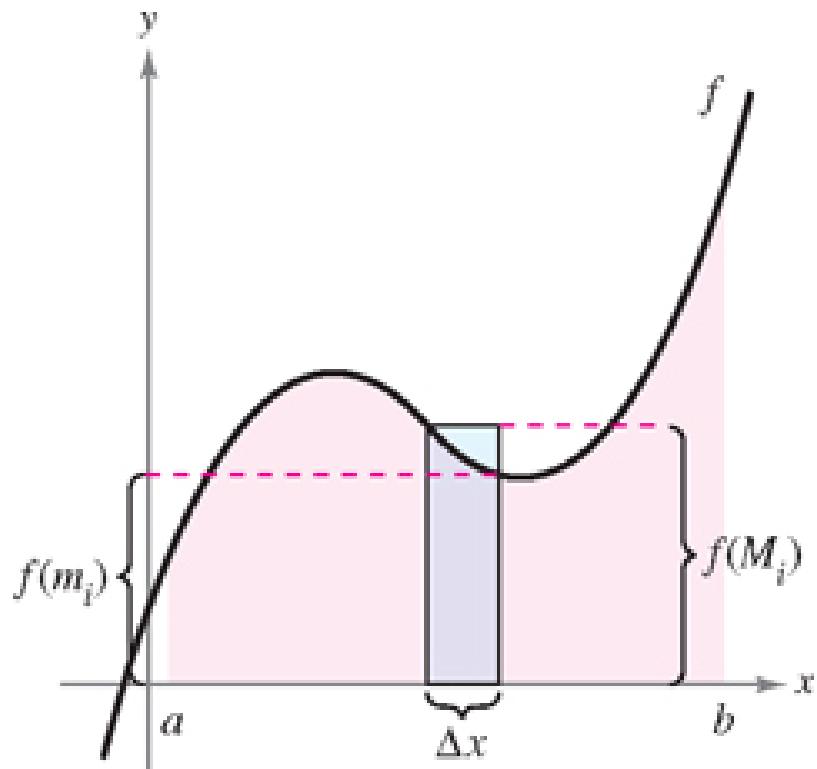
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Finding Area by the Limit Definition

Find the area of a plane region bounded above by the graph of a nonnegative, **continuous** function

$$y = f(x)$$

The region is bounded below by the x -axis and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$



- To approximate the area of the region, begin by subdividing the interval into subintervals, each of width

$$\Delta x = \frac{b - a}{n}$$

- The endpoints of the intervals are

$$\overbrace{a + 0(\Delta x)}^{a=x_0} < \overbrace{a + 1(\Delta x)}^{a=x_1} < \overbrace{a + 2(\Delta x)}^{a=x_2} < \cdots < \overbrace{a + n(\Delta x)}^{a=x_n}.$$

- Let

$$f(m_i) = \text{Minimum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval}$$

$$f(M_i) = \text{Maximum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval}$$

- Define an **inscribed rectangle** lying inside the i^{th} subregion
- Define an **circumscribed rectangle** lying outside the i^{th} subregion

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \leq f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

- The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of the circumscribed rectangles is called an **upper sum**.

$$\text{Lower sum} = s(n) = \sum_{i=1}^n f(m_i)\Delta x \quad \text{Area of inscribed rectangle}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^n f(M_i)\Delta x \quad \text{Area of circumscribed rectangle}$$

- The actual area of the region lies between these two sums.

$$s(n) \leq (\text{Area of region}) \leq S(n).$$

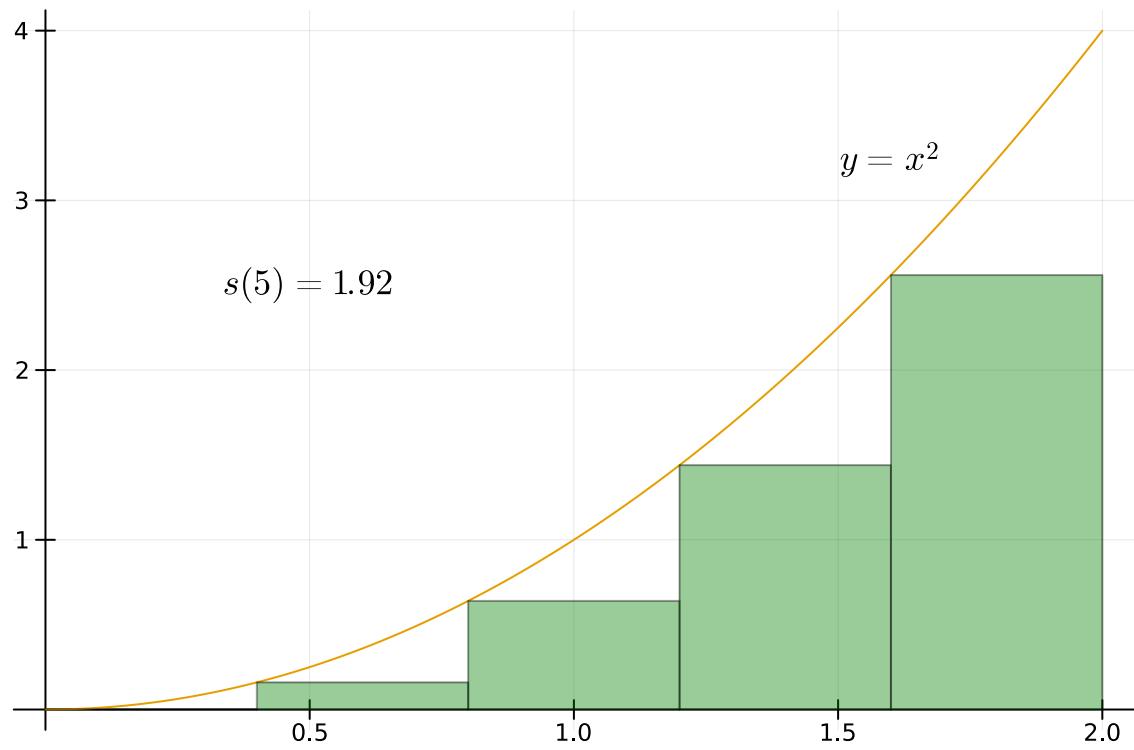
Example 4: Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x -axis between $x = 0$ and $x = 2$.

$n = 5$ $a = 0$ $b = 2$ method = **Left** ▾

f4 (generic function with 1 method)

1 f4(x) = x^2



Theorem Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \rightarrow \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n \rightarrow \infty} s(n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(m_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(M_i) \Delta x = \lim_{n \rightarrow \infty} S(n)$$

where

$$\Delta x = \frac{b - a}{n}$$

and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the i th subinterval.

Definition Area of a Region in the Plane

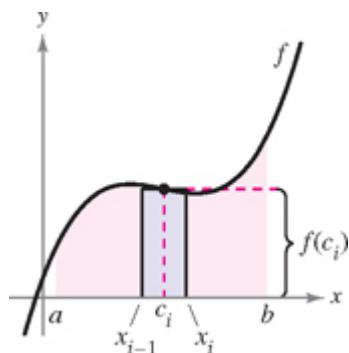
Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $y = b$ is

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$$

where

$$x_{i-1} \leq c_i \leq x_i \quad \text{and} \quad \Delta x = \frac{b-a}{n}.$$

See the graph



Example 5: Finding Area by the Limit Definition

Find the area of the region bounded by the graph of $f(x) = x^3$, the x -axis, and the vertical lines $x = 0$ and $x = 1$.

Example 7: A Region Bounded by the y -axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y -axis for $0 \leq y \leq 1$.)

Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x.$$

Example 8: Approximating Area with the Midpoint Rule

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = \sin x$ and the x -axis for $0 \leq x \leq \pi$.

2.0523443059540623

```
1 begin
2   f8(x)=sin(x)
3   Δx28 = π/4
4   A = Δx28*(f8(π/8)+f8(3π/8)+f8(5π/8)+f8(7π/8))
5 end
```

5.3: Riemann Sums and Definite Integrals

“ Objectives

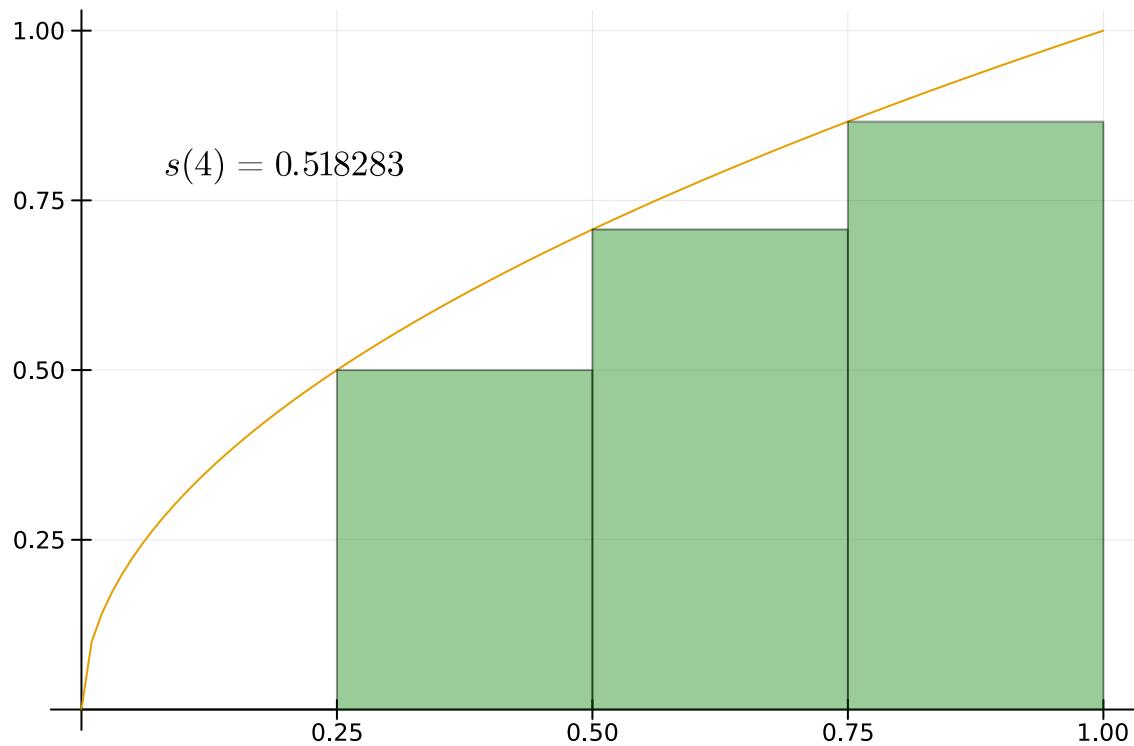
- 1 Understand the definition of a Riemann sum.
- 2 Evaluate a definite integral using limits and geometric formulas.
- 3 Evaluate a definite integral using properties of definite integrals.

Riemann Sums

g (generic function with 1 method)

```
1 g(x) = √x
```

n = a = b = method =



Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the i th subinterval

$$[x_{i-1}, x_i] \quad \text{ith subinterval}$$

If c_i is any point in the i th subinterval, then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ .

Remark

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$.

- If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n} \quad \text{Regular partition}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \quad \text{General partition}$$

- Note that

$$\|\Delta\| \rightarrow 0 \quad \text{implies that} \quad n \rightarrow \infty.$$

Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

exists, then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** of f from a to b . The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Theorem Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is,

$$\int_a^b f(x)dx \text{ exists.}$$

Theorem The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)dx$$

Example 3: Areas of Common Geometric Figures

Evaluate each integral using a geometric formula.

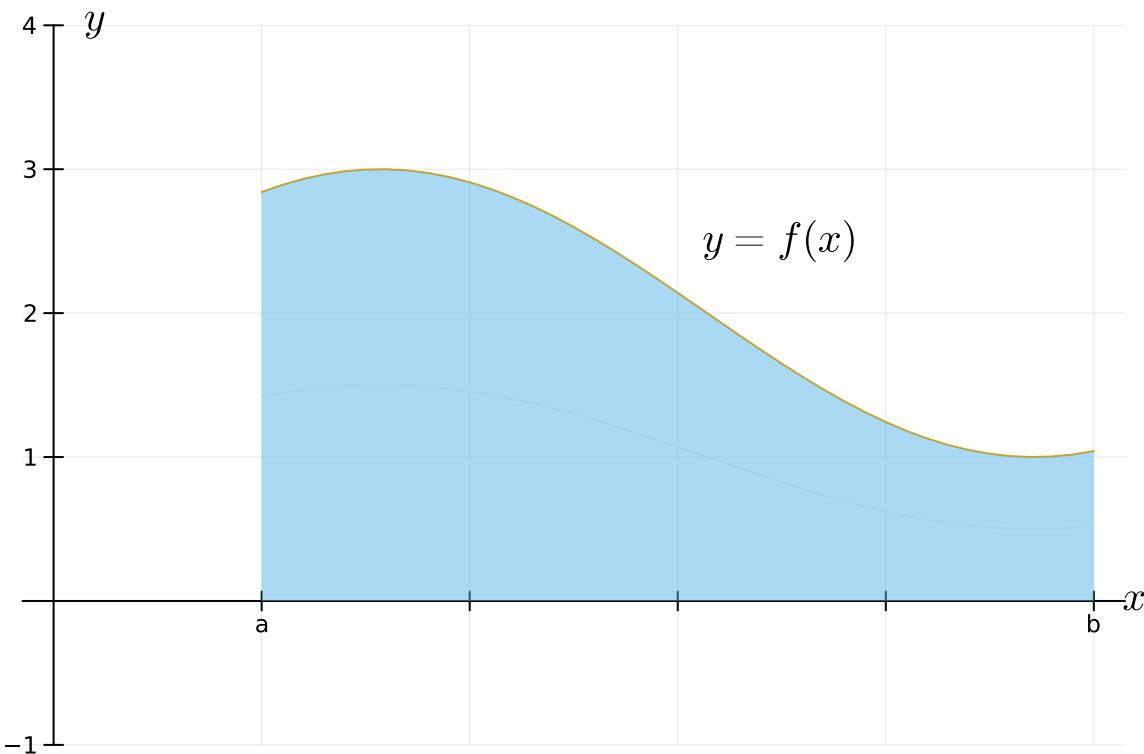
- $\int_1^3 4dx$
- $\int_0^3 (x + 2)dx$
- $\int_{-2}^2 \sqrt{4 - x^2}dx$

Remark The definite integral is a ****number****

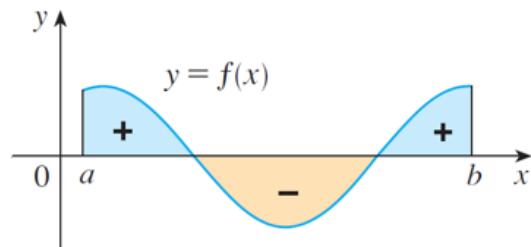
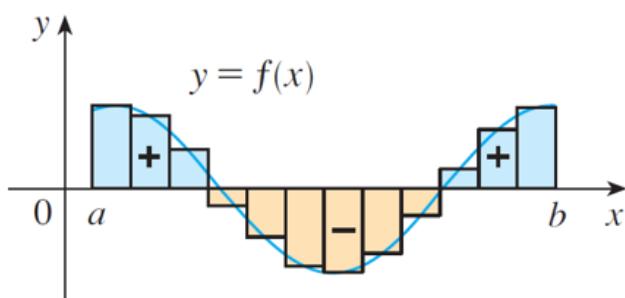
- It does not depend on x . In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(w)dw = \int_a^b f(\text{☺})d\text{☺}$$

- If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from a to b .



- $\int_a^b f(x)dx$ is the net area



Properties of Definite Integrals

Definitions Two Special Definite Integrals

- If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$.
- If f is integrable on $[a, b]$, then $\int_b^a f(x)dx = - \int_a^b f(x)dx$.

Theorem Additive Interval Property

If f is integrable on the three closed intervals determined by a, b and c , then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem Properties of Definite Integrals

- If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x)dx = k \int_a^b f(x)dx.$
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

Theorem Preservation of Inequality

- If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)dx.$$

- If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Examples:



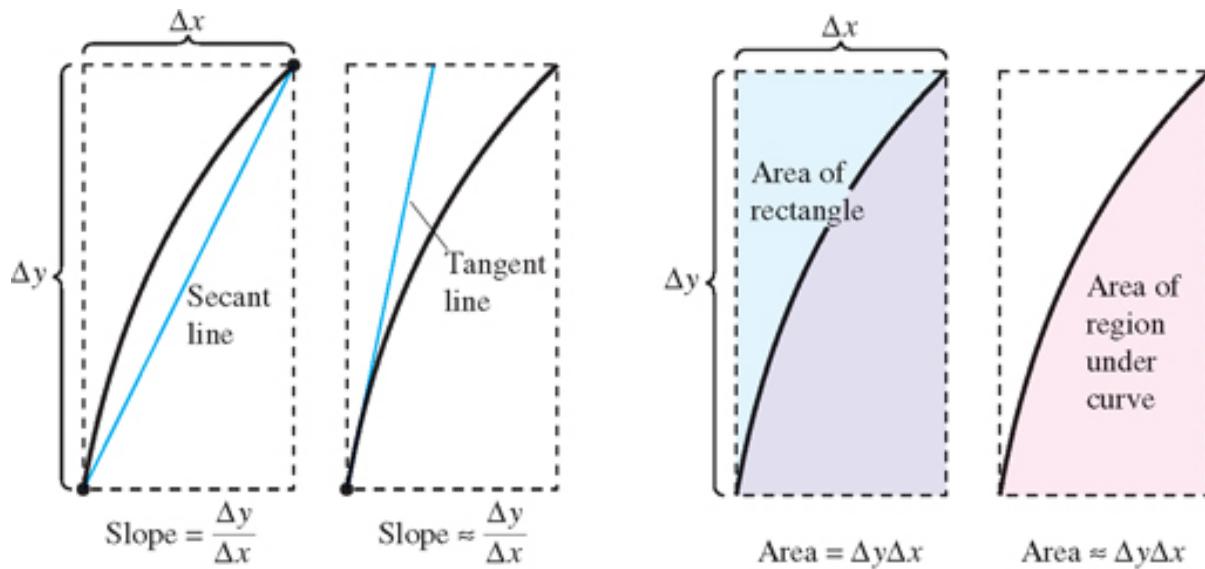
5.4: The Fundamental Theorem of Calculus

“ Objectives

- 1 Evaluate a definite integral using the Fundamental Theorem of Calculus.
- 2 Understand and use the Mean Value Theorem for Integrals.
- 3 Find the average value of a function over a closed interval.
- 4 Understand and use the Second Fundamental Theorem of Calculus.
- 5 Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus

Antidifferentiation and Definite Integration



(a) Differentiation

(b) Definite integration

- $\int_a^b f(x)dx$
 - definite integral
 - number
- $\int f(x)dx$
 - indefinite integral
 - function

Theorem The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark

We use the notation

$$\int_a^b f(x)dx = F(x) \Big|_a^b = F(b) - F(a) \quad \text{or} \quad \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Example 1: Evaluating a Definite Integral

Evaluate each definite integral.

$$\cdot \int_1^2 (x^2 - 3)dx$$

$$\cdot \int_1^4 3\sqrt{x}dx$$

$$\cdot \int_0^{\pi/4} \sec^2 x dx$$

$$\cdot \int_0^2 |2x - 1| dx$$

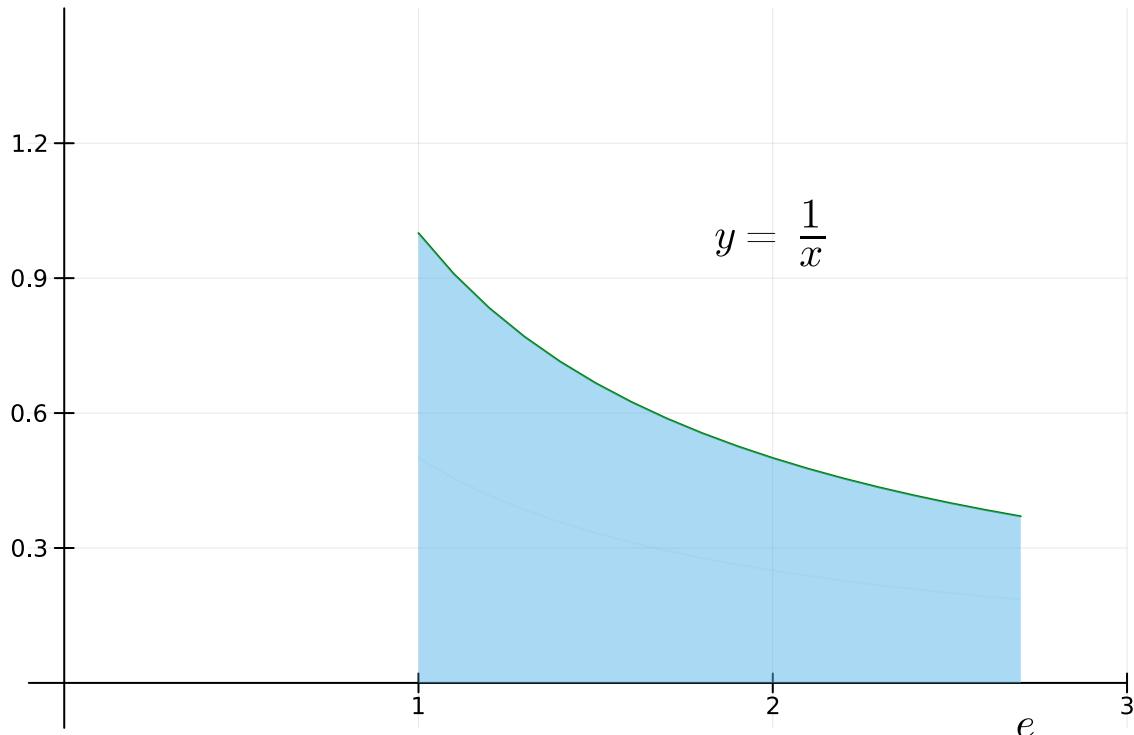


Example 3: Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of

$$y = \frac{1}{x}$$

the x -axis, and the vertical lines $x = 1$ and $x = e$.

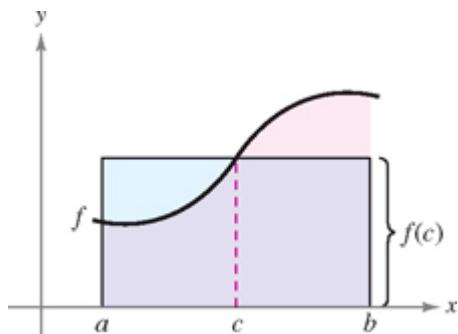


The Mean Value Theorem for Integrals

Theorem The Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$



Average Value of a Function

Definition the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

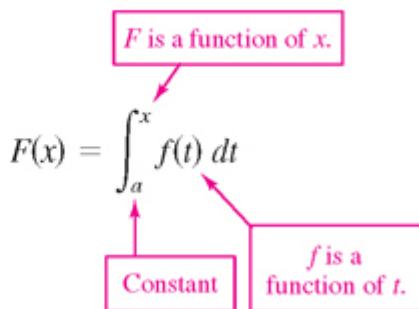
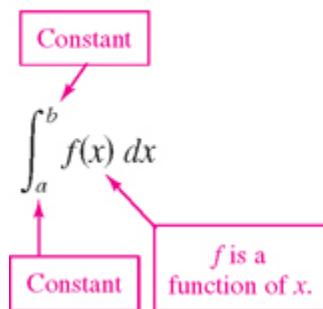
$$\text{Average value} = \frac{1}{b-a} \int_a^b f(x)dx$$

Example 4: Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

The Second Fundamental Theorem of Calculus

The Definite Integral as a Number The Definite Integral as a Function of x

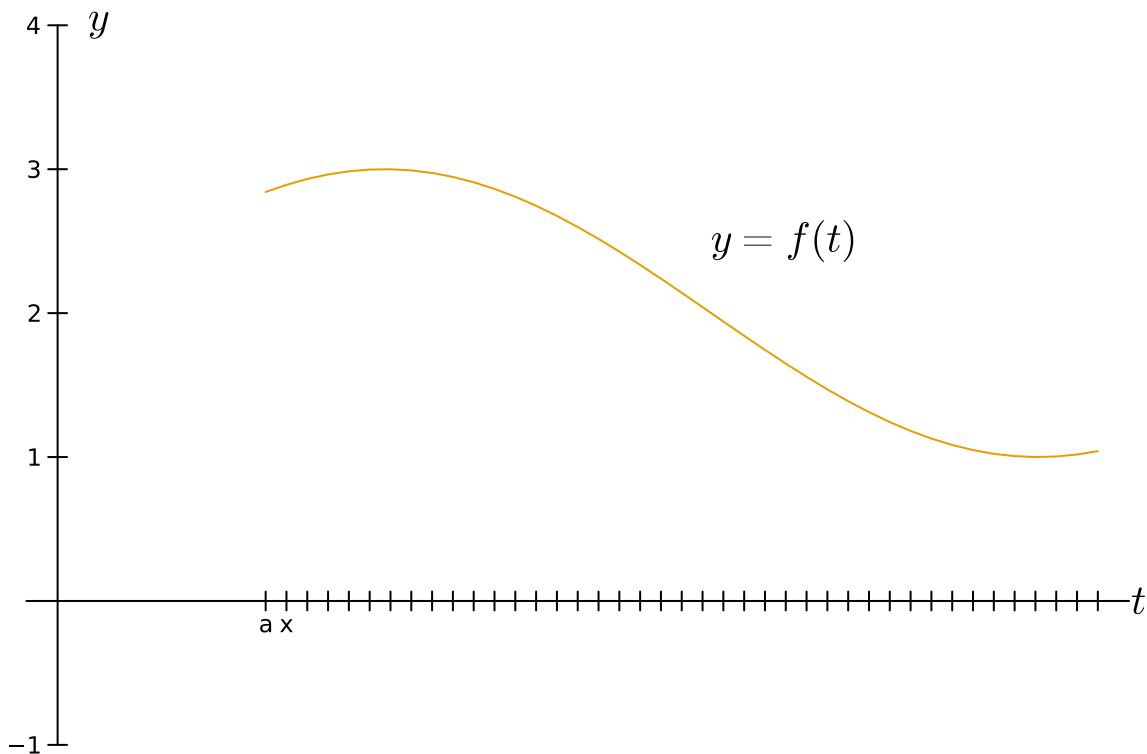


Consider the following function

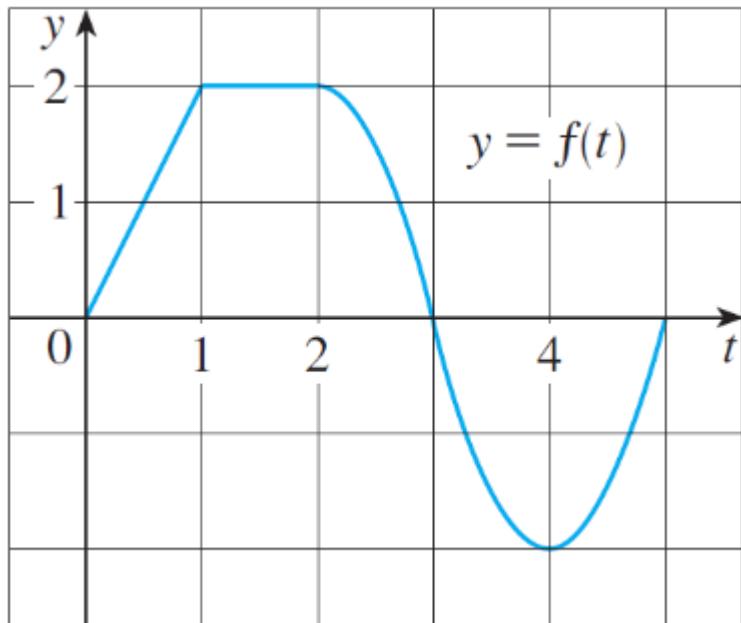
$$F(x) = \int_a^x f(t) dt$$

where f is a continuous function on the interval $[a, b]$ and $x \in [a, b]$.

$x =$



Example If $g(x) = \int_0^x f(t)dt$



Find $g(2)$

Theorem The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

Remarks

- $\frac{d}{dx} \left(\int_a^x f(u) du \right) = f(x)$
- $g(x)$ is an antiderivative of f

Examples

Find the derivative of

$$(1) g_1(x) = \int_0^x \sqrt{1+t} dt.$$

$$(2) g_2(x) = \int_x^0 \sqrt{1+t} dt.$$

$$(3) g_3(x) = \int_0^{x^2} \sqrt{1+t} dt.$$

$$(4) g_4(x) = \int_{\sin(x)}^{\cos(x)} \sqrt{1+t} dt.$$

💣 BE CAREFUL:

Evaluate $\int_{-3}^6 \frac{1}{x} dx$

Net Change Theorem

Question: If $y = F(x)$, then what does $F'(x)$ represents?

Theorem The Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x) dx = F(b) - F(a) \quad \text{Net change of } F(x)$$

- There are many applications, we will focus on one

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1)$$

- **Remarks**

$$\text{displacement} = \int_{t_1}^{t_2} v(t) dt$$

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)| dt$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t) dt = v(t_2) - v(t_1) \quad \text{is the change in velocity from time to time .}$$

Example 10: Solving a Particle Motion Problem

A particle is moving along a line. Its velocity function (in m/s^2) is given by

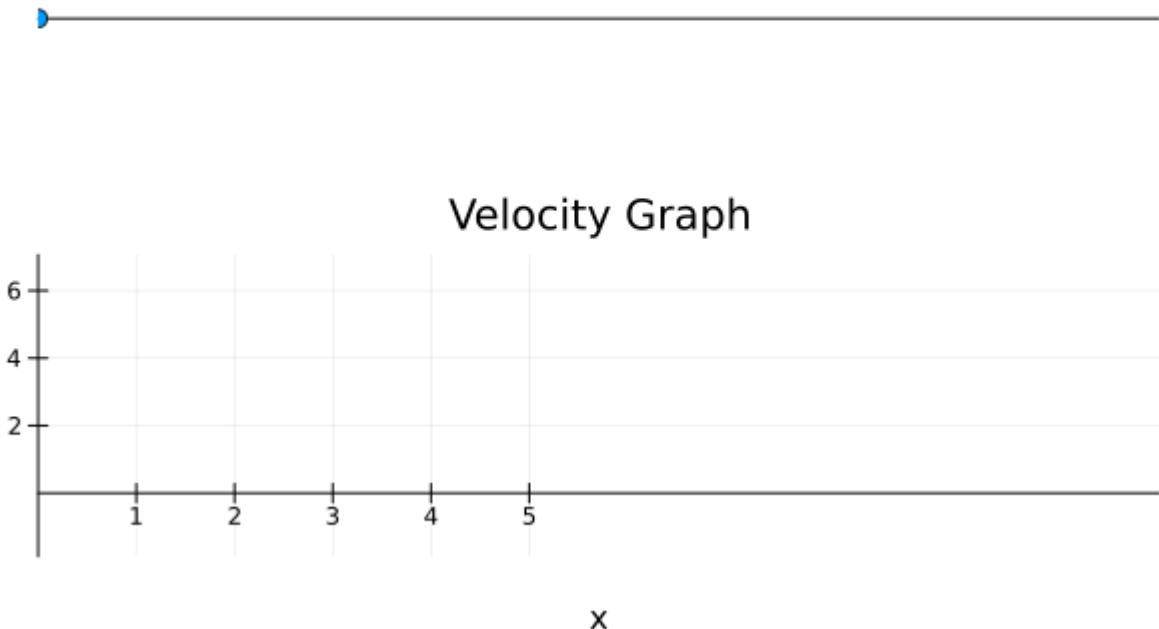
$$v(t) = t^3 - 10t^2 + 29t - 20,$$

- What is the **displacement** of the particle on the time interval $1 \leq t \leq 5$?
- What is the **total distance** traveled by the particle on the time interval $1 \leq t \leq 5$?

v (generic function with 1 method)

```
1 v(t) = t^3 - 10 * t^2 + 29 * t - 20
```

ne=1.0



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5.5: The Substitution Rule

“ Objectives

- 1 Use pattern recognition to find an indefinite integral.
- 2 Use a change of variables to find an indefinite integral.
- 3 Use the General Power Rule for Integration to find an indefinite integral.
- 4 Use a change of variables to evaluate a definite integral.
- 5 Evaluate a definite integral involving an even or odd function.

solve

$$\int 2x\sqrt{1+x^2} \ dx \quad \int \sqrt{u} \ du$$

Pattern Recognition

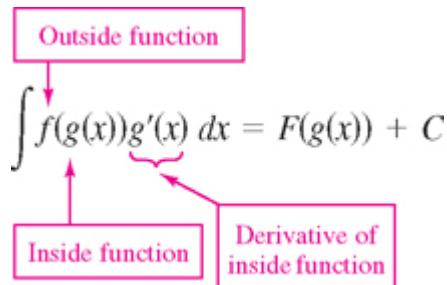
Theorem Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C.$$



Substitution Rule says: It is permissible to operate with dx and du after integral signs as if they were differentials.

Example Find

$$(i) \quad \int (x^2 + 1)^2 (2x) dx$$

$$(ii) \quad \int 5e^{5x} dx$$

$$(iii) \quad \int \frac{x}{\sqrt{1-4x^2}} dx$$

$$(iv) \quad \int \sqrt{1+x^2} x^5 dx$$

$$(v) \quad \int \tan x dx$$



Change of Variables for Indefinite Integrals

Example: Find

$$(i) \quad \int \sqrt{2x - 1} dx$$

$$(ii) \quad \int x \sqrt{2x - 1} dx$$

$$(iii) \quad \int \sin^2 3x \cos 3x dx$$

The General Power Rule for Integration

Theorem The General Power Rule for Integration

If g is a differentiable function of x , then

$$\int [g(x)]^n g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example: Find

$$(i) \quad \int 3(3x-1)^4 dx$$

$$(ii) \quad \int (e^x + 1)(e^x + x) dx$$

$$(iii) \quad \int 3x^2 \sqrt{x^3 - 2} dx$$

$$(iv) \quad \int \frac{-4x}{(1-2x^2)^2} dx$$

$$(v) \quad \int \cos^2 x \sin x dx$$

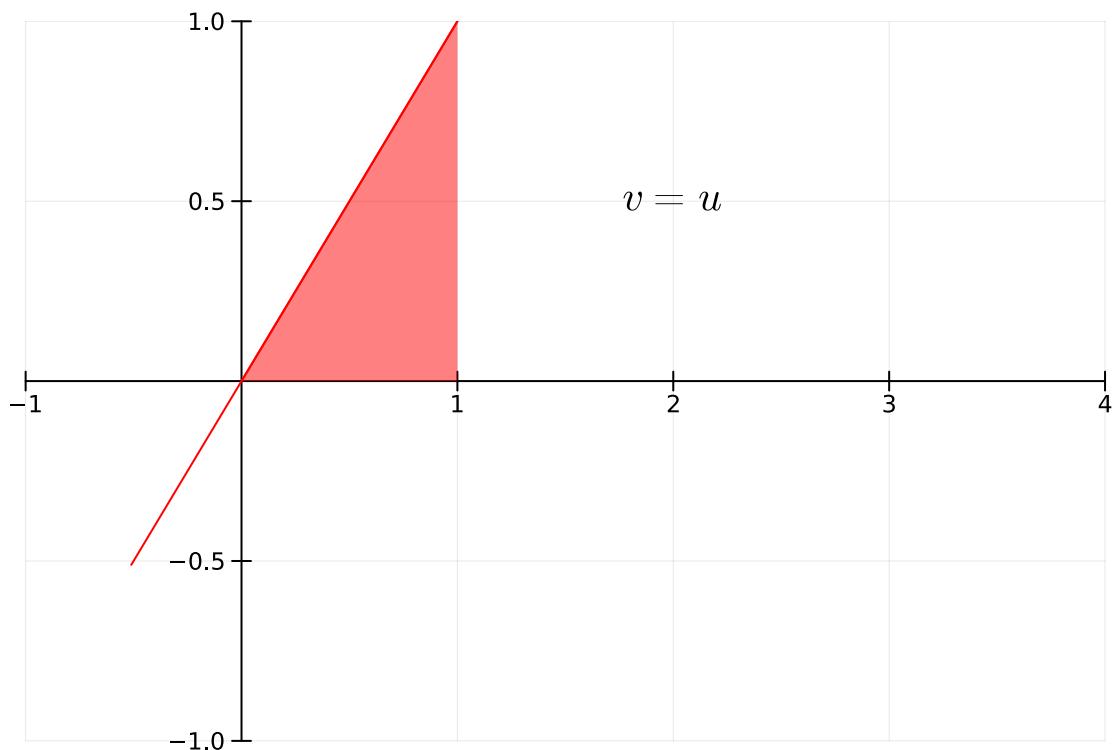
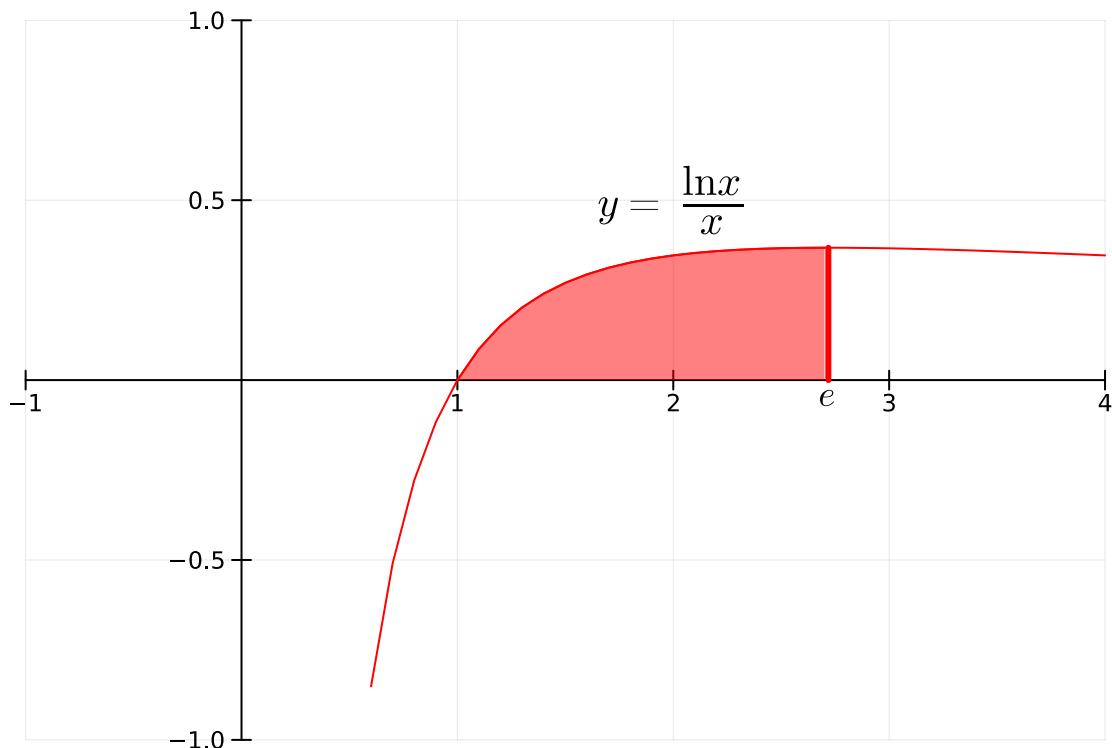


Change of Variables for Definite Integrals

Substitution: Definite Integrals

Example: Evaluate

$$\int_1^e \frac{\ln x}{x} dx$$



Example: Evaluate

$$(i) \quad \int_1^2 \frac{dx}{(3-5x)^2}$$

$$(iii) \quad \int_0^1 x(x^2 + 1)^3 \, dx$$

$$(iv) \quad \int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$$

Integration of Even and Odd Functions

Theorem Integration of Even and Odd Functions

Let f be integrable on $[-a, a]$.

- If f is even [$f(-x) = f(x)$], then

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$$

- If f is odd [$f(-x) = -f(x)$], then

$$\int_{-a}^a f(x) \, dx = 0$$

Example Find

$$\int_{-1}^1 \frac{\tan x}{1+x^2+x^4} \, dx$$

5.7: The Natural Logarithmic Function: Integration

“ Objectives

- 1 Use the Log Rule for Integration to integrate a rational function.
- 2 Integrate trigonometric functions.

Log Rule for Integration

Theorem Log Rule for Integration

Let u be a differentiable function of x .

$$(i) \quad \int \frac{1}{x} dx = \ln|x| + C$$

$$(ii) \quad \int \frac{1}{u} du = \ln|u| + C$$

Remark

$$\int \frac{u'}{u} dx = \ln|u| + C$$

Example 1: Using the Log Rule for Integration

$$\int \frac{2}{x} dx$$

Example 3: Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x -axis, and the line $x = 3$.

Example 5: Using Long Division Before Integrating

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx$$



Examples Find

$$(i) \quad \int \frac{1}{4x-1} dx$$

$$(ii) \quad \int \frac{3x^2+1}{x^3+x} dx$$

$$(iii) \quad \int \frac{\sec^2 x}{\tan x} dx$$

$$(iv) \quad \int \frac{x^2+x+1}{x^2+1} dx$$

$$(v) \quad \int \frac{2x}{(x+1)^2} dx$$

Example 7: Solve the differential equation

Solve

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

Integrals of Trigonometric Functions

Example 8: Using a Trigonometric Identity

$$\int \tan x dx$$

Example 9: Derivation of the Secant Formula

$$\int \sec x dx$$

5.8:Inverse Trigonometric Functions: Integration

 Objectives

- 1 Integrate functions whose antiderivatives involve inverse trigonometric functions.
- 2 Use the method of completing the square to integrate a function.
- 3 Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

Theorem Integrals Involving Inverse Trigonometric Functions

Let u be a differential function of x , and let $a > 0$.

$$1. \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$2. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$3. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

Examples Find

$$\rightarrow \int \frac{dx}{\sqrt{4 - x^2}},$$

$$\rightarrow \int \frac{dx}{2 + 9x^2},$$

$$\rightarrow \int \frac{dx}{x\sqrt{4x^2 - 9}},$$

$$\rightarrow \int \frac{dx}{\sqrt{e^{2x} - 1}},$$

$$\rightarrow \int \frac{x+2}{\sqrt{4-x^2}} dx.$$

Completing the Square

Example 5: Completing the Square

Find

$$\int \frac{dx}{x^2 - 4x + 7}.$$

Example 6: Completing the Square

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

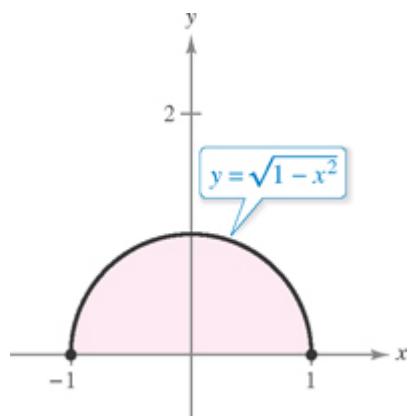
the x -axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

5.9: Hyperbolic Functions

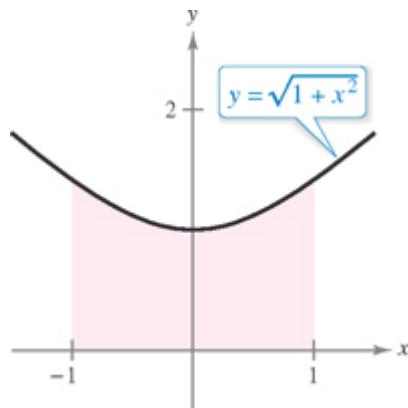
“ Objectives

- 1 Develop properties of hyperbolic functions (MATH101).
- 2 Differentiate (MATH101) and integrate hyperbolic functions.
- 3 Develop properties of inverse hyperbolic functions (Reading only).
- 4 Differentiate and integrate functions involving inverse hyperbolic functions. (Reading only).

Circle: $x^2 + y^2 = 1$



Hyperbola: $-x^2 + y^2 = 1$



Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

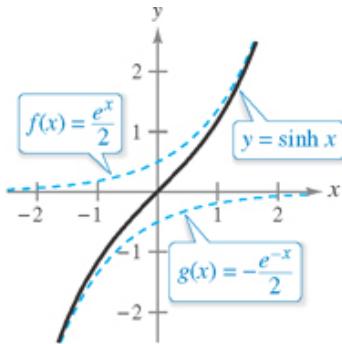
$$\csc x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

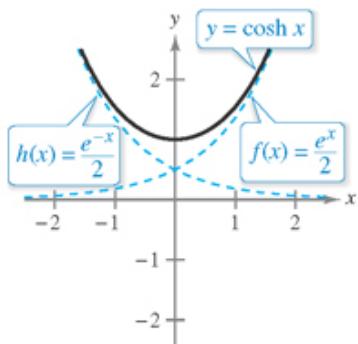
$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

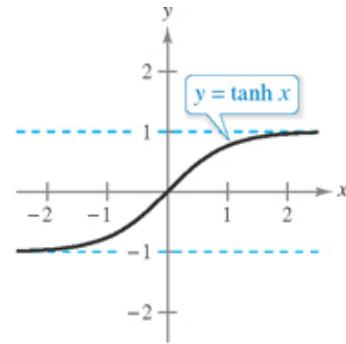
$$\coth x = \frac{1}{\tanh x}, \quad x \neq 0$$



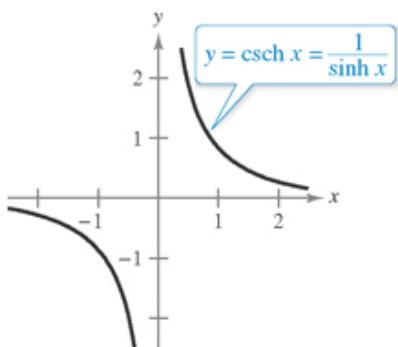
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



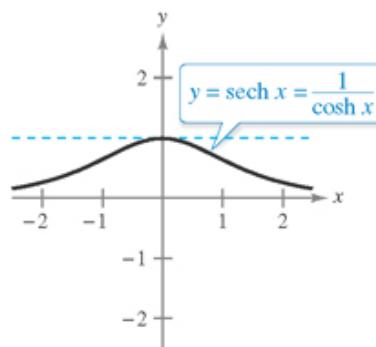
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



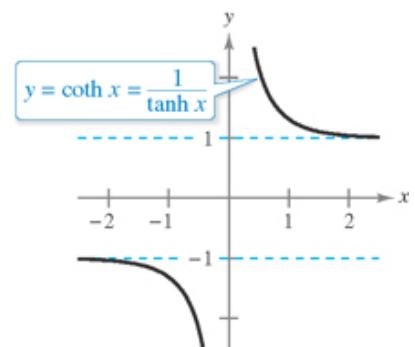
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1,$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh$$

$$\coth^2 x - \operatorname{csch}^2 x = 1,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh$$

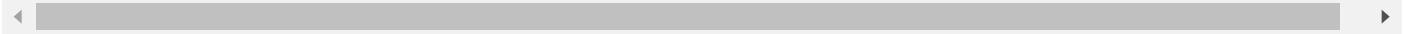
$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2},$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sin 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$



Theorem Differentiation and Integration of Hyperbolic Functions

Theorem Let u be a differentiable function of x .

$$\frac{d}{dx}(\sinh u) = (\cosh u)u', \quad \int \cosh u du = \sinh u + C$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)u', \quad \int \sinh u du = \cosh u + C$$

$$\frac{d}{dx}(\tanh u) = (\operatorname{sech}^2 u)u', \quad \int \operatorname{sech}^2 u du = \tanh u + C$$

$$\frac{d}{dx}(\coth u) = -(\operatorname{csch}^2 u)u', \quad \int \operatorname{csch}^2 u du = -\coth u + C$$

$$\frac{d}{dx}(\operatorname{sech} u) = -(\operatorname{sech} u \tanh u)u', \quad \int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}(\operatorname{csch} u) = -(\operatorname{csch} u \coth u)u', \quad \int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$$

Example 4: Integrating a Hyperbolic Function

Find

$$\int \cosh 2x \sinh^2 2x dx$$

7.1: Area of a Region Between Two Curves

Objectives

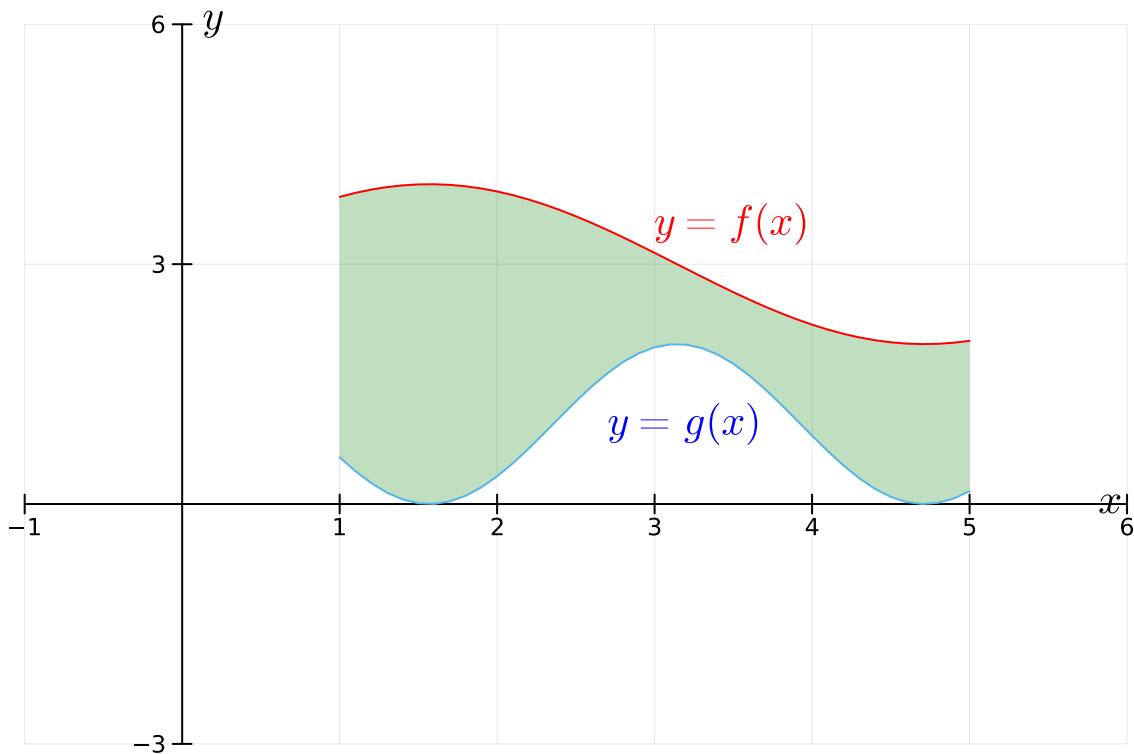
“

- 1 Find the area of a region between two curves using integration.
 - 2 Find the area of a region between intersecting curves using integration.
 - 3 Describe integration as an accumulation process.
-

Area of a Region Between Two Curves

move  $n =$  1

How can we find the area between the two curves?



$$\text{Area} = \int_a^b [f(x) - g(x)]dx$$

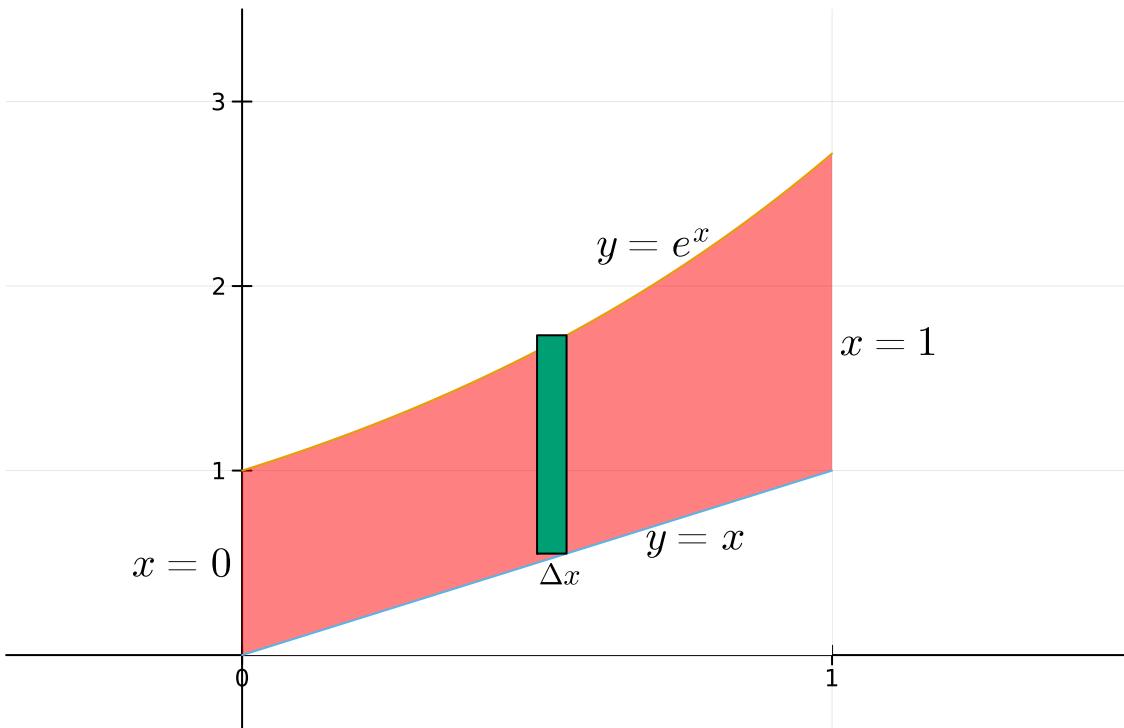
Remark

- Area = $y_{top} - y_{bottom}$.

Example 1: Finding the Area of a Region Between Two Curves

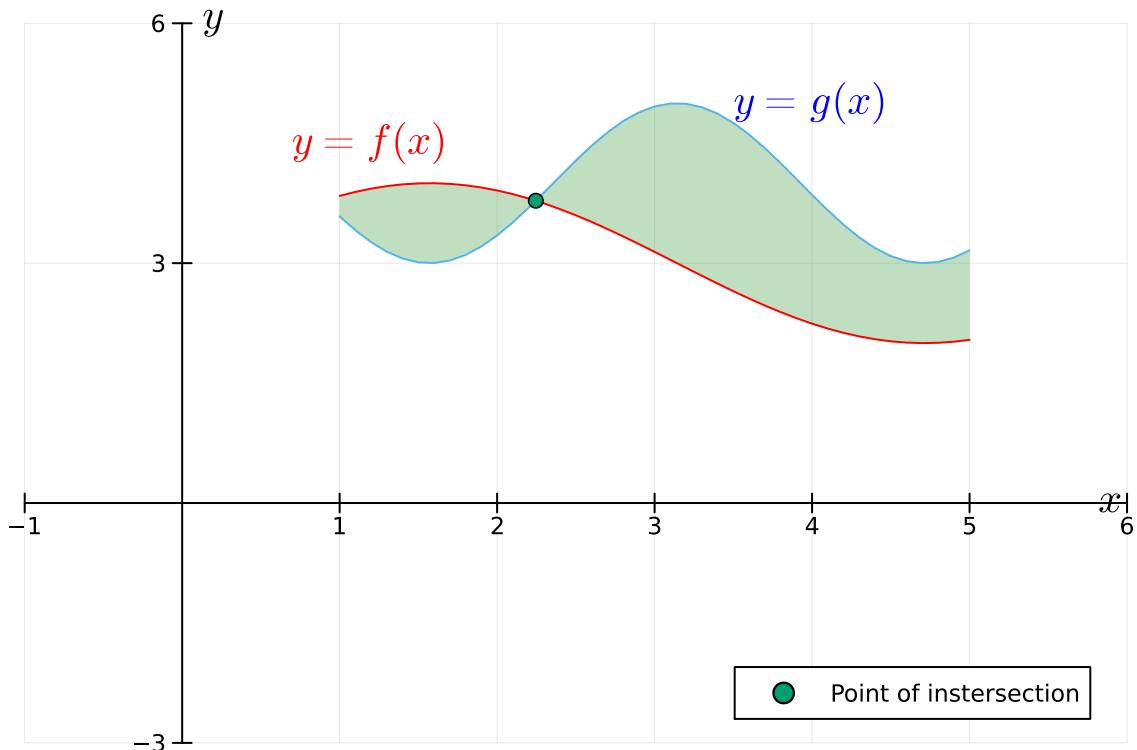
Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, bounded on the sides by $x = 0$ and $x = 1$.

Solution



Area of a Region Between Intersecting Curves

In general,



$$\text{Area} = \int_a^b |f(x) - g(x)| dx$$

Example 2: A Region Lying Between Two Intersecting Graphs

Find the area of the region enclosed by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution in class

Example 3: A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the curves

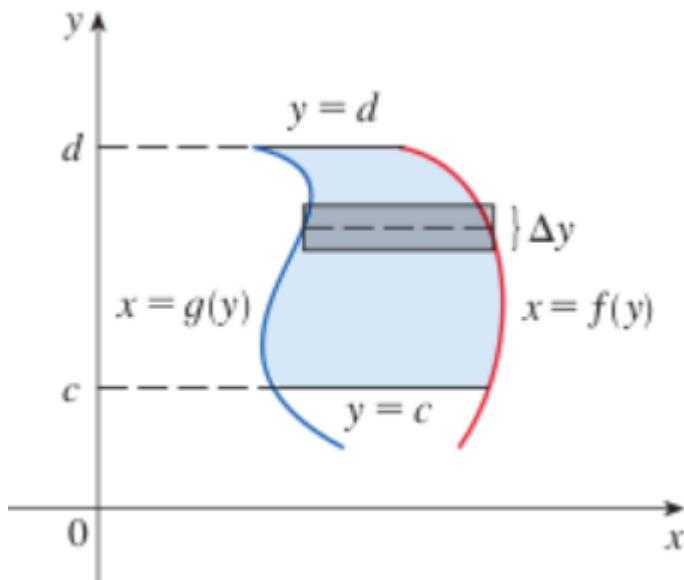
$$y = \cos(x), \quad y = \sin(x), \quad x = 0, \quad x = \frac{\pi}{2}$$

Example 4: Curves That Intersect at More than Two Points

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x, \quad g(x) = -x^2 + 2x.$$

Integrating with Respect to y



Example 5: Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

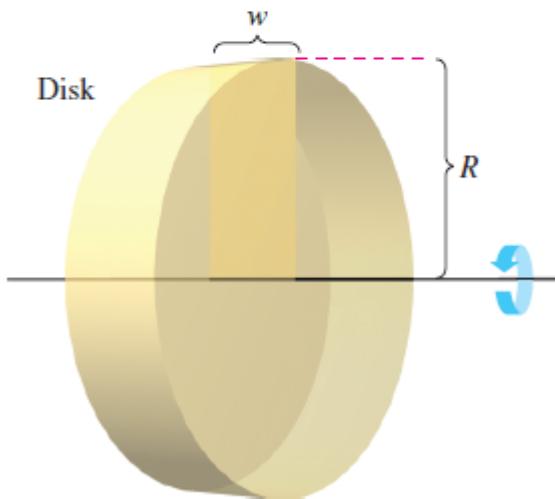
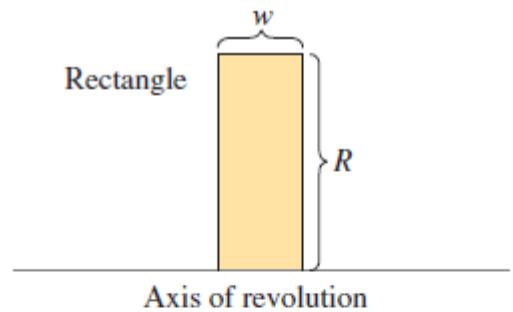
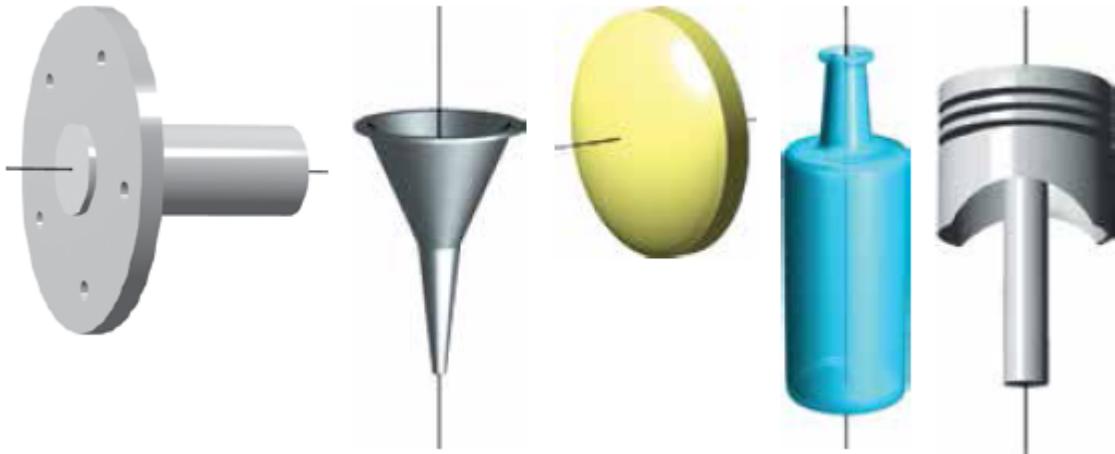
7.2: Volume: The Disk Method

“ Objectives

- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution using the washer method.
- Find the volume of a solid with known cross sections.

The Disk Method

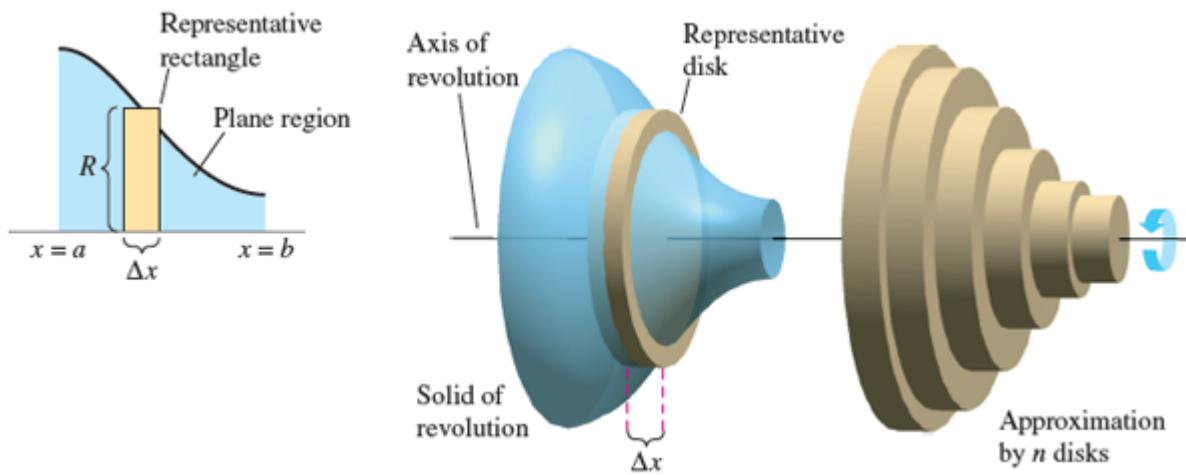
Solids of Revolution



Volume of a disk

$$V = \pi R^2 w$$

Disk Method



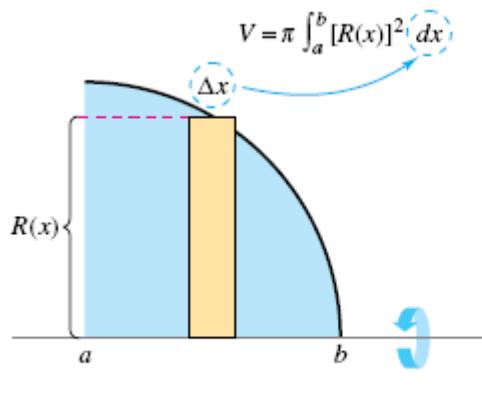
$$\begin{aligned}\text{Volume of solid} &\approx \sum_{i=1}^n \pi [R(x_i)]^2 \Delta x \\ &= \pi \sum_{i=1}^n [R(x_i)]^2 \Delta x\end{aligned}$$

Taking the limit $\|\Delta\| \rightarrow 0 (n \rightarrow \infty)$, we get

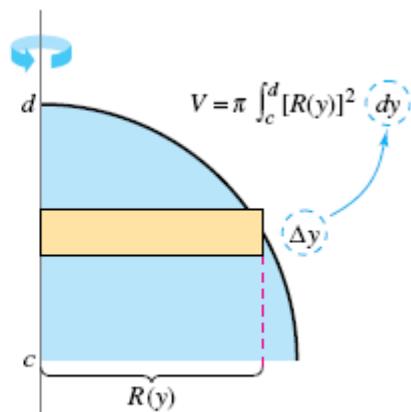
$$\text{Volume of solid} = \lim_{\|\Delta\| \rightarrow 0} \pi \sum_{i=1}^n [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 dx.$$

Disk Method

To find the volume of a solid of revolution with the disk method, use one of the formulas below



Horizontal axis of revolution



Vertical axis of revolution

Example 1: Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x -axis ($0 \leq x \leq \pi$) about the x -axis

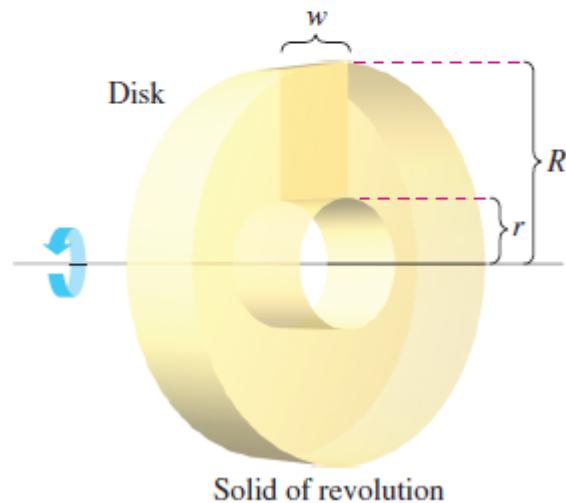
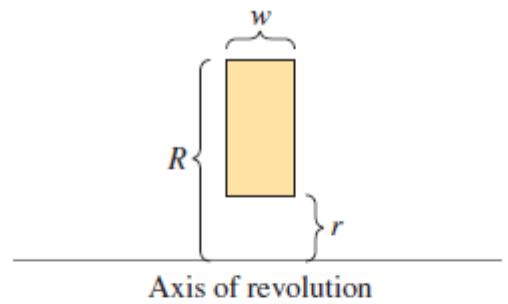
Example 2: Using a Line That Is Not a Coordinate Axis

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$f(x) = 2 - x^2$$

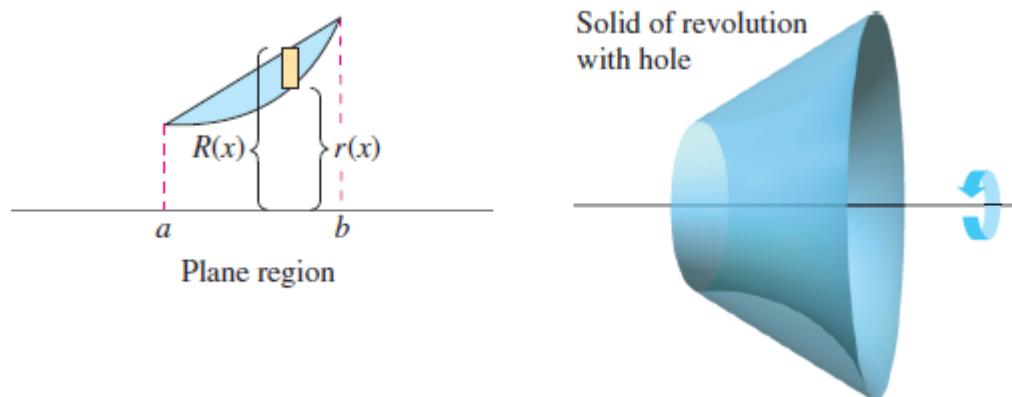
and $g(x) = 1$ about the line $y = 1$.

The Washer Method



$$\text{Volume of washer} = \pi(R^2 - r^2)w$$

Washer Method



$$V = \pi \int_a^b [(R[x])^2 - (r[x])^2] dx$$

Example 3: Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = \sqrt{x} \quad \text{and} \quad y = x^2$$

about the x -axis.

Example 4: Integrating with Respect to 'y': Two-Integral Case

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

about the y -axis

Solids with Known Cross Sections

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x -axis,

$$V = \int_a^b A(x)dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the y -axis,

$$V = \int_c^d A(y)dy$$

Example 6: Triangular Cross Sections

The base of a solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \quad g(x) = -1 + \frac{x}{2} \quad \text{and} \quad x = 0.$$

The cross sections perpendicular to the x -axis are equilateral triangles.

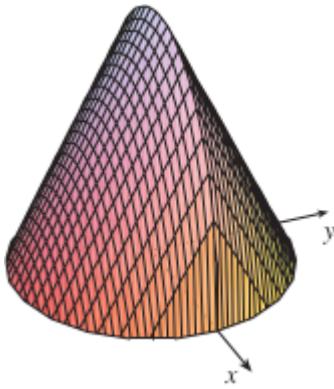
Exercise Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y -axis.

Exercise The region \mathcal{R} enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x -axis. Find the volume of the resulting solid.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $y = 2$.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $x = -1$.

Exercise Figure below shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



7.3: Volume: The Shell Method

“ Objectives

- 1 Find the volume of a solid of revolution using the shell method.
- 2 Compare the uses of the disk method and the shell method.

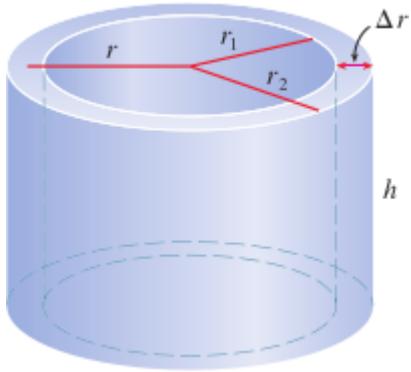
Problem Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y -axis.

Step 1: Step 2: Step 3:

""

The Shell Method

A shell is a hallow circular cylinder



$$V = 2\pi rh\Delta r = [\text{circumference}][\text{height}][\text{thickness}]$$

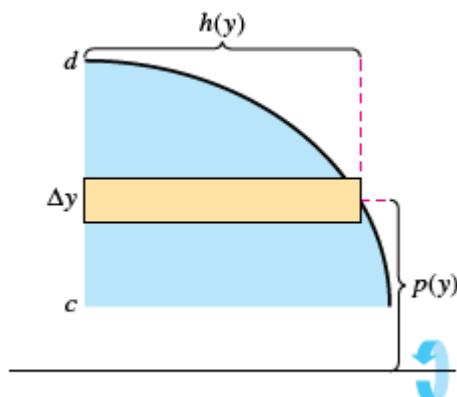
Cylindrical Shells Illustration

Shell Method



Horizontal Axis of Revolution

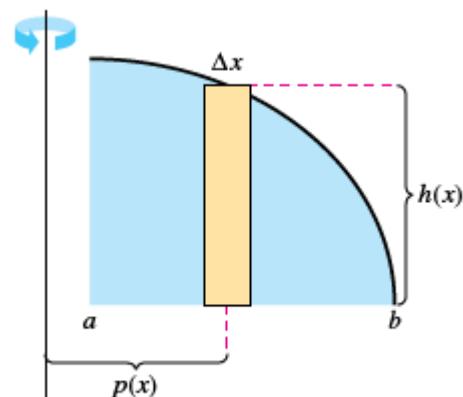
$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)dy$$



Horizontal axis of revolution

Vertical Axis of Revolution

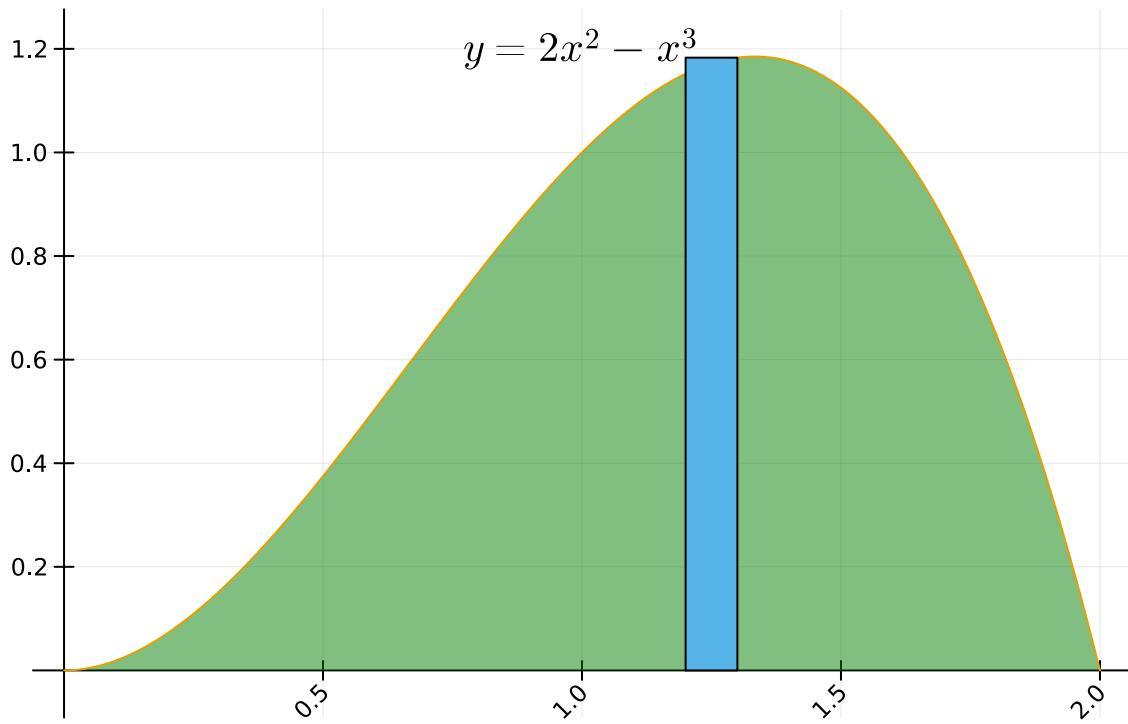
$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)dx$$



Vertical axis of revolution

Example: Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y -axis.

Solution:



Example : Find the volume of the solid obtained by rotating about the y -axis the region between $y = x$ and $y = x^2$.

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Example 4: Shell Method Preferable

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1.$$

about the y -axis.

Example 5: Shell Method Necessary

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$.

7.4: Arc Length and Surfaces of Revolution

“ Objectives

- 1 Find the arc length of a smooth curve.
- 2 Find the area of a surface of revolution.

Arc Length

Definition Arc Length

Let the function $y = f(x)$ represents a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 2: Finding Arc Length

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$.

Example 3: Finding Arc Length

Find the arc length of the graph of $(y - 1)^3 = x^2$ on the interval $[0, 8]$.

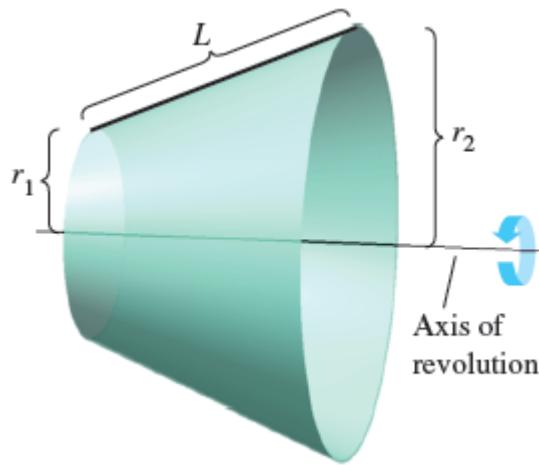
Example 4: Finding Arc Length

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$.

Area of a Surface of Revolution

Definition Surface of Revolution

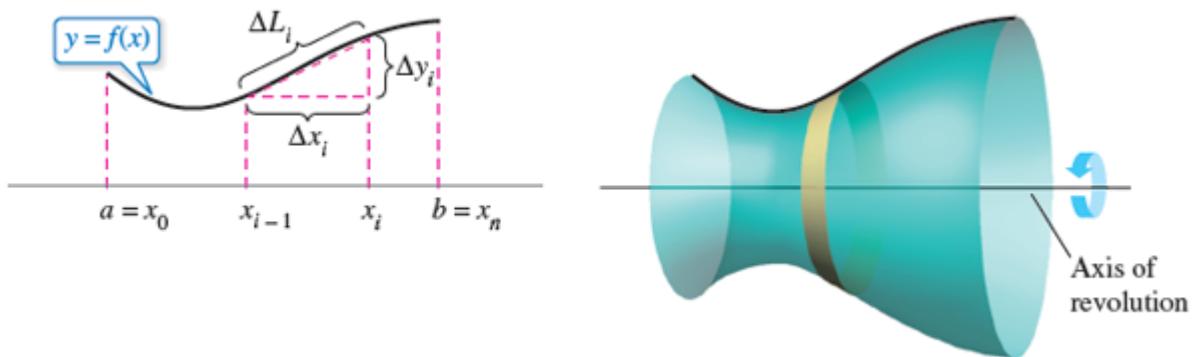
When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.



Surface Area of *frustum*

$$S = 2\pi rL, \quad \text{where} \quad r = \frac{r_1 + r_2}{2}$$

Consider a function f that has a continuous derivative on the interval $[a, b]$. The graph of f is revolved about the x -axis

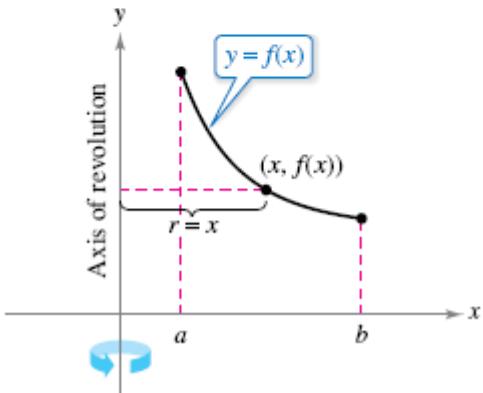
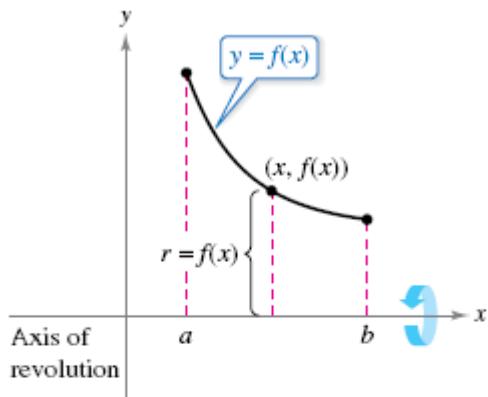


Surface Area Formula

$$S = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2} dx.$$

Definition Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$.



The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx, \quad \text{y is a function of x .}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution.

If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_a^b r(y) \sqrt{1 + [g'(y)]^2} dy, \quad \text{x is a function of y .}$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Remark

The formulas can be written as

$$S = 2\pi \int_a^b r(x)ds, \quad y \text{ is a function of } x.$$

and

$$S = 2\pi \int_c^d r(y)ds, \quad x \text{ is a function of } y.$$

where

$$ds = \sqrt{1 + [f'(x)]^2}dx \quad \text{and} \quad ds = \sqrt{1 + [g'(y)]^2}dy \quad \text{respectively.}$$

Example 6: The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x -axis.

Example 7: The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y -axis.

8.1: Basic Integration Rules

“ Objectives

- 1 Review procedures for fitting an integrand to one of the basic integration rules.

Review of Basic Integration Rules ()

$$1. \int kf(u) du = k \int f(u) du$$

$$2. \int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$$

$$3. \int du = u + C$$

$$4. \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

$$5. \int \frac{du}{u} = \ln|u| + C$$

$$6. \int e^u du = e^u + C$$

$$7. \int a^u du = \left(\frac{1}{\ln a} \right) a^u + C$$

$$8. \int \sin u du = -\cos u + C$$

$$9. \int \cos u du = \sin u + C$$

$$10. \int \tan u du = -\ln|\cos u| + C$$

$$11. \int \cot u du = \ln|\sin u| + C$$

$$12. \int \sec u du = \ln|\sec u + \tan u| + C$$

$$13. \int \csc u du = -\ln|\csc u + \cot u| + C$$

$$14. \int \sec^2 u du = \tan u + C$$

$$15. \int \csc^2 u du = -\cot u + C$$

$$16. \int \sec u \tan u du = \sec u + C$$

$$17. \int \csc u \cot u du = -\csc u + C$$

$$18. \int \frac{du}{\sqrt{1-u^2}} = \arcsin \frac{u}{1} + C$$

$$J = \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$19. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

Example 3:

Find $\int \frac{x^2}{\sqrt{16 - x^6}} dx.$

Example 4: A Disguised Form of the Log Rule

Find $\int \frac{dx}{1 + e^x}.$

8.2: Integration by Parts

Objectives

- 1 Find an antiderivative using integration by parts.

The integration rule that corresponds to the Product Rule for differentiation is called **integration by parts**.

Indefinite Integrals

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Theorem Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int udv = uv - \int vdu.$$

Example 1: Integration by Parts

Find $\int xe^x dx$.

Example 2: Integration by Parts

Find $\int x^2 \ln x dx$.

Example 3: An Integrand with a Single Term

Evaluate $\int_0^1 \arcsin x dx$.

Example 4: Repeated Use of Integration by Parts

Find $\int x^2 \sin x dx$.

Example 5: Integration by Parts

Find $\int \sec^3 x dx$.

Example 7: Using the Tabular Method

Find $\int x^2 \sin 4x dx$.

8.3: Trigonometric Integrals

“ Objectives

- 1 Solve trigonometric integrals involving powers of sine and cosine.
- 2 Solve trigonometric integrals involving powers of secant and tangent.
- 3 Solve trigonometric integrals involving sine-cosine products.

RECALL

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n) + \cos(m+n)],$$

$$\int \tan x dx = \ln |\sec x| + C, \quad \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \cot x dx = -\ln |\csc x| + C, \quad \int \csc x dx = \ln |\csc x - \cot x| + C$$

Integrals of Powers of Sine and Cosine

$$\int \sin^m x \cos^n x dx$$

- m is **odd**, write as $\int \sin^{m-1} x \cos^n x \sin x dx$. Example: $\int \sin^5 x \cos^2 x dx$
- n is **odd**, write as $\int \sin^m x \cos^{n-1} \cos x dx$. Example $\int \sin^5 x \cos^3 x dx$
- m and n are **even**, use formulae (Example $\int \cos^2 x dx$ and $\int \sin^4 x dx$)

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

Example 1: Power of Sine Is Odd and Positive

Find $\int \sin^3 x \cos^4 x dx$.

Example 2: Power of Cosine Is Odd and Positive

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$.

Example 3: Power of Cosine Is Even and Nonnegative

Find $\int \cos^4 x dx$.

Integrals of Powers of Secant and Tangent

$$\int \tan^m x \sec^n x dx$$

- n is even, write as $\int \tan^m x \sec^{n-2} \sec^2 x dx$. Example $\int \tan^6 x \sec^4 x dx$
- m is odd, write as $\int \tan^{m-1} x \sec^{n-1} \tan x \sec x dx$. Example $\int \tan^5 x \sec^7 x dx$.

Example 4: Power of Tangent Is Odd and Positive

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$.

Example 5: Power of Secant Is Even and Positive

Find $\int \sec^4 3x \tan^3 3x dx$.

Example 6: Power of Tangent Is Even

Evaluate $\int_0^{\pi/4} \tan^4 x dx$.

Example 7: Converting to Sines and Cosines

Find $\int \frac{\sec x}{\sqrt{\tan^2 x}} dx$.

Integrals Involving Sine-Cosine Products

Example 8: Using a Product-to-Sum Formula

Find $\int \sin 5x \cos 4x dx$.

8.4: Trigonometric Substitution

“ Objectives

- 1 Use trigonometric substitution to find an integral.
- 2 Use integrals to model and solve real-life applications.

Trigonometric Substitution

We use **trigonometric substitution** to find integrals involving the radicals

$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \sqrt{u^2 - a^2}.$$

Example 1: Trigonometric Substitution

Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Example 2: Trigonometric Substitution

Find $\int \frac{dx}{\sqrt{4x^2+1}}$.

Example 3: Trigonometric Substitution: Rational Powers

Find $\int \frac{dx}{(x^2+1)^{3/2}}$.

Example 4: Converting the Limits of Integration

Find $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x} dx$.

Applications

Example 5: Finding Arc Length

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

8.5: Partial Fractions

“ Objectives

- 1 Understand the concept of partial fraction decomposition.
- 2 Use partial fraction decomposition with linear factors to integrate rational functions.
- 3 Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions

We learn how to integrate rational function: quotient of polynomial.

$$f(x) = \frac{P(x)}{Q(x)}, \quad P, Q \text{ are polynomials}$$

How?

- **STEP 0** : if degree of P is greater than or equal to degree of Q goto **STEP 1**, else GOTO **STEP 2**.
- **STEP 1** : Perform long division of P by Q to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

and apply **STEP 2** on $\frac{R(x)}{Q(x)}$.

- **STEP 2** : Write the **partial fractions decomposition**
- **STEP 3** : Integrate

Partial Fractions Decomposition

We need to write $\frac{R(x)}{Q(x)}$ as sum of **partial fractions** by **factor $Q(x)$** . Based on the factors, we write the decomposition according to the following cases

case 1: $Q(x)$ is a product of distinct linear factors. we write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

then there exist constants A_1, A_2, \dots, A_k such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

case 2: $Q(x)$ is a product of linear factors, some of which are repeated. say first one

$$Q(x) = (a_1x + b_1)^r(a_2x + b_2) \cdots (a_kx + b_k)$$

then there exist constants $B_1, B_2, \dots, B_r, A_2, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \left[\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \cdots + \frac{B_r}{(a_1x + b_1)^r} \right] + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated. say we have (Note: the quadratic factor $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$). For example if

$$Q(x) = (ax^2 + bx + c)(a_1x + b_1)$$

then there exist constants A, B , and C such that

$$\frac{R(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{C}{a_1x + b_1}$$

case 4: $Q(x)$ contains irreducible quadratic factors, some of which are repeated. For example if

$$Q(x) = (ax^2 + bx + c)^r(a_1x + b_1)$$

then there exist constants $A_1, B_1, A_2, B_2, \dots, A_r, B_r$ and C such that

$$\frac{R(x)}{Q(x)} = \left[\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} \right] + \frac{C}{a_1x + b_1}$$

Example: Partial Fractions

Write out the form of the partial fractions decomposition of the function

$$\frac{x^3 + x + 1}{x(x-1)(x+1)^2(x^2+x+1)(x^2+4)^2}$$

More Examples

Find

- (1) $\int \frac{1}{x^2 - 5x + 6} dx.$
- (2) $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx.$
- (3) $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx.$
- (4) $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx.$
- (5) $\int \frac{x^3 + x}{x - 1} dx.$
- (6) $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx.$
- (7) $\int \frac{dx}{x^2 - a^2}, \text{ where } a \neq 0$
- (8) $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$
- (9) $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$
- (10) $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx$
- (11) $\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx$

Remarks

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$$
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

Rationalizing Substitutions Find

$$(1) \int \frac{\sqrt{x+4}}{x} dx.$$

$$(2) \int \frac{dx}{2\sqrt{x+3} + x}.$$

8.7: Rational Functions of Sine & Cosine

Special Substitution ($u = \tan(\frac{x}{2})$, $-\pi < x < \pi$) (for rational functions of $\sin x$ and $\cos x$)

$$dx = \frac{2}{1+u^2} du, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad \sin x = \frac{2u}{1+u^2}$$

$$(1) \int \frac{dx}{3 \sin x - 4 \cos x}.$$

$$(2) \int_0^{\frac{\pi}{2}} \frac{\sin 2x \, dx}{2 + \cos x}.$$

8.8: Improper Integrals

“ Objectives

- Evaluate an improper integral that has an infinite limit of integration.
- Evaluate an improper integral that has an infinite discontinuity.

Do you know how to evaluate the following?

$$(1) \int_1^\infty \frac{1}{x^2} dx \quad (\text{Type 1})$$

$$(2) \int_0^2 \frac{1}{x-1} dx \quad (\text{Type 2})$$

Improper Integrals with Infinite Limits of Integration

Definition of an Improper Integral of Type 1

- (a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

- (b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

- (c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

In part (c) any real number can be used

Example: Determine whether the following integrals are convergent or divergent.

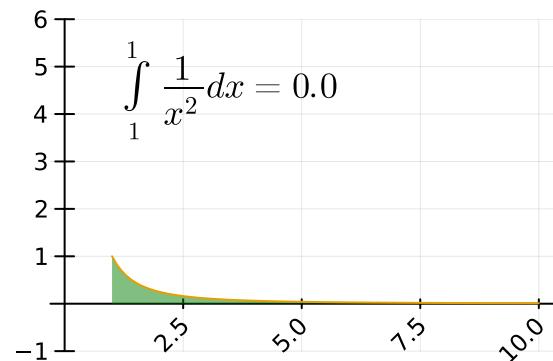
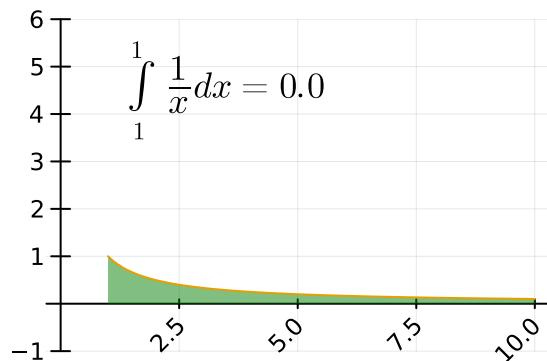
$$(1) \int_1^\infty \frac{1}{x^2} dx$$

$$(2) \int_1^\infty \frac{1}{x} dx$$

$$(3) \int_0^\infty e^{-x} dx$$

$$(4) \int_{-\infty}^\infty \frac{1}{1+x^2} dx$$

t = 1



p =

$$\int_1^\infty \frac{1}{x} dx = oo$$

Remark

$\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Improper Integrals with Infinite Discontinuities

Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b]$ and is discontinuous at b , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example:

$$(1) \quad \int_2^5 \frac{1}{\sqrt{x-2}} dx$$

$$(2) \quad \int_0^3 \frac{1}{1-x} dx$$

$$(3) \quad \int_0^1 \ln x dx$$

Example 9: Doubly Improper Integral

Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

9.1: Sequences

“ Objectives

- Write the terms of a sequence.
- Determine whether a sequence converges or diverges.
- Write a formula for the n^{th} term of a sequence.
- Use properties of monotonic sequences and bounded sequences.

Sequence: A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

- a_1 : first term,
- a_2 : second term,
- a_3 : third term,
- \vdots
- a_n : n^{th} term,

For example:

- 1, 2, 3, ...
- 1, 1/2, 1/3, ...
- -1, 1, -1, ...

Notation:

- $\{a_1, a_2, a_3, \dots, a_n, \dots\} = \{a_n\}$ or
- $\{a_1, a_2, a_3, \dots, a_n, \dots\} = \{a_n\}_{n=1}^\infty$

More examples

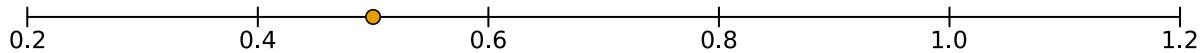
- $\left\{ \frac{n}{n+1} \right\}$
- $\left\{ \frac{(-1)^n(n+1)}{5^n} \right\}$
- $\left\{ \sqrt{n-4} \right\}_{n=4}^{\infty}$
- $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ (Fibonacci sequence)

n =

Example 1

● $a_n = \frac{n}{n+1}$

$$a_1 = \frac{1}{2} = 0.5$$

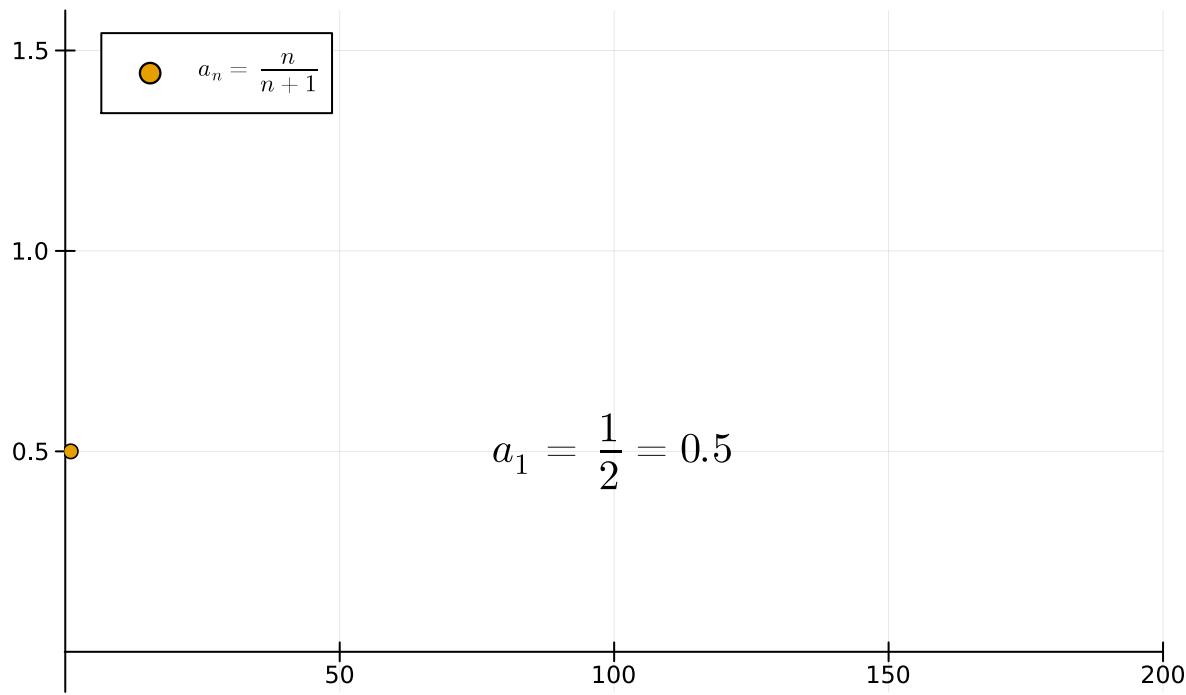


Visualization

1. On a number line (as above)
2. By plotting graph

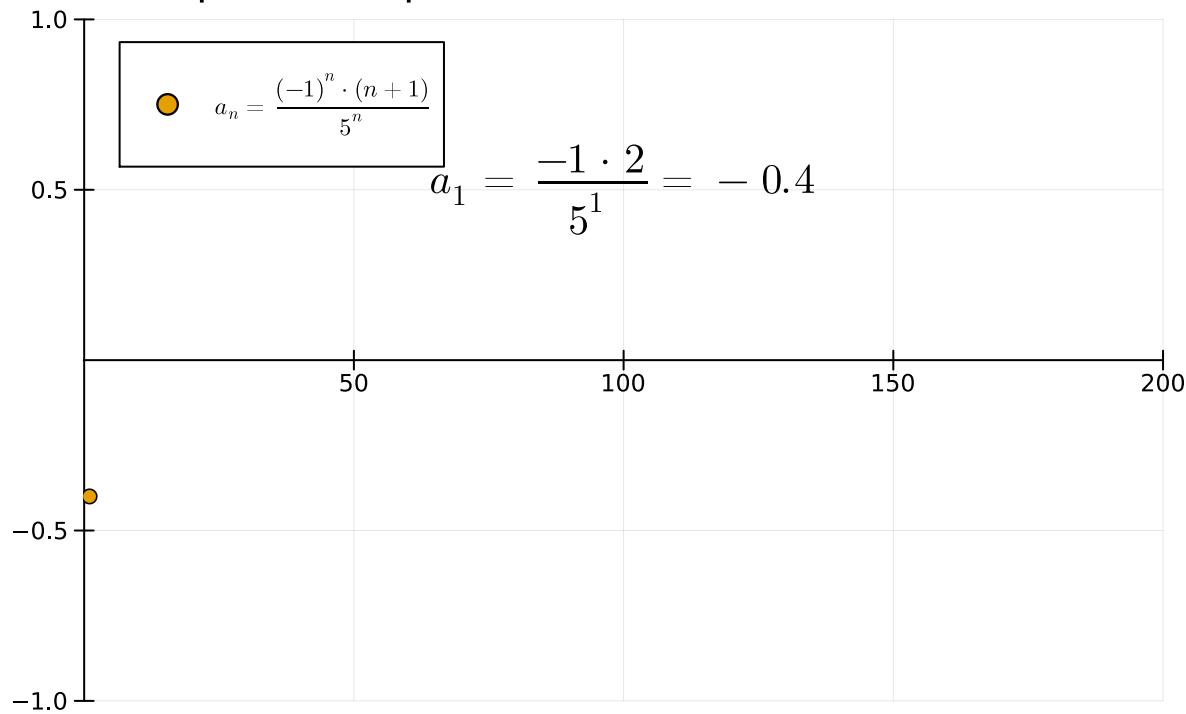
n =

Example 1 (Graph)



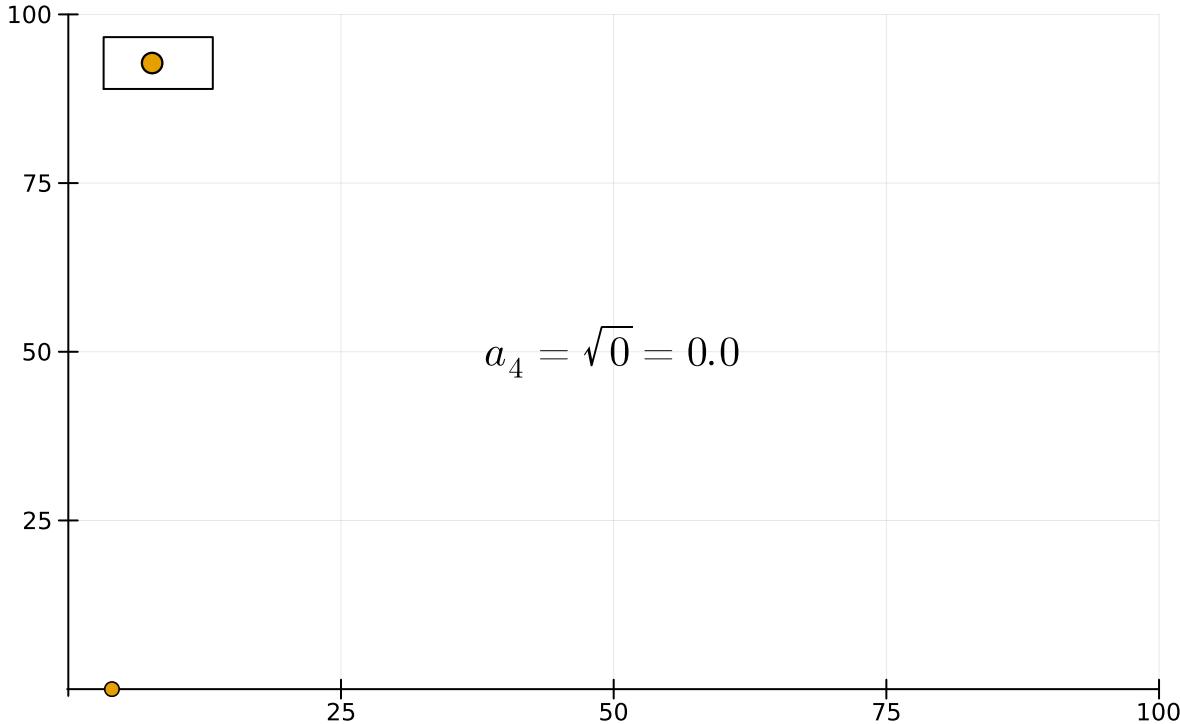
$n =$

Example 2 (Graph)



$n =$

Example 3 (Graph)



What are trying to study?

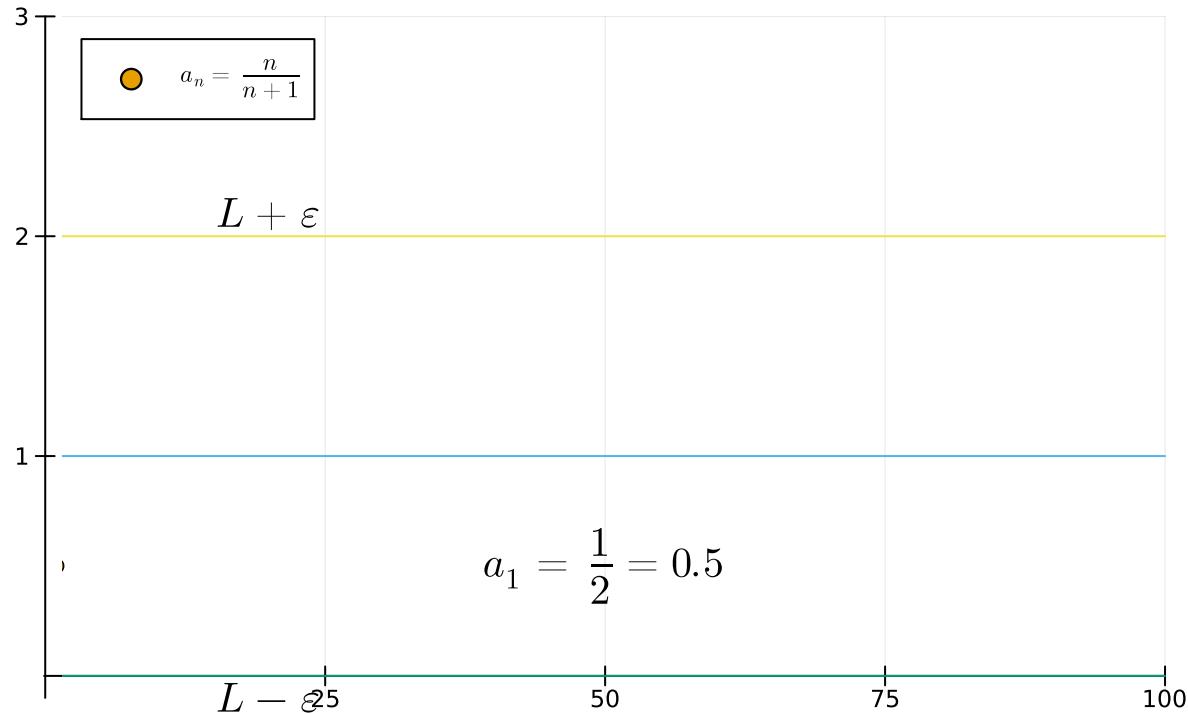
- **convergence** (what happened when n gets larger and larger $n \rightarrow \infty$)

For **Example 1**: $a_n = \frac{n}{n+1}$, it is fair to say and write

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

$\epsilon =$ $n =$

Example 1 (Graph)

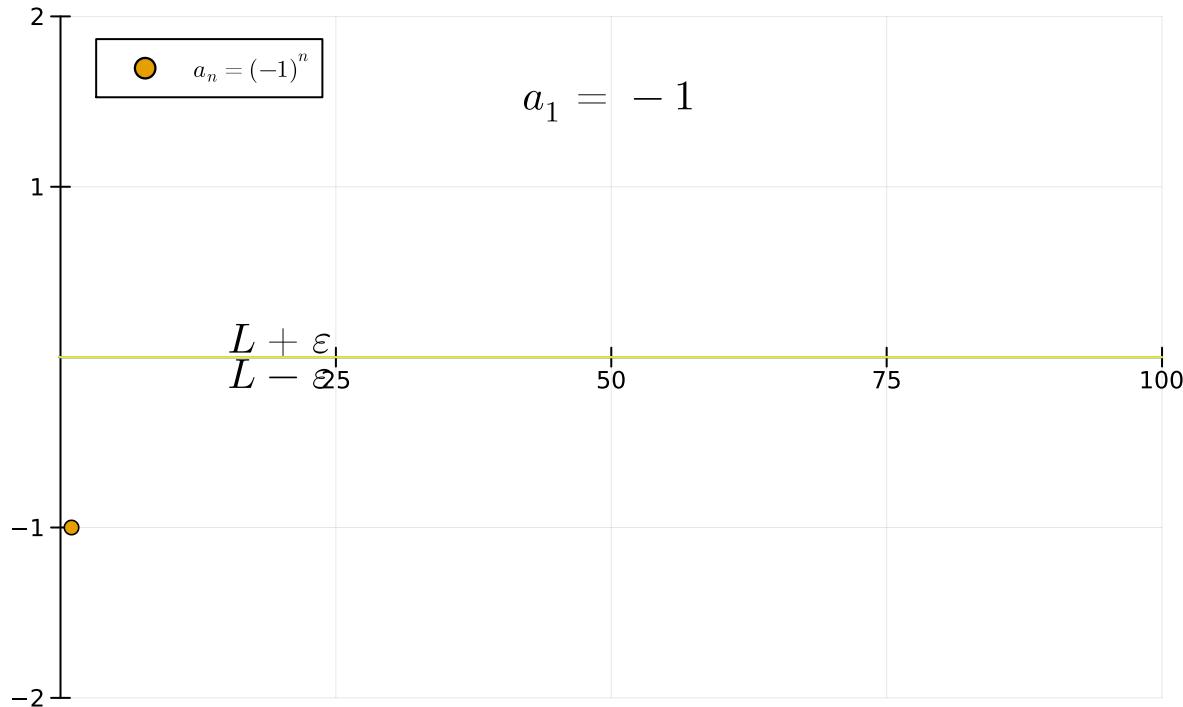


Example

$$\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \dots\}$$

$$\begin{array}{ll} \epsilon = & \boxed{0.0} \\ & n = \boxed{1} \\ L = & \boxed{0.0} \end{array}$$

Example



Remark:

$$\lim_{n \rightarrow \infty} (-1)^n \quad \text{DNE}$$

Limit of a Sequence

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L , written as

$$\lim_{n \rightarrow \infty} a_n = L$$

if for each $\epsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \epsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L . If the limit of a sequence does not exist, then the sequence **diverges**.

Theorem Limit of a Sequence

If

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{and} \quad f(n) = a_n \quad \text{when } n \text{ is an integer,}$$

then

$$\lim_{n \rightarrow \infty} a_n = L.$$

- **Remark**

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if} \quad r > 0$$

Limit Laws for Sequences Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. Sum Law

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

2. Difference Law

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

3. Constant Multiple Law

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

4. Product Law

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

5. Quotient Law

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0$$

Power Law

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p$$

Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then

$$\lim_{n \rightarrow \infty} b_n = L.$$

Theorem

If

$$\lim_{n \rightarrow \infty} |a_n| = 0,$$

then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Theorem

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L).$$

Remark

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1, \\ 1 & \text{if } r = 1 \end{cases}$$

Examples

Find

$$1. \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

$$2. \lim_{n \rightarrow \infty} \frac{n^2}{2^n - 1}.$$

$$3. \lim_{n \rightarrow \infty} \frac{n}{n+1}.$$

$$4. \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n+1}}.$$

$$5. \lim_{n \rightarrow \infty} \frac{\ln n}{n}.$$

$$6. \lim_{n \rightarrow \infty} \frac{(-1)^n}{n}.$$

$$7. \lim_{n \rightarrow \infty} \sin(\pi/n).$$

$$8. \lim_{n \rightarrow \infty} \frac{n!}{n^n}.$$

$$9. \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!}.$$

Exercise

$$\lim_{n \rightarrow \infty} \frac{n^n}{n!}.$$

Pattern Recognition for Sequences

Example

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \dots$$

and then determine whether the sequence you have chosen converges or diverges.

Example

Find a sequence $\{a_n\}$ whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \dots$$

and then determine whether the sequence you have chosen converges or diverges.

Monotonic and Bounded Sequences

Definition

- A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \dots$.
- It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
- A sequence is called **monotonic** if it is either increasing or decreasing.

Examples

Is the following increasing or decreasing?

1. $\left\{ \frac{3}{n+5} \right\}$.
2. $\left\{ \frac{n}{n^2+1} \right\}$.

Definition

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

A sequence is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

Example

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6), \quad \text{for } n = 1, 2, 3, \dots$$

9.2: Series and Convergence

“ Objectives

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the th-Term Test for Divergence of an infinite series.

Infinite Series

Consider the sequence $\{a_n\}_{n=1}^{\infty}$. The expression

$$a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series** (or simply **series**) and we use the notation

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

To make sense of this sum, we define a related **sequence** called the sequence of **partial sums** $\{s_n\}_{n=1}^{\infty}$ as

$$\begin{aligned}s_1 &= a_1 \\s_2 &= a_1 + a_2 \\s_3 &= a_1 + a_2 + a_3 \\\vdots & \\s_n &= a_1 + a_2 + \cdots + a_n = \sum_{i=1}^n a_i \\\vdots &\end{aligned}$$

and give the following definition

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the $\sum a_n$ series is called **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Remark

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

Exercise Assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence.

1. Find

$$\sum_{n=1}^{\infty} a_n \quad \text{if} \quad s_n = \sum_{i=1}^n a_i = \frac{n+2}{3n-5}$$

2. Can you find a_n ?

Solution

1. We find first

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n+2}{3n-5} = \frac{1}{3}$$

since the sequence $\{s_n\}$ converges to $\frac{1}{3}$, then the series converges and its sum is

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3}$$

2. Note that

$$\begin{aligned} a_n &= s_n - s_{n-1} = \frac{n+2}{3n-5} - \frac{(n-1)+2}{3(n-1)-5} \\ &= \frac{n+2}{3n-5} - \frac{n+1}{3n-8} \\ &= \frac{(n+2)(3n-8) - (n+1)(3n-5)}{(3n-5)(3n-8)} \end{aligned}$$

so,

$$a_n = \frac{-11}{(3n-5)(3n-8)}$$

Telescoping sum

Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution in class

Recall

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 (-1 < r < 1), \\ 1 & \text{if } r = 1, \end{cases}$$

So $\{r^n\}$ converges if $r \in (-1, 1]$ and diverges otherwise

Geometric Series

The series

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0$$

is called the **geometric series** with **common ratio r**

It is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

and divergent if $|r| \geq 1$.

Remark In words: the sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

Examples

1. Find the sum of the geometric series

$$4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$$

2. Is the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} \quad \text{convergent or divergent?}$$

3. Write $2.\bar{7}$ as rational number (ratio of integers).

4. Find the sum of the series

$$\sum_{n=0}^{\infty} x^n \quad \text{where} \quad |x| < 1.$$

Test for Divergence

Example Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent.

Theorem If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Proof

$$a_n = s_n - s_{n-1}$$

Divergence Test

If $\lim_{n \rightarrow \infty} a_n \neq 0$ or $\lim_{n \rightarrow \infty} a_n$ DNE then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

The series $\sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^2 + 5}$ is divergent.

Properties of Convergent Series

Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

$$(i) \quad \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

$$(ii) \quad \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$(iii) \quad \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Remark

If it can be shown that

$$\sum_{n=100}^{\infty} a_n$$

is convergent. Then

$$\sum_{n=1}^{\infty} a_n$$

is convergent.

9.3: The Integral Test and p -Series

“ Objectives

- Use the Integral Test to determine whether an infinite series converges or diverges.
- Use properties of -series and harmonic series.

The Integral Test and Estimates of Sums

Suppose f a function that is

1. continuous on $[1, \infty)$,
2. positive on $[1, \infty)$,
3. decreasing on $[1, \infty)$

and let $a_n = f(n)$. Then the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent if and only if the improper integral

$$\int_1^{\infty} f(x)dx$$

is convergent. In other words:

1. If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ convergent.
2. If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ divergent.

Examples

Test for convergence

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

Solution in class

Remark

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent but $\sum_{n=1}^{\infty} \frac{1}{n^2} \neq 1$.

Its sum is actually equal to $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

P-series and the Harmonic Series

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

1. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ is divergent; because it is a p -series with $p = \frac{1}{3} < 1$.
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent; because it is a p -series with $p = 3 > 1$.

Example

Show that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

is divergent.

Estimating the Sum of a Series

Suppose that the **integral test** is used to show that

$$\sum_{n=1}^{\infty} a_n$$

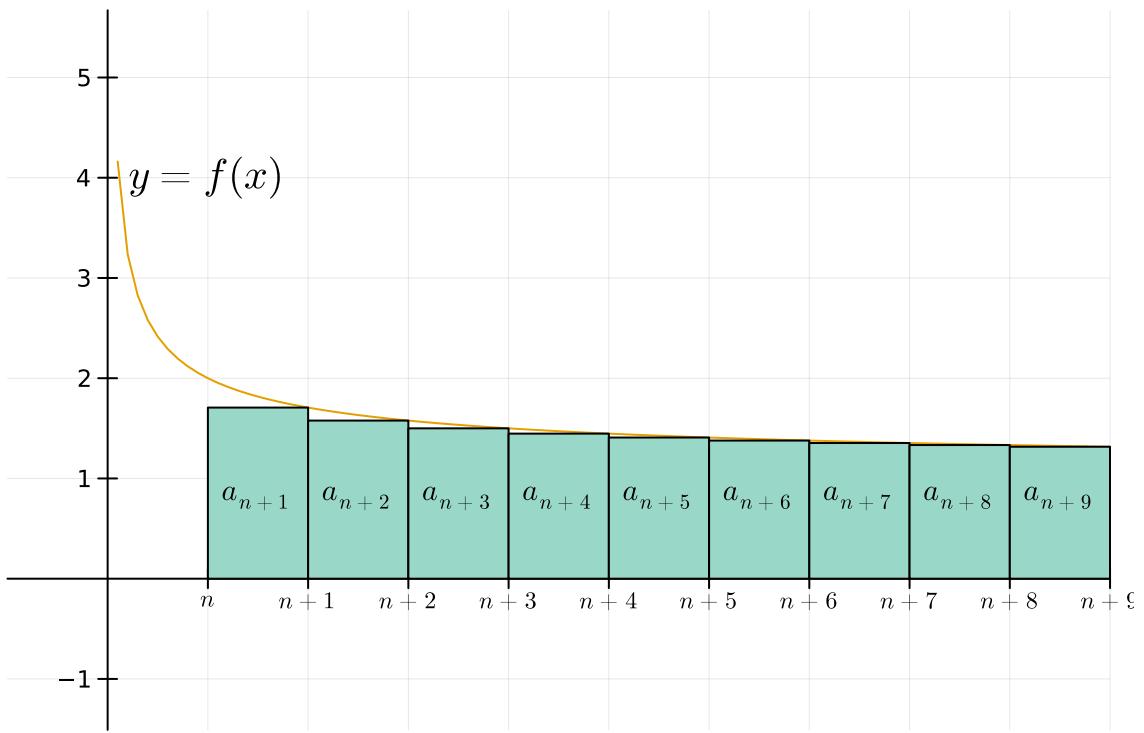
is **convergent**. So its sequence of partial sums $\{s_n = \sum_{i=1}^n a_i\}$ is convergent; that is

$$\lim_{n \rightarrow \infty} s_n = s.$$

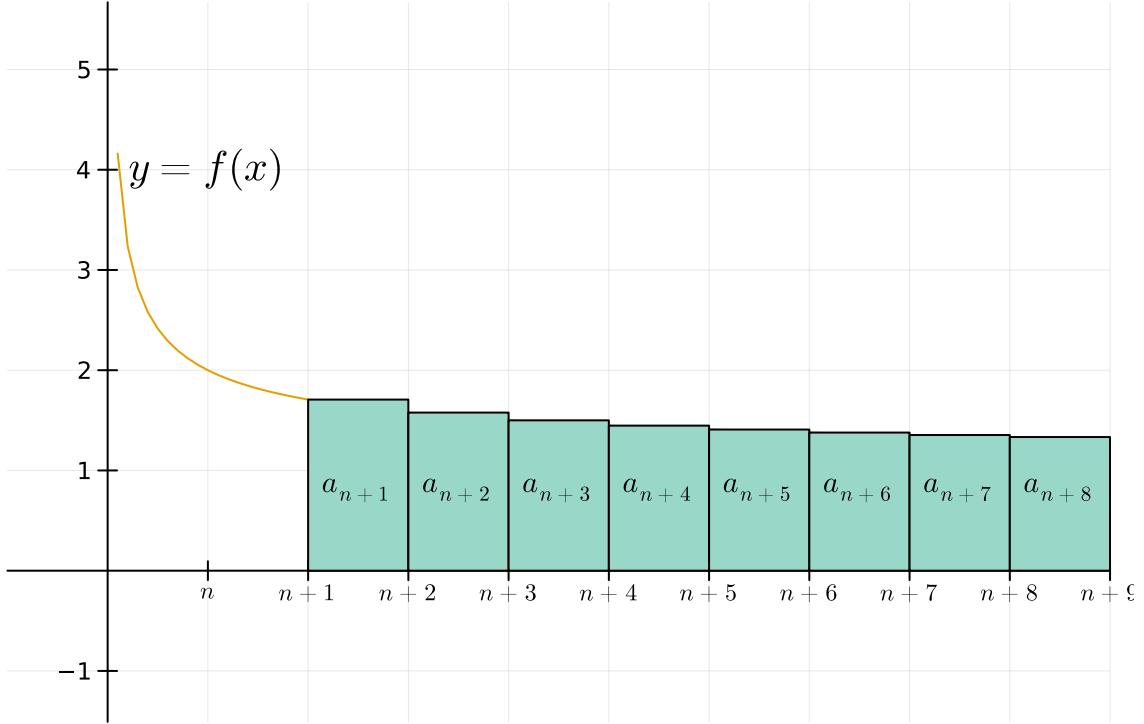
So we can write

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{s} = \underbrace{\sum_{i=1}^n a_i}_{s_n} + \underbrace{\sum_{i=n+1}^{\infty} a_i}_{R_n}$$

R_n is the **Remainder** or the error when s_n is used to approximate s .



$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x) dx$$



$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x) dx$$

Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

9.4: Comparisons of Series

“ Objectives

- Use the Direct Comparison Test to determine whether a series converges or diverges.
- Use the Limit Comparison Test to determine whether a series converges or diverges.

The Comparison Tests

The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent

Remarks

- Most of the time we use one of these series:

- p -series $\sum \frac{1}{n^p}$
- geometric series.

Examples

Test for convergence

$$(1) \quad \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Remark

Exercises 40 and 41 deal with the cases $c = 0$ and $c = \infty$.

Examples

Test for convergence

$$(3) \quad \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$(4) \quad \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}}$$

Exercises

Test for convergence

$$(5) \quad \sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$$

$$(6) \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

```
1 begin
2     using FileIO, ImageIO, ImageShow, ImageTransformations
3     using SymPy
4     using PlutoUI
5     using CommonMark
6     using Plots, PlotThemes, LaTeXStrings
7     using HypertextLiteral: @htl, @htl_str
8     using Colors
9     using Random
10 end
```

