# 5.2 The Definite Integral

**2 Definition of a Definite Integral** If $f$ is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

# Sigma Notation

## What do you think this notation means?

$$\sum_{i=1}^{5} i$$

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$

In [2]:
```
from sympy import Symbol, symbols, Sum, init_printing, Eq, Function, factor, Integr
al, integrate, Limit, oo, limit, exp
init_printing()
I, S, S_n, x, R_n, y= symbols("I,S, S_n, x, R_n, y")
i,k, n, m= symbols("i,k, n, m", positive=True, integer=True)
m=n
Sn=Sum(i,(i,1,m))
ex1 = Eq(Sn, factor(Sn.doit()))
ex1
```

Out[2]:

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

In [3]:
```
# more examples 2^i, 1/k , 3, start from 3
S=Sum(3,(i,1,5))
S
```

Out[3]: $\sum_{i=1}^{5} 3$

In [4]:
```
S = Function("S")
S=n*(n+1)/2
ex12=Eq(S,S.subs(n,m), evaluate=False)
ex12
```

Out[4]: $\frac{n}{2}(n+1) = \frac{n}{2}(n+1)$

## Three Formulae to memorize

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^{n} i^3 &= \left[\frac{n(n+1)}{2}\right]^2 \end{aligned}$$

## Note 1

## True or False?

$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(\Omega)d\Omega$$

## TRUE

**$x$ is a dummy variable.**

## Note 2

**The sum**

$$\sum_{i=1}^{n} f(x^*)\Delta x$$

**is called Reimann sum.**

## Note 3

**Geometric Meaning**

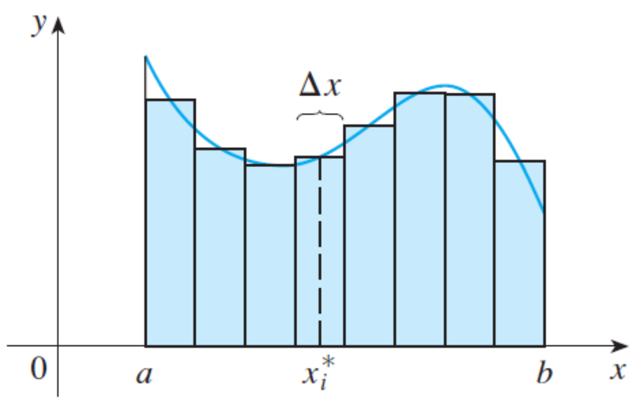



**FIGURE 1**
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

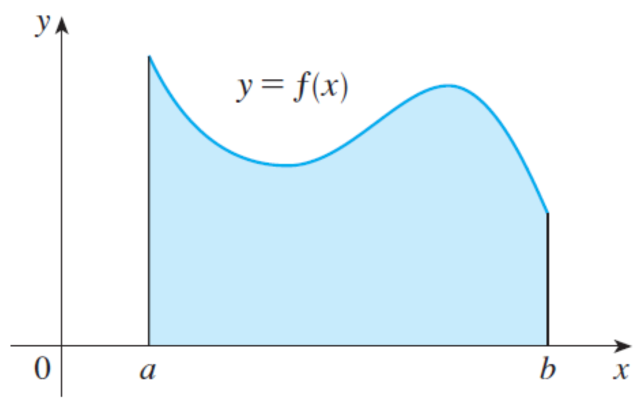
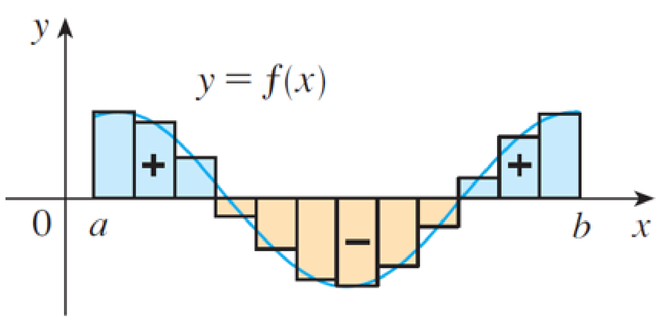
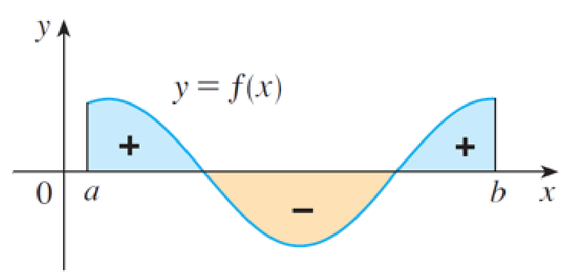


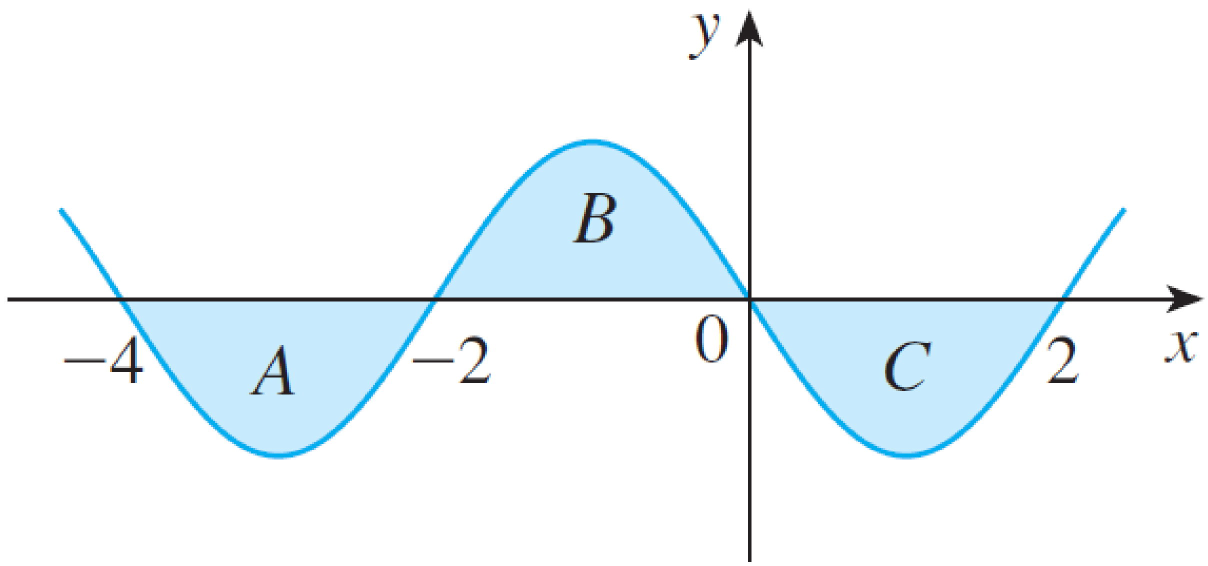

**FIGURE 2**
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from $a$ to $b$.

## Note 4



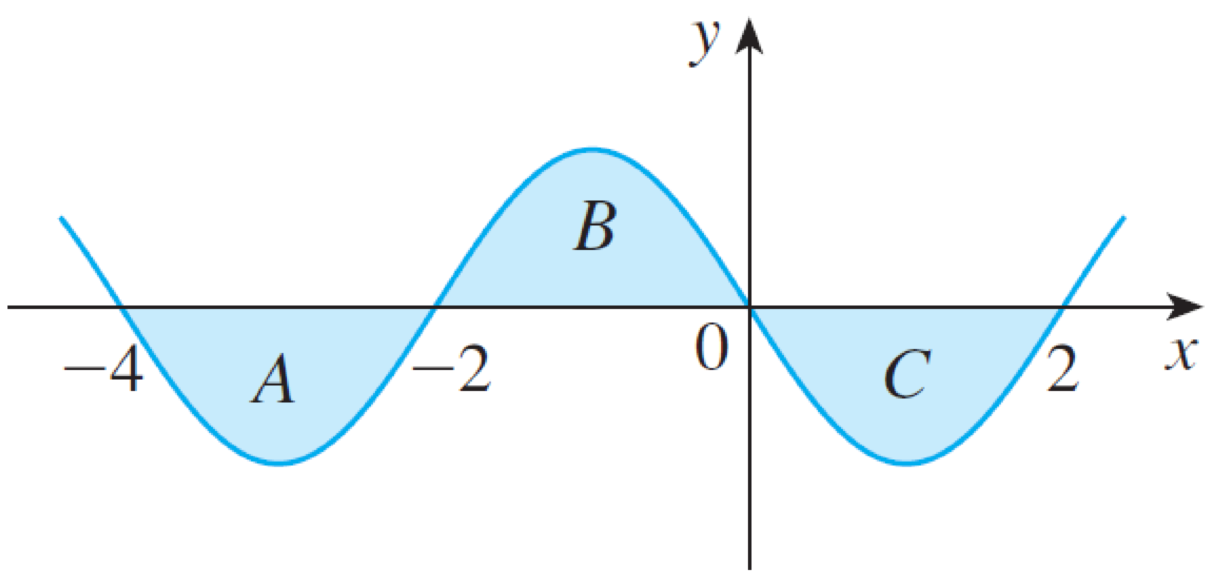
**Given the following graph of $f(x)$**




**where each of the regions $A, B$ and $C$ has area equal to 5, then**

$$\int_{-4}^{2} f(x)dx =$$

**Given the following graph of $f(x)$**




**where each of the regions $A, B$ and $C$ has area equal to 5, then the area between the graph and the x-axis from $x = -4$ to $x = 2$ is**

### Class of Integrable Functions

**3 Theorem** If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

## Useful theorem

**4 Theorem** If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\,\Delta x$$

## Question

**What do we mean when we say that a function $f$ is *integrable*?**

**(A) $f$ is continuous.**

**(B) $f$ is differentiable.**

**(C) $f$ has area.**

**(D) $f$ is discotinuous.**

**(E) none of the above.**

## How to Evaluate Integrals

### 1. Using the definition

### 2. Using a Computer Algebra System

### 3. Interpreting as areas

### 4. Approximating

### 5. Using integration techniques (tricks)

# 1. Using the definition

## Example: Find

$$\int_0^3 (4x^3 - 3x^2 + 2x)dx$$

In [5]:
```
S=Sum((4*(3*i/n)**3-3*(3*i/n)**2+2*(3*i/n))*(3/n),(i,1,n))
l=Limit(S,n,oo)
example2=Eq(R_n,Eq(factor(S),factor(S.doit())),evaluate=False)
example2
```

Out[5]:

$$R_n = \frac{9}{n^4}\sum_{i=1}^{n} i\left(36i^2 - 9in + 2n^2\right) = \frac{9}{2n^2}(n+1)(14n+15)$$

In [6]:
```
I=Integral(4*x**3-3*x**2+2*x,(x,0,3))
Rnlim=Limit(R_n,n,oo)
ll=Limit(factor(S),n,oo)
example1=Eq(Eq(I,Eq(Rnlim,Eq(ll,Limit(factor(S.doit()),n,oo),evaluate=False),evalua
te=False)),I.evalf(),evaluate=False)
example1
```

Out[6]:

$$\int_0^3 \left(4x^3 - 3x^2 + 2x\right)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{9}{n^4}\sum_{i=1}^{n} i\left(36i^2 - 9in + 2n^2\right)\right) = \lim_{n\to\infty}\left(\frac{9}{2n^2}(n -$$

## 2. Using a Computer Algebra System

## Example: (Reading Only)

### (i) Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.

### (ii) Use a computer algebra system to evaluate the expression

In [7]:
```
a = 1
b = 3
Δx=(b-a)/n
xi = a + i*Δx
f=exp(xi)
R = Sum(f*Δx,(i,1,n))
Rdoit=R.doit()
Rl=Limit(Rdoit,n,oo)
Rn=Eq(Eq(Limit(R_n,n,oo), Limit(R,n,oo),evaluate=False), Rl.doit(),evaluate=False)
Rdoit
```

Out[7]:

$$\frac{2e\left(e^{\frac{2}{n}} - e^{\frac{2}{n}(n+1)}\right)}{n\left(-e^{\frac{2}{n}} + 1\right)}$$

In [8]:
```
from sympy import plot_implicit, plot, sqrt
p1 = plot_implicit(Eq(x**2 + y**2, 6))
p1
```

```
<matplotlib.figure.Figure at 0x5448588>
```

Out[8]:
```
<sympy.plotting.plot.Plot at 0x5448630>
```

# Question

### The area $A$ of a circle of radius $r$ is

**(A)** $A = 2\pi r^2$

**(B)** $A = \pi r^2$

**(C)** $A = 2\pi r$

**(D)** $A = \pi^2 r^2$

**(E)** $A = \pi^2 r$

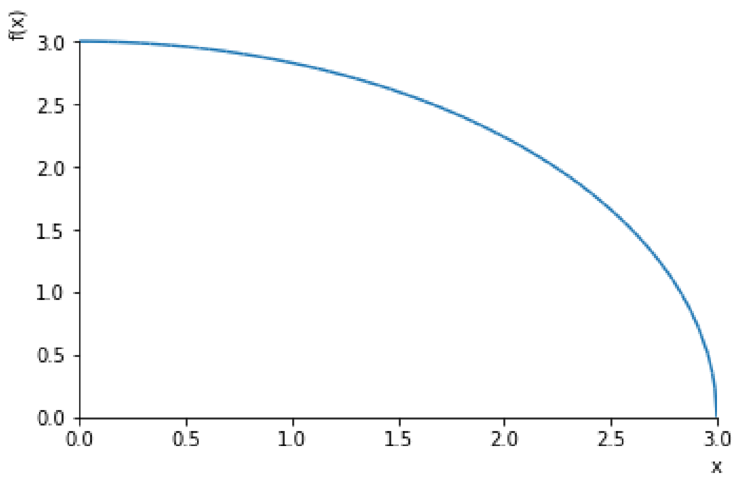
## 3. Interpreting as areas

### Example: Evaluate the following integrals by interpreting each in terms of areas

**(i)** $\int_0^3 \sqrt{9-x^2}dx$

**(ii)** $\int_{-2}^1 |x|dx$

```
In [9]: pi=plot(sqrt(9-x**2), (x,0,3))
```




## 4. Approximating

**Midpoint Rule**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

**Use the Midpoint Rule to approximate**

$$\int_1^2 \frac{1}{x}\,dx$$

**with $n = 5$.**

(SOLUTION IN CLASS)

## Properties of the Definite Integral

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

## Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where $c$ is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where $c$ is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

**EXAMPLE 7** If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

## Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

**EXAMPLE 8** Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

### Exercise 1

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cos\left(1+\frac{i}{n}\right)^2 =$$

| |
|---|
| **(A)** $\int_1^2 \cos(1+x^2)dx$. |
| **(B)** $\int_1^2 \cos(x^2)dx$. |
| **(C)** $\int_1^2 \cos^2(x)dx$. |
| **(D)** $\int_0^1 \cos(x^2)dx$. |
| **(E)** $\int_0^1 \cos(1+x^2)dx$. |

### Exercise 2

**If** $\int_{-5}^{7} f(x)dx = -17$, $\int_{-5}^{11} f(x)dx = 32$, **and** $\int_{8}^{7} f(x)dx = 5$, **then** $\int_{11}^{8} f(x)dx =$

| |
|---|
| **(A)** $-50$. |
| **(B)** $44$. |
| **(C)** $-60$. |
| **(D)** $19$. |
| **(E)** $-54$. |

## Exercise 3

**If $f$ is continuous function and**

$$2 \le f(x) \le 5 \quad \text{for} \quad 3 \le x \le 9,$$

**then ONE of the following statements is FALSE**

| |
|---|
| **(A)** $\int_3^9 \lvert f(x)\rvert \, dx \ge 12$ |
| **(B)** $\int_3^9 (3 - f(x)) \, dx \ge -12$ |
| **(C)** $\int_3^9 (1 - \lvert f(x)\rvert) \, dx \ge -10$ |
| **(D)** $\int_3^9 -2f(x) dx \le -24$ |
| **(E)** $\int_3^9 (f(x))^2 dx \ge 24$ |