Sec1.5

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1 1.5 Linear First-Order Equations

It is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

1.1 Integrating factor

An **integration factor** of a DE is a function $\rho(x, y)$ such that the multiplication of each side of the DE by $\rho(x, y)$ yeilds an equation where each side is recognizable as a derivative.

For Linear First-Order Equation, it is

$$\rho(x) = e^{\int P(x)dx}$$

METHOD: SOLUTION OF FIRST-ORDER EQUATIONS

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)\,dx}$.
2. Then multiply both sides of the differential equation by $\rho(x)$.
3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:
$$D_x\,[\rho(x)y(x)] = \rho(x)Q(x).$$
4. Finally, integrate this equation,
$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C,$$
then solve for y to obtain the general solution of the original differential equ tion.

1.2 Example

Solve the IVP

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \qquad y(0) = -1.$$

1.3 Example

Solve the DE

$$(x^2+1)\frac{dy}{dx}+3xy=6x.$$

THEOREM 1 **The Linear First-Order Equation**

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx}+P(x)y=Q(x), \quad y(x_0)=y_0 \tag{11}$$

has a unique solution $y(x)$ on I, given by the formula in Eq. (6) with an appropriate value of C.

1.4 Remarks

- Linear First-Order equations have *no singular solutions*.
- We can solve the initial value problem directly by computing

$$\rho(x)=e^{\int_{x_0}^{x} P(t)dt}$$

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2,\ y(0) = 0$

2. $y' - 2y = 3e^{2x},\ y(0) = 0$

3. $y' + 3y = 2xe^{-3x}$

4. $y' - 2xy = e^{x^2}$

5. $xy' + 2y = 3x,\ y(1) = 5$

6. $xy' + 5y = 7x^2,\ y(2) = 5$

7. $2xy' + y = 10\sqrt{x}$

8. $3xy' + y = 12x$

9. $xy' - y = x,\ y(1) = 7$

10. $2xy' - 3y = 9x^3$

11. $xy' + y = 3xy,\ y(1) = 0$

12. $xy' + 3y = 2x^5,\ y(2) = 1$

13. $y' + y = e^x,\ y(0) = 1$

14. $xy' - 3y = x^3,\ y(1) = 10$

15. $y' + 2xy = x,\ y(0) = -2$

16. $y' = (1 - y)\cos x,\ y(\pi) = 2$

17. $(1 + x)y' + y = \cos x,\ y(0) = 1$

18. $xy' = 2y + x^3 \cos x$

19. $y' + y \cot x = \cos x$

20. $y' = 1 + x + y + xy,\ y(0) = 0$

21. $xy' = 3y + x^4 \cos x,\ y(2\pi) = 0$

22. $y' = 2xy + 3x^2 \exp(x^2),\ y(0) = 5$

23. $xy' + (2x - 3)y = 4x^4$

24. $(x^2 + 4)y' + 3xy = x,\ y(0) = 1$

25. $(x^2 + 1)\dfrac{dy}{dx} + 3x^3 y = 6x \exp\left(-\tfrac{3}{2}x^2\right),\ y(0) = 1$