Interior Point Methods for Linear Programming Problems

Lecture 1

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Brief historical comments

- Klee and Minty (1972): How good is the simplex algorithm?
- Khachiyan (1979): Ellipsoid method
- Karmarkar (1984): Projective method

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- Path-following
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- Affine scaling
- Potential reduction

Interior point methods

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▷ For a list of pioneering articles and legends of IPM until 2001, visit

`www.mcs.anl.gov/research/projects/otc/InteriorPoint/`

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Logarithmic barrier method

Given LPP

$$
(\mathsf{P}) \quad \begin{cases} \min & c^T x \\ \text{subject to} & b - Ax = 0, x \geq 0. \end{cases}
$$

KKT system for (P) is

$$
\begin{cases} (\mathsf{PF}) & Ax = b, x \geq 0 \\ (\mathsf{St}\ \&\ \mathsf{DF}) & c - A^T\mu - \lambda = 0, \lambda \geq 0 \\ (\mathsf{CS}) & \lambda^T x = 0 \end{cases}
$$

Logarithmic barrier method

Given LPP

$$(\mathrm{P}) \quad \begin{cases} \min & c^T x \\ \text{subject to} & b - Ax = 0, x \geq 0. \end{cases}$$

Denoting $s = \lambda$ and $y = \mu$, the KKT system for (P) is

$$(\text{KKT P}) \begin{cases} Ax = b, \\ A^T y + s = c, \\ x \geq 0, s \geq 0, \\ s^T x = 0. \end{cases}$$

Logarithmic barrier method

Given LPP

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Denoting $s = \lambda$ and $y = \mu$, the KKT system for (P) is

$$
\text{(KKT P)} \begin{cases} Ax = b, \\ A^T y + s = c, \\ x \geq 0, s \geq 0, \\ s^T x = 0. \end{cases}
$$

(H.W.) KKT system of the dual of (P) is also (KKT P).

Logarithmic barrier method

KKT system for (P) is

$$
\text{(KKT P)} \begin{cases} Ax = b, & (1) \\ A^T y + s = c, & (2) \\ s^T x = 0, & (3) \\ x \geq 0, s \geq 0. & (4) \end{cases}
$$

Note

- (3) is the only nonlinear equation in (KKT P)
- Simplex method maintains (1), (2) & (3) and aims for (4).
- Interior-point methods maintain (1), (2) & (4) and aim for (3).

Logarithmic barrier method

Given LPP

$$
\text{(P)} \quad \begin{cases} \min & c^T x \\ \text{subject to} & Ax = b, x \geq 0. \end{cases}
$$

Denoting $\mathcal{X} = \{x \in \mathbb{R}^n : x \geq 0\}$, (P) can be equivalently written as follows:

$$
\text{(I)} \quad \begin{cases} \min & c^T x + I_{\mathcal{X}}(x) \\ \text{subject to} & Ax = b, \end{cases}
$$

where

$$
I_{\mathcal{X}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{X} \\ +\infty, & \text{if } x \notin \mathcal{X}. \end{cases}
$$

Logarithmic barrier method

Approximate (I) by

$$\text{(BP)} \quad \begin{cases} \min & c^T x + \frac{1}{t}\phi(x) \\ \text{subject to} & Ax = b, \end{cases}$$

where $\frac{1}{t}\phi$ is a continuous approximation of $I_{\mathcal{X}}$.

Logarithmic barrier method

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where $\frac{1}{t}\phi$ is a continuous approximation of $I_{\mathcal{X}}$.

By writing $\mu = \frac{1}{t}$, (BP) can be equivalently written as follows:

$$\text{(BP)} \quad \begin{cases} \min & \frac{c^T x}{\mu} + \phi(x) \\ \text{subject to} & Ax = b, \end{cases}$$

Logarithmic barrier method

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By writing $\mu = \frac{1}{t}$, (BP) can be equivalently written as follows:

$$\text{(BP)} \quad \begin{cases} \min & \frac{c^T x}{\mu} + \phi(x) \\ \text{subject to} & Ax = b, \end{cases}$$

> The function $\phi_B(x;\mu) = \frac{c^T x}{\mu} + \phi(x)$ is called a barrier function of the problem (P).

Logarithmic barrier method

Given LPP

$$\text{(P)} \quad \begin{cases} \min & c^T x \\ \text{subject to} & Ax = b, x \geq 0. \end{cases}$$

The log-barrier formulation of (P) is

$$\text{(LBP)} \quad \begin{cases} \min & \phi_B(x;\mu) \equiv \frac{c^T x}{\mu} - \sum_{i=1}^{n} \ln(x_i) \\ \text{subject to} & Ax = b. \end{cases}$$

The first order necessary and sufficient optimality conditions for (LBP) is

$$\text{(LBP OC)} \quad \begin{cases} \nabla \phi_B(x;\mu) + \sum_{i=1}^{m} \lambda_i \nabla h_j(x) = 0 \\ Ax = b \end{cases}$$

Logarithmic barrier method

The gradient and Hessian of ϕ_B is given by

$$\nabla \phi_B(x;\mu) = \frac{c}{\mu} - X^{-1}e \quad \text{and} \quad \nabla^2 \phi_B(x;\mu) = X^{-2},$$

where $X = \text{diag}(x_1, x_2, \ldots, x_n)$ and e is the n-vector $(1, 1, \ldots, 1)^T$.

(LBP OC) is given by

$$\text{(LBP OC)} \quad \begin{cases} \frac{c}{\mu} - X^{-1}e + A^T\lambda = 0 \\ Ax = b. \end{cases}$$

Logarithmic barrier method

(LBP OC) reduces to the following system by putting $y = -\lambda\mu$ and $s = \mu X^{-1}e$:

$$\text{(PD OCS)} \begin{cases} A^T y + s = c, \\ Ax = b, \\ Sx = \mu e, \end{cases}$$

where $S = \mathsf{diag}(s) = \mathsf{diag}(s_1, s_2, \ldots, s_n)$.

Logarithmic barrier method

(H.W.) The log-barrier formulation of the dual of (P) is

$$\begin{aligned} \min \quad & -\frac{b^T y}{\mu} - \sum_{i=1}^{n} \ln(s_i) \\ \text{subject to} \quad & A^T y + s = c. \end{aligned}$$

Show that the first order necessary and sufficient optimality conditions this problem is also (PD OCS).

Note

Log-barrier problem to the following problem does not have a solution:

$$\min 0 \ \text{ subject to } x \geq 0.$$

Logarithmic barrier method

Notations

- $\overset{\circ}{\mathcal{F}}(P) = \{x \in \mathbb{R}^n : Ax = b, x > 0\}$
- $\overset{\circ}{\mathcal{F}}(D) = \{(y,s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y + s = c, s > 0\}$

Logarithmic barrier method

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- $\overset{\circ}{\mathcal{F}}(P) = \{x \in \mathbb{R}^n : Ax = b, x > 0\}$
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Existence of a minimizer of $\phi_B(x; \mu)$

A necessary and sufficient condition for the existence of a minimizer $\phi_B(x; \mu)$ in $\overset{\circ}{\mathcal{F}}(P)$ is '$\overset{\circ}{\mathcal{F}}(P)$ and $\overset{\circ}{\mathcal{F}}(D)$ nonempty'.

For a given $\mu > 0$, let $(x(\mu), y(\mu), s(\mu))$ be the solution to (PD OCS)

Logarithmic barrier method

Central path

- $\{x(\mu) : \mu > 0\}$ is called the primal central path.
- $\{(x(\mu), y(\mu), s(\mu) : \mu > 0\}$ is called primal-dual central path.

Analytic center

If the feasible region of (P) is bounded, then the central path starts (i.e., when $\mu \to \infty$) from the unique point

$$\text{argmin} \sum_{i=1}^{n} -\ln(x_i),$$

which is called the analytic center of the feasible region.

Central path

If μ decreases, then $c^T x(\mu)$ decreases and $b^T y(\mu)$ increases.

Central path

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On the central path, $c^T x(\mu) - b^T y(\mu) = n\mu$.

Hence, as $\mu \to 0+$, duality gap becomes zero and we will reach at an optimum solution of both primal and dual.

References

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