1 Simulating the Moon's Orbit Around the Earth

Figure 1 shows the force diagram for the moon m at an instant as its orbiting Earth M. We will analyze the moon's motion in a reference frame wherein the Earth remains motionless and located at the origin. Since we know that the Moon will orbit in a plane (a direct result of the conservation of angular momentum), we define a two-dimensional coordinate system in the plane of the orbit, and do not have to worry about a third dimension of motion.

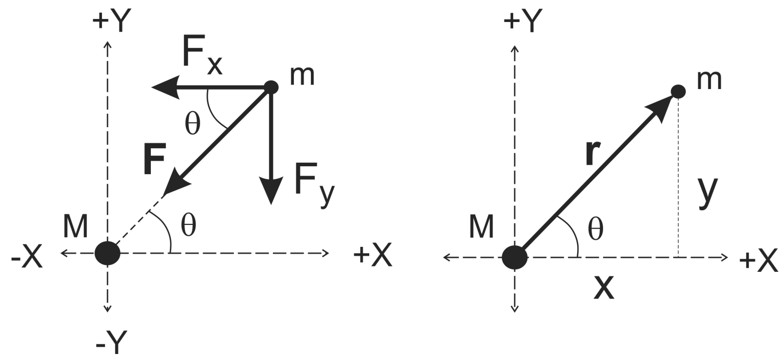

Figure 1: Force diagram and coordinate system for the moon orbiting Earth. The figure on the left shows the direction of the gravitational force acting on the moon due to the Earth, and the figure on the right shows the position vector of the moon relative to the Earth's center.

Starting with Newton's Law of Universal Gravitation,

$$|\vec{F}| = G\frac{m_1 m_2}{r^2} \tag{1}$$

the x- and y-components of the Earth's gravitational pull on the Moon are

$$F_x = F\cos\,\theta \tag{2}$$

$$F_y = F\sin\,\theta. \tag{3}$$

Combining these with Newton's Second Law $\Sigma\vec{F} = m\vec{a}$ yields

$$\sum F_x = -F_x = ma_x = -G\frac{mM_E}{r^2}\cos\,\theta \tag{4}$$

$$\sum F_y = -F_{gy} = ma_y = -G\frac{mM_E}{r^2}\sin\,\theta. \tag{5}$$

The components of the acceleration can then be calculated, and found to be

$$a_x = -G\frac{M_E}{r^2}\cos\,\theta \tag{6}$$

$$a_y = -G\frac{M_E}{r^2}\sin\,\theta. \tag{7}$$

In these equations we can use $x = r\,\cos\,\theta$ and $y = r\,\sin\,\theta$ (apply the proper trigonometric relations to the position vector of the right-hand-side of Figure 1) to get

$$a_x = -G\frac{M_E}{r^3}x \tag{8}$$

$$a_y = -G\frac{M_E}{r^3}y \tag{9}$$

Finally, applying the Pythagorean theorem,$r^2 = x^2 + y^2$, we can write

$$a_x = -G\frac{M_E}{(x^2+y^2)^{\frac{3}{2}}}x \tag{10}$$

$$a_y = -G\frac{M_E}{(x^2+y^2)^{\frac{3}{2}}}y \tag{11}$$

These acceleration equations are the main equations in the simulation, as they capture the physics of the Earth continually pulling the moon toward it.

2 Simple Euler Method

We wish to calculate the position of the moon for each time step of the simulation, and so from the Euler method we have

$$x(t+\Delta t) \approx x(t) + v_x(t)\Delta t \tag{12}$$

$$y(t+\Delta t) \approx y(t) + v_y(t)\Delta t \tag{13}$$

where Δt is the time step of the simulation. We now have equations for the position as a function of time, but these equations depend on the velocity. Again using the Euler method, we can define equations that give the velocity as a function of the accelerations defined above:

$$v_x(t+\Delta t) \approx v_x(t) + a_x(t)\Delta t \tag{14}$$

$$v_y(t+\Delta t) \approx v_y(t) + a_y(t)\Delta t \tag{15}$$

3 Euler-Cromer Method

In the Euler-Cromer method, we make a small change: we use $v(t+\Delta t)$ instead of $v(t)$ in Eqns.(12) and (13). The new set of equations is then

$$x(t+\Delta t) \approx x(t) + v_x(t+\Delta t)\Delta t \tag{16}$$

$$y(t+\Delta t) \approx y(t) + v_y(t+\Delta t)\Delta t \tag{17}$$

Notice that with this change, it is necessary to calculate $v_x(t+\Delta t)$ and $v_y(t+\Delta t)$ before calculating $x(t+\Delta t)$ and $y(t+\Delta t)$. With the Euler method, the order did not matter.