# 1 1D Free Fall with Air Resistance (Review of 1D case)

Figure 1 shows a mass falling vertically downward at an instant in time with instantaneous velocity $\boldsymbol{v}$. The mass is subject to two forces, the object's weight $m\boldsymbol{g}$, and a resistive force $R$. We assume that the resistive force is proportional to the square of the instantaneous velocity of the mass, a somewhat realistic model for a resistive force encountered by an object traveling at high speeds, and that its direction is opposite to that of the instantaneous velocity.

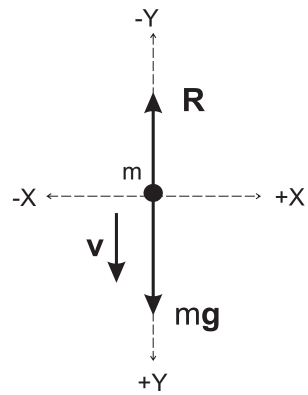


Figure 1: Free body diagram for a mass $m$ in free fall subject to a resistive force that is proportional to the square of the instantaneous velocity.

The magnitude of the resistive force is given by $R = \frac{D\varrho A}{2}v^2$, where $D$ is a constant called the drag coefficient of the projectile, $\varrho$ is the density of air, and $A$ is the cross-sectional area of the mass perpendicular to $\boldsymbol{v}$. Here we will assume that $D$ is a constant; but, $D$, depending on the physical situation, is most often a complicated function of the instantaneous velocity. Applying Newton's 2nd law in the y-direction (assuming down is positive) produces

$$\sum F_y = mg - \frac{D\varrho A}{2}v_y^2 = ma_y.$$

Solving for $a_y$, and using notation that emphasizes the time dependence of both velocity and acceleration, we arrive at

$$a_y(t) = g - \frac{D\varrho A}{2m}[v_y(t)]^2. \tag{1}$$

If we let $a_y = \frac{dv_y}{dt}$ then Equation 1 can be integrated to obtain an analytical function that describes the velocity as a function of time, assuming that the initial velocity is zero,

$$v_y(t) = \sqrt{\frac{2mg}{D\varrho A}}\tanh\left(\sqrt{\frac{D\varrho Ag}{2m}}t\right). \tag{2}$$

If we, in turn, let $v_y = \frac{dy}{dt}$ then Equation 2 can be integrated, assuming initial position at $y = 0$, to obtain the vertical position as a function of time,

$$y(t) = \frac{2m}{D\varrho A} \ln \left[ \cosh \left( \sqrt{\frac{D\varrho A g}{2m}} t \right) \right]. \quad (3)$$

Though the integrals that needed to be solved to produce Equations 2 and 3 are somewhat non-trivial, they do exist. They actually aren't that bad and the good student is encouraged, though it is not expected as part of this assignment, to at least try to set them up to see exactly where Equations 2 and 3 come from. If, however, we had employed a more complicated model for air resistance, or if we had used a realistic velocity-dependent function for $D$ then the integrals would have been far more difficult, and more than likely would not even exist.

It is prudent, therefore, to learn another approach, different from integral calculus in the continuum limit, to solve realistic physical problems. The approach we present here is known as a computational approach since we will make use of a computer to carry out the necessary calculations.

The computational approach is based on making a sound approximation to the time-dependent acceleration. If $\Delta t$ is sufficiently small, then $a_y(t)$ will be approximately equal to its average value over the time interval $\Delta t$: $a_y(t) \approx a_{y,avg}$. Then $a_y(t) \approx \frac{\Delta v_y}{\Delta t}$, or

$$a_y(t) \approx \frac{v_y(t + \Delta t) - v_y(t)}{\Delta t}. \quad (4)$$

Solving Equation 4 for $v_y(t + \Delta t)$, we arrive at

$$v_y(t + \Delta t) \approx v_y(t) + a_y(t)\Delta t. \quad (5)$$

In a similar manner, if we approximate $v_y(t)$ by its average value over a small time interval $\Delta t$ then the following equation results for the position at time $t + \Delta t$:

$$y(t + \Delta t) \approx y(t) + v_y(t)\Delta t. \quad (6)$$

This particular computational strategy is known as the *Euler method.* Equations 5 and 6 are called *finite difference* equations, and can be used to predict the values of the velocity and position at a later time $(t + \Delta t)$ if their respective values, and acceleration, are known at the previous time $t$. As long as $\Delta t$ is sufficiently small, then these equations can be iterated a sufficient number of times to produce accurate predictions for the velocity and position at any future time.

## 2 2D Projectile with Air Resistance

Figure 2 shows a projectile of mass $m$ traveling along its trajectory at some instant in time. The projectile is subject to two forces, the weight of the object $m\boldsymbol{g}$, acting in the downward (-y) direction, and a resistive force of magnitude

$R = \frac{D\varrho A}{2}v^2$ acting in a direction opposite to that of the instantaneous velocity $\boldsymbol{v}$. $D$ is a constant called the drag coefficient of the projectile, $\varrho$ is the density of air, and $A$ is the cross-sectional area of the mass perpendicular to $\boldsymbol{v}$. Our goal is to apply Newton's second law to obtain the acceleration in the x- and y-directions, and then use these expressions for the non-constant acceleration to numerically solve for the velocity and position as a function of time.

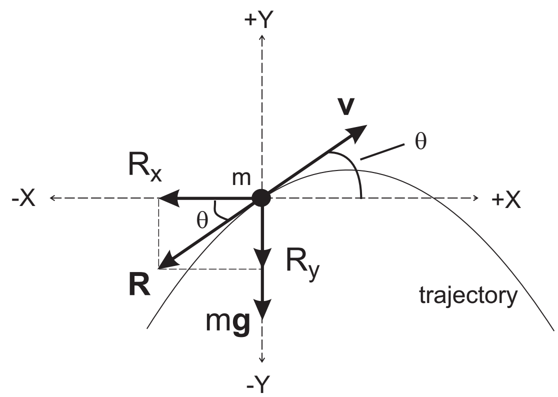


Figure 2: Free body diagram for a 2D projectile with a resistive force.

The *magnitudes* of the components of $\boldsymbol{R}$ are given by $R_x = R\cos\theta$ and $R_y = R\sin\theta$. Thus, applying Newton's second law in the x-direction produces

$$\sum F_x = -R\cos\theta = ma_x \tag{7}$$

and in the y-direction,

$$\sum F_y = -R\sin\theta - mg = ma_y. \tag{8}$$

As is obvious from Figure 2 the angle $\theta$ is also the angle between the direction of the instantaneous velocity $\boldsymbol{v}$ and the x-axis. Therefore, the *magnitudes* of the x- and y-components of the velocity vector can be written $v_x = v\cos\theta$ and $v_y = v\sin\theta$, respectively. Substituting $\cos\theta = \frac{v_x}{v}$ , $\sin\theta = \frac{v_y}{v}$, and $R = \frac{D\varrho A}{2}v^2$ into Equations (7) and (8), and a little rearranging yields

$$a_x(t) = -\frac{D\varrho A}{2m}v(t)v_x(t) \tag{9}$$

and

$$a_y(t) = -\frac{D\varrho A}{2m}v(t)v_y(t) - g, \tag{10}$$

where notation has been used to emphasize the time dependence of the velocity and acceleration, and $v(t)$ is related to its components via $v(t) = \sqrt{[v_x(t)]^2 + [v_y(t)]^2}$ .

The Euler method can now be applied to produce a numerical solution based on the iterative evaluation of the resulting finite difference equations. If a sufficiently small time interval $\Delta t$ is considered, then the time-dependent components of the acceleration in Equations (9) and (10) can be approximated by their average values over the time interval, $a_x(t) \approx a_{x,avg}$ and $a_y(t) \approx a_{y,avg}$. Thus, $a_x(t) \approx \frac{\Delta v_x}{\Delta t} = \frac{v_x(t+\Delta t)-v_x(t)}{\Delta t}$ and $a_y(t) \approx \frac{\Delta v_y}{\Delta t} = \frac{v_y(t+\Delta t)-v_y(t)}{\Delta t}$, or equivalently

$$v_x(t + \Delta t) \approx v_x(t) + a_x(t)\Delta t \tag{11}$$

and

$$v_y(t + \Delta t) \approx v_y(t) + a_y(t)\Delta t. \tag{12}$$

Similar equations, relating the x- and y-components of the position at a later time $t + \Delta t$ to those at the earlier time $t$, can be attained by approximating the x- and y-components of the instantaneous velocity with the average velocity over the time interval: $v_x(t) \approx v_{x,avg}$ and $v_y(t) \approx v_{y,avg}$, resulting in

$$x(t + \Delta t) \approx x(t) + v_x(t)\Delta t \tag{13}$$

and

$$y(t + \Delta t) \approx y(t) + v_y(t)\Delta t. \tag{14}$$

Equations (11), (12), (13), and (14) along with the expressions for $a_x$ and $a_y$ in Equations (9) and (10), can be iteratively solved to predict the velocity and position of the projectile for future times.