Project 2: Model of a Saturn V Rocket
Supplemental Notes

Equations of Motion for a Rocket

To describe the motion of a rocket we begin by defining our system of interest to be comprised of the rocket, the unburned fuel contained within the rocket, and the rocket's exhaust, and then apply the Impulse-Momentum Theorem,

$$\Delta \mathbf{p} = \int \mathbf{F}_{ext} dt = \mathbf{I}, \tag{1}$$

where $\mathbf{p}$ is the total momentum of the system, and $\mathbf{F}_{ext}$ is the net force exerted on the system by external agents. Equation (1) can also be written as

$$\mathbf{p}(t + dt) - \mathbf{p}(t) = \mathbf{F}_{ext} dt. \tag{2}$$

To a good approximation the infinitesimally small dt can be replaced by a finite Δt as long as Δt is kept sufficiently small. Thus Equation (2) can be approximated by

$$\mathbf{p}(t + \Delta t) - \mathbf{p}(t) \approx \mathbf{F}_{ext} \Delta t. \tag{3}$$

Figure 1 shows a schematic representation of a rocket at two different times, t and $t + \Delta t$. At time t the rocket has velocity $\mathbf{v}$, and the combined mass of the rocket and unburned fuel is m_r. After a time Δt has passed, the rocket has burned Δm_r of fuel and ejected the burnt fuel as exhaust in a direction opposite to its forward motion, causing the rocket's velocity to increase by $\Delta \mathbf{v}$. The exhaust Δm_e leaves the rocket with a velocity $\mathbf{v}_e$, measured relative to the rocket. The mass expelled as exhaust is equal to the amount of mass lost by the combined rocket and unburned fuel; thus, $\Delta m_r = -\Delta m_e$ where we choose $\Delta m_e > 0$.

At time t the total momentum of the system is given by

$$\boldsymbol{p}(t) = m_r \mathbf{v}, \tag{4}$$

while at time $t + \Delta t$ the momentum of the rocket-fuel-exhaust system is given by

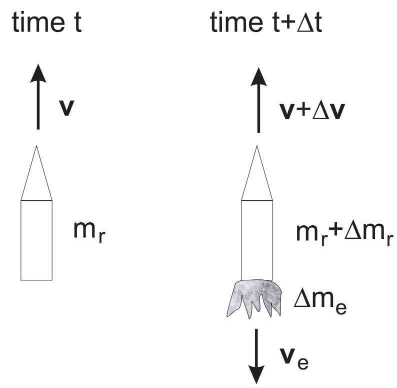

Figure 1: A rocket that has experienced a change in velocity $\Delta\mathbf{v}$ as a result of the burning and expulsion of a mass $\Delta m_e (> 0)$ of fuel over a time interval of Δt. The exhaust is expelled with a velocity $\mathbf{v}_e$ relative to the rocket. During the time interval the rocket loses an amount of mass $\Delta m_r = -\Delta m_e$.

$$\mathbf{p}(t + \Delta t) = (m_r + \Delta m_r)(\mathbf{v} + \Delta \boldsymbol{v}) + \Delta m_e(\mathbf{v} + \mathbf{v}_e), \tag{5}$$

where $(\mathbf{v} + \mathbf{v}_e)$ is the velocity of the exhaust relative to a fixed reference system (i.e. the Earth). Keep in mind that the exhaust moves with $\mathbf{v}_e$ relative to the rocket, but that the exhaust, as unburned fuel, was already moving relative to a fixed reference frame with velocity $\boldsymbol{v}$. It is, therefore, necessary to add the velocities $\mathbf{v}$ and $\mathbf{v}_e$ vectorially to come up with the velocity for the exhaust. Inserting these expressions for $\mathbf{p}(t)$ and $\mathbf{p}(t + \Delta t)$ into Equation (3), and carrying out all the multiplications on the left-hand-side, produces

$$m_r\mathbf{v} + m_r\Delta\mathbf{v} + \Delta m_r\boldsymbol{v} + \Delta m_r\Delta\boldsymbol{v} + \Delta m_e\mathbf{v} + \Delta m_e\mathbf{v}_e - m_r\mathbf{v} \approx \sum \boldsymbol{F}_{ext}\Delta t. \tag{6}$$

The first and seventh terms on the left-hand-side of Equation (6) cancel each other out. Furthermore, since $\Delta m_r = -\Delta m_e$, the third and fifth terms on the left-hand-side can be eliminated. Finally, if Δt is small then the associated changes in other physical quantities will also be small; thus, we can neglect the term $\Delta m_r\Delta\mathbf{v}$ compared with the other remaining terms on the left-hand-side of Equation (6) since it is the product of two small quantities. We are left with

$$m_r\Delta\mathbf{v} + \Delta m_e\mathbf{v}_e \approx \sum \mathbf{F}_{ext}\Delta t,$$

or, if we divide each side by Δt, we end up with

$$m_r\frac{\Delta\mathbf{v}}{\Delta t} + \frac{\Delta m_e}{\Delta t}\mathbf{v}_e \approx \sum \mathbf{F}_{ext}. \tag{7}$$

Next, we define an interesting quantity unique to rockets, the thrust $\mathbf{T}$. The thrust is the force that is responsible for pushing the rocket forward. The thrust is defined as $\boldsymbol{T} = -\frac{\Delta m_e}{\Delta t}\mathbf{v}_e$, where the negative sign is required since the thrust acts in a direction opposite to that of the velocity of the exhaust leaving the rocket. With one more approximation, $\frac{\Delta \mathbf{v}}{\Delta t} \approx \frac{d\mathbf{v}}{dt} = \mathbf{a}$ for small Δt, we arrive at an approximation to the "rocket equation"

$$m_r \mathbf{a} \approx \sum \mathbf{F}_{ext} + \mathbf{T}, \tag{8}$$

which can be used, as long as Δt is kept small, to predict the acceleration of a rocket subject to thrust $\mathbf{T}$ and a net external force $\sum \mathbf{F}_{ext}$.

Let us consider a rocket that is launched and moves only in a radial direction, outward from the Earth's surface. This radial direction coincides with "straight up" from our Earth-based perspective, and the rocket's motion can be considered to be all in one dimension. In unit vector notation, the vector quantities in Equation (8) are $\mathbf{a} = a\,\hat{\mathbf{j}}$, $\mathbf{T} = T\,\hat{\mathbf{j}}$, and $\sum \mathbf{F}_{ext} = -\frac{GM_E m_r}{(R_E+y)^2}\,\hat{\mathbf{j}}$, where we have assumed that the only external force acting on the rocket is the gravitational pull of the Earth. Accordingly, R_E is the radius of the Earth, M_E is the Earth's mass, G is the universal gravitational constant, and y is the altitude of the rocket measured relative to Earth's surface. Inserting these into Equation (8) we have, finally, an expression for the time-dependent acceleration of our rocket in 1D,

$$a(t) \approx -\frac{GM_E}{(R_E + y(t))^2} + \frac{T}{m_r(t)}. \tag{9}$$

In Equation (9) we have emphasized that m_r is time-dependent; that is, it decreases as the fuel in the rocket is used up. The time dependence of the mass can be found from the definition of thrust. The thrust has magnitude $T = \frac{\Delta m_e}{\Delta t} v_e$, or $T = -\frac{\Delta m_r}{\Delta t} v_e$, since $\Delta m_e = -\Delta m_r$. Using $\Delta m_r = m_r(t + \Delta t) - m_r(t)$, we arrive at

$$m_r(t + \Delta t) = m_r(t) - \frac{T}{v_e}\Delta t, \tag{10}$$

which relates the mass of the rocket and unburned fuel at time $t + \Delta T$ to that at the earlier time t. Equipped with Equations (9) and (10) we can make use of the Euler Method to approximate the rocket's velocity and position at each successive time step:

$$v(t + \Delta t) \approx v(t) + a(t)\Delta t \tag{11}$$

and

$$y(t + \Delta t) \approx y(t) + v(t)\Delta t. \tag{12}$$

Equations (9), (10), (11), and (12) constitute the set of equations that needs to be iteratively computed to predict the velocity and position of the rocket for future times.