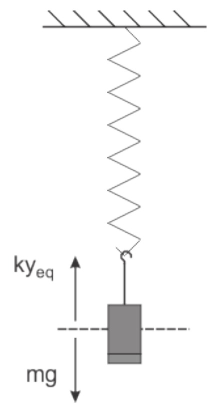
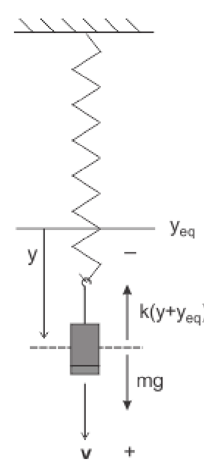
1 Simple Harmonic Oscillator



The figure above shows a mass-spring system in equilibrium. If the mass m is stationary, the system is in equilibrium, and the downward weight of the mass mg is balanced by an upward force ky_{eq} due to the spring, where k is the spring constant, and y_{eq} is the distance the spring has been stretched from its unstrained length. Applying Newton's 2nd law in the vertical direction produces a simple expression $y_{eq} = mg/k$, from which the equilibrium position can be calculated from the basic parameters of the system.

Now consider the next figure which shows a snapshot of the same spring-mass system at some instant after the mass has been set in motion.



At this instant the mass has a velocity of magnitude v in the downward direction, and is located a distance y below the equilibrium position. If the positive direction is taken to be vertically downward, applying Newton's 2nd Law produces

$$\sum F = mg - k(y + y_{eq}) = ma_y. \tag{1}$$

Since $y_{eq} = mg/k$, solving for a_y in Equation 1 yields

$$a_y = -\frac{k}{m}y, \tag{2}$$

where y is measured relative to the equilibrium position y_{eq}. The total energy of the system is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k\left(y + y_{eq}\right)^2 - mg\left(y + y_{eq}\right), \tag{3}$$

where the zero level for the gravitational potential energy was is taken to be at the position of the mass when the spring is in its unstrained configuration.

It is very easy to solve Equation 2 analytically for the displacement y as a function of time (via letting $a_y = \frac{d^2y}{dt^2}$ and solving the resulting second order differential equation). However, we will solve it computationally, and compare the results of our computational model with the exact analytical solution.

2 Euler Method

One computational approach is based on making a sound approximation to the time-dependent acceleration. If t is sufficiently small, then $a_y(t)$ will be approximately equal to its average value over the time interval Δt : $a_y\left(t\right) \approx a_{y,avg} = \frac{\Delta v_y}{\Delta t}$. Thus,

$$a_y\left(t\right) \approx \frac{v_y\left(t + \Delta t\right) - v_y\left(t\right)}{\Delta t}. \tag{4}$$

Solving Equation 4 for $v_y(t + \Delta t)$, we arrive at

$$v_y\left(t + \Delta t\right) \approx v_y\left(t\right) + a_y\left(t\right)\Delta t. \tag{5}$$

In a similar manner, if we approximate $v_y(t)$ by its average value over a small time interval Δt then the following equation results for the position at time $t + \Delta t$:

$$y\left(t + \Delta t\right) \approx y\left(t\right) + v_y\left(t\right)\Delta t. \tag{6}$$

This particular computational strategy is known as the *Euler method.* Equations 5 and 6 are called *finite difference equations*, and can be used to predict the values of the velocity and position at a later time $t + \Delta t$ if their respective values, and acceleration, are known at the previous time t. As long as Δt is sufficiently small, then these equations can be iterated a sufficient number of times to produce accurate predictions for the velocity and position at any future time.

For our specific consideration of the hanging SHO, we replace $a_y(t)$ in Equation 5 with the expression in Equation 2, obtained by applying Newton's 2nd Law. Thus,

$$v_y(t+\Delta t) \approx v_y(t) - \frac{k}{m} y(t) \Delta t. \tag{7}$$

Equations 6 and 7 constitute the Euler scheme for iteratively solving the hanging SHO. If we know the velocity and position of the mass m at time $t = 0$, then Equations 6 and 7 can be applied to predict their respective values at the later time $t + \Delta t$. We then use the values of velocity and position at time $t + \Delta t$ to predict those at time $t + 2\Delta t$ by again applying Equations 6 and 7. This process is repeated for a sufficient number of times (usually a very large number!) to attain predictions for the velocity and position at the desired future time t.

At this point it is convenient to change the notation from the explicit time dependence to an index-based format. Let $t_i = it$, where i is an integer known as the *time index.* $i = 0$ corresponds to $t = 0$, $i = 1$ to $t = \Delta t$,$i = 2$ to $t = 2\Delta t$, and so on. Then, instead of $v_y(t + \Delta t)$, we use the more economical form v_{i+1}, where we have also dropped the subscript y (since we are working in 1D we can get away with not explicitly labeling the component of the velocity). Using this index notation, Equations 6 and 7 become

$$y_{i+1} \approx y_i + v_i \Delta t, \tag{8}$$

and

$$v_{i+1} \approx v_i - \frac{k}{m} y_i \Delta t. \tag{9}$$

Equations 8 and 9 are the equations that need to be iterated (using small Δt) to predict the velocity and position of SHO for future times.

Without loss of generality, Equations 8 and 9 can also be written as

$$y_i \approx y_{i-1} + v_{i-1} \Delta t, \tag{10}$$

and

$$v_i \approx v_{i-1} - \frac{k}{m} y_{i-1} \Delta t. \tag{11}$$

3 Euler-Cromer Method

Another computational strategy is known as the modified Euler method, or the Euler-Cromer method. The Euler-Cromer algorithm can be obtained through a minor modification of the simpler Euler algorithm. Instead of using the velocity at the beginning of the time interval v_{i-1} to iteratively update the position, as in Equation 10, the value of the velocity at the end of the time interval v_i is used to update the position for each time step. Thus, the iterative equations for the Euler-Cromer method are

$$y_i \approx y_{i-1} + v_i \Delta t, \tag{12}$$

and

$$v_i \approx v_{i-1} - \frac{k}{m} y_{i-1} \Delta t. \tag{13}$$