Root Finding

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Introduction

In the simplest analytical case,

$$ax^2 + bx + c = 0$$

…produces the roots…

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We are essentially asking the question, “When does…,

$$f(x) = 0$$

Now it *is* possible to find roots graphically (i.e. where the function crosses the x–axis); however, it is difficult to find a precise solution.

The numerical techniques we will now discuss are conceptually simply, but can by tricky to implement properly in code.

Bracketing Methods

Bisection Method

{ First, choose brackets

$$x_l = 4$$
$$x_u = 6$$

Now perform the test, “does the function changes signs within the bracket?”, essentially,

$$f(x_u)f(x_l) < 0$$

If true, you can be certain if there is a root within the interval; however, if false, the interval can still contain a root.

Now approximate the root by **bisecting** the interval.

$$x_r = \frac{x_u + x_l}{2}$$

Now perform the bracket test between x_l and x_r; and x_u and x_r. Bisect the interval containing the root and continue.

False Position Method

A faster way to approach the root is with the **false position method**. This uses a line between x_u and x_l and chooses the next x_r where this line crosses zero.

Simplified:

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

The same bracket–checking conditions apply.

$$f(x_u)f(x_r) < 0$$
$$f(x_l)f(x_r) < 0$$

Simple Fixed Point Iteration

The **simple fixed point iteration** method is a better technique when an understanding of the function behavior is not available. First rearrange $f(x) = 0$ into the form $x = g(x)$. This can be done by simply adding x to both sides.

$$f(x) + x = 0 + x = x$$

Now iterate according to,

$$x_{i+1} = g(x_i)$$

While watching the **approximate error**,

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_i} \right|$$

If the error is getting smaller, the computation is converging on a root. If not, try a different initial point.

As stated above, this method is not guaranteed to converge.

Newton–Raphson Method

The **Newton–Raphson method** is similar to the simple fixed point iteration method, but uses the slope of the function (i.e. $f'(x_i)$) to find the next location to iterate.

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

Your next guess becomes,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$