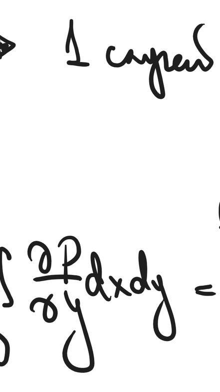
Основные формулы векторного анализа

Термины:

1. поле $\bar F=\{P(x,y,z),Q(x,y,z),R(x,y,z)\}\in C(G),\ G\in R^3$
2. $\operatorname{div}\bar F=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=(\nabla,F)$, где $\nabla=\left\{\frac{\partial}{\partial x};\frac{\partial}{\partial y};\frac{\partial}{\partial z}\right\}$ – оператор Гамильтона
   дивергенция поля
3. $\operatorname{rot}\bar F=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)i-\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)j+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)k=\nabla\times F$ – ротор поля
4. $\bar F$ – потенциальное, если $\exists\, u=u(x,y,z):\ \bar F=\operatorname{grad}u=\left\{\frac{\partial u}{\partial x};\frac{\partial u}{\partial y};\frac{\partial u}{\partial z}\right\}$
5. $\bar F$ – соленоидальным, если $\exists$ поле $\vec G$: $\bar F=\operatorname{rot}\bar G$
6. $\int_{\vec L}Pdx+Qdy+Rdz$ – работа поля $\bar F=\{P,Q,R\}$ вдоль ориентированной кривой
   $\int_{\vec L}Pdx+Qdy+Rdz=\int_{\vec L}(\bar F,d\bar r)$
7. $\iint_{\vec S}Pdydz+Qdzdx+Rdxdy=\iint_S(P\cos\alpha+Q\cos\beta+R\cos\gamma)ds=\iint_S(\bar F,\bar n_e)ds$ –, $\quad \bar n_e=\{\cos\alpha,\cos\beta,\cos\gamma\}$
   – поток векторного поля F через ориентир. поверхность S
8. Кривая $\bar r=\bar r(t)=\{x(t),y(t),z(t)\}$, $t\in[\alpha,\beta]$ наз. векторной линией поля $F=\{P,Q,R\}$, если
   $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=dt \qquad \begin{aligned}x'_t&=P\\ y'_t&=Q\\ z'_t&=R\end{aligned}$$

Формула Грина



$\bar F=\{P(x,y),Q(x,y)\},\ (x,y)\in\bar{\mathcal D},\ \bar F(x,y)\in C^1(\bar{\mathcal D})$

$\mathcal D$ – односвязный, измеримый компакт

Тогда $\int_{\vec L}P\,dx+Q\,dy=\iint_{\mathcal D}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy \quad (1)\ $ (Грина)

- 1 случай $\mathcal D=\mathcal D_{\text{ст.пох}}$ $=$

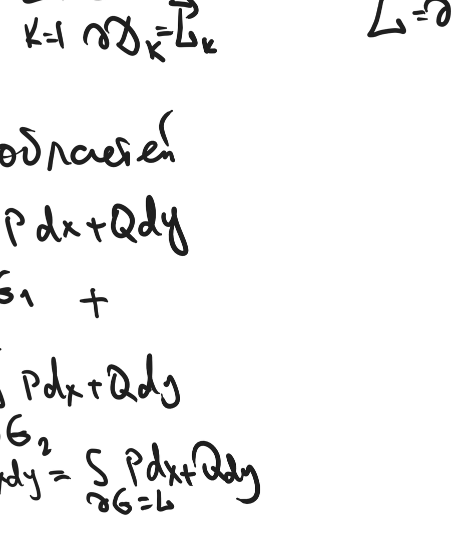
$$\iint_{\mathcal D}\frac{\partial P}{\partial y}dxdy=\int_a^b dx\int_{y=\varphi_1(x)}^{y=\varphi_2(x)}\frac{\partial P}{\partial y}dy=\int_a^b\big(P(x,\varphi_2(x))-P(x,\varphi_1(x))\big)dx=$$

$$=-\int_{\vec L_3}Pdx-\int_{\vec L_1}Pdx-\int_{\vec L_2}Pdx-\int_{\vec L_4}Pdx=-\int_{\vec L}Pdx \quad (*)$$

Аналогично, $\iint_{\mathcal D}\frac{\partial Q}{\partial x}dxdy=\int_L Q\,dy \quad **$

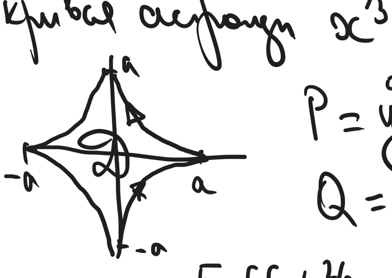
$$\iint_{\mathcal D}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_L Pdx+Qdy$$

Общий случай



$$\iint_{\mathcal D=\cup\mathcal D_k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\sum_k\iint_{D_k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\sum_{k=1}\int_{\partial\mathcal D_k=\vec L_k}Pdx+Qdy=\int_{L=\partial\mathcal D}Pdx+Qdy$$

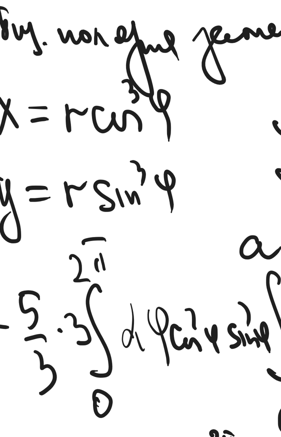
Формула Грина для многосвязных областей



G

$$\iint_{G_1}(Q'_x-P'_y)dxdy=\int_{\partial G_1}Pdx+Qdy$$
$$+\qquad +$$
$$\iint_{G_2}(Q'_x-P'_y)dxdy=\int_{\partial G_2}Pdx+Qdy$$
$$\iint_G(Q'_x-P'_y)dxdy=\int_{\partial G=L}Pdx+Qdy$$

Пример. Вычислить $I=\int_{\vec L}y^{5/3}dx-x^{5/3}dy$, где L – ориентированная кривая астроиды $x^{2/3}+y^{2/3}=a^{2/3}$



$$P=y^{5/3} \qquad \frac{\partial P}{\partial y}=\frac{5}{3}y^{2/3} \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-\frac{5}{3}\left(x^{2/3}+y^{2/3}\right)$$
$$Q=-x^{5/3} \qquad \frac{\partial Q}{\partial x}=-\frac{5}{3}x^{2/3}$$

$$I=-\frac{5}{3}\iint_{\mathcal D}\left(x^{2/3}+y^{2/3}\right)dxdy$$

Обобщ. полярные замены

$$\begin{cases}x=r\cos^3\varphi\\ y=r\sin^3\varphi\end{cases} \qquad J=3r\cos^2\varphi\sin^2\varphi$$

$x^{2/3}+y^{2/3}=a^{2/3}$

$r^{2/3}\cos^2\varphi+r^{2/3}\sin^2\varphi=a^{2/3}$

$r^{2/3}=a^{2/3}\to r=a$

$x^{2/3}+y^{2/3}=r^{2/3}$

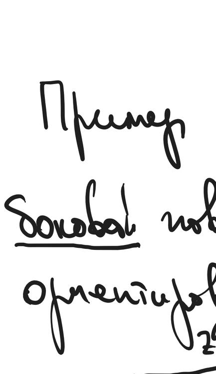
$$I=-\frac{5}{3}\cdot 3\int_0^{2\pi}d\varphi\cos^2\varphi\sin^2\varphi\int_0^a r^{2/3}\cdot r\,dr=$$

$$=-5\cdot\int_0^{2\pi}\sin^2\varphi\cos^2\varphi\,d\varphi\cdot 3\frac{r^{8/3}}{8}\Big|_0^a=-\frac{15}{8}a^{8/3}\cdot\frac{1}{4}\int_0^{2\pi}\underbrace{\sin^2 2\varphi}_{=\frac{1-\cancel{\cos 4\varphi}}{2}}d\varphi=-\frac{15}{32}a^{8/3}\pi$$

Формула Остроградского-Гаусса

$\partial G=\vec S$

n_e – внешняя нормаль

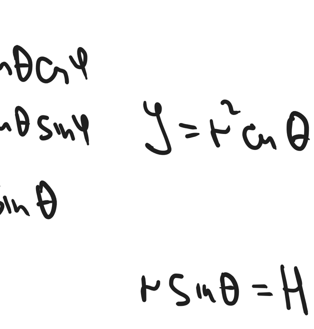


G

1) $\bar F=\{P,Q,R\}\in C^1(\bar G)$ 2) $\bar S=\partial G$ – гладкая граница

$$\iint_{\vec S}(\bar F,\bar n_e)ds=\iint_{\vec S}Pdydz+Qdzdx+Rdxdy=$$
$$=\iiint_G\operatorname{div}F\,dxdydz=\iiint_G\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$$

Пример. Найти поток векторного поля $\{x^3;y^3;z^3\}=\bar F$ по боковой поверхности конуса $H^2(x^2+y^2)=z^2R^2$, $0\le z\le H$, ориентированной внешней нормалью



$$\iint_{\vec S_1}(F,n_e^1)ds+\iint_{\vec S_2}(F,n_e^2)ds=\iiint_G\operatorname{div}F\,dxdydz \;-$$

$$\operatorname{div}F=3(x^2+y^2+z^2)$$

$$3\iiint_G(x^2+y^2+z^2)dxdydz=3\int_0^{2\pi}d\varphi\int_{\operatorname{arctg}\frac{H}{R}}^{\pi/2}\cos\theta\,d\theta\int_0^{H/\sin\theta}\underbrace{r^4dr}_{\frac{r^5}{5}}\;=\;\frac{3}{5}H^5\cdot 2\pi\int_{\operatorname{arctg}\frac{H}{R}}^{\pi/2}\frac{\cos\theta}{\sin^5\theta}\underbrace{d\theta}_{d\sin\theta}\;=$$

$x=r\cos\theta\cos\varphi$ $\qquad\varphi\in[0,2\pi)$

$y=r\cos\theta\sin\varphi$ $\quad J=r^2\cos\theta$ $\qquad\theta\in\left[\operatorname{arctg}\frac{H}{R};\frac{\pi}{2}\right]$

$z=r\sin\theta$ $\qquad r\in\left[0;\frac{H}{\sin\theta}\right]$

$r\sin\theta=H$

$$=\frac{3}{5}H^5\cdot 2\pi\cdot\left(-\frac{1}{4}\right)\frac{1}{\sin^4\theta}\Big|_{\operatorname{arctg}\frac{H}{R}}^{\pi/2}=-\frac{3}{10}\pi H^5\left[1-\frac{1}{\sin^4\left(\operatorname{arctg}\frac{H}{R}\right)}\right]=$$

$$=\left|\frac{1}{\sin^2\left(\operatorname{arctg}\frac{H}{R}\right)}=1+\operatorname{ctg}^2\left(\operatorname{arctg}\frac{H}{R}\right)=1+\frac{R^2}{H^2}=\frac{R^2+H^2}{H^2}\right|=-\frac{3}{10}\pi H^5\left[1-\frac{(R^2+H^2)^2}{H^4}\right]=$$

$$=-\frac{3\pi}{10}\pi H^5\left[\frac{\cancel{H^4}-R^4-2R^2H^2-\cancel{H^4}}{H^4}\right]=\frac{3\pi}{10}H\left(2H^2+R^2\right)R^2$$

Формула Стокса

$$\int_{\vec L}Pdx+Qdy+Rdz=\iint_{\vec S}(\operatorname{rot}\bar F,\bar n_e)ds \;-$$

– ф. Стокса

$\bar F \qquad L\in S$