Тройной интеграл.

$U_\delta(M) \setminus \partial G$ – связным многим $\Leftrightarrow M \in \partial G$

$u = f(x,y,z) \in C(\bar{G})$

G – измеримый, связный компакт.

ξ, G_ξ – ступенчатое тело, соотв. ξ, Π_σ, $\sigma \in \xi$, $d(\xi)$

$M\xi = \{M_\sigma \in \Pi_\sigma, \forall \sigma \in \xi\}$ – набор точек

$$S_{f,G}(\xi, M\xi) = \sum_{\sigma\in\xi} f(M_\sigma)\,\mu(\Pi_\sigma) \text{ – интегральные суммы}$$

$$\iiint_G f\,dxdydz = \lim_{d\xi\to 0} S_{f,G}(\xi, M\xi)$$

Т. (необх. условия интегрируемости f по G)

1) $f(x,y,z)$ – интегрируема на измеримом, связном компакте G

$\Rightarrow f(x,y,z)$ – ограничена на G: $\exists M>0$: $|f(x,y,z)| \le M$ $\forall (x,y,z)\in G$

Т. (дост. условия интегрируемости)

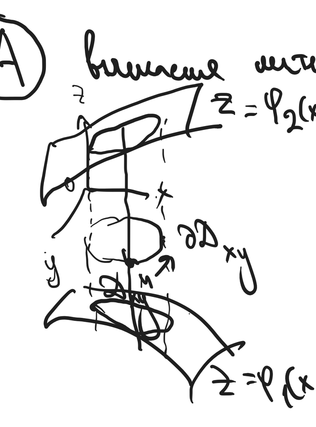
1) $f(x,y,z) \in C(G)$

2) G – измеримый, связный компакт.

$\Rightarrow$ Тогда $\exists \iiint_G f\,dxdydz$

Способы вычисления тройного интеграла

(A) вычисление интеграла для цилиндрических!



$z=\varphi_2(x,y) \in C(\mathcal{D}_{xy})$

$$\iiint_{G_y} f\,dxdydz = \iint_{\mathcal{D}_{xy}} dxdy \underbrace{\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz}_{\Phi(x,y)} \qquad (1)$$

$z=\varphi_1(x,y)$

1) $\varphi_1(x,y)=C_1$, $\varphi_2(x,y)=C_2$ – константы.

$[z_k; z_{k+1}]$

$C_1\ z_1\ z_2\ z_n\ C_2$

$$\iint_{\mathcal{D}_{xy}} dxdy \int_{C_1}^{C_2} f(x,y,z)dz = \sum_{k=1} \iint_{\mathcal{D}_{xy}} dxdy \underbrace{\int_{z_k}^{z_{k+1}} f(x,y,z)dz}_{\text{Т. о среднем}} = \sum_k \iint_{\mathcal{D}_{xy}} dxdy\, f(x,y,\bar{z}_k)\,\Delta z_k =$$

$\bar{z}_k \in [z_k, z_{k+1}]$

$$= \sum \Delta z_k \underbrace{\iint_{\mathcal{D}_{xy}} f(x,y,\bar{z}_k)dxdy}_{} = \sum_k \Delta z_k\Big(\iint_{\mathcal{D}\xi} f(x,y,\bar{z}_k)dxdy + \iint_{\mathcal{D}\setminus\mathcal{D}\xi} f(x,y,\bar{z}_k)\,dxdy\Big) =$$

$\mathcal{D}\xi = \cup\Pi_\sigma$

$$= \sum \Delta z_k \sum_{\sigma\in\xi} \underbrace{\iint_{\Pi_\sigma} f(x,y,\bar{z}_k)dxdy}_{\text{Т. о среднем}} + \underbrace{\sum \Delta z_k \iint_{\mathcal{D}\setminus\mathcal{D}\xi} f(x,y,\bar{z}_k)\,dxdy}_{\Phi\xi} =$$

$|f(x,y,z)| \le K$ $\forall (x,y,z)\in G_y$

$\exists \bar{M}_\sigma \in \Pi_\sigma$ $\bar{M}_\sigma = (\bar{x}_\sigma, \bar{y}_\sigma)$

$$= \sum_K \sum_\sigma \underbrace{f(\bar{x}_\sigma, \bar{y}_\sigma, \bar{z}_k)}_{f(\bar{M}_\sigma)}\cdot \underbrace{\mu(\Pi_\sigma)\cdot\Delta z_k}_{\mu(\bar{\Pi}_\sigma)} + \Phi\xi = S_{f,G_y}(\xi, \bar{M}\xi) + \Phi\xi - S_{f,G_y}(\xi,\bar{M}\xi) + o(1)$$

$S_{f,G_y}(\xi,\bar{M}\xi) \to \iiint_{G_y} f\,dxdydz$, $o(1) \to 0$

$$|\Phi\xi| \le \sum \Delta z_k \iint_{\mathcal{D}\setminus\mathcal{D}\xi} |f|\,dxdy \le K \cdot \sum_k \Delta z_k \cdot \mu(\mathcal{D}\setminus\mathcal{D}_\xi) = K\,\mu(\mathcal{D}\setminus\mathcal{D}_\xi)\,(C_2 - C_1) = o(1)$$

$\mu(\mathcal{D}\setminus\mathcal{D}_\xi) = o(1)$, $d\xi \to 0$

2. $\min_{D_{xy}} \varphi_1(x,y) = C_1$, $\max_{D_{xy}} \varphi_2(x,y) = C_2$

$\varphi_1(x,y)$ $\varphi_2(x,y)$ f

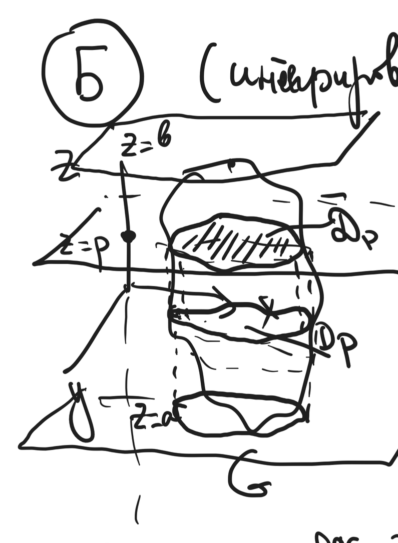
C_1 0 0 C_2

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z)\in G_y \\ 0, & (x,y,z)\in G_y^{C_1C_2}\setminus G_y \end{cases}$$

$$\iiint_{G_C^{C_1C_2}} \tilde{f}(x,y,z)\,dxdydz = \iint_{\mathcal{D}_{xy}} dxdy \int_{C_1}^{C_2} \tilde{f}(x,y,z)\,dz = \iint_{\mathcal{D}_{xy}} dxdy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz \quad (1)$$

$= \iiint_{G_C} f\,dxdydz$

(Б) (интегрирование по сечениям)



$(x,y,z)\in G$ $a \le z \le b$.

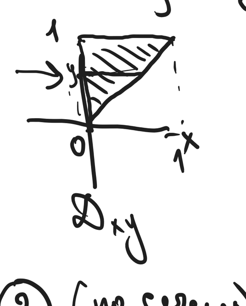
(x,y,p) – сечение $\bar{\mathcal{D}}_p$ $(x,y)\in\mathcal{D}_p$

$$\iiint_G f(x,y,z) = \int_a^b dp \iint_{\mathcal{D}p} f(x,y,p)\,dxdy \qquad (2)$$

<u>Пример</u> $\iiint_G x^2y^2\,dxdydz =$

$G = \{(x,y,z):\ \underbrace{0 \le x \le y}_{\mathcal{D}_{xy}} \le z \le 1\}$, $\varphi_1 = y$, $\varphi_2 = 1$

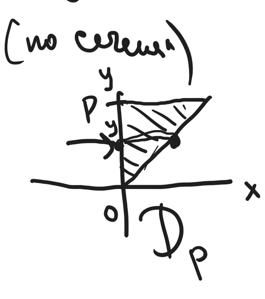
1) $= \iint_{\mathcal{D}_{xy}} dxdy \int_y^1 x^2y^2\,dz = \iint_{\mathcal{D}_{xy}} x^2y^2\,dxdy\,(1-y) = \iint_{\mathcal{D}_{xy}} x^2y^2(1-y)\,dxdy =$



$$= \int_0^1 y^2(1-y)dy \int_0^y x^2dx = \frac{1}{3}\int_0^1 y^5(1-y)dy = \frac{1}{3}\left[\frac{1}{6} - \frac{1}{7}\right] = \frac{1}{126}$$

2) (по сечениям) $\underbrace{0 \le x \le y \le p}_{} \le 1$

$0 \le p \le 1$



$$I = \int_0^1 dp \iint_{\mathcal{D}_p} x^2y^2\,dxdy = \int_0^1 dp \int_0^p dy \int_0^y x^2y^2\,dx = \int_0^1 dp \int_0^p y^2dy\, \frac{x^3}{3}\Big|_0^y =$$

$$= \frac{1}{3}\int_0^1 dp \int_0^p y^5\,dy = \frac{1}{3}\int_0^1 dp\, \frac{y^6}{6}\Big|_0^p = \frac{1}{3\cdot 6}\int_0^1 p^6\,dp = \frac{1}{3\cdot 6\cdot 7} = \frac{1}{126}$$

<u>Пример 2</u> $\iiint_G xyz\,dxdydz$, G – ограничено поверхностями:

$\underbrace{y=x^2,\ x=y^2}_{\mathcal{D}_{xy}}$, $\underbrace{z=xy}_{\varphi_2}$, $\underbrace{z=0}_{\varphi_1}$

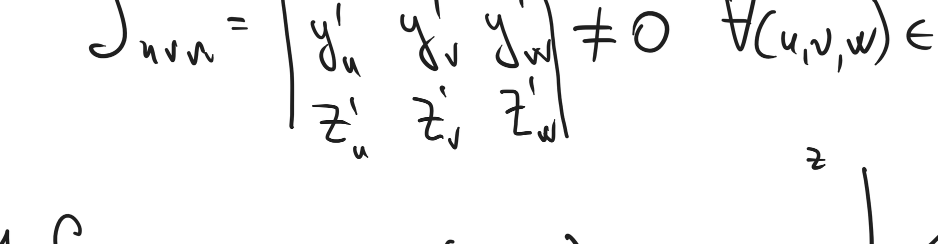
$$I = \iint_{\mathcal{D}_{xy}} dxdy \int_0^{xy} xyz\,dz =$$

$$= \frac{1}{2}\iint_{\mathcal{D}_{xy}} xy\,dxdy\ x^2y^2 = \frac{1}{2}\iint_{\mathcal{D}_{xy}} x^3y^3\,dxdy =$$

$$= \frac{1}{2}\int_0^1 dx\, x^3 \int_{y=x^2}^{y=\sqrt{x}} y^3\,dy = \frac{1}{2}\int_0^1 x^3 \cdot\frac{1}{4}\left(x^2 - x^8\right)dx =$$

$$= \frac{1}{8}\left(\frac{x^6}{6} - \frac{x^{12}}{12}\right)\Big|_0^1 = \frac{1}{8}\cdot\frac{1}{12} = \frac{1}{96}$$

Замена переменных в тройном интеграле



$$\bar{r}(u,v,w)\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \text{– замена перемен.}$$

G_{uvw} G_{xyz}

биекция $\bar{r}(\partial G_{uvw}) = \partial G_{xyz}$

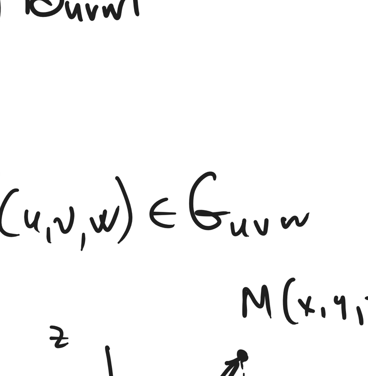
$\bar{r}(u,v) \in C^1(\mathcal{D}_{uv})$

$f(x,y,z) \in C(G_{xyz})$

$$\iiint_{G_{xyz}} f\,dxdydz = \iiint_{G_{uvw}} f(\bar{r}(u,v,w))\cdot|J_{uvw}|\,dudvdw$$

$$J_{uvw} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} \ne 0 \quad \forall (u,v,w)\in G_{uvw}$$

1. Сферическая замена (r,φ,θ)



$r \ge 0$, $0 \le \varphi < 2\pi$, $\theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases} \text{– сферическая} \qquad J_{r,\varphi,\theta} = r^2\cos\theta > 0$$

2. Цилиндрическая система (r,φ,h)

$$\begin{cases} 0 \le \varphi < 2\pi \\ r \ge 0 \\ h \in R \end{cases}$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \text{– цилиндр. замена} \qquad J_{r,\varphi,h} = r$$