

ANGULAR CORRELATIONS

MIKE MORAN

If we have an initial state with particles a and A with a channel spin of $(j^\pi)_1$, and the lighter particle a has an orbital angular momentum of ℓ_a , that creates a state J^π in a compound nucleus C that then decays into a second two-body system with particles b and B , where the new channel spin is $(j^\pi)_2$ and the lighter particle b has an orbital angular momentum ℓ_b , we can determine the angular correlation between the two light particles a and b in the center of mass frame.

To do this, we need to first look at the coupling between the various quantum states involved to see what possible quantum values there are for the orbital angular momenta. Note that, in most cases, only the two lowest values of ℓ_i are necessary, since higher values are drastically less likely to have an impact on the reaction. Following that, we can determine the contributions to the angular correlation arising from the mixed states (if they are present) and relate that to our differential cross section.

For the remainder of this discussion, I will be using the $^{27}\text{Al}(p, \alpha)^{24}\text{Mg}$ reaction and looking at two states identified in Andersen *et al.*, 1961: the isotropic $E_p = 1182$ keV state with $J^\pi = 2^+$, and the non-isotropic $E_p = 1315$ keV state with $J^\pi = 4^+$. The first of these states was measured during the February 2017 experiment, and the second state would not be reasonably detectable at 0° .

1. QUANTUM NUMBER SELECTION

Our quantum numbers are defined by the properties of the entrance and exit channels and the state within the compound nucleus we are trying to populate. We will approach these numbers by looking at them moving from the entrance channel to the compound nucleus to the exit channel. We are assuming here that all particles involved are nuclear particles and that there are no intervening γ s to consider.

1.1. Entrance Channel. The entrance channel has a spin-parity of $(j^\pi)_1$, determined by the spin and parity of the two separate particles. The channel spin is determined by

$$|j_a - j_A| \leq j_1 \leq j_a + j_A,$$

and the channel parity is determined by

$$\pi_1 = \pi_a \pi_A (-1)^{\ell_a}.$$

These equations also hold for the exit channel, after making the replacements $1 \rightarrow 2$, $a \rightarrow b$, and $A \rightarrow B$. The value of the relative angular momentum, taken here to be the orbital angular momentum of just the smaller particle, will be constrained by the compound nuclear state, but for now it can be considered a free parameter.

For our reaction, the proton has $J^\pi = 1/2^+$ and ^{27}Al has $J^\pi = 5/2^+$, leading to two possible channel spins: $j_1 = 2$ or 3 . Again, the parity cannot be determined, since we have not defined our orbital angular momentum.

1.2. Compound Nucleus. Within our compound nucleus, we select a single state with spin-parity J^π to populate. The energy kinematics would determine what that level's excitation energy is and thus what the required beam energy would be to populate the state, but that is not discussed here. Now that we know our intermediate state's properties, we can determine the final possible values for our entrance channel.

First, the entrance channel's ℓ_a can be set by following the triangular rule

$$|J - j_1| \leq \ell_a \leq J + j_1.$$

Since we also have the parity of our composite state, we can determine whether our ℓ_a values will be odd or even, since

$$\pi = \pi_a \pi_A (-1)^{\ell_a},$$

leading to the chain of possible values as $\ell, \ell+2, \dots, \ell+n$. As mentioned previous, we usually only have to worry about the two lowest angular momenta, ℓ and $\ell+2$, but in reality all would be considered. Again, these equations also hold for the exit channel, after making the same replacements as before.

For our example, the results of figuring out these possibilities are summarized in below. Note that all ℓ_p values are even, since our entrance channel and compound nuclear state parities are even.

$$\begin{aligned} J^\pi = 2^+ : j_1 = 2 \rightarrow \ell_p = 0, 2, \text{ and } 4, j_1 = 3 \rightarrow \ell_p = 2 \text{ and } 4 \\ J^\pi = 4^+ : j_1 = 2 \rightarrow \ell_p = 2, 4, \text{ and } 6, j_1 = 3 \rightarrow \ell_p = 2, 4, \text{ and } 6 \end{aligned}$$

1.3. Exit Channel. Our exit channel properties are determined the same way as our exit channel: finding the allowed j_2 based on the exit channel's two particles, determining the allowed ℓ_b values by looking at the triangle rule, and selecting states where the parities of the compound nuclear state and the exit channel are the same. From here, we have assigned all possible quantum numbers describing our reaction.

Our exit channel consists of our α particle and ^{24}Mg in its ground state, both of which are $J^\pi = 0^+$. If we were instead looking at the α_1 channel, populating the first excited state in ^{24}Mg , we would use the spin-parity (2^+) of that state and would need to handle the outgoing γ accordingly. In our case, our exit channel can only be $(j^\pi)_2 = 0^+$, which simplifies things greatly, by only allowing our α particle to have $\ell_\alpha = 2$ (for our 2^+ state) or $\ell_\alpha = 4$ (for our 4^+ state).

1.4. Final Description. To condense our quantum number representation down, I will introduce the following notation describing the sequence of angular momenta:

$$j_1(L_1)J(L_2)j_2,$$

where we can make the substitution $L_i = \ell_{a|b}$ since we are assuming that all of the orbital angular momentum is contained within the smaller particle within each channel. For our two example resonances, we can write our allowed quantum number sequences as

$$2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} 2(2)0 \text{ and } 3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} 2(2)0$$

for our $J^\pi = 2^+$ resonance (note that we have limited ourselves to just the two lowest ℓ_p numbers), and

$$2 \binom{2}{4} 4(4)0 \text{ and } 3 \binom{2}{4} 4(4)0$$

for our $J^\pi = 4^+$ resonance. Now that we have our allowed quantum numbers, we can move on to actually calculating the angular distributions.

2. ANGULAR CORRELATION

The angular correlation between two detected particles, one incoming and one outgoing, can be expressed as a sum of Legendre polynomials over the angle between the two particles, given by

$$(2.0.1) \quad W(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n P_n(\cos \theta),$$

where $P_n(x)$ are the Legendre polynomials. Due to symmetry concerns, we only have to worry about even- n terms within the sum, which reduces this down to fewer terms. The upper limit n_{\max} is based on the specific values of the various angular momenta involved, and the coefficients b_n will be discussed later. This distribution can be related to the differential and total cross section through

$$\left(\frac{d\sigma}{d\Omega} \right)_\theta = \frac{1}{4\pi} \sigma W(\theta).$$

If we have an isotropic angular distribution, which is the current status for our two resonances, we must have $W(\theta) = 1$.

The coefficients b_n are also based on the angular momentum states involved, which include terms describing the vector coupling between the various states. We will use the coefficients F_n defined in Biedenharn, 1960:

$$F_n(LL'jJ) = (-1)^{j-J-1} \sqrt{(2L+1)(2L'+1)(2J+1)} \langle L1L'-1|n0 \rangle W(JJLL';nj),$$

where $\langle L1m_1L2m_2|JM \rangle$ are Clebsch-Gordan coefficients, and $W(j_1j_2j_3j_4; j_5j_6)$ are Racah coefficients. These coefficients can either be found in a tabulated resource or calculated. For our expansion coefficients b_n , we'll have an F_n for the entrance and exit channels.

Since our entrance channel has two possible spin states, we will have to consider the mixing between those two states. The total angular correlation, taking that mixing into account, is given by

$$W(\theta) = W_{j_s}(\theta) + \delta_c^2 W_{j'_s}(\theta),$$

where $\delta_c^2 = P_{j'_s}/P_{j_s}$ is the channel spin mixing ratio, defined as the ratio between the probabilities for forming or decaying from the intermediate state through the two spins. To note, we take $j'_s > j_s$, which is the standard convention.

At this point, we have all of our scaffolding in place, so we just need to bring all of the pieces together. Since our entrance channel can form the compound state through two different orbital angular momenta ℓ_1 , we also need to consider coherent interference between those two possibilities.

For a simple two-step process, our expansion coefficients are given by

$$b_n = a_n(1)A_n(1)a_n(2)A_n(2),$$

where the factors $a_n(i)$ and $A_n(i)$ are determined by the angular momentum of the entrance and exit channels. In our case, our entrance channel has coherent interference between the various orbital angular momentum possibilities, while our exit channel progresses through pure radiation. I will discuss the simpler case first.

2.1. Pure Radiation. In a pure radiation, where only a single orbital angular momentum takes part, there are no alterations to the coefficients, and they are only determined by the properties of the particles involved. For $s = 0$ particles, which includes our α particle, the coefficients are

$$a_n(i) = \frac{2L_i(L_i + 1)}{2L_i(L_i + 1) - n(n + 1)}$$

and

$$A_n(i) = F_n(L_i j_i J) = F_n(L_i L_i j_i J).$$

If we had $s \neq 0$ particles, we would need to use the possible channel spin quantum numbers instead of just the initial state spin, something that we have already taken into account. The sum over the Legendre polynomials given in (2.0.1) is restricted to $0 \leq n \leq \min(2L_1, 2L_2, 2J)$.

2.2. Mixed Radiation. For a channel where multiple orbital angular momenta take part in the formation or decay from a composite state, we have an orbital angular momentum mixing ratio $\delta_i^2 = \Gamma_{\ell_1}/\Gamma_{\ell_2}$, where Γ_{ℓ} being the particle partial width for a given orbital angular momentum. For these channels, the coefficients $a_n(i)A_n(i)$ are replaced with

$$a_n(L_i L_i)F_n(L_i j_i J) + 2\delta_i a_n(L_i L'_i)F_n(L_i L'_i j_i J) + \delta_i^2 a_n(L'_i L'_i)F_n(L'_i j_i J),$$

where we are assuming that $L'_i > L_i$ and, as mentioned before, we use the channel spin j_i instead of the state spin.

The values of $a_n(i)$ are also changed to represent the mixing between the orbital angular momenta. Instead of the definition used previously, we use

$$\begin{aligned} a_n(L_i L'_i) &= \cos(\zeta_{L_i} - \zeta_{L'_i}) \frac{\langle L_i 0 L'_i 0 | n 0 \rangle}{\langle L_i 1 L'_i -1 | n 0 \rangle} \\ &= \cos(\zeta_{L_i} - \zeta_{L'_i}) \frac{2\sqrt{L_i L'_i (L_i + 1)(L'_i + 1)}}{L_i(L_i + 1) + L'_i(L'_i + 1) - n(n + 1)}. \end{aligned}$$

The charge particle phase shifts ζ_L are given by

$$\zeta_L = -\arctan\left(\frac{F_L}{G_L}\right) + \sum_{n=1}^L \arctan\left(\frac{\eta}{n}\right),$$

where F_L and G_L are the regular and irregular Coulomb wave functions, respectively, and $\eta = Z_a Z_B e^2 / (\hbar v)$ is the Sommerfeld parameter. The Coulomb wave functions can either be found in a tabulation or calculated using a program.

2.3. Combining Everything. Since the full equation for our reaction, combining all of the above pieces, would easily fill a half page and not be any more intelligible than the above breakdown, I will not lay out the entire thing here. Instead, I will describe how the various pieces fit together.

Looking at both resonances, our $^{27}\text{Al}(p, \alpha)^{24}\text{Mg}$ reaction consists of an entrance channel that exhibits mixing between orbital angular momentum states and between channel spin states, and an exit channel that progresses through a single

pure radiation channel. We can write out each part by hand if we desired, but since there are a large number of coefficients that we would need to calculate anyway, it is easier to do the entire calculation through code.

Since the Coulomb wave functions are radius-dependent, we can also leave the $\cos(\zeta_{L_i} - \zeta_{L'_i})$ term unevaluated. In addition, since higher orbital angular momentum terms decrease the penetration factor for the reaction, we could even simplify the orbital angular momentum mixing to not be included, meaning that the angular correlation would, in our case, just be based on the two different entrance channel spin states.

3. CODE

Since we need to calculate the various coefficients in order to determine if there is an angular correlation between the particles, and to see if fitted experimental data matches up with the theory, we will use a short piece of code to perform those calculations. The code below is written in **python**. In addition, a separate package **angular** was written, making use of additional mathematical packages: **mpmath**, **numpy**, and **sympy**, to perform all of the required coefficient calculations.

The code in place gives the coefficients present for each Legendre polynomial, as further expansion can be handled by the end user. In addition, the coefficients being displayed separately allow for an easier relation to previous experimental results.

```
import angular
```

```
# check against Iliadis - should give [1, 2]
coeff = angular.angular_correlation(0, 1, 1, 1, 0)
print(coeff)
```