MBDyn Theory
and Developer's Manual
Version develop

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Chapter 1

Introduction

This document describes the formulation MBDyn, the free general-purpose multibody dynamics software, relies on. It also describes implementation-related aspects.

The document is far from complete; in fact, its preparation started at a late stage of the project, when many parts of the software were already completed and in use. From that time on, many modifications and improvements have been steadily documented and, occasionally, the theory manual anticipated the development (as should always happen, I know).

The Developers committed themselves to keeping it at least up to date with new developments; undocumented stuff should be documented as soon as it needs modifications or refactoring.

Alessandro Fumagalli, Marco Morandini and Mattia Mattaboni contributed significantly to this document.

Chapter 2

Parsing

In MBDyn there are different levels of parsing. Input file parsing suffers from a scattered and occasionally outdated design. The need to preserve backwards compatibility with existing models restrained so far from entirely redesign it, although selected improvements occur over time.

MBDyn provides support for the implementation of new functionalities, and to parse their input. Parsing is delegated to a dedicated object, the MBDynParser, which inherits from the HighParser, a higher-level parsing object with bits of MBDyn's syntax built-in in a somewhat modular way. It exploits the functionalities of the LowParser, a lower-level parsing object that deals with tokenizing an input stream based on the expected tokens. Whenever appropriate (e.g. whenever a number is expected), control is delegated to the MathParser, which allows to parse and evaluate sequences of mathematical expressions, including variable declaration and definition. Variables are saved in a table as soon as they appear, and can be used by subsequent expressions when the mathematical parser is called again.

The traditional approach to data parsing consists in defining a table of keywords that are looked up and, based on the corresponding key code, by executing the appropriate code in a switch-case block.

The code is being gradually moved to a newer approach based on sets of associative arrays that map keywords to the functional objects that are used to parse the related items.

Although no significant improvement results in parsing of existing data types, this approach allows to register new data types run-time, e.g. from a run-time loaded module, thus easing the extension and the customization of the code.

2.1 HighParser

2.1.1 Traditional Usage

The `HighParser` class and its descendants use a `KeyTable` object containing a list of legal keywords to return a valid keyword index when `HighParser::GetWord()`, and significantly `HighParser::GetDescription()` are invoked. The `KeyTable` can be changed during parsing. `KeyTable` is a class. Its constructor takes a pointer to an array of strings and a reference to the `HighParser` object. The last string in the array must be null. The `KeyTable` class constructor keeps track of previous `KeyTable` objects in the `HighParser`, and restores them upon destruction.

The suggested usage inside a stacked call sequence of parsing functions is

```
Part *
read_part(HighParser& HP)
{
    // prepare names
```

```
    enum KeyWord { KEYWORD1, KEYWORD2, KEYWORD_LAST };
    char *key_table_array[] = { "keyword1", "keyword2", 0 };
    // build KeyTable class
    KeyTable k(HP, key_table_array);
    Part *returned_object = 0;
    // parse input
    do {
        switch (KeyWord(HP.GetWord())) {
        default:
            // do something...
            break;
        case KEYWORD1:
            // ...and build returned_object
            return returned_object;
        }
    } while (true);
}

void
read_all(HighParser& HP)
{
    // prepare names
    enum KeyWord { KEY1, KEY2, PART, KEY_LAST };
    char *keytable[] = { "key1", "key2", "part", 0 };
    // build KeyTable class
    KeyTable k(HP, keytable);
    // do something
    Part *part = 0;
    do {
        switch (KeyWord(HP.GetWord())) {
        default:
            // do something...
            break;
        case PART:
            // read part
            Part *part = read_part(HP);
            break;
        }
    } while (true);
}
```

Here the `KeyTable` set by function `read_all()` is automatically restored after the call to `read_part()`; `read_part()` temporarily changes the `KeyTable` used by the parser.

2.1.2 Table-Driven Usage

The table-driven usage is based on defining a functional object that is able to parse a data type:

```
    class Datum {
    public:
```

```
    virtual ~Datum(void) {};
};

struct DatumRead {
    virtual ~DatumRead(void) {};
    virtual Datum *Read(MBDynParser &HP) const = 0;
};
```

and a container for its descendants:

```
typedef std::string KeyType;
typedef std::map<KeyType, DatumRead *> DatumMapType;
DatumMapType DatumMap;
```

Then, a functional object for each specific datum type is derived from `DatumRead` ...

```
class MyDatum : public Datum {
    // ...
public:
    Datum(int);
    // ...
}

struct MyDatumRead : public DatumRead {
    Datum *Read(MBDynParser &HP) const {
        return new MyDatum(HP.GetInt());
    };
};
```

... and stored exactly once into the associative container by means of a dedicated helper:

```
bool
SetDatumRead(KeyType key, DatumRead *rf)
{
    return DatumMap.insert(DatumMapType::value_type(key, rf)).second;
}

// somewhere early in the code ...
SetDatumRead("mydatum", new MyDatumRead);

// note: somewhere else later, in the code, place
for (DatumMapType::iterator i = DatumMap.begin();
    i != DatumMap.end();
    i++)
{
    delete i->second;
}
DatumMap.clear();
```

The parsing function is something like

```
Datum *
ReadDatum(MBDynParser &HP)
```

```
{
    KeyType key(HP.GetStringWithDelims());
    if (key.c_str() == 0) {
        // error ...
        return 0;
    }
    DatumMapType::iterator i = DatumMap.find(key);
    if (i == DatumMap.end()) {
        // error ...
        return 0;
    }
    return i->second->Read(HP);
}
```

Only the `MyDatum` portions need be added for each new datum type; they can be declared, defined and registered anywhere in the code, including in run-time loaded modules.

Currently, drives, constitutive laws and scalar functions are handled according to this scheme; examples are provided in `modules/module-drive/`, `modules/module-constlaw/` and `modules/module-scalarfunc/`. More types will be reworked accordingly.

2.2 LowParser

...

2.3 MathParser

...

Chapter 3

Solvers

...

3.1 Matrix classes

3.2 Sparse matrices

3.3 Linear solvers

LinearSolver classes wraps linear solvers
SolutionManagers classes deals with the solution of linear systems. They own a pointer to a LinearSolver, where they allocate the underlying linear solver. SolutionManager::MatrInitialize() is called when the structure of the underlying spares matrix changes. SolutionManager::MatrReset(), is called to delete a factorization. It usually calls LinearSolver::Reset(). The matrix has to be explicitly zeroed before a Jacobian matrix assembly.

3.4 Non linear solvers

...

3.5 Parallel solver

3.5.1 Partitioning

`iTotVertices` is equal to the sum of nodes and elements. It is made in this way because we want the partitioner to generate a twofold subdivision:

- a subdivision related to elements; this subdivision is done in order to share the computational load during the assembly phase;
- a subdivision related to nodes, which is necessary for the solving phase with the substructuring method.

Of course this two partitions must be connected, so we build created a graph which is made of nodes and elements as vertex. The connection between vertexes are only between nodes and elements. There is no node to node or element to element connection.

`pVertexWgts` contains what we call the computational weight of each entity, so nodes have weight null while elements has a weight related to the dimension of the submatrix of the Jacobian matrix assembled by each one of them.

`pCommWgts` contains the communication weights (see metis documentation) which are a measure of the quantity of data which needs to be sent if the i-th vertex is part of an interface between different partitions. This means that nodes have a `CommWgts` equal to the number of DOFs, while elements have a weight equal to any internal DOFs they possess.

3.6 Convergence check

MDByn needs to solve a set of nonlinear equations

$$\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}, \tag{3.1}$$

where $\boldsymbol{f}$ is the so-called residual vector. The standard `norm` convergence check is based on the norm

$$f = \sqrt{\boldsymbol{f}^T \boldsymbol{f}} \tag{3.2}$$

of vector $\boldsymbol{f}$. While this often works in practice it has two shortcoming:

1. one doesn not know the generalized forces magnitude of the model, that is, a residual norm equal to 1 can be perfectly fine if the order of magnitude of the generalized forces transmitted by the structure is of 1E6, while is not ok if the order of magnitude is 10;

2. different components of vector $\boldsymbol{f}$ do have different physical dimensions, and it makes no sense to sum them together.

The `relnorm` test is designed to solve the first shorcoming. In addition to assembing the residual component f_i as

$$f_i = \sum_{e=1}^{\text{elements of the model}} f_{i(e)}(\boldsymbol{x}), \tag{3.3}$$

also the absolute values of the different contribution from the elements are assembled into a new vector $\tilde{\boldsymbol{f}}$ with components

$$\tilde{f}_i = \sum_{e=1}^{\text{elements of the model}} |f_{i(e)}(\boldsymbol{x})|. \tag{3.4}$$

The convergence test is then performed over

$$f = \frac{\sqrt{\boldsymbol{f}^T \boldsymbol{f}}}{\sqrt{\tilde{\boldsymbol{f}}^T \tilde{\boldsymbol{f}}}} \tag{3.5}$$

so that the residual is automatically scaled with respect to the generalized force magnitude appearing in the model.

The `sepnorm` test tries to solve both issues. Beside assembling the additional vector $\tilde{\boldsymbol{f}}$, just like the `relnorm` test, it partitions the equations into N independent set $\boldsymbol{f}_{[j]}$ and $\tilde{\boldsymbol{f}}_{[j]}$, with $j \in [1, N]$, one foreach physical dimension apperaring in the residual vector, so that every components $\boldsymbol{f}_{i[j]}$ of the subvector $\boldsymbol{f}_{[j]}$ has the same physical dimension. The relative residual test is then computed independently for each set

$$f_{[j]} = \frac{\sqrt{\boldsymbol{f}_{[j]}^T \boldsymbol{f}_{[j]}}}{\sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}}} \tag{3.6}$$

and the test returns the worst $f_{[j]}$, $\max_j(f_{[j]})$.

In order to identify the set of equations that are inactive (for which either $\sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}} = 0$ or $f_{[j]} \approx 1$ because the norms are very small and comes mostly from numerical errors),

1. $f_{[j]} = 0$ if $\sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}} = 0$
   AND
2. $f_{[j]} = 0$ if $\sqrt{\boldsymbol{f}_{[j]}^T \boldsymbol{f}_{[j]}} / \sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}} > \epsilon_1$ and $\sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}} < \epsilon_2 * \max_j \left(\sqrt{\tilde{\boldsymbol{f}}_{[j]}^T \tilde{\boldsymbol{f}}_{[j]}} \right)$

where the coefficients ϵ_1 and ϵ_2 are hard coded, $\epsilon_1 = 1.E - 1$ and $\epsilon_2 = 1.E - 5$. One could consider, in the future, to allow reading them from the input file.

Chapter 4

Orientation Handling

The orientation of structural nodes is handled by storing the orientation matrix of each node, and updating it during prediction and correction by incremental orientations.

The code uses Gibbs-Rodrigues parameters to account for incremental rotations. These parameters allow very efficient implementation, since they only require few algebraic operations to obtain the orientation matrix and related entities. However, they require the absolute value of the orientation to be limited; this requirement is usually overcome by accuracy requirements, so it is not viewed as a strong limitation.

One main advantage of this approach is that only three parameters are required to handle orientations. A main drawback of this approach is that it is quite difficult to keep track of how large a change in orientation occurred to a given node, since simply converting the orientation matrix into any other representation of a finite rotation typically results in the minimal rotation that yields that matrix.

MBDyn can output rotations in the following representations:

- Euler angles according to the 123 sequence;
- the orientation vector;
- the orientation matrix.

4.1 Euler Angles

Consider an orientation expressed by matrix $\boldsymbol{R}$. It can be expressed as the result of a precise sequence of rotations about three distinct axes. The 123 sequence means that matrix $\boldsymbol{R}$ is obtained as a sequence of three rotations: the first about axis 1; the second about axis 2 as it results after the first rotation; the third about axis 3 as it results after the second rotation, namely:

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \tag{4.1}$$

$$\boldsymbol{R}_2 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \tag{4.2}$$

$$\boldsymbol{R}_3 = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.3}$$

piled up, must equal $\boldsymbol{R}$:

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3 \qquad (4.4)\\ &= \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{bmatrix}.\end{aligned}$$

The angles can then be computed by means of a simple, partially recursive algorithm:

$$\alpha = -\tan^{-1}\left(\frac{\boldsymbol{R}_{23}}{\boldsymbol{R}_{33}}\right) \qquad (4.5)$$

$$\beta = \tan^{-1}\left(\frac{\boldsymbol{R}_{13}}{\cos\alpha\boldsymbol{R}_{33}-\sin\alpha\boldsymbol{R}_{23}}\right) \qquad (4.6)$$

$$\gamma = \tan^{-1}\left(\frac{\cos\alpha\boldsymbol{R}_{21}+\sin\alpha\boldsymbol{R}_{31}}{\cos\alpha\boldsymbol{R}_{22}+\sin\alpha\boldsymbol{R}_{32}}\right) \qquad (4.7)$$

The use of the function `atan2(double dSin, double dCos)` eliminates the risk of divisions by zero or of excessive loss of precision.

4.2 Orientation Vector

An orientation matrix can be represented as a finite rotation about an axis, in the form

$$\boldsymbol{R} = \boldsymbol{I} + \sin\phi\,\boldsymbol{n}\times + (1-\cos\phi)\,\boldsymbol{n}\times\boldsymbol{n}\times \qquad (4.8)$$

where ϕ is the amplitude of the rotation and $\boldsymbol{n}$ is the unit vector that represents the rotation axis.

Simple algebra manipulation shows that the trace of the orientation matrix corresponds to

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{R}) &= \mathrm{tr}\,(\boldsymbol{I}) + (1-\cos\phi)\,\mathrm{tr}\,(\boldsymbol{n}\times\boldsymbol{n}\times) \qquad (4.9)\\ &= 3 - 2\,(1-\cos\phi)\,, \qquad (4.10)\end{aligned}$$

since

$$\boldsymbol{n}\times\boldsymbol{n}\times = \boldsymbol{n}\boldsymbol{n}^T - \boldsymbol{I}, \qquad (4.11)$$

so

$$\cos\phi = \frac{\mathrm{tr}\,(\boldsymbol{R})-1}{2}. \qquad (4.12)$$

At the same time, the skew-symmetric portion of $\boldsymbol{R}$ is

$$\mathrm{skw}\,(\boldsymbol{R}) = \sin\phi\,\boldsymbol{n}\times\,, \qquad (4.13)$$

so

$$\mathrm{ax}\,(\mathrm{skw}\,(\boldsymbol{R})) = \sin\phi\,\boldsymbol{n} \qquad (4.14)$$

which yields the amplitude of the rotation,

$$\sin\phi = \mathrm{norm}\,(\mathrm{ax}\,(\mathrm{skw}\,(\boldsymbol{R})))\,, \qquad (4.15)$$

and the direction of the rotation axis,

$$\boldsymbol{n} = \frac{1}{\sin\phi}\operatorname{ax}\left(\operatorname{skw}\left(\boldsymbol{R}\right)\right), \tag{4.16}$$

while ϕ can be computed from its sine and cosine as

$$\phi = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi}\right). \tag{4.17}$$

Chapter 5

Integration

differential variable: a variable is declared differential in `SimulationEntity::GetDofType()` by returning `DofOrder::DIFFERENTIAL`. The increment of the value of a differential variable is equal to $\Delta x = \texttt{dCoef}\Delta\dot{x}$. When writing the Jacobian matrix, this must be considered; as a consequence, for an equation $f = 0$ (the residual is $-f$) the linearization is $f_{/\dot{x}} + f_{/x} * \texttt{dCoef}) * \Delta\dot{x} = -f$, as $\texttt{dCoef} * \Delta\dot{x} = \Delta x$.

algebraic variable: a variable is declared algebraic in `SimulationEntity::GetDofType()` by returning `DofOrder::ALGEBRAIC`. The increment of the value of an algebraic variable is the increment of the variable.

differential equation: an equation is declared differential in `SimulationEntity::GetEqType()` by returning `DofOrder::DIFFERENTIAL`. An equation $f = 0$ must be declared differential if $f_{/\dot{x}}$ is not null. The residual is $-f$, and its linearization is: $\left(f_{/\dot{x}} + f_{/x} * \texttt{dCoef}\right) * \Delta\dot{x} = -f$.

algebraic equation: an equation is declared algebraic in `SimulationEntity::GetEqType()` by returning `DofOrder::ALGEBRAIC`. An equation $f = 0$ can be declared algebraic iff $f_{/\dot{x}}$ is structurally null (e.g. regardless of the values the state may assume) and x is not algebraic. If an equation $f\left(x, t\right) = 0$ is declared differential, the residual is $-f$, and its linearization is: $f_{/x} * \texttt{dCoef} * \Delta\dot{x} = -f$. If the equation can be declared algebraic, it can be divided by `dCoef`: $f/\texttt{dCoef} = 0$, with residual $-f/\texttt{dCoef}$, and linearization $f_{/x} * \Delta\dot{x} = -f/\texttt{dCoef}$. This helps scaling the equations. Clearly, this has no sense if x is algebraic, or if $f_{/\dot{x}} \neq 0$.

5.1 Nodal rotation

The rotational DOF unknown during `AssRes()` and `AssJac()` are the increment of (Gibbs-Rodrigues) rotation parameters with respect to the reference configuration. Of course the increment of the parameter is $\Delta\boldsymbol{g} = \texttt{dCoef}\Delta\dot{\boldsymbol{g}}$. The increment of angular velocity is $\Delta\boldsymbol{\omega} = \boldsymbol{G}\Delta\dot{\boldsymbol{g}} + \Delta\boldsymbol{G}\dot{\boldsymbol{g}} - \boldsymbol{\omega}_{\text{ref}} \times \boldsymbol{G}\Delta\boldsymbol{g}$, where $\boldsymbol{G}(\boldsymbol{g})$ is the tensor relating $\boldsymbol{g}_\delta$ to $\delta\boldsymbol{g}$, and $\boldsymbol{\omega}_{\text{ref}}$ is the nodal reference angular velocity (Wref). We assume $\boldsymbol{G} = \boldsymbol{I}$ and $\Delta\boldsymbol{G} = \boldsymbol{0}$, so that $\Delta\boldsymbol{\omega} = \Delta\dot{\boldsymbol{g}} - \boldsymbol{\omega}_{\text{ref}} \times \Delta\boldsymbol{g}$ and so $\Delta\boldsymbol{\omega} = \Delta\dot{\boldsymbol{g}} - \boldsymbol{\omega}_{\text{ref}} \times \Delta\dot{\boldsymbol{g}} * \texttt{dCoef}$.

5.2 Integrators

Most of MBDyn's integrators are documented in [1, 2], that are distributed verbatim in Appendix **??**, respectively.

Chapter 6

Solution Phases

6.1 Initial Assembly

This phase only involves some of the structural elements. It is intended to ensure that the initial configuration and velocity complies with the constraint equations. It is not performed if the `control data` block contains the statement `skip initial joint assembly`.

To allow non-compliant system analysis, the initial configuration and velocity can be changed by this phase. To this purpose, the nodal positions, orientations, velocities and angular velocities are grounded by dummy springs, acting as penalty functions. The springs can be set on a node basis, and separately for configuration and velocity, to allow to selectively enforce the initial configuration.

The problem can be stated as follows:

$$\boldsymbol{K}\boldsymbol{x} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} + \boldsymbol{A}^T \boldsymbol{\lambda}_{\boldsymbol{A}} = \boldsymbol{K}\boldsymbol{x}_0 + \boldsymbol{f} \tag{6.1}$$

$$\boldsymbol{C}\boldsymbol{v} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{C}\boldsymbol{v}_0 \tag{6.2}$$

$$\boldsymbol{\phi}\left(\boldsymbol{x}, t\right) = \boldsymbol{0} \tag{6.3}$$

$$\boldsymbol{A}\left(\boldsymbol{x}, t\right)\boldsymbol{v} + \boldsymbol{b}\left(\boldsymbol{x}, t\right) = \boldsymbol{0} \tag{6.4}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}} v + \boldsymbol{\phi}_{/t} = \boldsymbol{0} \tag{6.5}$$

where Equation (6.3) and (6.4) respectively contain the holonomic and non-holonomic constraint equations. The solution of this problem leads to the direct determination of an initial configuration and velocity that complies with the constraints.

Unfortunately, this requires the implementation of more constraints than required by the regular solution phases, namely the time derivative of the algebraic constraints that depend only on the configuration.

NOTE: Rationale. The rationale of an initial assembly procedure is to drive the problem in a configuration that complies with the kinematic constraints, starting from the configuration that was defined during the input.

Constraint equations can be partitioned in holonomic,

$$\boldsymbol{\phi}(\boldsymbol{x}, t) = \boldsymbol{0} \tag{6.6}$$

and non-holonomic, which are usually expressed as linear in the coordinate derivatives,

$$\boldsymbol{A}(\boldsymbol{x}, t)\dot{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{0} \tag{6.7}$$

It is legitimate that $\boldsymbol{\phi}(\boldsymbol{x}_0^{(0)}, t_0) \neq \boldsymbol{0}$, or that $\boldsymbol{A}(\boldsymbol{x}_0^{(0)}, t_0)\boldsymbol{v}_0^{(0)} + \boldsymbol{b}(\boldsymbol{x}_0^{(0)}, t_0) \neq \boldsymbol{0}$, where the subscript 0 indicates the initial condition, and the superscript (0) indicates the values provided at input. In such cases, a procedure is sought that provides $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$. Since the problem in many cases is underdetermined (as the number of constraint equations is less than the number of degrees of freedom of the unconstrained problem), a criterion is needed to determine the correction to the input configuration.

A minimum norm correction is obtained by requiring that the solution complies with the constraints and, at the same time, departs as little as possible from the configuration that was provided at input, namely

$$\min_{\boldsymbol{x}_0, \boldsymbol{v}_0} \frac{1}{2} \left[\left(\boldsymbol{x}_0 - \boldsymbol{x}_0^{(0)}\right)^T \boldsymbol{W}_{\boldsymbol{x}} \left(\boldsymbol{x}_0 - \boldsymbol{x}_0^{(0)}\right) + \left(\boldsymbol{v}_0 - \boldsymbol{v}_0^{(0)}\right)^T \boldsymbol{W}_{\boldsymbol{v}} \left(\boldsymbol{v}_0 - \boldsymbol{v}_0^{(0)}\right) \right]$$

$$\text{subjected to} \quad \begin{array}{l} \boldsymbol{\phi}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \\ \dot{\boldsymbol{\phi}} = \boldsymbol{\phi}_{/\boldsymbol{x}}(\boldsymbol{x}_0, t_0)\boldsymbol{v}_0 + \boldsymbol{\phi}_{/t}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \\ \boldsymbol{A}(\boldsymbol{x}_0, t_0)\boldsymbol{v}_0 + \boldsymbol{b}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \end{array} \tag{6.8}$$

The problem can be reformulated as

$$\boldsymbol{W}_{\boldsymbol{x}} \left(\boldsymbol{x}_0 - \boldsymbol{x}_0^{(0)}\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} = \boldsymbol{0} \tag{6.9a}$$

$$\boldsymbol{W}_{\boldsymbol{v}} \left(\boldsymbol{v}_0 - \boldsymbol{v}_0^{(0)}\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\mu}_{\boldsymbol{\phi}} + \boldsymbol{A}^T \boldsymbol{\mu}_{\boldsymbol{\psi}} = \boldsymbol{0} \tag{6.9b}$$

$$\boldsymbol{\phi}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \tag{6.9c}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}}(\boldsymbol{x}_0, t_0)\boldsymbol{v}_0 + \boldsymbol{\phi}_{/t}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \tag{6.9d}$$

$$\boldsymbol{A}(\boldsymbol{x}_0, t_0)\boldsymbol{v}_0 + \boldsymbol{b}(\boldsymbol{x}_0, t_0) = \boldsymbol{0} \tag{6.9e}$$

After (incomplete) linearization, the problem can be solved as

$$\begin{bmatrix} \boldsymbol{W}_{\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}_{\boldsymbol{v}} & \boldsymbol{0} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{A}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{x}_0 \\ \Delta\boldsymbol{v}_0 \\ \Delta\boldsymbol{\lambda}_{\boldsymbol{\phi}} \\ \Delta\boldsymbol{\mu}_{\boldsymbol{\phi}} \\ \Delta\boldsymbol{\mu}_{\boldsymbol{\psi}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{W}_{\boldsymbol{x}} \left(\boldsymbol{x}_0^{(0)} - \boldsymbol{x}_0\right) - \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} \\ \boldsymbol{W}_{\boldsymbol{v}} \left(\boldsymbol{v}_0^{(0)} - \boldsymbol{v}_0\right) - \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\mu}_{\boldsymbol{\phi}} - \boldsymbol{A}^T \boldsymbol{\mu}_{\boldsymbol{\psi}} \\ -\boldsymbol{\phi} \\ -\dot{\boldsymbol{\phi}} \\ -\boldsymbol{A}\boldsymbol{v}_0 - \boldsymbol{b} \end{Bmatrix} \tag{6.10}$$

The problem can be decomposed in two formally independent subproblems,

$$\begin{bmatrix} \boldsymbol{W}_{\boldsymbol{x}} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{x}_0 \\ \Delta\boldsymbol{\lambda}_{\boldsymbol{\phi}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{W}_{\boldsymbol{x}} \left(\boldsymbol{x}_0^{(0)} - \boldsymbol{x}_0\right) - \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} \\ -\boldsymbol{\phi} \end{Bmatrix} \tag{6.11}$$

$$\begin{bmatrix} \boldsymbol{W}_{\boldsymbol{v}} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{A}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{v}_0 \\ \Delta\boldsymbol{\mu}_{\boldsymbol{\phi}} \\ \Delta\boldsymbol{\mu}_{\boldsymbol{\psi}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{W}_{\boldsymbol{v}} \left(\boldsymbol{v}_0^{(0)} - \boldsymbol{v}_0\right) - \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\mu}_{\boldsymbol{\phi}} - \boldsymbol{A}^T \boldsymbol{\mu}_{\boldsymbol{\psi}} \\ -\dot{\boldsymbol{\phi}} \\ -\boldsymbol{A}\boldsymbol{v}_0 - \boldsymbol{b} \end{Bmatrix} \tag{6.12}$$

where the latter depends on the former since $\boldsymbol{\phi}$, $\boldsymbol{A}$, and $\boldsymbol{b}$ may depend on $\boldsymbol{x}_0$.

The problems may couple again when, in order to influence the solution, one adds to the right hand side of Eq. (6.9a) some relatively arbitrary contributions that can be interpreted as the residual of an equilibrium,

$$\boldsymbol{W}_{\boldsymbol{x}} \left(\boldsymbol{x}_0 - \boldsymbol{x}_0^{(0)}\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} = \boldsymbol{r}(\boldsymbol{x}_0, \boldsymbol{v}_0, \boldsymbol{a}_0, t_0) \tag{6.13}$$

where $\boldsymbol{r}$ may include position, velocity, and even acceleration-dependent loads. By selectively activating what appears in $\boldsymbol{r}$, one would drive the minimization towards a solution that finds the desired trade-off between staying close to the input configuration and, at the same time, tries to minimize the strain energy, or the gravitational potential energy, or whatever is desired.

For example, one could set $\boldsymbol{r} = \boldsymbol{f}(t_0) - \boldsymbol{K}\boldsymbol{x}_0$ to account for a dead load $\boldsymbol{f}(t_0)$, e.g. the weight. By carefully crafting the weight matrix $\boldsymbol{W}_{\boldsymbol{x}}$ (e.g. by setting it to zero if the stiffness matrix $\boldsymbol{K}$ is not singular), one could allow the nodes to depart from their initial configuration and obtain the static equilibrium configuration directly with the initial assembly.

(OUTDATED?) NOTE: this is a work in progress. A new procedure is being considered, which requires only the use of the constraint equation ϕ and its Jacobian matrix $\phi_{/x}$.

The problem that is solved with the new procedure is:

- solve for configuration and holonomic constraints first

$$\boldsymbol{K}\boldsymbol{x} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda}_{\boldsymbol{\phi}} = \boldsymbol{K}\boldsymbol{x}_0 + \boldsymbol{f} \tag{6.14}$$

$$\boldsymbol{\phi}(\boldsymbol{x}, t) = \boldsymbol{0} \tag{6.15}$$

- then solve for non-holonomic ones,

$$\boldsymbol{C}\boldsymbol{v} + \boldsymbol{A}^T \boldsymbol{\lambda}_{\boldsymbol{A}} = \boldsymbol{C}\boldsymbol{v}_0 \tag{6.16}$$

$$\boldsymbol{A}(\boldsymbol{x}, t)\,\boldsymbol{v} + \boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{0} \tag{6.17}$$

  keeping the configuration $\boldsymbol{x}$ fixed, so that only the configuration is required to comply with the constraint equations, and the same constraint Jacobian matrix and residual of the regular steps are required;

- finally, after convergence, the following equation is considered:

$$\boldsymbol{\phi}_{/\boldsymbol{x}}\boldsymbol{v} + \boldsymbol{\phi}_{/t} = 0 \tag{6.18}$$

  which uses the Jacobian matrix of the constraint equations; only the derivative of the time-dependent constraints is required.

  The number of rows of matrix $\boldsymbol{\phi}_{/\boldsymbol{x}}$ is equal to the number of degrees of freedom that are constrained, so typically the matrix must be underdetermined. It can be decomposed as

$$\boldsymbol{\phi}_{/\boldsymbol{x}}^T = \boldsymbol{Q}\boldsymbol{R} = \left[\begin{array}{cc} \boldsymbol{Q}_1 & \boldsymbol{Q}_2 \end{array}\right] \left[\begin{array}{c} \boldsymbol{R}_1 \\ \boldsymbol{0} \end{array}\right] \tag{6.19}$$

  so

$$\boldsymbol{Q}_1^T \boldsymbol{v} + \boldsymbol{R}_1^{-T} \boldsymbol{\phi}_{/t} = 0 \tag{6.20}$$

  becomes a compatibility test for the initial velocities. There are two possible choices:

  1. the initial configuration assessment fails if the given initial velocities, after the correction occurring during the position and orientation assessment and correction phase, do not pass test (6.20);
  2. the initial velocities $\boldsymbol{v}$ are corrected into $\boldsymbol{v}_c$ by projecting them in the space that is compatible with the constraints, e.g.:

$$\boldsymbol{v}_c = \boldsymbol{v} - \boldsymbol{Q}_1 \left(\boldsymbol{Q}_1^T \boldsymbol{v} + \boldsymbol{R}_1^{-T} \boldsymbol{\phi}_{/t}\right) \tag{6.21}$$

  The initial assembly procedure can be repeated until test (6.20) passes.

A consistent implementation of this approach is not available yet; it requires the availability of the time derivatives of the constraint equations, which is an open issue since it is not well understood what is the physical meaning of knowing the time derivatives of a constraint. Implementation issues are still open as well.

6.2 Initial Value Problem

6.2.1 Initial Derivatives

The so-called "derivatives" phase can be thought as computing the initial value of the highest order derivatives at $t = t_0$ before any iteration starts. For simplicity, think of an explicit Ordinary Differential Equation (ODE) problem like

$$\dot{\boldsymbol{y}} = \hat{\boldsymbol{f}}(\boldsymbol{y}, t) \tag{6.22}$$

with the initial value of $t(0) = t_0$ and $\boldsymbol{y}(0) = \boldsymbol{y}_0$; then, the computation of $\dot{\boldsymbol{y}}(0)$ is trivial. Now, the actual problem is Differential Algebraic (DAE) and implicit, i.e. something like

$$\boldsymbol{f}(\dot{\boldsymbol{y}}, \boldsymbol{y}, t) = 0 \tag{6.23}$$

(actually, it's a bit more complicated, since it's index 3), and we still need to compute the derivatives $\dot{\boldsymbol{y}}(0)$ of the differential variables, and the algebraic variables as well, which in the above representation are hidden in the $\dot{\boldsymbol{y}}$. During the regular solution phase (i.e. the Newton iteration) we solve a problem of the form

$$\boldsymbol{f}_{/\dot{\boldsymbol{y}}} \Delta\dot{\boldsymbol{y}} + \boldsymbol{f}_{/\boldsymbol{y}} \Delta\boldsymbol{y} = -\boldsymbol{f} \tag{6.24}$$

and, according to the integration formula that we are using,

$$\Delta\boldsymbol{y} = \texttt{dCoef} \cdot \Delta\dot{\boldsymbol{y}} \tag{6.25}$$

where $\texttt{dCoef} = b_0 h$ is essentially the time step h times some coefficient b_0 specific for the integration formula; it is 1/2 for Crank-Nicolson, 2/3 for BDF and so on. So the actual iteration is

$$\left(\boldsymbol{f}_{/\dot{\boldsymbol{y}}} + \texttt{dCoef} \cdot \boldsymbol{f}_{/\boldsymbol{y}}\right) \Delta\dot{\boldsymbol{y}} = -\boldsymbol{f}. \tag{6.26}$$

To avoid the need to implement a dedicated routine to compute the initial value of $\dot{\boldsymbol{y}}$, we iterate over the above reported problem, ideally with a time step of 0, which means that $\boldsymbol{f}_{/\boldsymbol{y}}$ is not considered. However, since the problem is differential algebraic, the Jacobian matrix $\boldsymbol{f}_{/\dot{\boldsymbol{y}}}$ can be structurally singular, so the time step must be greater than 0, but small enough to let the correction

$$\Delta\boldsymbol{y} = \texttt{dCoef} \cdot \Delta\dot{\boldsymbol{y}} \tag{6.27}$$

be negligible with respect to $\Delta\dot{\boldsymbol{y}}$.

Assuming the above is true, during this phase the update procedure of the Newton iteration is modified. Instead of computing

$$\dot{\boldsymbol{y}}^{(i+1)} = \dot{\boldsymbol{y}}^{(i)} + \Delta\dot{\boldsymbol{y}} \tag{6.28a}$$
$$\boldsymbol{y}^{(i+1)} = \boldsymbol{y}^{(i)} + \texttt{dCoef} \cdot \Delta\dot{\boldsymbol{y}}, \tag{6.28b}$$

the "derivatives" update actually consists in

$$\dot{\boldsymbol{y}}^{(i+1)} = \dot{\boldsymbol{y}}^{(i)} + \Delta\dot{\boldsymbol{y}} \tag{6.29a}$$
$$\boldsymbol{y}^{(i+1)} = \boldsymbol{y}^{(i)}; \tag{6.29b}$$

namely, the initial value of $\boldsymbol{y}$ is preserved. This corresponds to performing a Newton iteration with an incomplete Jacobian matrix. For this purpose, nodes and elements that need to take this into account when updating internal states can provide a `DerivativesUpdate` method (declared in `SimulationEntity`).

To conclude, the “derivatives coefficient” defined in the input file is the “`dCoef`” of Eq. (6.26), which can be interpreted as the time step of a “fake” initial step that is used to compute the initial value of the highest order derivatives. It should be as small as possible, but too small makes the problem ill-conditioned (and 0 makes it structurally singular). The default value is usually fine (it rarely needs to be set, unless something really strange is going on).

If the initial derivatives phase requires more than one iteration this may mean that the system is impulsively loaded with inappropriate initial values of states; In this case, the number of iterations can be increased, by setting

```
derivatives coefficient: 1e-9;
derivatives tolerance: 1e-6;
derivatives max iterations: 10;
```

in the `initial value` block. If the solution does not converge, you should enable iterations output, adding

```
output: iterations;
```

in the `initial value` block. If you notice that the error settled to some value, and does not reduce as iterations progress, this means that the incomplete update of Eqs. (6.29) is preventing the problem from converging, because the initial value of $\boldsymbol{y}$ is compatible with the constraints but does not satisfy equilibrium.

In this case, one can play with the `derivatives coefficient` parameter, to emphasize the “inertia” terms instead of the “elastic” ones, or simply increase the `derivatives tolerance` so that the solution with the current `derivatives coefficient` is considered converged. The latter choice may result in a “rough” behavior of the solution during the initial steps of the simulation.

6.2.2 Dummy Steps

TODO

6.2.3 Regular Steps

TODO

6.3 Inverse Dynamics Problem

In principle, an inverse dynamics problem consists in computing the ‘torques’ (actuator forces) required to comply with equilibrium for a given configuration (position, velocity and acceleration) of the entire system.

In practice, in many cases the motion of the system is not directly known in terms of the coordinates that are used to describe it, but rather in terms of the prescribed motion of the ‘joints’ (the actuators) that apply the unknown torques. As a consequence, an inverse kinematics problem needs to be solved first, to determine the configuration of the system up to the acceleration level as a function of the motion prescribed to the end effector up to the acceleration level. Section 6.3.2 deals with the case of a fully actuated problem, i.e. a problem whose motion is completely prescribed in terms of joint coordinates. Section 6.3.4 deals with the case of an underactuated problem, i.e. a problem whose motion is only partially prescribed in terms of joint coordinates.

In many cases, even the motion of the actuators is not directly known. On the contrary, the motion of another part of the system (called the ‘end effector’) is prescribed, and the actuators’ motion required to move the end effector as prescribed needs to be computed. As a consequence, the inverse kinematics problem is driven by the motion prescribed to the end effector rather than directly to the joints. This case is specifically dealt with in Section 6.3.5.

6.3.1 Nomenclature

In the following, $\boldsymbol{x}$ are the n coordinates that describe the motion of the system;

$$\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0} \tag{6.30}$$

are b so-called 'passive' constraints, namely those that describe the assembly of the system;

$$\boldsymbol{\vartheta}(\boldsymbol{x}) = \boldsymbol{\theta} \tag{6.31}$$

are j so-called 'joint coordinates', namely those that describe the motion $\boldsymbol{\theta}$ of the joints, with $\boldsymbol{\phi}_{/\boldsymbol{x}}\boldsymbol{\vartheta}^{+}_{/\boldsymbol{x}} \equiv \boldsymbol{0}$;

$$\boldsymbol{\psi}(\boldsymbol{x}) = \boldsymbol{\alpha}(t) \tag{6.32}$$

are c so-called 'control constraints', namely equations that prescribe the motion of the end effector, with $\boldsymbol{\phi}_{/\boldsymbol{x}}\boldsymbol{\psi}^{+}_{/\boldsymbol{x}} \equiv \boldsymbol{0}$. When the joint motion is directly prescribed, $\boldsymbol{\theta} \equiv \boldsymbol{\alpha}(t)$ and thus $\boldsymbol{\vartheta}(\boldsymbol{x}) \equiv \boldsymbol{\psi}(\boldsymbol{x})$. In some cases, a partial overlap may exist between the two sets of equations.

6.3.2 Fully Actuated, Collocated Problem

In this case, $\boldsymbol{\psi}(\boldsymbol{x}) \equiv \boldsymbol{\vartheta}(\boldsymbol{x})$ and $c = j = n - b$.

Inverse Kinematics: Position Subproblem

$$\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0} \tag{6.33a}$$
$$\boldsymbol{\psi}(\boldsymbol{x}) = \boldsymbol{\alpha} \tag{6.33b}$$

using Newton-Raphson:

$$\begin{bmatrix} \boldsymbol{\phi}_{/\boldsymbol{x}} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} \end{bmatrix} \Delta\boldsymbol{x} = \left\{ \begin{array}{c} -\boldsymbol{\phi}(\boldsymbol{x}) \\ \boldsymbol{\alpha} - \boldsymbol{\psi}(\boldsymbol{x}) \end{array} \right\}. \tag{6.34}$$

Inverse Kinematics: Velocity Subproblem

$$\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}} = \boldsymbol{0} \tag{6.35a}$$
$$\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}} = \dot{\boldsymbol{\alpha}} \tag{6.35b}$$

or

$$\begin{bmatrix} \boldsymbol{\phi}_{/\boldsymbol{x}} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} \end{bmatrix} \dot{\boldsymbol{x}} = \left\{ \begin{array}{c} \boldsymbol{0} \\ \dot{\boldsymbol{\alpha}} \end{array} \right\}. \tag{6.36}$$

Inverse Kinematics: Acceleration Subproblem

$$\boldsymbol{\phi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}} = -\left(\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \tag{6.37a}$$
$$\boldsymbol{\psi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{\alpha}} - \left(\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \tag{6.37b}$$

or

$$\begin{bmatrix} \boldsymbol{\phi}_{/\boldsymbol{x}} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} \end{bmatrix} \ddot{\boldsymbol{x}} = \left\{ \begin{array}{c} -\left(\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{\alpha}} - \left(\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \end{array} \right\}. \tag{6.38}$$

Inverse Dynamics Subproblem

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{c} = \boldsymbol{f} \tag{6.39}$$

or

$$\begin{bmatrix} \boldsymbol{\phi}_{/\boldsymbol{x}} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} \end{bmatrix}^T \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{c} \end{Bmatrix} = \boldsymbol{f} - \boldsymbol{M}\ddot{\boldsymbol{x}}. \tag{6.40}$$

The matrix of the Newton-Raphson problem for the position is identical to the matrices of the linear problems for the velocity and the acceleration, while the inverse dynamics problem uses its transpose, so the same factorization can be easily reused.

6.3.3 Fully Actuated, Non-Collocated Problem

In this case, $\boldsymbol{\psi}(\boldsymbol{x}) \neq \boldsymbol{\vartheta}(\boldsymbol{x})$, but still $c = j = n - b$.

Inverse Kinematics: Position, Velocity and Acceleration Subproblems

The position, velocity and acceleration subproblems that define the inverse kinematics problem are identical to those of the collocated case of Section 6.3.2.

Inverse Dynamics Subproblem

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\vartheta}_{/\boldsymbol{x}}^T \boldsymbol{c} = \boldsymbol{f} \tag{6.41}$$

or

$$\begin{bmatrix} \boldsymbol{\phi}_{/\boldsymbol{x}} \\ \boldsymbol{\vartheta}_{/\boldsymbol{x}} \end{bmatrix}^T \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{c} \end{Bmatrix} = \boldsymbol{f} - \boldsymbol{M}\ddot{\boldsymbol{x}}. \tag{6.42}$$

The inverse dynamics problem uses a different matrix from the transpose of that that was used for the inverse kinematics subproblems.

6.3.4 Underdetermined, Underactuated but Collocated Problem

In this case, again $\boldsymbol{\psi}(\boldsymbol{x}) \equiv \boldsymbol{\vartheta}(\boldsymbol{x})$ but $c = j < n - b$.

Inverse Kinematics: Position Subproblem

Constraint equations:

$$\boldsymbol{\phi}\left(\boldsymbol{x}\right) = \boldsymbol{0} \tag{6.43a}$$

$$\boldsymbol{\psi}\left(\boldsymbol{x}\right) = \boldsymbol{\alpha}(t) \tag{6.43b}$$

with $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\psi}_{/\boldsymbol{x}}$ rectangular, full row rank. The problem is underdetermined; as a consequence, some criteria are needed to find an optimal solution.

The problem can be augmented by

$$\boldsymbol{K}\left(\boldsymbol{x} - \boldsymbol{x}_0\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{0} \tag{6.44}$$

This nonlinear problem is solved for $\boldsymbol{x}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to convergence ($\boldsymbol{K}$, $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\psi}_{/\boldsymbol{x}}$ may further depend on $\boldsymbol{x}$). It corresponds to a least-squares solution for $\boldsymbol{x} - \boldsymbol{x}_0$, where the quadratic form

$$J_{\boldsymbol{x}} = \frac{1}{2} \left(\boldsymbol{x} - \boldsymbol{x}_0\right)^T \boldsymbol{K} \left(\boldsymbol{x} - \boldsymbol{x}_0\right) \tag{6.45}$$

is minimized subjected to $\boldsymbol{\phi} = \boldsymbol{0}$ and $\boldsymbol{\psi} = \boldsymbol{\alpha}$, and weighted by matrix $\boldsymbol{K}$; $\boldsymbol{x}_0$ is a reference solution. Matrix $\boldsymbol{K}$ can further depend on $\boldsymbol{x}$.

The reference solution $\boldsymbol{x}_0$ can be used to further control the quality of the solution. For example, it may represent a prescribed tentative, although possibly incompatible, trajectory. Another option consists in augmenting $J_{\boldsymbol{x}}$ with another quadratic form

$$J_{\boldsymbol{x}} = \frac{1}{2} \left(\boldsymbol{x} - \boldsymbol{x}_0\right)^T \boldsymbol{K} \left(\boldsymbol{x} - \boldsymbol{x}_0\right) + w_{\boldsymbol{x}} \frac{1}{2} \left(\boldsymbol{x} - \boldsymbol{x}_{\text{prev}}\right)^T \boldsymbol{M} \left(\boldsymbol{x} - \boldsymbol{x}_{\text{prev}}\right) \tag{6.46}$$

where $\boldsymbol{x}_{\text{prev}}$ is the value of $\boldsymbol{x}$ at the previous time step. This modified quadratic form weights the rate of change of $\boldsymbol{x}$ within two consecutive steps. As a consequence, minimal position changes (i.e. velocities), weighted by the mass of the system, are sought. The corresponding problem is

$$\boldsymbol{K} \left(\boldsymbol{x} - \boldsymbol{x}_0\right) + w_{\boldsymbol{x}} \boldsymbol{M} \left(\boldsymbol{x} - \boldsymbol{x}_{\text{prev}}\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{0} \tag{6.47a}$$

$$\boldsymbol{\phi}\left(\boldsymbol{x}\right) = \boldsymbol{0} \tag{6.47b}$$

$$\boldsymbol{\psi}\left(\boldsymbol{x}\right) = \boldsymbol{\alpha}, \tag{6.47c}$$

or

$$\left(\boldsymbol{K} + w_{\boldsymbol{x}} \boldsymbol{M}\right) \boldsymbol{x} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{K} \boldsymbol{x}_0 + w_{\boldsymbol{x}} \boldsymbol{M} \boldsymbol{x}_{\text{prev}} \tag{6.48a}$$

$$\boldsymbol{\phi}\left(\boldsymbol{x}\right) = \boldsymbol{0} \tag{6.48b}$$

$$\boldsymbol{\psi}\left(\boldsymbol{x}\right) = \boldsymbol{\alpha}. \tag{6.48c}$$

Using Newton-Raphson:

$$\left[\begin{array}{ccc} \boldsymbol{K} + w_{\boldsymbol{x}} \boldsymbol{M} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{\psi}_{/\boldsymbol{x}}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \Delta \boldsymbol{x} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{K}\left(\boldsymbol{x}_0 - \boldsymbol{x}\right) + w_{\boldsymbol{x}} \boldsymbol{M}\left(\boldsymbol{x}_{\text{prev}} - \boldsymbol{x}\right) - \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} - \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} \\ -\boldsymbol{\phi}\left(\boldsymbol{x}\right) \\ \boldsymbol{\alpha}(t) - \boldsymbol{\psi}\left(\boldsymbol{x}\right) \end{array}\right\} \tag{6.49}$$

Inverse Kinematics: Velocity Subproblem

Constraint first derivative:

$$\boldsymbol{R}\left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right) + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{0} \tag{6.50a}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}} = \boldsymbol{0} \tag{6.50b}$$

$$\boldsymbol{\psi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}} = \dot{\boldsymbol{\alpha}}. \tag{6.50c}$$

In analogy with the position constraint case, it corresponds to minimizing the quadratic form

$$J_{\dot{\boldsymbol{x}}} = \frac{1}{2} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right)^T \boldsymbol{R} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right), \tag{6.51}$$

which can be augmented as well, resulting in

$$J_{\dot{\boldsymbol{x}}} = \frac{1}{2} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right)^T \boldsymbol{R} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right) + w_{\dot{\boldsymbol{x}}} \frac{1}{2} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\text{prev}}\right)^T \boldsymbol{M} \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\text{prev}}\right), \tag{6.52}$$

to minimize the velocity increment between two consecutive time steps. The corresponding problem is

$$\left(\boldsymbol{R} + w_{\dot{\boldsymbol{x}}} \boldsymbol{M}\right) \dot{\boldsymbol{x}} + \boldsymbol{\phi}_{/\boldsymbol{x}}^T \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}}^T \boldsymbol{\mu} = \boldsymbol{R} \dot{\boldsymbol{x}}_0 + w_{\dot{\boldsymbol{x}}} \boldsymbol{M} \dot{\boldsymbol{x}}_{\text{prev}} \tag{6.53a}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}} = \boldsymbol{0} \tag{6.53b}$$

$$\boldsymbol{\psi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}} = \dot{\boldsymbol{\alpha}}, \tag{6.53c}$$

or

$$\left[\begin{array}{ccc} \boldsymbol{R} + w_{\dot{\boldsymbol{x}}} \boldsymbol{M} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{\psi}_{/\boldsymbol{x}}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \dot{\boldsymbol{x}} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{R} \dot{\boldsymbol{x}}_0 + w_{\dot{\boldsymbol{x}}} \boldsymbol{M} \dot{\boldsymbol{x}}_{\text{prev}} \\ \boldsymbol{0} \\ \dot{\boldsymbol{\alpha}} \end{array}\right\}. \tag{6.54}$$

The constraint derivative problem is linear in $\dot{\boldsymbol{x}}$; the same symbols $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are used for the multipliers since they are ineffective (their value is never used). The problems can be solved sequentially. Only in case $\boldsymbol{K} \equiv \boldsymbol{R}$ and $w_{\boldsymbol{x}} \equiv w_{\dot{\boldsymbol{x}}}$ both problems use the same matrix and thus the same factorization can be reused.

Inverse Dynamics Subproblem

The second derivative of the constraint cannot be resolved as in the fully actuated case, otherwise it could yield accelerations that cannot be imposed by the constraints. On the contrary, the inverse dynamics problem of Eq. (6.39) is directly solved, yielding both the accelerations and the torques,

$$\boldsymbol{M} \ddot{\boldsymbol{x}} + \boldsymbol{\phi}_{/\boldsymbol{x}} \boldsymbol{\lambda} + \boldsymbol{\psi}_{/\boldsymbol{x}} \boldsymbol{c} = \boldsymbol{f} \tag{6.55a}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}} \ddot{\boldsymbol{x}} = -\left(\boldsymbol{\phi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}} \dot{\boldsymbol{x}} \tag{6.55b}$$

$$\boldsymbol{\psi}_{/\boldsymbol{x}} \ddot{\boldsymbol{x}} = \ddot{\boldsymbol{\alpha}} - \left(\boldsymbol{\psi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}} \dot{\boldsymbol{x}} \tag{6.55c}$$

or

$$\left[\begin{array}{ccc} \boldsymbol{M} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{\psi}_{/\boldsymbol{x}}^T \\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\psi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \ddot{\boldsymbol{x}} \\ \boldsymbol{\lambda} \\ \boldsymbol{c} \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{f} \\ -\left(\boldsymbol{\phi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{\alpha}} - \left(\boldsymbol{\psi}_{/\boldsymbol{x}} \dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}} \dot{\boldsymbol{x}} \end{array}\right\} \tag{6.56}$$

As a consequence, the same structure of the inverse kinematic problems is achieved, and the last problem of Eq. (6.56) directly yields both accelerations and multipliers.

6.3.5 Underdetermined, Overcontrolled Problem

In this case, $\boldsymbol{\psi}(\boldsymbol{x}) \neq \boldsymbol{\vartheta}(\boldsymbol{x})$, $c < n - b$, $j = n - b$, which implies $c < j$. This is the case, for example, of an inverse biomechanics problem, where each degree of freedom is a joint commanded by a set of muscles. The inverse kinematics problem is solved by prescribing the motion of some part (e.g. a hand or a foot) in order to determine an ‘optimal’ (e.g. in terms of maximal ergonomy) motion. Then, an inverse dynamics problem is computed by freeing the part whose motion was initially prescribed, and by computing the torque required by all joints.

Inverse Kinematics: Position and Velocity Subproblems

The constraint and its first derivative are dealt with as in Section 6.3.4.

Inverse Kinematics: Acceleration Subproblem

Constraint second derivative:

$$\boldsymbol{M}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_0\right)+\boldsymbol{\phi}_{/\boldsymbol{x}}^T\boldsymbol{\lambda}+\boldsymbol{\psi}_{/\boldsymbol{x}}^T\boldsymbol{\mu}=\boldsymbol{0} \tag{6.57a}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}}=-\left(\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \tag{6.57b}$$

$$\boldsymbol{\psi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}}=\ddot{\boldsymbol{\alpha}}-\left(\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}}. \tag{6.57c}$$

In analogy with the position constraint case, it corresponds to minimizing the quadratic form

$$J_{\ddot{\boldsymbol{x}}}=\frac{1}{2}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_0\right)^T\boldsymbol{M}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_0\right), \tag{6.58}$$

which can be augmented as well, resulting in

$$J_{\ddot{\boldsymbol{x}}}=\frac{1}{2}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_0\right)^T\boldsymbol{M}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_0\right)+w_{\ddot{\boldsymbol{x}}}\frac{1}{2}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_{\text{prev}}\right)^T\boldsymbol{M}\left(\ddot{\boldsymbol{x}}-\ddot{\boldsymbol{x}}_{\text{prev}}\right), \tag{6.59}$$

to minimize the acceleration increment between two consecutive time steps. The corresponding problem is

$$\left(1+w_{\dot{\boldsymbol{x}}}\right)\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{\phi}_{/\boldsymbol{x}}^T\boldsymbol{\lambda}+\boldsymbol{\psi}_{/\boldsymbol{x}}^T\boldsymbol{\mu}=\boldsymbol{M}\ddot{\boldsymbol{x}}_0+w_{\ddot{\boldsymbol{x}}}\boldsymbol{M}\ddot{\boldsymbol{x}}_{\text{prev}} \tag{6.60a}$$

$$\boldsymbol{\phi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}}=-\left(\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}} \tag{6.60b}$$

$$\boldsymbol{\psi}_{/\boldsymbol{x}}\ddot{\boldsymbol{x}}=\ddot{\boldsymbol{\alpha}}-\left(\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}}, \tag{6.60c}$$

or

$$\left[\begin{array}{ccc}\left(1+w_{\ddot{\boldsymbol{x}}}\right)\boldsymbol{M} & \boldsymbol{\phi}_{/\boldsymbol{x}}^T & \boldsymbol{\psi}_{/\boldsymbol{x}}^T\\ \boldsymbol{\phi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{\psi}_{/\boldsymbol{x}} & \boldsymbol{0} & \boldsymbol{0}\end{array}\right]\left\{\begin{array}{c}\ddot{\boldsymbol{x}}\\ \boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{array}\right\}=\left\{\begin{array}{c}\boldsymbol{M}\ddot{\boldsymbol{x}}_0+w_{\ddot{\boldsymbol{x}}}\boldsymbol{M}\ddot{\boldsymbol{x}}_{\text{prev}}\\ -\left(\boldsymbol{\phi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{\alpha}}-\left(\boldsymbol{\psi}_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\right)_{/\boldsymbol{x}}\dot{\boldsymbol{x}}\end{array}\right\}. \tag{6.61}$$

Inverse Dynamics Subproblem

The inverse dynamics subproblem is identical to that of the fully determined, non-collocated case, Eq. (6.41).

Implementation

- during 'position' inverse kinematics substep:
  - assemble $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\phi}_{/\boldsymbol{x}}^T$ for passive constraints
  - assemble $\boldsymbol{\psi}_{/\boldsymbol{x}}$ and $\boldsymbol{\psi}_{/\boldsymbol{x}}^T$ for prescribed motion
  - assemble equations related to $\boldsymbol{\vartheta}_{/\boldsymbol{x}}$ as $\boldsymbol{c}=\boldsymbol{0}$ to neutralize them
  - assemble dummy springs
  - optionally assemble mass matrix contribution weighted by $w_{/\boldsymbol{x}}$
- during 'velocity' inverse kinematics substep:
  - assemble $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\phi}_{/\boldsymbol{x}}^T$ for passive constraints
  - assemble $\boldsymbol{\psi}_{/\boldsymbol{x}}$ and $\boldsymbol{\psi}_{/\boldsymbol{x}}^T$ for prescribed motion

- assemble equations related to $\boldsymbol{\vartheta}_{/\boldsymbol{x}}$ as $\boldsymbol{c} = \mathbf{0}$ to neutralize them
- assemble dummy dampers
- optionally assemble mass matrix contribution weighted by $w_{/\dot{\boldsymbol{x}}}$

- during 'acceleration' inverse kinematics substep:
  - assemble $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\phi}^T_{/\boldsymbol{x}}$ for passive constraints
  - assemble $\boldsymbol{\psi}_{/\boldsymbol{x}}$ and $\boldsymbol{\psi}^T_{/\boldsymbol{x}}$ for prescribed motion
  - assemble equations related to $\boldsymbol{\vartheta}_{/\boldsymbol{x}}$ as $\boldsymbol{c} = \mathbf{0}$ to neutralize them
  - assemble mass matrix contribution, optionally weighted by $(1 + w_{/\ddot{\boldsymbol{x}}})$

- during inverse dynamics substep:
  - assemble $\boldsymbol{\phi}_{/\boldsymbol{x}}$ and $\boldsymbol{\phi}^T_{/\boldsymbol{x}}$ for passive constraints
  - assemble $\boldsymbol{\vartheta}_{/\boldsymbol{x}}$ and $\boldsymbol{\vartheta}^T_{/\boldsymbol{x}}$ for torques
  - assemble equations related to $\boldsymbol{\psi}_{/\boldsymbol{x}}$ as $\boldsymbol{\mu} = \mathbf{0}$ to neutralize them
  - assemble other elements (external forces, springs, etc.)

Chapter 7

Data Structure

(This chapter is a mess.)

7.1 Constitutive Laws

Code that uses constitutive laws requires

$$\boldsymbol{f} = \boldsymbol{f}\left(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}, t, \ldots\right) \tag{7.1}$$

where $\boldsymbol{f}$ can be 1, 3, and 6 dimensional. Similarly, $\boldsymbol{\epsilon}$ and $\dot{\boldsymbol{\epsilon}}$ respectively are 1, 3, and 6 dimensional (scalar, `Vec3`, and `Vec6`).

The `ConstitutiveLaw` provides

$$\texttt{GetF()} = \boldsymbol{f}\left(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}, t, \ldots\right) \tag{7.2a}$$

$$\texttt{GetFDE()} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\epsilon}} \tag{7.2b}$$

$$\texttt{GetFDEPrime()} = \frac{\partial \boldsymbol{f}}{\partial \dot{\boldsymbol{\epsilon}}} \tag{7.2c}$$

The values returned by these methods are updated by a call to `Update()`.

7.2 ExpandableRowVector

`ExpandableRowVector` is a class that supports the computation of nested Jacobian matrices.

Consider a scalar y that depends on a set of variables $\boldsymbol{k}$, namely $y = y\left(\boldsymbol{k}\right)$. The generic variable k_i, in turn, may be directly a subset of the variables of the problem, $\boldsymbol{x}$, with $k_i = x_j$, or depend on them either directly, namely $k_i = k_i\left(\boldsymbol{x}\right)$, or indirectly, namely $k_i = k_i\left(\boldsymbol{k}'\left(\boldsymbol{x}\right)\right)$ and so on, with as many levels of recursion as needed, namely

$$\frac{\partial y}{\partial x_j} = \sum_i \frac{\partial y}{\partial k_i}\left(\sum_{i'} \frac{\partial k_i}{\partial k'_{i'}}\left(\cdots \sum_j \frac{\partial k^{(n)}_{i^{(n)}}}{\partial x_j}\right)\right). \tag{7.3}$$

The computation of the contribution of y to the Jacobian matrix of the problem requires the capability to assemble the partial derivatives of y with respect to the variables $\boldsymbol{x}$ in the appropriate locations.

The `ExpandableRowVector` allows to associate each partial derivative of y with respect to the variables $\boldsymbol{k}$ it depends on, namely $y_{/k_i}$, to either

a) its index, if k_i is directly a variable of the problem, $k_i \rightarrow x_j$, or

b) to another `ExpandableRowVector` that contains the partial derivatives of the corresponding k_i with respect to the variables it depends on.

Since the assembly of the contributions to the Jacobian matrix of the problem is usually done using submatrices, within an element the association between a coefficient and a problem variable is not done based on the absolute numbering of the variable, but rather on a local numbering within the submatrix. The submatrix in turn will take care of mapping local indexing to global indexing when it is assembled to the global Jacobian matrix of the problem.

When the variable k_i directly corresponds to the problem variable x_j, the `ExpandableRowVector` provides a method to associate one of its elements to the problem variable indexing within the submatrix. So, if $k_i \rightarrow x_j$, and the global index j corresponds to index `ip` in the submatrix,

v.Set($y_{/k_i}$, i, ip)

simultaneously sets the value $y_{/k_i}$ and the variable subindex `ip` of the `i`-th element of vector `v`.

Alternatively, the method

```
v.SetIdx(i, ip)
```

associates the variable subindex `ip` to the `i`-th element of the vector.

When k_i is a function of a set of variables, and its partial derivatives with respect to those variables are stored in another `ExpandableRowVector` $\boldsymbol{w}$, the method

```
v.Link(i, &w)
```

allows to associate vector `w` to the `i`-th element of vector `v`.

As soon as $\boldsymbol{v}$ is assembled, it can be contributed to a specific equation (row) of a Jacobian matrix.

For example, calling the method

```
void
Add(FullSubMatrixHandler& WM,
    const integer eq,
    const doublereal c = 1.) const
```

of vector `v` corresponds to

```
for (integer j = 1; j <= n; j++) {
    WM(eq, idx(j)) += c*v(j);
}
```

where `idx` is the vector that contains the submatrix indexes of the elements of $\boldsymbol{v}$, as set by the `SetIdx()` method.

If another `ExpandableRowVector` is linked to the `i`-th element of vector $\boldsymbol{v}$, the related operation on the submatrix is propagated recursively as

```
for (integer j = 1; j <= n; j++) {
    if ( /* linked to another ExpandableRowVector */ ) {
        // propagate operation
        xm(j)->Add(WM, eq, c*v(j));
    } else {
        WM(eq, idx(j)) += c*v(j);
    }
}
```

where `xm` is the container of the pointers to the linked `ExpandableRowVector`.

Blocks of equations can be added simultaneously using the method

```
void
Add(FullSubMatrixHandler& WM,
    const std::vector<integer>& eq,
    const std::vector<doublereal>& cc,
    const doublereal c = 1.) const
```

The same operation illustrated above is performed on each equation whose subindex is stored in the array `eq`, weighed by the coefficients stored in the array `cc`.

A similar method allows to build a subvector. Its use is currently undocumented.

Example.

```
ExpandableRowVector v(4);
ExpandableRowVector w(2);
FullSubMatrixHandler m(6, 9);

w.Set(10., 1, 8);
w.Set(20., 2, 9);

v.Set(1., 1, 4);
v.Set(2., 2, 5);
v.Set(3., 3, 6);
v.Set(10., 4);
v.Link(4, &w);

v.Add(m, 5, 1.);

//       col.  1   2   3   4   5   6   7    8    9
// m(5, :) = { 0., 0., 0., 1., 2., 3., 0., 100., 200. }

std::vector<integer> eq(3);
std::vector<doublereal> cc(3);
eq[0] = 1; cc[0] = 1.;
eq[1] = 2; cc[1] = 2.;
eq[2] = 3; cc[2] = 3.;

v.Add(m, eq, cc, 1.);

//       col.  1   2   3   4   5   6   7    8    9
// m(1, :) = { 0., 0., 0., 1., 2., 3., 0., 100., 200. }
// m(2, :) = { 0., 0., 0., 2., 4., 6., 0., 200., 400. }
// m(3, :) = { 0., 0., 0., 3., 6., 9., 0., 300., 600. }
```

Chapter 8

Nodes

8.1 Structural Nodes

Structural nodes provide the kinematics unknowns and the corresponding equilibrium equations.

8.1.1 Dynamic Structural Nodes

The dynamic structural node also provides momentum and momenta moment unknowns, and the equations that represent their definition in terms of the inertia properties and the node's kinematics:

$$\begin{aligned}\boldsymbol{\beta} &= m\dot{\boldsymbol{x}}_{\mathrm{CM}} \\ &= m\left(\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}\right) \\ &= m\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{s} \qquad (8.1a)\\ \boldsymbol{\gamma} &= \boldsymbol{\gamma}_{\mathrm{CM}} + \boldsymbol{b} \times \boldsymbol{\beta} \\ &= \boldsymbol{J}_{\mathrm{CM}}\boldsymbol{\omega} + \boldsymbol{b} \times \boldsymbol{\beta} \\ &= \boldsymbol{s} \times \dot{\boldsymbol{x}} + \boldsymbol{J}\boldsymbol{\omega}, \qquad (8.1b)\end{aligned}$$

where

$$\boldsymbol{b} = \frac{\boldsymbol{s}}{m} \qquad (8.2)$$

is the distance between the node and the CM of the body, which is constant in the reference frame of the node for a rigid body, and $\boldsymbol{J} = \boldsymbol{J}_{\mathrm{CM}} + m\boldsymbol{b} \times \boldsymbol{b} \times^T$.

The contribution to the equations of motion (Newton-Euler) is

$$\begin{aligned}\boldsymbol{f}_{\mathrm{in}} &= \dot{\boldsymbol{\beta}} \qquad (8.3a)\\ \boldsymbol{m}_{\mathrm{in}} &= \dot{\boldsymbol{\gamma}}_{\mathrm{CM}} + \boldsymbol{b} \times \boldsymbol{f}_{\mathrm{in}} \\ &= \dot{\boldsymbol{\gamma}} - \left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \times \boldsymbol{\beta} \\ &= \dot{\boldsymbol{\gamma}} - m\left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \times \left(\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}\right) \\ &= \dot{\boldsymbol{\gamma}} + m\dot{\boldsymbol{x}} \times \left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \\ &= \dot{\boldsymbol{\gamma}} + m\dot{\boldsymbol{x}} \times \left(\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}\right) \\ &= \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{x}} \times \boldsymbol{\beta} \qquad (8.3b)\end{aligned}$$

The perturbation of the momentum and momenta moment definitions yields

$$\delta\boldsymbol{\beta} = m\delta\dot{\boldsymbol{x}} - \boldsymbol{s} \times \delta\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{s} \times \boldsymbol{\theta}_\delta \tag{8.4a}$$

$$\delta\boldsymbol{\gamma} = \boldsymbol{s} \times \delta\dot{\boldsymbol{x}} + \boldsymbol{J}\delta\boldsymbol{\omega} + \left(\dot{\boldsymbol{x}} \times \boldsymbol{s} \times\ - (\boldsymbol{J}\boldsymbol{\omega}) \times\ + \boldsymbol{J}\boldsymbol{\omega} \times\ \right)\boldsymbol{\theta}_\delta \tag{8.4b}$$

The perturbation of the inertia force and moment yields

$$\delta\boldsymbol{f}_{\text{in}} = \delta\dot{\boldsymbol{\beta}} \tag{8.5a}$$

$$\delta\boldsymbol{m}_{\text{in}} = \delta\dot{\boldsymbol{\gamma}} - \boldsymbol{\beta} \times \delta\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \times \delta\boldsymbol{\beta} \tag{8.5b}$$

Accelerations

Since momentum and momenta moments are used as the unknowns that take care of inertia, linear and angular accelerations are not directly available from the simulation. They are computed, on demand, as postprocessing from the derivatives of the definitions of the momentum and the momenta moment.

The differentiation of Equations (8.1) yields

$$\dot{\boldsymbol{\beta}} = m\ddot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{s} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{s} \tag{8.6a}$$

$$\dot{\boldsymbol{\gamma}} = (\boldsymbol{\omega} \times \boldsymbol{s}) \times \dot{\boldsymbol{x}} + \boldsymbol{s} \times \ddot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{J}\dot{\boldsymbol{\omega}}. \tag{8.6b}$$

As a consequence, the linear and angular accelerations are computed as

$$\dot{\boldsymbol{\omega}} = \boldsymbol{J}_{\text{CM}}^{-1}\left(\dot{\boldsymbol{\gamma}} - \boldsymbol{b} \times \dot{\boldsymbol{\beta}} + \dot{\boldsymbol{x}} \times \boldsymbol{\beta} - \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega}\right) \tag{8.7}$$

$$\begin{aligned}\ddot{\boldsymbol{x}} &= \frac{1}{m}\left(\dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\omega}} \times \boldsymbol{s} - \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{s}\right) \\ &= \left(\frac{1}{m}\boldsymbol{I} + \boldsymbol{b} \times^T \boldsymbol{J}_{\text{CM}}^{-1}\boldsymbol{b} \times\right)\dot{\boldsymbol{\beta}} - \boldsymbol{b} \times^T \boldsymbol{J}_{\text{CM}}^{-1}\dot{\boldsymbol{\gamma}} - \boldsymbol{b} \times^T \boldsymbol{J}_{\text{CM}}^{-1}\left(\dot{\boldsymbol{x}} \times \boldsymbol{\beta} - \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega}\right),\end{aligned} \tag{8.8}$$

namely

$$\left\{\begin{array}{c}\ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{\omega}}\end{array}\right\} = \boldsymbol{M}^{-1}\left\{\begin{array}{c}\dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}}\end{array}\right\} + \left[\begin{array}{c}-\boldsymbol{b}\times^T \\ \boldsymbol{I}\end{array}\right]\boldsymbol{J}_{\text{CM}}^{-1}\left(\dot{\boldsymbol{x}} \times \boldsymbol{\beta} - \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega}\right) \tag{8.9}$$

Note on the inverse of the mass matrix

The mass matrix is

$$\boldsymbol{M} = \left[\begin{array}{cc}m\boldsymbol{I} & \boldsymbol{s} \times^T \\ \boldsymbol{s} \times & \boldsymbol{J}\end{array}\right] \tag{8.10}$$

Its inverse is

$$\begin{aligned}\boldsymbol{M}^{-1} &= \left[\begin{array}{cc}\left(m\boldsymbol{I} - \boldsymbol{s} \times^T \boldsymbol{J}^{-1}\boldsymbol{s} \times\right)^{-1} & -\frac{1}{m}\boldsymbol{s} \times^T \left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{s} \times \boldsymbol{s} \times^T\right)^{-1} \\ -\frac{1}{m}\left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{s} \times \boldsymbol{s} \times^T\right)^{-1}\boldsymbol{s} \times & \left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{s} \times \boldsymbol{s} \times^T\right)^{-1}\end{array}\right] \\ &= \left[\begin{array}{cc}\frac{1}{m}\boldsymbol{I} + \boldsymbol{b} \times^T \boldsymbol{J}_{\text{CM}}^{-1}\boldsymbol{b} \times & -\boldsymbol{b} \times^T \boldsymbol{J}_{\text{CM}}^{-1} \\ -\boldsymbol{J}_{\text{CM}}^{-1}\boldsymbol{b} \times & \boldsymbol{J}_{\text{CM}}^{-1}\end{array}\right].\end{aligned} \tag{8.11}$$

In fact, according to the matrix inversion lemma,

$$\left(m\boldsymbol{I} - \boldsymbol{s}\times{}^T\boldsymbol{J}^{-1}\boldsymbol{s}\times\right)^{-1} = \frac{1}{m}\boldsymbol{I} + \boldsymbol{b}\times{}^T\boldsymbol{J}_{\mathrm{CM}}^{-1}\boldsymbol{b}\times \tag{8.12}$$

(NOTE: simplifications when using updated-updated formulas).

Perturbation of momentum and momenta moment derivatives

According to Eqs. (8.4), the perturbation of the linear and angular velocity is

$$\left\{\begin{array}{c}\delta\dot{\boldsymbol{x}}\\ \delta\boldsymbol{\omega}\end{array}\right\} = \boldsymbol{M}^{-1}\left(\left\{\begin{array}{c}\delta\boldsymbol{\beta}\\ \delta\gamma\end{array}\right\} - \left[\begin{array}{c}-\boldsymbol{\omega}\times\boldsymbol{s}\times\\ \dot{\boldsymbol{x}}\times\boldsymbol{s}\times-(\boldsymbol{J}\boldsymbol{\omega})\times+\boldsymbol{J}\boldsymbol{\omega}\times\end{array}\right]\boldsymbol{\theta}_\delta\right) \tag{8.13a}$$

The perturbation of the momentum and momenta moment derivatives is

$$\delta\dot{\boldsymbol{\beta}} = m\delta\ddot{\boldsymbol{x}} - \boldsymbol{s}\times\delta\dot{\boldsymbol{\omega}} - (\boldsymbol{\omega}\times\boldsymbol{s})\times\delta\boldsymbol{\omega} - \boldsymbol{\omega}\times\boldsymbol{s}\times\delta\boldsymbol{\omega} - \dot{\boldsymbol{\omega}}\times\boldsymbol{s}\times\boldsymbol{\theta}_\delta - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{s}\times\boldsymbol{\theta}_\delta \tag{8.14a}$$

$$\begin{aligned}\delta\dot{\boldsymbol{\gamma}} &= \boldsymbol{s}\times\delta\ddot{\boldsymbol{x}} + \boldsymbol{J}\delta\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega}\times\boldsymbol{s})\times\delta\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}\times\boldsymbol{s}\times\delta\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{J}\delta\boldsymbol{\omega} - (\boldsymbol{J}\boldsymbol{\omega})\times\delta\boldsymbol{\omega}\\ &\quad + \dot{\boldsymbol{x}}\times\boldsymbol{\omega}\times\boldsymbol{s}\times\boldsymbol{\theta}_\delta + \ddot{\boldsymbol{x}}\times\boldsymbol{s}\times\boldsymbol{\theta}_\delta - \boldsymbol{\omega}\times(\boldsymbol{J}\boldsymbol{\omega})\times\boldsymbol{\theta}_\delta + \boldsymbol{\omega}\times\boldsymbol{J}\boldsymbol{\omega}\times\boldsymbol{\theta}_\delta - (\boldsymbol{J}\dot{\boldsymbol{\omega}})\times\boldsymbol{\theta}_\delta + \boldsymbol{J}\dot{\boldsymbol{\omega}}\times\boldsymbol{\theta}_\delta.\end{aligned} \tag{8.14b}$$

As a consequence,

$$\begin{aligned}\left\{\begin{array}{c}\delta\ddot{\boldsymbol{x}}\\ \delta\dot{\boldsymbol{\omega}}\end{array}\right\} &= \boldsymbol{M}^{-1}\left(\left\{\begin{array}{c}\dot{\boldsymbol{\beta}}\\ \dot{\gamma}\end{array}\right\} - \left[\begin{array}{c}\boldsymbol{0}\\ (\boldsymbol{\omega}\times\boldsymbol{s})\times\end{array}\right]\delta\dot{\boldsymbol{x}} - \left[\begin{array}{c}-(\boldsymbol{\omega}\times\boldsymbol{s})\times-\boldsymbol{\omega}\times\boldsymbol{s}\times\\ \dot{\boldsymbol{x}}\times\boldsymbol{s}\times+\boldsymbol{\omega}\times\boldsymbol{J}-(\boldsymbol{J}\boldsymbol{\omega})\times\end{array}\right]\delta\boldsymbol{\omega}\right.\\ &\quad\left. - \left[\begin{array}{c}-\dot{\boldsymbol{\omega}}\times\boldsymbol{s}\times-\boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{s}\times\\ \dot{\boldsymbol{x}}\times\boldsymbol{\omega}\times\boldsymbol{s}\times+\ddot{\boldsymbol{x}}\times\boldsymbol{s}\times-\boldsymbol{\omega}\times(\boldsymbol{J}\boldsymbol{\omega})\times+\boldsymbol{\omega}\times\boldsymbol{J}\boldsymbol{\omega}\times-(\boldsymbol{J}\dot{\boldsymbol{\omega}})\times+\boldsymbol{J}\dot{\boldsymbol{\omega}}\times\end{array}\right]\boldsymbol{\theta}_\delta\right)\end{aligned} \tag{8.15}$$

(NOTE: simplifications when using updated-updated formulas).

Variable Mass Momentum:

$$\boldsymbol{\beta} = m\dot{\boldsymbol{x}}_{\mathrm{CM}} \tag{8.16}$$

with

$$\dot{\boldsymbol{x}}_{\mathrm{CM}} = \dot{\boldsymbol{x}} + \boldsymbol{\omega}\times\boldsymbol{b} + \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}. \tag{8.17}$$

$$\dot{\boldsymbol{\beta}} = m\ddot{\boldsymbol{x}}_{\mathrm{CM}} + \dot{m}\dot{\boldsymbol{x}}_{\mathrm{CM}} \tag{8.18}$$

Eq. (8.3a) becomes

$$\boldsymbol{f}_{\mathrm{in}} = \dot{\boldsymbol{\beta}} - \dot{m}\dot{\boldsymbol{x}}_{\mathrm{CM}}. \tag{8.19}$$

Only the portion of $\dot{\boldsymbol{\beta}}$ related to the change in mass is subtracted from the inertia forces; the portion related to the change in mass distribution, $\dot{\tilde{\boldsymbol{b}}}$, is retained.

Momenta moment:

$$\begin{aligned}\boldsymbol{\gamma} &= \boldsymbol{\gamma}_{\mathrm{CM}} + \boldsymbol{b}\times\boldsymbol{\beta}\\ &= \boldsymbol{J}_{\mathrm{CM}}\boldsymbol{\omega} + \boldsymbol{b}\times m\dot{\boldsymbol{x}}_{\mathrm{CM}}\end{aligned} \tag{8.20}$$

$$\dot{\boldsymbol{\gamma}}_{\text{CM}} = \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega} + \boldsymbol{J}_{\text{CM}}\dot{\boldsymbol{\omega}} + \left(\boldsymbol{J}_{\text{CM}\ \ \text{variable geometry}/t} + \boldsymbol{J}_{\text{CM}\ \ \text{variable mass}/t}\right)\boldsymbol{\omega} \tag{8.21}$$

$$\begin{aligned}
\dot{\boldsymbol{\gamma}} &= \dot{\boldsymbol{\gamma}}_{\text{CM}} + \dot{\boldsymbol{b}} \times \boldsymbol{\beta} + \boldsymbol{b} \times \dot{\boldsymbol{\beta}} \\
&= \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega} + \boldsymbol{J}_{\text{CM}}\dot{\boldsymbol{\omega}} + \left(\boldsymbol{J}_{\text{CM variable geometry}/t} + \boldsymbol{J}_{\text{CM variable mass}/t}\right)\boldsymbol{\omega} \\
&+ \left(\boldsymbol{\omega} \times \boldsymbol{b} + \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \times \boldsymbol{\beta} + \boldsymbol{b} \times \left(m\ddot{\boldsymbol{x}}_{\text{CM}} + \dot{m}\dot{\boldsymbol{x}}_{\text{CM}}\right)
\end{aligned} \tag{8.22}$$

Inertia moment of the whole body

$$\begin{aligned}
\boldsymbol{m}_{\text{in}} &= \dot{\boldsymbol{\gamma}}_{\text{CM}} + \boldsymbol{b} \times \dot{\boldsymbol{\beta}} \\
&= \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{b}} \times \boldsymbol{\beta} \\
&= \boldsymbol{\omega} \times \boldsymbol{J}_{\text{CM}}\boldsymbol{\omega} + \boldsymbol{J}_{\text{CM}}\dot{\boldsymbol{\omega}} + \left(\boldsymbol{J}_{\text{CM variable geometry}/t} + \boldsymbol{J}_{\text{CM variable mass}/t}\right)\boldsymbol{\omega} \\
&+ \boldsymbol{b} \times \left(m\ddot{\boldsymbol{x}}_{\text{CM}} + \dot{m}\dot{\boldsymbol{x}}_{\text{CM}}\right)
\end{aligned} \tag{8.23}$$

Inertia moment of the whole body minus the mass that is lost

$$\boldsymbol{m}_{\text{in}} = \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{b}} \times \boldsymbol{\beta} - \boldsymbol{J}_{\text{CM variable mass}/t}\boldsymbol{\omega} - \boldsymbol{b} \times \dot{m}\dot{\boldsymbol{x}}_{\text{CM}} \tag{8.24}$$

Only the portion of $\dot{\boldsymbol{\gamma}}$ related to the change in mass is subtracted from the inertia forces; the portion related to the change in mass distribution, $\boldsymbol{J}_{\text{CM variable geometry}/t}$, is retained.

Actually, in the constant mass code, the term $\dot{\boldsymbol{b}} \times \boldsymbol{\beta}$ is written as $-\boldsymbol{v} \times \boldsymbol{\beta}$, exploiting its structure:

$$\begin{aligned}
\dot{\boldsymbol{b}} \times \boldsymbol{\beta} &= m\left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \times \left(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{b}\right) \\
&= m\left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \times \boldsymbol{v} \\
&= m\left(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{b}\right) \times \boldsymbol{v} \\
&= \boldsymbol{\beta} \times \boldsymbol{v}
\end{aligned} \tag{8.25}$$

In the variable mass case, this can be written as

$$\begin{aligned}
\dot{\boldsymbol{b}} \times \boldsymbol{\beta} &= m\left(\boldsymbol{\omega} \times \boldsymbol{b} + \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \times \left(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{b} + \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \\
&= \boldsymbol{\beta} \times \boldsymbol{v} + m\left(\boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \times \left(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{b} + \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) + m\left(\boldsymbol{\omega} \times \boldsymbol{b}\right) \times \boldsymbol{R}\dot{\tilde{\boldsymbol{b}}} \\
&= \boldsymbol{\beta} \times \boldsymbol{v} + m\left(\boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \times \boldsymbol{v}
\end{aligned} \tag{8.26}$$

The correction term $\boldsymbol{\beta} \times \boldsymbol{v}$ is already included in the equations of motion by the `AutomaticStructuralElem` element associated with the underlying node; the correction term $m\left(\boldsymbol{R}\dot{\tilde{\boldsymbol{b}}}\right) \times \boldsymbol{v}$ must be explicitly applied by the `DynamicVariableBody` element.

Linearization: currently the `DynamicVariableBody` element does not contribute to the Jacobian matrix with respect to the variable inertia properties; only the original contribution of the constant inertia properties element is provided. This implies that the residual is exact to the required tolerance, while the Jacobian matrix is not; convergence should not be compromised as soon as moderate inertia variations are used.

8.1.2 Static Structural Nodes

Static structural nodes represent a degeneration of the dynamic ones, when the inertia is structurally null. In that case, no momentum nor momenta moment definition equations are instantiated, and their derivatives are always null. As a consequence, static nodes only instantiate their equilibrium equations.

8.1.3 Dummy Nodes

Dummy nodes are special structural nodes that do not directly participate in the analysis. In fact, they do not provide degrees of freedom; they rather present information associated with other structural nodes in a different manner, which is output in the `.mov` file much like if they were regular nodes.

Offset Dummy Node. The `offset` dummy structural node outputs the configuration of a reference frame that is rigidly offset from a base node (subscript b) by $\boldsymbol{b}_h$ and oriented by matrix $\boldsymbol{R}_h$ according to the transformation

$$\boldsymbol{x} = \boldsymbol{x}_b + \boldsymbol{R}_b \boldsymbol{b}_h \tag{8.27a}$$
$$\boldsymbol{R} = \boldsymbol{R}_b \boldsymbol{R}_h \tag{8.27b}$$
$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_b + \boldsymbol{\omega}_b \times \boldsymbol{R}_b \boldsymbol{b}_h \tag{8.27c}$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_b \tag{8.27d}$$
$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_b + \left(\dot{\boldsymbol{\omega}}_b \times + \boldsymbol{\omega}_b \times \boldsymbol{\omega}_b \times\right) \boldsymbol{R}_b \boldsymbol{b}_h \tag{8.27e}$$
$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_b \tag{8.27f}$$

Reference Frame Dummy Node. The `reference frame` dummy structural node outputs the configuration of the base node (subscript b) in the reference frame of a reference node (subscript r), optionally offset by $\boldsymbol{b}_h$ and with an orientation relative to r provided by matrix $\boldsymbol{R}_h$, according to the transformation

$$\overline{\boldsymbol{x}} = \boldsymbol{R}_h^T \left(\boldsymbol{R}_r^T \left(\boldsymbol{x}_b - \boldsymbol{x}_r\right) - \boldsymbol{b}_h\right) \tag{8.28a}$$
$$\overline{\boldsymbol{R}} = \boldsymbol{R}_h^T \boldsymbol{R}_r^T \boldsymbol{R}_b \tag{8.28b}$$
$$\dot{\overline{\boldsymbol{x}}} = \boldsymbol{R}_h^T \boldsymbol{R}_r^T \left(\dot{\boldsymbol{x}}_b - \dot{\boldsymbol{x}}_r - \boldsymbol{\omega}_r \times \left(\boldsymbol{x}_b - \boldsymbol{x}_r\right)\right) \tag{8.28c}$$
$$\overline{\boldsymbol{\omega}} = \boldsymbol{R}_h^T \boldsymbol{R}_r^T \left(\boldsymbol{\omega}_b - \boldsymbol{\omega}_r\right) \tag{8.28d}$$
$$\ddot{\overline{\boldsymbol{x}}} = \boldsymbol{R}_h^T \boldsymbol{R}_r^T \left(\ddot{\boldsymbol{x}}_b - \ddot{\boldsymbol{x}}_r - \left(\dot{\boldsymbol{\omega}}_r \times + \boldsymbol{\omega}_r \times \boldsymbol{\omega}_r \times\right) \left(\boldsymbol{x}_b - \boldsymbol{x}_r\right) - 2\boldsymbol{\omega}_r \times \left(\dot{\boldsymbol{x}}_b - \dot{\boldsymbol{x}}_r\right)\right) \tag{8.28e}$$
$$\dot{\overline{\boldsymbol{\omega}}} = \boldsymbol{R}_h^T \boldsymbol{R}_r^T \left(\dot{\boldsymbol{\omega}}_b - \dot{\boldsymbol{\omega}}_r - \boldsymbol{\omega}_r \times \boldsymbol{\omega}_b\right) \tag{8.28f}$$

Pivot Reference Frame Dummy Node. The `pivot reference frame` dummy node is a variant of the `reference frame` dummy node that outputs the configuration of the base node, expressed in the reference frame of the reference node, as if it were attached to the pivot node (subscript p), optionally offset by $\boldsymbol{b}_k$ and with an orientation relative to p provided by matrix $\boldsymbol{R}_k$, according to the transformation

$$\hat{\boldsymbol{x}} = \boldsymbol{x}_p + \boldsymbol{R}_p \left(\boldsymbol{R}_k \overline{\boldsymbol{x}} + \boldsymbol{b}_k\right) \tag{8.29a}$$
$$\hat{\boldsymbol{R}} = \boldsymbol{R}_p \boldsymbol{R}_k \overline{\boldsymbol{R}} \tag{8.29b}$$
$$\dot{\hat{\boldsymbol{x}}} = \dot{\boldsymbol{x}}_p + \boldsymbol{\omega}_p \times \boldsymbol{R}_p \left(\boldsymbol{R}_k \overline{\boldsymbol{x}} + \boldsymbol{b}_k\right) + \boldsymbol{R}_p \boldsymbol{R}_k \dot{\overline{\boldsymbol{x}}} \tag{8.29c}$$
$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_p + \boldsymbol{R}_p \boldsymbol{R}_k \overline{\boldsymbol{\omega}} \tag{8.29d}$$
$$\ddot{\hat{\boldsymbol{x}}} = \ddot{\boldsymbol{x}}_p + \left(\dot{\boldsymbol{\omega}}_p \times + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p \times\right) \boldsymbol{R}_p \left(\boldsymbol{R}_k \overline{\boldsymbol{x}} + \boldsymbol{b}_k\right) + 2\boldsymbol{\omega}_p \times \boldsymbol{R}_p \boldsymbol{R}_k \dot{\overline{\boldsymbol{x}}} + \boldsymbol{R}_p \boldsymbol{R}_k \ddot{\overline{\boldsymbol{x}}} \tag{8.29e}$$
$$\dot{\hat{\boldsymbol{\omega}}} = \dot{\boldsymbol{\omega}}_p + \boldsymbol{\omega}_p \times \boldsymbol{R}_p \boldsymbol{R}_k \overline{\boldsymbol{\omega}} + \boldsymbol{R}_p \boldsymbol{R}_k \dot{\overline{\boldsymbol{\omega}}} \tag{8.29f}$$

8.1.4 Relative Motion

MBDyn uses the absolute coordinates of the nodes, and their absolute orientation, to define the motion. Whenever the representation of the motion in a relative reference frame is required or convenient, it needs to be computed from the absolute one that is computed inside the code.

This is possible either run-time, by adding `dummy` structural nodes, or off-line, as a post-processing. For this purpose, the script `abs2rel.awk` is provided to convert the contents of the `.mov` file into the corresponding motion relative to a given node.

Absolute motion as function of relative (tilde) and reference (0) motion:

$$\boldsymbol{R} = \boldsymbol{R}_0 \tilde{\boldsymbol{R}} \tag{8.30a}$$

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{R}_0 \tilde{\boldsymbol{x}} \tag{8.30b}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}} \tag{8.30c}$$

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} \tag{8.30d}$$

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{\omega}}} \tag{8.30e}$$

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_0 + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \ddot{\tilde{\boldsymbol{x}}} \tag{8.30f}$$

Relative motion (tilde) as function of absolute and reference (0) motion:

$$\tilde{\boldsymbol{R}} = \boldsymbol{R}_0^T \boldsymbol{R} \tag{8.31a}$$

$$\tilde{\boldsymbol{x}} = \boldsymbol{R}_0^T \left(\boldsymbol{x} - \boldsymbol{x}_0\right) \tag{8.31b}$$

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{R}_0^T \left(\boldsymbol{\omega} - \boldsymbol{\omega}_0\right) \tag{8.31c}$$

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \boldsymbol{R}_0^T \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0 - \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}}\right) \\ &= \boldsymbol{R}_0^T \left(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0\right) - \left(\boldsymbol{R}_0^T \boldsymbol{\omega}_0\right) \times \tilde{\boldsymbol{x}} \end{aligned} \tag{8.31d}$$

$$\begin{aligned} \dot{\tilde{\boldsymbol{\omega}}} &= \boldsymbol{R}_0^T \left(\dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_0 - \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}}\right) \\ &= \boldsymbol{R}_0^T \left(\dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_0\right) - \left(\boldsymbol{R}_0^T \boldsymbol{\omega}_0\right) \times \tilde{\boldsymbol{\omega}} \end{aligned} \tag{8.31e}$$

$$\begin{aligned} \ddot{\tilde{\boldsymbol{x}}} &= \boldsymbol{R}_0^T \left(\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_0 - \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} - \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} - 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}}\right) \\ &= \boldsymbol{R}_0^T \left(\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_0\right) - \left(\boldsymbol{R}_0^T \dot{\boldsymbol{\omega}}_0\right) \times \tilde{\boldsymbol{x}} \\ &\quad - \left(\boldsymbol{R}_0^T \boldsymbol{\omega}_0\right) \times \left(\boldsymbol{R}_0^T \boldsymbol{\omega}_0\right) \times \tilde{\boldsymbol{x}} - 2\left(\boldsymbol{R}_0^T \boldsymbol{\omega}_0\right) \times \dot{\tilde{\boldsymbol{x}}} \end{aligned} \tag{8.31f}$$

Perturbation, assuming reference motion is imposed:

$$\boldsymbol{\theta}_\delta = \boldsymbol{R}_0 \tilde{\boldsymbol{\theta}}_\delta \tag{8.32a}$$

$$\delta \boldsymbol{x} = \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} \tag{8.32b}$$

$$\delta \boldsymbol{\omega} = \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} \tag{8.32c}$$

$$\delta \dot{\boldsymbol{x}} = \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} \tag{8.32d}$$

$$\delta \dot{\boldsymbol{\omega}} = \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{\omega}}} \tag{8.32e}$$

$$\begin{aligned} \delta \ddot{\boldsymbol{x}} &= \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \delta \ddot{\tilde{\boldsymbol{x}}} \\ &= \left(\dot{\boldsymbol{\omega}}_0 \times {} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times {}\right) \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \delta \ddot{\tilde{\boldsymbol{x}}} \end{aligned} \tag{8.32f}$$

Perturbation, when reference motion is independent:

$$\begin{aligned}
\boldsymbol{\theta}_\delta &= \boldsymbol{\theta}_{0\delta} + \boldsymbol{R}_0 \tilde{\boldsymbol{\theta}}_\delta && \text{(8.33a)}\\
\delta \boldsymbol{x} &= \delta \boldsymbol{x}_0 + \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} \\
&= \delta \boldsymbol{x}_0 - (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \boldsymbol{\theta}_{0\delta} + \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} && \text{(8.33b)}\\
\delta \boldsymbol{\omega} &= \delta \boldsymbol{\omega}_0 + \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} \\
&= \delta \boldsymbol{\omega}_0 - (\boldsymbol{R}_0 \tilde{\boldsymbol{\omega}}) \times \boldsymbol{\theta}_{0\delta} + \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} && \text{(8.33c)}\\
\delta \dot{\boldsymbol{x}} &= \delta \dot{\boldsymbol{x}}_0 + \delta \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} \\
&\quad + \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} \\
&= \delta \dot{\boldsymbol{x}}_0 - (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \delta \boldsymbol{\omega}_0 - \left(\boldsymbol{\omega}_0 \times (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \ + \left(\boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} \right) \times \right) \boldsymbol{\theta}_{0\delta} \\
&\quad + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} && \text{(8.33d)}\\
\delta \dot{\boldsymbol{\omega}} &= \delta \dot{\boldsymbol{\omega}}_0 + \delta \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}} + \boldsymbol{\omega}_0 \times \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{\omega}} + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} \\
&\quad + \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{\omega}}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{\omega}}} \\
&= \delta \dot{\boldsymbol{\omega}}_0 - (\boldsymbol{R}_0 \tilde{\boldsymbol{\omega}}) \times \delta \boldsymbol{\omega}_0 - \left(\boldsymbol{\omega}_0 \times (\boldsymbol{R}_0 \tilde{\boldsymbol{\omega}}) \times \ + \left(\boldsymbol{R}_0 \dot{\tilde{\boldsymbol{\omega}}} \right) \times \right) \boldsymbol{\theta}_{0\delta} \\
&\quad + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{\omega}}} && \text{(8.33e)}\\
\delta \ddot{\boldsymbol{x}} &= \delta \ddot{\boldsymbol{x}}_0 \\
&\quad + \delta \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} \\
&\quad + \delta \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \delta \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} \\
&\quad + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} \\
&\quad + 2 \delta \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} + 2 \boldsymbol{\omega}_0 \times \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} + 2 \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} \\
&\quad + \boldsymbol{\theta}_{0\delta} \times \boldsymbol{R}_0 \ddot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \delta \ddot{\tilde{\boldsymbol{x}}} \\
&= \delta \ddot{\boldsymbol{x}}_0 - (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \delta \dot{\boldsymbol{\omega}}_0 \\
&\quad - \left((\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \ + \boldsymbol{\omega}_0 \times (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \ + 2 \left(\boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} \right) \times \right) \delta \boldsymbol{\omega}_0 \\
&\quad - \left((\dot{\boldsymbol{\omega}}_0 \times \ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \,) (\boldsymbol{R}_0 \tilde{\boldsymbol{x}}) \times \ + 2 \boldsymbol{\omega}_0 \times \left(\boldsymbol{R}_0 \dot{\tilde{\boldsymbol{x}}} \right) \times \ + \left(\boldsymbol{R}_0 \ddot{\tilde{\boldsymbol{x}}} \right) \times \right) \boldsymbol{\theta}_{0\delta} \\
&\quad + (\dot{\boldsymbol{\omega}}_0 \times \ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \,) \boldsymbol{R}_0 \delta \tilde{\boldsymbol{x}} + 2 \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \delta \dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}_0 \delta \ddot{\tilde{\boldsymbol{x}}} && \text{(8.33f)}
\end{aligned}$$

8.1.5 Motion Expressed in a Relative Reference Frame

An alternative, useful case is given by representing the absolute motion in a relative reference frame:

$$\begin{aligned}
\tilde{\boldsymbol{R}} &= \boldsymbol{R}_0^T \boldsymbol{R} && \text{(8.34a)}\\
\tilde{\boldsymbol{x}} &= \boldsymbol{R}_0^T (\boldsymbol{x} - \boldsymbol{x}_0) && \text{(8.34b)}\\
\overline{\boldsymbol{\omega}} &= \boldsymbol{R}_0^T \boldsymbol{\omega} && \text{(8.34c)}\\
\dot{\overline{\boldsymbol{x}}} &= \boldsymbol{R}_0^T \dot{\boldsymbol{x}} && \text{(8.34d)}\\
\dot{\overline{\boldsymbol{\omega}}} &= \boldsymbol{R}_0^T \dot{\boldsymbol{\omega}} && \text{(8.34e)}\\
\ddot{\overline{\boldsymbol{x}}} &= \boldsymbol{R}_0^T \ddot{\boldsymbol{x}} && \text{(8.34f)}
\end{aligned}$$

8.1.6 Dynamics in a Relative Reference Frame

The rationale is that MBDyn is based on the assumption that the kinematics of the nodes and their equations of motion are formulated in the global reference frame. However, in some cases one may want to express the motion of the nodes in a relative reference frame.

The following discussion is intended to modify the inertial contribution to the equilibrium in order to reformulate the dynamical problem in a relative reference frame, whose motion is imposed. The constraints and the deformable components are unaffected, as they already intrinsically based on relative kinematics.

Momentum and momenta moments

$$\boldsymbol{\beta} = m\dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{s}_0 + \boldsymbol{R}_0\tilde{\boldsymbol{\beta}} \tag{8.35a}$$
$$\boldsymbol{\gamma} = \boldsymbol{s} \times (\dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{x}}) + \boldsymbol{J}\boldsymbol{\omega}_0 + \boldsymbol{R}_0\tilde{\boldsymbol{\gamma}} \tag{8.35b}$$

where

$$\boldsymbol{s}_0 = \boldsymbol{R}_0\tilde{\boldsymbol{s}}_0 \tag{8.36}$$
$$\tilde{\boldsymbol{s}}_0 = \tilde{\boldsymbol{s}} + m\tilde{\boldsymbol{x}} \tag{8.37}$$
$$\boldsymbol{s} = \boldsymbol{R}_0\tilde{\boldsymbol{s}} \tag{8.38}$$
$$\boldsymbol{J} = \boldsymbol{R}_0\tilde{\boldsymbol{J}}\boldsymbol{R}_0^T \tag{8.39}$$

Momentum and momenta moments derivative

$$\dot{\boldsymbol{\beta}} = m\ddot{\boldsymbol{x}}_0 + (\dot{\boldsymbol{\omega}}_0 \times\ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times)\, \boldsymbol{R}_0\tilde{\boldsymbol{s}}_0 + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\beta} + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{\beta}}} \tag{8.40a}$$
$$\begin{aligned}\dot{\boldsymbol{\gamma}} &= \dot{\boldsymbol{s}} \times (\dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{x}}) \\ &\quad + \boldsymbol{s} \times \left(\ddot{\boldsymbol{x}}_0 + (\dot{\boldsymbol{\omega}}_0 \times\ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times)\, \boldsymbol{R}_0\tilde{\boldsymbol{x}} + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\dot{\tilde{\boldsymbol{x}}}\right) \\ &\quad + \boldsymbol{\omega}_0 \times \boldsymbol{J}\boldsymbol{\omega}_0 + \boldsymbol{J}\dot{\boldsymbol{\omega}}_0 + \boldsymbol{R}_0\left(\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{J}} - \tilde{\boldsymbol{J}}\tilde{\boldsymbol{\omega}} \times\right)\boldsymbol{R}_0^T\boldsymbol{\omega}_0 \\ &\quad + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{\gamma}} + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{\gamma}}}\end{aligned} \tag{8.40b}$$

Inertia forces and moments projected in the relative frame

$$\begin{aligned}\overline{\boldsymbol{f}}_{\text{in}} &= \boldsymbol{R}_0^T\dot{\boldsymbol{\beta}} \\ &= m\ddot{\overline{\boldsymbol{x}}}_0 + \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times\ + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right)\tilde{\boldsymbol{s}}_0 + 2\overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\beta}} + \underbrace{\dot{\tilde{\boldsymbol{\beta}}}}_{\text{relative momentum}}\end{aligned} \tag{8.41a}$$
$$\begin{aligned}\overline{\boldsymbol{m}}_{\text{in}} &= \boldsymbol{R}_0^T(\dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{x}} \times \boldsymbol{\beta}) \\ &= \tilde{\boldsymbol{s}} \times \left(\ddot{\overline{\boldsymbol{x}}}_0 + \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times\ + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right)\tilde{\boldsymbol{x}} + \overline{\boldsymbol{\omega}}_0 \times \dot{\tilde{\boldsymbol{x}}}\right) \\ &\quad + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{J}}\overline{\boldsymbol{\omega}}_0 + \tilde{\boldsymbol{J}}\dot{\overline{\boldsymbol{\omega}}}_0 + \left(\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{J}} - \tilde{\boldsymbol{J}}\tilde{\boldsymbol{\omega}} \times\right)\overline{\boldsymbol{\omega}}_0 \\ &\quad + \dot{\tilde{\boldsymbol{x}}} \times \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{s}} + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\gamma}} + \underbrace{\dot{\tilde{\boldsymbol{x}}} \times \tilde{\boldsymbol{\beta}} + \dot{\tilde{\boldsymbol{\gamma}}}}_{\text{relative momenta moment}}\end{aligned} \tag{8.41b}$$

The contributions

$$\overline{\boldsymbol{f}}^*_{\text{in}} = 2\overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\beta}} + \dot{\tilde{\boldsymbol{\beta}}} \tag{8.42a}$$
$$\overline{\boldsymbol{m}}^*_{\text{in}} = \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\gamma}} + \dot{\tilde{\boldsymbol{x}}} \times \tilde{\boldsymbol{\beta}} + \dot{\tilde{\boldsymbol{\gamma}}} \tag{8.42b}$$

belong to the `automatic structural` element associated to each dynamic node. As such, they are computed once for all irrespective of the number of rigid bodies that are connected to the node. The remaining contributions are computed by each rigid body.

Perturbation, assuming reference motion is imposed:

$$\delta \overline{\boldsymbol{f}}_{\text{in}} = \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \left(m \delta \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{s}} \times \tilde{\boldsymbol{\theta}}_\delta\right) \tag{8.43a}$$

$$+ 2 \overline{\boldsymbol{\omega}}_0 \times \delta \tilde{\boldsymbol{\beta}} + \delta \dot{\tilde{\boldsymbol{\beta}}} \tag{8.43b}$$

$$\begin{aligned}
\delta \overline{\boldsymbol{m}}_{\text{in}} = \Big(& \left(\ddot{\overline{\boldsymbol{x}}}_0 + \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \tilde{\boldsymbol{x}} + \overline{\boldsymbol{\omega}}_0 \times \dot{\tilde{\boldsymbol{x}}}\right) \times \tilde{\boldsymbol{s}} \times \\
& + \left(\overline{\boldsymbol{\omega}}_0 + \tilde{\boldsymbol{\omega}}\right) \times \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0 \times - \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0\right) \times\right) \\
& + \tilde{\boldsymbol{J}} \left(\dot{\overline{\boldsymbol{\omega}}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\omega}}\right) \times - \left(\tilde{\boldsymbol{J}} \left(\dot{\overline{\boldsymbol{\omega}}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\omega}}\right)\right) \times \\
& - \dot{\tilde{\boldsymbol{x}}} \times \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{s}} \times \Big) \tilde{\boldsymbol{\theta}}_\delta \\
& + \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0 \times - \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0\right) \times\right) \delta \tilde{\boldsymbol{\omega}} \\
& + \tilde{\boldsymbol{s}} \times \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \delta \tilde{\boldsymbol{x}} \\
& + \left(\tilde{\boldsymbol{s}} \times \overline{\boldsymbol{\omega}}_0 \times - \left(\overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{s}}\right) \times - \tilde{\boldsymbol{\beta}} \times\right) \delta \dot{\tilde{\boldsymbol{x}}} \\
& + \overline{\boldsymbol{\omega}}_0 \times \delta \tilde{\boldsymbol{\gamma}} + \dot{\tilde{\boldsymbol{x}}} \times \delta \tilde{\boldsymbol{\beta}} + \delta \dot{\tilde{\boldsymbol{\gamma}}}
\end{aligned} \tag{8.43c}$$

To summarize:

$$\begin{aligned}
\left\{ \begin{array}{c} \delta \overline{\boldsymbol{f}}_{\text{in}} \\ \delta \overline{\boldsymbol{m}}_{\text{in}} \end{array} \right\} = & \left[\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{array} \right] \left\{ \begin{array}{c} \delta \dot{\tilde{\boldsymbol{\beta}}} \\ \delta \dot{\tilde{\boldsymbol{\gamma}}} \end{array} \right\} + \left[\begin{array}{cc} 2 \overline{\boldsymbol{\omega}}_0 \times & \boldsymbol{0} \\ \dot{\tilde{\boldsymbol{x}}} \times & \overline{\boldsymbol{\omega}} \times \end{array} \right] \left\{ \begin{array}{c} \delta \tilde{\boldsymbol{\beta}} \\ \delta \tilde{\boldsymbol{\gamma}} \end{array} \right\} \\
& + \left[\begin{array}{cc} \boldsymbol{0} & \boldsymbol{0} \\ \left(\tilde{\boldsymbol{s}} \times \overline{\boldsymbol{\omega}}_0 \times - \left(\overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{s}}\right) \times - \tilde{\boldsymbol{\beta}} \times\right) & \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0 \times - \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0\right) \times\right) \end{array} \right] \left\{ \begin{array}{c} \delta \dot{\tilde{\boldsymbol{x}}} \\ \delta \tilde{\boldsymbol{\omega}} \end{array} \right\} \\
& + \left[\begin{array}{c} m \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \\ \tilde{\boldsymbol{s}} \times \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \end{array} \right] \delta \tilde{\boldsymbol{x}} \\
& + \left[\begin{array}{c} - \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \tilde{\boldsymbol{s}} \times \\ \left(\begin{array}{c} \left(\ddot{\overline{\boldsymbol{x}}}_0 + \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \tilde{\boldsymbol{x}} + \overline{\boldsymbol{\omega}}_0 \times \dot{\tilde{\boldsymbol{x}}}\right) \times \tilde{\boldsymbol{s}} \times \\ - \dot{\tilde{\boldsymbol{x}}} \times \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{s}} \times + \overline{\boldsymbol{\omega}} \times \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0 \times - \left(\tilde{\boldsymbol{J}} \overline{\boldsymbol{\omega}}_0\right) \times\right) \\ + \tilde{\boldsymbol{J}} \left(\dot{\overline{\boldsymbol{\omega}}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\omega}}\right) \times - \left(\tilde{\boldsymbol{J}} \left(\dot{\overline{\boldsymbol{\omega}}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\omega}}\right)\right) \times \end{array} \right) \end{array} \right] \tilde{\boldsymbol{\theta}}_\delta
\end{aligned} \tag{8.44}$$

8.1.7 Airstream Velocity

Aerodynamic elements need the absolute velocity of specific points in order to evaluate the boundary conditions for the computation of the aerodynamic forces.

The velocity of the airstream at a given absolute location $\boldsymbol{x}$ is

$$\boldsymbol{v}_{\text{as}} = \boldsymbol{v}_{\text{as}}(\boldsymbol{x}, t), \tag{8.45}$$

which results from the superimposition of a time-dependent airstream velocity and of a time- and space-dependent gust velocity,

$$\boldsymbol{v}_{\text{as}} = \boldsymbol{v}_{\text{as0}}(t) + \boldsymbol{v}_{\text{g}}(\boldsymbol{x}, t). \tag{8.46}$$

The velocity of point $\boldsymbol{x}$ with respect to the air at point $\boldsymbol{x}$ itself is thus

$$\boldsymbol{v}_{\mathrm{a}} = \dot{\boldsymbol{x}} - \boldsymbol{v}_{\mathrm{as}}. \tag{8.47}$$

When the motion of the point is expressed in the relative reference frame, the position of the point is expressed by Eq. (8.30d). The velocity of Eq. (8.47) becomes

$$\boldsymbol{v}_{\mathrm{a}} = \dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{x}} + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{x}}} - \boldsymbol{v}_{\mathrm{as}}. \tag{8.48}$$

This velocity, projected in the relative reference frame, is

$$\begin{aligned}\overline{\boldsymbol{v}}_{\mathrm{a}} &= \boldsymbol{R}_0^T \boldsymbol{v}_{\mathrm{a}} \\ &= \dot{\overline{\boldsymbol{x}}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{x}} + \dot{\tilde{\boldsymbol{x}}} - \overline{\boldsymbol{v}}_{\mathrm{as}}.\end{aligned} \tag{8.49}$$

When referring aerodynamic elements to the relative frame, the equivalent airstream speed is thus

$$\boldsymbol{v}_{\mathrm{as\ equivalent}} = \overline{\boldsymbol{v}}_{\mathrm{as}} - \dot{\overline{\boldsymbol{x}}}_0 - \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{x}}, \tag{8.50}$$

so that the velocity of the air at point $\tilde{\boldsymbol{x}}$ is computed as

$$\overline{\boldsymbol{v}}_{\mathrm{a}} = \dot{\tilde{\boldsymbol{x}}} - \boldsymbol{v}_{\mathrm{as\ equivalent}}, \tag{8.51}$$

in analogy with Eq. (8.47). This allows to confine the relative reference frame modification of the aerodynamic forces in the computation of the airstream velocity.

8.1.8 Implementation Notes

Kinematics

The relative reference frame dynamics is under development. In the current design, the whole model is subjected to a single relative reference frame. The "ground" is intended to be rigidly connected to the relative reference frame (e.g. all "pin" joints and the "clamp" joint). Deformable elements and kinematic constraints do not discriminate between absolute and relative reference frame, since they only deal with the relative kinematics of the nodes they connect.

The relative frame motion is delegated to the `DataManager`. The `StructNode` is responsible for providing a pointer to a `RigidBodyKinematics` object that describes the relative frame motion.

The kinematic parameters received in input with the `rigid body kinematics` card in the `control data` section are intended expressed in the global reference frame. Note however that the calls to the `Get*()` methods of the `RigidBodyKinematics` class return them in the relative reference frame, namely pre-multiplied by the transpose of the orientation matrix:

$$\texttt{GetR()}: \quad \boldsymbol{R}_0 = \texttt{\{}\ \exp(\texttt{abs\_orientation\_vector} \times) \ \ \texttt{|}\ \texttt{abs\_orientation}\ \texttt{\}} \tag{8.52a}$$

$$\texttt{GetX()}: \quad \overline{\boldsymbol{x}}_0 = \boldsymbol{R}_0^T \texttt{abs\_position} \tag{8.52b}$$

$$\texttt{GetW()}: \quad \overline{\boldsymbol{\omega}}_0 = \boldsymbol{R}_0^T \texttt{abs\_angular\_velocity} \tag{8.52c}$$

$$\texttt{GetV()}: \quad \dot{\overline{\boldsymbol{x}}}_0 = \boldsymbol{R}_0^T \texttt{abs\_velocity} \tag{8.52d}$$

$$\texttt{GetWP()}: \quad \dot{\overline{\boldsymbol{\omega}}}_0 = \boldsymbol{R}_0^T \texttt{abs\_angular\_acceleration} \tag{8.52e}$$

$$\texttt{GetXPP()}: \quad \ddot{\overline{\boldsymbol{x}}}_0 = \boldsymbol{R}_0^T \texttt{abs\_acceleration} \tag{8.52f}$$

Dynamics

The equations of motion (force and moment equilibrium) and the definitions of the momentum and momenta moment are not altered.

The `AutomaticStructElem` element takes care of the portion of relative frame inertia forces that depends on the momentum and on the momenta moment. The `DynamicBody` element takes care of the remaining portion of relative frame inertia forces. The `StaticBody` element has been modified accordingly.

Air Properties

The `AirProperties` element takes care of the relative frame motion in order to project the absolute reference frame airstream velocity into the relative reference frame, according to Section 8.1.7.

All other built-in aerodynamic elements should naturally inherit the correct air velocity at the reference points.

Future Development

Future development may allow to have multiple relative reference frame submodels, connected to each other and to the absolute reference frame submodel, if any, by dedicated algebraic constraints.

8.1.9 Pseudo-Velocities Approach

NOTE: this is not currently implemented, nor foreseen.

The pseudo-velocities approach consists in replacing the momentum and momenta moment definitions with the definitions of the angular and linear velocities,

$$\boldsymbol{v} = \dot{\boldsymbol{x}} \tag{8.53a}$$

$$\boldsymbol{\omega} = \mathrm{ax}\left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right) \tag{8.53b}$$

An advantage is that this approach allows an easier implementation of constraint stabilization.

Kinematics:

$$\boldsymbol{R} := \text{node orientation matrix} \tag{8.54a}$$

$$\boldsymbol{x} := \text{node position} \tag{8.54b}$$

Angular and linear velocity, $\boldsymbol{\omega}$ and $\boldsymbol{v}$:

$$\boldsymbol{\omega} = \mathrm{ax}\left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right) \tag{8.55a}$$

$$\boldsymbol{v} = \dot{\boldsymbol{x}} \tag{8.55b}$$

Inertia forces and moments:

$$\boldsymbol{f}_{\mathrm{in}} = m\dot{\boldsymbol{v}}_{\mathrm{CM}} \tag{8.56a}$$

$$\boldsymbol{m}_{\mathrm{in}} = \boldsymbol{\omega} \times \boldsymbol{J}_{\mathrm{CM}}\boldsymbol{\omega} + \boldsymbol{J}_{\mathrm{CM}}\dot{\boldsymbol{\omega}} + \boldsymbol{s} \times \dot{\boldsymbol{v}}_{\mathrm{CM}} \tag{8.56b}$$

where

$$\boldsymbol{s} \qquad \text{static moment with respect to the node position} \tag{8.57a}$$

$$\boldsymbol{J}_{\mathrm{CM}} \qquad \text{inertia tensor with respect to the center of mass} \tag{8.57b}$$

$$\boldsymbol{x}_{\mathrm{CM}} = \boldsymbol{x} + \frac{1}{m}\boldsymbol{s} \qquad \text{position of the center of mass} \tag{8.57c}$$

$$\boldsymbol{v}_{\mathrm{CM}} = \boldsymbol{v} + \frac{1}{m}\boldsymbol{\omega} \times \boldsymbol{s} \qquad \text{velocity of the center of mass} \tag{8.57d}$$

$$\dot{\boldsymbol{v}}_{\mathrm{CM}} = \dot{\boldsymbol{v}} + \frac{1}{m}\left(\dot{\boldsymbol{\omega}} \times\ + \boldsymbol{\omega} \times \boldsymbol{\omega} \times\right)\boldsymbol{s} \tag{8.57e}$$

Perturbation

$$\delta \boldsymbol{s} = \boldsymbol{\theta}_\delta \times \boldsymbol{s} \tag{8.58a}$$

$$\delta \boldsymbol{J}_{\mathrm{CM}} = \boldsymbol{\theta}_\delta \times \boldsymbol{J}_{\mathrm{CM}} - \boldsymbol{J}_{\mathrm{CM}}\boldsymbol{\theta}_\delta \times \tag{8.58b}$$

$$\begin{aligned}\delta \boldsymbol{x}_{\mathrm{CM}} &= \delta \boldsymbol{x} + \frac{1}{m}\delta \boldsymbol{s} \\ &= \delta \boldsymbol{x} - \frac{1}{m}\boldsymbol{s} \times \boldsymbol{\theta}_\delta\end{aligned} \tag{8.58c}$$

$$\begin{aligned}\boldsymbol{v}_{\mathrm{CM}} &= \delta \boldsymbol{v} + \frac{1}{m}\delta\boldsymbol{\omega} \times \boldsymbol{s} + \frac{1}{m}\boldsymbol{\omega} \times \delta \boldsymbol{s} \\ &= \delta \boldsymbol{v} - \frac{1}{m}\boldsymbol{s} \times \delta\boldsymbol{\omega} - \frac{1}{m}\boldsymbol{\omega} \times \boldsymbol{s} \times \boldsymbol{\theta}_\delta\end{aligned} \tag{8.58d}$$

$$\begin{aligned}\delta \dot{\boldsymbol{v}}_{\mathrm{CM}} &= \delta \dot{\boldsymbol{v}} + \frac{1}{m}\left(\delta\dot{\boldsymbol{\omega}} \times \boldsymbol{s} + \delta\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{s} + \boldsymbol{\omega} \times \delta\boldsymbol{\omega} \times \boldsymbol{s} + \left(\dot{\boldsymbol{\omega}} \times\ + \boldsymbol{\omega} \times \boldsymbol{\omega} \times\right)\delta \boldsymbol{s}\right) \\ &= \delta \dot{\boldsymbol{v}} - \frac{1}{m}\boldsymbol{s} \times \delta\dot{\boldsymbol{\omega}} - \frac{1}{m}\left((\boldsymbol{\omega} \times \boldsymbol{s}) \times\ + \boldsymbol{\omega} \times \boldsymbol{s} \times\right)\delta\boldsymbol{\omega} \\ &\quad - \frac{1}{m}\left(\dot{\boldsymbol{\omega}} \times\ + \boldsymbol{\omega} \times \boldsymbol{\omega} \times\right)\boldsymbol{s} \times \boldsymbol{\theta}_\delta\end{aligned} \tag{8.58e}$$

Inertia forces and moments with respect to nodal quantities:

$$\begin{aligned}\boldsymbol{f}_{\mathrm{in}} &= m\dot{\boldsymbol{v}}_{\mathrm{CM}} \\ &= m\dot{\boldsymbol{v}} + \left(\dot{\boldsymbol{\omega}} \times\ + \boldsymbol{\omega} \times \boldsymbol{\omega} \times\right)\boldsymbol{s}\end{aligned} \tag{8.59a}$$

$$\begin{aligned}\boldsymbol{m}_{\mathrm{in}} &= \boldsymbol{\omega} \times \boldsymbol{J}_{\mathrm{CM}}\boldsymbol{\omega} + \boldsymbol{J}_{\mathrm{CM}}\dot{\boldsymbol{\omega}} + \boldsymbol{s} \times \dot{\boldsymbol{v}}_{\mathrm{CM}} \\ &= \boldsymbol{s} \times \dot{\boldsymbol{v}} + \boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega}\end{aligned} \tag{8.59b}$$

with

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{CM}} - \frac{1}{m}\boldsymbol{s} \times \boldsymbol{s} \times \tag{8.60}$$

the inertia tensor with respect to the node.

Perturbation:

$$\begin{aligned}\delta \boldsymbol{f}_{\mathrm{in}} &= m\delta\dot{\boldsymbol{v}} - \boldsymbol{s} \times \delta\dot{\boldsymbol{\omega}} - \left((\boldsymbol{\omega} \times \boldsymbol{s}) \times\ + \boldsymbol{\omega} \times \boldsymbol{s} \times\right)\delta\boldsymbol{\omega} \\ &\quad - \left(\dot{\boldsymbol{\omega}} \times\ + \boldsymbol{\omega} \times \boldsymbol{\omega} \times\right)\boldsymbol{s} \times \boldsymbol{\theta}_\delta\end{aligned} \tag{8.61a}$$

$$\begin{aligned}\delta \boldsymbol{m}_{\mathrm{in}} &= \boldsymbol{s} \times \delta\dot{\boldsymbol{v}} + \boldsymbol{J}\delta\dot{\boldsymbol{\omega}} + \left(\boldsymbol{\omega} \times \boldsymbol{J} - (\boldsymbol{J}\boldsymbol{\omega}) \times\right)\delta\boldsymbol{\omega} \\ &\quad + \left(\boldsymbol{\omega} \times \left(\boldsymbol{J}\boldsymbol{\omega} \times\ - (\boldsymbol{J}\boldsymbol{\omega}) \times\right) + \boldsymbol{J}\dot{\boldsymbol{\omega}} \times\ - (\boldsymbol{J}\dot{\boldsymbol{\omega}}) \times\ + \dot{\boldsymbol{v}} \times \boldsymbol{s} \times\right)\boldsymbol{\theta}_\delta\end{aligned} \tag{8.61b}$$

Relative frame kinematics:

$$\boldsymbol{R} = \boldsymbol{R}_0\tilde{\boldsymbol{R}} \tag{8.62a}$$
$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{R}_0\tilde{\boldsymbol{x}} \tag{8.62b}$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{R}_0\tilde{\boldsymbol{\omega}} \tag{8.62c}$$
$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{x}} + \boldsymbol{R}_0\tilde{\boldsymbol{v}} \tag{8.62d}$$
$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{\omega}}} \tag{8.62e}$$
$$\dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}_0 + (\dot{\boldsymbol{\omega}}_0 \times\ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times)\, \boldsymbol{R}_0\tilde{\boldsymbol{x}} + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0\tilde{\boldsymbol{v}} + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{v}}} \tag{8.62f}$$

Perturbation:

$$\boldsymbol{\theta}_\delta = \boldsymbol{R}_0\tilde{\boldsymbol{\theta}}_\delta \tag{8.63a}$$
$$\delta\boldsymbol{x} = \boldsymbol{R}_0\delta\tilde{\boldsymbol{x}} \tag{8.63b}$$
$$\delta\boldsymbol{\omega} = \boldsymbol{R}_0\delta\tilde{\boldsymbol{\omega}} \tag{8.63c}$$
$$\delta\boldsymbol{v} = \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\delta\tilde{\boldsymbol{x}} + \boldsymbol{R}_0\delta\tilde{\boldsymbol{v}} \tag{8.63d}$$
$$\delta\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\delta\tilde{\boldsymbol{\omega}} + \boldsymbol{R}_0\delta\dot{\tilde{\boldsymbol{\omega}}} \tag{8.63e}$$
$$\delta\dot{\boldsymbol{v}} = (\dot{\boldsymbol{\omega}}_0 \times\ + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times)\, \boldsymbol{R}_0\delta\tilde{\boldsymbol{x}} + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0\delta\tilde{\boldsymbol{v}} + \boldsymbol{R}_0\delta\dot{\tilde{\boldsymbol{v}}} \tag{8.63f}$$

Relative frame kinematics projected in the relative frame:

$$\overline{\boldsymbol{R}} = \tilde{\boldsymbol{R}} \tag{8.64a}$$
$$\overline{\boldsymbol{x}} = \boldsymbol{R}_0^T\boldsymbol{x}_0 + \tilde{\boldsymbol{x}} \tag{8.64b}$$
$$\overline{\boldsymbol{\omega}} = \boldsymbol{R}_0^T\boldsymbol{\omega}_0 + \tilde{\boldsymbol{\omega}} \tag{8.64c}$$
$$\overline{\boldsymbol{v}} = \boldsymbol{R}_0^T\boldsymbol{v}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{v}} \tag{8.64d}$$
$$\dot{\overline{\boldsymbol{\omega}}} = \boldsymbol{R}_0^T\dot{\boldsymbol{\omega}}_0 + \overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{\omega}} + \dot{\tilde{\boldsymbol{\omega}}} \tag{8.64e}$$
$$\dot{\overline{\boldsymbol{v}}} = \boldsymbol{R}_0^T\dot{\boldsymbol{v}}_0 + \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times\ + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \tilde{\boldsymbol{x}} + 2\overline{\boldsymbol{\omega}}_0 \times \tilde{\boldsymbol{v}} + \dot{\tilde{\boldsymbol{v}}} \tag{8.64f}$$

Perturbation projected in the relative frame:

$$\overline{\boldsymbol{\theta}}_\delta = \tilde{\boldsymbol{\theta}}_\delta \tag{8.65a}$$
$$\delta\overline{\boldsymbol{x}} = \delta\tilde{\boldsymbol{x}} \tag{8.65b}$$
$$\delta\overline{\boldsymbol{\omega}} = \delta\tilde{\boldsymbol{\omega}} \tag{8.65c}$$
$$\delta\overline{\boldsymbol{v}} = \overline{\boldsymbol{\omega}}_0 \times \delta\tilde{\boldsymbol{x}} + \delta\tilde{\boldsymbol{v}} \tag{8.65d}$$
$$\delta\dot{\overline{\boldsymbol{\omega}}} = \overline{\boldsymbol{\omega}}_0 \times \delta\tilde{\boldsymbol{\omega}} + \delta\dot{\tilde{\boldsymbol{\omega}}} \tag{8.65e}$$
$$\delta\dot{\overline{\boldsymbol{v}}} = \left(\dot{\overline{\boldsymbol{\omega}}}_0 \times\ + \overline{\boldsymbol{\omega}}_0 \times \overline{\boldsymbol{\omega}}_0 \times\right) \delta\tilde{\boldsymbol{x}} + 2\overline{\boldsymbol{\omega}}_0 \times \delta\tilde{\boldsymbol{v}} + \delta\dot{\tilde{\boldsymbol{v}}} \tag{8.65f}$$

Inertia forces and moments with respect to nodal quantities, in a relative frame:

$$\overline{\boldsymbol{f}}_{\text{in}} = m\dot{\overline{\boldsymbol{v}}}_{\text{CM}} \tag{8.66a}$$
$$\overline{\boldsymbol{m}}_{\text{in}} = \tilde{\boldsymbol{s}} \times \dot{\overline{\boldsymbol{v}}}_{\text{CM}} + \overline{\boldsymbol{\omega}} \times \tilde{\boldsymbol{J}}_{\text{CM}}\overline{\boldsymbol{\omega}} + \tilde{\boldsymbol{J}}_{\text{CM}}\dot{\overline{\boldsymbol{\omega}}} \tag{8.66b}$$

Perturbation

$$
\begin{aligned}
\delta\overline{\boldsymbol{f}}_{\text{in}} &= m\delta\dot{\overline{\boldsymbol{v}}}_{\text{CM}} \\
&= m\delta\dot{\tilde{\boldsymbol{v}}} + 2m\overline{\boldsymbol{\varpi}}_0\times\delta\tilde{\boldsymbol{v}} + m\left(\dot{\overline{\boldsymbol{\varpi}}}_0\times\, + \overline{\boldsymbol{\varpi}}_0\times\overline{\boldsymbol{\varpi}}_0\times\right)\delta\tilde{\boldsymbol{x}} \\
&\quad - \tilde{\boldsymbol{s}}\times\delta\dot{\tilde{\boldsymbol{\omega}}} - \left(\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}}_0\times\, + \overline{\boldsymbol{\varpi}}\times\tilde{\boldsymbol{s}}\times\, - (\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}})\times\right)\delta\tilde{\boldsymbol{\omega}} \\
&\quad - \left(\dot{\overline{\boldsymbol{\varpi}}}\times\, + \overline{\boldsymbol{\varpi}}\times\overline{\boldsymbol{\varpi}}\times\right)\tilde{\boldsymbol{s}}\times\tilde{\boldsymbol{\theta}}_\delta
\end{aligned}
\tag{8.67a}
$$

$$
\begin{aligned}
\delta\overline{\boldsymbol{m}}_{\text{in}} &= \tilde{\boldsymbol{s}}\times\delta\dot{\overline{\boldsymbol{v}}}_{\text{CM}} + \delta\tilde{\boldsymbol{s}}\times\dot{\overline{\boldsymbol{v}}}_{\text{CM}} \\
&\quad + \tilde{\boldsymbol{J}}_{\text{CM}}\delta\dot{\overline{\boldsymbol{\varpi}}} + \left(\overline{\boldsymbol{\varpi}}\times\tilde{\boldsymbol{J}}_{\text{CM}} - \left(\tilde{\boldsymbol{J}}_{\text{CM}}\overline{\boldsymbol{\varpi}}\right)\times\right)\delta\overline{\boldsymbol{\varpi}} \\
&\quad + \left(\overline{\boldsymbol{\varpi}}\times\left(\tilde{\boldsymbol{J}}_{\text{CM}}\overline{\boldsymbol{\varpi}}\times\, - \left(\tilde{\boldsymbol{J}}_{\text{CM}}\overline{\boldsymbol{\varpi}}\right)\times\right) + \tilde{\boldsymbol{J}}_{\text{CM}}\dot{\overline{\boldsymbol{\varpi}}}\times\, - \left(\tilde{\boldsymbol{J}}_{\text{CM}}\dot{\overline{\boldsymbol{\varpi}}}\right)\times\right)\tilde{\boldsymbol{\theta}}_\delta \\
&= \tilde{\boldsymbol{s}}\times\delta\dot{\tilde{\boldsymbol{v}}} \\
&\quad + 2\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}}_0\times\delta\tilde{\boldsymbol{v}} \\
&\quad + \tilde{\boldsymbol{s}}\times\left(\dot{\overline{\boldsymbol{\varpi}}}_0\times\, + \overline{\boldsymbol{\varpi}}_0\times\overline{\boldsymbol{\varpi}}_0\times\right)\delta\tilde{\boldsymbol{x}} \\
&\quad + \tilde{\boldsymbol{J}}\delta\dot{\tilde{\boldsymbol{\omega}}} \\
&\quad + \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}_0\times\, + \overline{\boldsymbol{\varpi}}\times\tilde{\boldsymbol{J}} - \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\right)\times\right)\delta\tilde{\boldsymbol{\omega}} \\
&\quad + \left(\tilde{\boldsymbol{J}}\dot{\overline{\boldsymbol{\varpi}}}\times\, - \left(\tilde{\boldsymbol{J}}\dot{\overline{\boldsymbol{\varpi}}}\right)\times\, + \overline{\boldsymbol{\varpi}}\times\left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\times\, - \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\right)\times\right) + \dot{\overline{\boldsymbol{v}}}\times\tilde{\boldsymbol{s}}\times\right)\tilde{\boldsymbol{\theta}}_\delta
\end{aligned}
\tag{8.67b}
$$

To summarize:

$$
\begin{aligned}
&\left\{\begin{array}{c}\delta\overline{\boldsymbol{f}}_{\text{in}}\\ \delta\overline{\boldsymbol{m}}_{\text{in}}\end{array}\right\} = \left[\begin{array}{cc} m\boldsymbol{I} & -\tilde{\boldsymbol{s}}\times \\ \tilde{\boldsymbol{s}}\times & \tilde{\boldsymbol{J}}\end{array}\right]\left\{\begin{array}{c}\delta\dot{\tilde{\boldsymbol{v}}}\\ \delta\dot{\tilde{\boldsymbol{\omega}}}\end{array}\right\} \\
&+\left[\begin{array}{cc} 2m\overline{\boldsymbol{\varpi}}_0\times & -\left(\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}}_0\times\, + \overline{\boldsymbol{\varpi}}\times\tilde{\boldsymbol{s}}\times\, - (\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}})\times\right) \\ 2\tilde{\boldsymbol{s}}\times\overline{\boldsymbol{\varpi}}_0\times & \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}_0\times\, + \overline{\boldsymbol{\varpi}}\times\tilde{\boldsymbol{J}} - \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\right)\times\right)\end{array}\right]\left\{\begin{array}{c}\delta\tilde{\boldsymbol{v}}\\ \delta\tilde{\boldsymbol{\omega}}\end{array}\right\} \\
&+\left[\begin{array}{c} m\left(\dot{\overline{\boldsymbol{\varpi}}}_0\times\, + \overline{\boldsymbol{\varpi}}_0\times\overline{\boldsymbol{\varpi}}_0\times\right) \\ \tilde{\boldsymbol{s}}\times\left(\dot{\overline{\boldsymbol{\varpi}}}_0\times\, + \overline{\boldsymbol{\varpi}}_0\times\overline{\boldsymbol{\varpi}}_0\times\right)\end{array}\right]\delta\tilde{\boldsymbol{x}} \\
&+\left[\begin{array}{c} -\left(\dot{\overline{\boldsymbol{\varpi}}}\times\, + \overline{\boldsymbol{\varpi}}\times\overline{\boldsymbol{\varpi}}\times\right)\tilde{\boldsymbol{s}}\times \\ \left(\tilde{\boldsymbol{J}}\dot{\overline{\boldsymbol{\varpi}}}\times\, - \left(\tilde{\boldsymbol{J}}\dot{\overline{\boldsymbol{\varpi}}}\right)\times\, + \overline{\boldsymbol{\varpi}}\times\left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\times\, - \left(\tilde{\boldsymbol{J}}\overline{\boldsymbol{\varpi}}\right)\times\right) + \dot{\overline{\boldsymbol{v}}}\times\tilde{\boldsymbol{s}}\times\right)\end{array}\right]\tilde{\boldsymbol{\theta}}_\delta
\end{aligned}
\tag{8.68}
$$

Chapter 9

Constraints (aka Joints)

9.1 Algebraic Constraints

Consider a holonomic constraint equation of the form

$$\boldsymbol{\phi}\left(\boldsymbol{q}, t\right) = 0 \tag{9.1}$$

Its time derivative and perturbation yields

$$\begin{aligned}
\delta \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\mu}^T \boldsymbol{\phi}\right) &= \delta \dot{\boldsymbol{\mu}}^T \boldsymbol{\phi} + \delta \boldsymbol{\phi}^T \dot{\boldsymbol{\mu}} + \delta \boldsymbol{\mu}^T \dot{\boldsymbol{\phi}} + \delta \dot{\boldsymbol{\phi}}^T \boldsymbol{\mu} \\
&= \delta \dot{\boldsymbol{\mu}}^T \boldsymbol{\phi} + \delta \boldsymbol{q}^T \boldsymbol{\phi}_{/\boldsymbol{q}}^T \dot{\boldsymbol{\mu}} + \delta \boldsymbol{\mu}^T \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right) \\
&\quad + \left(\delta \dot{\boldsymbol{q}}^T \boldsymbol{\phi}_{/\boldsymbol{q}}^T + \delta \boldsymbol{q}^T \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^T\right) \boldsymbol{\mu} \\
&= \delta \dot{\boldsymbol{q}}^T \boldsymbol{\phi}_{/\boldsymbol{q}}^T \boldsymbol{\mu} \\
&\quad + \delta \boldsymbol{q}^T \left(\left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^T \boldsymbol{\mu} + \boldsymbol{\phi}_{/\boldsymbol{q}}^T \dot{\boldsymbol{\mu}}\right) \\
&\quad + \delta \dot{\boldsymbol{\mu}}^T \boldsymbol{\phi} \\
&\quad + \delta \boldsymbol{\mu}^T \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)
\end{aligned} \tag{9.2}$$

Its linearization yields

$$\begin{aligned}
\Delta\left(\delta \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\mu}^T \boldsymbol{\phi}\right)\right) &= \delta \dot{\boldsymbol{q}}^T \left(\boldsymbol{\phi}_{/\boldsymbol{q}}^T \Delta \boldsymbol{\mu} + \left(\boldsymbol{\phi}_{/\boldsymbol{q}}^T \boldsymbol{\mu}\right)_{/\boldsymbol{q}} \Delta \boldsymbol{q}\right) \\
&\quad + \delta \boldsymbol{q}^T \left(\left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^T \Delta \boldsymbol{\mu} + \boldsymbol{\phi}_{/\boldsymbol{q}} \Delta \dot{\boldsymbol{\mu}} + \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \Delta \dot{\boldsymbol{q}}\right)_{/\boldsymbol{q}}^T \boldsymbol{\mu} \right. \\
&\qquad \left. + \left(\left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^T \boldsymbol{\mu} + \boldsymbol{\phi}_{/\boldsymbol{q}}^T \dot{\boldsymbol{\mu}}\right)_{/\boldsymbol{q}} \Delta \boldsymbol{q}\right) \\
&\quad + \delta \dot{\boldsymbol{\mu}}^T \boldsymbol{\phi}_{/\boldsymbol{q}} \Delta \boldsymbol{q} \\
&\quad + \delta \boldsymbol{\mu}^T \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \Delta \dot{\boldsymbol{q}} + \left(\boldsymbol{\phi}_{/\boldsymbol{q}} \dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}} \Delta \boldsymbol{q}\right)
\end{aligned} \tag{9.3}$$

An unconstrained problem of the form

$$\boldsymbol{M} \dot{\boldsymbol{q}} - \boldsymbol{p} = \boldsymbol{0} \tag{9.4a}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{F}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \tag{9.4b}$$

becomes

$$\begin{aligned}
\boldsymbol{M}\dot{\boldsymbol{q}} - \boldsymbol{p} + \boldsymbol{\phi}_{/\boldsymbol{q}}^{T}\boldsymbol{\mu} &= \boldsymbol{0} && (9.5a)\\
\dot{\boldsymbol{p}} + \left(\boldsymbol{\phi}_{/\boldsymbol{q}}\dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^{T}\boldsymbol{\mu} + \boldsymbol{\phi}_{/\boldsymbol{q}}^{T}\dot{\boldsymbol{\mu}} &= \boldsymbol{F}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) && (9.5b)\\
\boldsymbol{\phi} &= \boldsymbol{0} && (9.5c)\\
\boldsymbol{\phi}_{/\boldsymbol{q}}\boldsymbol{M}^{-1}\boldsymbol{p} + \boldsymbol{\phi}_{/t} &= \boldsymbol{0}, && (9.5d)
\end{aligned}$$

where $\dot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\boldsymbol{p}$ has been used in Eq. (9.5d). Actually, $\dot{\boldsymbol{\mu}}$ is independent of $\boldsymbol{\mu}$; $\boldsymbol{\mu}$ by definition is zero if the exact solution is considered. So $\dot{\boldsymbol{\mu}}$ is redefined as $\dot{\boldsymbol{\mu}} = \boldsymbol{\lambda}$:

$$\begin{aligned}
\boldsymbol{M}\dot{\boldsymbol{q}} - \boldsymbol{p} + \boldsymbol{\phi}_{/\boldsymbol{q}}^{T}\boldsymbol{\mu} &= \boldsymbol{0} && (9.6a)\\
\dot{\boldsymbol{p}} + \left(\boldsymbol{\phi}_{/\boldsymbol{q}}\dot{\boldsymbol{q}} + \boldsymbol{\phi}_{/t}\right)_{/\boldsymbol{q}}^{T}\boldsymbol{\mu} + \boldsymbol{\phi}_{/\boldsymbol{q}}^{T}\boldsymbol{\lambda} &= \boldsymbol{F}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) && (9.6b)\\
\boldsymbol{\phi} &= \boldsymbol{0} && (9.6c)\\
\boldsymbol{\phi}_{/\boldsymbol{q}}\boldsymbol{M}^{-1}\boldsymbol{p} + \boldsymbol{\phi}_{/t} &= \boldsymbol{0}. && (9.6d)
\end{aligned}$$

If $\boldsymbol{\mu} = 0$ then a conventional DAE of index 3 results, i.e.

$$\begin{aligned}
\boldsymbol{M}\dot{\boldsymbol{q}} - \boldsymbol{p} &= \boldsymbol{0} && (9.7a)\\
\dot{\boldsymbol{p}} + \boldsymbol{\phi}_{/\boldsymbol{q}}^{T}\boldsymbol{\lambda} &= \boldsymbol{F}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) && (9.7b)\\
\boldsymbol{\phi} &= \boldsymbol{0}, && (9.7c)
\end{aligned}$$

while the previous form is known as stabilized index 2 form. The latter is typically used throughout MBDyn, while the former is used in the initial assembly and, occasionally, in specific constraints.

9.1.1 Beam Slider Joint

The beam slider joint allows a point attache to a node, $\boldsymbol{x}_c = \boldsymbol{x}_b + \boldsymbol{R}_b\boldsymbol{f}_b$ to slide along a sequence of three-node beam elements. The current contact position on the beam is $\boldsymbol{x} = \sum_{i=1}^{3} N_i(s)(\boldsymbol{x}_i + \boldsymbol{R}_i\boldsymbol{f}_i)$.

Files. It is implemented in files
`mbdyn/struct/beamslider.h`
`mbdyn/struct/beamslider.cc`

Unknowns.

- s position along the beam
- $\boldsymbol{F}$ reaction force vector (three components)
- $\boldsymbol{M}$ reaction moment vector (no components for the *spherical* slider, two components for the *classical* slider, three components for the *spline* slider, currently not implemented)

If friction is present additional unknowns depend on the chosen friction model.

Definitions.

- $\boldsymbol{x}_{ti} = \boldsymbol{x}_i + \boldsymbol{R}_i\boldsymbol{f}_i$
- $\boldsymbol{l} = \sum_{i=1}^{3} N_{i/s}(s)\boldsymbol{x}_{ti}$ beam tangent

- $\boldsymbol{e}_{3a} = \boldsymbol{l}/\sqrt{\boldsymbol{l}\cdot\boldsymbol{l}}$ unit beam tangent
- $\boldsymbol{x}_c = \boldsymbol{x}_b + \boldsymbol{R}_b\boldsymbol{f}_b$ body sliding point position
- $\boldsymbol{x} = \sum_{i=1}^{3} N_i(s)(\boldsymbol{x}_i + \boldsymbol{R}_i\boldsymbol{f}_i)$ contact point position on the beam
- $\boldsymbol{v}_c = \dot{\boldsymbol{x}}_b + \boldsymbol{\omega}_b \times \boldsymbol{R}_b\boldsymbol{f}_b$ body sliding point velocity
- $\boldsymbol{v} = \sum_{i=1}^{3} N_i(s)(\dot{\boldsymbol{x}}_i + \boldsymbol{\omega}_i \times \boldsymbol{R}_i\boldsymbol{f}_i)$ contact point velocity on the beam

Equations.

- 1: $\boldsymbol{F}\cdot\boldsymbol{l}/\texttt{dCoef} = 0$
- 2–4: $(\boldsymbol{x}_c - \boldsymbol{x})/\texttt{dCoef} = \boldsymbol{0}$

If *classical* two additiona equation:

- 5: $\boldsymbol{R}_b(2)\cdot\boldsymbol{l} = 0$
- 6: $\boldsymbol{R}_b(3)\cdot\boldsymbol{l} = 0$

If *spline*: not implemented

Friction If friction is present the friction coefficient f is computed as a function of the relative sliding velocity $v_{rel} = \boldsymbol{e}_{3a}\cdot\boldsymbol{v}_{rel}$ where $\boldsymbol{v}_{rel} = \boldsymbol{v}_c - \boldsymbol{v}$. The tangent friction force component is $F_3 = f * S_{hc} * \sqrt{(}\boldsymbol{F}\cdot\boldsymbol{F})$ and his direction is that of $\boldsymbol{e}_{3a}$, so that the resultant reaction force is $\boldsymbol{F}_{res} = \boldsymbol{F} - \boldsymbol{e}_{3a}F_3$

Friction linearization.

$$\Delta v_{rel} = \boldsymbol{e}_{3a}\cdot\Delta\boldsymbol{v}_{rel} + \boldsymbol{v}_{rel}\cdot\Delta\boldsymbol{e}_{3a}$$

$$\begin{aligned}
\boldsymbol{e}_{3a}\cdot\Delta\boldsymbol{v}_{rel} = & \boldsymbol{e}_{3a}\cdot\Delta\boldsymbol{x}_c \\
& - \sum_{i=1}^{3} \boldsymbol{e}_{3a}\cdot(\dot{\boldsymbol{x}}_i + \boldsymbol{\omega}_i \times \boldsymbol{R}_i\boldsymbol{f}_i)N_{i/s}\Delta s \\
& - \sum_{i=1}^{3} N_i\boldsymbol{e}_{3a}\cdot\Delta\dot{\boldsymbol{x}}_i \\
& - \sum_{i=1}^{3} N_i(\boldsymbol{R}_i\boldsymbol{f}_i)\times\boldsymbol{e}_{3a}\Delta\boldsymbol{\omega}_i \\
& - \sum_{i=1}^{3} N_i(\boldsymbol{R}_i\boldsymbol{f}_i)\times\boldsymbol{\omega}_i\times\boldsymbol{e}_{3a}\texttt{dCoef}\Delta\dot{\boldsymbol{g}}
\end{aligned}$$

where, as explained in Sec. 5.1, $\Delta\boldsymbol{\omega} = \Delta\dot{\boldsymbol{g}} - \boldsymbol{\omega}_{\mathrm{ref}}\times\Delta\dot{\boldsymbol{g}} * \texttt{dCoef}$. Furthermore,

$$\boldsymbol{v}\Delta\boldsymbol{e}_{3a} = \left(\frac{\boldsymbol{v}}{\sqrt{\boldsymbol{l}\cdot\boldsymbol{l}}} - \frac{v_{rel}\boldsymbol{l}}{(\boldsymbol{l}\cdot\boldsymbol{l})^{3/2}}\right)\cdot\Delta\boldsymbol{l}$$

with

$$
\begin{aligned}
\Delta \boldsymbol{l} = & \sum_{i=1}^{3} N_{i/ss} \boldsymbol{x}_{ti} \Delta \dot{s} \\
& + \sum_{i=1}^{3} N_{i/s} \boldsymbol{I} \texttt{dCoef} \Delta \dot{\boldsymbol{x}}_i \\
& + \sum_{i=1}^{3} N_{i/s} (-\boldsymbol{R}_i \boldsymbol{f}_i) \times \texttt{dCoef} \Delta \dot{\boldsymbol{g}}_i
\end{aligned}
$$

9.1.2 Clamp Joint

The clamp joint (`Clamp`) imposes the position and the orientation of a node.

Files. It is implemented in files
`mbdyn/struct/genj.h`
`mbdyn/struct/genj.cc`

Definitions.

$$\boldsymbol{d}_c = \boldsymbol{x} - \boldsymbol{x}_0 \tag{9.8}$$

$$\boldsymbol{\theta}_c = \text{ax}\left(\exp^{-1}\left(\boldsymbol{R}\boldsymbol{R}_0^T\right)\right) \tag{9.9}$$

Constraint Equation.

$$\boldsymbol{d}_c = \boldsymbol{0} \tag{9.10}$$

$$\boldsymbol{\theta}_c = \boldsymbol{0} \tag{9.11}$$

Perturbation.

$$\delta \boldsymbol{d}_c = \delta \boldsymbol{x} \tag{9.12}$$

$$\delta \boldsymbol{\theta}_c = \boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_c)\, \boldsymbol{\theta}_\delta \tag{9.13}$$

Forces.

$$\boldsymbol{f} = \boldsymbol{\lambda} \tag{9.14a}$$

$$\boldsymbol{m} = \boldsymbol{\Gamma}^{-T}(\boldsymbol{\theta}_c)\, \overline{\boldsymbol{\mu}} \stackrel{\text{def}}{=} \boldsymbol{\mu} \tag{9.14b}$$

Linearization.

$$
\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}
\left\{ \begin{array}{c} \Delta \boldsymbol{x} \\ \boldsymbol{\theta}_\Delta \end{array} \right\}
+
\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}
\left\{ \begin{array}{c} \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{array} \right\}
= -
\left\{ \begin{array}{c} -\boldsymbol{\lambda} \\ -\boldsymbol{\mu} \\ -\boldsymbol{d}_c \\ -\boldsymbol{\theta}_c \end{array} \right\}
\tag{9.15}
$$

since $\boldsymbol{\Gamma}(\boldsymbol{\theta}_c)\, \boldsymbol{\theta}_c = \boldsymbol{\theta}_c$.

9.1.3 Distance Joint

The distance joint (`DistanceJoint`) imposes the distance between two nodes, which may depend on time and other states of the problem.

Variants. There is a variant with offsets (`DistanceJointWithOffset`), discussed in Section 9.1.4.

Files. It is implemented in files
`mbdyn/struct/distance.h`
`mbdyn/struct/distance.cc`

Definitions.

$$\boldsymbol{d} = \boldsymbol{x}_2 - \boldsymbol{x}_1 \tag{9.16}$$

$$d = \sqrt{\boldsymbol{d}^T \boldsymbol{d}} \tag{9.17}$$

$$\boldsymbol{u} = \frac{\boldsymbol{d}}{d} \tag{9.18}$$

Limitations.

$$d > 0 \tag{9.19}$$

Constraint Equation.

$$d\sqrt{\boldsymbol{u}^T \boldsymbol{u}} = d \tag{9.20}$$

Forces.

$$\boldsymbol{F}_1 = \alpha \boldsymbol{u} \tag{9.21a}$$
$$\boldsymbol{F}_2 = -\alpha \boldsymbol{u} \tag{9.21b}$$

Linearization.

$$\begin{bmatrix} \frac{\alpha}{d}\boldsymbol{I} & -\frac{\alpha}{d}\boldsymbol{I} & -\boldsymbol{u} \\ -\frac{\alpha}{d}\boldsymbol{I} & \frac{\alpha}{d}\boldsymbol{I} & \boldsymbol{u} \\ -\boldsymbol{u}^T & \boldsymbol{u}^T & 0 \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\alpha \end{Bmatrix} = \begin{Bmatrix} \alpha\boldsymbol{u} \\ -\alpha\boldsymbol{u} \\ d\left(1 - \sqrt{\boldsymbol{u}^T\boldsymbol{u}}\right) \end{Bmatrix} \tag{9.22}$$

Constraint Equation Derivative.

$$d\boldsymbol{u}^T \dot{\boldsymbol{u}} = 0 \tag{9.23}$$

Force Derivatives.

$$\dot{\boldsymbol{F}}_1 = \alpha\dot{\boldsymbol{u}} + \dot{\alpha}\boldsymbol{u} \tag{9.24a}$$

$$\dot{\boldsymbol{F}}_2 = -\alpha\dot{\boldsymbol{u}} - \dot{\alpha}\boldsymbol{u} \tag{9.24b}$$

where

$$\dot{\boldsymbol{u}} = \frac{\dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_1}{d} - \boldsymbol{u}\frac{\dot{d}}{d} \tag{9.25}$$

Linearization.

$$\left[\begin{array}{cccccc} \frac{\dot{\alpha}}{d}\boldsymbol{I} & \frac{\alpha}{d}\boldsymbol{I} & -\frac{\dot{\alpha}}{d}\boldsymbol{I} & -\frac{\alpha}{d}\boldsymbol{I} & -\dot{\boldsymbol{u}} & -\boldsymbol{u} \\ -\frac{\dot{\alpha}}{d}\boldsymbol{I} & -\frac{\alpha}{d}\boldsymbol{I} & \frac{\dot{\alpha}}{d}\boldsymbol{I} & \frac{\alpha}{d}\boldsymbol{I} & \dot{\boldsymbol{u}} & \boldsymbol{u} \\ -\dot{\boldsymbol{u}}^T & -\boldsymbol{u}^T & \dot{\boldsymbol{u}}^T & \boldsymbol{u}^T & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \delta\boldsymbol{x}_1 \\ \delta\dot{\boldsymbol{x}}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\dot{\boldsymbol{x}}_2 \\ \delta\alpha \\ \delta\dot{\alpha} \end{array}\right\} = \left\{\begin{array}{c} \alpha\dot{\boldsymbol{u}} + \dot{\alpha}\boldsymbol{u} \\ -\alpha\dot{\boldsymbol{u}} - \dot{\alpha}\boldsymbol{u} \\ d\boldsymbol{u}^T\dot{\boldsymbol{u}} \end{array}\right\} \tag{9.26}$$

9.1.4 Distance Joint With Offsets

The distance joint with offset (`DistanceJointWithOffset`) imposes the distance between two points that rigidly offset from the respective nodes. It is a variant of the distance joint (`DistanceJoint`) discussed in Section 9.1.3.

Files. It is implemented in files
`mbdyn/struct/distance.h`
`mbdyn/struct/distance.cc`

Definitions.

$$\boldsymbol{d} = \boldsymbol{x}_2 + \boldsymbol{b}_2 - \boldsymbol{x}_1 - \boldsymbol{b}_1 \tag{9.27}$$

$$d = \sqrt{\boldsymbol{d}^T\boldsymbol{d}} \tag{9.28}$$

$$\boldsymbol{u} = \frac{\boldsymbol{d}}{d} \tag{9.29}$$

Limitations:

$$d > 0 \tag{9.30}$$

Constraint equation

$$d\sqrt{\boldsymbol{u}^T\boldsymbol{u}} = d \tag{9.31}$$

Forces:

$$\boldsymbol{F}_1 = \alpha\boldsymbol{u} \tag{9.32a}$$

$$\boldsymbol{M}_1 = \alpha\boldsymbol{b}_1 \times \boldsymbol{u} \tag{9.32b}$$

$$\boldsymbol{F}_2 = -\alpha\boldsymbol{u} \tag{9.32c}$$

$$\boldsymbol{M}_2 = -\alpha\boldsymbol{b}_2 \times \boldsymbol{u} \tag{9.32d}$$

Linearization:

$$\begin{bmatrix} \frac{\alpha}{d}\boldsymbol{I} & -\frac{\alpha}{d}\boldsymbol{b}_1 \times & -\frac{\alpha}{d}\boldsymbol{I} & \frac{\alpha}{d}\boldsymbol{b}_2 \times & -\boldsymbol{u} \\ \frac{\alpha}{d}\boldsymbol{b}_1 \times & -\frac{\alpha}{d}(\boldsymbol{b}_1 + \boldsymbol{d}) \times \boldsymbol{b}_1 \times & -\frac{\alpha}{d}\boldsymbol{b}_1 \times & \frac{\alpha}{d}\boldsymbol{b}_1 \times \boldsymbol{b}_2 \times & -\boldsymbol{b}_1 \times \boldsymbol{u} \\ -\frac{\alpha}{d}\boldsymbol{I} & \frac{\alpha}{d}\boldsymbol{b}_1 \times & \frac{\alpha}{d}\boldsymbol{I} & -\frac{\alpha}{d}\boldsymbol{b}_2 \times & \boldsymbol{u} \\ -\frac{\alpha}{d}\boldsymbol{b}_2 \times & \frac{\alpha}{d}\boldsymbol{b}_2 \times \boldsymbol{b}_1 \times & \frac{\alpha}{d}\boldsymbol{b}_2 \times & -\frac{\alpha}{d}(\boldsymbol{b}_2 - \boldsymbol{d}) \times \boldsymbol{b}_2 \times & \boldsymbol{b}_2 \times \boldsymbol{u} \\ -\boldsymbol{u}^T & -(\boldsymbol{b}_1 \times \boldsymbol{u})^T & \boldsymbol{u}^T & (\boldsymbol{b}_2 \times \boldsymbol{u})^T & 0 \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \\ \delta\alpha \end{Bmatrix} = \begin{Bmatrix} \alpha\boldsymbol{u} \\ \alpha\boldsymbol{b}_1 \times \boldsymbol{u} \\ -\alpha\boldsymbol{u} \\ -\alpha\boldsymbol{b}_2 \times \boldsymbol{u} \\ d\left(1 - \sqrt{\boldsymbol{u}^T\boldsymbol{u}}\right) \end{Bmatrix} \tag{9.33}$$

Constraint Equation Derivative

$$d\boldsymbol{u}^T\dot{\boldsymbol{u}} = 0 \tag{9.34}$$

Forces:

$$\dot{\boldsymbol{F}}_1 = \alpha\dot{\boldsymbol{u}} + \dot{\alpha}\boldsymbol{u} \tag{9.35a}$$

$$\dot{\boldsymbol{M}}_1 = \alpha(\boldsymbol{\omega}_1 \times \boldsymbol{b}_1) \times \boldsymbol{u} + \alpha\boldsymbol{b}_1 \times \dot{\boldsymbol{u}} + \dot{\alpha}\boldsymbol{b}_1 \times \boldsymbol{u} \tag{9.35b}$$

$$\dot{\boldsymbol{F}}_2 = -\alpha\dot{\boldsymbol{u}} - \dot{\alpha}\boldsymbol{u} \tag{9.35c}$$

$$\dot{\boldsymbol{M}}_2 = -\alpha(\boldsymbol{\omega}_2 \times \boldsymbol{b}_2) \times \boldsymbol{u} - \alpha\boldsymbol{b}_2 \times \dot{\boldsymbol{u}} - \dot{\alpha}\boldsymbol{b}_2 \times \boldsymbol{u} \tag{9.35d}$$

Linearization: TODO©.

9.1.5 Spherical hinge

The spherical hinge joint (`SphericalHingeJoint`) constrains the positions of two points, that may be rigidly offset from two nodes, to be coincident.

Variants. There exists a pinned version (`SphericalPinJoint`, TODO©).

Files. It is implemented in files
`mbdyn/struct/spherj.h`
`mbdyn/struct/spherj.cc`

Joint data.

$$\tilde{\boldsymbol{b}}_1, \tilde{\boldsymbol{b}}_2 \tag{9.36}$$

where:
$\tilde{\boldsymbol{b}}_1$, $\tilde{\boldsymbol{b}}_2$ offset of connection point from nodes 1, 2 in node reference;

Constraint equations (normalized[1] by `dCoef`)

$$(\boldsymbol{x}_2 + \boldsymbol{b}_2) - (\boldsymbol{x}_1 + \boldsymbol{b}_1) = \boldsymbol{0} \tag{9.37}$$

where:

$\boldsymbol{x}_1$, $\boldsymbol{x}_2$	position of nodes 1, 2;
$\boldsymbol{R}_1$, $\boldsymbol{R}_2$	orientation of nodes 1, 2.
$\boldsymbol{b}_1 = \boldsymbol{R}_1\tilde{\boldsymbol{b}}_1$, $\boldsymbol{b}_2 = \boldsymbol{R}_2\tilde{\boldsymbol{b}}_2$	offset of connection point from nodes 1, 2 in global frame

Residual vector:

$$\begin{aligned}
\text{node1 momentum}: \ 1-3 \quad & -= \quad \boldsymbol{F} \\
\text{node1 angular momentum}: \ 4-6 \quad & -= \quad (\boldsymbol{R}_1 \cdot \boldsymbol{d}_1) \times \boldsymbol{F} \\
\text{node2 momentum}: \ 7-9 \quad & += \quad \boldsymbol{F} \\
\text{node2 angular momentum}: \ 10-12 \quad & += \quad (\boldsymbol{R}_2 \cdot \boldsymbol{d}_2) \times \boldsymbol{F} \\
\text{constraint}: \ 13-15 \quad & = \quad ((\boldsymbol{x}_1 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_1) - (\boldsymbol{x}_2 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_2))/\texttt{dCoef}
\end{aligned}$$

where:

$\boldsymbol{F}$ constraint reaction force.

9.1.6 Revolute hinge

The revolute hinge joint (`PlaneHingeJoint`) constrains the positions of two points, that may be rigidly offset from two nodes, to be coincident. It also constrains their orientations to keep the respective axis 3 parallel.

Variants. There exist a pinned version (`RevolutePinJoint`, TODO©), a version that does not constrain the position (`RevoluteRotationJoint`), and a version which imposes the relative angular velocity about axis 3 (`AxialRotationJoint`).

Files. It is implemented in files

```
mbdyn/struct/planej.h
mbdyn/struct/planej.cc
```

Joint data.

$$\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}_{h1}, \boldsymbol{R}_{h2} \tag{9.38}$$

where:
$\boldsymbol{d}_1$, $\boldsymbol{d}_2$: offset of nodes 1,2 in node reference;
$\boldsymbol{R}_{h1}$, $\boldsymbol{R}_{h2}$: joint relative orientation wrt. nodes 1,2 (FIXME).

Constraint equations (normalized by `dCoef`)

$$\begin{aligned}
(\boldsymbol{x}_1 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_1) - (\boldsymbol{x}_2 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_2) &= 0 \\
(\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2] &= 0 \\
(\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1] &= 0
\end{aligned}$$

[1] When purely algebraic constraints are considered, to improve the scaling of the matrix, the constraint equation can be divided by `dCoef`, the coefficient related to the integration method illustrated in Equation (6.27).

where:
$\boldsymbol{x}_1$, $\boldsymbol{x}_2$: positions of nodes 1,2;
$\boldsymbol{R}_1$, $\boldsymbol{R}_2$: orientation of nodes 1, 2.

Residual vector:

$$
\begin{aligned}
\text{node1 momentum}: \; 1-3 \quad & -= \quad \boldsymbol{F} \\
\text{node1 angular momentum}: \; 4-6 \quad & -= \quad (\boldsymbol{R}_1 \cdot \boldsymbol{d}_1) \times \boldsymbol{F} + \\
& \quad\quad (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2] \times (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[1] + \\
& \quad\quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \times (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1] * \boldsymbol{M}[2] \\
\text{node2 momentum}: \; 7-9 \quad & += \quad \boldsymbol{F} \\
\text{node2 angular momentum}: \; 10-12 \quad & += \quad (\boldsymbol{R}_2 \cdot \boldsymbol{d}_2) \times \boldsymbol{F} + \\
& \quad\quad (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2] \times (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[1] + \\
& \quad\quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \times (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1] * \boldsymbol{M}[2] \\
\text{constraint}: \; 13-15 \quad & = \quad ((\boldsymbol{x}_1 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_1) - (\boldsymbol{x}_2 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_2))/\texttt{dCoef} \\
\text{constraint}: \; 16 \quad & = \quad ((\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2])/\texttt{dCoef} \\
\text{constraint}: \; 17 \quad & = \quad ((\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1])/\texttt{dCoef}
\end{aligned}
$$

where:
$\boldsymbol{F}$: constraint reaction force;
$\boldsymbol{M}$: constraint moment reaction (third component null).

Friction:

- add third component of constraint moment $\boldsymbol{M}$
- add a constraint equation; this could be one of the following:
  - direct definition of $\boldsymbol{M}[3]$ in function of relative velocity, friction coefficient and $\boldsymbol{F}$
  - impose null relative velocity
- optionally add internal states dynamic z (for friction)

add third component of constraint moment M

Residual vector:

$$
\begin{aligned}
\text{node1 momentum}: \ 1-3 \quad -= & \quad \boldsymbol{F} \\
\text{node1 angular momentum}: \ 4-6 \quad -= & \quad (\boldsymbol{R}_1 \cdot \boldsymbol{d}_1) \times \boldsymbol{F} + \\
& \quad (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2] \times (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[1] + \\
& \quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \times (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1] * \boldsymbol{M}[2] + \\
& \quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[3] \\
\text{node2 momentum}: \ 7-9 \quad += & \quad \boldsymbol{F} \\
\text{node2 angular momentum}: \ 10-12 \quad += & \quad (\boldsymbol{R}_2 \cdot \boldsymbol{d}_2) \times \boldsymbol{F} + \\
& \quad (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2] \times (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[1] + \\
& \quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \times (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1] * \boldsymbol{M}[2] + \\
& \quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * \boldsymbol{M}[3] \\
\text{constraint}: \ 13-15 \quad = & \quad ((\boldsymbol{x}_1 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_1) - (\boldsymbol{x}_2 + \boldsymbol{R}_1 \cdot \boldsymbol{d}_2))/\texttt{dCoef} \\
\text{constraint}: \ 16 \quad = & \quad ((\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[2])/\texttt{dCoef} \\
\text{constraint}: \ 17 \quad = & \quad ((\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\boldsymbol{R}_2 \cdot \boldsymbol{R}_{h2})[1])/\texttt{dCoef} \\
\text{friction}: 18 \quad = & \quad \boldsymbol{M}[3] - f(\boldsymbol{F}, v, \text{friction coef}) \\
= & \quad (\text{rel velocity})[3] \\
\text{(optional) friction states}: 19... \quad = & \quad \dot{z} - g(z, v)
\end{aligned}
$$

where v is the relative velocity $v = r * (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \cdot (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]$. The constraint is based on positions. This means that during integration $(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)$ will NOT have exactly the direction $(\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]$. We choose to disregard this error. The friction moment should be along $(\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]$.
r is the joint radius.
CHECK THIS!!!
Explicit form of friction moment contribution (FIXME: remove vector $(\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]$ and write scalar equation???):
$f(\boldsymbol{F}, \text{rel velocity})$: $\boldsymbol{M}[3] = \text{sh\_c}(||\boldsymbol{F}||, f_\text{c}(v, z, \dot{z})) * (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * ||\boldsymbol{F}|| * f_\text{c}(v, z, \dot{z})$,
with $\boldsymbol{R}_{h1}$ constant.
Variation of friction moment contribution:

$$
\begin{aligned}
\delta(\boldsymbol{M}[3](\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]) &= \boldsymbol{M}[3] * (\boldsymbol{R}_{\delta 1} \times \boldsymbol{R}_1)[3] + \\
&\quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]) * \delta \boldsymbol{M}[3] \\
&= \boldsymbol{M}[3] * \boldsymbol{R}_1^T \cdot (\boldsymbol{R}_{\delta 1} \times)[3] + \\
&\quad (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]) * \delta \boldsymbol{M}[3]
\end{aligned}
$$

Variation of friction moment contribution component:

$$
\begin{aligned}
\delta \boldsymbol{M}[3] &= \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) * f_{\text{c}}(v,z,\dot{z}) * \frac{\boldsymbol{F}}{||\boldsymbol{F}||} \cdot \delta \boldsymbol{F} + \\
&\quad \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) * (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] * ||\boldsymbol{F}|| * \frac{\partial f_{\text{c}}}{\partial v} * \delta v + \\
&\quad ||\boldsymbol{F}|| * f_{\text{c}}(v,z,\dot{z}) * \frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial ||\boldsymbol{F}||} \frac{\partial ||\boldsymbol{F}||}{\partial \boldsymbol{F}} \cdot \delta \boldsymbol{F} + \\
&\quad ||\boldsymbol{F}|| * f_{\text{c}}(v,z,\dot{z}) * \frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial f_{\text{c}}(v,z,\dot{z})} * \frac{\partial f_{\text{c}}}{\partial v} * \delta v \\
&= \left(\begin{array}{l} \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) * f_{\text{c}}(v,z,\dot{z}) \frac{\boldsymbol{F}}{||\boldsymbol{F}||} \\ ||\boldsymbol{F}|| * f_{\text{c}}(v,z,\dot{z}) * \frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial ||\boldsymbol{F}||} \end{array} \right) \cdot \delta \boldsymbol{F} + \\
&\quad \left(\begin{array}{l} \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) * ||\boldsymbol{F}|| \\ ||\boldsymbol{F}|| * f_{\text{c}}(v,z,\dot{z}) * \frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial f_{\text{c}}(v,z,\dot{z})} \end{array} \right) * \left\{ \begin{array}{c} \frac{\partial f_{\text{c}}}{\partial v} * \delta v \\ \frac{\partial f_{\text{c}}}{\partial z} * \delta z \\ \frac{\partial f_{\text{c}}}{\partial \dot{z}} * \delta \dot{z} \end{array} \right\}
\end{aligned}
$$

where

- $v = r * (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \cdot (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3]$ so that

$$
\begin{aligned}
\delta v &= d * (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\delta \boldsymbol{\omega}_1 - \delta \boldsymbol{\omega}_2) + \\
&\quad d * (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \cdot (\boldsymbol{R}_{\delta 1} \times \boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \\
&= d * (\boldsymbol{R}_1 \cdot \boldsymbol{R}_{h1})[3] \cdot (\delta \boldsymbol{\omega}_1 - \delta \boldsymbol{\omega}_2) + \\
&\quad d * (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \cdot \boldsymbol{R}_{h1}^T \cdot \boldsymbol{R}_1^T \cdot (\boldsymbol{R}_{\delta 1} \times)[3]
\end{aligned}
$$

- $\text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) = r * C_\alpha(\alpha(\text{constants}, f_{\text{c}}(v,z,\dot{z}), ||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))) \frac{1}{\sqrt{1 + f_{\text{c}}^2(v,z,\dot{z})}}$

- $\alpha(\text{constants}, f_{\text{c}}(v,z,\dot{z}), ||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) = \sin^{-1}\left(\sqrt{\frac{2 * 31 * ||\boldsymbol{F}||}{E * b * \sqrt{1 + f_{\text{c}}^2(v,z,\dot{z})}} \frac{r'/r}{r' - r}} \right)$

- for very low joint loads (angle of contact $\alpha < 20^o$, i.e. about less that 1% of the joint allowable load) we can safely assume $C_\alpha \approx 1$ so that
  $\text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z})) \approx r * \frac{1}{\sqrt{1 + f_{\text{c}}^2(v,z,\dot{z})}}$,
  $\frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial ||\boldsymbol{F}||} = \boldsymbol{0}$
  and
  $\frac{\partial \text{sh\_c}(||\boldsymbol{F}||, f_{\text{c}}(v,z,\dot{z}))}{\partial \text{f}_{\text{c}}(\text{v,z,\dot{z}})} = -r * (1 + f_{\text{c}}^2(v,z,\dot{z}))^{-3/2} * f_{\text{c}}(v,z,\dot{z})$

9.1.7 Inline

This joint forces a point rigidly attached to node b to slide along a line rigidly attached to node a, thus removing 2 degrees of freedom.

Files. It is implemented in files
`mbdyn/struct/inline.h`
`mbdyn/struct/inline.cc`

Definitions. Orientation of line

$$\boldsymbol{R}_v = \boldsymbol{R}_a \tilde{\boldsymbol{R}}_v \tag{9.39}$$

the axis is along local axis 3 of matrix $\tilde{\boldsymbol{R}}_v$, $\tilde{\boldsymbol{e}}_3$, while directions $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$ are constrained.

Origin of line

$$\boldsymbol{x}_p = \boldsymbol{x}_a + \boldsymbol{p} = \boldsymbol{x}_a + \boldsymbol{R}_a \tilde{\boldsymbol{p}} \tag{9.40}$$

Sliding point

$$\boldsymbol{x}_q = \boldsymbol{x}_b + \boldsymbol{q} = \boldsymbol{x}_b + \boldsymbol{R}_b \tilde{\boldsymbol{q}} \tag{9.41}$$

Constraint equations: distance between sliding point $\boldsymbol{x}_q$ and axis null in local directions $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$, namely

$$\boldsymbol{e}_i^T \boldsymbol{d} = 0, \tag{9.42}$$

with $i = 1, 2$ and $\boldsymbol{d} = \boldsymbol{x}_q - \boldsymbol{x}_p$. After simple manipulation,

$$\boldsymbol{e}_i^T \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a \right) - \tilde{\boldsymbol{e}}_i^T \tilde{\boldsymbol{p}} = 0. \tag{9.43}$$

Perturbation:

$$\left(\boldsymbol{e}_i \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a \right) \right)^T \boldsymbol{\theta}_{a\delta} + \boldsymbol{e}_i^T \left(\delta \boldsymbol{x}_b - \boldsymbol{q} \times \boldsymbol{\theta}_{b\delta} - \delta \boldsymbol{x}_a \right) = 0 \tag{9.44}$$

Forces and moments:

$$\boldsymbol{f}_a = - \sum_{i=1,2} \boldsymbol{e}_i \lambda_i \tag{9.45a}$$

$$\boldsymbol{m}_a = \sum_{i=1,2} \boldsymbol{e}_i \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a \right) \lambda_i \tag{9.45b}$$

$$\boldsymbol{f}_b = \sum_{i=1,2} \boldsymbol{e}_i \lambda_i \tag{9.45c}$$

$$\boldsymbol{m}_b = - \sum_{i=1,2} \boldsymbol{e}_i \times \boldsymbol{q} \lambda_i \tag{9.45d}$$

Linearization

$$\left[\begin{array}{cccc} -\boldsymbol{e}_i^T & \left(\boldsymbol{e}_i \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a \right) \right)^T & \boldsymbol{e}_i^T & - \left(\boldsymbol{e}_i \times \boldsymbol{q} \right)^T \end{array} \right] \left\{ \begin{array}{c} \delta \boldsymbol{x}_a \\ \boldsymbol{\theta}_{a\delta} \\ \delta \boldsymbol{x}_b \\ \boldsymbol{\theta}_{b\delta} \end{array} \right\} = 0 \tag{9.46}$$

$$\delta \boldsymbol{f}_a = -\sum_{i=1,2} \left(\boldsymbol{e}_i \delta\lambda_i - \lambda_i \boldsymbol{e}_i \times \boldsymbol{\theta}_{a\delta}\right) \tag{9.47a}$$

$$\delta \boldsymbol{m}_a = \sum_{i=1,2} \left(\boldsymbol{e}_i \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a\right) \delta\lambda_i + \lambda_i \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a\right) \times \boldsymbol{e}_i \times \boldsymbol{\theta}_{a\delta} + \lambda_i \boldsymbol{e}_i \times \left(\delta\boldsymbol{x}_b - \boldsymbol{q} \times \boldsymbol{\theta}_{b\delta} - \delta\boldsymbol{x}_a\right)\right) \tag{9.47b}$$

$$\delta \boldsymbol{f}_b = \sum_{i=1,2} \left(\boldsymbol{e}_i \delta\lambda_i - \lambda_i \boldsymbol{e}_i \times \boldsymbol{\theta}_{a\delta}\right) \tag{9.47c}$$

$$\delta \boldsymbol{m}_b = -\sum_{i=1,2} \left(\boldsymbol{e}_i \times \boldsymbol{q} \delta\lambda_i + \lambda_i \boldsymbol{q} \times \boldsymbol{e}_i \times \boldsymbol{\theta}_{a\delta} - \lambda_i \boldsymbol{e}_i \times \boldsymbol{q} \times \boldsymbol{\theta}_{b\delta}\right) \tag{9.47d}$$

or

$$\begin{aligned}
\left\{\begin{array}{c} \delta\boldsymbol{f}_a \\ \delta\boldsymbol{m}_a \\ \delta\boldsymbol{f}_b \\ \delta\boldsymbol{m}_b \end{array}\right\} = & \left[\begin{array}{cc} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 \\ \boldsymbol{e}_1 \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a\right) & \boldsymbol{e}_2 \times \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a\right) \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ -\boldsymbol{e}_1 \times \boldsymbol{q} & -\boldsymbol{e}_2 \times \boldsymbol{q} \end{array}\right] \left\{\begin{array}{c} \delta\lambda_1 \\ \delta\lambda_2 \end{array}\right\} \\
& + \left[\begin{array}{cccc} \boldsymbol{0} & \sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times & \boldsymbol{0} & \boldsymbol{0} \\ -\sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times & \sum_{i=1,2} \lambda_i \left(\boldsymbol{x}_b + \boldsymbol{q} - \boldsymbol{x}_a\right) \times \boldsymbol{e}_i \times & \sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times & -\sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times \boldsymbol{q} \times \\ \boldsymbol{0} & -\sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\sum_{i=1,2} \lambda_i \boldsymbol{q} \times \boldsymbol{e}_i \times & \boldsymbol{0} & \sum_{i=1,2} \lambda_i \boldsymbol{e}_i \times \boldsymbol{q} \times \end{array}\right] \left\{\begin{array}{c} \delta\boldsymbol{x}_a \\ \boldsymbol{\theta}_{a\delta} \\ \delta\boldsymbol{x}_b \\ \boldsymbol{\theta}_{b\delta} \end{array}\right\}
\end{aligned} \tag{9.48}$$

9.1.8 Drive Hinge

This joint (`DriveHingeJoint`) forces two nodes to assume a relative orientation given by a rotation vector $\boldsymbol{\theta}$, whose direction with respect to node 1 represents the rotation axis, and whose amplitude represents the magnitude of the rotation.

Variants. There exist no variants, although a pinned version may be of use (TODO©).

Files. It is implemented in files
`mbdyn/struct/drvhinge.h`
`mbdyn/struct/drvhinge.cc`

Definitions.

$$\begin{aligned}
\boldsymbol{R}_{\text{rel}} &= \boldsymbol{R}_1^T \boldsymbol{R}_2 \\
\boldsymbol{\theta} &= \text{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{\text{rel}}\right)\right)
\end{aligned}$$

Limitations:

$$\|\boldsymbol{\theta}\| < \pi \tag{9.49}$$

Constraint equation

$$\boldsymbol{\theta} - \boldsymbol{\theta}_0 = \boldsymbol{0} \tag{9.50}$$

Couples:

$$\begin{aligned} \boldsymbol{M}_1 &= -\boldsymbol{R}_1\boldsymbol{\alpha} \\ \boldsymbol{M}_2 &= \boldsymbol{R}_1\boldsymbol{\alpha} \end{aligned}$$

Linearization:

$$\begin{bmatrix} (\boldsymbol{R}_1\boldsymbol{\alpha})\times & 0 & -\boldsymbol{R}_1 \\ -(\boldsymbol{R}_1\boldsymbol{\alpha})\times & 0 & \boldsymbol{R}_1 \\ -\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T & \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T & 0 \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \\ \delta\boldsymbol{\alpha} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{R}_1\boldsymbol{\alpha} \\ -\boldsymbol{R}_1\boldsymbol{\alpha} \\ \boldsymbol{\theta}_0 - \boldsymbol{\theta} \end{Bmatrix} \tag{9.51}$$

The linearization of the reaction moments contribution to the moment equilibrium equations of the nodes is straightforward. The linearization of the constraint equation is a bit more complicated. According to the definition of $\boldsymbol{\theta}$, its linearization yields

$$\begin{aligned} \delta\boldsymbol{\theta} &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\,\mathrm{ax}\left(\delta\boldsymbol{R}_{\mathrm{rel}}\boldsymbol{R}_{\mathrm{rel}}^T\right) \\ &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\,\mathrm{ax}\left(\delta\boldsymbol{R}_1^T\boldsymbol{R}_1 + \boldsymbol{R}_1^T\delta\boldsymbol{R}_2\boldsymbol{R}_2^T\boldsymbol{R}_1\right) \\ &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\,\mathrm{ax}\left(\boldsymbol{R}_1^T\boldsymbol{\theta}_{1\delta}\times{}^T\boldsymbol{R}_1 + \boldsymbol{R}_1^T\boldsymbol{\theta}_{2\delta}\times\boldsymbol{R}_1\right) \\ &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\,\mathrm{ax}\left(-\boldsymbol{R}_1^T\boldsymbol{\theta}_{1\delta}\times\boldsymbol{R}_1 + \boldsymbol{R}_1^T\boldsymbol{\theta}_{2\delta}\times\boldsymbol{R}_1\right) \\ &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\,\mathrm{ax}\left(\boldsymbol{R}_1^T\left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right)\times\boldsymbol{R}_1\right) \\ &= \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T\left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right) \end{aligned}$$

according to the updated-updated[2] simplifications, $\boldsymbol{\theta}_{i\delta} \overset{\mathrm{uu}}{=} \delta\boldsymbol{g}_i$, thus resulting in

$$\boldsymbol{\theta}_\delta = \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T\left(\delta\boldsymbol{g}_2 - \delta\boldsymbol{g}_1\right) \tag{9.52}$$

note that $\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}$ does not simplify to $\boldsymbol{I}$ because $\boldsymbol{\theta}$ in general is a finite rotation.

9.1.9 Drive Displacement

This joint (`DriveDisplacementJoint`) constrains the relative position of a point optionally offset from node b with respect to a point optionally offset from node a, so that their relative position matches a value that is imposed in node a reference frame; the imposed vector may depend on time and other states during the simulation.

Variants. There exists a pinned version (`DriveDisplacementPinJoint`), discussed in Section 9.1.10.

Files. It is implemented in files
`mbdyn/struct/drvdisp.h`
`mbdyn/struct/drvdisp.cc`

Constraint Equation. The constraint equation is

$$\boldsymbol{\phi} = \boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a - \boldsymbol{f}_a - \boldsymbol{v} = \boldsymbol{0} \tag{9.53}$$

or, based on the initial assumptions,

$$\boldsymbol{\phi} = \boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a - \boldsymbol{R}_a\left(\tilde{\boldsymbol{f}}_a + \tilde{\boldsymbol{v}}\right) = \boldsymbol{0} \tag{9.54}$$

[2]In the following, the operator $\overset{\mathrm{uu}}{=}$ will be used to indicate the updated-updated approximation.

The perturbation of the constraint equation yields

$$\delta\boldsymbol{\phi} = \delta\boldsymbol{x}_b - \boldsymbol{f}_b \times \boldsymbol{\theta}_{b\delta} - \delta\boldsymbol{x}_a + \boldsymbol{d} \times \boldsymbol{\theta}_{a\delta} \tag{9.55}$$

where $\boldsymbol{d} = \boldsymbol{R}_a \left(\tilde{\boldsymbol{f}}_a + \tilde{\boldsymbol{v}}\right)$. The contribution of the constraint to the equilibrium equations is

$$\left\{\begin{array}{c} \boldsymbol{F}_a \\ \boldsymbol{C}_a \\ \boldsymbol{F}_b \\ \boldsymbol{C}_b \end{array}\right\} = \left\{\begin{array}{c} -\boldsymbol{\lambda} \\ -\boldsymbol{d} \times \boldsymbol{\lambda} \\ \boldsymbol{\lambda} \\ \boldsymbol{f}_b \times \boldsymbol{\lambda} \end{array}\right\} \tag{9.56}$$

where $\boldsymbol{\lambda}$ is the reaction force in the global reference frame.

The contribution of forces and moments and of the constraint equation to the Jacobian matrix is

$$\left[\begin{array}{ccccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{0} & -\boldsymbol{\lambda} \times \boldsymbol{d} \times & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{d} \times \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\lambda} \times \boldsymbol{f}_b \times & \boldsymbol{f}_b \times \\ -\boldsymbol{I} & \boldsymbol{d} \times & \boldsymbol{I} & -\boldsymbol{f}_b \times & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \delta\boldsymbol{x}_a \\ \boldsymbol{\theta}_{a\delta} \\ \delta\boldsymbol{x}_b \\ \boldsymbol{\theta}_{b\delta} \\ \delta\boldsymbol{\lambda} \end{array}\right\} \tag{9.57}$$

The derivative of the constraint equation is

$$\dot{\boldsymbol{\phi}} = \dot{\boldsymbol{x}}_b + \boldsymbol{\omega}_b \times \boldsymbol{f}_b - \dot{\boldsymbol{x}}_a - \boldsymbol{\omega}_a \times \boldsymbol{d} - \dot{\boldsymbol{d}} \tag{9.58}$$

where $\dot{\boldsymbol{d}} = \boldsymbol{R}_a \dot{\tilde{\boldsymbol{v}}}$; its perturbation yields

$$\delta\dot{\boldsymbol{\phi}} = \delta\dot{\boldsymbol{x}}_b + \delta\boldsymbol{\omega}_b \times \boldsymbol{f}_b + \boldsymbol{\omega}_b \times \boldsymbol{\theta}_{b\delta} \times \boldsymbol{f}_b - \delta\dot{\boldsymbol{x}}_a - \delta\boldsymbol{\omega}_a \times \boldsymbol{d} - \boldsymbol{\omega}_a \times \boldsymbol{\theta}_{a\delta} \times \boldsymbol{d} - \boldsymbol{\theta}_{a\delta} \times \dot{\boldsymbol{d}}. \tag{9.59}$$

The contributions to the Jacobian matrix and residual of the initial assembly are

$$
\left[\begin{array}{ccccc}
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \\
\boldsymbol{0} & -\boldsymbol{\lambda}\times\boldsymbol{d}\times\ -\boldsymbol{\mu}\times\dot{\boldsymbol{d}}\times+(\boldsymbol{\omega}_a\times\boldsymbol{\mu})\times\boldsymbol{d}\times & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{d}\times \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\lambda}\times\boldsymbol{f}_b\times\ -(\boldsymbol{\omega}_b\times\boldsymbol{\mu})\times\boldsymbol{f}_b\times & \boldsymbol{f}_b\times \\
-\boldsymbol{I} & \boldsymbol{d}\times & \boldsymbol{I} & -\boldsymbol{f}_b\times & \boldsymbol{0} \\
\hline
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{\mu}\times\boldsymbol{d}\times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mu}\times\boldsymbol{f}_b\times & \boldsymbol{0} \\
\boldsymbol{0} & \dot{\boldsymbol{d}}\times\ +\boldsymbol{\omega}_a\times\boldsymbol{d}\times & \boldsymbol{0} & -\boldsymbol{\omega}_b\times\boldsymbol{f}_b\times & \boldsymbol{0}
\end{array}\right]
\left\{\begin{array}{c}
\delta\boldsymbol{x}_a \\ \boldsymbol{\theta}_{a\delta} \\ \delta\boldsymbol{x}_b \\ \boldsymbol{\theta}_{b\delta} \\ \delta\boldsymbol{\lambda}
\end{array}\right\}
$$

$$
+\left[\begin{array}{ccccc}
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{d}\times\boldsymbol{\mu}\times & \boldsymbol{0} & \boldsymbol{0} & -\dot{\boldsymbol{d}}\times\ +\boldsymbol{d}\times\boldsymbol{\omega}_a\times \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{f}_b\times\boldsymbol{\mu}\times & -\boldsymbol{f}_b\times\boldsymbol{\omega}_b\times \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\hline
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{d}\times \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{f}_b\times \\
-\boldsymbol{I} & \boldsymbol{d}\times & \boldsymbol{I} & -\boldsymbol{f}_b\times & \boldsymbol{0}
\end{array}\right]
\left\{\begin{array}{c}
\delta\dot{\boldsymbol{x}}_a \\ \delta\boldsymbol{\omega}_a \\ \delta\dot{\boldsymbol{x}}_b \\ \delta\boldsymbol{\omega}_b \\ \delta\boldsymbol{\mu}
\end{array}\right\}
$$

$$
=\left\{\begin{array}{c}
\boldsymbol{\lambda} \\
\boldsymbol{d}\times\boldsymbol{\lambda}-\boldsymbol{d}\times\boldsymbol{\omega}_a\times\boldsymbol{\mu}+\dot{\boldsymbol{d}}\times\boldsymbol{\mu} \\
-\boldsymbol{\lambda} \\
-\boldsymbol{f}_b\times\boldsymbol{\lambda}+\boldsymbol{f}_b\times\boldsymbol{\omega}_b\times\boldsymbol{\mu} \\
\boldsymbol{x}_a+\boldsymbol{d}-\boldsymbol{x}_b-\boldsymbol{f}_b \\
\hline
\boldsymbol{\mu} \\
\boldsymbol{d}\times\boldsymbol{\mu} \\
-\boldsymbol{\mu} \\
-\boldsymbol{f}_b\times\boldsymbol{\mu} \\
\dot{\boldsymbol{x}}_a+\boldsymbol{\omega}_a\times\boldsymbol{d}+\dot{\boldsymbol{d}}-\dot{\boldsymbol{x}}_b-\boldsymbol{\omega}_b\times\boldsymbol{f}_b
\end{array}\right\} \tag{9.60}
$$

9.1.10 Drive Displacement Pin

This joint (`DriveDisplacementPinJoint`) is the "pinned" variant of the drive displacement joint (`DriveDisplacementJoint`) discussed in Section 9.1.9. It imposes the absolute position $\boldsymbol{v}$ of a point optionally connected to a node by a rigid offset $\boldsymbol{f}$:

$$
\boldsymbol{\phi}=\boldsymbol{x}+\boldsymbol{f}-\boldsymbol{x}_0-\boldsymbol{v}=\boldsymbol{0} \tag{9.61}
$$

The perturbation of the constraint equation yields

$$
\delta\boldsymbol{\phi}=\delta\boldsymbol{x}-\boldsymbol{f}\times\boldsymbol{\theta}_\delta \tag{9.62}
$$

The contribution of the constraint to the equilibrium equations is

$$
\left\{\begin{array}{c}\boldsymbol{F} \\ \boldsymbol{C}\end{array}\right\}=\left\{\begin{array}{c}\boldsymbol{\lambda} \\ \boldsymbol{f}\times\boldsymbol{\lambda}\end{array}\right\} \tag{9.63}
$$

where $\boldsymbol{\lambda}$ is the reaction force in the absolute reference frame.

The contribution of forces and moments and of the constraint equation to the Jacobian matrix is

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{\lambda} \times \boldsymbol{f} \times & \boldsymbol{f} \times \\ \boldsymbol{I} & -\boldsymbol{f} \times & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{x} \\ \boldsymbol{\theta}_\delta \\ \delta \boldsymbol{\lambda} \end{Bmatrix} \tag{9.64}$$

The derivative of the constraint equation is

$$\dot{\boldsymbol{\phi}} = \dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{f} - \dot{\boldsymbol{v}}; \tag{9.65}$$

its perturbation yields

$$\delta \dot{\boldsymbol{\phi}} = \delta \dot{\boldsymbol{x}} - \boldsymbol{f} \times \delta \boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{f} \times \boldsymbol{\theta}_\delta \tag{9.66}$$

The contributions to the Jacobian matrix and residual of the initial assembly are

$$\left[\begin{array}{ccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{\lambda} \times \boldsymbol{f} \times - (\boldsymbol{\omega} \times \boldsymbol{\mu}) \times \boldsymbol{f} \times & \boldsymbol{f} \times \\ \boldsymbol{I} & -\boldsymbol{f} \times & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mu} \times \boldsymbol{f} \times & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\omega} \times \boldsymbol{f} \times & \boldsymbol{0} \end{array}\right] \begin{Bmatrix} \delta \boldsymbol{x} \\ \boldsymbol{\theta}_\delta \\ \delta \boldsymbol{\lambda} \end{Bmatrix}$$
$$\left[\begin{array}{ccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{f} \times \boldsymbol{\mu} \times & -\boldsymbol{f} \times \boldsymbol{\omega} \times \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{f} \times \\ \boldsymbol{I} & -\boldsymbol{f} \times & \boldsymbol{0} \end{array}\right] \begin{Bmatrix} \delta \dot{\boldsymbol{x}} \\ \delta \boldsymbol{\omega} \\ \delta \boldsymbol{\mu} \end{Bmatrix} = \left\{\begin{array}{c} -\boldsymbol{\lambda} \\ -\boldsymbol{f} \times \boldsymbol{\lambda} + \boldsymbol{f} \times \boldsymbol{\omega} \times \boldsymbol{\mu} \\ \boldsymbol{x}_0 + \boldsymbol{v} - \boldsymbol{x} - \boldsymbol{f} \\ \hline -\boldsymbol{\mu} \\ -\boldsymbol{f} \times \boldsymbol{\mu} \\ \dot{\boldsymbol{v}} - \dot{\boldsymbol{x}} - \boldsymbol{\omega} \times \boldsymbol{f} \end{array}\right\} \tag{9.67}$$

9.1.11 Imposed Displacement

The imposed displacement joint (`ImposedDisplacementJoint`) is similar to the drive displacement joint (`DriveDisplacementPinJoint`), discussed in Section 9.1.10, but the relative displacement v, a scalar possibly dependent on the time and on other states of the problem, is imposed only in direction $\tilde{\boldsymbol{e}}_a$, relative to node a.

Variants. There exists a pinned version (`ImposedDisplacementPinJoint`), discussed in Section 9.1.12.

Files. It is implemented in files
`mbdyn/struct/impdisp.h`
`mbdyn/struct/impdisp.cc`

Constraint equation.

$$\phi = \boldsymbol{e}_a^T \left(\boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a - \boldsymbol{f}_a \right) - v = 0, \tag{9.68}$$

or, based on the initial assumptions,

$$\phi = \boldsymbol{e}_a^T \left(\boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a \right) - \left(\tilde{\boldsymbol{e}}_a^T \tilde{\boldsymbol{f}}_a + v \right) = 0, \tag{9.69}$$

or

$$\phi = \boldsymbol{e}_a^T \boldsymbol{d} - \left(\tilde{\boldsymbol{e}}_a^T \tilde{\boldsymbol{f}}_a + v \right) = 0, \tag{9.70}$$

where $\boldsymbol{d} = \boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a$ and $\tilde{\boldsymbol{e}}_a$ is constant in the reference frame of node a. The perturbation of the constraint equation yields

$$\delta\phi = \boldsymbol{e}_a^T \left(\delta\boldsymbol{x}_b - \boldsymbol{f}_b \times \boldsymbol{\theta}_{b\delta} - \delta\boldsymbol{x}_a\right) + \left(\boldsymbol{e}_a \times \left(\boldsymbol{x}_b + \boldsymbol{f}_b - \boldsymbol{x}_a\right)\right)^T \boldsymbol{\theta}_{a\delta}, \tag{9.71}$$

or

$$\delta\phi = \boldsymbol{e}_a^T \left(\delta\boldsymbol{x}_b - \boldsymbol{f}_b \times \boldsymbol{\theta}_{b\delta} - \delta\boldsymbol{x}_a\right) + \left(\boldsymbol{e}_a \times \boldsymbol{d}\right)^T \boldsymbol{\theta}_{a\delta}. \tag{9.72}$$

The contribution of the constraint to the equilibrium equations is

$$\left\{\begin{array}{c} \boldsymbol{F}_a \\ \boldsymbol{C}_a \\ \boldsymbol{F}_b \\ \boldsymbol{C}_b \end{array}\right\} = \left\{\begin{array}{c} -\boldsymbol{e}_a\lambda \\ -\boldsymbol{d} \times \boldsymbol{e}_a\lambda \\ \boldsymbol{e}_a\lambda \\ \boldsymbol{f}_b \times \boldsymbol{e}_a\lambda \end{array}\right\} \tag{9.73}$$

where λ is the reaction force (a scalar).

The contribution of forces and moments and of the constraint equation to the Jacobian matrix is

$$\left[\begin{array}{ccccc} \boldsymbol{0} & \lambda\boldsymbol{e}_a \times & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e}_a \\ -\lambda\boldsymbol{e}_a \times & \lambda\boldsymbol{d} \times \boldsymbol{e}_a \times & \lambda\boldsymbol{e}_a \times & -\lambda\boldsymbol{e}_a \times \boldsymbol{f}_b \times & -\boldsymbol{d} \times \boldsymbol{e}_a \\ \boldsymbol{0} & -\lambda\boldsymbol{e}_a \times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_a \\ \boldsymbol{0} & -\lambda\boldsymbol{f}_b \times \boldsymbol{e}_a \times & \boldsymbol{0} & \lambda\boldsymbol{e}_a \times \boldsymbol{f}_b \times & \boldsymbol{f}_b \times \boldsymbol{e}_a \\ -\boldsymbol{e}_a^T & -\left(\boldsymbol{d} \times \boldsymbol{e}_a\right)^T & \boldsymbol{e}_a^T & \left(\boldsymbol{f}_b \times e_a\right)^T & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \delta\boldsymbol{x}_a \\ \boldsymbol{\theta}_{a\delta} \\ \delta\boldsymbol{x}_b \\ \boldsymbol{\theta}_{b\delta} \\ \delta\lambda \end{array}\right\} \tag{9.74}$$

The constraint derivative is

$$\dot{\phi} = \boldsymbol{e}_a^T \left(\dot{\boldsymbol{x}}_b + \boldsymbol{\omega}_b \times \boldsymbol{f}_b - \dot{\boldsymbol{x}}_a\right) + \boldsymbol{d}^T \left(\boldsymbol{\omega}_a \times \boldsymbol{e}_a\right) - \dot{v} = 0, \tag{9.75}$$

or

$$\dot{\phi} = \boldsymbol{e}_a^T \dot{\boldsymbol{d}} + \boldsymbol{d}^T \left(\boldsymbol{\omega}_a \times \boldsymbol{e}_a\right) - \dot{v} = 0, \tag{9.76}$$

with $\dot{\boldsymbol{d}} = \dot{\boldsymbol{x}}_b + \boldsymbol{\omega}_b \times \boldsymbol{f}_b - \dot{\boldsymbol{x}}_a$. Its linearization yields

$$\begin{aligned} \delta\dot{\phi} = {} & \boldsymbol{e}_a^T \delta\dot{\boldsymbol{x}}_b - \boldsymbol{e}_a^T \delta\dot{\boldsymbol{x}}_a + \left(\boldsymbol{f}_b \times \boldsymbol{e}_a\right)^T \delta\boldsymbol{\omega}_b - \left(\boldsymbol{d} \times \boldsymbol{e}_a\right)^T \delta\boldsymbol{\omega}_a \\ & + \left(\boldsymbol{\omega}_a \times \boldsymbol{e}_a\right)^T \delta\boldsymbol{x}_b - \left(\boldsymbol{\omega}_a \times \boldsymbol{e}_a\right)^T \delta\boldsymbol{x}_a \\ & + \left(\boldsymbol{f}_b \times \boldsymbol{e}_a \times \left(\boldsymbol{\omega}_b - \boldsymbol{\omega}_a\right)\right)^T \boldsymbol{\theta}_{b\delta} + \Big(\boldsymbol{e}_a \times \Big(\dot{\boldsymbol{d}} - \boldsymbol{\omega}_a \times \boldsymbol{d}\Big)\Big)^T \boldsymbol{\theta}_{a\delta} \end{aligned} \tag{9.77}$$

9.1.12 Imposed Displacement Pin

The imposed displacement pin joint (`ImposedDisplacementPinJoint`) is the "pinned" variant of the imposed displacement joint (`ImposedDisplacementJoint`) discussed in Section 9.1.11.

In this case, the direction $\boldsymbol{e}$ is referred to the absolute frame, so it does no longer depend on the state of the problem.

Variants. There exists a version that imposes the relative displacement of two nodes along a direction that is fixed with respect to one of them (`ImposedDisplacementJoint`), discussed in Section 9.1.11.

Files. It is implemented in files
`mbdyn/struct/impdisp.h`
`mbdyn/struct/impdisp.cc`

The constraint equation is

$$\phi = \boldsymbol{e}^T \left(\boldsymbol{x} + \boldsymbol{f} - \boldsymbol{x}_0\right) - v = 0, \tag{9.78}$$

or

$$\phi = \boldsymbol{e}^T \boldsymbol{d} - v = 0, \tag{9.79}$$

where $\boldsymbol{d} = \boldsymbol{x} + \boldsymbol{f} - \boldsymbol{x}_0$. The perturbation of the constraint equation yields

$$\delta\phi = \boldsymbol{e}^T \left(\delta\boldsymbol{x}_b - \boldsymbol{f}_b \times \boldsymbol{\theta}_{b\delta}\right). \tag{9.80}$$

The contribution of the constraint to the equilibrium equations is

$$\left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{C} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{e}\lambda \\ \boldsymbol{f} \times \boldsymbol{e}\lambda \end{array} \right\} \tag{9.81}$$

where λ is the reaction force (a scalar).

The contribution of forces and moments and of the constraint equation to the Jacobian matrix is

$$\left[\begin{array}{ccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e} \\ \boldsymbol{0} & \lambda\boldsymbol{e} \times \boldsymbol{f} \times & \boldsymbol{f} \times \boldsymbol{e} \\ \boldsymbol{e}^T & \left(\boldsymbol{f} \times e\right)^T & \boldsymbol{0} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{x} \\ \boldsymbol{\theta}_\delta \\ \delta\lambda \end{array} \right\} \tag{9.82}$$

The constraint derivative is

$$\dot{\phi} = \boldsymbol{e}^T \left(\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{f}\right) - \dot{v} = 0, \tag{9.83}$$

or

$$\dot{\phi} = \boldsymbol{e}^T \dot{\boldsymbol{d}} - \dot{v} = 0, \tag{9.84}$$

Its linearization yields

$$\left[\begin{array}{cccccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e} & \boldsymbol{0} \\ \boldsymbol{0} & \lambda\boldsymbol{e} \times \boldsymbol{f} \times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{f} \times \boldsymbol{e} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e} \\ \boldsymbol{0} & \mu\boldsymbol{e} \times \boldsymbol{f} \times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{f} \times \boldsymbol{e} \\ \boldsymbol{e}^T & \left(\boldsymbol{f} \times e\right)^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\left(\boldsymbol{f} \times \boldsymbol{\omega} \times \boldsymbol{e}\right)^T & \boldsymbol{e}^T & \left(\boldsymbol{f} \times \boldsymbol{e}\right)^T & \boldsymbol{0} & \boldsymbol{0} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{x} \\ \boldsymbol{\theta}_\delta \\ \delta\dot{\boldsymbol{x}} \\ \delta\boldsymbol{\omega} \\ \delta\lambda \\ \delta\mu \end{array} \right\} \tag{9.85}$$

9.1.13 Total Joint

This joint (`TotalJoint`) forces two nodes to assume an imposed relative position and orientation about selected axes. Both imposed position and orientation are specified in the reference frame of node 1. The joint consists in 6 equations, 3 for positions and 3 for orientation, that can be selectively turned on or off to enforce a constraint on that DOF.

Variants. There exists a pinned version of this joint that imposes the absolute position and orientation of a node (`TotalPinJoint`). It is discussed in Section 9.1.14.

Files. It is implemented in files
`mbdyn/struct/totalj.h`
`mbdyn/struct/totalj.cc`

Definitions. The position of the nodes with respect to the global reference frame are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The matrices that describe the orientation of the nodes with respect to the global reference frame are $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. An offset is allowed for the location of the points where the relative position is constrained; the offsets $\tilde{\boldsymbol{f}}_1$ and $\tilde{\boldsymbol{f}}_2$ are rigidly connected to the nodes, so in the global reference frame their expression is

$$\boldsymbol{f}_1 = \boldsymbol{R}_1 \tilde{\boldsymbol{f}}_1 \tag{9.86a}$$
$$\boldsymbol{f}_2 = \boldsymbol{R}_2 \tilde{\boldsymbol{f}}_2. \tag{9.86b}$$

The position constraint may be expressed in a reference frame that is rigidly connected to node 1, $\tilde{\boldsymbol{R}}_{1h}$, whose orientation in the global reference frame is

$$\boldsymbol{R}_{1h} = \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1h}. \tag{9.87}$$

Similarly, the orientation constraint may be expressed in a reference frame that is rigidly connected to both nodes by matrices $\tilde{\boldsymbol{R}}_{1hr}$ and $\tilde{\boldsymbol{R}}_{2hr}$, whose orientation in the global reference frame is

$$\boldsymbol{R}_{1hr} = \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1hr} \tag{9.88a}$$
$$\boldsymbol{R}_{2hr} = \boldsymbol{R}_2 \tilde{\boldsymbol{R}}_{2hr}. \tag{9.88b}$$

The relative position and orientation may refer to different local reference frames to increase the versatility of the element.

It is essential to notice that the differentiation of the constraint orientation matrices does not affect the relative orientation between the constraint and each node, namely matrices $\tilde{\boldsymbol{R}}_{1hr}$ and $\tilde{\boldsymbol{R}}_{2hr}$, which are constant. As a consequence,

$$\begin{aligned} \delta \boldsymbol{R}_{1hr} &= \delta \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1hr} \\ &= \boldsymbol{\theta}_{1\delta} \times \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1hr} \\ &= \boldsymbol{\theta}_{1\delta} \times \boldsymbol{R}_{1hr} \end{aligned} \tag{9.89a}$$
$$\delta \boldsymbol{R}_{2hr} = \boldsymbol{\theta}_{2\delta} \times \boldsymbol{R}_{2hr}, \tag{9.89b}$$

and

$$\dot{\boldsymbol{R}}_{1hr} = \boldsymbol{\omega}_1 \times \boldsymbol{R}_{1hr} \tag{9.90}$$
$$\dot{\boldsymbol{R}}_{2hr} = \boldsymbol{\omega}_2 \times \boldsymbol{R}_{2hr}. \tag{9.91}$$

The same is true for the relative position orientation matrix, $\boldsymbol{R}_{1h}$.

Orientation Constraint. The relative orientation is expressed by matrix

$$\boldsymbol{R}_{\text{rel}} = \boldsymbol{R}_{1hr}^T \boldsymbol{R}_{2hr} \tag{9.92}$$

The desired relative orientation can be expressed by means of a rotation vector $\boldsymbol{\theta}_0$; the corresponding rotation matrix is

$$\boldsymbol{R}_0 = \exp\left(\boldsymbol{\theta}_0 \times\right). \tag{9.93}$$

This operation can always be performed with no ambiguity, regardless of the amplitude of $\boldsymbol{\theta}_0$. However, the same is not true for the inverse operation, namely

$$\boldsymbol{\theta}_0 = \mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}_0\right)\right), \tag{9.94}$$

which can only yield the minimal $\boldsymbol{\theta}_0$ corresponding to $\boldsymbol{R}_0$, with the indetermination corresponding to an arbitrary amount of complete revolutions.

The equality

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 \tag{9.95}$$

implies that

$$\boldsymbol{R}_{\mathrm{rel}} = \boldsymbol{R}_0. \tag{9.96}$$

Equation (9.96) can be rewritten as

$$\boldsymbol{R}_{\mathrm{rel}}\boldsymbol{R}_0^T = \boldsymbol{I}, \tag{9.97}$$

and thus Eq. (9.95), after defining

$$\boldsymbol{R}^\delta = \boldsymbol{R}_{\mathrm{rel}}\boldsymbol{R}_0^T, \tag{9.98}$$

is equivalent to

$$\mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}^\delta\right)\right) = \mathbf{0}. \tag{9.99}$$

In general, the difference between the desired rotation $\boldsymbol{\theta}_0$ and the current relative rotation vector between nodes 1 and 2,

$$\boldsymbol{\theta} = \mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{1hr}^T\boldsymbol{R}_{2hr}\right)\right) \tag{9.100}$$

through the previous redefinition can be expressed as

$$\boldsymbol{\theta}^\delta = \mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{1hr}^T\boldsymbol{R}_{2hr}\boldsymbol{R}_0^T\right)\right) \tag{9.101}$$

and is equivalent to

$$\boldsymbol{\theta}^\delta = \boldsymbol{\theta} - \boldsymbol{\theta}_0 \tag{9.102}$$

when $\boldsymbol{\theta}^\delta = \mathbf{0}$.

Position Constraint. The desired relative position is imposed by means of a vector $\boldsymbol{x}_0$ defined in the relative position constraint reference frame of node 1. The relative position is represented by vector $\boldsymbol{x}$, defined as

$$\boldsymbol{x} = \boldsymbol{R}_{1h}^T\left(\boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1\right); \tag{9.103}$$

the constraint equations for the relative position are:

$$\boldsymbol{x} = \boldsymbol{x}_0. \tag{9.104}$$

In analogy with the definition of $\boldsymbol{\theta}^\delta$, a vector $\boldsymbol{x}^\delta$ can be used to express the constraint on the position DOFs,

$$\boldsymbol{x}^\delta = \boldsymbol{x} - \boldsymbol{x}_0. \tag{9.105}$$

Constraint Equations. The resulting constraint equations are:

$$\boldsymbol{x}^{\delta} = \mathbf{0} \tag{9.106a}$$

$$\boldsymbol{\theta}^{\delta} = \mathbf{0}. \tag{9.106b}$$

Any component of $\boldsymbol{x}^{\delta}$ and $\boldsymbol{\theta}^{\delta}$ can be selectively set to zero to enforce a constraint on that degree of freedom. The axis about which the constraint is applied can be arbitrarily defined with respect to the node orientation by appropriately setting the constant relative orientation matrix of each node, both for position ($\tilde{\boldsymbol{R}}_{1h}$) and for orientation ($\tilde{\boldsymbol{R}}_{1hr}$ and $\tilde{\boldsymbol{R}}_{2hr}$). For imposed rotations, an arbitrary amplitude of the rotation about the constraint axis can be imposed by means of the components of vector $\boldsymbol{\theta}_0$. The same is true for positions, by means of $\tilde{\boldsymbol{v}}$.

Alternatively, equations related to any of the components of $\boldsymbol{x}^{\delta}$ and $\boldsymbol{\theta}^{\delta}$ can be omitted, thus omitting the constraint along or about the corresponding axis. The corresponding value of $\boldsymbol{x}^{\delta}$ and $\boldsymbol{\theta}^{\delta}$ could be used to apply moments based on the relative position/orientation of the nodes, e.g. linear/rotational springs or dampers. In this case, the singularity problem on rotations is back: the norm of the relative rotation must not exceed π, i.e. $\left|\boldsymbol{\theta}^{\delta}\right| < \pi$. Note that the relative rotation is computed with respect to $\boldsymbol{\theta}_0$, so an anelastic (or imposed) rotation can be easily applied to the spring, resulting in an imposed rotation with stiffness.

Linearization. First, the perturbation of $\boldsymbol{x}^{\delta}$,

$$\delta\boldsymbol{x}^{\delta} = \boldsymbol{R}_{1h}^{T}\left(\delta\boldsymbol{x}_2 - \boldsymbol{b}_2 \times \boldsymbol{\theta}_{2\delta} - \delta\boldsymbol{x}_1 + \boldsymbol{b}_1 \times \boldsymbol{\theta}_{1\delta}\right) \tag{9.107}$$

is computed, where

$$\boldsymbol{b}_1 = \boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 \tag{9.108a}$$

$$\boldsymbol{b}_2 = \boldsymbol{f}_2. \tag{9.108b}$$

Note that, since the constraint equations are defined by setting to zero any components of the vector $\boldsymbol{x}^{\delta}$, the related Jacobian matrix contribution is obtained by selecting the corresponding columns of matrix $\boldsymbol{R}_{1h}$.

Then, the perturbation of $\boldsymbol{\theta}^{\delta}$ results from

$$\begin{aligned}\boldsymbol{\theta}_{\delta}^{\delta}\times &= \delta\boldsymbol{R}^{\delta}\boldsymbol{R}^{\delta T}\\ &= \boldsymbol{R}_{1hr}^{T}\boldsymbol{\theta}_{1\delta}\times^{T}\boldsymbol{R}_{1hr} + \boldsymbol{R}_{1hr}^{T}\boldsymbol{\theta}_{2\delta}\times\boldsymbol{R}_{1hr}\\ &= \boldsymbol{R}_{1hr}^{T}\left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right)\times\boldsymbol{R}_{1hr},\end{aligned} \tag{9.109}$$

which implies

$$\boldsymbol{\theta}_{\delta}^{\delta} = \boldsymbol{R}_{1hr}^{T}\left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right). \tag{9.110}$$

Note that, since the constraint equations are defined by setting to zero any components of the vector $\boldsymbol{\theta}^{\delta}$, the related Jacobian matrix contribution is obtained by selecting the corresponding columns of matrix $\boldsymbol{R}_{1hr}$.

After calling $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}}$ the Lagrange multipliers respectively related to the relative position and orientation constraints, and noticing that in the global reference frame they become

$$\boldsymbol{\lambda}_{\boldsymbol{x}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}} \tag{9.111a}$$

$$\boldsymbol{\lambda}_{\boldsymbol{\theta}} = \boldsymbol{R}_{1hr}\tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}}, \tag{9.111b}$$

the contribution of the constraint reactions to the equilibrium equations of the participating nodes is

$$\boldsymbol{F}_1 = -\boldsymbol{\lambda}_{\boldsymbol{x}} \tag{9.112a}$$
$$\boldsymbol{C}_1 = -\boldsymbol{b}_1 \times \boldsymbol{\lambda}_{\boldsymbol{x}} - \boldsymbol{\lambda}_{\boldsymbol{\theta}} \tag{9.112b}$$
$$\boldsymbol{F}_2 = \boldsymbol{\lambda}_{\boldsymbol{x}} \tag{9.112c}$$
$$\boldsymbol{C}_2 = \boldsymbol{b}_2 \times \boldsymbol{\lambda}_{\boldsymbol{x}} + \boldsymbol{\lambda}_{\boldsymbol{\theta}}. \tag{9.112d}$$

Due to the simplifications related to the updated-updated approach, the contribution of the constraint equations and of the equilibrium to the Jacobian matrix is:

$$\left[\begin{array}{cccc|cc}
\boldsymbol{0} & \boldsymbol{\lambda}_{\boldsymbol{x}} \times & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_{1h} & \boldsymbol{0} \\
-\boldsymbol{\lambda}_{\boldsymbol{x}} \times & \boldsymbol{b}_1 \times \boldsymbol{\lambda}_{\boldsymbol{x}} \times + \boldsymbol{\lambda}_{\boldsymbol{\theta}} \times & \boldsymbol{\lambda}_{\boldsymbol{x}} \times & -\boldsymbol{\lambda}_{\boldsymbol{x}} \times \boldsymbol{b}_2 \times & -\boldsymbol{b}_1 \times \boldsymbol{R}_{1h} & -\boldsymbol{R}_{1hr} \\
\boldsymbol{0} & -\boldsymbol{\lambda}_{\boldsymbol{x}} \times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_{1h} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{b}_2 \times \boldsymbol{\lambda}_{\boldsymbol{x}} \times - \boldsymbol{\lambda}_{\boldsymbol{\theta}} \times & \boldsymbol{0} & \boldsymbol{\lambda}_{\boldsymbol{x}} \times \boldsymbol{b}_2 \times & \boldsymbol{b}_2 \times \boldsymbol{R}_{1h} & \boldsymbol{R}_{1hr} \\
\hline
-\boldsymbol{R}_{1h}^T & \boldsymbol{R}_{1h}^T \boldsymbol{b}_1 \times & \boldsymbol{R}_{1h}^T & -\boldsymbol{R}_{1h}^T \boldsymbol{b}_2 \times & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{R}_{1hr}^T & \boldsymbol{0} & \boldsymbol{R}_{1hr}^T & \boldsymbol{0} & \boldsymbol{0}
\end{array}\right]
\left\{\begin{array}{c}
\delta \boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \\ \delta \boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \\ \hline \delta \tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}} \\ \delta \tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}}
\end{array}\right\}$$
$$= \left\{\begin{array}{c}
\boldsymbol{\lambda}_{\boldsymbol{x}} \\ \boldsymbol{b}_1 \times \boldsymbol{\lambda}_{\boldsymbol{x}} + \boldsymbol{\lambda}_{\boldsymbol{\theta}} \\ -\boldsymbol{\lambda}_{\boldsymbol{x}} \\ -\boldsymbol{b}_2 \times \boldsymbol{\lambda}_{\boldsymbol{x}} - \boldsymbol{\lambda}_{\boldsymbol{\theta}} \\ \hline -\boldsymbol{x}^\delta \\ -\boldsymbol{\theta}^\delta
\end{array}\right\}. \tag{9.113}$$

The updated-updated form is obtained by simply replacing $\boldsymbol{\theta}_{1\delta}$ and $\boldsymbol{\theta}_{2\delta}$ with $\delta \boldsymbol{g}_1$ and $\delta \boldsymbol{g}_2$, respectively.

Summary

- each component of the relative position/orientation can be subjected to separate constraint conditions;
- an ideal constraint results from activating the constraint equation on that degree of freedom and setting to zero, or to any desired, time dependent value the relative position/orientation of that component;
- a deformable constraint results from deactivating the constraint equation on that degree of freedom, possibly adding a constitutive law based on that error and possibly on its derivative (currently, this requires a separate instance of a deformable element); in this case, the norm of the amplitude of the relative orientation is limited to π;
- the imposed relative orientation goes in matrix $\boldsymbol{R}_0$ by means of a rotation vector $\boldsymbol{\theta}_0$ which can be time dependent; there is no limitation on the amplitude of that rotation;
- friction can be added as well on those components that are not constrained;
- an interesting option, left as a future development, consists in leaving all the constraint equations always in place, and activate/deactivate them based on some trigger.

Future Development: Velocity/Acceleration

The time derivative of vector $\boldsymbol{x}^\delta$ yields

$$\boldsymbol{v}^\delta = \dot{\boldsymbol{x}} - \boldsymbol{v}_0 = \boldsymbol{R}_{1h}^T \left(\dot{\boldsymbol{x}}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 - \dot{\boldsymbol{x}}_1 - \boldsymbol{\omega}_1 \times (\boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1) \right) - \boldsymbol{v}_0. \tag{9.114}$$

Its linearization yields

$$\begin{aligned}
\delta\boldsymbol{v}^{\delta} &= \delta\dot{\boldsymbol{x}} \\
&= \boldsymbol{R}_{1h}^{T}\delta\dot{\boldsymbol{x}}_2 \\
&- \boldsymbol{R}_{1h}^{T}\delta\dot{\boldsymbol{x}}_1 \\
&- \boldsymbol{R}_{1h}^{T}\boldsymbol{f}_2\times\delta\boldsymbol{\omega}_2 \\
&+ \boldsymbol{R}_{1h}^{T}\left(\boldsymbol{x}_2+\boldsymbol{f}_2-\boldsymbol{x}_1\right)\times\delta\boldsymbol{\omega}_1 \\
&- \boldsymbol{R}_{1h}^{T}\boldsymbol{\omega}_1\times\delta\boldsymbol{x}_2 \\
&+ \boldsymbol{R}_{1h}^{T}\boldsymbol{\omega}_1\times\delta\boldsymbol{x}_1 \\
&- \boldsymbol{R}_{1h}^{T}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\boldsymbol{f}_2\times\boldsymbol{\theta}_{2\delta} \\
&+ \boldsymbol{R}_{1h}^{T}\left(\dot{\boldsymbol{x}}_2+\boldsymbol{\omega}_2\times\boldsymbol{f}_2-\dot{\boldsymbol{x}}_1-\boldsymbol{\omega}_1\times\left(\boldsymbol{x}_2+\boldsymbol{f}_2-\boldsymbol{x}_1\right)\right)\times\boldsymbol{\theta}_{1\delta} \\
&\overset{\mathrm{uu}}{=} \boldsymbol{R}_{1h}^{T}\delta\dot{\boldsymbol{x}}_2 \\
&- \boldsymbol{R}_{1h}^{T}\delta\dot{\boldsymbol{x}}_1 \\
&- \boldsymbol{R}_{1h}^{T}\boldsymbol{f}_2\times\delta\dot{\boldsymbol{g}}_2 \\
&+ \boldsymbol{R}_{1h}^{T}\left(\boldsymbol{x}_2+\boldsymbol{f}_2-\boldsymbol{x}_1\right)\times\delta\dot{\boldsymbol{g}}_1 \\
&- \boldsymbol{R}_{1h}^{T}\boldsymbol{\omega}_1\times\delta\boldsymbol{x}_2 \\
&+ \boldsymbol{R}_{1h}^{T}\boldsymbol{\omega}_1\times\delta\boldsymbol{x}_1 \\
&+ \boldsymbol{R}_{1h}^{T}\left(\left(\boldsymbol{f}_2\times\boldsymbol{\omega}_2\right)\times\ +\boldsymbol{\omega}_1\times\boldsymbol{f}_2\times\ \right)\boldsymbol{\theta}_{2\delta} \\
&+ \boldsymbol{R}_{1h}^{T}\left(\left(\dot{\boldsymbol{x}}_2+\boldsymbol{\omega}_2\times\boldsymbol{f}_2-\dot{\boldsymbol{x}}_1\right)\times\ -\boldsymbol{\omega}_1\times\left(\boldsymbol{x}_2+\boldsymbol{f}_2-\boldsymbol{x}_1\right)\times\ \right)\boldsymbol{\theta}_{1\delta}.
\end{aligned} \tag{9.115}$$

The time derivative of matrix $\boldsymbol{R}^{\delta}$ yields the constraint equation on the angular velocity

$$\begin{aligned}
\boldsymbol{\omega}^{\delta}\times\ &= \dot{\boldsymbol{R}}^{\delta}\boldsymbol{R}^{\delta T} \\
&= \boldsymbol{R}_{1hr}^{T}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\boldsymbol{R}_{1hr}-\boldsymbol{R}^{\delta}\boldsymbol{\omega}_0\times\boldsymbol{R}^{\delta T},
\end{aligned} \tag{9.116}$$

which results in

$$\boldsymbol{\omega}^{\delta} = \boldsymbol{R}_{1hr}^{T}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)-\boldsymbol{R}^{\delta}\boldsymbol{\omega}_0, \tag{9.117}$$

where $\boldsymbol{\omega}_0$ is the derivative of the imposed orientation,

$$\boldsymbol{\omega}_0 = \dot{\boldsymbol{\theta}}_0. \tag{9.118}$$

Note that if the angular velocity constraint is the derivative of an orientation constraint, $\boldsymbol{R}^{\delta} = \boldsymbol{I}$, so, by setting $\boldsymbol{\omega}^{\delta} = \boldsymbol{0}$, Equation (9.117) becomes

$$\boldsymbol{\omega}_0 = \boldsymbol{R}_{1hr}^{T}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right). \tag{9.119}$$

Otherwise, Equation (9.117) represents a non-holonomic constraint. In this case, $\boldsymbol{R}^{\delta}$ is again the identity matrix, although resulting from an unknown $\boldsymbol{R}_0$, which is the result of integrating the dynamics of the problem with the non-holonomic constraint represented by Equation (9.119). In fact,

$$\boldsymbol{R}_0 = \boldsymbol{R}_{1hr}^{T}\boldsymbol{R}_{2hr} \tag{9.120}$$

is the definition of the relative orientation resulting from imposing the constraint of Equation (9.119).

The perturbation of Equation (9.119) results in

$$\begin{aligned}\delta\left(\boldsymbol{R}_{1hr}^T\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\right) &= \boldsymbol{R}_{1hr}^T\left(\delta\boldsymbol{\omega}_2-\delta\boldsymbol{\omega}_1+\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\boldsymbol{\theta}_{1\delta}\right)\\ &\overset{\text{uu}}{=} \boldsymbol{R}_{1hr}^T\left(\delta\dot{\boldsymbol{g}}_2-\delta\dot{\boldsymbol{g}}_1-\boldsymbol{\omega}_2\times\delta\boldsymbol{g}_2+\boldsymbol{\omega}_2\times\delta\boldsymbol{g}_1\right)\end{aligned} \tag{9.121}$$

Further time differentiation of the constraint equation results in

$$\dot{\boldsymbol{\omega}}^\delta = \boldsymbol{R}_{1hr}^T\left(\dot{\boldsymbol{\omega}}_2-\dot{\boldsymbol{\omega}}_1-\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2\right)-\boldsymbol{\omega}^\delta\times\boldsymbol{R}^\delta\boldsymbol{\omega}_0-\boldsymbol{R}^\delta\dot{\boldsymbol{\omega}}_0. \tag{9.122}$$

Similarly, if the angular acceleration constraint is the derivative of an orientation or an angular velocity constraint, then $\boldsymbol{R}^\delta=\boldsymbol{I}$, $\boldsymbol{\omega}^\delta=\boldsymbol{0}$, respectively, and, by setting $\dot{\boldsymbol{\omega}}^\delta=\boldsymbol{0}$, Equation (9.122) becomes

$$\dot{\boldsymbol{\omega}}_0 = \boldsymbol{R}_{1hr}^T\left(\dot{\boldsymbol{\omega}}_2-\dot{\boldsymbol{\omega}}_1-\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2\right). \tag{9.123}$$

9.1.14 Total Pin Joint

This joint (`TotalPinJoint`) forces a node to assume an imposed absolute position and orientation about selected axes. Both imposed position and orientation are specified in the global reference frame, optionally re-oriented by a constant orientation. The joint consists in 6 equations, 3 for positions and 3 for orientation, that can be selectively turned on or off to enforce a constraint on that DOF.

Variants. This joint is a variant of the `TotalJoint`, described in Section 9.1.13.

Files. It is implemented in files
`mbdyn/struct/totalj.h`
`mbdyn/struct/totalj.cc`

Definitions. The position of the node with respect to the global reference frame is $\boldsymbol{x}_n$, while the absolute position of the constraint is $\boldsymbol{x}_c$. The matrix that expresses the orientation of the node with respect to the global reference frame is $\boldsymbol{R}_n$. An offset is allowed for the location of the points where the absolute position is constrained; the offset $\tilde{\boldsymbol{f}}_n$ is rigidly connected to the node, so in the global reference frame its expression is

$$\boldsymbol{f}_n = \boldsymbol{R}_n\tilde{\boldsymbol{f}}_n \tag{9.124}$$

The position constraint may be expressed in a reference frame, whose orientation in the global reference frame is $\boldsymbol{R}_{ch}$. Later on, it will occasionally be convenient to express the absolute position of the constraint, $\boldsymbol{x}_c$, in the reference frame $\boldsymbol{R}_{ch}$, namely

$$\tilde{\boldsymbol{x}}_c = \boldsymbol{R}_{ch}^T\boldsymbol{x}_c. \tag{9.125}$$

Similarly, the orientation constraint may be expressed in a reference frame that is rigidly connected to the node by matrix $\tilde{\boldsymbol{R}}_{nhr}$, whose orientation in the global reference frame is

$$\boldsymbol{R}_{nhr} = \boldsymbol{R}_n\tilde{\boldsymbol{R}}_{nhr}, \tag{9.126}$$

while the absolute orientation is described by the constant matrix $\boldsymbol{R}_{chr}$. The absolute position and orientation may refer to different local reference frames to increase the versatility of the element.

It is essential to notice that the differentiation of the constraint orientation matrices does not affect the relative orientation between the constraint and the node, namely matrix $\tilde{\boldsymbol{R}}_{nhr}$, which is constant. As a consequence,

$$\begin{aligned} \delta \boldsymbol{R}_{nhr} &= \delta \boldsymbol{R}_n \tilde{\boldsymbol{R}}_{nhr} \\ &= \boldsymbol{\theta}_{n\delta} \times \boldsymbol{R}_n \tilde{\boldsymbol{R}}_{nhr} \\ &= \boldsymbol{\theta}_{n\delta} \times \boldsymbol{R}_{nhr}, \end{aligned} \tag{9.127}$$

and

$$\dot{\boldsymbol{R}}_{nhr} = \boldsymbol{\omega}_n \times \boldsymbol{R}_{nhr}. \tag{9.128}$$

The same is true for the relative position orientation matrix, $\boldsymbol{R}_{nh}$.

Orientation Constraint. The matrix that expresses the relative orientation is

$$\boldsymbol{R}_{\text{rel}} = \boldsymbol{R}_{chr}^T \boldsymbol{R}_{nhr} \tag{9.129}$$

The desired relative orientation is expressed in analogy with the `TotalJoint`.

In general, the difference between the desired rotation $\boldsymbol{\theta}_0$ and the current absolute orientation of the node,

$$\boldsymbol{\theta} = \text{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{chr}^T \boldsymbol{R}_{nhr}\right)\right) \tag{9.130}$$

through the redefinition of Eq. (9.98) can be expressed as

$$\boldsymbol{\theta}^\delta = \text{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{chr}^T \boldsymbol{R}_{nhr} \boldsymbol{R}_0^T\right)\right) \tag{9.131}$$

and is equivalent to

$$\boldsymbol{\theta}^\delta = \boldsymbol{\theta} - \boldsymbol{\theta}_0 \tag{9.132}$$

when $\boldsymbol{\theta}^\delta = \boldsymbol{0}$.

Position Constraint. The desired absolute position is imposed by means of a vector $\boldsymbol{x}_0$ defined in the absolute position constraint reference frame $\boldsymbol{R}_{ch}$. The absolute position is represented by vector $\boldsymbol{x}$, defined as

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{R}_{ch}^T \left(\boldsymbol{x}_n + \boldsymbol{f}_n - \boldsymbol{x}_c\right) \\ &= \boldsymbol{R}_{ch}^T \left(\boldsymbol{x}_n + \boldsymbol{f}_n\right) - \tilde{\boldsymbol{x}}_c; \end{aligned} \tag{9.133}$$

the constraint equations for the relative position are:

$$\boldsymbol{x} = \boldsymbol{x}_0. \tag{9.134}$$

In analogy with the definition of $\boldsymbol{\theta}^\delta$, a vector $\boldsymbol{x}^\delta$ can be used to express the constraint on the position DOFs,

$$\boldsymbol{x}^\delta = \boldsymbol{x} - \boldsymbol{x}_0. \tag{9.135}$$

Constraint Equations. The resulting constraint equations are identical to Eqs. (9.106a, 9.106b), related to the `TotalJoint`, as illustrated at page 68; the same considerations apply.

Linearization. First, the perturbation of $\boldsymbol{x}^\delta$,

$$\delta \boldsymbol{x}^\delta = \boldsymbol{R}_{ch}^T \left(\delta \boldsymbol{x}_n - \boldsymbol{f}_n \times \boldsymbol{\theta}_{n\delta}\right) \tag{9.136}$$

is computed. Note that, since the constraint equations are defined by setting to zero any components of the vector $\boldsymbol{x}^\delta$, the related Jacobian matrix contribution is obtained by selecting the corresponding columns of matrix $\boldsymbol{R}_{ch}$.

Then, the perturbation of $\boldsymbol{\theta}^\delta$ results from

$$\begin{aligned}\boldsymbol{\theta}_\delta^\delta \times &= \delta \boldsymbol{R}^\delta \boldsymbol{R}^{\delta T} \\ &= \boldsymbol{R}_{chr}^T \boldsymbol{\theta}_{n\delta} \times \boldsymbol{R}_{chr},\end{aligned} \tag{9.137}$$

which implies

$$\boldsymbol{\theta}_\delta^\delta = \boldsymbol{R}_{chr}^T \boldsymbol{\theta}_{n\delta}. \tag{9.138}$$

Note that, since the constraint equations are defined by setting to zero any components of the vector $\boldsymbol{\theta}^\delta$, the related Jacobian matrix contribution is obtained by selecting the corresponding columns of matrix $\boldsymbol{R}_{chr}$.

After calling $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}}$ the Lagrange multipliers respectively related to the relative position and orientation constraints, and noticing that in the global reference frame they become

$$\boldsymbol{\lambda}_{\boldsymbol{x}} = \boldsymbol{R}_{ch} \tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}} \tag{9.139a}$$

$$\boldsymbol{\lambda}_{\boldsymbol{\theta}} = \boldsymbol{R}_{chr} \tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}}, \tag{9.139b}$$

the contribution of the constraint reactions to the equilibrium equations of the participating nodes is

$$\boldsymbol{F}_n = \boldsymbol{\lambda}_{\boldsymbol{x}} \tag{9.140a}$$

$$\boldsymbol{C}_n = \boldsymbol{f}_n \times \boldsymbol{\lambda}_{\boldsymbol{x}} + \boldsymbol{\lambda}_{\boldsymbol{\theta}}. \tag{9.140b}$$

Due to the simplifications related to the updated-updated approach, the contribution of the constraint equations and of the equilibrium to the Jacobian matrix is:

$$\left[\begin{array}{cc|cc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_{ch} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\lambda}_{\boldsymbol{x}} \times \boldsymbol{f}_n \times & \boldsymbol{f}_n \times \boldsymbol{R}_{ch} & \boldsymbol{R}_{chr} \\ \hline \boldsymbol{R}_{ch}^T & -\boldsymbol{R}_{ch}^T \boldsymbol{f}_n \times & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{chr}^T & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left\{\begin{array}{c} \delta \boldsymbol{x}_n \\ \boldsymbol{\theta}_{n\delta} \\ \hline \delta \tilde{\boldsymbol{\lambda}}_{\boldsymbol{x}} \\ \delta \tilde{\boldsymbol{\lambda}}_{\boldsymbol{\theta}} \end{array}\right\} = \left\{\begin{array}{c} -\boldsymbol{\lambda}_{\boldsymbol{x}} \\ -\boldsymbol{f}_n \times \boldsymbol{\lambda}_{\boldsymbol{x}} - \boldsymbol{\lambda}_{\boldsymbol{\theta}} \\ \hline -\boldsymbol{x}^\delta \\ -\boldsymbol{\theta}^\delta \end{array}\right\}. \tag{9.141}$$

The updated-updated form is obtained by simply replacing $\boldsymbol{\theta}_{n\delta}$ with $\delta \boldsymbol{g}_n$.

Stabilized Form

Equations.

$$\boldsymbol{R}_{ch} \boldsymbol{\mu}_{\boldsymbol{x}} = \boldsymbol{0} \tag{9.142a}$$

$$\boldsymbol{f}_n \times \boldsymbol{R}_{ch} \boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{R}_{chr} \boldsymbol{\mu}_{\boldsymbol{\theta}} = \boldsymbol{0} \tag{9.142b}$$

$$\boldsymbol{R}_{ch} \boldsymbol{\lambda}_{\boldsymbol{x}} = \boldsymbol{0} \tag{9.142c}$$

$$\begin{aligned} -\boldsymbol{f}_n \times \boldsymbol{\omega}_n \times \boldsymbol{R}_{ch} \boldsymbol{\mu}_{\boldsymbol{x}} - \boldsymbol{R}_{chr} \left(\boldsymbol{R}^\delta \boldsymbol{\omega}_0\right) \times \boldsymbol{\mu}_{\boldsymbol{\theta}} & \\ + \boldsymbol{f}_n \times \boldsymbol{R}_{ch} \boldsymbol{\lambda}_{\boldsymbol{x}} + \boldsymbol{R}_{chr} \boldsymbol{\lambda}_{\boldsymbol{\theta}} &= \boldsymbol{0} \end{aligned} \tag{9.142d}$$

$$\boldsymbol{R}_{ch}^T \left(\boldsymbol{x}_n + \boldsymbol{f}_n\right) - \boldsymbol{x}_0 = \boldsymbol{0} \tag{9.142e}$$

$$\mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}^\delta\right)\right) = \boldsymbol{0} \tag{9.142f}$$

$$\boldsymbol{R}_{ch}^T \left(\dot{\boldsymbol{x}}_n + \boldsymbol{\omega}_n \times \boldsymbol{f}_n\right) - \dot{\boldsymbol{x}}_0 = \boldsymbol{0} \tag{9.142g}$$

$$\boldsymbol{R}_{chr}^T \boldsymbol{\omega}_n - \boldsymbol{R}^\delta \boldsymbol{\omega}_0 = \boldsymbol{0} \tag{9.142h}$$

Linearization.

$$
\begin{aligned}
\boldsymbol{R}_{ch}\delta\boldsymbol{\mu}_{\boldsymbol{x}} &= \boldsymbol{0} && (9.143\text{a})\\
-\left(\boldsymbol{f}_n \times \boldsymbol{R}_{ch}\boldsymbol{\mu}_{\boldsymbol{x}}\right) \times \boldsymbol{\theta}_{n\delta} + \boldsymbol{f}_n \times \boldsymbol{R}_{ch}\delta\boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{R}_{chr}\delta\boldsymbol{\mu}_{\boldsymbol{\theta}} &= \boldsymbol{0} && (9.143\text{b})\\
\boldsymbol{R}_{ch}\delta\boldsymbol{\lambda}_{\boldsymbol{x}} &= \boldsymbol{0} && (9.143\text{c})\\
\left(\boldsymbol{f}_n \times \boldsymbol{\omega}_n \times \boldsymbol{R}_{ch}\boldsymbol{\mu}_{\boldsymbol{x}}\right) \times \boldsymbol{\theta}_{n\delta} + \boldsymbol{f}_n \times \left(\boldsymbol{R}_{ch}\boldsymbol{\mu}_{\boldsymbol{x}}\right) \times \delta\boldsymbol{\omega}_n &\\
- \boldsymbol{f}_n \times \boldsymbol{\omega}_n \times \boldsymbol{R}_{ch}\delta\boldsymbol{\mu}_{\boldsymbol{x}} &\\
-\left(\boldsymbol{R}_{chr}\boldsymbol{\mu}_{\boldsymbol{\theta}}\right) \times \left(\boldsymbol{R}_{nhr}\boldsymbol{R}_0^T\boldsymbol{\omega}_0\right) \times \boldsymbol{\theta}_{n\delta} - \boldsymbol{R}_{chr}\left(\boldsymbol{R}^{\delta}\boldsymbol{\omega}_0\right) \times \delta\boldsymbol{\mu}_{\boldsymbol{\theta}} &\\
-\left(\boldsymbol{f}_n \times \boldsymbol{R}_{ch}\boldsymbol{\lambda}_{\boldsymbol{x}}\right) \times \boldsymbol{\theta}_{n\delta} + \boldsymbol{f}_n \times \boldsymbol{R}_{ch}\delta\boldsymbol{\lambda}_{\boldsymbol{x}} + \boldsymbol{R}_{chr}\delta\boldsymbol{\lambda}_{\boldsymbol{\theta}} &= \boldsymbol{0} && (9.143\text{d})\\
\boldsymbol{R}_{ch}^T\delta\boldsymbol{x}_n - \boldsymbol{R}_{ch}^T\boldsymbol{f}_n \times \boldsymbol{\theta}_{n\delta} &= \boldsymbol{0} && (9.143\text{e})\\
\boldsymbol{R}_{chr}^T\boldsymbol{\theta}_{n\delta} &= \boldsymbol{0} && (9.143\text{f})\\
\boldsymbol{R}_{ch}^T\delta\dot{\boldsymbol{x}}_n - \boldsymbol{R}_{ch}^T\boldsymbol{f}_n \times \delta\boldsymbol{\omega}_n - \boldsymbol{R}_{ch}^T\boldsymbol{\omega}_n \times \boldsymbol{f}_n \times \boldsymbol{\theta}_{n\delta} &= \boldsymbol{0} && (9.143\text{g})\\
\boldsymbol{R}_{chr}^T\delta\boldsymbol{\omega}_n + \boldsymbol{R}_{chr}^T\left(\boldsymbol{R}_{nhr}\boldsymbol{R}_0^T\boldsymbol{\omega}_0\right) \times \boldsymbol{\theta}_{n\delta} &= \boldsymbol{0} && (9.143\text{h})
\end{aligned}
$$

Summary

The same considerations formulated for the `TotalJoint` apply.

9.1.15 Gimbal

This element implements an ideal gimbal joint. It is discussed in [3]. A gimbal is a joint that allows the rotation between two nodes about two orthogonal axes. The angular velocity about the remaining axis is preserved regardless of the relative angle between the two nodes. It is essentially equivalent to two Cardano joints[3] each of which accounts for half of the relative orientation.

Variants. This joint is only defined in the variant that constrains the relative rotation between two nodes. It may need to be combined with another joint that constrains the (relative) position of the related nodes, and with a deformable hinge to provide some stiffness or damping.

Files. It is implemented in files
`mbdyn/struct/gimbal.h`
`mbdyn/struct/gimbal.cc`

Definitions. The relative orientation between the nodes is made of three steps, described by

$$
\begin{aligned}
\boldsymbol{R}_{\text{rel}} &= \boldsymbol{R}_a^T\boldsymbol{R}_b && (= \boldsymbol{R}_{ab}) && (9.144\text{a})\\
&= \exp\left(\vartheta\boldsymbol{e}_2 \times\right)\exp\left(\varphi\boldsymbol{e}_1 \times\right)\exp\left(\vartheta\boldsymbol{e}_2 \times\right) && (= \boldsymbol{R}\left(\vartheta, \varphi\right)) && (9.144\text{b})
\end{aligned}
$$

where

$$
\begin{aligned}
\exp\left(\vartheta\boldsymbol{e}_2 \times\right) &= \boldsymbol{I} + \sin\vartheta\boldsymbol{e}_2 \times + \left(1 - \cos\vartheta\right)\boldsymbol{e}_2 \times \boldsymbol{e}_2 \times && (9.145\text{a})\\
\exp\left(\varphi\boldsymbol{e}_1 \times\right) &= \boldsymbol{I} + \sin\varphi\boldsymbol{e}_1 \times + \left(1 - \cos\varphi\right)\boldsymbol{e}_1 \times \boldsymbol{e}_1 \times && (9.145\text{b})
\end{aligned}
$$

and $\boldsymbol{e}_i$ are the unit vectors in directions $i = 1, 2, 3$.

[3]Or "universal joints"; this is also known as a double Hooke joint.

This represents a sequence of mutually orthogonal rotations, symmetrical with respect to the mid of the central rotation, and thus consisting in a sequence of two Cardano joints in the "W" arrangement, which is homokinetic.
The constraint equation is obtained by equating the relative orientation matrices of Equations (9.144a, 9.144b):

$$\boldsymbol{R}_{ab} = \boldsymbol{R}\left(\vartheta, \varphi\right), \tag{9.146}$$

or, which is equivalent,

$$\boldsymbol{R}_{ab}\boldsymbol{R}\left(\vartheta, \varphi\right)^T = \boldsymbol{I}. \tag{9.147}$$

As a consequence, the rotation vector

$$\boldsymbol{\theta} = \mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{ab}\boldsymbol{R}\left(\vartheta, \varphi\right)^T\right)\right), \tag{9.148}$$

corresponding to the orientation matrix at the left-hand side of Equation (9.147), must vanish:

$$\boldsymbol{\theta} = \boldsymbol{0}. \tag{9.149}$$

Lagrange multipliers approach:

$$\begin{aligned} 0 &= \delta\left(\boldsymbol{\lambda}^T\boldsymbol{\theta}\right) \\ &= \delta\boldsymbol{\lambda}^T\boldsymbol{\theta} + \delta\boldsymbol{\theta}^T\boldsymbol{\lambda}, \end{aligned} \tag{9.150}$$

where

$$\delta\boldsymbol{\theta} = \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{\theta}_\delta. \tag{9.151}$$

Note, that by definition $\boldsymbol{\theta}$ must be zero, so $\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right) \cong \boldsymbol{I}$. Without any simplification, Equation (9.150) results in

$$0 = \delta\left(\boldsymbol{\lambda}^T\boldsymbol{\theta}\right) \tag{9.152a}$$

$$ = \delta\boldsymbol{\lambda}^T\boldsymbol{\theta} + \boldsymbol{\theta}_\delta^T\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-T}\boldsymbol{\lambda}. \tag{9.152b}$$

To eliminate the $\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-T}$ term, the multipliers can be redefined as

$$\hat{\boldsymbol{\lambda}} = \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-T}\boldsymbol{\lambda}. \tag{9.153}$$

The perturbation $\boldsymbol{\theta}_\delta$ results from

$$\begin{aligned} \boldsymbol{\theta}_\delta \times &= \delta\exp\left(\boldsymbol{\theta}\times\right)\exp\left(\boldsymbol{\theta}\times\right)^T \\ &= \delta\left(\boldsymbol{R}_a^T\boldsymbol{R}_b\exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\exp\left(\varphi\boldsymbol{e}_1\times\right)^T\exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\right)\cdot \\ &\qquad \left(\boldsymbol{R}_a^T\boldsymbol{R}_b\exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\exp\left(\varphi\boldsymbol{e}_1\times\right)^T\exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\right)^T \\ &= \boldsymbol{R}_a^T\left(\boldsymbol{\theta}_{b\delta} - \boldsymbol{\theta}_{a\delta}\right)\times\boldsymbol{R}_a \\ &- \delta\vartheta\boldsymbol{R}_a^T\boldsymbol{R}_b\left(\left(\boldsymbol{I} + \exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\exp\left(\varphi\boldsymbol{e}_1\times\right)^T\right)\boldsymbol{e}_2\right)\times\boldsymbol{R}_b^T\boldsymbol{R}_a \\ &- \delta\varphi\boldsymbol{R}_a^T\boldsymbol{R}_b\left(\exp\left(\vartheta\boldsymbol{e}_2\times\right)^T\boldsymbol{e}_1\right)\times\boldsymbol{R}_b^T\boldsymbol{R}_a \end{aligned} \tag{9.154}$$

which yields

$$\boldsymbol{\theta}_\delta = \boldsymbol{R}_a^T \left(\boldsymbol{\theta}_{b\delta} - \boldsymbol{\theta}_{a\delta}\right) - \boldsymbol{R}_a^T \boldsymbol{R}_b \left(\delta\vartheta \boldsymbol{w}_\vartheta + \delta\varphi \boldsymbol{w}_\varphi\right) \tag{9.155}$$

with

$$\begin{aligned}
\boldsymbol{w}_\vartheta &= \left(\boldsymbol{I} + \exp\left(\vartheta \boldsymbol{e}_2 \times\right)^T \exp\left(\varphi \boldsymbol{e}_1 \times\right)^T\right) \boldsymbol{e}_2 \\
&= \sin\varphi \sin\vartheta \boldsymbol{e}_1 + \left(1 + \cos\varphi\right) \boldsymbol{e}_2 - \sin\varphi \cos\vartheta \boldsymbol{e}_3 && \text{(9.156a)} \\
\boldsymbol{w}_\varphi &= \exp\left(\vartheta \boldsymbol{e}_2 \times\right)^T \boldsymbol{e}_1 \\
&= \cos\vartheta \boldsymbol{e}_1 + \sin\vartheta \boldsymbol{e}_3. && \text{(9.156b)}
\end{aligned}$$

Linearization

$$\begin{aligned}
&\begin{bmatrix}
-\left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right) \times & \boldsymbol{0} & \boldsymbol{R}_a \\
\left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right) \times & \boldsymbol{0} & -\boldsymbol{R}_a \\
\boldsymbol{R}_a^T & -\boldsymbol{R}_a^T & \boldsymbol{0} \\
-\left(\left(\boldsymbol{R}_b \boldsymbol{w}_\vartheta\right) \times \left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right)\right)^T & \left(\left(\boldsymbol{R}_b \boldsymbol{w}_\vartheta\right) \times \left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right)\right)^T & \left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\vartheta\right)^T \\
-\left(\left(\boldsymbol{R}_b \boldsymbol{w}_\varphi\right) \times \left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right)\right)^T & \left(\left(\boldsymbol{R}_b \boldsymbol{w}_\varphi\right) \times \left(\boldsymbol{R}_a \hat{\boldsymbol{\lambda}}\right)\right)^T & \left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\varphi\right)^T
\end{bmatrix}
\left\{\begin{array}{c} \boldsymbol{\theta}_{a\Delta} \\ \boldsymbol{\theta}_{b\Delta} \\ \Delta\hat{\boldsymbol{\lambda}} \end{array}\right\} \\
&\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\vartheta & \boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\varphi \\
\left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_{\vartheta/\vartheta}\right)^T \hat{\boldsymbol{\lambda}} & \left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_{\vartheta/\varphi}\right)^T \hat{\boldsymbol{\lambda}} \\
\left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_{\varphi/\vartheta}\right)^T \hat{\boldsymbol{\lambda}} & \left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_{\varphi/\varphi}\right)^T \hat{\boldsymbol{\lambda}}
\end{bmatrix}
\left\{\begin{array}{c} \Delta\vartheta \\ \Delta\varphi \end{array}\right\} \\
&= \left\{\begin{array}{c}
-\boldsymbol{R}_a \hat{\boldsymbol{\lambda}} \\
\boldsymbol{R}_a \hat{\boldsymbol{\lambda}} \\
\boldsymbol{\theta} \\
-\left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\vartheta\right)^T \hat{\boldsymbol{\lambda}} \\
-\left(\boldsymbol{R}_a^T \boldsymbol{R}_b \boldsymbol{w}_\varphi\right)^T \hat{\boldsymbol{\lambda}}
\end{array}\right\}
\end{aligned} \tag{9.157}$$

where

$$\begin{aligned}
\boldsymbol{w}_{\vartheta/\vartheta} &= \sin\varphi \cos\vartheta \boldsymbol{e}_1 + \sin\varphi \sin\vartheta \boldsymbol{e}_3 && \text{(9.158a)} \\
\boldsymbol{w}_{\vartheta/\varphi} &= \cos\varphi \sin\vartheta \boldsymbol{e}_1 - \sin\varphi \boldsymbol{e}_2 - \cos\varphi \cos\vartheta \boldsymbol{e}_3 && \text{(9.158b)} \\
\boldsymbol{w}_{\varphi/\vartheta} &= -\sin\vartheta \boldsymbol{e}_1 + \cos\vartheta \boldsymbol{e}_3 && \text{(9.158c)} \\
\boldsymbol{w}_{\varphi/\varphi} &= \boldsymbol{0}. && \text{(9.158d)}
\end{aligned}$$

9.1.16 Screw Joint

Author: Mauro Manetti and Marco Morandini

The screw joint is a combination of:

- an in line joint, that constrains one point to move along a line attached to another point:

$$\begin{aligned}
\boldsymbol{e}_{1hx}^T \left(\boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1\right) = 0 \\
\boldsymbol{e}_{1hy}^T \left(\boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1\right) = 0
\end{aligned}$$

where $\boldsymbol{e}_{1hi}$ is the i-th column of matrix $\boldsymbol{R}_1\tilde{\boldsymbol{R}}_{1h}$

$$\boldsymbol{e}_{jhi} = \left(\boldsymbol{R}_j\tilde{\boldsymbol{R}}_{jh}\right)[i] \tag{9.159}$$

$\boldsymbol{x}_j$ is the position of node j and $\boldsymbol{f}_j$ is the offset of the joint with respect node j, both in the absolute coordinate system; the offset is constant with respect to the reference frame of the joint, so $\boldsymbol{f}_j = \boldsymbol{R}_j\tilde{\boldsymbol{f}}_j$.

- a revolute rotation joint, that constrains the relative orientation of two nodes to be a rotation about an axis that is fixed with respect to the two bodies:

$$\boldsymbol{e}_{2hx}^T\boldsymbol{e}_{1hz} = 0$$
$$\boldsymbol{e}_{2hy}^T\boldsymbol{e}_{1hz} = 0$$

- a linear relationship between the relative rotation about the common axis and the relative position along the common axis:

$$\frac{p}{2\pi}(\theta - \theta_0) - \boldsymbol{e}_{1hz}^T(\boldsymbol{d} - \boldsymbol{d}_0) = 0 \tag{9.160}$$

where the nodes distance vectors $\boldsymbol{d}$ and $\boldsymbol{d}_0$ can be written as:

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1 \\ \boldsymbol{d}_0 &= \boldsymbol{x}_{2_0} + \boldsymbol{f}_{2_0} - \boldsymbol{x}_{1_0} - \boldsymbol{f}_{1_0} \end{aligned}$$

while p represents the distance between the nodes along axis $\boldsymbol{e}_{1hz}$ corresponding to one revolution about the same axis.

The rotation θ is formulated as

$$\begin{aligned} \theta =&\tilde{\boldsymbol{R}}_{1h3}^T\operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_1^T\boldsymbol{R}_2\right)\right) \\ =&\tilde{\boldsymbol{R}}_{1h3}^T\boldsymbol{R}_1^T\operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_2\boldsymbol{R}_1^T\right)\right) \\ =&\boldsymbol{e}_{1hz}^T\operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_2\boldsymbol{R}_1^T\right)\right) \end{aligned} \tag{9.161}$$

where $\tilde{\boldsymbol{R}}_{1h3}$ is the third axis of the constant relative orientation of node 1, while $\boldsymbol{e}_{1hz}$ is the screw joint axis. To overcome the limitation $|\theta| < \pi$, an appropriate unwrap angle function has been implemented in the code, so allowing to compute the right residual of Equation (9.160).

The virtual perturbation of Equation (9.161) can be expressed as

$$\delta\theta = \tilde{\boldsymbol{R}}_{1h3}^T\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) \tag{9.162}$$

so the virtual perturbation of Equation (9.160)

$$\begin{aligned} &\frac{p}{2\pi}\boldsymbol{e}_{1hz}^T\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) - \boldsymbol{e}_{1hz}^T\boldsymbol{d} \times \boldsymbol{\theta}_{1\delta} \\ &-\boldsymbol{e}_{1hz}^T(\delta\boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta\boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta}) = 0 \end{aligned} \tag{9.163}$$

yields the contribution of the constraint to the forces and moments acting on the constrained nodes:

$$\boldsymbol{F}_1 = \boldsymbol{e}_{1hz}\lambda \tag{9.164}$$

$$\boldsymbol{C}_1 = \left(-\frac{p}{2\pi}\boldsymbol{R}_1\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T}\tilde{\boldsymbol{R}}_{1h3} + \boldsymbol{f}_1 \times \boldsymbol{e}_{1hz}\right)\lambda \tag{9.165}$$

$$\boldsymbol{F}_2 = -\boldsymbol{e}_{1hz}\lambda \tag{9.166}$$

$$\boldsymbol{C}_2 = \left(\frac{p}{2\pi}\boldsymbol{R}_1\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T}\tilde{\boldsymbol{R}}_{1h3} - \boldsymbol{f}_2 \times \boldsymbol{e}_{1hz}\right)\lambda \tag{9.167}$$

where λ is the Lagrange multiplier that here assumes the meaning of reaction force along the screw axis.

The problem linearization requires the already computed virtual perturbation of the constraint equation (see Equation (9.163)) and the constraint forces and couples virtual perturbation:

$$\begin{aligned}
\delta \boldsymbol{F}_1 =& -\lambda \boldsymbol{e}_{1hz} \times \boldsymbol{\theta}_{1\delta} + \boldsymbol{e}_{1hz} \delta\lambda && (9.168)\\
\delta \boldsymbol{C}_1 =& \lambda \left[\frac{p}{2\pi} \left(\boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3} \right) \times \boldsymbol{\theta}_{1\delta} - \right.\\
& \frac{p}{2\pi} \boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{L}\left(-\boldsymbol{\theta}, \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3}\right) \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1} \boldsymbol{R}_1^T (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) + \\
& \left. \boldsymbol{e}_{1hz} \times \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} - \boldsymbol{f}_1 \times \boldsymbol{e}_{1hz} \times \boldsymbol{\theta}_{1\delta} \right] + \\
& \frac{\boldsymbol{C}_1}{\lambda} \delta\lambda && (9.169)\\
\delta \boldsymbol{F}_2 =& \lambda \boldsymbol{e}_{1hz} \times \boldsymbol{\theta}_{1\delta} - \boldsymbol{e}_{1hz} \delta\lambda && (9.170)\\
\delta \boldsymbol{C}_2 =& \lambda \left[-\frac{p}{2\pi} \left(\boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3} \right) \times \boldsymbol{\theta}_{1\delta} + \right.\\
& \frac{p}{2\pi} \boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{L}\left(-\boldsymbol{\theta}, \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3}\right) \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1} \boldsymbol{R}_1^T (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) - \\
& \left. \boldsymbol{e}_{1hz} \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} + \boldsymbol{f}_2 \times \boldsymbol{e}_{1hz} \times \boldsymbol{\theta}_{1\delta} \right] + \\
& \frac{\boldsymbol{C}_2}{\lambda} \delta\lambda && (9.171)
\end{aligned}$$

where the operator $\boldsymbol{L}()$ has been introduced according to the following definition

$$\delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^T \boldsymbol{a} = -\boldsymbol{L}(-\boldsymbol{\theta}, \boldsymbol{a}) \delta\boldsymbol{\theta} \tag{9.172}$$

which can be manipulated to obtain the needed relation

$$\begin{aligned}
\delta(\boldsymbol{\Gamma}(\boldsymbol{\theta})^T \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{a}) =& \delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^T \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{a} + \\
& \boldsymbol{\Gamma}(\boldsymbol{\theta})^T \delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{a} + \\
& \boldsymbol{\Gamma}(\boldsymbol{\theta})^T \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \delta \boldsymbol{a} = \delta \boldsymbol{a} && (9.173)\\
\delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{a} =& \delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{L}\left(-\boldsymbol{\theta}, \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{a}\right) \delta\boldsymbol{\theta} && (9.174)
\end{aligned}$$

Now a contribution useful to obtain previous results will be reported

$$\begin{aligned}
\delta \left(\boldsymbol{R}_1 \Gamma(\boldsymbol{\theta})^{-1} \tilde{\boldsymbol{R}}_{1h3} \right) =& \delta \boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3} + \boldsymbol{R}_1 \delta \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3} \\
=& -\left(\boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3} \right)_{\times} \boldsymbol{\theta}_{1\delta} + \boldsymbol{R}_1 \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \boldsymbol{L}(-\boldsymbol{\theta}, \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-T} \tilde{\boldsymbol{R}}_{1h3}) \delta\boldsymbol{\theta}
\end{aligned}$$

This joint formulation can be improved to take into account the presence of friction. Figures (9.1) and (9.2) show sketches useful to identify the forces acting on the screw thread. In the figures the case of a screw which is raising a load $\boldsymbol{F}$ is presented, the equilibrium equation can be written for the vertical and the horizontal directions to obtain

$$\boldsymbol{F} + \mu \boldsymbol{F}_n \sin\alpha = \boldsymbol{F}_n \cos\gamma_n cos\alpha \tag{9.175}$$

$$\boldsymbol{P} = \mu \boldsymbol{F}_n \cos\alpha + \boldsymbol{F}_n \cos\gamma_n \sin\alpha \tag{9.176}$$

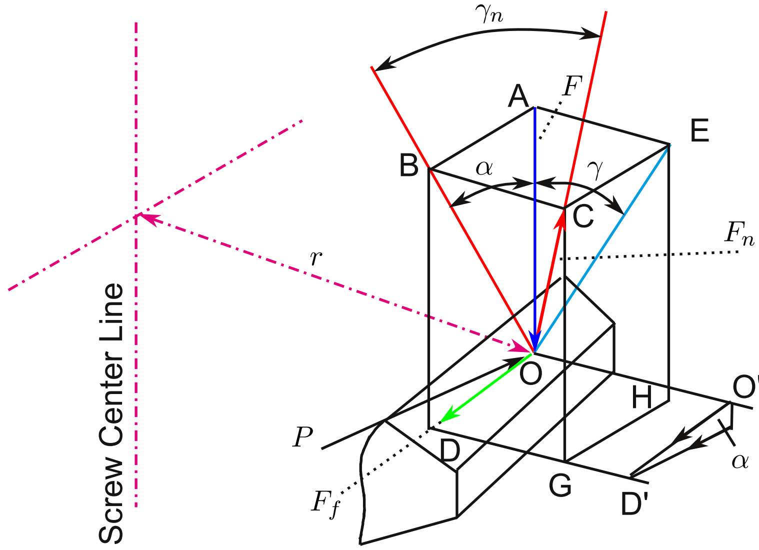

Figure 9.1: 3D screw thread sketch.

where $\boldsymbol{F}$ is the load, or the force, along the screw axis, $\boldsymbol{P}$ the force acting to rotate the screw, i.e. the torque moment over the mean thread radius r, $\boldsymbol{F}_n$ is the thread normal reaction, α the lead angle, and γ_n the angle between the vectors $\boldsymbol{OC}$ and $\boldsymbol{OB}$ represented in Figure (9.1). Inside the code all the screw friction formulation is developed in function of the angle γ_n for convenience, anyway the usual screw informations relate the thread angle 2γ, so the joint requires as input the half thread angle γ. Looking at the Figure 9.1 it is possible to find the relation between these two angles

$$BC = AE = OA \tan\gamma = OB \cos\alpha \tan\gamma \tag{9.177}$$

so

$$\tan\gamma_n = \frac{BC}{OB} = \cos\alpha \tan\gamma \quad \Rightarrow \quad \gamma_n = tan^{-1}(\cos\alpha \tan\gamma). \tag{9.178}$$

From Equations (9.175) and (9.176) is possible to retrive the relation between the screw raising torque $\boldsymbol{C}_r$ and the axial force $\boldsymbol{F}$

$$\boldsymbol{C}_r = r\boldsymbol{F}\left(\frac{\cos\gamma_n \sin\alpha + \mu\cos\alpha}{\cos\gamma_n \cos\alpha - \mu \sin\alpha}\right). \tag{9.179}$$

It is easy to find the previous relation in presence of a lowering torque

$$\boldsymbol{C}_l = r\boldsymbol{F}\left(\frac{\cos\gamma_n \sin\alpha - \mu\cos\alpha}{\cos\gamma_n \cos\alpha + \mu \sin\alpha}\right). \tag{9.180}$$

from which it is possible to understand the torque dependency from the versus of the friction force $\mu\boldsymbol{F}_n$. A general discriminant to apply the right formula can be linked to the versus of the relative velocity v between the internal and external thread and the sign of the constraint Lagrange multiplier λ which

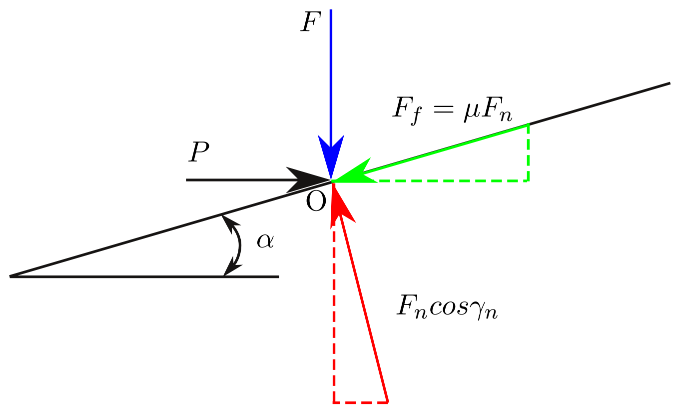

Figure 9.2: 2D screw thread sketch

represents the versus of the axial screw force

$$\boldsymbol{C} = r\boldsymbol{F}\left(\frac{\cos\gamma_n \sin\alpha + sign(v\lambda)\,\mu\cos\alpha}{\cos\gamma_n\cos\alpha - sign(v\lambda)\,\mu\sin\alpha}\right). \tag{9.181}$$

Inside the code $sign(v\lambda)$ will be embedded in the friction coefficient computation $\mu = \mu(v, \boldsymbol{F})$, so in the following for brevity it will be removed from equations introducing the notation $\boldsymbol{C} = \boldsymbol{C}(\mu)$.

The torque acting on the screw for the friction presence can now be computed as

$$\boldsymbol{C}_{frc}(\mu) = \boldsymbol{C}(\mu) - \boldsymbol{C}(0) = r\boldsymbol{F}\left[\frac{\mu\sec\gamma_n(1+\tan^2\alpha)}{1-\mu\sec\gamma_n\tan\alpha}\right]. \tag{9.182}$$

Equation (9.182) can be used to add the torque friction contribution to the couples acting on the constrained nodes reported in Equations (9.165) and (9.167)

$$\boldsymbol{C}_{1_{frc}} = \boldsymbol{C}_1 - \boldsymbol{C}_{frc}(\mu) = \boldsymbol{C}_1 - r\boldsymbol{F}_1\left[\frac{\mu\sec\gamma_n(1+\tan^2\alpha)}{1-\mu\sec\gamma_n\tan\alpha}\right] \tag{9.183}$$

$$\boldsymbol{C}_{2_{frc}} = \boldsymbol{C}_2 - \boldsymbol{C}_{frc}(\mu) = \boldsymbol{C}_2 - r\boldsymbol{F}_2\left[\frac{\mu\sec\gamma_n(1+\tan^2\alpha)}{1-\mu\sec\gamma_n\tan\alpha}\right] \tag{9.184}$$

At this point the only further required informations regard the relative velocity v and the virtual variation of the friction torque contribution. Starting from the already used formula

$$\delta\boldsymbol{\theta} = \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) \tag{9.185}$$

it is possible to write the following relation

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T(\boldsymbol{\Gamma}(\boldsymbol{\theta}_2)\dot{\boldsymbol{\theta}}_2 - \boldsymbol{\Gamma}(\boldsymbol{\theta}_1)\dot{\boldsymbol{\theta}}_1). \tag{9.186}$$

The angular velocity vector is defined as

$$\boldsymbol{\omega}_\theta = \boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \boldsymbol{\Gamma}(\boldsymbol{\theta})\boldsymbol{\Gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{R}_1^T(\boldsymbol{\Gamma}(\boldsymbol{\theta}_2)\dot{\boldsymbol{\theta}}_2 - \boldsymbol{\Gamma}(\boldsymbol{\theta}_1)\dot{\boldsymbol{\theta}}_1) = \boldsymbol{R}_1^T(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \tag{9.187}$$

from which the angular velocity modulus with sign can be obtained

$$\omega_\theta = \tilde{\boldsymbol{R}}_{1h3}^T \boldsymbol{\omega}_\theta = \tilde{\boldsymbol{R}}_{1h3}^T \boldsymbol{R}_1^T (\boldsymbol{\omega_2} - \boldsymbol{\omega}_1). \tag{9.188}$$

Now the relative velocity on the screw thread, along the friction force direction, can be written as

$$v = \omega_\theta r \cos\alpha = \tilde{\boldsymbol{R}}_{1h3}^T \boldsymbol{R}_1^T (\boldsymbol{\omega_2} - \boldsymbol{\omega}_1) r \cos\alpha. \tag{9.189}$$

The virtual perturbation of the friction couple is required to write the linearized problem

$$\begin{aligned}
\delta \boldsymbol{C}_{frc} =& \delta \boldsymbol{F} r \left[\frac{\mu \sec\gamma_n (1 + \tan^2\alpha)}{1 - \mu \sec\gamma_n \tan\alpha} \right] + \\
& \boldsymbol{F} r \left[\frac{\sec\gamma_n (1 + \tan^2\alpha)}{1 - \mu \sec\gamma_n \tan\alpha} + \frac{\mu \sec\gamma_n (1 + \tan^2\alpha)}{(1 - \mu \sec\gamma_n \tan\alpha)^2} \sec\gamma_n \tan\alpha \right] \delta\mu \\
=& r \left[\frac{\mu \sec\gamma_n (1 + \tan^2\alpha)}{1 - \mu \sec\gamma_n \tan\alpha} \right] \delta \boldsymbol{F} + \boldsymbol{F} r \left[\frac{\mu \sec\gamma_n (1 + \tan^2\alpha)}{(1 - \mu \sec\gamma_n \tan\alpha)^2} \right] \delta\mu
\end{aligned} \tag{9.190}$$

where the friction coefficient virtual variation is automatically computed by the code, once provided the F and μ virtual perturbations

$$\delta\mu = \delta\mu(\delta \boldsymbol{F}, \delta v). \tag{9.191}$$

Anyway the screw axial forces virtual variations have already been reported in Equations 9.168 and 9.170, so to complete the problem description is just required the definition of the relative velocity virtual perturbation

$$\begin{aligned}
\delta v =& \tilde{\boldsymbol{R}}_{1h3}^T \delta \boldsymbol{R}_1^T (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) r \cos\alpha + \tilde{\boldsymbol{R}}_{1h3}^T \boldsymbol{R}_1^T (\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1) r \cos\alpha \\
=& r \cos\alpha \tilde{\boldsymbol{R}}_{1h3}^T \boldsymbol{R}_1^T (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1)_\times \boldsymbol{\theta}_{1\delta} + r \cos\alpha \tilde{\boldsymbol{R}}_{1h3}^T (\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1).
\end{aligned} \tag{9.192}$$

Physics. In other words, node 1 is the screw and node 2 is the bolt; neglecting the offset, the force along the screw axis is related to the couple about the same axis by the relationship

$$C = -\frac{p}{2\pi} F \tag{9.193}$$

which results from a power balance

$$C\omega_\theta + F v_{lin} = 0 \tag{9.194}$$

in terms of relative linear (v_{lin}) and angular (ω_θ) velocity, with the kinematic relationship

$$v_{lin} = \frac{p}{2\pi} \omega_\theta \tag{9.195}$$

9.1.17 Strapdown Sensor

This constraint imposes the acceleration and the angular velocity of a node in a relative frame attached to the node itself.

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_n + \dot{\boldsymbol{\omega}}_n \times \boldsymbol{b} + \boldsymbol{\omega}_n \times \boldsymbol{\omega}_n \times \boldsymbol{b} \tag{9.196a}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n \tag{9.196b}$$

where $\boldsymbol{b} = \boldsymbol{R}_n \tilde{\boldsymbol{b}}$ is the offset of the accelerometer from the node.

The acceleration and the angular velocity in the reference frame of the node, as output from the strapdown sensor, respectively are $\overline{\ddot{\boldsymbol{x}}}$ and $\overline{\boldsymbol{\omega}}$. The constraint equations are

$$\ddot{\boldsymbol{x}} = \boldsymbol{R}_n \overline{\ddot{\boldsymbol{x}}} \tag{9.197a}$$

$$\boldsymbol{\omega} = \boldsymbol{R}_n \overline{\boldsymbol{\omega}} \tag{9.197b}$$

The acceleration of the point where the strapdown sensor is placed is

$$\begin{aligned}
\ddot{\boldsymbol{x}} = {} & \left(\frac{1}{m}\boldsymbol{I} + \boldsymbol{f} \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \boldsymbol{f} \times\right) \dot{\boldsymbol{\beta}} - \boldsymbol{f} \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \dot{\gamma} - \boldsymbol{f} \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \left(\dot{\boldsymbol{x}}_n \times \boldsymbol{\beta} - \boldsymbol{\omega}_n \times \boldsymbol{J}_{\mathrm{CM}} \boldsymbol{\omega}_n\right) \\
& + \boldsymbol{b} \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \left(\dot{\gamma} - \boldsymbol{f} \times \dot{\boldsymbol{\beta}} + \dot{\boldsymbol{x}}_n \times \boldsymbol{\beta} - \boldsymbol{\omega}_n \times \boldsymbol{J}_{\mathrm{CM}} \boldsymbol{\omega}_n\right) + \boldsymbol{\omega}_n \times \boldsymbol{\omega}_n \times \boldsymbol{b} \\
= {} & \left(\frac{1}{m}\boldsymbol{I} + (\boldsymbol{f} - \boldsymbol{b}) \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \boldsymbol{f} \times\right) \dot{\boldsymbol{\beta}} - (\boldsymbol{f} - \boldsymbol{b}) \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \dot{\gamma} \\
& - (\boldsymbol{f} - \boldsymbol{b}) \times {}^T\boldsymbol{J}_{\mathrm{CM}}^{-1} \left(\dot{\boldsymbol{x}}_n \times \boldsymbol{\beta} - \boldsymbol{\omega}_n \times \boldsymbol{J}_{\mathrm{CM}} \boldsymbol{\omega}_n\right) + \boldsymbol{\omega}_n \times \boldsymbol{\omega}_n \times \boldsymbol{b}
\end{aligned} \tag{9.198}$$

Note: this constraint is extremely simpler when the node is in the center of mass ($\boldsymbol{f} = \boldsymbol{0}$), and the strapdown sensor is in the node ($\boldsymbol{b} = \boldsymbol{0}$).

9.1.18 Rigid body displacement joint

Author: Reinhard Resch

A rigid body displacement joint is a load distributing rigid body coupling constraint similar to NASTRAN's `RBE3` element.

Files. It is implemented in files
`mbdyn/struct/rbdispjad.h`
`mbdyn/struct/rbdispjad.cc`

Kinematics The purpose of a rigid body displacement joint is, to apply a position and orientation constraint to a master node in such a way, that the master node is following the weighted average rigid body motion of a set of displacement only slave nodes. In contrast to NASTRAN's `RBE3` element, large deformations and large rotations are supported. For each slave node the difference $\boldsymbol{u}_j$, between the actual position of the slave node $\boldsymbol{x}_j$ and the corresponding nominal offset from the master node, can be written as:

$$\boldsymbol{u}_j \quad = \quad \boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j \tag{9.199}$$

$\boldsymbol{x}_m$ Position of the master node

$\boldsymbol{R}_m$ Orientation matrix of the master node

$\boldsymbol{o}_j$ Initial distance between the slave node and the master node

$\boldsymbol{x}_j$ Position of the slave node

In order to derive the expressions for the kinematic constraints, the virtual perturbation of $\boldsymbol{u}_j$ is needed.

$$\delta\boldsymbol{u}_j \quad = \quad \delta\boldsymbol{x}_m + \delta\boldsymbol{R}_m \, \boldsymbol{o}_j - \delta\boldsymbol{x}_j \tag{9.200}$$

$$\delta\boldsymbol{R}_m \quad \approx \quad \delta\boldsymbol{\varphi}_m \times \boldsymbol{R}_m \tag{9.201}$$

$$\delta\boldsymbol{u}_j \quad = \quad \delta\boldsymbol{x}_m + \delta\boldsymbol{\varphi}_m \times (\boldsymbol{R}_m \, \boldsymbol{o}_j) - \underbrace{\delta\boldsymbol{x}_j}_{=\boldsymbol{0}} \tag{9.202}$$

$\delta \boldsymbol{u}_j$ Virtual perturbation of $\boldsymbol{u}_j$

$\delta \boldsymbol{x}_m$ Virtual perturbation of the position of the master node

$\delta \boldsymbol{R}_m$ Virtual perturbation of the orientation matrix of the master node

$\delta \boldsymbol{\varphi}_m$ Virtual perturbation of the rotation vector of the master node

$\delta \boldsymbol{x}_j$ Virtual perturbation of $\boldsymbol{x}_j$

Since the kinematic constraints should be applied only to the master node, $\delta \boldsymbol{x}_j = \boldsymbol{0}$ is assumed.

Fictitious strain energy In the following equation, the virtual perturbation of the fictitious strain energy $U^\star$ is defined.

$$\delta U^\star = \sum_{j=1}^{N} w_j \, \delta \boldsymbol{u}_j^T \, \boldsymbol{u}_j \tag{9.203}$$

$\delta U^\star$ Virtual perturbation of the fictitious strain energy

w_j Weighting factor for slave node j

N Number of slave nodes

It is not required, that every weighting factor w_j is greater than zero. However the following condition must hold:

$$\sum_{j=1}^{N} w_j > 0 \tag{9.204}$$

Physical interpretation One could give the following physical interpretation of $U^\star$: Between each slave node and the master node, a fictitious linear isotropic spring with stiffness w_j is attached at an offset $\boldsymbol{o}_j$ with respect to the master node. Then $U^\star$ is the overall strain energy of all fictitious springs. In order to fulfill the constraint equations, the fictitious strain energy $U^\star$ must be minimized. For that reason, we may write:

$$\delta U^\star \equiv \boldsymbol{0} \tag{9.205}$$

Within this context it is not obvious why $w_j < 0$ should be allowed. However, this condition is required only for second order hexahedral elements which require negative weights at the corner nodes. Otherwise it would not be possible to achieve a smooth stress distribution for such elements.

Constraint equations Now equation 9.199 and equation 9.202 are substituted into equation 9.203:

$$\delta U^\star = \sum_{j=1}^{N} w_j \left(\delta \boldsymbol{x}_m^T + \delta \boldsymbol{\varphi}_m^T \times \boldsymbol{R}_m \, \boldsymbol{o}_j\right) (\boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j) \tag{9.206}$$

$$= \sum_{j=1}^{N} w_j \left[\delta \boldsymbol{x}_m^T \, (\boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j) + \delta \boldsymbol{\varphi}_m^T \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j)\right] \tag{9.207}$$

Since equation 9.207 and equation 9.205 must be valid for any $\delta \boldsymbol{x}_m$ and any $\delta \boldsymbol{\varphi}_m$ we can directly derive the constraint equations $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Phi}_r$:

$$\boldsymbol{\Phi}_t = \sum_{j=1}^{N} w_j \, (\boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j) = \boldsymbol{0} \tag{9.208}$$

$$\boldsymbol{\Phi}_r = \sum_{j=1}^{N} w_j \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\boldsymbol{x}_m + \boldsymbol{R}_m \, \boldsymbol{o}_j - \boldsymbol{x}_j) = \boldsymbol{0} \tag{9.209}$$

$\boldsymbol{\Phi}_t$ Constraint equations for the position of the master node

$\boldsymbol{\Phi}_r$ Constraint equations for the orientation of the master node

Because $(\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\boldsymbol{R}_m \, \boldsymbol{o}_j) = \boldsymbol{0}$, the expression for $\boldsymbol{\Phi}_r$ can be simplified.

$$\boldsymbol{\Phi}_r = \sum_{j=1}^{N} w_j \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\boldsymbol{x}_m - \boldsymbol{x}_j) = \boldsymbol{0} \tag{9.210}$$

Virtual perturbation of constraint equations In order to include Lagrange multipliers in the equations of motion, it is required to derive the virtual perturbation of the constraint equations $\delta\boldsymbol{\Phi}_t$ and $\delta\boldsymbol{\Phi}_r$.

$$\delta\boldsymbol{\Phi}_t = \sum_{j=1}^{N} w_j \, [\delta\boldsymbol{x}_m + \delta\boldsymbol{\varphi}_m \times (\boldsymbol{R}_m \, \boldsymbol{o}_j) - \delta\boldsymbol{x}_j] \tag{9.211}$$

$$= \sum_{j=1}^{N} w_j \, [\delta\boldsymbol{x}_m - (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times \delta\boldsymbol{\varphi}_m - \delta\boldsymbol{x}_j] \tag{9.212}$$

$$\delta\boldsymbol{\Phi}_t^T = \sum_{j=1}^{N} w_j \, \left[\delta\boldsymbol{x}_m^T + \delta\boldsymbol{\varphi}_m^T \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times - \delta\boldsymbol{x}_j^T\right] \tag{9.213}$$

Now equation 9.213 can be rearranged and written in matrix form:

$$\delta\boldsymbol{\Phi}_t^T = \begin{pmatrix} \delta\boldsymbol{x}_m^T & \delta\boldsymbol{\varphi}_m^T & \delta\boldsymbol{x}_1^T & \dots & \delta\boldsymbol{x}_N^T \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{N} w_j \, \boldsymbol{I} \\ \sum_{j=1}^{N} w_j \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times \\ -w_1 \, \boldsymbol{I} \\ \vdots \\ -w_N \, \boldsymbol{I} \end{pmatrix} \tag{9.214}$$

The same procedure is also applied to $\boldsymbol{\Phi}_r$.

$$\delta\boldsymbol{\Phi}_r = \sum_{j=1}^{N} w_j \, [-(\boldsymbol{x}_m - \boldsymbol{x}_j) \times \delta\boldsymbol{R}_m \, \boldsymbol{o}_j + (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\delta\boldsymbol{x}_m - \delta\boldsymbol{x}_j)] \tag{9.215}$$

$$= \sum_{j=1}^{N} w_j \, [-(\boldsymbol{x}_m - \boldsymbol{x}_j) \times \delta\boldsymbol{\varphi}_m \times (\boldsymbol{R}_m \, \boldsymbol{o}_j) + (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\delta\boldsymbol{x}_m - \delta\boldsymbol{x}_j)] \tag{9.216}$$

$$= \sum_{j=1}^{N} w_j \, [(\boldsymbol{x}_m - \boldsymbol{x}_j) \times (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times \delta\boldsymbol{\varphi}_m + (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\delta\boldsymbol{x}_m - \delta\boldsymbol{x}_j)] \tag{9.217}$$

$$\delta\boldsymbol{\Phi}_r^T = \sum_{j=1}^{N} w_j \, \left[\delta\boldsymbol{\varphi}_m^T \, (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times (\boldsymbol{x}_m - \boldsymbol{x}_j) \times - \left(\delta\boldsymbol{x}_m^T - \delta\boldsymbol{x}_j^T\right) (\boldsymbol{R}_m \, \boldsymbol{o}_j) \times\right] \tag{9.218}$$

Also equation 9.218 can be rearranged and written in matrix form:

$$\delta\boldsymbol{\Phi}_r^T = \begin{pmatrix}\delta\boldsymbol{x}_m^T & \delta\boldsymbol{\varphi}_m^T & \delta\boldsymbol{x}_1^T & \dots & \delta\boldsymbol{x}_N^T\end{pmatrix} \begin{pmatrix} -\sum_{j=1}^N w_j\ (\boldsymbol{R}_m\,\boldsymbol{o}_j)\times \\ \sum_{j=1}^N w_j\ (\boldsymbol{R}_m\,\boldsymbol{o}_j)\times(\boldsymbol{x}_m-\boldsymbol{x}_j)\times \\ w_1\ (\boldsymbol{R}_m\,\boldsymbol{o}_1)\times \\ \vdots \\ w_N\ (\boldsymbol{R}_m\,\boldsymbol{o}_N)\times \end{pmatrix} \tag{9.219}$$

Adding contributions due to Lagrange multipliers In order to enforce the constraint equations $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Phi}_r$, Lagrange multipliers $\boldsymbol{\lambda}_t$ and $\boldsymbol{\lambda}_r$ are required.

$$\delta W_\lambda = \delta\boldsymbol{\Phi}_t^T\,\boldsymbol{\lambda}_t + \delta\boldsymbol{\Phi}_r^T\,\boldsymbol{\lambda}_r \tag{9.220}$$

$$= \begin{pmatrix}\delta\boldsymbol{x}_m \\ \delta\boldsymbol{\varphi}_m \\ \delta\boldsymbol{x}_1 \\ \vdots \\ \delta\boldsymbol{x}_N\end{pmatrix}^T \begin{pmatrix} \sum_{j=1}^N w_j\ [\boldsymbol{\lambda}_t-\ (\boldsymbol{R}_m\,\boldsymbol{o}_j)\times\boldsymbol{\lambda}_r] \\ \sum_{j=1}^N w_j\,[(\boldsymbol{R}_m\,\boldsymbol{o}_j)\times\boldsymbol{\lambda}_t+(\boldsymbol{R}_m\,\boldsymbol{o}_j)\times(\boldsymbol{x}_m-\boldsymbol{x}_j)\times\boldsymbol{\lambda}_r] \\ w_1\,[(\boldsymbol{R}_m\,\boldsymbol{o}_1)\times\boldsymbol{\lambda}_r-\boldsymbol{\lambda}_t] \\ \vdots \\ w_N\,[(\boldsymbol{R}_m\,\boldsymbol{o}_N)\times\boldsymbol{\lambda}_r-\boldsymbol{\lambda}_t] \end{pmatrix} \tag{9.221}$$

$$= \begin{pmatrix}\delta\boldsymbol{x}_m \\ \delta\boldsymbol{\varphi}_m \\ \delta\boldsymbol{x}_1 \\ \vdots \\ \delta\boldsymbol{x}_N\end{pmatrix}^T \begin{pmatrix}\boldsymbol{F}_m \\ \boldsymbol{M}_m \\ \boldsymbol{F}_1 \\ \vdots \\ \boldsymbol{F}_N\end{pmatrix} \tag{9.222}$$

$\boldsymbol{\lambda}_t$ Lagrange multipliers for position constraints $\boldsymbol{\Phi}_t$

$\boldsymbol{\lambda}_r$ Lagrange multipliers for orientation constraints $\boldsymbol{\Phi}_r$

$\boldsymbol{F}_m$ Reaction force at the master node

$\boldsymbol{M}_m$ Reaction couple at the master node

$\boldsymbol{F}_j$ Reaction force at slave node j

Contributions of the element Finally the contributions of the rigid body displacement joint can be summarized as:

$$\begin{pmatrix}\texttt{dCoef}^{-1}\,\boldsymbol{\Phi}_t \\ \texttt{dCoef}^{-1}\,\boldsymbol{\Phi}_r \\ \boldsymbol{F}_m \\ \boldsymbol{M}_m \\ \boldsymbol{F}_1 \\ \vdots \\ \boldsymbol{F}_N\end{pmatrix} = \begin{pmatrix} \texttt{dCoef}^{-1}\sum_{j=1}^N w_j\ (\boldsymbol{x}_m+\boldsymbol{R}_m\,\boldsymbol{o}_j-\boldsymbol{x}_j) \\ \texttt{dCoef}^{-1}\sum_{j=1}^N w_j\ (\boldsymbol{R}_m\,\boldsymbol{o}_j)\times(\boldsymbol{x}_m-\boldsymbol{x}_j) \\ \sum_{j=1}^N w_j\ [\boldsymbol{\lambda}_t-\ (\boldsymbol{R}_m\,\boldsymbol{o}_j)\times\boldsymbol{\lambda}_r] \\ \sum_{j=1}^N w_j\,(\boldsymbol{R}_m\,\boldsymbol{o}_j)\times[\boldsymbol{\lambda}_t+(\boldsymbol{x}_m-\boldsymbol{x}_j)\times\boldsymbol{\lambda}_r] \\ w_1\,[(\boldsymbol{R}_m\,\boldsymbol{o}_1)\times\boldsymbol{\lambda}_r-\boldsymbol{\lambda}_t] \\ \vdots \\ w_N\,[(\boldsymbol{R}_m\,\boldsymbol{o}_N)\times\boldsymbol{\lambda}_r-\boldsymbol{\lambda}_t] \end{pmatrix} \tag{9.223}$$

In equation 9.223 the constraint equations $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Phi}_r$ are divided by `dCoef` in order to improve the condition number of the Jacobian matrix.

9.2 Deformable Constraints

Definitions

$$
\begin{aligned}
\boldsymbol{f}_1 &= \boldsymbol{R}_1 \tilde{\boldsymbol{f}}_1 \\
\boldsymbol{f}_2 &= \boldsymbol{R}_2 \tilde{\boldsymbol{f}}_2 \\
\boldsymbol{R}_{1h} &= \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1h} \\
\boldsymbol{R}_{2h} &= \boldsymbol{R}_2 \tilde{\boldsymbol{R}}_{2h}
\end{aligned}
$$

where $\boldsymbol{R}_i$ is the current orientation of node i, $\tilde{\boldsymbol{R}}_{ih}$ is a constant re-orientation of the joint with respect to node i, so $\boldsymbol{R}_{ih}$ is the orientation of the joint with respect to the global frame; $\tilde{\boldsymbol{f}}_i$ is the offset of the joint with respect to node i in the node reference frame, so $\boldsymbol{f}_i$ is the offset of the joint with respect to node i in the global frame, and $\boldsymbol{x}_i + \boldsymbol{f}_i$ is the position of the joint with respect to the global frame.

Equilibrium equations are obtained by means of the Virtual Work Principle (VWP). While the unknown orientation parameters depend on the parametrization in use, and typically are Gibbs-Rodrigues parameters according to the updated-updated approach, equilibrium is written in terms of the equations conjugated to perturbations of relative rotation, namely:

$$\delta\mathcal{L} = \sum \delta\boldsymbol{x}_i^T \boldsymbol{F}_i + \boldsymbol{\theta}_{i\delta} \boldsymbol{M}_i = 0 \tag{9.224}$$

9.2.1 Rod With Offsets

Distance vector between pin points

$$\boldsymbol{l} = \boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1 \tag{9.225}$$

scalar distance

$$l = \sqrt{\boldsymbol{l}^T \boldsymbol{l}} \tag{9.226}$$

strain

$$\varepsilon = \frac{l}{l_0} - 1 \tag{9.227}$$

strain rate

$$\dot{\varepsilon} = \frac{\dot{l}}{l_0} \tag{9.228}$$

where

$$\dot{l} = \frac{1}{l} \boldsymbol{l}^T \dot{\boldsymbol{l}} \tag{9.229a}$$

$$\dot{\boldsymbol{l}} = \dot{\boldsymbol{x}}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 - \dot{\boldsymbol{x}}_1 - \boldsymbol{\omega}_1 \times \boldsymbol{f}_1 \tag{9.229b}$$

scalar force

$$f = f\left(\varepsilon, \dot{\varepsilon}\right) \tag{9.230}$$

nodal forces and moments result from the VWP according to

$$\delta\mathcal{L} = \delta l^T f \tag{9.231a}$$

$$= \delta\boldsymbol{l}^T \frac{\boldsymbol{l}}{l} f \tag{9.231b}$$

$$= \left(\delta\boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta\boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta}\right)^T \frac{\boldsymbol{l}}{l} f \tag{9.231c}$$

force vector

$$\boldsymbol{F} = \frac{\boldsymbol{l}}{l} f \tag{9.232}$$

nodal forces and moments

$$\boldsymbol{F}_1 = -\boldsymbol{F} \tag{9.233a}$$
$$\boldsymbol{M}_1 = -\boldsymbol{f}_1 \times \boldsymbol{F} \tag{9.233b}$$
$$\boldsymbol{F}_2 = \boldsymbol{F} \tag{9.233c}$$
$$\boldsymbol{M}_2 = \boldsymbol{f}_2 \times \boldsymbol{F} \tag{9.233d}$$

equation linearization

$$\delta \boldsymbol{F}_1 = -\delta \boldsymbol{F} \tag{9.234a}$$
$$\delta \boldsymbol{M}_1 = -\boldsymbol{f}_1 \times \delta \boldsymbol{F} - \boldsymbol{F} \times \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \tag{9.234b}$$
$$\delta \boldsymbol{F}_2 = \delta \boldsymbol{F} \tag{9.234c}$$
$$\delta \boldsymbol{M}_2 = \boldsymbol{f}_2 \times \delta \boldsymbol{F} + \boldsymbol{F} \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} \tag{9.234d}$$

force linearization

$$\delta \boldsymbol{F} = \frac{f}{l} \left(\boldsymbol{I} - \frac{\boldsymbol{l}\boldsymbol{l}^T}{l^2} \right) \delta \boldsymbol{l} + \frac{\boldsymbol{l}}{l} \delta f \tag{9.235}$$

scalar force linearization

$$\delta f = \frac{\partial f}{\partial \varepsilon} \delta \varepsilon + \frac{\partial f}{\partial \dot{\varepsilon}} \delta \dot{\varepsilon} \tag{9.236}$$

strain linearization

$$\delta \varepsilon = \frac{1}{l_0} \delta l \tag{9.237}$$

where

$$\delta l = \frac{1}{l} \boldsymbol{l}^T \delta \boldsymbol{l} \tag{9.238a}$$
$$\delta \boldsymbol{l} = \delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \tag{9.238b}$$

strain rate linearization

$$\delta \dot{\varepsilon} = \frac{1}{l_0} \delta \dot{l} \tag{9.239}$$

where

$$\delta \dot{l} = \frac{\boldsymbol{l}^T}{l} \delta \dot{\boldsymbol{l}} + \frac{\dot{\boldsymbol{l}}^T}{l} \left(\boldsymbol{I} - \frac{\boldsymbol{l}\boldsymbol{l}^T}{l^2} \right) \delta \boldsymbol{l} \tag{9.240a}$$
$$\begin{aligned} \delta \dot{\boldsymbol{l}} = {} & \delta \dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \times \delta \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} \\ & - \delta \dot{\boldsymbol{x}}_1 + \boldsymbol{f}_1 \times \delta \boldsymbol{\omega}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \end{aligned} \tag{9.240b}$$

to summarize:

$$\delta \boldsymbol{F} = \boldsymbol{K}_{\boldsymbol{l}} \delta \boldsymbol{l} + \boldsymbol{K}_{\dot{\boldsymbol{l}}} \delta \dot{\boldsymbol{l}} \tag{9.241}$$

with

$$\boldsymbol{K}_{\boldsymbol{l}} = \frac{f}{l}\boldsymbol{I} + \left(\frac{1}{l^2 l_0}\frac{\partial f}{\partial \varepsilon} - \frac{\dot{\varepsilon}}{l^3}\frac{\partial f}{\partial \dot{\varepsilon}} - \frac{f}{l^3}\right)\boldsymbol{l}\boldsymbol{l}^T + \frac{1}{l^2 l_0}\frac{\partial f}{\partial \dot{\varepsilon}}\boldsymbol{l}\dot{\boldsymbol{l}}^T \tag{9.242a}$$

$$\boldsymbol{K}_{\dot{\boldsymbol{l}}} = \frac{1}{l^2 l_0}\frac{\partial f}{\partial \dot{\varepsilon}}\boldsymbol{l}\boldsymbol{l}^T \tag{9.242b}$$

according to the simplifications of the updated-updated approach,

$$\boldsymbol{\theta}_{i\delta} \stackrel{\text{uu}}{=} \delta \boldsymbol{g}_i \tag{9.243a}$$

$$\delta\boldsymbol{\omega}_i \stackrel{\text{uu}}{=} \delta\dot{\boldsymbol{g}}_i - \boldsymbol{\omega} \times \delta\boldsymbol{g}_i \tag{9.243b}$$

recalling that

$$\delta z = c\delta\dot{z} \tag{9.244}$$

the linearization of the force becomes

$$\begin{aligned}
\delta\boldsymbol{F} = {} & \left(c\boldsymbol{K}_{\boldsymbol{l}} + \boldsymbol{K}_{\dot{\boldsymbol{l}}}\right)\delta\dot{\boldsymbol{x}}_2 \\
& + \left(-\left(c\boldsymbol{K}_{\boldsymbol{l}} + \boldsymbol{K}_{\dot{\boldsymbol{l}}}\right)\boldsymbol{f}_2 \times {} + c\boldsymbol{K}_{\dot{\boldsymbol{l}}}\left(\boldsymbol{f}_2 \times \boldsymbol{\omega}_2\right)\times\right)\delta\dot{\boldsymbol{g}}_2 \\
& - \left(c\boldsymbol{K}_{\boldsymbol{l}} + \boldsymbol{K}_{\dot{\boldsymbol{l}}}\right)\delta\dot{\boldsymbol{x}}_1 \\
& - \left(-\left(c\boldsymbol{K}_{\boldsymbol{l}} + \boldsymbol{K}_{\dot{\boldsymbol{l}}}\right)\boldsymbol{f}_1 \times {} + c\boldsymbol{K}_{\dot{\boldsymbol{l}}}\left(\boldsymbol{f}_1 \times \boldsymbol{\omega}_1\right)\times\right)\delta\dot{\boldsymbol{g}}_1
\end{aligned} \tag{9.245}$$

9.2.2 Deformable Hinge

The deformable hinge applies to two nodes an internal moment that may depend on their relative orientation and angular velocity by means of a 3D constitutive law. It is discussed in [4]. Two variants of this joint are presented:

- the one historically implemented in MBDyn, called "attached", considers the constitutive law and the resulting moment attached to node 1;
- the other one, called "invariant", defines an intermediate orientation that is halfway between that of the two nodes, and considers the constitutive law and the resulting moment attached to that intermediate orientation; as a consequence, the resulting internal moment does not depend on the node sequence even when anisotropic constitutive laws are considered.

The relative rotation vector is computed in analogy with the `drive hinge` joint (`DriveHingeJoint`).

$$\boldsymbol{\theta} = \operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}\right)\right) \tag{9.246}$$

where $\boldsymbol{R}_{1h}$, $\boldsymbol{R}_{2h}$ are the matrices that express the orientation of each side of the hinge in the global reference frame, defined as

$$\boldsymbol{R}_{1h} = \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1h} \tag{9.247a}$$

$$\boldsymbol{R}_{2h} = \boldsymbol{R}_2 \tilde{\boldsymbol{R}}_{2h} \tag{9.247b}$$

and $\tilde{\boldsymbol{R}}_{1h}$, $\tilde{\boldsymbol{R}}_{2h}$ are the matrices that express the orientation of each side of the hinge with respect to the corresponding node. The perturbation of $\boldsymbol{R}_{1h}$, $\boldsymbol{R}_{2h}$ yields

$$\delta\boldsymbol{R}_{1h} = \boldsymbol{\theta}_{1\delta} \times \boldsymbol{R}_1 \tilde{\boldsymbol{R}}_{1h} = \boldsymbol{\theta}_{1\delta} \times \boldsymbol{R}_{1h} \tag{9.248a}$$

$$\delta\boldsymbol{R}_{2h} = \boldsymbol{\theta}_{2\delta} \times \boldsymbol{R}_2 \tilde{\boldsymbol{R}}_{2h} = \boldsymbol{\theta}_{2\delta} \times \boldsymbol{R}_{2h} \tag{9.248b}$$

since matrices $\tilde{\boldsymbol{R}}_{1h}$, $\tilde{\boldsymbol{R}}_{2h}$ are constant.

Since the relative orientation $\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}$ is a rotation about an axis parallel to $\boldsymbol{\theta}$, $\boldsymbol{\theta}$ itself does not change when it is referred to any intermediate orientation between $\boldsymbol{I}$ and $\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}$, namely

$$\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h} \boldsymbol{\theta} = \boldsymbol{\theta}. \tag{9.249}$$

The same applies for any relative orientation tensor built from a rotation vector $\alpha\boldsymbol{\theta}$ parallel to $\boldsymbol{\theta}$, whatever value the scalar α assumes.

The relative angular velocity is defined as the derivative of the relative orientation matrix

$$\begin{aligned}
\boldsymbol{\omega} \times &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}\right) \left(\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}\right)^T \\
&= \left(\dot{\boldsymbol{R}}_{1h}^T \boldsymbol{R}_{2h} + \boldsymbol{R}_{1h}^T \dot{\boldsymbol{R}}_{2h}\right) \boldsymbol{R}_{2h}^T \boldsymbol{R}_{1h} \\
&= \boldsymbol{R}_{1h}^T \boldsymbol{\omega}_1 \times^T \boldsymbol{R}_{1h} + \boldsymbol{R}_{1h}^T \boldsymbol{\omega}_2 \times \boldsymbol{R}_{1h} \\
&= \boldsymbol{R}_{1h}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \times \boldsymbol{R}_{1h}
\end{aligned} \tag{9.250}$$

so

$$\boldsymbol{\omega} = \boldsymbol{R}_{1h}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \tag{9.251}$$

Attached Deformable Hinge

The constitutive law is defined attached to the reference frame of node 1; the value of the relative rotation vector, $\boldsymbol{\theta}$, does not vary, according to what stated earlier.

The perturbation of $\boldsymbol{\theta}$ yields

$$\boldsymbol{\theta}_\delta = \boldsymbol{R}_{1h}^T \left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right) \tag{9.252}$$

which, according to the simplifications of the updated-updated approach, becomes

$$\boldsymbol{\theta}_\delta \overset{\mathrm{uu}}{=} \boldsymbol{R}_{1h}^T \left(\delta\boldsymbol{g}_2 - \delta\boldsymbol{g}_1\right) \tag{9.253}$$

The perturbation of the relative angular velocity of Equation (9.251), according to the simplifications of the updated-updated approach, yields

$$\begin{aligned}
\delta\boldsymbol{\omega} &= \boldsymbol{R}_{1h}^T \left(\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \times \boldsymbol{\theta}_{1\delta} + \delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1\right) \\
&\overset{\mathrm{uu}}{=} \boldsymbol{R}_{1h}^T \left(\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \times \delta\boldsymbol{g}_1 + \delta\dot{\boldsymbol{g}}_2 - \boldsymbol{\omega}_2 \times \delta\boldsymbol{g}_2 - \delta\dot{\boldsymbol{g}}_1 + \boldsymbol{\omega}_1 \times \delta\boldsymbol{g}_1\right) \\
&= \boldsymbol{R}_{1h}^T \left(\delta\dot{\boldsymbol{g}}_2 - \delta\dot{\boldsymbol{g}}_1 - \boldsymbol{\omega}_2 \times \left(\delta\boldsymbol{g}_2 - \delta\boldsymbol{g}_1\right)\right).
\end{aligned} \tag{9.254}$$

The internal moment $\tilde{\boldsymbol{M}}$ is computed as function of the relative rotation and velocity

$$\tilde{\boldsymbol{M}} = \tilde{\boldsymbol{M}} \left(\boldsymbol{\theta}, \boldsymbol{\omega}\right). \tag{9.255}$$

The nodal moments result from the VWP as

$$\begin{aligned}
\delta\mathcal{L} &= \boldsymbol{\theta}_\delta^T \tilde{\boldsymbol{M}} \\
&= \left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right)^T \boldsymbol{R}_{1h} \tilde{\boldsymbol{M}},
\end{aligned} \tag{9.256}$$

which states that the internal moment is applied to each node after pre-multiplication by $\boldsymbol{R}_{1h}$

$$\boldsymbol{M}_i = \left(-1\right)^i \boldsymbol{R}_{1h} \tilde{\boldsymbol{M}} \left(\boldsymbol{\theta}, \boldsymbol{\omega}\right). \tag{9.257}$$

Its linearization yields

$$\delta \boldsymbol{M}_i = (-1)^i \boldsymbol{R}_{1h} \left(\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}} \delta\boldsymbol{\theta} + \tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}} \delta\boldsymbol{\omega} \right) - \boldsymbol{M}_i \times \boldsymbol{\theta}_{1\delta} \tag{9.258}$$

The complete linearized problem is

$$\begin{aligned}
&\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\omega}} & -\boldsymbol{M}_{/\boldsymbol{\omega}} \\ -\boldsymbol{M}_{/\boldsymbol{\omega}} & \boldsymbol{M}_{/\boldsymbol{\omega}} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{\omega}_1 \\ \delta\boldsymbol{\omega}_2 \end{array} \right\} \\
&+ \left[\begin{array}{cc} -\boldsymbol{M}_{/\boldsymbol{\omega}} \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times & \boldsymbol{0} \\ \boldsymbol{M}_{/\boldsymbol{\omega}} \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times & \boldsymbol{0} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array} \right\} \\
&+ \left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\theta}} & -\boldsymbol{M}_{/\boldsymbol{\theta}} \\ -\boldsymbol{M}_{/\boldsymbol{\theta}} & \boldsymbol{M}_{/\boldsymbol{\theta}} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array} \right\} \\
&+ \left[\begin{array}{cc} \boldsymbol{M} \times & \boldsymbol{0} \\ -\boldsymbol{M} \times & \boldsymbol{0} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{M} \\ -\boldsymbol{M} \end{array} \right\},
\end{aligned} \tag{9.259}$$

where

$$\boldsymbol{M}_{/\boldsymbol{\theta}} = \boldsymbol{R}_{1h} \tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}} \boldsymbol{\Gamma} \left(\boldsymbol{\theta} \right)^{-1} \boldsymbol{R}_{1h}^T \tag{9.260a}$$

$$\boldsymbol{M}_{/\boldsymbol{\omega}} = \boldsymbol{R}_{1h} \tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}} \boldsymbol{R}_{1h}^T. \tag{9.260b}$$

According to the simplifications of the updated-updated approach, the linearization becomes

$$\begin{aligned}
&\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\omega}} & -\boldsymbol{M}_{/\boldsymbol{\omega}} \\ -\boldsymbol{M}_{/\boldsymbol{\omega}} & \boldsymbol{M}_{/\boldsymbol{\omega}} \end{array} \right] \left\{ \begin{array}{c} \delta\dot{\boldsymbol{g}}_1 \\ \delta\dot{\boldsymbol{g}}_2 \end{array} \right\} \\
&+ \left[\begin{array}{cc} -\boldsymbol{M}_{/\boldsymbol{\omega}} \boldsymbol{\omega}_2 \times & \boldsymbol{M}_{/\boldsymbol{\omega}} \boldsymbol{\omega}_2 \times \\ \boldsymbol{M}_{/\boldsymbol{\omega}} \boldsymbol{\omega}_2 \times & -\boldsymbol{M}_{/\boldsymbol{\omega}} \boldsymbol{\omega}_2 \times \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array} \right\} \\
&+ \left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\theta}} & -\boldsymbol{M}_{/\boldsymbol{\theta}} \\ -\boldsymbol{M}_{/\boldsymbol{\theta}} & \boldsymbol{M}_{/\boldsymbol{\theta}} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array} \right\} \\
&+ \left[\begin{array}{cc} \boldsymbol{M} \times & \boldsymbol{0} \\ -\boldsymbol{M} \times & \boldsymbol{0} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array} \right\} \overset{\text{uu}}{=} \left\{ \begin{array}{c} \boldsymbol{M} \\ -\boldsymbol{M} \end{array} \right\}.
\end{aligned} \tag{9.261}$$

Invariant Deformable Hinge

The rotation $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}/2$ is used to define an intermediate reference frame for the joint, whose orientation with respect to node 1 is

$$\begin{aligned}
\tilde{\boldsymbol{R}} &= \exp \left(\frac{1}{2} \exp^{-1} \left(\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h} \right) \right) \\
&= \exp \left(\tilde{\boldsymbol{\theta}} \times \right)
\end{aligned} \tag{9.262}$$

The orientation of the intermediate reference frame with respect to node 2 is defined by $-\tilde{\boldsymbol{\theta}}$, i.e. $\tilde{\boldsymbol{R}}^T$. The orientation of the intermediate frame with respect to the global frame is thus

$$\hat{\boldsymbol{R}} = \boldsymbol{R}_{1h} \tilde{\boldsymbol{R}} \tag{9.263a}$$

$$= \boldsymbol{R}_{2h} \tilde{\boldsymbol{R}}^T \tag{9.263b}$$

The perturbation of $\tilde{\boldsymbol{R}}$ is

$$\delta \tilde{\boldsymbol{R}} = \tilde{\boldsymbol{\theta}}_\delta \times \tilde{\boldsymbol{R}} \tag{9.264}$$

where

$$\delta\tilde{\boldsymbol{\theta}} = \frac{1}{2}\delta\boldsymbol{\theta} \tag{9.265}$$

and thus

$$\tilde{\boldsymbol{\theta}}_\delta = \boldsymbol{\Gamma}\left(\tilde{\boldsymbol{\theta}}\right)\frac{1}{2}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{\theta}_\delta \tag{9.266}$$

Another interesting result is obtained by considering that the perturbation of the relative orientation matrix must be equal to the perturbation of the square of the half-relative orientation, namely

$$\begin{aligned}\boldsymbol{\theta}_\delta\times &= \delta\left(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{R}}\right)\left(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{R}}\right)^T \\ &= \tilde{\boldsymbol{\theta}}_\delta\times + \tilde{\boldsymbol{R}}\tilde{\boldsymbol{\theta}}_\delta\times\tilde{\boldsymbol{R}}^T\end{aligned} \tag{9.267}$$

and thus

$$\boldsymbol{\theta}_\delta = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)\tilde{\boldsymbol{\theta}}_\delta \tag{9.268}$$

The matrix $\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)$ has very interesting properties.

Property 1.

$$\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)\tilde{\boldsymbol{R}}^T = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) \tag{9.269}$$

Property 2.

$$\tilde{\boldsymbol{R}}^T\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) \tag{9.270}$$

Property 3.

$$\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)\tilde{\boldsymbol{R}} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) \tag{9.271}$$

Property 4.

$$\tilde{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) \tag{9.272}$$

Property 5. As a consequence of Equations (9.269, 9.272),

$$\tilde{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)\tilde{\boldsymbol{R}}^T = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) \tag{9.273}$$

which basically means that the transformation $\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)$ is coaxial with the rotation described by matrix $\tilde{\boldsymbol{R}}$. A similar property exists for matrix $\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)$ (e.g. by transposing Equation (9.273)), while the following

Property 6.

$$\tilde{\boldsymbol{R}}^T\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)\tilde{\boldsymbol{R}} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) \tag{9.274}$$

is true as a consequence of Equations (9.270, 9.271).

Property 7. Another interesting property is

$$\boldsymbol{I} - \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1}. \tag{9.275}$$

This can be easily proved by considering that, according to Equation (9.272),

$$\begin{aligned}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} &= \left(\tilde{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)\right)^{-1} \\ &= \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1} \tilde{\boldsymbol{R}}^T,\end{aligned} \tag{9.276}$$

and, as a consequence,

$$\boldsymbol{I} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) \tag{9.277}$$

which reduces again to the identity matrix.

Property 8.

$$\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^T = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) \tag{9.278}$$

Property 9.

$$\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-T} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1} \tag{9.279}$$

Property 10. As a consequence of Equations (9.269–9.272),

$$\hat{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right) = \boldsymbol{R}_{1h}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right) \tag{9.280}$$

and

$$\hat{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1} = \boldsymbol{R}_{1h}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} \tag{9.281}$$

and all other combinations.

It is convenient to define

$$\hat{\boldsymbol{I}} = \hat{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} \hat{\boldsymbol{R}}^T. \tag{9.282}$$

According to Equations (9.268) and (9.252), the perturbation of the intermediate relative rotation vector yields

$$\begin{aligned}\tilde{\boldsymbol{\theta}}_\delta &= \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} \boldsymbol{\theta}_\delta \\ &= \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1} \boldsymbol{R}_{1h}^T \left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right) \\ &= \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1} \hat{\boldsymbol{R}}^T \left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right).\end{aligned} \tag{9.283}$$

The perturbation of $\hat{\boldsymbol{R}}$, according to Equations (9.263a) and (9.263b), respectively yields

$$\begin{aligned}
\delta\hat{\boldsymbol{R}}\hat{\boldsymbol{R}}^T &= \boldsymbol{\theta}_{1\delta}\times + \boldsymbol{R}_{1h}\tilde{\boldsymbol{\theta}}_\delta \times \boldsymbol{R}_{1h}^T \\
&= \boldsymbol{\theta}_{2\delta}\times - \hat{\boldsymbol{R}}\tilde{\boldsymbol{\theta}}_\delta \times \hat{\boldsymbol{R}}^T,
\end{aligned} \tag{9.284}$$

so

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_\delta &= \boldsymbol{\theta}_{1\delta} + \boldsymbol{R}_{1h}\tilde{\boldsymbol{\theta}}_\delta && \text{Eq. (9.263a)} \\
&= \boldsymbol{\theta}_{1\delta} + \boldsymbol{R}_{1h}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{R}_{1h}^T\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right) \\
&= \boldsymbol{\theta}_{1\delta} + \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right) \\
&= \boldsymbol{\theta}_{2\delta} - \hat{\boldsymbol{R}}\tilde{\boldsymbol{\theta}}_\delta && \text{Eq. (9.263b)} \\
&= \boldsymbol{\theta}_{2\delta} - \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{R}_{1h}^T\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right) \\
&= \boldsymbol{\theta}_{2\delta} - \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}^T\right)^{-1}\hat{\boldsymbol{R}}^T\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right) \\
&= \hat{\boldsymbol{I}}\boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}^T\boldsymbol{\theta}_{1\delta}
\end{aligned} \tag{9.285}$$

The perturbation of matrix $\hat{\boldsymbol{I}}$ yields

$$\begin{aligned}
\delta\hat{\boldsymbol{I}} &= \delta\hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T \\
&\quad + \hat{\boldsymbol{R}}\,\delta\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T \\
&\quad + \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\delta\hat{\boldsymbol{R}}^T
\end{aligned} \tag{9.286a}$$

$$\begin{aligned}
&= \hat{\boldsymbol{\theta}}_\delta\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\tilde{\boldsymbol{\theta}}_\delta\times\tilde{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T + \hat{\boldsymbol{I}}\hat{\boldsymbol{\theta}}_\delta\times{}^T \\
&= \hat{\boldsymbol{\theta}}_\delta\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\underbrace{\boldsymbol{R}_{1h}^T\boldsymbol{R}_{1h}}_{\boldsymbol{I}}\tilde{\boldsymbol{\theta}}_\delta\times\underbrace{\boldsymbol{R}_{1h}^T\boldsymbol{R}_{1h}}_{\boldsymbol{I}}\tilde{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T - \hat{\boldsymbol{I}}\hat{\boldsymbol{\theta}}_\delta\times \\
&= \hat{\boldsymbol{\theta}}_\delta\times\hat{\boldsymbol{I}} - \underbrace{\hat{\boldsymbol{R}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{R}_{1h}^T}_{\hat{\boldsymbol{I}}^T}\underbrace{\boldsymbol{R}_{1h}\tilde{\boldsymbol{\theta}}_\delta\times\boldsymbol{R}_{1h}^T}_{\left(\boldsymbol{R}_{1h}\tilde{\boldsymbol{\theta}}_\delta\right)\times}\underbrace{\overbrace{\boldsymbol{R}_{1h}\tilde{\boldsymbol{R}}}^{\hat{\boldsymbol{R}}}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T}_{\hat{\boldsymbol{I}}} - \hat{\boldsymbol{I}}\hat{\boldsymbol{\theta}}_\delta\times \\
&= \hat{\boldsymbol{\theta}}_\delta\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}^T\left(\underbrace{\boldsymbol{R}_{1h}\left(\boldsymbol{I}+\tilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{R}_{1h}^T}_{\hat{\boldsymbol{I}}}\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right)\right)\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}\hat{\boldsymbol{\theta}}_\delta\times \\
&= \left(\hat{\boldsymbol{I}}\boldsymbol{\theta}_{2\delta}+\hat{\boldsymbol{I}}^T\boldsymbol{\theta}_{1\delta}\right)\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}^T\left(\hat{\boldsymbol{I}}\left(\boldsymbol{\theta}_{2\delta}-\boldsymbol{\theta}_{1\delta}\right)\right)\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}\left(\hat{\boldsymbol{I}}\boldsymbol{\theta}_{2\delta}+\hat{\boldsymbol{I}}^T\boldsymbol{\theta}_{1\delta}\right)\times
\end{aligned} \tag{9.286b}$$

The perturbation of matrix $\hat{\boldsymbol{I}}$ is generally useful when multiplying a generic vector $\boldsymbol{v}$, resulting in

$$\begin{aligned}
\delta\hat{\boldsymbol{I}}\boldsymbol{v} &= \left(\left(\hat{\boldsymbol{I}}\boldsymbol{v}\times - \left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\right)\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\hat{\boldsymbol{I}}\right)\boldsymbol{\theta}_{1\delta} \\
&\quad + \left(\left(\hat{\boldsymbol{I}}\boldsymbol{v}\times - \left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\right)\hat{\boldsymbol{I}} + \hat{\boldsymbol{I}}^T\left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\hat{\boldsymbol{I}}\right)\boldsymbol{\theta}_{2\delta} \\
&= \hat{\boldsymbol{I}}_{1(\boldsymbol{v})}\boldsymbol{\theta}_{1\delta} + \hat{\boldsymbol{I}}_{2(\boldsymbol{v})}\boldsymbol{\theta}_{2\delta} \qquad (9.287a) \\
\delta\left(\hat{\boldsymbol{I}}^T\right)\boldsymbol{v} &= \left(\left(\hat{\boldsymbol{I}}^T\boldsymbol{v}\times - \left(\hat{\boldsymbol{I}}^T\boldsymbol{v}\right)\times\right)\hat{\boldsymbol{I}}^T + \hat{\boldsymbol{I}}^T\left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\hat{\boldsymbol{I}}\right)\boldsymbol{\theta}_{1\delta} \\
&\quad + \left(\left(\hat{\boldsymbol{I}}^T\boldsymbol{v}\times - \left(\hat{\boldsymbol{I}}^T\boldsymbol{v}\right)\times\right)\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}^T\left(\hat{\boldsymbol{I}}\boldsymbol{v}\right)\times\hat{\boldsymbol{I}}\right)\boldsymbol{\theta}_{2\delta} \\
&= \left(\hat{\boldsymbol{I}}^T\right)_{1(\boldsymbol{v})}\boldsymbol{\theta}_{1\delta} + \left(\hat{\boldsymbol{I}}^T\right)_{2(\boldsymbol{v})}\boldsymbol{\theta}_{2\delta}. \qquad (9.287b)
\end{aligned}$$

Relative Rotation Vector. The relative rotation vector is $\boldsymbol{\theta}$; by definition, this does not depend on the reference frame that is considered, among those intermediate between node 1 and 2, since it is coaxial to any rotation intermediate between the orientations of the two nodes; in detail

$$\tilde{\boldsymbol{R}}^T\boldsymbol{\theta} = \boldsymbol{\theta}, \qquad (9.288)$$

since $\tilde{\boldsymbol{\theta}}$ is co-axial to $\boldsymbol{\theta}$ by construction.

This implies that

$$\begin{aligned}
\delta\boldsymbol{\theta} &= \delta\left(\tilde{\boldsymbol{R}}^T\boldsymbol{\theta}\right) \\
&= \tilde{\boldsymbol{R}}^T\left(\boldsymbol{\theta}\times\tilde{\boldsymbol{\theta}}_\delta + \delta\boldsymbol{\theta}\right) \\
&= \tilde{\boldsymbol{R}}^T\left(\boldsymbol{\theta}\times\boldsymbol{\Gamma}\left(\tilde{\boldsymbol{\theta}}\right)\delta\tilde{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}\right) \\
&= \tilde{\boldsymbol{R}}^T\left(2\tilde{\boldsymbol{\theta}}\times\boldsymbol{\Gamma}\left(\tilde{\boldsymbol{\theta}}\right)\frac{\delta\boldsymbol{\theta}}{2} + \delta\boldsymbol{\theta}\right) \\
&= \tilde{\boldsymbol{R}}^T\left(\tilde{\boldsymbol{\theta}}\times\boldsymbol{\Gamma}\left(\tilde{\boldsymbol{\theta}}\right) + \boldsymbol{I}\right)\delta\boldsymbol{\theta} \\
&= \delta\boldsymbol{\theta}, \qquad (9.289)
\end{aligned}$$

since, by definition,

$$\boldsymbol{I} + \tilde{\boldsymbol{\theta}}\times\boldsymbol{\Gamma}\left(\tilde{\boldsymbol{\theta}}\right) = \tilde{\boldsymbol{R}}. \qquad (9.290)$$

The perturbation of the relative rotation vector $\boldsymbol{\theta}$,

$$\delta\overline{\boldsymbol{\theta}} = \delta\boldsymbol{\theta}, \qquad (9.291)$$

is used as perturbation of the measure of the straining of the joint.

Relative Angular Velocity. The relative angular velocity between the two nodes, in the intermediate reference frame $\hat{\boldsymbol{R}}$, is

$$\overline{\boldsymbol{\omega}} = \hat{\boldsymbol{R}}^T\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right), \qquad (9.292)$$

which corresponds to that of the attached case, Eq. (9.251), but projected in the material frame by $\hat{\boldsymbol{R}}^T$ instead of the reference frame of node 1, $\boldsymbol{R}_{1h}$. The linearization of the angular velocity of Equation (9.292)

yields

$$\begin{aligned}
\delta\overline{\boldsymbol{\omega}} &= \hat{\boldsymbol{R}}^T \left((\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \left(\boldsymbol{\theta}_{1\delta} + \boldsymbol{R}_{1h} \tilde{\boldsymbol{\theta}}_\delta \right) + \delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 \right) \\
&= \hat{\boldsymbol{R}}^T \left(\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 \right) \\
&\quad + \hat{\boldsymbol{R}}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \boldsymbol{R}_{1h} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \boldsymbol{R}_{1h}^T \boldsymbol{\theta}_{2\delta} \\
&\quad + \hat{\boldsymbol{R}}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \left(\boldsymbol{I} - \boldsymbol{R}_{1h} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \boldsymbol{R}_{1h}^T \right) \boldsymbol{\theta}_{1\delta} \\
&= \hat{\boldsymbol{R}}^T \left(\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 \right) \\
&\quad + \hat{\boldsymbol{R}}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \hat{\boldsymbol{R}} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{2\delta} \\
&\quad + \hat{\boldsymbol{R}}^T \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \hat{\boldsymbol{R}} \left(\boldsymbol{I} - \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \right) \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{1\delta} \\
&= \hat{\boldsymbol{R}}^T \left(\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 \right) \\
&\quad + \overline{\boldsymbol{\omega}} \times \left(\left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{2\delta} + \left(\boldsymbol{I} - \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \right) \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{1\delta} \right) \\
&= \hat{\boldsymbol{R}}^T \left(\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 \right) \\
&\quad + \overline{\boldsymbol{\omega}} \times \left(\left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{2\delta} + \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T \right)^{-1} \hat{\boldsymbol{R}}^T \boldsymbol{\theta}_{1\delta} \right) \\
&= \hat{\boldsymbol{R}}^T \left(\delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1 + \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \left(\hat{\boldsymbol{I}} \boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}^T \boldsymbol{\theta}_{1\delta} \right) \right)
\end{aligned} \tag{9.293}$$

since

$$\tilde{\boldsymbol{R}} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right) \tilde{\boldsymbol{R}}^T = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right) \tag{9.294}$$

and thus

$$\begin{aligned}
\left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} &= \tilde{\boldsymbol{R}}^{-T} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \tilde{\boldsymbol{R}}^{-1} \\
&= \tilde{\boldsymbol{R}} \left(\boldsymbol{I} + \tilde{\boldsymbol{R}} \right)^{-1} \tilde{\boldsymbol{R}}^T,
\end{aligned} \tag{9.295}$$

where Equation (9.275) has been used.

Intermediate Angular Velocity. Consider now

$$\begin{aligned}
\hat{\boldsymbol{\omega}} \times &= \dot{\hat{\boldsymbol{R}}} \hat{\boldsymbol{R}}^T \\
&= \boldsymbol{\omega}_1 \times + \boldsymbol{R}_{1h} \tilde{\boldsymbol{\omega}} \times \boldsymbol{R}_{1h}^T \\
&= \boldsymbol{\omega}_2 \times - \hat{\boldsymbol{R}} \tilde{\boldsymbol{\omega}} \times \hat{\boldsymbol{R}}^T,
\end{aligned} \tag{9.296}$$

where $\tilde{\boldsymbol{\omega}}$ is the angular velocity associated to the differentiation of matrix $\tilde{\boldsymbol{R}}$ with respect to time:

$$\tilde{\boldsymbol{\omega}} = \dot{\tilde{\boldsymbol{R}}} \tilde{\boldsymbol{R}}^T. \tag{9.297}$$

A relationship between that derivative and the remaining angular velocities results from

$$\begin{aligned}
\boldsymbol{\omega} \times &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\tilde{\boldsymbol{R}} \tilde{\boldsymbol{R}} \right) \left(\tilde{\boldsymbol{R}} \tilde{\boldsymbol{R}} \right)^T \\
&= \tilde{\boldsymbol{\omega}} \times + \tilde{\boldsymbol{R}} \tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{R}}^T,
\end{aligned} \tag{9.298}$$

so

$$\boldsymbol{\omega} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)\tilde{\boldsymbol{\omega}}; \tag{9.299}$$

as a consequence,

$$\tilde{\boldsymbol{\omega}} = \left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{R}_{1h}^{T}\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right). \tag{9.300}$$

The absolute velocity of the intermediate orientation results in

$$\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{I}}\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1; \tag{9.301}$$

its perturbation becomes

$$\begin{aligned}
\delta\hat{\boldsymbol{\omega}} &= \delta\left(\hat{\boldsymbol{I}}\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1\right) \\
&= \hat{\boldsymbol{I}}\delta\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\delta\boldsymbol{\omega}_1 \\
&\quad + \begin{pmatrix} \left(\hat{\boldsymbol{I}}\boldsymbol{\omega}_2 \times + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1 \times\right)\hat{\boldsymbol{I}} \\ -\left(\hat{\boldsymbol{I}}\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}} \\ +\hat{\boldsymbol{I}}\left(\hat{\boldsymbol{I}}^T\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right)\right)\times\hat{\boldsymbol{I}}^T \end{pmatrix}\boldsymbol{\theta}_{2\delta} \\
&\quad + \begin{pmatrix} \left(\hat{\boldsymbol{I}}\boldsymbol{\omega}_2 \times + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1 \times\right)\hat{\boldsymbol{I}}^T \\ -\left(\hat{\boldsymbol{I}}\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}}^T \\ -\hat{\boldsymbol{I}}\left(\hat{\boldsymbol{I}}^T\left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right)\right)\times\hat{\boldsymbol{I}}^T \end{pmatrix}\boldsymbol{\theta}_{1\delta} \\
&= \hat{\boldsymbol{I}}\delta\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\delta\boldsymbol{\omega}_1 + \hat{\boldsymbol{I}}_2\boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}_1\boldsymbol{\theta}_{1\delta} \\
&\overset{\text{uu}}{=} \hat{\boldsymbol{I}}\delta\dot{\boldsymbol{g}}_2 + \hat{\boldsymbol{I}}^T\delta\dot{\boldsymbol{g}}_1 + \left(\hat{\boldsymbol{I}}_2 - \hat{\boldsymbol{I}}\boldsymbol{\omega}_2\times\right)\delta\boldsymbol{g}_2 + \left(\hat{\boldsymbol{I}}_1 - \hat{\boldsymbol{I}}^T\boldsymbol{\omega}_1\times\right)\delta\boldsymbol{g}_1
\end{aligned} \tag{9.302}$$

Equilibrium Equations. The nodal moment results from the VWP, considering the perturbation of the relative orientation $\overline{\boldsymbol{\theta}}_\delta$,

$$\begin{aligned}
\delta\mathcal{L} &= \overline{\boldsymbol{\theta}}_\delta^T\tilde{\boldsymbol{M}} \\
&= \left(\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}\right)^T\hat{\boldsymbol{R}}\tilde{\boldsymbol{M}},
\end{aligned} \tag{9.303}$$

which corresponds to

$$\boldsymbol{M}_i = (-1)^i\,\hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}\left(\overline{\boldsymbol{\theta}}, \overline{\boldsymbol{\varpi}}\right). \tag{9.304}$$

It depends on the relative rotation vector $\overline{\boldsymbol{\theta}} = \tilde{\boldsymbol{R}}^T\boldsymbol{\theta} = \boldsymbol{\theta}$, and on the relative angular velocity $\overline{\boldsymbol{\varpi}}$, illustrated by Equation (9.292), expressed in the intermediate reference frame.

Equilibrium Perturbation. The perturbation of the moment then yields

$$\begin{aligned}
\delta\boldsymbol{M}_i &= (-1)^i\,\hat{\boldsymbol{R}}\left(\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}}\delta\overline{\boldsymbol{\theta}} + \tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}}\delta\overline{\boldsymbol{\varpi}}\right) \\
&\quad - \boldsymbol{M}_i \times \hat{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}\right)^{-1}\hat{\boldsymbol{R}}^T\boldsymbol{\theta}_{2\delta} \\
&\quad - \boldsymbol{M}_i \times \hat{\boldsymbol{R}}\left(\boldsymbol{I} + \tilde{\boldsymbol{R}}^T\right)^{-1}\hat{\boldsymbol{R}}^T\boldsymbol{\theta}_{1\delta} \\
&= (-1)^i\,\hat{\boldsymbol{R}}\left(\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}}\delta\overline{\boldsymbol{\theta}} + \tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}}\delta\overline{\boldsymbol{\varpi}}\right) - \boldsymbol{M}_i \times \left(\hat{\boldsymbol{I}}\boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}^T\boldsymbol{\theta}_{1\delta}\right).
\end{aligned} \tag{9.305}$$

The complete linearized problem is

$$
\begin{aligned}
&\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\omega}} & -\boldsymbol{M}_{/\boldsymbol{\omega}} \\ -\boldsymbol{M}_{/\boldsymbol{\omega}} & \boldsymbol{M}_{/\boldsymbol{\omega}} \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{\omega}_1 \\ \delta\boldsymbol{\omega}_2 \end{array}\right\} \\
&+\left[\begin{array}{cc} -\boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}}^T & -\boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}} \\ \boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}}^T & \boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\right)\times\hat{\boldsymbol{I}} \end{array}\right]\left\{\begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array}\right\} \\
&+\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\theta}} & -\boldsymbol{M}_{/\boldsymbol{\theta}} \\ -\boldsymbol{M}_{/\boldsymbol{\theta}} & \boldsymbol{M}_{/\boldsymbol{\theta}} \end{array}\right]\left\{\begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array}\right\} \\
&+\left[\begin{array}{cc} \boldsymbol{M}\times\hat{\boldsymbol{I}}^T & \boldsymbol{M}\times\hat{\boldsymbol{I}} \\ -\boldsymbol{M}\times\hat{\boldsymbol{I}}^T & -\boldsymbol{M}\times\hat{\boldsymbol{I}} \end{array}\right]\left\{\begin{array}{c} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{array}\right\}=\left\{\begin{array}{c} \boldsymbol{M} \\ -\boldsymbol{M} \end{array}\right\}
\end{aligned} \tag{9.306}
$$

where

$$\boldsymbol{M}_{/\boldsymbol{\theta}}=\hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_{1h}^T \tag{9.307a}$$

$$\boldsymbol{M}_{/\boldsymbol{\omega}}=\hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}}\hat{\boldsymbol{R}}^T. \tag{9.307b}$$

Accounting for the simplifications of the updated-updated approach, it becomes

$$
\begin{aligned}
&\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\omega}} & -\boldsymbol{M}_{/\boldsymbol{\omega}} \\ -\boldsymbol{M}_{/\boldsymbol{\omega}} & \boldsymbol{M}_{/\boldsymbol{\omega}} \end{array}\right]\left\{\begin{array}{c} \delta\dot{\boldsymbol{g}}_1 \\ \delta\dot{\boldsymbol{g}}_2 \end{array}\right\} \\
&+\left[\begin{array}{cc} -\boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2\times\hat{\boldsymbol{I}}^T+\boldsymbol{\omega}_1\times\hat{\boldsymbol{I}}\right) & \boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2\times\hat{\boldsymbol{I}}^T+\boldsymbol{\omega}_1\times\hat{\boldsymbol{I}}\right) \\ \boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2\times\hat{\boldsymbol{I}}^T+\boldsymbol{\omega}_1\times\hat{\boldsymbol{I}}\right) & -\boldsymbol{M}_{/\boldsymbol{\omega}}\left(\boldsymbol{\omega}_2\times\hat{\boldsymbol{I}}^T+\boldsymbol{\omega}_1\times\hat{\boldsymbol{I}}\right) \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array}\right\} \\
&+\left[\begin{array}{cc} \boldsymbol{M}_{/\boldsymbol{\theta}} & -\boldsymbol{M}_{/\boldsymbol{\theta}} \\ -\boldsymbol{M}_{/\boldsymbol{\theta}} & \boldsymbol{M}_{/\boldsymbol{\theta}} \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array}\right\} \\
&+\left[\begin{array}{cc} \boldsymbol{M}\times\hat{\boldsymbol{I}}^T & \boldsymbol{M}\times\hat{\boldsymbol{I}} \\ -\boldsymbol{M}\times\hat{\boldsymbol{I}}^T & -\boldsymbol{M}\times\hat{\boldsymbol{I}} \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{array}\right\}\overset{\text{uu}}{=}\left\{\begin{array}{c} \boldsymbol{M} \\ -\boldsymbol{M} \end{array}\right\}
\end{aligned} \tag{9.308}
$$

Note on attached vs. invariant deformable hinge

The moment applied by the attached formulation to node 1, in the global reference frame, is

$$\boldsymbol{M}_a=\boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}_a\left(\boldsymbol{\theta}\right), \tag{9.309}$$

while the moment applied by the invariant formulation to node 1, in the global reference frame, is

$$
\begin{aligned}
\boldsymbol{M}_i&=\hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_i\left(\boldsymbol{\theta}\right) \\
&=\boldsymbol{R}_{1h}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{M}}_i\left(\tilde{\boldsymbol{R}}^T\boldsymbol{\theta}\right),
\end{aligned} \tag{9.310}
$$

since $\tilde{\boldsymbol{R}}^T\boldsymbol{\theta}=\boldsymbol{\theta}$. This means that whatever formula is used for the constitutive law, in the invariant case, it is equivalent to using an attached formula with a constitutive law re-oriented by $\tilde{\boldsymbol{R}}$,

$$\tilde{\boldsymbol{M}}_a\left(\boldsymbol{\theta}\right)=\tilde{\boldsymbol{R}}\tilde{\boldsymbol{M}}_i\left(\tilde{\boldsymbol{R}}^T\boldsymbol{\theta}\right); \tag{9.311}$$

similar considerations apply to the viscous portion of the constitutive law. This transformation should be kept in mind when determining the properties of the deformable component. In fact, exchanging the locations where the constitutive law is evaluated implies a transformation of the constitutive matrix; in case a linear elastic constitutive law,

$$\tilde{\boldsymbol{M}}_i=\tilde{\boldsymbol{K}}_i\boldsymbol{\theta}, \tag{9.312}$$

with a constant (symmetric, positive definite) $\tilde{\boldsymbol{K}}_i$ matrix, is used for the invariant formula, which appears to be a natural solution for simple elastic hinges, it is equivalent to a nonlinear elastic constitutive law when transposing it into the attached formulation. The two formulas are coincident, and independent from $\boldsymbol{\theta}$, only in case of an isotropic spring, namely $\tilde{\boldsymbol{K}}_i = k\boldsymbol{I}$.

A "natural" solution, for a geometrically and materially symmetrical component, behaves the same when the order of the nodes is swapped. This corresponds to the "invariant" formula for the deformable hinge. However, typical experiments to determine the mechanical behavior of a component would rather consist in straining it while measuring the resulting loads at one or both ends, not in the intermediate location where the constitutive law would be naturally applied. As a consequence, a "natural" procedure for the determination of the constitutive law would consist in determining the $\boldsymbol{M}_a(\boldsymbol{\theta})$ law first; then, assuming the moment can be expressed in the form

$$\boldsymbol{R}_{1h}^T \boldsymbol{M}_a(\boldsymbol{\theta}) = \tilde{\boldsymbol{M}}_a(\boldsymbol{\theta}), \tag{9.313}$$

matrix $\tilde{\boldsymbol{R}}$ would be computed from the measured relative rotation of the extremities of the component, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}/2$, and the constitutive law $\tilde{\boldsymbol{M}}_i(\boldsymbol{\theta})$ would be computed as

$$\tilde{\boldsymbol{M}}_i\left(\tilde{\boldsymbol{R}}^T\boldsymbol{\theta}\right) = \tilde{\boldsymbol{R}}^T\tilde{\boldsymbol{M}}_a(\boldsymbol{\theta}) \tag{9.314}$$

to yield the invariant constitutive law.

9.2.3 Deformable Displacement Joint

The deformable displacement joint applies an internal force to two nodes at a specified point that may be offset from the nodes. The force may depend on the relative position and velocity of the nodes at the point of application through a 3D constitutive law.

The points whose relative displacement represents the measure of the straining can be offset from the nodes by rigid offsets $\tilde{\boldsymbol{f}}_1$ and $\tilde{\boldsymbol{f}}_2$, which are defined in the reference frame of the respective nodes. So the offsets in the global reference frame are

$$\boldsymbol{f}_1 = \boldsymbol{R}_1\tilde{\boldsymbol{f}}_1 \tag{9.315a}$$

$$\boldsymbol{f}_2 = \boldsymbol{R}_2\tilde{\boldsymbol{f}}_2. \tag{9.315b}$$

The constitutive law is expressed in a reference frame that may be rotated from that of the node by rigid rotations $\tilde{\boldsymbol{R}}_{1h}$ and $\tilde{\boldsymbol{R}}_{2h}$. So the orientations of the constitutive law in the global reference frame are

$$\boldsymbol{R}_{1h} = \boldsymbol{R}_1\tilde{\boldsymbol{R}}_{1h} \tag{9.316a}$$

$$\boldsymbol{R}_{2h} = \boldsymbol{R}_2\tilde{\boldsymbol{R}}_{2h}. \tag{9.316b}$$

As for the `deformable hinge`, the constitutive law of this joint may be either attached to one node or defined in an intermediate reference frame that accounts for the relative orientation of the two nodes, so that the order in which the nodes are defined becomes irrelevant.

The relative position in the absolute reference frame, $\boldsymbol{d}$, is

$$\boldsymbol{d} = \boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 - \boldsymbol{f}_1. \tag{9.317}$$

The relative velocity is

$$\dot{\boldsymbol{d}} = \dot{\boldsymbol{x}}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 - \dot{\boldsymbol{x}}_1 - \boldsymbol{\omega}_1 \times \boldsymbol{f}_1. \tag{9.318}$$

Attached Deformable Displacement Joint

The "attached" form of the deformable displacement joint is obtained by projecting the relative position $\boldsymbol{d}$ in a reference frame attached to node 1:

$$\tilde{\boldsymbol{d}} = \boldsymbol{R}_{1h}^T \boldsymbol{d} \tag{9.319}$$

The relative velocity is

$$\dot{\tilde{\boldsymbol{d}}} = \boldsymbol{R}_{1h}^T \left(\dot{\boldsymbol{d}} - \boldsymbol{\omega}_1 \times \boldsymbol{d} \right), \tag{9.320}$$

but it can be replaced by

$$\dot{\tilde{\boldsymbol{d}}} = \boldsymbol{R}_{1h}^T \left(\dot{\boldsymbol{d}}_1 - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \right), \tag{9.321}$$

where

$$\boldsymbol{d}_1 = \boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 \tag{9.322a}$$

$$\dot{\boldsymbol{d}}_1 = \dot{\boldsymbol{x}}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 - \dot{\boldsymbol{x}}_1, \tag{9.322b}$$

since the relative velocity associated with the rotation of $\boldsymbol{f}_1$, the offset of the reference point attached to node 1, is zero by definition. The linearization of the distance yields

$$\begin{aligned}
\delta \tilde{\boldsymbol{d}} &= \boldsymbol{R}_{1h}^T \left(\delta \boldsymbol{d} - \boldsymbol{d} \times \boldsymbol{\theta}_{1\delta} \right) \\
&= \boldsymbol{R}_{1h}^T \left(\delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \right) + \boldsymbol{R}_{1h}^T \boldsymbol{d} \times \boldsymbol{\theta}_{1\delta} \\
&= \boldsymbol{R}_{1h}^T \left(\delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \left(\boldsymbol{x}_2 + \boldsymbol{f}_2 - \boldsymbol{x}_1 \right) \times \boldsymbol{\theta}_{\delta} 1 \right) \\
&= \boldsymbol{R}_{1h}^T \left(\delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \boldsymbol{d}_1 \times \boldsymbol{\theta}_{1\delta} \right) \\
&\overset{\text{uu}}{=} \boldsymbol{R}_{1h}^T \left(\delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \delta \boldsymbol{g}_2 - \delta \boldsymbol{x}_1 + \boldsymbol{d}_1 \times \delta \boldsymbol{g}_1 \right)
\end{aligned} \tag{9.323}$$

while the linearization of the relative velocity yields

$$\begin{aligned}
\delta \dot{\tilde{\boldsymbol{d}}} &= \boldsymbol{R}_{1h}^T \left(\delta \dot{\boldsymbol{d}} + \boldsymbol{d} \times \delta \boldsymbol{\omega}_1 - \boldsymbol{\omega}_1 \times \delta \boldsymbol{d} + \left(\dot{\boldsymbol{d}} - \boldsymbol{\omega}_1 \times \boldsymbol{d} \right) \times \boldsymbol{\theta}_{1\delta} \right) \\
&= \boldsymbol{R}_{1h}^T \left(\begin{array}{c} \delta \dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \times \delta \boldsymbol{\omega}_2 - \delta \dot{\boldsymbol{x}}_1 + \boldsymbol{d}_1 \times \delta \boldsymbol{\omega}_1 \\ - \boldsymbol{\omega}_1 \times \delta \boldsymbol{x}_2 - \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \right) \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} \\ + \boldsymbol{\omega}_1 \times \delta \boldsymbol{x}_1 + \left(\dot{\boldsymbol{d}} - \boldsymbol{\omega}_1 \times \boldsymbol{d} \right) \times \boldsymbol{\theta}_{1\delta} \end{array} \right) \\
&\overset{\text{uu}}{=} \boldsymbol{R}_{1h}^T \left(\begin{array}{c} \delta \dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \times \delta \dot{\boldsymbol{g}}_2 - \delta \dot{\boldsymbol{x}}_1 + \boldsymbol{d}_1 \times \delta \dot{\boldsymbol{g}}_1 \\ - \boldsymbol{\omega}_1 \times \delta \boldsymbol{x}_2 - \left(\left(\boldsymbol{\omega}_2 \times \boldsymbol{f}_2 \right) \times \ - \boldsymbol{\omega}_1 \times \boldsymbol{f}_2 \times \ \right) \delta \boldsymbol{g}_2 \\ + \boldsymbol{\omega}_1 \times \delta \boldsymbol{x}_1 + \left(\dot{\boldsymbol{d}}_1 \times \ - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times \ \right) \delta \boldsymbol{g}_1 \end{array} \right)
\end{aligned} \tag{9.324}$$

The nodal forces and moments result from the VWP

$$\begin{aligned}
\delta \mathcal{L} &= \delta \tilde{\boldsymbol{d}}^T \tilde{\boldsymbol{F}} \\
&= \left(\delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \boldsymbol{d}_1 \times \boldsymbol{\theta}_{1\delta} \right)^T \boldsymbol{R}_{1h} \tilde{\boldsymbol{F}},
\end{aligned} \tag{9.325}$$

which corresponds to

$$\boldsymbol{F}_i = \left(-1 \right)^i \boldsymbol{R}_{1h} \tilde{\boldsymbol{F}} \left(\tilde{\boldsymbol{d}}, \dot{\tilde{\boldsymbol{d}}} \right) \tag{9.326a}$$

$$\boldsymbol{M}_i = \boldsymbol{d}_i \times \boldsymbol{F}_i \tag{9.326b}$$

where $\boldsymbol{d}_1$ has already been defined, and

$$\boldsymbol{d}_2 = \boldsymbol{f}_2; \tag{9.327}$$

note that

$$\begin{aligned}\boldsymbol{\omega}_2 \times \boldsymbol{f}_2 &= \boldsymbol{\omega}_2 \times \boldsymbol{d}_2 \\ &= \dot{\boldsymbol{d}}_2. \end{aligned} \tag{9.328}$$

Their linearization yields

$$\delta \boldsymbol{F}_i = (-1)^i \boldsymbol{R}_{1h} \left(\tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}} \delta \tilde{\boldsymbol{d}} + \tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}} \delta \dot{\tilde{\boldsymbol{d}}} \right) - \boldsymbol{F}_i \times \boldsymbol{\theta}_{1\delta} \tag{9.329a}$$

$$\delta \boldsymbol{M}_i = \boldsymbol{d}_i \times \delta \boldsymbol{F}_i - \boldsymbol{F}_i \times \delta \boldsymbol{d}_i \tag{9.329b}$$

The complete linearized problem, after applying the updated-updated approximation, is

$$\begin{aligned}
&\begin{bmatrix}
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_1 \times & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_2 \times \\
\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_1 \times & -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_2 \times \\
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_1 \times & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_2 \times \\
-\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_1 \times & \boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{d}_2 \times
\end{bmatrix}
\begin{Bmatrix} \delta \dot{\boldsymbol{x}}_1 \\ \delta \dot{\boldsymbol{g}}_1 \\ \delta \dot{\boldsymbol{x}}_2 \\ \delta \dot{\boldsymbol{g}}_2 \end{Bmatrix} \\
&\quad + \begin{bmatrix}
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
-\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times
\end{bmatrix} \delta \boldsymbol{x}_1 +
\begin{bmatrix}
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times \right) \\
-\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times \right) \\
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times \right) \\
\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times \right)
\end{bmatrix} \delta \boldsymbol{g}_1 \\
&\quad + \begin{bmatrix}
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times \\
-\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{\omega}_1 \times
\end{bmatrix} \delta \boldsymbol{x}_2 +
\begin{bmatrix}
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times \right) \\
\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times \right) \\
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times \right) \\
-\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times \right)
\end{bmatrix} \delta \boldsymbol{g}_2 \\
&+ \begin{bmatrix}
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_1 \times & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_2 \times \\
\boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_1 \times & -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_2 \times \\
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_1 \times & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_2 \times \\
-\boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_1 \times & \boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{d}_2 \times
\end{bmatrix}
\begin{Bmatrix} \delta \boldsymbol{x}_1 \\ \delta \boldsymbol{g}_1 \\ \delta \boldsymbol{x}_2 \\ \delta \boldsymbol{g}_2 \end{Bmatrix} \\
&+ \begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F} \times & \boldsymbol{0} & \boldsymbol{0} \\
-\boldsymbol{F} \times & \boldsymbol{d}_1 \times \boldsymbol{F} \times & \boldsymbol{F} \times & -\boldsymbol{F} \times \boldsymbol{d}_2 \times \\
\boldsymbol{0} & -\boldsymbol{F} \times & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{d}_2 \times \boldsymbol{F} \times & \boldsymbol{0} & \boldsymbol{F} \times \boldsymbol{d}_2 \times
\end{bmatrix}
\begin{Bmatrix} \delta \boldsymbol{x}_1 \\ \delta \boldsymbol{g}_1 \\ \delta \boldsymbol{x}_2 \\ \delta \boldsymbol{g}_2 \end{Bmatrix}
= \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{d}_1 \times \boldsymbol{F} \\ -\boldsymbol{F} \\ -\boldsymbol{d}_2 \times \boldsymbol{F} \end{Bmatrix}
\end{aligned} \tag{9.330}$$

where

$$\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} = \boldsymbol{R}_{1h} \tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{R}_{1h}^T \tag{9.331a}$$

$$\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} = \boldsymbol{R}_{1h} \tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}} \boldsymbol{R}_{1h}^T \tag{9.331b}$$

Invariant Deformable Displacement Joint

The invariant form of the `deformable displacement` joint assumes that the constitutive properties of the component, that explicitly depend only on the relative position of the nodes, is defined in a reference frame that is intermediate between those of node 1 and 2, much like the invariant form of the `deformable hinge` joint.

The distance between the two reference points, expressed in the global reference frame, is pulled back in the intermediate material reference frame by the transpose of matrix $\hat{\boldsymbol{R}}$

$$\tilde{\boldsymbol{d}} = \hat{\boldsymbol{R}}^T \boldsymbol{d} \tag{9.332}$$

The relative velocity is

$$\begin{aligned} \dot{\tilde{\boldsymbol{d}}} &= \dot{\hat{\boldsymbol{R}}}^T \boldsymbol{d} + \hat{\boldsymbol{R}}^T \dot{\boldsymbol{d}} \\ &= \hat{\boldsymbol{R}}^T \left(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d} \right); \end{aligned} \tag{9.333}$$

for a discussion of the properties of the entities that appear in Eq. (9.301) see Section 9.2.2.

The perturbation of the relative position yields

$$\delta \tilde{\boldsymbol{d}} = \hat{\boldsymbol{R}}^T \left(\delta \boldsymbol{d} + \boldsymbol{d} \times \hat{\boldsymbol{\theta}}_\delta \right) \tag{9.334}$$

where

$$\begin{aligned} \delta \boldsymbol{d} &= \delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \\ &\stackrel{\text{uu}}{=} \delta \boldsymbol{x}_2 - \boldsymbol{f}_2 \times \delta \boldsymbol{g}_2 - \delta \boldsymbol{x}_1 + \boldsymbol{f}_1 \times \delta \boldsymbol{g}_1 \end{aligned} \tag{9.335a}$$

$$\begin{aligned} \hat{\boldsymbol{\theta}}_\delta &= \hat{\boldsymbol{I}} \boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}^T \boldsymbol{\theta}_{1\delta} \\ &\stackrel{\text{uu}}{=} \hat{\boldsymbol{I}} \delta \boldsymbol{g}_2 + \hat{\boldsymbol{I}}^T \delta \boldsymbol{g}_1, \end{aligned} \tag{9.335b}$$

and thus

$$\begin{aligned} \delta \tilde{\boldsymbol{d}} &= \hat{\boldsymbol{R}}^T \left(\delta \boldsymbol{x}_2 + \left(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times \right) \boldsymbol{\theta}_{2\delta} - \delta \boldsymbol{x}_1 + \left(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times \right) \boldsymbol{\theta}_{1\delta} \right) \\ &\stackrel{\text{uu}}{=} \hat{\boldsymbol{R}}^T \left(\delta \boldsymbol{x}_2 + \left(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times \right) \delta \boldsymbol{g}_2 - \delta \boldsymbol{x}_1 + \left(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times \right) \delta \boldsymbol{g}_1 \right). \end{aligned} \tag{9.336}$$

The perturbation of the relative velocity yields

$$\delta \dot{\tilde{\boldsymbol{d}}} = \hat{\boldsymbol{R}}^T \left(\delta \dot{\boldsymbol{d}} + \boldsymbol{d} \times \delta \hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}} \times \delta \boldsymbol{d} + \left(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d} \right) \times \hat{\boldsymbol{\theta}}_\delta \right) \tag{9.337}$$

where

$$\begin{aligned} \delta \dot{\boldsymbol{d}} &= \delta \dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \times \delta \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta \dot{\boldsymbol{x}}_1 + \boldsymbol{f}_1 \times \delta \boldsymbol{\omega}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \\ &\stackrel{\text{uu}}{=} \delta \dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \delta \dot{\boldsymbol{g}}_2 + (\boldsymbol{f}_2 \times \boldsymbol{\omega}_2) \times \delta \boldsymbol{g}_2 - \delta \dot{\boldsymbol{x}}_1 + \boldsymbol{f}_1 \delta \dot{\boldsymbol{g}}_1 - (\boldsymbol{f}_1 \times \boldsymbol{\omega}_1) \times \delta \boldsymbol{g}_1. \end{aligned} \tag{9.338}$$

The computation of $\delta\hat{\boldsymbol{\omega}}$ has been illustrated in Section 9.2.2. Thus,

$$
\begin{aligned}
\delta\dot{\tilde{\boldsymbol{d}}} &= \hat{\boldsymbol{R}}^T \Big(\delta\dot{\boldsymbol{x}}_2 - \boldsymbol{f}_2 \times \delta\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2 \times \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta\dot{\boldsymbol{x}}_1 + \boldsymbol{f}_1 \times \delta\boldsymbol{\omega}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta} \\
&\qquad + \boldsymbol{d} \times \Big(\hat{\boldsymbol{I}}\delta\boldsymbol{\omega}_2 + \hat{\boldsymbol{I}}^T\delta\boldsymbol{\omega}_1 + \hat{\boldsymbol{I}}_2\boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}_1\boldsymbol{\theta}_{1\delta}\Big) \\
&\qquad - \hat{\boldsymbol{\omega}} \times (\delta\boldsymbol{x}_2 - \boldsymbol{f}_2 \times \boldsymbol{\theta}_{2\delta} - \delta\boldsymbol{x}_1 + \boldsymbol{f}_1 \times \boldsymbol{\theta}_{1\delta}) \\
&\qquad + \Big(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d}\Big) \times \Big(\hat{\boldsymbol{I}}\boldsymbol{\theta}_{2\delta} + \hat{\boldsymbol{I}}^T\boldsymbol{\theta}_{1\delta}\Big)\Big) \\
&= \hat{\boldsymbol{R}}^T \Big(\delta\dot{\boldsymbol{x}}_2 - \delta\dot{\boldsymbol{x}}_1 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times\Big)\delta\boldsymbol{\omega}_2 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times\Big)\delta\boldsymbol{\omega}_1 \\
&\qquad - \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{x}_2 + \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{x}_1 \\
&\qquad + \Big((\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_2) \times \boldsymbol{f}_2 \times + \boldsymbol{d} \times \hat{\boldsymbol{I}}_2 + \Big(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d}\Big) \times \hat{\boldsymbol{I}}\Big)\boldsymbol{\theta}_{2\delta} \\
&\qquad + \Big(-(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_1) \times \boldsymbol{f}_1 \times + \boldsymbol{d} \times \hat{\boldsymbol{I}}_1 + \Big(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d}\Big) \times \hat{\boldsymbol{I}}^T\Big)\boldsymbol{\theta}_{1\delta}\Big) \\
&\overset{\mathrm{uu}}{=} \hat{\boldsymbol{R}}^T \Big(\delta\dot{\boldsymbol{x}}_2 - \delta\dot{\boldsymbol{x}}_1 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times\Big)\delta\dot{\boldsymbol{g}}_2 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times\Big)\delta\dot{\boldsymbol{g}}_1 \\
&\qquad - \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{x}_2 + \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{x}_1 \\
&\qquad + \left(\begin{array}{c} \Big(\hat{\boldsymbol{I}}^T(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\Big) \times \boldsymbol{f}_2 \times - \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times\Big)\boldsymbol{\omega}_2 \times \\ + \boldsymbol{d} \times \hat{\boldsymbol{I}}_2 + \Big(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d}\Big) \times \hat{\boldsymbol{I}} \end{array}\right)\delta\boldsymbol{g}_2 \\
&\qquad + \left(\begin{array}{c} \Big(\hat{\boldsymbol{I}}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\Big) \times \boldsymbol{f}_1 \times - \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times\Big)\boldsymbol{\omega}_1 \times \\ + \boldsymbol{d} \times \hat{\boldsymbol{I}}_1 + \Big(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d}\Big) \times \hat{\boldsymbol{I}}^T \end{array}\right)\delta\boldsymbol{g}_1\Big)
\end{aligned} \tag{9.339}
$$

The force and the moment result from the VWP

$$
\begin{aligned}
\delta\mathcal{L} &= \delta\tilde{\boldsymbol{d}}^T\tilde{\boldsymbol{F}} \\
&= \Big(\delta\boldsymbol{d} + \boldsymbol{d} \times \hat{\boldsymbol{\theta}}_\delta\Big)^T \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}} \\
&= \Big(\delta\boldsymbol{x}_2 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}} - \boldsymbol{f}_2 \times\Big)\boldsymbol{\theta}_{2\delta} - \delta\boldsymbol{x}_1 + \Big(\boldsymbol{d} \times \hat{\boldsymbol{I}}^T + \boldsymbol{f}_1 \times\Big)\boldsymbol{\theta}_{1\delta}\Big)^T \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}
\end{aligned} \tag{9.340}
$$

and, after defining

$$
\boldsymbol{F} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}\left(\tilde{\boldsymbol{d}}, \dot{\tilde{\boldsymbol{d}}}\right), \tag{9.341}
$$

are

$$
\boldsymbol{F}_1 = -\boldsymbol{F} \tag{9.342a}
$$

$$
\boldsymbol{M}_1 = -\Big(\boldsymbol{f}_1 \times + \hat{\boldsymbol{I}}\boldsymbol{d} \times\Big)\boldsymbol{F} \tag{9.342b}
$$

$$
\boldsymbol{F}_2 = \boldsymbol{F} \tag{9.342c}
$$

$$
\boldsymbol{M}_2 = \Big(\boldsymbol{f}_2 \times - \hat{\boldsymbol{I}}^T\boldsymbol{d} \times\Big)\boldsymbol{F}. \tag{9.342d}
$$

Their linearization, after defining

$$
\delta\boldsymbol{F} = \hat{\boldsymbol{R}}\Big(\tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}}\delta\tilde{\boldsymbol{d}} + \tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}}\delta\dot{\tilde{\boldsymbol{d}}}\Big) - \boldsymbol{F} \times \hat{\boldsymbol{\theta}}_\delta, \tag{9.343}
$$

yields

$$\delta \boldsymbol{F}_1 = -\delta \boldsymbol{F} \tag{9.344a}$$

$$\delta \boldsymbol{M}_1 = \left(\boldsymbol{f}_1 \times + \hat{\boldsymbol{I}} \boldsymbol{d} \times \right) \delta \boldsymbol{F} - \delta \boldsymbol{f}_1 \times \boldsymbol{F} - \delta \hat{\boldsymbol{I}} \boldsymbol{d} \times \boldsymbol{F} - \hat{\boldsymbol{I}} \delta \boldsymbol{d} \times \boldsymbol{F} \tag{9.344b}$$

$$\delta \boldsymbol{F}_2 = \delta \boldsymbol{F} \tag{9.344c}$$

$$\delta \boldsymbol{M}_2 = \left(\boldsymbol{f}_2 \times - \hat{\boldsymbol{I}}^T \boldsymbol{d} \times \right) \delta \boldsymbol{F} + \delta \boldsymbol{f}_2 \times \boldsymbol{F} - \delta \hat{\boldsymbol{I}}^T \boldsymbol{d} \times \boldsymbol{F} - \hat{\boldsymbol{I}}^T \delta \boldsymbol{d} \times \boldsymbol{F} \tag{9.344d}$$

After setting

$$\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} = \hat{\boldsymbol{R}} \tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}} \hat{\boldsymbol{R}}^T \tag{9.345a}$$

$$\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} = \hat{\boldsymbol{R}} \tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{R}}^T \tag{9.345b}$$

and

$$\boldsymbol{A}_1 = \left(\boldsymbol{f}_1 \times + \hat{\boldsymbol{I}} \boldsymbol{d} \times \right) \tag{9.346a}$$

$$\boldsymbol{A}_2 = \left(\boldsymbol{f}_2 \times - \hat{\boldsymbol{I}}^T \boldsymbol{d} \times \right) \tag{9.346b}$$

$$\boldsymbol{B}_1 = \left(\left(\hat{\boldsymbol{I}} \left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \right) \right) \times \boldsymbol{f}_1 \times + \boldsymbol{d} \times \hat{\boldsymbol{I}}_1 + \left(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d} \right) \times \hat{\boldsymbol{I}}^T \right) \tag{9.346c}$$

$$\boldsymbol{B}_2 = \left(\left(\hat{\boldsymbol{I}}^T \left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \right) \right) \times \boldsymbol{f}_2 \times + \boldsymbol{d} \times \hat{\boldsymbol{I}}_2 + \left(\dot{\boldsymbol{d}} - \hat{\boldsymbol{\omega}} \times \boldsymbol{d} \right) \times \hat{\boldsymbol{I}} \right) \tag{9.346d}$$

the complete linearized problem is

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_1^T & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_2^T \\ \boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_1^T & -\boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_2^T \\ -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_1^T & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_2^T \\ -\boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_1^T & \boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{A}_2^T \end{bmatrix} \begin{Bmatrix} \delta \dot{\boldsymbol{x}}_1 \\ \delta \boldsymbol{\omega}_1 \\ \delta \dot{\boldsymbol{x}}_2 \\ \delta \boldsymbol{\omega}_2 \end{Bmatrix} \\
&+ \begin{bmatrix} -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_1 & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_2 \\ -\boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & -\boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_1 & \boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & -\boldsymbol{A}_1 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_2 \\ \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_1 & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_2 \\ \boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & \boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_1 & -\boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \hat{\boldsymbol{\omega}} \times & \boldsymbol{A}_2 \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \boldsymbol{B}_2 \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \\ \delta \boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{Bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_1^T & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_2^T \\ \boldsymbol{A}_1 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_1 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_1^T & -\boldsymbol{A}_1 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_1 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_2^T \\ -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_1^T & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_2^T \\ -\boldsymbol{A}_2 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_2 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_1^T & \boldsymbol{A}_2 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_2 \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \boldsymbol{A}_2^T \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \\ \delta \boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{Bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{F} \times \hat{\boldsymbol{I}}^T \\ -\hat{\boldsymbol{I}} \boldsymbol{F} \times & \boldsymbol{A}_1 \boldsymbol{F} \times \hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T \boldsymbol{F} \times \boldsymbol{f}_1 \times - \hat{\boldsymbol{I}}_{1(\boldsymbol{d} \times \boldsymbol{F})} \\ \boldsymbol{0} & -\boldsymbol{F} \times \hat{\boldsymbol{I}}^T \\ -\hat{\boldsymbol{I}}^T \boldsymbol{F} \times & -\boldsymbol{A}_2 \boldsymbol{F} \times \hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T \boldsymbol{F} \times \boldsymbol{f}_1 \times - \left(\hat{\boldsymbol{I}}^T \right)_{1(\boldsymbol{d} \times \boldsymbol{F})} \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \end{Bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{F} \times \hat{\boldsymbol{I}} \\ \hat{\boldsymbol{I}} \boldsymbol{F} \times & \boldsymbol{A}_1 \boldsymbol{F} \times \hat{\boldsymbol{I}} - \hat{\boldsymbol{I}} \boldsymbol{F} \times \boldsymbol{f}_2 \times - \hat{\boldsymbol{I}}_{2(\boldsymbol{d} \times \boldsymbol{F})} \\ \boldsymbol{0} & -\boldsymbol{F} \times \hat{\boldsymbol{I}} \\ \hat{\boldsymbol{I}}^T \boldsymbol{F} \times & -\boldsymbol{A}_2 \boldsymbol{F} \times \hat{\boldsymbol{I}} + \hat{\boldsymbol{I}} \boldsymbol{F} \times \boldsymbol{f}_2 \times + \left(\hat{\boldsymbol{I}}^T \right)_{2(\boldsymbol{d} \times \boldsymbol{F})} \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{Bmatrix} \\
&= \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{A}_1 \boldsymbol{F} \\ -\boldsymbol{F} \\ -\boldsymbol{A}_2 \boldsymbol{F} \end{Bmatrix}
\end{aligned} \tag{9.347}$$

When the updated-updated approach is considered, after defining

$$\hat{\boldsymbol{B}}_1 = \boldsymbol{B}_1 + \boldsymbol{A}_1^T \boldsymbol{\omega}_1 \times \tag{9.348a}$$

$$\hat{\boldsymbol{B}}_2 = \boldsymbol{B}_2 - \boldsymbol{A}_2^T \boldsymbol{\omega}_2 \times , \tag{9.348b}$$

one obtains

$$\begin{aligned}
&\begin{bmatrix}
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_1^T & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_2^T \\
\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_1^T & -\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_2^T \\
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_1^T & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_2^T \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & -\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_1^T & \boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} & \boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{A}_2^T
\end{bmatrix}
\begin{Bmatrix} \delta\dot{\boldsymbol{x}}_1 \\ \delta\dot{\boldsymbol{g}}_1 \\ \delta\dot{\boldsymbol{x}}_2 \\ \delta\dot{\boldsymbol{g}}_2 \end{Bmatrix} \\
&+ \begin{bmatrix}
-\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_1 & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_2 \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_1 & \boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_2 \\
\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_1 & -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_2 \\
\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_1 & -\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{B}}_2
\end{bmatrix}
\begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \\
&+ \begin{bmatrix}
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T \\
\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T & -\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T \\
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T & \boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T
\end{bmatrix}
\begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \\
&+ \begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}\boldsymbol{F}\times & \boldsymbol{A}_1\boldsymbol{F}\times\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \hat{\boldsymbol{I}}_{1(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\boldsymbol{A}_2\boldsymbol{F}\times\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \left(\hat{\boldsymbol{I}}^T\right)_{1(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \end{Bmatrix} \\
&+ \begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}\boldsymbol{F}\times & \boldsymbol{A}_1\boldsymbol{F}\times\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times - \hat{\boldsymbol{I}}_{2(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\boldsymbol{A}_2\boldsymbol{F}\times\hat{\boldsymbol{I}} + \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times + \left(\hat{\boldsymbol{I}}^T\right)_{2(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\begin{Bmatrix} \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \\
&\stackrel{\text{uu}}{=} \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{A}_1\boldsymbol{F} \\ -\boldsymbol{F} \\ -\boldsymbol{A}_2\boldsymbol{F} \end{Bmatrix}
\end{aligned} \tag{9.349}$$

9.2.4 Deformable Joint

The deformable joint applies an internal force and an internal moment to two nodes at a specified point that may be offset from the nodes; the force and the moment may depend on the relative position and velocity at the point of application and on the relative rotation vector and angular velocity of the nodes through a 6D constitutive law.

Although it may appear as a combination of the Deformable Hinge and the Deformable Displacement Joint, and despite some commonality of code, it is a bit more general because the internal force and moment may depend on the relative displacement and orientation, i.e. the displacements and the orientations are coupled. However, if a simple linear diagonal constitutive law is used, the same result with a bit less overhead is obtained by using a combination of a deformable hinge and a deformable displacement joint.

The deformable joint equations can be computed by combining those of the deformable hinge (Section 9.2.2) and of the deformable displacement joint (Section 9.2.3), considering that both the force $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ simultaneously depend on the relative displacement, $\tilde{\boldsymbol{d}}$, the relative orientation, $\boldsymbol{\theta}$, and their time derivatives, $\dot{\tilde{\boldsymbol{d}}}$ and $\boldsymbol{\omega}$.

Attached Deformable Joint

Internal force and moment:

$$\boldsymbol{F}_i = (-1)^i \boldsymbol{R}_{1h}\tilde{\boldsymbol{F}}\left(\tilde{\boldsymbol{d}}, \boldsymbol{\theta}, \dot{\tilde{\boldsymbol{d}}}, \boldsymbol{\omega}\right) \tag{9.350a}$$

$$\boldsymbol{M}_i = \boldsymbol{d}_i \times \boldsymbol{F}_i + (-1)^i \boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}\left(\tilde{\boldsymbol{d}}, \boldsymbol{\theta}, \dot{\tilde{\boldsymbol{d}}}, \boldsymbol{\omega}\right). \tag{9.350b}$$

Their linearization, after defining

$$\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}}\boldsymbol{R}_{1h}^T \tag{9.351a}$$

$$\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{R}_{1h}^T \tag{9.351b}$$

$$\boldsymbol{F}_{/\boldsymbol{\theta}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{F}}_{/\boldsymbol{\theta}}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_{1h}^T \tag{9.351c}$$

$$\boldsymbol{F}_{/\boldsymbol{\omega}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{F}}_{/\boldsymbol{\omega}}\boldsymbol{R}_{1h}^T \tag{9.351d}$$

$$\boldsymbol{M}_{/\tilde{\boldsymbol{d}}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}_{/\tilde{\boldsymbol{d}}}\boldsymbol{R}_{1h}^T \tag{9.351e}$$

$$\boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{R}_{1h}^T \tag{9.351f}$$

$$\boldsymbol{M}_{/\boldsymbol{\theta}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_{1h}^T \tag{9.351g}$$

$$\boldsymbol{M}_{/\boldsymbol{\omega}} = \boldsymbol{R}_{1h}\tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}}\boldsymbol{R}_{1h}^T, \tag{9.351h}$$

yields the viscous contribution to perturbation,

$$
\begin{aligned}
&\begin{bmatrix} \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \\ \boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} \\ -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \\ -\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} - \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} \end{bmatrix} \delta\dot{\boldsymbol{x}}_1 + \begin{bmatrix} -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times + \boldsymbol{F}_{/\boldsymbol{\omega}} \\ -\boldsymbol{d}_1 \times \left(\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times + \boldsymbol{M}_{/\boldsymbol{\omega}} \\ \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\omega}} \\ \boldsymbol{d}_2 \times \left(\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_1 \times - \boldsymbol{M}_{/\boldsymbol{\omega}} \end{bmatrix} \delta\dot{\boldsymbol{g}}_1 \\
&+ \begin{bmatrix} -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \\ -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} - \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} \\ \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} \\ \boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} \end{bmatrix} \delta\dot{\boldsymbol{x}}_2 + \begin{bmatrix} \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\omega}} \\ \boldsymbol{d}_1 \times \left(\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times - \boldsymbol{M}_{/\boldsymbol{\omega}} \\ -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times + \boldsymbol{F}_{/\boldsymbol{\omega}} \\ -\boldsymbol{d}_2 \times \left(\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{d}_2 \times + \boldsymbol{M}_{/\boldsymbol{\omega}} \end{bmatrix} \delta\dot{\boldsymbol{g}}_2 \\
&+ \begin{bmatrix} -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{\omega}_1 \times \\ -\left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\boldsymbol{\omega}_1 \times \\ \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{\omega}_1 \times \\ \left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\boldsymbol{\omega}_1 \times \end{bmatrix} \delta\boldsymbol{x}_1 \\
&+ \begin{bmatrix} -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times\right) - \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{\omega}_2 \times \\ -\left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times\right) - \left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\boldsymbol{\omega}} + \boldsymbol{M}_{/\boldsymbol{\omega}}\right)\boldsymbol{\omega}_2 \times \\ \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times\right) + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{\omega}_2 \times \\ \left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\left(\dot{\boldsymbol{d}}_1 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_1 \times\right) + \left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\boldsymbol{\omega}} + \boldsymbol{M}_{/\boldsymbol{\omega}}\right)\boldsymbol{\omega}_2 \times \end{bmatrix} \delta\boldsymbol{g}_1 \\
&+ \begin{bmatrix} \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{\omega}_1 \times \\ \left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\boldsymbol{\omega}_1 \times \\ -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\boldsymbol{\omega}_1 \times \\ -\left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\boldsymbol{\omega}_1 \times \end{bmatrix} \delta\boldsymbol{x}_2 \\
&+ \begin{bmatrix} \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times\right) + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{\omega}_2 \times \\ \left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times\right) + \left(\boldsymbol{d}_1 \times \boldsymbol{F}_{/\boldsymbol{\omega}} + \boldsymbol{M}_{/\boldsymbol{\omega}}\right)\boldsymbol{\omega}_2 \times \\ -\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}}\left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times\right) - \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{\omega}_2 \times \\ -\left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} + \boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}}\right)\left(\dot{\boldsymbol{d}}_2 \times - \boldsymbol{\omega}_1 \times \boldsymbol{d}_2 \times\right) + \left(\boldsymbol{d}_2 \times \boldsymbol{F}_{/\boldsymbol{\omega}} + \boldsymbol{M}_{/\boldsymbol{\omega}}\right)\boldsymbol{\omega}_2 \times \end{bmatrix} \delta\boldsymbol{g}_2
\end{aligned}
\tag{9.352}
$$

the elastic contribution to perturbation,

$$
\begin{aligned}
&\begin{bmatrix} \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \\ \boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}} \\ -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \\ -\boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}} \end{bmatrix} \delta\boldsymbol{x}_1 + \begin{bmatrix} -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times + \boldsymbol{F}_{/\boldsymbol{\theta}} \\ -\boldsymbol{d}_1 \times \left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times + \boldsymbol{M}_{/\boldsymbol{\theta}} \\ \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\theta}} \\ \boldsymbol{d}_2 \times \left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times - \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_1 \times - \boldsymbol{M}_{/\boldsymbol{\theta}} \end{bmatrix} \delta\boldsymbol{g}_1 \\
&+ \begin{bmatrix} -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \\ -\boldsymbol{d}_1 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}} \\ \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} \\ \boldsymbol{d}_2 \times \boldsymbol{F}_{/\tilde{\boldsymbol{d}}} + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}} \end{bmatrix} \delta\boldsymbol{x}_2 + \begin{bmatrix} \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\theta}} \\ \boldsymbol{d}_1 \times \left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times - \boldsymbol{M}_{/\boldsymbol{\theta}} \\ -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times + \boldsymbol{F}_{/\boldsymbol{\theta}} \\ -\boldsymbol{d}_2 \times \left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times - \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{d}_2 \times + \boldsymbol{M}_{/\boldsymbol{\theta}} \end{bmatrix} \delta\boldsymbol{g}_2
\end{aligned}
\tag{9.353}
$$

and a common contribution to perturbation,

$$
\begin{bmatrix} \boldsymbol{0} & \boldsymbol{F} \times & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{F} \times & \boldsymbol{d}_1 \times \boldsymbol{F} \times + \boldsymbol{M} \times & \boldsymbol{F} \times & -\boldsymbol{F} \times \boldsymbol{d}_2 \times \\ \boldsymbol{0} & -\boldsymbol{F} \times & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{d}_2 \times \boldsymbol{F} \times - \boldsymbol{M} \times & \boldsymbol{0} & \boldsymbol{F} \times \boldsymbol{d}_2 \times \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \overset{\text{uu}}{=} \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{d}_1 \times \boldsymbol{F} + \boldsymbol{M} \\ -\boldsymbol{F} \\ -\boldsymbol{d}_2 \times \boldsymbol{F} - \boldsymbol{M} \end{Bmatrix}
\tag{9.354}
$$

The contribution of Eq. (9.354) is always present. The contribution of Eq. (9.353) is present in the elastic and in the viscoelastic variants of the joint. The contribution of Eq. (9.352) is present in the viscous and in the viscoelastic variants of the joint.

Invariant Deformable Joint

The force and the moment, after defining

$$\boldsymbol{F} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}\left(\tilde{\boldsymbol{d}}, \dot{\tilde{\boldsymbol{d}}}, \boldsymbol{\theta}, \overline{\boldsymbol{\varpi}}\right) \tag{9.355a}$$

$$\boldsymbol{M} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}\left(\tilde{\boldsymbol{d}}, \dot{\tilde{\boldsymbol{d}}}, \boldsymbol{\theta}, \overline{\boldsymbol{\varpi}}\right), \tag{9.355b}$$

are

$$\boldsymbol{F}_1 = -\boldsymbol{F} \tag{9.356a}$$

$$\boldsymbol{M}_1 = -\left(\boldsymbol{f}_1 \times + \hat{\boldsymbol{I}}\boldsymbol{d} \times\right)\boldsymbol{F} - \boldsymbol{M} \tag{9.356b}$$

$$\boldsymbol{F}_2 = \boldsymbol{F} \tag{9.356c}$$

$$\boldsymbol{M}_2 = \left(\boldsymbol{f}_2 \times - \hat{\boldsymbol{I}}^T\boldsymbol{d} \times\right)\boldsymbol{F} + \boldsymbol{M}. \tag{9.356d}$$

Their linearization, after defining

$$\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}_{/\tilde{\boldsymbol{d}}}\hat{\boldsymbol{R}}^T \tag{9.357a}$$

$$\boldsymbol{F}_{/\dot{\tilde{\boldsymbol{d}}}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{R}}^T \tag{9.357b}$$

$$\boldsymbol{F}_{/\boldsymbol{\theta}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}_{/\boldsymbol{\theta}}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_{1h}^T \tag{9.357c}$$

$$\boldsymbol{F}_{/\boldsymbol{\omega}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{F}}_{/\boldsymbol{\omega}}\hat{\boldsymbol{R}}^T \tag{9.357d}$$

$$\boldsymbol{M}_{/\tilde{\boldsymbol{d}}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\tilde{\boldsymbol{d}}}\hat{\boldsymbol{R}}^T \tag{9.357e}$$

$$\boldsymbol{M}_{/\dot{\tilde{\boldsymbol{d}}}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\dot{\tilde{\boldsymbol{d}}}}\hat{\boldsymbol{R}}^T \tag{9.357f}$$

$$\boldsymbol{M}_{/\boldsymbol{\theta}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\boldsymbol{\theta}}\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_{1h}^T \tag{9.357g}$$

$$\boldsymbol{M}_{/\boldsymbol{\omega}} = \hat{\boldsymbol{R}}\tilde{\boldsymbol{M}}_{/\boldsymbol{\omega}}\hat{\boldsymbol{R}}^T, \tag{9.357h}$$

and exploiting the definitions of Eqs (9.346) and

$$\boldsymbol{C}_1 = \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \times \hat{\boldsymbol{I}}^T \tag{9.358a}$$

$$\boldsymbol{C}_2 = \left(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\right) \times \hat{\boldsymbol{I}}, \tag{9.358b}$$

yields

$$
\begin{aligned}
&\begin{bmatrix}
\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}} \\
\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{M}_{/\boldsymbol{\omega}} \\
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{F}_{/\boldsymbol{\omega}} \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{M}_{/\boldsymbol{\omega}}
\end{bmatrix}
\left\{\begin{matrix} \delta\dot{\boldsymbol{x}}_1 \\ \delta\boldsymbol{\omega}_1 \end{matrix}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{F}_{/\boldsymbol{\omega}} \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{M}_{/\boldsymbol{\omega}} \\
\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}} \\
\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{M}_{/\boldsymbol{\omega}}
\end{bmatrix}
\left\{\begin{matrix} \delta\dot{\boldsymbol{x}}_2 \\ \delta\boldsymbol{\omega}_2 \end{matrix}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 - \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_1 \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_1\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 - \boldsymbol{M}_{/\boldsymbol{\omega}}\boldsymbol{C}_1 \\
\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_1 \\
\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_1\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_1 + \boldsymbol{M}_{/\boldsymbol{\omega}}\boldsymbol{C}_1
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \end{matrix}\right\} \\
&+\begin{bmatrix}
\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 - \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_2 \\
\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_2\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 - \boldsymbol{M}_{/\boldsymbol{\omega}}\boldsymbol{C}_2 \\
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_2 \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\boldsymbol{C}_2\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{B}_2 + \boldsymbol{M}_{/\boldsymbol{\omega}}\boldsymbol{C}_2
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{matrix}\right\} \\
&+\begin{bmatrix}
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}} \\
\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_1\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{M}_{/\boldsymbol{\theta}} \\
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{F}_{/\boldsymbol{\theta}} \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_2\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{M}_{/\boldsymbol{\theta}}
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \end{matrix}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{F}_{/\boldsymbol{\theta}} \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{M}_{/\boldsymbol{\theta}} \\
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}} \\
\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{M}_{/\boldsymbol{\theta}}
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{matrix}\right\} \\
&+\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}\boldsymbol{F}\times & \left(\boldsymbol{A}_1\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \hat{\boldsymbol{I}}_{1(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\left(\boldsymbol{A}_2\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \left(\hat{\boldsymbol{I}}^T\right)_{1(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_1 \\ \boldsymbol{\theta}_{1\delta} \end{matrix}\right\} \\
&+\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}\boldsymbol{F}\times & \left(\boldsymbol{A}_1\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times - \hat{\boldsymbol{I}}_{2(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\left(\boldsymbol{A}_2\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}} + \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times + \left(\hat{\boldsymbol{I}}^T\right)_{2(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\left\{\begin{matrix} \delta\boldsymbol{x}_2 \\ \boldsymbol{\theta}_{2\delta} \end{matrix}\right\} \\
&= \left\{\begin{matrix} \boldsymbol{F} \\ \boldsymbol{A}_1\boldsymbol{F} + \boldsymbol{M} \\ -\boldsymbol{F} \\ -\boldsymbol{A}_2\boldsymbol{F} - \boldsymbol{M} \end{matrix}\right\}
\end{aligned}
\tag{9.359}
$$

The updated-updated form is

$$
\begin{aligned}
&\begin{bmatrix}
\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}} \\
\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{M}_{/\boldsymbol{\omega}} \\
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{F}_{/\boldsymbol{\omega}} \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{M}_{/\boldsymbol{\omega}}
\end{bmatrix}
\left\{\begin{array}{c} \delta\dot{\boldsymbol{x}}_1 \\ \delta\dot{\boldsymbol{g}}_1 \end{array}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{F}_{/\boldsymbol{\omega}} \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{M}_{/\boldsymbol{\omega}} \\
\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}} \\
\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}} & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\omega}}\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{M}_{/\boldsymbol{\omega}}
\end{bmatrix}
\left\{\begin{array}{c} \delta\dot{\boldsymbol{x}}_2 \\ \delta\dot{\boldsymbol{g}}_2 \end{array}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 - \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1 \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 - \boldsymbol{M}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1 \\
\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1 \\
\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_1 + \boldsymbol{M}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_1
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \end{array}\right\} \\
&+\begin{bmatrix}
\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 - \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2 \\
\boldsymbol{A}_1\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2\right) - \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 - \boldsymbol{M}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2 \\
-\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2 \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{\omega}}\times & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 + \boldsymbol{F}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2\right) + \boldsymbol{M}_{/\dot{\boldsymbol{d}}}\hat{\boldsymbol{B}}_2 + \boldsymbol{M}_{/\boldsymbol{\omega}}\hat{\boldsymbol{C}}_2
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{array}\right\} \\
&+\begin{bmatrix}
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}} \\
\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_1\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{M}_{/\boldsymbol{\theta}} \\
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{F}_{/\boldsymbol{\theta}} \\
-\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_2\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_1^T - \boldsymbol{M}_{/\boldsymbol{\theta}}
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \end{array}\right\} \\
&+\begin{bmatrix}
-\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{F}_{/\boldsymbol{\theta}} \\
-\boldsymbol{A}_1\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & -\boldsymbol{A}_1\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) - \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T - \boldsymbol{M}_{/\boldsymbol{\theta}} \\
\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}} \\
\boldsymbol{A}_2\boldsymbol{F}_{/\tilde{\boldsymbol{d}}} & \boldsymbol{A}_2\left(\boldsymbol{F}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{F}_{/\boldsymbol{\theta}}\right) + \boldsymbol{M}_{/\tilde{\boldsymbol{d}}}\boldsymbol{A}_2^T + \boldsymbol{M}_{/\boldsymbol{\theta}}
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{array}\right\} \\
&+\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}\boldsymbol{F}\times & \left(\boldsymbol{A}_1\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \hat{\boldsymbol{I}}_{1(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}}^T \\
-\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\left(\boldsymbol{A}_2\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}}^T - \hat{\boldsymbol{I}}^T\boldsymbol{F}\times\boldsymbol{f}_1\times - \left(\hat{\boldsymbol{I}}^T\right)_{1(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_1 \\ \delta\boldsymbol{g}_1 \end{array}\right\} \\
&+\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}\boldsymbol{F}\times & \left(\boldsymbol{A}_1\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}} - \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times - \hat{\boldsymbol{I}}_{2(\boldsymbol{d}\times\boldsymbol{F})} \\
\boldsymbol{0} & -\boldsymbol{F}\times\hat{\boldsymbol{I}} \\
\hat{\boldsymbol{I}}^T\boldsymbol{F}\times & -\left(\boldsymbol{A}_2\boldsymbol{F}\times + \boldsymbol{M}\times\right)\hat{\boldsymbol{I}} + \hat{\boldsymbol{I}}\boldsymbol{F}\times\boldsymbol{f}_2\times + \left(\hat{\boldsymbol{I}}^T\right)_{2(\boldsymbol{d}\times\boldsymbol{F})}
\end{bmatrix}
\left\{\begin{array}{c} \delta\boldsymbol{x}_2 \\ \delta\boldsymbol{g}_2 \end{array}\right\} \\
&= \left\{\begin{array}{c} \boldsymbol{F} \\ \boldsymbol{A}_1\boldsymbol{F} + \boldsymbol{M} \\ -\boldsymbol{F} \\ -\boldsymbol{A}_2\boldsymbol{F} - \boldsymbol{M} \end{array}\right\}
\end{aligned}
\tag{9.360}
$$

with

$$\hat{\boldsymbol{C}}_1 = \boldsymbol{C}_1 + \boldsymbol{\omega}_1\times \tag{9.361a}$$

$$\hat{\boldsymbol{C}}_2 = \boldsymbol{C}_2 - \boldsymbol{\omega}_2\times\,. \tag{9.361b}$$

Note that

$$\hat{\boldsymbol{C}}_1 = (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \hat{\boldsymbol{I}}^T + \boldsymbol{\omega}_1 \times \qquad = \boldsymbol{\omega}_1 \times \hat{\boldsymbol{I}} + \boldsymbol{\omega}_2 \times \hat{\boldsymbol{I}}^T \tag{9.362a}$$
$$\hat{\boldsymbol{C}}_2 = (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \hat{\boldsymbol{I}} - \boldsymbol{\omega}_2 \times \qquad = -\boldsymbol{\omega}_1 \times \hat{\boldsymbol{I}} - \boldsymbol{\omega}_2 \times \hat{\boldsymbol{I}}^T, \tag{9.362b}$$

so $\hat{\boldsymbol{C}}_2 = -\hat{\boldsymbol{C}}_1$.

9.2.5 Deformable Axial Joint

Strain:

$$\boldsymbol{\theta} = \mathrm{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{1h}^T \boldsymbol{R}_{2h}\right)\right) \tag{9.363}$$
$$\epsilon = \tilde{\boldsymbol{e}}_z \cdot \boldsymbol{\theta} \tag{9.364}$$

Strain rate:

$$\dot{\epsilon} = \tilde{\boldsymbol{e}}_z \cdot \boldsymbol{R}_{1h}^T (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \tag{9.365}$$

Virtual perturbation/linearization of strain:

$$\delta\epsilon = \tilde{\boldsymbol{e}}_z \cdot \boldsymbol{R}_{1h}^T (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) = \boldsymbol{e}_z^T (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta}) \tag{9.366}$$

Linearization of strain rate:

$$\delta\dot{\epsilon} = \tilde{\boldsymbol{e}}_z \cdot \boldsymbol{R}_{1h}^T ((\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \boldsymbol{\theta}_{1\delta} + \delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1) = \boldsymbol{e}_z^T ((\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \boldsymbol{\theta}_{1\delta} + \delta\boldsymbol{\omega}_2 - \delta\boldsymbol{\omega}_1) \tag{9.367}$$

Virtual work:

$$\delta\mathcal{L} = \delta\epsilon m(\epsilon, \dot{\epsilon}) = (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta})^T \boldsymbol{R}_{1h} \tilde{\boldsymbol{e}}_z m = (\boldsymbol{\theta}_{2\delta} - \boldsymbol{\theta}_{1\delta})^T \boldsymbol{e}_z m \tag{9.368}$$

Loads (residual contribution):

$$\boldsymbol{m}_1 = \boldsymbol{e}_z m \tag{9.369a}$$
$$\boldsymbol{m}_2 = -\boldsymbol{e}_z m \tag{9.369b}$$

Linearization:

$$\delta(\boldsymbol{e}_z m) = -m\boldsymbol{e}_z \times \boldsymbol{\theta}_{1\delta} + \boldsymbol{e}_z \left(m_{/\epsilon}\delta\epsilon + m_{/\dot{\epsilon}}\delta\dot{\epsilon}\right) \tag{9.370}$$

Jacobian matrix contributions:

$$\begin{aligned} & m \begin{bmatrix} \boldsymbol{e}_z \times \\ -\boldsymbol{e}_z \times \end{bmatrix} \boldsymbol{\theta}_{1\delta} + m_{/\epsilon} \begin{bmatrix} \boldsymbol{e}_z\boldsymbol{e}_z^T & -\boldsymbol{e}_z\boldsymbol{e}_z^T \\ -\boldsymbol{e}_z\boldsymbol{e}_z^T & \boldsymbol{e}_z\boldsymbol{e}_z^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{\theta}_{1\delta} \\ \boldsymbol{\theta}_{2\delta} \end{Bmatrix} \\ & + m_{/\dot{\epsilon}} \left(\begin{bmatrix} \boldsymbol{e}_z\boldsymbol{e}_z^T & -\boldsymbol{e}_z\boldsymbol{e}_z^T \\ -\boldsymbol{e}_z\boldsymbol{e}_z^T & \boldsymbol{e}_z\boldsymbol{e}_z^T \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{\omega}_1 \\ \delta\boldsymbol{\omega}_2 \end{Bmatrix} + \begin{bmatrix} -\boldsymbol{e}_z\boldsymbol{e}_z^T (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \\ \boldsymbol{e}_z\boldsymbol{e}_z^T (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \end{bmatrix} \boldsymbol{\theta}_{1\delta} \right) = \begin{Bmatrix} \boldsymbol{e}_z \\ -\boldsymbol{e}_z \end{Bmatrix} m \end{aligned} \tag{9.371}$$

With the updated-updated approximation:

$$\begin{aligned} & m \begin{bmatrix} \boldsymbol{e}_z \times \\ -\boldsymbol{e}_z \times \end{bmatrix} \delta\boldsymbol{g}_1 + m_{/\epsilon} \begin{bmatrix} \boldsymbol{e}_z\boldsymbol{e}_z^T & -\boldsymbol{e}_z\boldsymbol{e}_z^T \\ -\boldsymbol{e}_z\boldsymbol{e}_z^T & \boldsymbol{e}_z\boldsymbol{e}_z^T \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \\ & + m_{/\dot{\epsilon}} \left(\begin{bmatrix} \boldsymbol{e}_z\boldsymbol{e}_z^T & -\boldsymbol{e}_z\boldsymbol{e}_z^T \\ -\boldsymbol{e}_z\boldsymbol{e}_z^T & \boldsymbol{e}_z\boldsymbol{e}_z^T \end{bmatrix} \begin{Bmatrix} \delta\dot{\boldsymbol{g}}_1 \\ \delta\dot{\boldsymbol{g}}_2 \end{Bmatrix} + \begin{bmatrix} \boldsymbol{e}_z\boldsymbol{e}_z^T\boldsymbol{\omega}_2 \times & -\boldsymbol{e}_z\boldsymbol{e}_z^T\boldsymbol{\omega}_2 \times \\ -\boldsymbol{e}_z\boldsymbol{e}_z^T\boldsymbol{\omega}_2 \times & \boldsymbol{e}_z\boldsymbol{e}_z^T\boldsymbol{\omega}_2 \times \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{g}_1 \\ \delta\boldsymbol{g}_2 \end{Bmatrix} \right) \overset{\mathrm{uu}}{=} \begin{Bmatrix} \boldsymbol{e}_z \\ -\boldsymbol{e}_z \end{Bmatrix} m \end{aligned} \tag{9.372}$$

Note: $\boldsymbol{e}_z\boldsymbol{e}_z^T\boldsymbol{\omega}_2 \times \; = \boldsymbol{e}_z (\boldsymbol{e}_z \times \boldsymbol{\omega}_2)^T$.

9.3 Viscous Body

This element implements the behavior of a viscous body, namely a force and a moment that depend on the absolute linear and angular velocity of a node, projected in the reference frame of the node. This element allows, for example, to implement the aerodynamics of a flight-mechanics rigid-body model, whose aerodynamic forces and moments depend on the absolute linear and angular velocity, projected in the reference frame of the body, by means of an appropriate constitutive law.

The force and moment are defined as

$$\boldsymbol{f} = \boldsymbol{R}\boldsymbol{R}_h\tilde{\boldsymbol{f}} \tag{9.373a}$$

$$\boldsymbol{m} = \boldsymbol{R}\boldsymbol{R}_h\tilde{\boldsymbol{m}} + \boldsymbol{o}\times\boldsymbol{f} \tag{9.373b}$$

where

$$\boldsymbol{o} = \boldsymbol{R}\tilde{\boldsymbol{o}}, \tag{9.374}$$

$$\tilde{\boldsymbol{f}} = \tilde{\boldsymbol{f}}\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}\right) \tag{9.375a}$$

$$\tilde{\boldsymbol{m}} = \tilde{\boldsymbol{m}}\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}\right) \tag{9.375b}$$

and

$$\tilde{\boldsymbol{v}} = \boldsymbol{R}_h^T\boldsymbol{R}^T\left(\dot{\boldsymbol{x}} + \boldsymbol{\omega}\times\boldsymbol{o}\right) \tag{9.376a}$$

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{R}_h^T\boldsymbol{R}^T\boldsymbol{\omega}. \tag{9.376b}$$

The linearization of the force and moment yields

$$\delta\tilde{\boldsymbol{f}} = \tilde{\boldsymbol{f}}_{/\tilde{\boldsymbol{v}}}\delta\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{f}}_{/\tilde{\boldsymbol{\omega}}}\delta\tilde{\boldsymbol{\omega}} \tag{9.377a}$$

$$\delta\tilde{\boldsymbol{m}} = \tilde{\boldsymbol{m}}_{/\tilde{\boldsymbol{v}}}\delta\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{m}}_{/\tilde{\boldsymbol{\omega}}}\delta\tilde{\boldsymbol{\omega}} \tag{9.377b}$$

with

$$\delta\tilde{\boldsymbol{v}} = \boldsymbol{R}_h^T\boldsymbol{R}^T\left(\delta\dot{\boldsymbol{x}} + \delta\boldsymbol{\omega}\times\boldsymbol{o} + \boldsymbol{\omega}\times\boldsymbol{\theta}_\delta\times\boldsymbol{o}\right) \quad \stackrel{\text{uu}}{=} \boldsymbol{R}_h^T\boldsymbol{R}^T\left(\delta\dot{\boldsymbol{x}} - \boldsymbol{o}\times\delta\dot{\boldsymbol{g}} + \left(\boldsymbol{o}\times\boldsymbol{\omega}\right)\times\delta\boldsymbol{g}\right) \tag{9.378a}$$

$$\delta\tilde{\boldsymbol{\omega}} = \boldsymbol{R}_h^T\boldsymbol{R}^T\left(\delta\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{\theta}_\delta\right) \quad \stackrel{\text{uu}}{=} \boldsymbol{R}_h^T\boldsymbol{R}^T\delta\dot{\boldsymbol{g}} \tag{9.378b}$$

The linearization of the nodal forces and moments yields

$$\delta\boldsymbol{f} = \boldsymbol{\theta}_\delta\times\boldsymbol{f} + \boldsymbol{R}\boldsymbol{R}_h\delta\tilde{\boldsymbol{f}} \tag{9.379a}$$

$$\delta\boldsymbol{m} = \boldsymbol{\theta}_\delta\times\boldsymbol{m} + \boldsymbol{R}\boldsymbol{R}_h\delta\tilde{\boldsymbol{m}} + \boldsymbol{o}\times\boldsymbol{R}\boldsymbol{R}_h\delta\tilde{\boldsymbol{f}}, \tag{9.379b}$$

namely

$$\begin{aligned}
&\left[\begin{array}{cc} \boldsymbol{f}_{/\tilde{v}} & \boldsymbol{f}_{/\tilde{\omega}} - \boldsymbol{f}_{/\tilde{v}}\boldsymbol{o}\times \\ \boldsymbol{m}_{/\tilde{v}} + \boldsymbol{o}\times\boldsymbol{f}_{/\tilde{v}} & \boldsymbol{m}_{/\tilde{\omega}} - \boldsymbol{m}_{/\tilde{v}}\boldsymbol{o}\times + \boldsymbol{o}\times\left(\boldsymbol{f}_{/\tilde{\omega}} - \boldsymbol{f}_{/\tilde{v}}\boldsymbol{o}\times\right) \end{array}\right]\left\{\begin{array}{c} \delta\dot{\boldsymbol{x}} \\ \delta\boldsymbol{\omega} \end{array}\right\} \\
&\quad + \left[\begin{array}{c} \boldsymbol{f}_{/\tilde{\omega}}\boldsymbol{\omega}\times - \boldsymbol{f}_{/\tilde{v}}\boldsymbol{\omega}\times\boldsymbol{o}\times \\ \boldsymbol{m}_{/\tilde{\omega}}\boldsymbol{\omega}\times - \boldsymbol{m}_{/\tilde{v}}\boldsymbol{\omega}\times\boldsymbol{o}\times + \boldsymbol{o}\times\left(\boldsymbol{f}_{/\tilde{\omega}}\boldsymbol{\omega}\times - \boldsymbol{f}_{/\tilde{v}}\boldsymbol{\omega}\times\boldsymbol{o}\times\right) \end{array}\right]\boldsymbol{\theta}_\delta \\
&\qquad + \left[\begin{array}{c} -\boldsymbol{f}\times \\ -\boldsymbol{m}\times \end{array}\right]\boldsymbol{\theta}_\delta = \left\{\begin{array}{c} \delta\boldsymbol{f} \\ \delta\boldsymbol{m} \end{array}\right\}
\end{aligned} \tag{9.380}$$

where

$$\boldsymbol{f}_{/\tilde{\boldsymbol{v}}} = \boldsymbol{R}\boldsymbol{R}_h \tilde{\boldsymbol{f}}_{/\tilde{\boldsymbol{v}}} \boldsymbol{R}_h^T \boldsymbol{R}^T \tag{9.381a}$$

$$\boldsymbol{f}_{/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}\boldsymbol{R}_h \tilde{\boldsymbol{f}}_{/\tilde{\boldsymbol{\omega}}} \boldsymbol{R}_h^T \boldsymbol{R}^T \tag{9.381b}$$

$$\boldsymbol{m}_{/\tilde{\boldsymbol{v}}} = \boldsymbol{R}\boldsymbol{R}_h \tilde{\boldsymbol{m}}_{/\tilde{\boldsymbol{v}}} \boldsymbol{R}_h^T \boldsymbol{R}^T \tag{9.381c}$$

$$\boldsymbol{m}_{/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}\boldsymbol{R}_h \tilde{\boldsymbol{m}}_{/\tilde{\boldsymbol{\omega}}} \boldsymbol{R}_h^T \boldsymbol{R}^T \tag{9.381d}$$

The updated-updated approximation yields

$$\begin{aligned} &\left[\begin{array}{cc} \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} & \boldsymbol{f}_{/\tilde{\boldsymbol{\omega}}} - \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} \boldsymbol{o} \times \\ \boldsymbol{m}_{/\tilde{\boldsymbol{v}}} + \boldsymbol{o} \times \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} & \boldsymbol{m}_{/\tilde{\boldsymbol{\omega}}} - \boldsymbol{m}_{/\tilde{\boldsymbol{v}}} \boldsymbol{o} \times + \boldsymbol{o} \times \left(\boldsymbol{f}_{/\tilde{\boldsymbol{\omega}}} - \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} \boldsymbol{o} \times \right) \end{array} \right] \left\{ \begin{array}{c} \delta \dot{\boldsymbol{x}} \\ \delta \dot{\boldsymbol{g}} \end{array} \right\} \\ &\qquad + \left[\begin{array}{c} \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} \left(\boldsymbol{o} \times \boldsymbol{\omega} \right) \times \\ \left(\boldsymbol{m}_{/\tilde{\boldsymbol{v}}} + \boldsymbol{o} \times \boldsymbol{f}_{/\tilde{\boldsymbol{v}}} \right) \left(\boldsymbol{o} \times \boldsymbol{\omega} \right) \times \end{array} \right] \delta \boldsymbol{g} + \left[\begin{array}{c} -\boldsymbol{f} \times \\ -\boldsymbol{m} \times \end{array} \right] \delta \boldsymbol{g} = \left\{ \begin{array}{c} \delta \boldsymbol{f} \\ \delta \boldsymbol{m} \end{array} \right\} \end{aligned} \tag{9.382}$$

9.4 Modal Element

9.4.1 Kinematics

Position of an arbitrary point P

$$\boldsymbol{x}_P = \boldsymbol{x}_0 + \boldsymbol{f}_P + \boldsymbol{u}_P \tag{9.383}$$

where $\boldsymbol{x}_0$ is the position of the point that describes the global motion of the body, $\boldsymbol{f}_P$ is the relative position of the point when the body is undeformed, and $\boldsymbol{u}_P$ is the relative displacement of the point when the body is deformed.

It can be rewritten as

$$\boldsymbol{x}_P = \boldsymbol{x}_0 + \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \tilde{\boldsymbol{u}}_P \right) \tag{9.384}$$

where $\boldsymbol{R}_0$ is the global orientation matrix of the body, and the *tilde* $(\tilde{\cdot})$ indicates entities expressed in the reference frame attached to the body.

The deformation of the body is expressed by a linear combination of M displacement (and rotation, for those models that consider them, like beam trusses) shapes

$$\tilde{\boldsymbol{u}}_P = \sum_{j=1,M} \boldsymbol{U}_{Pj} q_j = \boldsymbol{U}_P \boldsymbol{q} \tag{9.385}$$

where $\boldsymbol{U}_{Pj}$ is the vector containing the components of the j-th displacement shape related to point P, and q_j is the j-th mode multiplier.

The orientation of the generic point P is

$$\boldsymbol{R}_P = \boldsymbol{R}_0 \tilde{\boldsymbol{R}}_P \tag{9.386}$$

and, assuming a representation of the relative orientation by a linear combination of rotation shapes

$$\tilde{\boldsymbol{\phi}} = \sum_{j=1,M} \boldsymbol{V}_{Pj} q_j = \boldsymbol{V}_P \boldsymbol{q} \tag{9.387}$$

it results in a linearized orientation

$$\boldsymbol{R}_P \cong \boldsymbol{R}_0 \left(\boldsymbol{I} + \left(\boldsymbol{V}_P \boldsymbol{q} \right) \times \right) \tag{9.388}$$

which is no longer orthogonal, because of matrix

$$\tilde{\boldsymbol{R}}_P = \boldsymbol{I} + (\boldsymbol{V}_P \boldsymbol{q}) \times \tag{9.389}$$

which represents a linearized rotation.

The first and second derivatives of position and orientation yield:

$$\dot{\boldsymbol{x}}_P = \dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{R}_0 \boldsymbol{U}_P \dot{\boldsymbol{q}} \tag{9.390a}$$

$$\boldsymbol{\omega}_P = \boldsymbol{\omega}_0 + \boldsymbol{R}_0 \boldsymbol{V}_P \dot{\boldsymbol{q}} \tag{9.390b}$$

$$\begin{aligned}\ddot{\boldsymbol{x}}_P = {} & \ddot{\boldsymbol{x}}_0 + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) \\ & + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_P \dot{\boldsymbol{q}} + \boldsymbol{R}_0 \boldsymbol{U}_P \ddot{\boldsymbol{q}}\end{aligned} \tag{9.390c}$$

$$\dot{\boldsymbol{\omega}}_P = \dot{\boldsymbol{\omega}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_P \dot{\boldsymbol{q}} + \boldsymbol{R}_0 \boldsymbol{V}_P \ddot{\boldsymbol{q}} \tag{9.390d}$$

The virtual perturbation of the position and orientation of the generic point P are:

$$\delta \boldsymbol{x}_P = \delta \boldsymbol{x}_0 + \delta \boldsymbol{\phi}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{R}_0 \boldsymbol{U}_P \delta \boldsymbol{q} \tag{9.391a}$$

$$\delta \boldsymbol{\phi}_P = \delta \boldsymbol{\phi}_0 + \boldsymbol{R}_0 \boldsymbol{V}_P \delta \boldsymbol{q} \tag{9.391b}$$

Without significant losses in generality, from now on it is assumed that the structure of the problem is given in form of lumped inertia parameters in specific points, corresponding to FEM nodes, and that the position of each node corresponds to the center of mass of each lumped mass. A model made of N FEM nodes is considered. The nodal mass of the i-th FEM node is

$$\boldsymbol{M}_i = \begin{bmatrix} m_i \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_i \end{bmatrix} \tag{9.392}$$

There is no strict requirement for matrix $\boldsymbol{J}_i$ to be diagonal.

The inertia forces and moments acting on each FEM node are:

$$\boldsymbol{F}_i = -m_i \ddot{\boldsymbol{x}}_i \tag{9.393a}$$

$$\boldsymbol{C}_i = -\boldsymbol{R}_i \boldsymbol{J}_i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \tag{9.393b}$$

and the virtual work done by the inertia forces is

$$\delta L = \sum_{i=1,N} \left(\delta \boldsymbol{x}_i^T \boldsymbol{F}_i + \delta \boldsymbol{\phi}_i^T \boldsymbol{C}_i \right) \tag{9.394}$$

which results in

$$\delta L = \left\{ \begin{array}{c} \delta \boldsymbol{x}_0 \\ \delta \boldsymbol{\phi}_0 \\ \delta \boldsymbol{q} \end{array} \right\}^T \left(\begin{bmatrix} \boldsymbol{M}_{xx} & \boldsymbol{M}_{x\phi} & \boldsymbol{M}_{xq} \\ & \boldsymbol{M}_{\phi\phi} & \boldsymbol{M}_{\phi q} \\ \text{sym.} & & \boldsymbol{M}_{qq} \end{bmatrix} \left\{ \begin{array}{c} \ddot{\boldsymbol{x}}_0 \\ \dot{\boldsymbol{\omega}}_0 \\ \ddot{\boldsymbol{q}} \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{F}_x \\ \boldsymbol{F}_\phi \\ \boldsymbol{F}_q \end{array} \right\} \right) \tag{9.395}$$

with

$$\boldsymbol{M}_{xx} = \boldsymbol{I} \sum_{i=1,N} m_i \tag{9.396a}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0 \sum_{i=1,N} m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times^T \boldsymbol{R}_0^T \tag{9.396b}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0 \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{9.396c}$$

$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0 \sum_{i=1,N} \left(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times^T + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \right) \boldsymbol{R}_0^T \tag{9.396d}$$

$$\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0 \sum_{i=1,N} \left(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \boldsymbol{U}_i + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{V}_i \right) \tag{9.396e}$$

$$\boldsymbol{M}_{qq} = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{V}_i \right) \tag{9.396f}$$

$$\boldsymbol{F}_x = \sum_{i=1,N} m_i \left(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \right) \tag{9.396g}$$

$$\begin{aligned} \boldsymbol{F}_\phi = \sum_{i=1,N} \boldsymbol{R}_0 \Big(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \left(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \right) \\ + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{R}_0^T \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_i \dot{\boldsymbol{q}} \Big) \end{aligned} \tag{9.396h}$$

$$\begin{aligned} \boldsymbol{F}_q = \sum_{i=1,N} \Big(m_i \boldsymbol{U}_i^T \boldsymbol{R}_0^T \Big(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \\ + \boldsymbol{V}_i^T \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{R}_0^T \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_i \dot{\boldsymbol{q}} \Big) \end{aligned} \tag{9.396i}$$

The $\boldsymbol{M}_{jk}$ terms can be rewritten to highlight contributions of order 0, 1, and higher:

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\left(\sum_{i=1,N} m_i\right) \tag{9.397a}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0\left(\left(\sum_{i=1,N} m_i\tilde{\boldsymbol{f}}_i\right) + \left(\left(\sum_{i=1,N} m_i\boldsymbol{U}_i\right)\boldsymbol{q}\right)\right) \times^T \boldsymbol{R}_0^T \tag{9.397b}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0\left(\sum_{i=1,N} m_i\boldsymbol{U}_i\right) \tag{9.397c}$$

$$\begin{aligned}\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0\Bigg(&\sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i\right)\\ &+ \sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i \times (\boldsymbol{U}_i\boldsymbol{q}) \times^T + m_i(\boldsymbol{U}_i\boldsymbol{q}) \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T + (\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i\right)\\ &+ \sum_{i=1,N}\left(m_i(\boldsymbol{U}_i\boldsymbol{q}) \times (\boldsymbol{U}_i\boldsymbol{q}) \times^T + (\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T\right)\Bigg)\boldsymbol{R}_0^T\end{aligned} \tag{9.397d}$$

$$\begin{aligned}\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0\Bigg(&\sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+ \sum_{i=1,N}\left(m_i(\boldsymbol{U}_i\boldsymbol{q}) \times \boldsymbol{U}_i + \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T \boldsymbol{V}_i + (\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+ \sum_{i=1,N}(\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T \boldsymbol{V}_i\Bigg)\end{aligned} \tag{9.397e}$$

$$\begin{aligned}\boldsymbol{M}_{qq} = &\sum_{i=1,N}\left(m_i\boldsymbol{U}_i^T\boldsymbol{U}_i + \boldsymbol{V}_i^T\boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+ \sum_{i=1,N}\left(\boldsymbol{V}_i^T\boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T \boldsymbol{V}_i + \boldsymbol{V}_i^T(\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+ \sum_{i=1,N}\boldsymbol{V}_i^T(\boldsymbol{V}_i\boldsymbol{q}) \times \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q}) \times^T \boldsymbol{V}_i\end{aligned} \tag{9.397f}$$

9.4.2 Physics: Orthogonality

Some noteworthy entities appear in the above equations, which may partially simplify under special circumstances.

The overall mass of the body

$$m = \sum_{i=1,N} m_i \tag{9.398}$$

The static (first order) inertia moment

$$\boldsymbol{S}_{x\phi} = \sum_{i=1,N} m_i\tilde{\boldsymbol{f}}_i \tag{9.399}$$

vanishes if point $\boldsymbol{x}_0$ is the center of mass of the undeformed body.

Similarly, the static (first order) inertia moment computed with the modal displacement shapes

$$\boldsymbol{S}_{xq} = \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{9.400}$$

vanishes if the mode shapes have been inertially decoupled from the rigid body displacements. In fact, the decoupling of the rigid and the deformable modes is expressed by

$$\begin{aligned} &\sum_{i=1,N} \boldsymbol{x}_r^T m_i \boldsymbol{U}_i = \\ &\boldsymbol{x}_r^T \sum_{i=1,N} m_i \boldsymbol{U}_i = 0 \end{aligned} \tag{9.401}$$

where $\boldsymbol{x}_r^T$ describes three independent rigid translations, which, for the arbitrariety of $\boldsymbol{x}_r$, implies the above Equation (9.400).

In the same case, also the zero-order terms of the coupling between the rigid body rotations and the modal variables,

$$\boldsymbol{S}_{\phi q} = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{9.402}$$

also vanishes. In fact, the decoupling of the rigid and the deformable modes is expressed by

$$\begin{aligned} &\sum_{i=1,N} \left(m_i \boldsymbol{\phi}_r^T \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{\phi}_r^T \boldsymbol{J}_i \boldsymbol{V}_i \right) = \\ &\quad \boldsymbol{\phi}_r^T \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) = 0 \end{aligned} \tag{9.403}$$

where $\boldsymbol{\phi}_r^T$ describes three independent rigid rotations, and $\boldsymbol{\phi}_r^T \tilde{\boldsymbol{f}}_i \times$ describes the corresponding displacements, which, for the arbitrariety of $\boldsymbol{\phi}_r$, implies the above Equation (9.402).

The second order inertia moment is

$$\boldsymbol{J} = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i \right) \tag{9.404}$$

It results in a diagonal matrix if the orientation of the body is aligned with the principal inertia axes.

The modal mass matrix is

$$\boldsymbol{m} = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{9.405}$$

It is diagonal if only the normal modes are considered.

9.4.3 Simplifications

The problem, as stated up to now, already contains some simplifications. First of all, those related to the lumped inertia model of a continuum; moreover, those related to the mode superposition to describe the straining of the body which, in the case of the FEM node rotation, yields a non-orthogonal linearized rotation matrix.

Further simplifications are usually accepted in common modeling practice, where some of the higher order terms are simply discarded.

When only the 0-th order coefficients are used in matrices M_{uv}, the dynamics of the body are written referred to the undeformed shape. This approximation can be quite drastic, but in some cases it may be reasonable, if the reference straining, represented by $\boldsymbol{q}$, remains very small throughout the simulation. This approximation is also required when the only available data are the global inertia properties (e.g. m, the postion of the center of mass and the inertia matrix $\boldsymbol{J}$), and the modal mass matrix $\boldsymbol{m}$.

More refined approximations include higher order terms: for example, the first and second order contributions illustrated before. This corresponds to using finer and finer descriptions of the inertia properties of the system, corresponding to computing the inertia properties in the deformed condition with first and second order accuracy, respectively.

9.4.4 Invariants

The dynamics of the deformed body can be written without any detailed knowledge of the mass distribution, provided some aggregate information can be gathered in so-called *invariants*. They are:

1. Total mass (scalar)

$$\mathcal{I}_1 = \sum_{i=1,N} m_i \tag{9.406}$$

where m_i is the mass of the i-th FEM node[4].

2. Static moment (matrix 3×1)

$$\mathcal{I}_2 = \sum_{i=1,N} m_i \tilde{\boldsymbol{f}}_i \tag{9.407}$$

3. Static coupling between rigid body and FEM node displacements (matrix $3 \times M$)

$$\mathcal{I}_3 = \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{9.408}$$

where the portion related to the k-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the three components of the modal displacement $\boldsymbol{U}_{ik}$ of the k-th mode.

4. Static coupling between rigid body rotations and FEM node displacements (matrix $3 \times M$)

$$\mathcal{I}_4 = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{9.409}$$

where the portion related to the k-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the cross product of the FEM node position $\tilde{\boldsymbol{f}}_i$ and the three components of the modal displacement $\boldsymbol{U}_{ik}$ of the k-th mode.

5. Static coupling between FEM node displacements (matrix $3 \times M \times M$)

$$\mathcal{I}_{5j} = \sum_{i=1,N} m_i \boldsymbol{U}_{ij} \times \boldsymbol{U}_i \tag{9.410}$$

where the portion related to the j-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the cross product of the three components of the FEM node j-th modal displacement $\boldsymbol{U}_{ij}$ and the three components of the k-th modal displacement $\boldsymbol{U}_{ik}$.

[4] Although the input format, because of NASTRAN legacy, allows each global direction to have a separate mass value, invariants assume that the same value is given, and only use the one associated with component 1.

6. Modal mass matrix (matrix $M \times M$)

$$\mathcal{I}_6 = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{9.411}$$

7. Inertia matrix (matrix 3×3)

$$\mathcal{I}_7 = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i \right) \tag{9.412}$$

8. (matrix $3 \times M \times 3$)

$$\mathcal{I}_{8j} = \sum_{i=1,N} m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_{ij} \times^T \tag{9.413}$$

9. (matrix $3 \times M \times M \times 3$)

$$\mathcal{I}_{9jk} = \sum_{i=1,N} m_i \boldsymbol{U}_{ij} \times \boldsymbol{U}_{ik} \times \tag{9.414}$$

10. (matrix $3 \times M \times 3$)

$$\mathcal{I}_{10j} = \sum_{i=1,N} \boldsymbol{V}_{ij} \times \boldsymbol{J}_i \tag{9.415}$$

11. (matrix $3 \times M$)

$$\mathcal{I}_{11} = \sum_{i=1,N} \boldsymbol{J}_i \boldsymbol{V}_i \tag{9.416}$$

Using the invariants, the contributions to the inertia matrix of the body become

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\, \mathcal{I}_1 \tag{9.417a}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0 \left(\mathcal{I}_2 + \mathcal{I}_3 \boldsymbol{q} \right) \times \boldsymbol{R}_0^T \tag{9.417b}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0 \mathcal{I}_3 \tag{9.417c}$$

$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0 \left(\mathcal{I}_7 + \left(\mathcal{I}_{8j} + \mathcal{I}_{8j}^T \right) q_j + \mathcal{I}_{9jk} q_j q_k \right) \boldsymbol{R}_0^T \tag{9.417d}$$

$$\begin{aligned} \boldsymbol{M}_{\phi q} = \boldsymbol{R}_0 \Bigg(& \mathcal{I}_4 + \mathcal{I}_{5j} q_j + \sum_{i=1,N} \left(\boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times^T \boldsymbol{V}_i + \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times \boldsymbol{J}_i \boldsymbol{V}_i \right) \\ & + \sum_{i=1,N} \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times^T \boldsymbol{V}_i \Bigg) \end{aligned} \tag{9.417e}$$

$$\begin{aligned} \boldsymbol{M}_{qq} = {} & \mathcal{I}_6 \\ & + \sum_{i=1,N} \left(\boldsymbol{V}_i^T \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times^T \boldsymbol{V}_i + \boldsymbol{V}_i^T \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times \boldsymbol{J}_i \boldsymbol{V}_i \right) \\ & + \sum_{i=1,N} \boldsymbol{V}_i^T \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q} \right) \times^T \boldsymbol{V}_i \end{aligned} \tag{9.417f}$$

where summation over repeated indices is assumed. The remaining summation terms could be also cast into some invariant form; however, in common practice (e.g. in ADAMS) they are simply neglected, under the assumption that the finer the discretization, the smaller the FEM node inertia, so that linear and quadratic terms in the nodal rotation become reasonably small, yielding

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\,\mathcal{I}_1 \tag{9.418a}$$
$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0\left(\mathcal{I}_2 + \mathcal{I}_3\boldsymbol{q}\right)\times\boldsymbol{R}_0^T \tag{9.418b}$$
$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0\mathcal{I}_3 \tag{9.418c}$$
$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0\left(\mathcal{I}_7 + \left(\mathcal{I}_{8j} + \mathcal{I}_{8j}^T\right)q_j + \mathcal{I}_{9jk}q_jq_k\right)\boldsymbol{R}_0^T \tag{9.418d}$$
$$\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0\left(\mathcal{I}_4 + \mathcal{I}_{5j}q_j\right) \tag{9.418e}$$
$$\boldsymbol{M}_{qq} = \mathcal{I}_6 \tag{9.418f}$$

In some cases, the only remaining quadratic term in $\mathcal{I}_{9jk}$ is neglected as well.

9.4.5 Interfacing

The basic interface between the FEM and the multibody world occurs by clamping regular multibody nodes to selected nodes on the FEM mesh. Whenever more sophisticated interfacing is required, for example connecting a multibody node to a combination of FEM nodes, an FEM node equivalent to the desired aggregate of nodes should either be prepared at the FEM side, for example by means of RBEs, or at the FEM database side, for example by averaging existing mode shapes according to the desired pattern, into an equivalent FEM node[5].

The clamping is imposed by means of a coincidence and a parallelism constraint between the locations and the orientations of the two points: the multibody node N and the FEM node P, according to the expressions

$$\boldsymbol{x}_N + \boldsymbol{f}_N = \boldsymbol{x}_P \tag{9.419}$$
$$\operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_N^T\boldsymbol{R}_P\right)\right) = \boldsymbol{0} \tag{9.420}$$

which becomes

$$\boldsymbol{x}_N + \boldsymbol{R}_N\tilde{\boldsymbol{f}}_N - \boldsymbol{x}_0 - \boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P\boldsymbol{q}\right) = \boldsymbol{0} \tag{9.421}$$
$$\operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_N^T\boldsymbol{R}_0\left(\boldsymbol{I} + (\boldsymbol{V}_P\boldsymbol{q})\times\right)\right)\right) = \boldsymbol{0} \tag{9.422}$$

The reaction forces exchanged are $\boldsymbol{\lambda}$ in the global frame, while the reaction moments are $\boldsymbol{R}_N\boldsymbol{\alpha}$ in the reference frame of node N:

$$\boldsymbol{F}_N = \boldsymbol{\lambda} \tag{9.423}$$
$$\boldsymbol{M}_N = \boldsymbol{f}_N\times\boldsymbol{\lambda} + \boldsymbol{R}_N\boldsymbol{\alpha} \tag{9.424}$$
$$\boldsymbol{F}_P = -\boldsymbol{\lambda} \tag{9.425}$$
$$\boldsymbol{M}_P = -\boldsymbol{R}_N\boldsymbol{\alpha} \tag{9.426}$$

The force and the moment apply on the rigid body displacement and rotation, and on the modal equations as well, according to

$$\begin{matrix}\delta\boldsymbol{x}_P^T\boldsymbol{F}_P \\ +\,\delta\boldsymbol{\phi}_P^T\boldsymbol{M}_P\end{matrix} = \left\{\begin{matrix}\delta\boldsymbol{x}_0 \\ \delta\boldsymbol{\phi}_0 \\ \delta\boldsymbol{q}\end{matrix}\right\}^T\left\{\begin{matrix}-\boldsymbol{\lambda} \\ -\left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P\boldsymbol{q}\right)\right)\times\boldsymbol{\lambda} - \boldsymbol{R}_N\boldsymbol{\alpha} \\ -\boldsymbol{U}_P^T\boldsymbol{R}_0^T\boldsymbol{\lambda} - \boldsymbol{V}_P^T\boldsymbol{R}_0^T\boldsymbol{R}_N\boldsymbol{\alpha}\end{matrix}\right\} \tag{9.427}$$

[5] For example, to constrain the displacement of a FEM node P that represents the weighing of the displacement of a set of FEM nodes according to a constant weighing matrix $\boldsymbol{W}_P \in \mathbb{R}^{3n\times 3}$, simply use $\boldsymbol{U}_P = \boldsymbol{W}_P^T\boldsymbol{U}$.

The linearization of the constraint yields

$$\left[\begin{array}{ccccc} -\boldsymbol{I} & \boldsymbol{f}_N \times & \boldsymbol{I} & -\left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P+\boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & \boldsymbol{R}_0\boldsymbol{U}_P \\ \boldsymbol{0} & -\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T & \boldsymbol{0} & \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T & \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T\boldsymbol{R}_0\boldsymbol{V}_P \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{x}_N \\ \delta\boldsymbol{g}_N \\ \delta\boldsymbol{x}_0 \\ \delta\boldsymbol{g}_0 \\ \delta\boldsymbol{q} \end{array}\right\} = \left\{\begin{array}{c} (9.421) \\ (9.422) \end{array}\right\} \tag{9.428}$$

Note that $\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1} \cong \boldsymbol{I}$ since $\boldsymbol{\theta} \to \boldsymbol{0}$ when the constraint is satisfied. The linearization of forces and moments yields

$$\begin{aligned} &\left[\begin{array}{cc} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\lambda}\times\boldsymbol{f}_N\times+\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \\ \boldsymbol{0} & -\boldsymbol{V}_P^T\boldsymbol{R}_0^T\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{x}_N \\ \delta\boldsymbol{g}_N \end{array}\right\} \\ +&\left[\begin{array}{ccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\lambda}\times\left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P+\boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & -\boldsymbol{\lambda}\times\boldsymbol{R}_0\boldsymbol{U}_P \\ \boldsymbol{0} & \boldsymbol{U}_P^T\boldsymbol{R}_0^T\boldsymbol{\lambda}\times+\boldsymbol{V}_P^T\boldsymbol{R}_0^T\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times & \boldsymbol{0} \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{x}_0 \\ \delta\boldsymbol{g}_0 \\ \delta\boldsymbol{q} \end{array}\right\} \\ +&\left[\begin{array}{cc} -\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{f}_N\times & -\boldsymbol{R}_N \\ \boldsymbol{I} & \boldsymbol{0} \\ \left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P+\boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & \boldsymbol{R}_N \\ \boldsymbol{U}_P^T\boldsymbol{R}_0^T & \boldsymbol{V}_P^T\boldsymbol{R}_0^T\boldsymbol{R}_N \end{array}\right]\left\{\begin{array}{c} \delta\boldsymbol{\lambda} \\ \delta\boldsymbol{\alpha} \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{\lambda} \\ \boldsymbol{f}_N\times\boldsymbol{\lambda}+\boldsymbol{R}_N\boldsymbol{\alpha} \\ \\ (9.427) \\ \\ \end{array}\right\} \end{aligned} \tag{9.429}$$

9.4.6 Virtual strain energy

Author: Reinhard Resch The modal element is based on the assumption of small elastic deformations but arbitrary superimposed rigid body motions. Since elastic deformations are small, the virtual strain energy δW may be approximated as a linear function of the modal displacement $\boldsymbol{q}$.

$$\delta W \approx \delta\boldsymbol{q}^T\,\boldsymbol{K}_{qq}\,\boldsymbol{q} \tag{9.430}$$

$\boldsymbol{K}_{qq}$ reduced order linear stiffness matrix

Quasi static corrections for stress stiffening

However, there are applications of structures subject to small elastic deformations and linear elastic material, where those linear approximation of the virtual strain energy according equation 9.430 is not applicable due to significant pre-stress effects. See also [5]. Typical examples are:

1. rotor blades of helicopters, wind turbines and gas turbines
2. propellers of airplanes
3. rotors of turbo chargers for internal combustion engines
4. connecting rods of high speed reciprocating engines

5. slender flexible arms of robots

According to the following assumptions, which are described in [5], it is possible to apply a quasi static correction to the virtual strain energy.

1. The flexible body should have low stiffness in some directions but high stiffness in other directions.
2. There are only low loads applied in directions of low stiffness. So, the deformations will remain small despite of low stiffness in those directions.
3. There are also high loads applied in directions of high stiffness. Because of such a high stiffness in those directions, the deformations will remain small. However, those high loads will cause high stresses which are affecting the stiffness in other directions.

Based on those assumptions, the corrected expression for the virtual strain energy according [5] is:

$$\delta W \approx \delta \boldsymbol{q}^T \left(\boldsymbol{K}_{qq} + \boldsymbol{K}_{qq_{geo}}\right) \boldsymbol{q} \tag{9.431}$$

In equation 9.431 the matrix $\boldsymbol{K}_{qq_{geo}}$ is called "geometrical stiffness matrix" or "stress stiffening matrix". According to [5], $\boldsymbol{K}_{qq_{geo}}$ can be written as a linear combination of a set of constant matrices.

$$\begin{aligned}\boldsymbol{K}_{qq_{geo}} = &\sum_{i=1,3} \left(\tilde{\ddot{x}}_{0_i} - \tilde{g}_i\right) \boldsymbol{K}_{0t_i} + \sum_{i=1,3} \tilde{\dot{\omega}}_{0_i} \boldsymbol{K}_{0r_i} + \sum_{m=1,6} \tilde{\omega}_{q_m} \boldsymbol{K}_{0\omega_m} \\ &+ \sum_{i=1,3} \sum_{j=1,M} \tilde{F}_{P_{ij}} \boldsymbol{K}_{0F_{ij}} + \sum_{i=1,3} \sum_{j=1,M} \tilde{M}_{P_{ij}} \boldsymbol{K}_{0M_{ij}}\end{aligned} \tag{9.432}$$

So, $\boldsymbol{K}_{qq_{geo}}$ is a function of the rigid body motion of the modal node, and also a function of the Lagrange multipliers applied at the interface nodes.

$$\tilde{\ddot{\boldsymbol{x}}}_0 = \boldsymbol{R}_0^T \ddot{\boldsymbol{x}}_0 \tag{9.433}$$

$$\tilde{\dot{\boldsymbol{\omega}}}_0 = \boldsymbol{R}_0^T \dot{\boldsymbol{\omega}}_0 \tag{9.434}$$

$$\tilde{\boldsymbol{\omega}}_q = \begin{pmatrix}\tilde{\omega}_{0_1}^2 & \tilde{\omega}_{0_2}^2 & \tilde{\omega}_{0_3}^2 & \tilde{\omega}_{0_1}\tilde{\omega}_{0_2} & \tilde{\omega}_{0_2}\tilde{\omega}_{0_3} & \tilde{\omega}_{0_1}\tilde{\omega}_{0_3}\end{pmatrix}^T \tag{9.435}$$

$$\tilde{\boldsymbol{\omega}}_0 = \boldsymbol{R}_0^T \boldsymbol{\omega}_0 \tag{9.436}$$

$$\tilde{\boldsymbol{F}}_{P_j} = - \boldsymbol{R}_0^T \boldsymbol{\lambda}_j \tag{9.437}$$

$$\tilde{\boldsymbol{M}}_{P_j} = - \boldsymbol{R}_0^T \boldsymbol{R}_N \boldsymbol{\alpha}_j \tag{9.438}$$

Those matrices are also called "pre-stress stiffness matrices" and can be a determined a priori based on the stress field for various static unit loads.

$\boldsymbol{K}_{0t_i}$ A unit linear acceleration is applied in direction x_i

$\boldsymbol{K}_{0r_i}$ A unit angular acceleration is applied along the x_i axis

$\boldsymbol{K}_{0\omega_m}$ Can be computed from six angular velocity loads

$\boldsymbol{K}_{0F_{ij}}$ A unit force is applied in x_i direction at interface node j

$\boldsymbol{K}_{0M_{ij}}$ A unit moment is applied along the x_i axis at interface node j

In theory, any structural Finite Element solver like ANSYS, ABAQUS or NASTRAN could be used to generate those matrices. For a comprehensive description of the underlying procedure, the reader is referred to [6]. Probably, the most convenient solution to get those matrices and all the modal element data is, to use a dedicated toolkit like mboct-fem-pkg with builtin support for MBDyn's input files.

Chapter 10

Beam Element

TODO

10.1 Generalized strains, strain rates and their linearization

Authors: Marco Morandini and Wenguo Zhu

This a draft, but hopefully consistent and correct documentation of the beam strains, strain rates and of their derivatives. Arbitrary point on the beam reference line, and the rotation parameter:

$$p(\xi) = N_i(\xi)(x_i + R_i \tilde{f}_i)$$

$$g(\xi) = N_i(\xi) g_i$$

The generalized strain:

$$\varepsilon(\xi) = R^T(\xi) N_i'(\xi)(x_i + R_i \tilde{f}_i) - \overline{p}'$$

$$\kappa(\xi) = R^T(\xi)\left(G(\xi) N_i'(\xi) g_i\right) + \kappa_r$$

Rotation:

$$R = R(N_i g_i) R_r$$

Derivative of Rotation:

$$\dot{R} = \omega \times R$$

$$\dot{R}^T = -R^T \omega \times$$

Section angular velocity:

$$\omega = G(\xi) N_i'(\xi) \dot{g}_i - \omega_r \times N_i g_i$$

And the strain rate can be expressed as:

$$
\begin{aligned}
l &= N_i'(\xi)(x_i + R_i\tilde{f}_i)
\end{aligned}
$$

$$
\begin{aligned}
\dot{l} &= N_{Jk}'(\dot{x}_k + \dot{R}_k\tilde{f}_k) \\
&= N_{Jk}'(\dot{x}_k + \omega_k \times R_k\tilde{f}_k)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\bar{\varepsilon}} &= R^T(\dot{l} - \omega \times l) \\
&= R^T N_{Jk}'(\dot{x}_k + \omega_k \times R_k\tilde{f}_k) \\
&\quad - R^T\omega \times N_{Ji}'(x_i + R_i\tilde{f}_i)
\end{aligned}
$$

$$
\boldsymbol{G} = \frac{4}{4 + \boldsymbol{g}^T\boldsymbol{g}}(\boldsymbol{I} + \frac{1}{2}\boldsymbol{g}\times)
$$

$$
\dot{G} = -\frac{4}{(4 + \boldsymbol{g}^T\boldsymbol{g})^2}(\dot{\boldsymbol{g}} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes \dot{\boldsymbol{g}})(\boldsymbol{I} + \frac{1}{2}\boldsymbol{g}\times) + \frac{4}{4 + \boldsymbol{g}^T\boldsymbol{g}}(\frac{1}{2}\dot{\boldsymbol{g}}\times)
$$

$$
\dot{\bar{\kappa}} = R^T(G(\xi)N_i'(\xi)\dot{g}_i + \dot{G}(\xi)N_i'(\xi)g_i - \omega \times (G(\xi)N_i'(\xi)g_i)) + \dot{\kappa}_r
$$

$$
\omega = G(N_i g_i)N_i'\dot{g}_i + R(N_i g_i)\omega_r
$$

Angular velocity linearization at an evaluation point J:

$$
\begin{aligned}
\triangle\omega &= \triangle G N_{Ji}\dot{g}_i \\
&\quad + \cancel{G}^{G\approx I} N_{Ji}\Delta\dot{g}_i \\
&\quad + \Delta R\omega_r \\
&= \frac{1}{2}N_{jk}\Delta g_k \times N_{Ji}'\dot{g}_i \\
&\quad + N_{Ji}\Delta\dot{g}_i \\
&\quad - \omega_r \times N_{Ji}\Delta g_i \\
&= \cancel{-\frac{1}{2}N_{Ji}'\dot{g}_i \times N_{jk}\Delta g_k}^{\dot{g}\approx 0} \\
&\quad + N_{Ji}\Delta\dot{g}_i \\
&\quad - \omega_r \times N_{Ji}\Delta g_i \\
&= N_{Ji}\Delta\dot{g}_i - \omega_r \times N_{Ji}\Delta g_i
\end{aligned}
$$

Rotation linearization at an evaluation point J:

$$
\begin{aligned}
\triangle R &= \triangle g \times R_r \\
&= (N_{Ji}\Delta g_i) \times R_r
\end{aligned}
$$

$$\Delta R^T = -R_r^T (N_{Ji}\Delta g_i)\times$$

Rotation rate linearization at an evaluation point J:

$$\begin{aligned}
\dot{R} &= \cancel{\dot{R}_\delta R_r = (G_\delta \dot{g}) \times R_r} \\
&= \omega \times R
\end{aligned}$$

$$\dot{R}^T = -R^T \omega \times$$

$$\begin{aligned}
\Delta \dot{R} &= \Delta\omega \times R \\
&\quad + \omega \times \Delta R \\
&= (N_{Ji}\Delta \dot{g}_i - \omega_r \times N_{Ji}\Delta g_i) \times R \\
&\quad + \omega \times N_{Ji}\Delta g_i \times \\
&= N_{Ji}\Delta \dot{g}_i \times R \\
&\quad - N_{Ji}\Delta g_i \otimes \omega_r R \\
&\quad + \omega_r \otimes N_{Ji} R^T \Delta g_i \\
&\quad + \omega \times N_{Ji}\Delta g_i \times R_r
\end{aligned}$$

$$\begin{aligned}
\Delta \dot{R}_i &= \Delta \dot{g}_i \times R_i \\
&\quad - \Delta g_i \otimes \omega_{ri} R_i \\
&\quad + \omega_{ri} \otimes R_i^T \Delta g_i \\
&\quad + \omega_i \times \Delta g_i \times R_{ri}
\end{aligned}$$

$$\begin{aligned}
\triangle l &= N'_{Ji}(\triangle x_i + \triangle R_i \tilde{f}_i) \\
&= N'_{Ji}(\triangle x_i - (R_i \tilde{f}_i) \times \triangle g_i)
\end{aligned}$$

Strain linearization at an evaluation point J:

$$\begin{aligned}
\Delta \varepsilon &= \triangle R^T N'_{Jk}(x_k + R_k \tilde{f}_k) \\
&\quad + R^T N'_{Jk}(\Delta x_k + \Delta R_k \tilde{f}_k) \\
&= R_r^T N'_{Jk}(x_k + R_k \tilde{f}_k) \times (N_{Jm}\Delta g_m) \\
&\quad + R^T N'_{Jk}(\Delta x_k + \Delta g_k \times R_k \tilde{f}_k) \\
&= R_r^T N'_{Jk}(x_k + R_k \tilde{f}_k) \times (N_{Jm}\Delta g_m) \\
&\quad + R_r^T N'_{Jk}(\Delta x_k - R_k \tilde{f}_k \times \Delta g_k)
\end{aligned}$$

$$\begin{aligned}
\Delta \kappa &= \cancel{\Delta R^T G N'_{Ji} g_i}^{g_i \approx 0} \\
&\quad \cancel{+ R^T \Delta G N'_{Ji} g_i}^{g_i \approx 0} \\
&\quad + R^T G N'_{Ji} \Delta g_i \\
&= R_r^T N'_{Ji} \Delta g_i
\end{aligned}$$

$$\dot{l} = N'_{Ji}(\dot{x}_i + \dot{R}_i\tilde{f}_i)$$

$$\begin{aligned}
\triangle\dot{l} &= N'_{Ji}(\triangle\dot{x}_i + \triangle\dot{R}_i\tilde{f}_i) \\
&= N'_{Ji}(\triangle\dot{x}_i + \left(\Delta\dot{g}_i \times R_i - \Delta g_i \otimes \omega_{ri}R_i + \omega_{ri} \otimes R_i^T\Delta g_i + \omega_i \times \Delta g_i \times R_{ri}\right)\tilde{f}_i) \\
&= N'_{Ji}\triangle\dot{x}_i \\
&\quad -N'_{Ji}(R_i\tilde{f}_i) \times \Delta\dot{g}_i \\
&\quad -N'_{Ji}(\omega_{ri}R_i\tilde{f}_i)\Delta g_i \\
&\quad +N'_{Ji}\omega_{ri} \otimes \tilde{f}_iR_i^T\Delta g_i \\
&\quad -N'_{Ji}\omega_i \times (R_{ri}\tilde{f}_i) \times \Delta g_i
\end{aligned}$$

Strain rate linearization at an evaluation point J:

$$\begin{aligned}
\triangle\dot{\tilde{\varepsilon}}_J &= \Delta R^T(\dot{l} - \omega \times l) \\
&\quad +R^T\Delta\dot{l} \\
&\quad -R^T\Delta\omega \times l \\
&\quad -R^T\omega \times \Delta l \\
&= R_r^T(\dot{l} - \omega \times l) \times N_{Ji}\Delta g_i \\
&\quad +R^TN'_{Ji}\triangle\dot{x}_i + \\
&\quad -R^TN'_{Ji}(R_i\tilde{f}_i) \times \Delta\dot{g}_i \\
&\quad -R^TN'_{Ji}(\omega_{ri}R_i\tilde{f}_i)\Delta g_i \\
&\quad +R^TN'_{Ji}\omega_{ri} \otimes \tilde{f}_iR_i^T\Delta g_i \\
&\quad -R^TN'_{Ji}\omega_i \times (R_{ri}\tilde{f}_i) \times \Delta g_i \\
&\quad +R^Tl \times N_{Ji}\Delta\dot{g}_i \\
&\quad \cancel{+R^T(\omega_r \times N_{Ji}\Delta g_i) \times l}^{EXPANDED\ IN\ THE\ TWO\ LINES\ BELOW} \\
&\quad +(\omega_r \cdot l)R^TN_{Ji}\Delta g_i \\
&\quad -R^T\omega_r \otimes l \cdot N_{Ji}\Delta g_i \\
&\quad -R^T\omega \times N'_{Ji}\Delta x_i \\
&\quad +R^T\omega \times N'_{Ji}(R_i\tilde{f}_i) \times \triangle g_i \\
&= R^TN'_{Ji}\triangle\dot{x}_i \\
&\quad -R^T\omega \times N'_{Ji}\Delta x_i \\
&\quad +R_r^T(\dot{l} - \omega \times l) \times N_{Ji}\Delta g_i \\
&\quad -R^TN'_{Ji}(R_i\tilde{f}_i) \times \Delta\dot{g}_i \\
&\quad -R^TN'_{Ji}(\omega_{ri}R_i\tilde{f}_i)\Delta g_i \\
&\quad +R^TN'_{Ji}\omega_{ri} \otimes \tilde{f}_iR_i^T\Delta g_i \\
&\quad -R^TN'_{Ji}\omega_i \times (R_{ri}\tilde{f}_i) \times \Delta g_i \\
&\quad +R^Tl \times N_{Ji}\Delta\dot{g}_i \\
&\quad +R^T(\omega_r \cdot l)N_{Ji}\Delta g_i \\
&\quad -R^T\omega_r \otimes l \cdot N_{Ji}\Delta g_i \\
&\quad +R^T\omega \times N'_{Ji}(R_i\tilde{f}_i) \times \triangle g_i
\end{aligned}$$

$$
\begin{aligned}
\triangle\dot{\kappa}_J(\xi) &= \triangle R^T \left(\cancel{G}^{\approx I} \cancel{N'_{Jk}\dot{g}_k}^{\dot{g}_k \approx 0} \cancel{+\Delta\dot{G}N'_{Jk}g_k}^{g_k \approx 0} \cancel{-\omega_J \times (\cancel{G}^{\approx I} N'_{Jk}g_k)}^{g_k \approx 0} \right) + \dot{\kappa}_r \\
&\quad \cancel{+R^T(\Delta G N'_{Jk}\dot{g}_k)}^{\dot{g}_k \approx 0} \\
&\quad +R^T(\cancel{G}^{\approx I} N'_{Jk}\Delta\dot{g}_k) \\
&\quad \cancel{+R^T(\cancel{G}^{\approx I} N'_{Ji}g_i)}^{g_i \approx 0} \times \Delta\omega \\
&\quad \cancel{+R^T(\Delta G N'_{Ji}g_i) \times \omega}^{g_i \approx 0} \\
&\quad +R^T(\cancel{G}^{\approx I} N'_{Ji}\Delta g_i) \times \omega \\
&= +R^T N'_{Jk}\Delta\dot{g}_k \\
&\quad +R^T(N'_{Ji}\Delta g_i) \times \omega \\
&= +R^T(N'_{Jk}\Delta\dot{g}_k) \\
&\quad -R^T\omega \times (N'_{Ji}\Delta g_i) \\
&= +R^T N'_{Jk}\Delta\dot{g}_k \\
&\quad -R^T\omega \times (N'_{Jm}\Delta g_m)
\end{aligned}
$$

10.2 Fully coupled piezoelectric beam

A fully coupled piezoelectric beam, differently from a piezoelectric beam, do contribute to the electri equations of the abstrc nodes it is linked to. Each abstract node do represent an electrode. The constitutive law at any given evaluation point is

$$
\left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{Q} \end{array} \right\} = \left[\begin{array}{cc} \boldsymbol{E}_{kk} & \boldsymbol{E}_{kV} \\ \boldsymbol{E}_{Vk} & \boldsymbol{E}_{VV} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{k} \\ \boldsymbol{V} \end{array} \right\} \tag{10.1}
$$

where $\boldsymbol{F}$ are the six beam internal actions at the evaluation point, $\boldsymbol{Q}$ the vector of electrodes charges per unit of beam length, $\boldsymbol{k}$ the six beam generalized deformations and $\boldsymbol{V}$ the electric difference of potential at the electrodes. The equation contribution to the structural nodes are the same of the piezoelectric beam. The difference with respect to the piezoelectric beam is the equations contribution to the asbtract nodes. The unknowns of the abstract nodes do represent the electric difference of potential at the electrodes. The equations at the abstract nodes states that the sum of charge flux (the electric current) flowing out from each node must be equal to zero:

$$
\sum_i I_i = 0 \tag{10.2}
$$

where I_i is the current flowing out from the abstract nodedue to the connected element i. For this element the current flowing out from the electrodes is

$$
\boldsymbol{I} = -\int_l \dot{\boldsymbol{Q}} \mathrm{d}s = -\int_l \boldsymbol{E}_{Vk}\dot{\boldsymbol{k}} + \boldsymbol{E}_{VV}\dot{\boldsymbol{V}} \mathrm{d}s \tag{10.3}
$$

The fully copled beam element thus turns out to be viscoelastic, as its contribution to the electric equations do depend on the time derivative of the beam generalized measure. The integral $\int () \mathrm{d}s$ is computed with the same gauss quadrature scheme used for the evaluation of the beam stiffness matrix. The assembled residual contribution is equal to Eq. (10.3) with sign changed.

Chapter 11

Shell Element

Authors: Marco Morandini and Riccardo Vescovini

11.1 Variational principle

The formulation refers to the modified Hu-Washizu variational functional. The linearization of the functional is written as:

$$\delta W + \Delta \delta W = 0 \tag{11.1}$$

with:

$$\delta W = \int_A \left(\delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} + \delta \hat{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} \right) \mathrm{d}A \tag{11.2}$$

with

$$\boldsymbol{\sigma} = \boldsymbol{\sigma} \left(\boldsymbol{\epsilon} + \hat{\boldsymbol{\epsilon}} \right) \tag{11.3}$$

A linear (visco)elastic stress-strain relationship is considered,

$$\boldsymbol{\sigma} = \boldsymbol{C} \left(\boldsymbol{\epsilon} + \hat{\boldsymbol{\epsilon}} \right) + \boldsymbol{E} \left(\dot{\boldsymbol{\epsilon}} + \dot{\hat{\boldsymbol{\epsilon}}} \right) \tag{11.4}$$

FIXME: $\dot{\hat{\epsilon}}$? Direi di no! and:

$$\Delta \delta W = \int_A \left(\delta \boldsymbol{\epsilon}^T \boldsymbol{C} \left(\Delta \boldsymbol{\epsilon} + \Delta \hat{\boldsymbol{\epsilon}} \right) + \delta \hat{\boldsymbol{\epsilon}}^T \boldsymbol{C} \left(\Delta \boldsymbol{\epsilon} + \Delta \hat{\boldsymbol{\epsilon}} \right) + \Delta \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \right) \mathrm{d}A \tag{11.5}$$

where:

- $\boldsymbol{C}$ and $\boldsymbol{E}$ are the linear viscoelastic constitutive law matrices (integrated along the shell thickness)
- $\boldsymbol{\epsilon}$ is the vector of the compatible strains (resulting from the strain-displacement relations)
- $\hat{\boldsymbol{\epsilon}}$ is the vector of the enhancing strains (EAS)
- $\boldsymbol{\sigma}$ is the vector of forces and moments per unit length

The enhancing strains $\hat{\boldsymbol{\epsilon}}$ are interpolated within the element.
The compatible strains $\boldsymbol{\epsilon}$ are defined as:

$$\underset{12\times 1}{\boldsymbol{\epsilon}} = \left\{ \begin{array}{c} \tilde{\boldsymbol{\epsilon}}_1 \\ \tilde{\boldsymbol{\epsilon}}_2 \\ \tilde{\boldsymbol{\kappa}}_1 \\ \tilde{\boldsymbol{\kappa}}_2 \end{array} \right\} \tag{11.6}$$

where:

$$\tilde{\boldsymbol{\epsilon}}_k = \boldsymbol{T}^T \boldsymbol{y}_{,k} - \boldsymbol{e}_k \tag{11.7}$$

where the comma followed by the index represents the derivation with respect to the k-th arc-length coordinate.
$\boldsymbol{T}$ is the orientation and $\boldsymbol{y}$ is the position vector both in the deformed configuration, and $\boldsymbol{e}_k = \{e_{1k}, e_{2k}, e_{3k}\}$, with $e_{ik} = \delta_{ik}$.

$$\tilde{\boldsymbol{\kappa}}_k = \boldsymbol{T}^T \boldsymbol{\kappa}_k - \boldsymbol{T}_0^T \boldsymbol{\kappa}_k^0 \tag{11.8}$$

with:

$$\boldsymbol{\kappa}_k \times = \boldsymbol{T}_{,k} \boldsymbol{T}^T \tag{11.9}$$

11.1.1 Strain Rate

The compatible strain derivatives $\dot{\boldsymbol{\epsilon}}$ are defined as:

$$\underset{12\times 1}{\dot{\boldsymbol{\epsilon}}} = \left\{ \begin{array}{c} \dot{\tilde{\boldsymbol{\epsilon}}}_1 \\ \dot{\tilde{\boldsymbol{\epsilon}}}_2 \\ \dot{\tilde{\boldsymbol{\kappa}}}_1 \\ \dot{\tilde{\boldsymbol{\kappa}}}_2 \end{array} \right\} \tag{11.10}$$

where:

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\boldsymbol{\epsilon}}_k = \dot{\boldsymbol{T}}^T \boldsymbol{y}_{,k} + \boldsymbol{T}^T \dot{\boldsymbol{y}}_{,k} = -\boldsymbol{T}^T \boldsymbol{\omega} \times \boldsymbol{y}_{,k} + \boldsymbol{T}^T \dot{\boldsymbol{y}}_{,k} \tag{11.11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\boldsymbol{\kappa}}_k = \dot{\boldsymbol{T}}^T \boldsymbol{\kappa}_k + \boldsymbol{T}^T \dot{\boldsymbol{\kappa}}_k = -\boldsymbol{T}^T \boldsymbol{\omega} \times \boldsymbol{\kappa}_k + \boldsymbol{T}^T \dot{\boldsymbol{\kappa}}_k . \tag{11.12}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{T}_{,k}) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\kappa}_k \times \boldsymbol{T}) = \dot{\boldsymbol{\kappa}}_k \times \boldsymbol{T} + \boldsymbol{\kappa}_k \times \boldsymbol{\omega} \times \boldsymbol{T} \tag{11.13}$$

$$(\dot{\boldsymbol{T}})_{,k} = (\boldsymbol{\omega} \times \boldsymbol{T})_{,k} = \boldsymbol{\omega}_{,k} \times \boldsymbol{T} + \boldsymbol{\omega} \times \boldsymbol{\kappa}_k \times \boldsymbol{T} \tag{11.14}$$

then

$$\dot{\boldsymbol{\kappa}}_k \times + \boldsymbol{\kappa}_k \times \boldsymbol{\omega} \times = \boldsymbol{\omega}_{,k} \times + \boldsymbol{\omega} \times \boldsymbol{\kappa}_k \times \tag{11.15}$$

i.e.

$$\dot{\boldsymbol{\kappa}}_k = \boldsymbol{\omega}_{,k} + \boldsymbol{\omega} \times \boldsymbol{\kappa}_k \tag{11.16}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{\kappa}}_k = \boldsymbol{T}^T \boldsymbol{\omega}_{,k} \tag{11.17}$$

The vector $\boldsymbol{\sigma}$ is:

$$\boldsymbol{\sigma} = \left\{ \begin{array}{c} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{array} \right\} \tag{11.18}$$

11.2 Finite element discretization and notation

The shell elements has four nodes, numbered counterclockwise starting from the upper right node, as shown in Figure 11.1. In the natural domain, the coordinate system is identified by the coordinates ξ_1

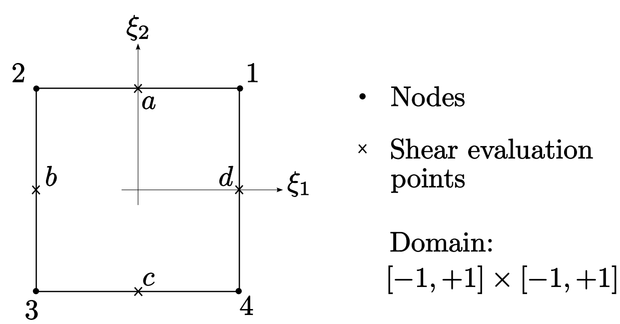

Figure 11.1: Shell finite element

and ξ_2, both varying in the range $[-1, 1]$.
In the following, the notation

$$\boldsymbol{\xi} = (\xi_1, \xi_2) \tag{11.19}$$

will be used. Four points are defined at the element's mid-sides. They are used in the context of the ANS approach. In particular, they are denoted by the set of coordinates:

$$\begin{aligned} \boldsymbol{\xi}_a &= (\xi_{1a}, \xi_{2a}) \\ \boldsymbol{\xi}_b &= (\xi_{1b}, \xi_{2b}) \\ \boldsymbol{\xi}_c &= (\xi_{1c}, \xi_{2c}) \\ \boldsymbol{\xi}_d &= (\xi_{1d}, \xi_{2d}) \end{aligned} \tag{11.20}$$

The following convention is adopted:

- the subscript n denotes a nodal variable
- $\boldsymbol{\xi}_i$ means $\boldsymbol{\xi}$ evaluated in a generic integration point
- $\boldsymbol{\xi}_A$ means $\boldsymbol{\xi}$ evaluated in one of the four shear evaluation points
- $\boldsymbol{\xi}_0$ means $\boldsymbol{\xi}$ evaluated at the origin

According to the numbering of the nodes, the bilinear shape functions defined at element level are:

$$L_1(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi_1)(1+\xi_2) \tag{11.21a}$$

$$L_2(\boldsymbol{\xi}) = \frac{1}{4}(1-\xi_1)(1+\xi_2) \tag{11.21b}$$

$$L_3(\boldsymbol{\xi}) = \frac{1}{4}(1-\xi_1)(1-\xi_2) \tag{11.21c}$$

$$L_4(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi_1)(1-\xi_2) \tag{11.21d}$$

The degrees of freedom associated with each element are 24 nodal position and orientation parameter components, collected in vector $\boldsymbol{q}$:

$$\underset{24\times 1}{\delta\boldsymbol{q}} = \left\{\begin{array}{c} \delta\boldsymbol{y}_n \\ \boldsymbol{\varphi}_{\delta n}\end{array}\right\} = \left\{\begin{array}{c} \delta\boldsymbol{y}_1 \\ \boldsymbol{\varphi}_{\delta 1} \\ \delta\boldsymbol{y}_2 \\ \boldsymbol{\varphi}_{\delta 2} \\ \delta\boldsymbol{y}_3 \\ \boldsymbol{\varphi}_{\delta 3} \\ \delta\boldsymbol{y}_4 \\ \boldsymbol{\varphi}_{\delta 4}\end{array}\right\} \qquad \underset{24\times 1}{\Delta\boldsymbol{q}} = \left\{\begin{array}{c} \delta\boldsymbol{y}_n \\ \boldsymbol{\varphi}_{\Delta n}\end{array}\right\} = \left\{\begin{array}{c} \Delta\boldsymbol{y}_1 \\ \boldsymbol{\varphi}_{\Delta 1} \\ \Delta\boldsymbol{y}_2 \\ \boldsymbol{\varphi}_{\Delta 2} \\ \Delta\boldsymbol{y}_3 \\ \boldsymbol{\varphi}_{\Delta 3} \\ \Delta\boldsymbol{y}_4 \\ \boldsymbol{\varphi}_{\Delta 4}\end{array}\right\} \tag{11.22}$$

and 7 internal degrees of freedom related to the assumed strain (EAS) and collected in vector $\boldsymbol{\beta}$.
The total number of degrees of freedom is 31.

11.3 Linearization

The virtual variation of the compatible strains $\boldsymbol{\epsilon}$ is

$$\delta\boldsymbol{\epsilon} = \left\{\begin{array}{c} \delta\tilde{\boldsymbol{\epsilon}}_1 \\ \delta\tilde{\boldsymbol{\epsilon}}_2 \\ \delta\tilde{\boldsymbol{\kappa}}_1 \\ \delta\tilde{\boldsymbol{\kappa}}_2 \end{array}\right\} \tag{11.23}$$

In particular, the virtual variations of the linear deformation $\tilde{\boldsymbol{\epsilon}}_k$ is:

$$\delta\tilde{\boldsymbol{\epsilon}}_k = \delta\left(\boldsymbol{T}^T\boldsymbol{y}_{,k}\right) = \sum_{i=1}^{n}\boldsymbol{T}^T\boldsymbol{y}_{,k}\times\boldsymbol{\Phi}_n L_n\boldsymbol{\varphi}_{\delta n} + \sum_{i=1}^{n}\boldsymbol{T}^T L_{n,k}\delta\boldsymbol{y}_n \tag{11.24}$$

where:

$$\boldsymbol{\Phi}_n = \overline{\boldsymbol{T}}\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\Gamma}}_n^{-1}\overline{\boldsymbol{T}}^T \tag{11.25}$$

The virtual variations of the curvatures $\tilde{\boldsymbol{\kappa}}_k$ is:

$$\delta\tilde{\boldsymbol{\kappa}}_k = \delta\left(\boldsymbol{T}^T\boldsymbol{\kappa}_k\right) = \sum_{i=1}^{n}\boldsymbol{T}^T\boldsymbol{\kappa}_k\times\boldsymbol{\Phi}_n L_n\boldsymbol{\varphi}_{\delta n} + \sum_{i=1}^{n}\boldsymbol{T}^T\boldsymbol{K}_{kn}\boldsymbol{\varphi}_{\delta n} \tag{11.26}$$

where:

$$\boldsymbol{K}_{kn} = \overline{\boldsymbol{T}}\mathcal{L}\left(\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\varphi}}_{,k}\right)\tilde{\boldsymbol{\Gamma}}_n^{-1}\overline{T}^T L_n + \boldsymbol{\Phi}_n L_{n,k} \tag{11.27}$$

the term $L_{n,k}$ is the derivative of the n-th shape function with respect to the k-th arc-length coordinate. The last term of Eq. 11.5 requires the evaluation of the terms:

$$\begin{aligned}\boldsymbol{n}_k \cdot \Delta\delta\tilde{\boldsymbol{\epsilon}}_k = \boldsymbol{n}_k \cdot \Delta\delta\left(\boldsymbol{T}^T\boldsymbol{y}_{,k}\right) = &\sum_{m,n=1}^{4} \boldsymbol{\varphi}_{\delta n}\boldsymbol{\Phi}_n^T L_n \boldsymbol{y}_{,k} \times (\boldsymbol{T}\boldsymbol{n}_k) \times \boldsymbol{\Phi}_m L_m \boldsymbol{\varphi}_{\Delta n} + \\ &+ \sum_{m,n=1}^{4} \boldsymbol{\varphi}_{\delta n}\boldsymbol{\Phi}_n^T L_n (\boldsymbol{T}\boldsymbol{n}_k) \times L_{m,k}\Delta\boldsymbol{y}_n + \\ &- \sum_{m,n=1}^{4} \delta\boldsymbol{y}_n L_{n,k} (\boldsymbol{T}\boldsymbol{n}_k) \times \boldsymbol{\Phi}_m L_m \boldsymbol{\varphi}_{\Delta m}\end{aligned} \tag{11.28}$$

$$\begin{aligned}\boldsymbol{m}_k \cdot \Delta\delta\tilde{\boldsymbol{\kappa}}_k = \boldsymbol{m}_k \cdot \Delta\delta\left(\boldsymbol{T}^T\boldsymbol{\kappa}_k\right) = &\sum_{m,n=1}^{4} \boldsymbol{\varphi}_{\delta n} L_n \boldsymbol{\Phi}_n^T \boldsymbol{\kappa}_k \times (\boldsymbol{T}\boldsymbol{m}_k) \times \boldsymbol{\Phi}_{\boldsymbol{m}} L_m \boldsymbol{\varphi}_{\Delta m} + \\ &+ \sum_{m,n=1}^{4} \boldsymbol{\varphi}_{\delta n} L_n \boldsymbol{\Phi}_n^T (\boldsymbol{T}\boldsymbol{m}_k) \times \boldsymbol{\Phi}_m L_{m,k} \boldsymbol{\varphi}_{\Delta m}\end{aligned} \tag{11.29}$$

11.4 Structural Damping

Consistently formulated structural damping requires to express the time derivatives of strain and curvature. A linear contribution to internal force and moment fluxes is considered,

$$\boldsymbol{\sigma} \mathrel{+}= \boldsymbol{E}\dot{\boldsymbol{\epsilon}} \tag{11.30}$$

The discretized strain rate is

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{\epsilon}}_k = \boldsymbol{T}^T\boldsymbol{y}_{,k} \times \sum_{i=1}^{n} \boldsymbol{\Phi}_i L_i \boldsymbol{\omega}_i + \boldsymbol{T}^T \sum_{i=1}^{n} L_{n,k}\dot{\boldsymbol{y}}_i \tag{11.31}$$

The discretized curvature rate is

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{\kappa}}_k = \boldsymbol{T}^T \sum_{i=1}^{n} L_{n,k}\boldsymbol{\omega}_i \tag{11.32}$$

11.5 Implementation

Having shown the variational principle and the linearization of strains and curvatures, it is then possible to develop the procedure to derive the Jacobian matrix and the residual.

11.5.1 Orientation interpolation

In the initial configuration it is built the matrix:

$$\boldsymbol{iTa}_n = \boldsymbol{R}_n^T \left[\boldsymbol{t}_{1n}\boldsymbol{t}_{2n}\boldsymbol{t}_{3n}\right] \tag{11.33}$$

with $\boldsymbol{R}_n$ node orientation, and $\boldsymbol{t}_{in}$ vectors tangent to the element surface.
So:

$$\boldsymbol{T}_n = \boldsymbol{R}_n \boldsymbol{iTa}_n \tag{11.34}$$

It is then calculated:

$$\boldsymbol{T}_{\text{avg}} = \frac{1}{4}\sum_{n=1}^{4}\boldsymbol{T}_n \tag{11.35}$$

which is in general a not orthogonal matrix.
An orthogonal matrix can be obtained by applying:

$$\overline{\boldsymbol{T}} = \texttt{Rot}\left(\texttt{VecRot}\left(\boldsymbol{T}_{\text{avg}}\right)\right) \tag{11.36}$$

which is then used to calculate the nodal rotation:

$$\tilde{\boldsymbol{R}}_n = \overline{\boldsymbol{T}}^T \boldsymbol{T}_n \tag{11.37}$$

and:

$$\tilde{\boldsymbol{\varphi}}_n = \texttt{VecRot}\left(\tilde{\boldsymbol{R}}_n\right) \tag{11.38}$$

The nodal values of $\tilde{\boldsymbol{\varphi}}_n$ are interpolated by means of the shape functions:

$$\tilde{\boldsymbol{\varphi}}\left(\boldsymbol{\xi}_i\right) = \sum_{i=1}^{4} L_n\left(\boldsymbol{\xi}_i\right)\tilde{\boldsymbol{\varphi}}_n \tag{11.39}$$

from which is obtained the rotation matrix at the integration point:

$$\tilde{\boldsymbol{R}}\left(\boldsymbol{\xi}_i\right) = \texttt{Rot}\left(\tilde{\boldsymbol{\varphi}}\left(\boldsymbol{\xi}_i\right)\right) \tag{11.40}$$

and finally the orientation at the integration point:

$$\boldsymbol{T}\left(\boldsymbol{\xi}_i\right) = \overline{\boldsymbol{T}}\tilde{\boldsymbol{R}}\left(\boldsymbol{\xi}_i\right) \tag{11.41}$$

The virtual variation of $\boldsymbol{\varphi}$ at the integration point is related to the nodal values by:

$$\boldsymbol{\varphi}_\delta\left(\boldsymbol{\xi}_i\right) = \sum_{n=1}^{4}\boldsymbol{\Phi}_n\left(\boldsymbol{\xi}_i\right) L_n\left(\boldsymbol{\xi}_i\right)\boldsymbol{\varphi}_{\delta n} \tag{11.42}$$

with:

$$\boldsymbol{\Phi}_n\left(\boldsymbol{\xi}_i\right) = \overline{\boldsymbol{T}}\tilde{\boldsymbol{\Gamma}}\left(\tilde{\boldsymbol{\varphi}}\left(\boldsymbol{\xi}_i\right)\right)\tilde{\boldsymbol{\Gamma}}^{-1}\left(\tilde{\boldsymbol{\varphi}}_n\right)\overline{\boldsymbol{T}}^T \tag{11.43}$$

Similarly the virtual variation of $\boldsymbol{\varphi}$ can be obtained at the shear evaluation points as:

$$\boldsymbol{\varphi}_\delta\left(\boldsymbol{\xi}_A\right) = \sum_{n=1}^{4}\boldsymbol{\Phi}_n\left(\boldsymbol{\xi}_A\right) L_n\left(\boldsymbol{\xi}_A\right)\boldsymbol{\varphi}_{\delta n} \tag{11.44}$$

11.5.2 Position interpolation

The nodal positions are so interpolated as:

$$\boldsymbol{y} = \sum_{n=1}^{4} L_n \boldsymbol{y}_n \tag{11.45}$$

The derivatives with respect to the generic coordinate ξ_k at the integration points is:

$$\frac{\partial \boldsymbol{x}(\boldsymbol{\xi}_i)}{\partial \xi_k} = \sum_{n=1}^{4} \frac{\partial L_n(\boldsymbol{\xi}_i)}{\partial \xi_k} \boldsymbol{x}_n \tag{11.46}$$

It is then possible to build the matrix:

$$\underset{2\times 2}{\boldsymbol{S}_{\alpha\beta}}(\boldsymbol{\xi}_i) = \begin{bmatrix} \underset{1\times 3}{\boldsymbol{t}_1^0}(\boldsymbol{\xi}_i) \underset{3\times 1}{\frac{\partial \boldsymbol{x}(\boldsymbol{\xi}_i)}{\partial \xi_1}} & \boldsymbol{t}_1^0(\boldsymbol{\xi}_i) \frac{\partial \boldsymbol{x}(\boldsymbol{\xi}_i)}{\partial \xi_2} \\ \boldsymbol{t}_2^0(\boldsymbol{\xi}_i) \frac{\partial \boldsymbol{x}(\boldsymbol{\xi}_i)}{\partial \xi_1} & \boldsymbol{t}_2^0(\boldsymbol{\xi}_i) \frac{\partial \boldsymbol{x}(\boldsymbol{\xi}_i)}{\partial \xi_2} \end{bmatrix} \tag{11.47}$$

observing that $\boldsymbol{T}_0(\boldsymbol{\xi}_i) = \left[\boldsymbol{t}_1^0(\boldsymbol{\xi}_i)\, \boldsymbol{t}_2^0(\boldsymbol{\xi}_i)\, \boldsymbol{t}_3^0(\boldsymbol{\xi}_i)\right]$.
With the same procedure are derived $\boldsymbol{S}_{\alpha\beta}(\boldsymbol{\xi}_0)$ and $\boldsymbol{S}_{\alpha\beta}(\boldsymbol{\xi}_A)$.

$$\underset{4\times 2}{\boldsymbol{L}_{\alpha B}}(\boldsymbol{\xi}_i) = \begin{bmatrix} \frac{\partial L_1(\boldsymbol{\xi}_i)}{\partial \xi_1} & \frac{\partial L_1(\boldsymbol{\xi}_i)}{\partial \xi_2} \\ \frac{\partial L_2(\boldsymbol{\xi}_i)}{\partial \xi_1} & \frac{\partial L_2(\boldsymbol{\xi}_i)}{\partial \xi_2} \\ \frac{\partial L_3(\boldsymbol{\xi}_i)}{\partial \xi_1} & \frac{\partial L_3(\boldsymbol{\xi}_i)}{\partial \xi_2} \\ \frac{\partial L_4(\boldsymbol{\xi}_i)}{\partial \xi_1} & \frac{\partial L_4(\boldsymbol{\xi}_i)}{\partial \xi_2} \end{bmatrix} \tag{11.48}$$

$$\underset{4\times 2}{\boldsymbol{L}_{\alpha\beta}}(\boldsymbol{\xi}_i) = \boldsymbol{L}_{\alpha B}(\boldsymbol{\xi}_i)\, \boldsymbol{S}_{\alpha\beta}^{-1}(\boldsymbol{\xi}_i) \tag{11.49}$$

which is the matrix giving the variation of the shape functions L_n with respect the the arc-length coordinates:

$$\boldsymbol{L}_{\alpha\beta} = \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \\ L_{3,1} & L_{3,2} \\ L_{4,1} & L_{4,2} \end{bmatrix} \tag{11.50}$$

In order to perform the integration in the isoparametric domain the area element need to be calculated. In particular, the determinant of the Jacobian of the transformation between the physical and the natural domain has to be calculated. It is:

$$\alpha(\boldsymbol{\xi}_i) = \det \boldsymbol{S}_{\alpha\beta}(\boldsymbol{\xi}_i) \qquad \alpha(\boldsymbol{\xi}_A) = \det \boldsymbol{S}_{\alpha\beta}(\boldsymbol{\xi}_A) \tag{11.51}$$

11.5.3 Enhancing strains interpolation

The virtual variation and increments of the enhancing strains $\hat{\boldsymbol{\epsilon}}$ are interpolated as:

$$\underset{12\times 1}{\delta\hat{\boldsymbol{\epsilon}}}(\boldsymbol{\xi}_i) = \underset{12\times 7}{\boldsymbol{P}}(\boldsymbol{\xi}_i)\, \underset{7\times 1}{\delta\boldsymbol{\beta}} \qquad \Delta\hat{\boldsymbol{\epsilon}}(\boldsymbol{\xi}_i) = \boldsymbol{P}(\boldsymbol{\xi}_i)\, \Delta\boldsymbol{\beta} \tag{11.52}$$

where $\boldsymbol{\beta}$ is the vector collecting the the strains parameters, while the expression of the matrix $\boldsymbol{P}$ is derived in the next.
The EAS approach here adopted considers the enhancing of the membrane strains only, i.e. ϵ_{11}, ϵ_{12}, ϵ_{21} and ϵ_{22}. Shear strains and bending strains are not enhanced.
The interpolation is performed in the natural domain adopting the shape functions:

$$\underset{4\times 7}{\boldsymbol{H}}(\boldsymbol{\xi}_i) = \begin{bmatrix} \xi_{1i} & \xi_{1i}\xi_{2i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{2i} & \xi_{1i}\xi_{2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_{1i} & \xi_{1i}\xi_{2i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_{1i}\xi_{2i} & \xi_{2i} \end{bmatrix} \tag{11.53}$$

The interpolation is reported in the physical domain with a push-forward operation with the transformation matrix $\boldsymbol{M}_0$, defined as:

$$\underset{4\times 4}{\boldsymbol{M}_0} = \begin{bmatrix} s_{1,1}s_{1,1} & s_{1,2}s_{1,2} & s_{1,2}s_{1,1} & s_{1,1}s_{1,2} \\ s_{2,1}s_{2,1} & s_{2,2}s_{2,2} & s_{2,2}s_{2,1} & s_{2,1}s_{2,2} \\ s_{1,1}s_{2,1} & s_{1,2}s_{2,2} & s_{1,1}s_{2,2} & s_{1,2}s_{2,1} \\ s_{1,1}s_{2,1} & s_{1,2}s_{2,2} & s_{1,2}s_{2,1} & s_{1,1}s_{2,2} \end{bmatrix} \tag{11.54}$$

where the generic term $s_{i,k}$ is the element (i,k) of the matrix $\boldsymbol{S}_{\alpha\beta}(\boldsymbol{\xi}_0)$.

$$\underset{12\times 7}{\boldsymbol{P}(\boldsymbol{\xi}_i)} = \frac{\alpha(\boldsymbol{\xi}_0)}{\alpha(\boldsymbol{\xi}_i)} \underset{12\times 4}{\boldsymbol{\mathcal{P}}} \underset{4\times 4}{\boldsymbol{M}_0^{-T}} \underset{4\times 7}{\boldsymbol{H}(\boldsymbol{\xi}_i)} \tag{11.55}$$

where $\boldsymbol{\mathcal{P}}$ is a permutation matrix defined as:

$$\underset{12\times 4}{\boldsymbol{\mathcal{P}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11.56}$$

which has the places the enhanced strains in the corresponding positions of the global vector of membrane and bending strains $\boldsymbol{\epsilon}$.

11.5.4 Compatible strains interpolation

The virtual variations of the linear deformation $\tilde{\boldsymbol{\epsilon}}_k$ at the i-th integration point is obtained from Eq. 11.24:

$$\begin{gathered} \delta\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\xi}_i) = \delta\left(\boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{y}_{,k}(\boldsymbol{\xi}_i)\right) = \\ \sum_{i=1}^{n} \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{y}_{,k}(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) \boldsymbol{\varphi}_{n\delta} + \sum_{i=1}^{n} \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,k)(\boldsymbol{\xi}_i)\,\delta\boldsymbol{y}_n \end{gathered} \tag{11.57}$$

where $\boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,k)$ denotes the element (n,k) of the matrix $\boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)$.
The virtual variation of the curvatures $\tilde{\boldsymbol{\kappa}}_k$ at the i-th integration point is derived from Eq. 11.26:

$$\begin{gathered} \delta\tilde{\boldsymbol{\kappa}}_k(\boldsymbol{\xi}_i) = \delta\left(\boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{\kappa}_k(\boldsymbol{\xi}_i)\right) = \\ \sum_{i=1}^{n} \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{\kappa}_k(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) \boldsymbol{\varphi}_{n\delta} + \sum_{i=1}^{n} \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{K}_{kn}(\boldsymbol{\xi}_i) \boldsymbol{\varphi}_{n\delta} \end{gathered} \tag{11.58}$$

where:

$$\boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) = \overline{\boldsymbol{T}}\tilde{\boldsymbol{\Gamma}}(\tilde{\boldsymbol{\varphi}}(\boldsymbol{\xi}_i))\,\tilde{\boldsymbol{\Gamma}}_n^{-1}\overline{\boldsymbol{T}}^T \tag{11.59}$$

and:

$$\boldsymbol{K}_{kn}(\boldsymbol{\xi}_i) = \overline{\boldsymbol{T}}\mathcal{L}(\tilde{\boldsymbol{\varphi}}(\boldsymbol{\xi}_i), \tilde{\boldsymbol{\varphi}}_{,k}(\boldsymbol{\xi}_i))\tilde{\boldsymbol{\Gamma}}_n^{-1}\overline{T}^T L_n(\boldsymbol{\xi}_i) + \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i)\boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,k)(\boldsymbol{\xi}_i) \tag{11.60}$$

The derivative of the the vector $\tilde{\boldsymbol{\varphi}}$ with respect to the k-th arc-length coordinate can be expressed as function of the nodal values as:

$$\tilde{\boldsymbol{\varphi}}_{,k}(\boldsymbol{\xi}_i) = \sum_{n=1}^{4} \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,k)\boldsymbol{\varphi}_n \tag{11.61}$$

The curvatures are given by:

$$\boldsymbol{\kappa}_k(\boldsymbol{\xi}_i) = \overline{\boldsymbol{T}}\tilde{\boldsymbol{\Gamma}}(\tilde{\boldsymbol{\varphi}}(\boldsymbol{\xi}_i))\tilde{\boldsymbol{\varphi}}_{,k}(\boldsymbol{\xi}_i) \tag{11.62}$$

and the derivative of the position vector is:

$$\boldsymbol{y}_{,k}(\boldsymbol{\xi}_i) = \sum_{i=1}^{4} \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,k)\boldsymbol{y}_n \tag{11.63}$$

The compatible strains can so be expressed as function of the nodal variables $\boldsymbol{q}$ by considering Eqs. 11.57 and 11.58:

$$\delta\boldsymbol{\epsilon} = \overline{\boldsymbol{B}}\delta\boldsymbol{q} \tag{11.64}$$

with:

$$\underset{12\times 24}{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_i) = \begin{bmatrix} \overline{\boldsymbol{B}}_1(\boldsymbol{\xi}_i) & \overline{\boldsymbol{B}}_2(\boldsymbol{\xi}_i) & \overline{\boldsymbol{B}}_3(\boldsymbol{\xi}_i) & \overline{\boldsymbol{B}}_4(\boldsymbol{\xi}_i) \end{bmatrix} \tag{11.65}$$

and:

$$\underset{12\times 6}{\overline{\boldsymbol{B}}_n}(\boldsymbol{\xi}_i) = \begin{bmatrix} \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,1) & \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{y}_{,1}(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) \\ \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_i)(n,2) & \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{y}_{,2}(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) \\ 0 & \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{\kappa}_1(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) + \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{K}_{1n}(\boldsymbol{\xi}_i) \\ 0 & \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{\kappa}_2(\boldsymbol{\xi}_i) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_i) L_n(\boldsymbol{\xi}_i) + \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{K}_{2n}(\boldsymbol{\xi}_i) \end{bmatrix} \tag{11.66}$$

11.5.5 ANS

The ANS technique is applied to prevent shear locking. Here the ANS is applied to shear strains only, i.e. ϵ_1 and ϵ_2 The approach consists in interpolating the shear strains in the mid-sides of the element in the natural domain. Such interpolated strains are then used to derive the shear strains at the integration points.
The use of the ANS technique leads a re-definition of:

- the two rows of the $\overline{\boldsymbol{B}}$ matrix relative to shear strains
- the shear strains

The compatible strains are:

$$\tilde{\boldsymbol{\epsilon}}_1 = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_1 \end{Bmatrix} \qquad \tilde{\boldsymbol{\epsilon}}_2 = \begin{Bmatrix} \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_2 \end{Bmatrix} \tag{11.67}$$

At first, the compatible strains are evaluated at the shear evaluation points of Figure 11.1. This is done referring to Eq. 11.7

$$\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\xi}_A) = \boldsymbol{T}(\boldsymbol{\xi}_A)^T \boldsymbol{y}_{,k}(\boldsymbol{\xi}_A) - \boldsymbol{e}_k \tag{11.68}$$

The matrix $\overline{\boldsymbol{B}}$ of Eq. 11.65 is built also in the shear evaluation points for the membrane strains only, and is indicated as $\overline{\overline{\boldsymbol{B}}}$

$$\underset{6\times 24}{\overline{\overline{\boldsymbol{B}}}}(\boldsymbol{\xi}_A) = \left[\begin{array}{cccc} \overline{\overline{\boldsymbol{B}}}_1(\boldsymbol{\xi}_A) & \overline{\overline{\boldsymbol{B}}}_2(\boldsymbol{\xi}_A) & \overline{\overline{\boldsymbol{B}}}_3(\boldsymbol{\xi}_A) & \overline{\overline{\boldsymbol{B}}}_4(\boldsymbol{\xi}_A) \end{array} \right] \tag{11.69}$$

with:

$$\underset{6\times 6}{\overline{\overline{\boldsymbol{B}}}_n}(\boldsymbol{\xi}_A) = \left[\begin{array}{cc} \boldsymbol{T}(\boldsymbol{\xi}_A)^T \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_A)(n,1) & \boldsymbol{T}(\boldsymbol{\xi}_A)^T \boldsymbol{y}_{,1}(\boldsymbol{\xi}_A) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_A) L_n(\boldsymbol{\xi}_A) \\ \boldsymbol{T}(\boldsymbol{\xi}_A)^T \boldsymbol{L}_{\alpha\beta}(\boldsymbol{\xi}_A)(n,2) & \boldsymbol{T}(\boldsymbol{\xi}_A)^T \boldsymbol{y}_{,2}(\boldsymbol{\xi}_A) \times \boldsymbol{\Phi}_n(\boldsymbol{\xi}_A) L_n(\boldsymbol{\xi}_A) \end{array} \right] \tag{11.70}$$

The rows relative to the shear strains ϵ_1 and ϵ_2 are collected in:

$$\underset{4\times 24}{\overline{\overline{\boldsymbol{B3}}}}(\boldsymbol{\xi}_A) = \left[\begin{array}{c} \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_a)(3,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_b)(3,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_c)(3,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_d)(3,:) \end{array} \right] \qquad \underset{4\times 24}{\overline{\overline{\boldsymbol{B6}}}}(\boldsymbol{\xi}_A) = \left[\begin{array}{c} \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_a)(6,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_b)(6,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_c)(6,:) \\ \overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_d)(6,:) \end{array} \right] \tag{11.71}$$

where $\overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_a)(k,:)$ is used to denote all the element of the k-th row of $\overline{\overline{\boldsymbol{B}}}(\boldsymbol{\xi}_a)$.
The matrices $\overline{\overline{\boldsymbol{B3}}}$ and $\overline{\overline{\boldsymbol{B6}}}$ are interpolated to obtain the values at the integration points.

$$\overline{\boldsymbol{B3}}(\boldsymbol{\xi}_i) = \frac{1}{2} \left[\begin{array}{cccc} 1+\xi_{2i} & 0 & 1-\xi_{2i} & 0 \end{array} \right] \overline{\overline{\boldsymbol{B3}}}(\boldsymbol{\xi}_A) \tag{11.72}$$

$$\overline{\boldsymbol{B6}}(\boldsymbol{\xi}_i) = \frac{1}{2} \left[\begin{array}{cccc} 1+\xi_{1i} & 0 & 1-\xi_{1i} & 0 \end{array} \right] \overline{\overline{\boldsymbol{B6}}}(\boldsymbol{\xi}_A) \tag{11.73}$$

The resulting matrices $\overline{\boldsymbol{B3}}$ and $\overline{\boldsymbol{B6}}$ replace the corresponding rows of the matrix $\overline{\boldsymbol{B}}$ of Eq. 11.65.
The second step in the application of the ANS technique is the interpolation of the shear strains:

$$\epsilon_1(\boldsymbol{\xi}_i) = \frac{1}{2}(1+\xi_{2i})\,\epsilon_1(\boldsymbol{\xi}_a) + \frac{1}{2}(1-\xi_{2i})\,\epsilon_1(\boldsymbol{\xi}_c) \tag{11.74a}$$

$$\epsilon_2(\boldsymbol{\xi}_i) = \frac{1}{2}(1-\xi_{1i})\,\epsilon_2(\boldsymbol{\xi}_b) + \frac{1}{2}(1+\xi_{1i})\,\epsilon_2(\boldsymbol{\xi}_d) \tag{11.74b}$$

The so interpolated strains ϵ_1 and ϵ_2 are then inserted into the proper position in the vector $\boldsymbol{\epsilon}$ respectively.

11.5.6 Forces

Strains and curvatures can be calculated as:

$$\tilde{\boldsymbol{\epsilon}}_k(\boldsymbol{\xi}_i) = \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{y}_{,k}(\boldsymbol{\xi}_i) - \boldsymbol{e}_k \tag{11.75}$$

$$\tilde{\boldsymbol{\kappa}}_k = \boldsymbol{T}(\boldsymbol{\xi}_i)^T \boldsymbol{\kappa}_k(\boldsymbol{\xi}_i) - \boldsymbol{T}_0^T(\boldsymbol{\xi}_i)\,\boldsymbol{\kappa}_k^0(\boldsymbol{\xi}_i) \tag{11.76}$$

where $\boldsymbol{T}_0$ and $\boldsymbol{\kappa}_k^0$ are calculated at the first step.
The total strain is the sum of the compatible and the enhancing strains:

$$\boldsymbol{\epsilon}_t(\boldsymbol{\xi}_i) = \boldsymbol{\epsilon}(\boldsymbol{\xi}_i) + \hat{\boldsymbol{\epsilon}}(\boldsymbol{\xi}_i) \tag{11.77}$$

The resulting forces are easily computed by means of the constitutive law:

$$\boldsymbol{\sigma}(\boldsymbol{\xi}_i) = \boldsymbol{C}(\boldsymbol{\xi}_i)\,\boldsymbol{\epsilon}_t(\boldsymbol{\xi}_i) \tag{11.78}$$

which is:

$$\boldsymbol{\sigma}(\boldsymbol{\xi}_i) = \left\{ \begin{array}{c} \boldsymbol{n}_1(\boldsymbol{\xi}_i) \\ \boldsymbol{n}_2(\boldsymbol{\xi}_i) \\ \boldsymbol{m}_1(\boldsymbol{\xi}_i) \\ \boldsymbol{m}_2(\boldsymbol{\xi}_i) \end{array} \right\} \tag{11.79}$$

11.5.7 Jacobian matrix

The second variation of the variational principle of Eq. 11.5 gives the Jacobian matrix. In particular, the finite element discretization is here presented for the five different terms of Eq. 11.5.
The first term is:

$$\begin{aligned}
&\int_A \delta\boldsymbol{\epsilon}^T \boldsymbol{C} \Delta\boldsymbol{\epsilon} \mathrm{d}A = \\
&\qquad = \delta\boldsymbol{q}^T \sum_{i=1}^{4} \overline{\boldsymbol{B}}(\boldsymbol{\xi}_i)^T \boldsymbol{C}(\boldsymbol{\xi}_i)\, \overline{\boldsymbol{B}}(\boldsymbol{\xi}_i)\, \alpha(\boldsymbol{\xi}_i)\, w_i \Delta\boldsymbol{q} \\
&\qquad = \delta\boldsymbol{q}^T \boldsymbol{K}_m \Delta\boldsymbol{q}
\end{aligned} \tag{11.80}$$

The second term is:

$$\begin{aligned}
&\int_A \delta\hat{\boldsymbol{\epsilon}}^T \boldsymbol{C} \Delta\boldsymbol{\epsilon} \mathrm{d}A = \\
&\qquad = \delta\boldsymbol{\beta}^T \sum_{i=1}^{4} \boldsymbol{P}(\boldsymbol{\xi}_i)^T \boldsymbol{C}(\boldsymbol{\xi}_i)\, \overline{\boldsymbol{B}}(\boldsymbol{\xi}_i)\, \alpha(\boldsymbol{\xi}_i)\, w_i \Delta\boldsymbol{q} \\
&\qquad = \delta\boldsymbol{\beta}^T \boldsymbol{K}_{\beta q} \Delta\boldsymbol{q}
\end{aligned} \tag{11.81}$$

The third term is:

$$\begin{aligned}
&\int_A \delta\boldsymbol{\epsilon}^T \boldsymbol{C} \Delta\hat{\boldsymbol{\epsilon}} \mathrm{d}A = \\
&\qquad = \delta\boldsymbol{q}^T \boldsymbol{K}_{\beta q}^T \Delta\boldsymbol{\beta}
\end{aligned} \tag{11.82}$$

The fourth term is:

$$\begin{aligned}
&\int_A \delta\hat{\boldsymbol{\epsilon}}^T \boldsymbol{C} \Delta\hat{\boldsymbol{\epsilon}} \mathrm{d}A = \\
&\qquad = \delta\boldsymbol{\beta}^T \sum_{i=1}^{4} \boldsymbol{P}(\boldsymbol{\xi}_i)^T \boldsymbol{C}(\boldsymbol{\xi}_i)\, \boldsymbol{P}(\boldsymbol{\xi}_i)\, \alpha(\boldsymbol{\xi}_i)\, w_i \Delta\boldsymbol{\beta} \\
&\qquad = \delta\boldsymbol{\beta}^T \boldsymbol{K}_{\beta\beta} \Delta\boldsymbol{\beta}
\end{aligned} \tag{11.83}$$

The fifth term is:

$$\begin{aligned}
&\int_A \Delta\delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma} \mathrm{d}A = \\
&\qquad = \delta\boldsymbol{q}^T \sum_{i=1}^{4} \overline{\boldsymbol{D}}(\boldsymbol{\xi}_i)^T \boldsymbol{G}(\boldsymbol{\xi}_i)\, \overline{\boldsymbol{D}}(\boldsymbol{\xi}_i)\, \alpha(\boldsymbol{\xi}_i)\, w_i \Delta\boldsymbol{q} \\
&\qquad = \delta\boldsymbol{q}^T \boldsymbol{K}_g \Delta\boldsymbol{q}
\end{aligned} \tag{11.84}$$

where this last expression is obtained starting from Eqs. 11.28 and 11.29, and where:

$$\underset{15\times 24}{\overline{\boldsymbol{D}}}\left(\boldsymbol{\xi}_i\right) = \left[\begin{array}{cccc} \overline{\boldsymbol{D}}_1\left(\boldsymbol{\xi}_i\right) & \overline{\boldsymbol{D}}_2\left(\boldsymbol{\xi}_i\right) & \overline{\boldsymbol{D}}_3\left(\boldsymbol{\xi}_i\right) & \overline{\boldsymbol{D}}_4\left(\boldsymbol{\xi}_i\right) \end{array}\right] \tag{11.85}$$

with:

$$\underset{15\times 6}{\overline{\boldsymbol{D}}_n}\left(\boldsymbol{\xi}_i\right) = \begin{bmatrix} \boldsymbol{L}_{\alpha\beta}\left(\boldsymbol{\xi}_i\right)(n,1)\,\boldsymbol{I} & \mathbf{0} \\ \boldsymbol{L}_{\alpha\beta}\left(\boldsymbol{\xi}_i\right)(n,2)\,\boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{K}_{1n}\left(\boldsymbol{\xi}_i\right) \\ \mathbf{0} & \boldsymbol{K}_{2n}\left(\boldsymbol{\xi}_i\right) \\ \mathbf{0} & \boldsymbol{\Phi}_n\left(\boldsymbol{\xi}_i\right) L_n\left(\boldsymbol{\xi}_i\right) \end{bmatrix} \tag{11.86}$$

$$\begin{aligned} \boldsymbol{Hh}\left(\boldsymbol{\xi}_i\right) = {} & \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_1\left(\boldsymbol{\xi}_i\right) \otimes \boldsymbol{y}_{,1}\left(\boldsymbol{\xi}_i\right) - \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_1 \cdot \boldsymbol{y}_{,1}\left(\boldsymbol{\xi}_i\right)\boldsymbol{I} + \\ & + \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_2\left(\boldsymbol{\xi}_i\right) \otimes \boldsymbol{y}_{,2}\left(\boldsymbol{\xi}_i\right) - \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_2 \cdot \boldsymbol{y}_{,2}\left(\boldsymbol{\xi}_i\right)\boldsymbol{I} + \\ & + \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_1\left(\boldsymbol{\xi}_i\right) \otimes \boldsymbol{\kappa}_1\left(\boldsymbol{\xi}_i\right) - \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_1 \cdot \boldsymbol{\kappa}_1\left(\boldsymbol{\xi}_i\right)\boldsymbol{I} + \\ & + \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_2\left(\boldsymbol{\xi}_i\right) \otimes \boldsymbol{\kappa}_2\left(\boldsymbol{\xi}_i\right) - \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_2 \cdot \boldsymbol{\kappa}_2\left(\boldsymbol{\xi}_i\right)\boldsymbol{I} \end{aligned} \tag{11.87}$$

$$\underset{15\times 15}{\boldsymbol{G}}\left(\boldsymbol{\xi}_i\right) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_1\left(\boldsymbol{\xi}_i\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_2\left(\boldsymbol{\xi}_i\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_1\left(\boldsymbol{\xi}_i\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_1\left(\boldsymbol{\xi}_i\right) & \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{n}_2\left(\boldsymbol{\xi}_i\right) & \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_1\left(\boldsymbol{\xi}_i\right) & \boldsymbol{T}\left(\boldsymbol{\xi}_i\right)\boldsymbol{m}_2\left(\boldsymbol{\xi}_i\right) & \boldsymbol{Hh}\left(\boldsymbol{\xi}_i\right) \end{bmatrix} \tag{11.88}$$

11.5.8 Residual

The residual comes from the finite element approximation of Eq. 11.2.
The first term is:

$$\begin{aligned} \int_A \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\,\mathrm{d}A = & \\ & = \delta\boldsymbol{q}^T \sum_{i=1}^{4} \overline{\boldsymbol{B}}\left(\boldsymbol{\xi}_i\right)^T \boldsymbol{\sigma}\left(\boldsymbol{\xi}_i\right)\alpha\left(\boldsymbol{\xi}_i\right) w_i \\ & = \delta\boldsymbol{q}^T \boldsymbol{r}_d \end{aligned} \tag{11.89}$$

The second term is:

$$\begin{aligned} \int_A \delta\hat{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma}\,\mathrm{d}A = & \\ & = \delta\boldsymbol{\beta}^T \sum_{i=1}^{4} \boldsymbol{P}\left(\boldsymbol{\xi}_i\right)^T \boldsymbol{\sigma}\left(\boldsymbol{\xi}_i\right)\alpha\left(\boldsymbol{\xi}_i\right) w_i \\ & = \delta\boldsymbol{\beta}^T \boldsymbol{r}_\beta \end{aligned} \tag{11.90}$$

The resulting governing equations are then:

$$\left(\boldsymbol{K}_m + \boldsymbol{K}_g\right)\Delta\boldsymbol{q} + \boldsymbol{K}_{\beta q}^T \Delta\boldsymbol{\beta} = -\boldsymbol{r}_d \tag{11.91a}$$

$$\boldsymbol{K}_{\beta q}\Delta\boldsymbol{q} + \boldsymbol{K}_{\beta\beta}\Delta\boldsymbol{\beta} = -\boldsymbol{r}_\beta \tag{11.91b}$$

Chapter 12

Solid Element

Author: Reinhard Resch
Solid elements are intended to model general three dimensional structures with complex shapes, subject to large deformations, large strain and nonlinear constitutive laws. Even fully incompressible constitutive laws are supported.

Files. It is implemented in files
`mbdyn/struct/solid.h`
`mbdyn/struct/solid.cc`
`mbdyn/struct/solidshape.h`
`mbdyn/struct/solidshape.cc`
`mbdyn/struct/solidshapetest.cc`
`mbdyn/struct/solidinteg.h`
`mbdyn/struct/solidinteg.cc`
`mbdyn/struct/solidpress.h`
`mbdyn/struct/solidpress.cc`
`mbdyn/struct/solidcsl.h`
`mbdyn/struct/solidcsl.cc`

12.0.1 Kinematics

MBDyn's solid elements are based on the classical displacement based isoparametric Finite Element formulation [6]. For enhancements, required for nearly or fully incompressible constitutive laws, see section 12.0.4.

Interpolation The three dimensional global Cartesian coordinates $\boldsymbol{x}$ and the displacements $\boldsymbol{u} = \boldsymbol{x} - {}^{0}\boldsymbol{x}$, of a particle inside an element, are interpolated between coordinates $\hat{\boldsymbol{x}}$ and displacements $\hat{\boldsymbol{u}}$ at element nodes by means of shape functions $\boldsymbol{h} = f(\boldsymbol{r})$ [6], [7].

$$\boldsymbol{u}_i = \sum_{k=1}^{N_n} \hat{\boldsymbol{u}}_{ik}\, \boldsymbol{h}_k \tag{12.1}$$

$$\boldsymbol{x}_i = \sum_{k=1}^{N_n} \hat{\boldsymbol{x}}_{ik}\, \boldsymbol{h}_k \tag{12.2}$$

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix}^T \tag{12.3}$$

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{pmatrix}^T \tag{12.4}$$

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 \end{pmatrix}^T \tag{12.5}$$

$$\hat{\boldsymbol{u}} = \begin{pmatrix} \hat{\boldsymbol{u}}_{1,1} & \hat{\boldsymbol{u}}_{1,2} & \dots & \hat{\boldsymbol{u}}_{1,N_n} \\ \hat{\boldsymbol{u}}_{2,1} & \hat{\boldsymbol{u}}_{2,2} & \dots & \hat{\boldsymbol{u}}_{2,N_n} \\ \hat{\boldsymbol{u}}_{3,1} & \hat{\boldsymbol{u}}_{2,2} & \dots & \hat{\boldsymbol{u}}_{3,N_n} \end{pmatrix} \tag{12.6}$$

$$\boldsymbol{h} = \begin{pmatrix} \boldsymbol{h}_1 & \boldsymbol{h}_2 & \dots & \boldsymbol{h}_{N_n} \end{pmatrix}^T \tag{12.7}$$

$\boldsymbol{x} = f(\boldsymbol{r})$ global Cartesian coordinates of a particle inside the deformed body

${}^0\boldsymbol{x} = f(\boldsymbol{r})$ global Cartesian coordinates of a particle inside the undeformed body

$\boldsymbol{u} = \boldsymbol{x} - {}^0\boldsymbol{x}$ deformation of a particle within the body

$\boldsymbol{r}$ curvilinear natural coordinates of a particle within the body

$\boldsymbol{h} = f(\boldsymbol{r})$ shape functions of a particular Finite Element type

N_n number of nodes per element

Shape functions All solid elements in MBDyn are based on templates and are using standard shape functions based on [6], [8] and [9]. In order to implement a new solid element type in MBDyn, it is sufficient to provide a new C++ class which defines it's shape functions $\boldsymbol{h}$, derivatives of the shape functions versus the natural coordinates $\boldsymbol{h}_d = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}}$ and also the integration points $\boldsymbol{r}|_g$ and weights $\alpha|_g$. The following types of elements are currently implemented:

- Hexahedrons with linear and quadratic shape functions [6], [8]
- Pentahedrons with quadratic shape functions [9]
- Tetrahedrons with quadratic shape functions [9]

See also MBDyn's *input-manual* for further information.

Deformation gradient Because MBDyn's solid elements are based on a Total Lagrangian Formulation, the deformation gradient $\boldsymbol{F}$ is evaluated with respect to the undeformed state ${}^0\boldsymbol{x}$. As a consequence, the

Jacobian matrix ${}^0\boldsymbol{J}$ and the gradient operator $\boldsymbol{h}_{0d}$ are constant.

$$ {}^0\boldsymbol{J}_{ij} = \frac{\partial {}^0\boldsymbol{x}_j}{\partial \boldsymbol{r}_i} = \sum_{k=1}^{N_n} {}^0\hat{\boldsymbol{x}}_{jk} \, \underbrace{\frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{r}_i}}_{\boldsymbol{h}_{d_{ki}}} = \left({}^0\boldsymbol{x}\,\boldsymbol{h}_d\right)^T \tag{12.8} $$

$$ \left\{{}^0\boldsymbol{J}^{-1}\right\}_{ij} = \frac{\partial \boldsymbol{r}_j}{\partial {}^0\boldsymbol{x}_i} \tag{12.9} $$

$$ \boldsymbol{F}_{ij} = \frac{\partial \boldsymbol{x}_i}{\partial {}^0\boldsymbol{x}_j} = \delta_{ij} + \frac{\partial \boldsymbol{u}_i}{\partial {}^0\boldsymbol{x}_j} = \delta_{ij} + \sum_{k=1}^{N_n} \hat{\boldsymbol{u}}_{ik} \, \underbrace{\frac{\partial \boldsymbol{h}_k}{\partial {}^0\boldsymbol{x}_j}}_{\boldsymbol{h}_{0d_{kj}}} \tag{12.10} $$

$$ \dot{\boldsymbol{F}}_{ij} = \sum_{k=1}^{N_n} \dot{\hat{\boldsymbol{u}}}_{ik}\,\boldsymbol{h}_{0d_{kj}} \tag{12.11} $$

$$ \boldsymbol{h}_{0d_{ij}} = \frac{\partial \boldsymbol{h}_i}{\partial {}^0\boldsymbol{x}_j} = \sum_{k=1}^{3} \frac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{r}_k}\,\frac{\partial \boldsymbol{r}_k}{\partial {}^0\boldsymbol{x}_j} = \left\{\boldsymbol{h}_d\,{}^0\boldsymbol{J}^{-T}\right\}_{ij} \tag{12.12} $$

$$ \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{12.13} $$

$\boldsymbol{F}$ deformation gradient

${}^0\boldsymbol{J}$ Jacobian matrix $\det\left({}^0\boldsymbol{J}\right) > 0$

$\boldsymbol{h}_{0d} = \frac{\partial \boldsymbol{h}}{\partial {}^0\boldsymbol{x}}$ derivative of shape functions versus global Cartesian coordinates

$\boldsymbol{h}_d = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}}$ derivative of shape functions versus natural coordinates

δ_{ij} Kronecker delta

If elements become excessively distorted It is required that the deformation gradient $\boldsymbol{F}$ and the Jacobian matrix ${}^0\boldsymbol{J}$ are always invertible [6], [7]. For that reason $\det(\boldsymbol{F}) > 0$ will be checked by the solver at every iteration, and $\det\left({}^0\boldsymbol{J}\right) > 0$ will be checked once, when the mesh is loaded. An exception will be thrown by the solver, if any element becomes excessively distorted and those conditions are not valid any more.

Strain tensor and stress tensor In order to derive the expressions for virtual strain energy, the Green-Lagrange strain tensor $\boldsymbol{G}$ and the 2nd Piola-Kirchhoff stress tensor $\boldsymbol{S}$ are used. It is important to note, that the 2nd Piola-Kirchhoff stress tensor is work conjugate with the Green-Lagrange strain tensor [5], [6], [7]. Since MBDyn's solid element is pure displacement based, the stress tensor $\boldsymbol{S}$ is a function of the strain tensor $\boldsymbol{G}$ and optionally it's time derivative $\dot{\boldsymbol{G}}$ and the strain history. In case of viscoelastic constitutive laws, the strain rates $\dot{\boldsymbol{G}}$ are scaled according to equation 12.17 in order to make the effect of viscous damping independent on the strain [7].

$$\begin{aligned}
\boldsymbol{C}_{ij} &= \sum_{k=1}^{3} \boldsymbol{F}_{ki}\, \boldsymbol{F}_{kj} && (12.14)\\
\boldsymbol{G}_{ij} &= \frac{1}{2}\left(\boldsymbol{C}_{ij} - \delta_{ij}\right) = \frac{1}{2}\left(\sum_{k=1}^{3} \boldsymbol{F}_{ki}\, \boldsymbol{F}_{kj} - \delta_{ij}\right) && (12.15)\\
\dot{\boldsymbol{G}}_{ij} &= \frac{1}{2}\left[\sum_{k=1}^{3}\left(\dot{\boldsymbol{F}}_{ki}\, \boldsymbol{F}_{kj} + \boldsymbol{F}_{ki}\, \dot{\boldsymbol{F}}_{kj}\right)\right] && (12.16)\\
\dot{\boldsymbol{G}}^{\star} &= \boldsymbol{C}^{-1}\, \dot{\boldsymbol{G}}\, \boldsymbol{C}^{-1}\, \det\left(\boldsymbol{F}\right) && (12.17)\\
\boldsymbol{S} &= f\left(\boldsymbol{G}, \dot{\boldsymbol{G}}^{\star}\right) && (12.18)
\end{aligned}$$

$\boldsymbol{C}$ Right Cauchy-Green strain tensor

$\boldsymbol{G}$ Green-Lagrange strain tensor

$\dot{\boldsymbol{G}}^{\star}$ scaled Green-Lagrange strain rates

$\boldsymbol{S}$ 2nd Piola-Kirchhoff stress tensor

Constitutive laws The relationship between stress tensor $\boldsymbol{S}$ and strain tensor $\boldsymbol{G}$ is determined by a specific constitutive law. Right now, the following types of constitutive laws are implemented:

- linear elastic generic (isotropic or anisotropic)
- linear Kelvin-Voigt viscoelastic generic (isotropic or anisotropic) [7]
- Hookean linear elastic isotropic
- Hookean linear Kelvin-Voigt viscoelastic isotropic
- linear viscoelastic generalized Maxwell (isotropic or anisotropic) [10]
- nonlinear hyperelastic Neo-Hookean [7]
- nonlinear Kelvin-Voigt viscoelastic Neo-Hookean [7]
- nonlinear hyperelastic Mooney-Rivlin [6]
- bilinear elasto-plastic with isotropic hardening [6]

See MBDyn's *input-manual* for further information about constitutive laws usable for solid elements.

12.0.2 The principle of virtual displacements

The equations of motion for displacement based Finite Element methods can be derived from the principle of virtual displacements [5], [6], [7].

$$\underbrace{\bigcup_{e=1}^{N_e} \int_{{}^0V} {}^0\rho \left[\sum_{i=1}^{3} \delta \boldsymbol{u}_i \; (\ddot{\boldsymbol{u}}_i - \boldsymbol{b}_i)\right] d^0V}_{\delta^m W} + \underbrace{\bigcup_{e=1}^{N_e} \int_{{}^0V} \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \delta \boldsymbol{G}_{ij}\, \boldsymbol{S}_{ij}\right) d^0V}_{\delta^i W}$$

$$= \underbrace{\bigcup_{e=1}^{N_e} \int_{A} \sum_{i=1}^{3} \delta \boldsymbol{u}_i \, {}^A\boldsymbol{f}_i \, dA}_{\delta^e W} \tag{12.19}$$

${}^0\rho$ Density of the undeformed body

${}^0V$ Volume of the undeformed body

$\boldsymbol{b}$ Body loads due to gravity and rigid body kinematics

${}^A\boldsymbol{f}$ Surface loads due to pressure and surface traction's

${}^mW$ Virtual work of body loads (e.g. inertia terms and gravity loads)

${}^iW$ Internal virtual work (e.g. virtual strain energy)

${}^eW$ External virtual work (e.g. due to surface loads ${}^A\boldsymbol{f}$)

N_e Number of solid elements

$\bigcup$ Summation over all elements

12.0.3 Virtual strain energy

In order to get the expression for the internal force vector at the element nodes ${}^i\hat{\boldsymbol{k}}$, the following equivalence is used:

$$\delta^i W \;=\; \bigcup_{e=1}^{N_e} \int_{{}^0V} \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \delta \boldsymbol{G}_{ij}\, \boldsymbol{S}_{ij}\right) d^0V = \bigcup_{e=1}^{N_e} \sum_{k=1}^{3}\sum_{l=1}^{N_n} \delta \hat{\boldsymbol{u}}_{kl} \, {}^i\hat{\boldsymbol{k}}_{kl} \tag{12.20}$$

Virtual perturbation of the strain tensor As a first step, the virtual perturbation of the strain tensor $\delta\boldsymbol{G}$ must be expressed in terms of the vector of virtual nodal displacements $\delta\hat{\boldsymbol{u}}$. For that purpose, the virtual perturbation of equation 12.15 must be derived.

$$\delta \boldsymbol{G}_{ij} \;=\; \frac{1}{2} \sum_{k=1}^{3} (\delta \boldsymbol{F}_{ki}\, \boldsymbol{F}_{kj} + \boldsymbol{F}_{ki}\, \delta \boldsymbol{F}_{kj}) \tag{12.21}$$

Virtual perturbation of the deformation gradient $\delta\boldsymbol{F}$ In the same way, the virtual perturbation of equation 12.10 is derived. Since we are using a Total Lagrangian Formulation, $\boldsymbol{h}_{0d}$ is a constant matrix.

$$\delta \boldsymbol{F}_{ij} \;=\; \sum_{k=1}^{N_n} \delta \hat{\boldsymbol{u}}_{ik}\, \boldsymbol{h}_{0d_{kj}} \tag{12.22}$$

Virtual perturbation of the strain tensor expressed by virtual displacements $\delta\hat{\boldsymbol{u}}$ Now equation 12.22 is substituted into equation 12.21.

$$\delta\boldsymbol{G}_{ij} \quad = \quad \frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{N_n}\delta\hat{\boldsymbol{u}}_{kl}\,\left(\boldsymbol{h}_{0d_{li}}\,\boldsymbol{F}_{kj}+\boldsymbol{h}_{0d_{lj}}\,\boldsymbol{F}_{ki}\right) \tag{12.23}$$

Virtual strain energy expressed by virtual displacements $\delta\hat{\boldsymbol{u}}$ Finally equation 12.23 is substituted into equation 12.20.

$$\delta^{i}W \quad = \quad \frac{1}{2}\bigcup_{e=1}^{N_e}\int_{^0V}\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{N_n}\delta\hat{\boldsymbol{u}}_{kl}\left(\boldsymbol{h}_{0d_{li}}\,\boldsymbol{F}_{kj}+\boldsymbol{h}_{0d_{lj}}\,\boldsymbol{F}_{ki}\right)\,\boldsymbol{S}_{ij}\right]\,d^{0}V \tag{12.24}$$

Internal elastic reactions due to internal stress Because equation 12.24 and equation 12.20 must be valid for arbitrary virtual displacements $\delta\hat{\boldsymbol{u}}$, we can get ${}^{i}\hat{\boldsymbol{k}}$ just by comparing the coefficients of those two equations.

$$^{i}\hat{\boldsymbol{k}}_{kl} \quad = \quad \frac{1}{2}\int_{^0V}\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\boldsymbol{h}_{0d_{li}}\,\boldsymbol{F}_{kj}+\boldsymbol{h}_{0d_{lj}}\,\boldsymbol{F}_{ki}\right)\,\boldsymbol{S}_{ij}\right]\,d^{0}V \tag{12.25}$$

Numerical integration Finally standard numerical integration schemes (e.g. Gauss-Legendre) are applied, which sum up the weighted integrand at several integration points [6], [7]. Also in this case the values of $\det\left({}^{0}\boldsymbol{J}\right)$ for each integration point g are constant with respect to time, because a Total Lagrangian Formulation is used.

$$^{i}\hat{\boldsymbol{k}}_{kl} \quad \approx \quad \frac{1}{2}\sum_{g=1}^{N_g}\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\boldsymbol{h}_{0d_{li}}\,\boldsymbol{F}_{kj}+\boldsymbol{h}_{0d_{lj}}\,\boldsymbol{F}_{ki}\right)\,\boldsymbol{S}_{ij}\,\alpha\,\det\left({}^{0}\boldsymbol{J}\right)\right]\Bigg|_{g} \tag{12.26}$$

α Weighting factor

N_g Number of integration points

$|_g$ Expression is evaluated at integration point g

12.0.4 Displacement/Pressure (u/p-c) formulation

In case of nearly incompressible constitutive laws, pure displacement based formulations are not effective. For that reason, it is necessary to introduce the hydrostatic pressure $\tilde{p}$ as an additional unknown. For a detailed discussion see also [6] section 4.4.3. In order to derive the u/p-c formulation for large displacements, the following modified strain energy potential per unit volume is assumed for isotropic materials [6]:

$$^{i}\mathbb{W} = ^{i}\bar{\mathbb{W}} - \frac{1}{2\kappa}\left(\bar{p}-\tilde{p}\right)^{2} \tag{12.27}$$

See also [6] section 6.4.1, equation 6.136.

Strain energy potential per unit volume Within this section, ${}^{i}\mathbb{W}$ denotes strain energy potential per unit volume, whereas ${}^{i}W$ denotes overall strain energy potential. They are related as follows:

$$ {}^{i}W = \bigcup_{e=1}^{N_e} \int_{{}^{0}V} {}^{i}\mathbb{W}\, d^{0}V \tag{12.28} $$

${}^{i}W$ modified strain energy potential for the u/p-c formulation

${}^{i}\bar{W}$ strain energy potential for the pure displacement based formulation which is equal to equation 12.20

${}^{i}\mathbb{W}$ modified strain energy potential per unit volume for the u/p-c formulation

${}^{i}\bar{\mathbb{W}}$ strain energy potential per unit volume for the pure displacement based formulation equivalent to equation 12.20

$\bar{p}$ hydrostatic pressure obtained from the pure displacement based formulation

$\tilde{p}$ hydrostatic pressure interpolated from the element nodes

κ bulk modulus of the material

In order to enhance the principle of virtual-displacements equation 12.19, the virtual perturbation of the modified strain-energy potential ${}^{i}\mathbb{W}$ is derived with respect to the new unknowns (displacement $\hat{\boldsymbol{u}}$ and hydrostatic pressure $\hat{\boldsymbol{p}}$ at element nodes). This is equivalent to the application of the "Hu-Washizu" variational-principle. See also [6] section 4.4.3.

$$ \delta^{i}\mathbb{W} \quad = \quad \delta^{i}\bar{\mathbb{W}} - \frac{1}{\kappa}\,(\bar{p}-\tilde{p})\,(\delta\bar{p}-\delta\tilde{p}) \tag{12.29} $$

As a next step, the virtual perturbations of the hydrostatic pressure $\bar{p}$ and $\tilde{p}$ are derived.

$$ \delta\bar{p} \quad = \quad \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial\bar{p}}{\partial\boldsymbol{G}_{ij}}\,\delta\boldsymbol{G}_{ij} \tag{12.30} $$

$$ \delta\tilde{p} \quad = \quad \sum_{i=1}^{N_p} \frac{\partial\tilde{p}}{\partial\hat{\boldsymbol{p}}_i}\,\delta\hat{\boldsymbol{p}}_i \tag{12.31} $$

N_p number element nodes for hydrostatic pressure

The displacement based hydrostatic pressure $\bar{p}$ is depending only on the strain tensor $\boldsymbol{G}$ whereas the hydrostatic pressure $\tilde{p}$ is depending only on the assumed hydrostatic pressure at element nodes $\hat{\boldsymbol{p}}$.

Reformulation in terms of the modified stress tensor In the following step, equation 12.30 and equation 12.31 are substituted into equation 12.29.

$$ \delta^{i}\mathbb{W} \quad = \quad \delta^{i}\bar{\mathbb{W}} - \frac{1}{\kappa}\,(\bar{p}-\tilde{p}) \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial\bar{p}}{\partial\boldsymbol{G}_{ij}}\,\delta\boldsymbol{G}_{ij} - \sum_{i=1}^{N_p} \frac{\partial\tilde{p}}{\partial\hat{\boldsymbol{p}}_i}\,\delta\hat{\boldsymbol{p}}_i \right) \tag{12.32} $$

Furthermore we can use the expression for the virtual perturbation of the displacement based strain energy from equation 12.20. Within this section, $\bar{\boldsymbol{S}}$ denotes the displacement based stress tensor and $\boldsymbol{S}$ denotes the modified stress tensor.

$$ \delta^{i}\bar{\mathbb{W}} \quad = \quad \sum_{i=1}^{3}\sum_{j=1}^{3} \delta\boldsymbol{G}_{ij}\,\bar{\boldsymbol{S}}_{ij} \tag{12.33} $$

When comparing equation 12.32 to equation 12.33, it is obvious that the modified stress tensor $\boldsymbol{S}$ may be defined as:

$$\boldsymbol{S}_{ij} \quad = \quad \bar{\boldsymbol{S}}_{ij} - \frac{1}{\kappa} (\bar{p} - \tilde{p}) \frac{\partial \bar{p}}{\partial \boldsymbol{G}_{ij}} \tag{12.34}$$

The derivation of equation 12.34 is specific to a particular type of constitutive law. At the moment, the following types of constitutive laws area available for u/p-c formulations:

- Hookean linear elastic isotropic
- nonlinear hyperelastic Mooney-Rivlin
- bilinear-elastoplastic with isotropic hardening

Since the modified stress tensor is depending not only on the strain tensor $\boldsymbol{G}$, but also on the hydrostatic pressure $\tilde{p}$, specialized types of constitutive laws are required. Furthermore, it is required that those constitutive laws must return also the condition of volumetric compatibility $\frac{1}{\kappa} (\bar{p} - \tilde{p})$. See also MBDyn's *input-manual* for further information about constitutive laws usable for solid elements based on u/p-c formulations.

Using the modified stress tensor and the condition of volumetric compatibility Now equation 12.34 and equation 12.33 are substituted into equation 12.32.

$$\delta^i \mathbb{W} \quad = \quad \sum_{i=1}^{3} \sum_{j=1}^{3} \delta \boldsymbol{G}_{ij} \, \boldsymbol{S}_{ij} + \frac{1}{\kappa} (\bar{p} - \tilde{p}) \sum_{i=1}^{N_p} \frac{\partial \tilde{p}}{\partial \hat{\boldsymbol{p}}_i} \, \delta \hat{\boldsymbol{p}}_i \tag{12.35}$$

Since we are using isoparametric shape functions $\boldsymbol{g}$ in order to interpolate the hydrostatic pressure $\tilde{p}$, the following simplification can be applied:

$$\tilde{p} \quad = \quad \sum_{i=1}^{N_p} \boldsymbol{g}_i \, \hat{\boldsymbol{p}}_i \tag{12.36}$$

$$\frac{\partial \tilde{p}}{\partial \hat{\boldsymbol{p}}_i} \quad = \quad \boldsymbol{g}_i \tag{12.37}$$

If the order of interpolation for the hydrostatic pressure $\hat{\boldsymbol{p}}$ is too high compared to the order of interpolation for displacements $\hat{\boldsymbol{u}}$, then the resulting element may behave like an displacement based element which will not be effective [6]. So, a lower order of interpolation is used for the hydrostatic pressure in MBDyn. For example, linear elements are using a constant hydrostatic pressure, whereas quadratic elements are using a linear interpolation of the hydrostatic pressure based on the pressure at corner nodes. See also MBDyn's *input-manual* for further information about the node layout.

Integration over the volume of all elements In the next step, we integrate the virtual perturbation of the modified strain energy over the element volume, and sum up the contributions of all the elements.

$$\delta^i W \quad = \quad \bigcup_{e=1}^{N_e} \int_{^0V} \delta^i \mathbb{W} \, d^0V \tag{12.38}$$

Finally we can substitute equation 12.35 into equation 12.38.

$$\delta^i W \quad = \quad \bigcup_{e=1}^{N_e} \int_{^0V} \sum_{i=1}^{3} \sum_{j=1}^{3} \delta \boldsymbol{G}_{ij} \, \boldsymbol{S}_{ij} \, d^0V + \bigcup_{e=1}^{N_e} \int_{^0V} \frac{1}{\kappa} (\bar{p} - \tilde{p}) \sum_{i=1}^{N_p} \boldsymbol{g}_i \, \delta \hat{\boldsymbol{p}}_i \, d^0V \tag{12.39}$$

In the end, we can use the same formula for the internal force vector ${}^{i}\hat{\boldsymbol{k}}$ as equation 12.25, just by replacing the displacement based stress tensor $\bar{\boldsymbol{S}}$ by the modified stress tensor $\boldsymbol{S}$. In addition to that, equation 12.41 is required, which represents the weak form of the condition of volumetric compatibility.

$$
\begin{aligned}
{}^{i}\hat{\boldsymbol{k}}_{kl} &= \frac{1}{2}\int_{{}^{0}V}\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\boldsymbol{h}_{0d_{li}}\,\boldsymbol{F}_{kj}+\boldsymbol{h}_{0d_{lj}}\,\boldsymbol{F}_{ki}\right)\,\boldsymbol{S}_{ij}\right]d^{0}V && (12.40)\\
\hat{\boldsymbol{c}}_{i} &= \int_{{}^{0}V}\frac{1}{\kappa}\left(\bar{p}-\tilde{p}\right)\boldsymbol{g}_{i}\,d^{0}V && (12.41)\\
\delta^{i}W &= \bigcup_{e=1}^{N_e}\sum_{k=1}^{3}\sum_{l=1}^{N_n}\delta\hat{\boldsymbol{u}}_{kl}\,{}^{i}\hat{\boldsymbol{k}}_{kl}+\bigcup_{e=1}^{N_e}\sum_{i=1}^{N_p}\delta\hat{\boldsymbol{p}}_{i}\,\hat{\boldsymbol{c}}_{i} && (12.42)
\end{aligned}
$$

Those expressions will be used in order to formulate the global equations of motion 12.54 and 12.55 in section 12.0.6.

12.0.5 Virtual work of inertia terms and body loads

Also accelerations $\ddot{\hat{\boldsymbol{u}}}$ and virtual displacements $\delta\hat{\boldsymbol{u}}$ are interpolated by the same shape functions $\boldsymbol{h}$.

$$
\begin{aligned}
\delta^{m}W &= \bigcup_{e=1}^{N_e}\int_{{}^{0}V}\sum_{i=1}^{3}{}^{0}\rho\,\delta\boldsymbol{u}_{i}\,\left(\ddot{\boldsymbol{u}}_{i}-\boldsymbol{b}_{i}\right)\,d^{0}V && (12.43)\\
\delta\boldsymbol{u}_{i} &= \sum_{k=1}^{N_n}\delta\hat{\boldsymbol{u}}_{ik}\,\boldsymbol{h}_{k} && (12.44)\\
\ddot{\boldsymbol{u}}_{i} &= \sum_{k=1}^{N_n}\ddot{\hat{\boldsymbol{u}}}_{ik}\,\boldsymbol{h}_{k} && (12.45)
\end{aligned}
$$

Body loads In MBDyn, body loads due to gravity $\boldsymbol{g}$ and rigid body kinematics are considered. See also section 8.1.6. As a consequence, solid elements may be used also for typical rotor-dynamic analysis.

$$
\boldsymbol{b} = \boldsymbol{g}-\ddot{\overline{\boldsymbol{x}}}_{0}-\overline{\boldsymbol{\omega}}_{0}\times\left(\overline{\boldsymbol{\omega}}_{0}\times\boldsymbol{x}\right)-\dot{\overline{\boldsymbol{\omega}}}_{0}\times\boldsymbol{x}\underbrace{-2\,\overline{\boldsymbol{\omega}}_{0}\times\dot{\boldsymbol{u}}}_{a_c} \qquad (12.46)
$$

$\boldsymbol{g}$ Gravity.

$\ddot{\overline{\boldsymbol{x}}}_{0}$ Acceleration of the global reference frame.

$\overline{\boldsymbol{\omega}}_{0}$ Angular velocity of the global reference frame.

$\dot{\overline{\boldsymbol{\omega}}}_{0}$ Angular acceleration of the global reference frame.

$\boldsymbol{a}_{c}$ Acceleration of Coriolis (contributed by the “AutomaticStructDispElem” element)

Mass matrix and body load vector When equation 12.44 and equation 12.45 are substituted into equation 12.43, the virtual work $\delta^{m}W$ can be written in terms of the consistent mass matrix $\boldsymbol{M}$ and body load vector ${}^{b}\hat{\boldsymbol{f}}$. Because of the isoparametric approach, the consistent mass matrix $\boldsymbol{M}$ is constant with respect to time and needs to be evaluated only once [6], [7]. In contradiction to that, the actual magnitudes of

$\boldsymbol{b}$ and ${}^{b}\hat{\boldsymbol{f}}$ may be time dependent.

$$\hat{\boldsymbol{M}}_{kl} = \int_{{}^{0}V} {}^{0}\rho\, \boldsymbol{h}_k\, \boldsymbol{h}_l\, d^{0}V \approx \sum_{g=1}^{N_g} \left[{}^{0}\rho\, \boldsymbol{h}_k\, \boldsymbol{h}_l\, \alpha\, \det\left({}^{0}\boldsymbol{J}\right)\right]\Big|_g \tag{12.47}$$

$${}^{b}\hat{\boldsymbol{f}}_{ik} = \int_{{}^{0}V} {}^{0}\rho\, \boldsymbol{h}_k\, \boldsymbol{b}_i\, d^{0}V \approx \sum_{g=1}^{N_g} \left[{}^{0}\rho\, \boldsymbol{h}_k\, \boldsymbol{b}_i\, \alpha\, \det\left({}^{0}\boldsymbol{J}\right)\right]\Big|_g \tag{12.48}$$

$$\delta^{m}W = \bigcup_{e=1}^{N_e} \left(\sum_{i=1}^{3}\sum_{k=1}^{N_n}\sum_{l=1}^{N_n} \delta\hat{\boldsymbol{u}}_{ik}\, \ddot{\hat{\boldsymbol{u}}}_{il}\, \hat{\boldsymbol{M}}_{kl} - \sum_{i=1}^{3}\sum_{k=1}^{N_n} \delta\hat{\boldsymbol{u}}_{ik}\, {}^{b}\hat{\boldsymbol{f}}_{ik}\right) = \bigcup_{e=1}^{N_e} \delta\hat{\boldsymbol{u}}^{T} \left(\boldsymbol{M}\, \ddot{\hat{\boldsymbol{u}}} - {}^{b}\hat{\boldsymbol{f}}\right) \tag{12.49}$$

$\hat{\boldsymbol{M}}$ $N_n \times N_n$ element mass matrix in compact storage

$\boldsymbol{M}$ $3N_n \times 3N_n$ element mass matrix in redundant storage (e.g. $\boldsymbol{M}_{iijk} = \hat{\boldsymbol{M}}_{jk}$)

${}^{b}\hat{\boldsymbol{f}}$ Element body load vector

Lumped mass matrix In addition to a consistent mass matrix, also a lumped mass matrix can be used in MBDyn. For that purpose, the diagonal of the consistent mass matrix is scaled, so that the overall mass is conserved.

$$\hat{\boldsymbol{M}}^{\star}_{kk} = \int_{{}^{0}V} {}^{0}\rho\, \boldsymbol{h}_k^2\, d^{0}V \approx \sum_{g=1}^{N_g} \left[{}^{0}\rho\, \boldsymbol{h}_k^2\, \alpha\, \det\left({}^{0}\boldsymbol{J}\right)\right]\Big|_g \tag{12.50}$$

$$\hat{\boldsymbol{M}}_{kk} = \hat{\boldsymbol{M}}^{\star}_{kk} \frac{m}{\sum_{k=1}^{N_n} \hat{\boldsymbol{M}}^{\star}_{kk}} \tag{12.51}$$

$$m = \int_{{}^{0}V} {}^{0}\rho\, d^{0}V \approx \sum_{g=1}^{N_g} \left[{}^{0}\rho\, \alpha\, \det\left({}^{0}\boldsymbol{J}\right)\right]\Big|_g \tag{12.52}$$

If a lumped mass matrix is used, the virtual work of the inertia terms becomes:

$$\delta^{m}W = \bigcup_{e=1}^{N_e} \left(\sum_{i=1}^{3}\sum_{k=1}^{N_n} \delta\hat{\boldsymbol{u}}_{ik}\, \ddot{\hat{\boldsymbol{u}}}_{ik}\, \hat{\boldsymbol{M}}_{kk} - \sum_{i=1}^{3}\sum_{k=1}^{N_n} \delta\hat{\boldsymbol{u}}_{ik}\, {}^{b}\hat{\boldsymbol{f}}_{ik}\right) \tag{12.53}$$

12.0.6 Reformulation of inertia terms required for a first order DAE solver

The principle of virtual displacements equation 12.19 leads to the following system of second order ordinary differential equations.

$$\bigcup_{e=1}^{N_e} \left(\boldsymbol{M}\, \ddot{\hat{\boldsymbol{u}}} + {}^{i}\hat{\boldsymbol{k}} - {}^{b}\hat{\boldsymbol{f}} - {}^{e}\hat{\boldsymbol{f}}\right) = \boldsymbol{0} \tag{12.54}$$

The special case of u/p-c formulations In case of an u/p-c formulation, an additional set of algebraic equations 12.41 is required in order to determine the hydrostatic pressure $\hat{\boldsymbol{p}}$.

$$\bigcup_{e=1}^{N_e} \hat{\boldsymbol{c}} = \boldsymbol{0} \tag{12.55}$$

As a consequence, the whole system of equations becomes Differential Algebraic, even if it is unconstrained.

Implementation issues in MBDyn Since MBDyn does not use accelerations for the solution of equations of motion, a reformulation of the equations of motion is required for solid elements. It is required also because rigid bodies might be present in the same model, and those rigid bodies might share common nodes with solid elements. So, equation 12.54 cannot be implemented directly.

Introducing the momentum Since the mass matrix $\boldsymbol{M}$ is constant with respect to time, equation 12.54 can be easily reformulated as a system of two first order ODE's using the momentum $\hat{\boldsymbol{\beta}}$ as a new unknown. This approach is strictly consistent with MBDyn's conventions for rigid bodies.

$$\hat{\boldsymbol{\beta}} = \boldsymbol{M}\,\dot{\hat{\boldsymbol{u}}} \tag{12.56}$$

$$\dot{\hat{\boldsymbol{\beta}}} = \boldsymbol{M}\,\ddot{\hat{\boldsymbol{u}}} \tag{12.57}$$

$\hat{\boldsymbol{\beta}}$ Momentum at element nodes

With equation 12.56 and equation 12.57 the global system of equations is formulated as:

$$\bigcup_{e=1}^{N_e} \left(\hat{\boldsymbol{\beta}} - \boldsymbol{M}\,\dot{\hat{\boldsymbol{u}}}\right) = \boldsymbol{0} \tag{12.58}$$

$$-\bigcup_{e=1}^{N_e} \left(\dot{\hat{\boldsymbol{\beta}}} + {}^{i}\hat{\boldsymbol{k}} - {}^{b}\hat{\boldsymbol{f}} - {}^{e}\hat{\boldsymbol{f}}\right) = \boldsymbol{0} \tag{12.59}$$

$$-\bigcup_{e=1}^{N_e} \hat{\boldsymbol{c}} = \boldsymbol{0} \tag{12.60}$$

In equation 12.58 to 12.60, MBDyn's sign conventions are applied in order to ensure compatiblity with other elements. Equation 12.60 is present only in case of u/p-c based formulations.

Limitations The only remaining limitation of this approach is, that it is not possible to compute accelerations for a model using solid elements, unless a lumped mass matrix is used for all solid elements in the model. See the MBDyn's *input-manual* for further information on how to activate a lumped mass matrix instead of a consistent mass matrix.

12.0.7 Elements for surface loads

In order to apply surface loads at solid elements, dedicated surface elements are required. Basically the same isoparametric approach known from section 12.0.1 is applied also to surface elements. Instead of the determinant of the Jacobian matrix, equation 12.63 is used to transform infinitesimal surface areas between natural curvilinear coordinates $\boldsymbol{r}$ and global Cartesian coordinates $\boldsymbol{x}$.

$$\boldsymbol{x}_i = \sum_{k=1}^{N_n} \boldsymbol{h}_k \, \hat{\boldsymbol{x}}_{ik} \tag{12.61}$$

$$\boldsymbol{u}_i = \sum_{k=1}^{N_n} \boldsymbol{h}_k \, \hat{\boldsymbol{u}}_{ik} \tag{12.62}$$

$$dA = \left\| \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}_1} \times \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}_2} \right\| d\boldsymbol{r}_1 \, d\boldsymbol{r}_2 \tag{12.63}$$

$$\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{r}_1} = \sum_{k=1}^{N_n} \frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{r}_1} \, \hat{\boldsymbol{x}}_i \tag{12.64}$$

$$\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{r}_2} = \sum_{k=1}^{N_n} \frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{r}_2} \, \hat{\boldsymbol{x}}_i \tag{12.65}$$

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 \end{pmatrix}^T \tag{12.66}$$

Shape functions Also surface elements are based on C++ templates in order to simplify the implementation of new element types. The following types of elements are currently implemented:

- Quadrangular elements with linear and quadratic shape functions [6], [8]
- Triangular elements with quadratic shape functions [9]

See also MBDyn's *input-manual* for further information about the node layout of different surface elements.

Virtual work of surface loads In order to derive the expressions for external virtual work, surface loads must be expressed in terms of virtual displacements $\delta\hat{\boldsymbol{u}}$ at the element nodes.

$$\delta^e W = \bigcup_{e=1}^{N_e} \int_A \left(\sum_{i=1}^{3} \delta \boldsymbol{u}_i \, {}^A\boldsymbol{f}_i \right) dA \tag{12.67}$$

$$= \bigcup_{e=1}^{N_e} \int_A \left(\sum_{i=1}^{3} \sum_{k=1}^{N_n} \delta \hat{\boldsymbol{u}}_{ik} \, \boldsymbol{h}_k \, {}^A\boldsymbol{f}_i \right) dA \tag{12.68}$$

$$= \bigcup_{e=1}^{N_e} \sum_{i=1}^{3} \sum_{k=1}^{N_n} \delta \hat{\boldsymbol{u}}_{ik} \, {}^e\hat{\boldsymbol{f}}_{ik} \tag{12.69}$$

Pressure loads In case of pressure loads, ${}^A\boldsymbol{f}$ will always act normal to the surface. It should be emphasized, that the surface normal- and tangent vectors must be evaluated with respect to the deformed state.

$${}^A\boldsymbol{f} = -\frac{1}{\|{}^A\boldsymbol{n}\|} \, {}^A\boldsymbol{n} \, p \tag{12.70}$$

$$p = \sum_{k=1}^{N_n} \boldsymbol{h}_k \, \hat{\boldsymbol{p}}_k \tag{12.71}$$

$${}^A\boldsymbol{n} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}_1} \times \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}_2} \tag{12.72}$$

p pressure at a point $\boldsymbol{x}$ at the surface

$\hat{\boldsymbol{p}}$ pressures at the element nodes

${}^{A}\boldsymbol{n}$ outward surface normal vector at point $\boldsymbol{x}$

Finally, the virtual work of pressure loads can be evaluated by numerical integration across the surface area. Due to the fact, that the surface may be moved or deformed during the simulation, pressure loads will not be constant, even if the nodal pressures are constant values. In addition to that, pressure loads do contribute to the global stiffness matrix, and they will directly (e.g. due to the deformation of the surface) and indirectly (e.g. due to the stress in solid elements) affect mode shapes and natural frequencies of the structures they are applied to.

$$\begin{aligned}
\delta^{e}W &= -\bigcup_{e=1}^{N_e}\int_A \left(\sum_{i=1}^{3}\sum_{k=1}^{N_n}\delta\hat{\boldsymbol{u}}_{ik}\,\boldsymbol{h}_k\,{}^{A}\boldsymbol{n}_i\,\frac{p}{\|{}^{A}\boldsymbol{n}\|}\right) dA && (12.73)\\
&\approx -\bigcup_{e=1}^{N_e}\sum_{g=1}^{N_g}\left.\left(\sum_{i=1}^{3}\sum_{k=1}^{N_n}\delta\hat{\boldsymbol{u}}_{ik}\,\boldsymbol{h}_k\,p\,{}^{A}\boldsymbol{n}_i\,\alpha\right)\right|_g && (12.74)
\end{aligned}$$

Surface traction's Surface traction's are similar to pressure loads, but they do not necessarily act normal to the surface. Instead the initial orientation of traction loads will be specified by the user, by means of an orientation matrix $\boldsymbol{R}_f$. In that way, shear stresses may be imposed at the surface of a body.

$$\begin{aligned}
\boldsymbol{e}_1^{\star} &= \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{r}_1} && (12.75)\\
\boldsymbol{e}_2^{\star} &= \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{r}_2} && (12.76)\\
\boldsymbol{e}_3^{\star} &= \boldsymbol{e}_1^{\star}\times\boldsymbol{e}_2^{\star} && (12.77)\\
\boldsymbol{e}_2^{\star\star} &= \boldsymbol{e}_3^{\star}\times\boldsymbol{e}_1^{\star} && (12.78)\\
{}^{A}\boldsymbol{R} &= \begin{pmatrix}\frac{1}{\|\boldsymbol{e}_1^{\star}\|}\boldsymbol{e}_1^{\star} & \frac{1}{\|\boldsymbol{e}_2^{\star\star}\|}\boldsymbol{e}_2^{\star\star} & \frac{1}{\|\boldsymbol{e}_3^{\star}\|}\boldsymbol{e}_3^{\star}\end{pmatrix} && (12.79)\\
{}^{A}\boldsymbol{f} &= {}^{A}\boldsymbol{R}\,{}^{A_0}\boldsymbol{R}^{T}\,\boldsymbol{R}_f\,{}^{A}\bar{\boldsymbol{f}} && (12.80)\\
{}^{A}\bar{\boldsymbol{f}}_i &= \sum_{k=1}^{N_n}\boldsymbol{h}_k\,{}^{A}\hat{\bar{\boldsymbol{f}}}_{ik} && (12.81)\\
\delta^{e}W &= \bigcup_{e=1}^{N_e}\int_A\left(\sum_{i=1}^{3}\sum_{k=1}^{N_n}\delta\hat{\boldsymbol{u}}_{ik}\,\boldsymbol{h}_k\,{}^{A}\boldsymbol{f}_i\right) dA && (12.82)\\
&\approx \bigcup_{e=1}^{N_e}\sum_{g=1}^{N_g}\left.\left(\sum_{i=1}^{3}\sum_{k=1}^{N_n}\delta\hat{\boldsymbol{u}}_{ik}\,\boldsymbol{h}_k\,{}^{A}\boldsymbol{f}_i\,\|\boldsymbol{e}_3^{\star}\|\,\alpha\right)\right|_g && (12.83)
\end{aligned}$$

${}^{A}\boldsymbol{R}$ relative orientation matrix at point $\boldsymbol{x}$ at the deformed state

${}^{A_0}\boldsymbol{R}$ relative orientation matrix at point $\boldsymbol{x}$ at the undeformed state

$\boldsymbol{R}_f$ absolute orientation matrix for the applied traction load with respect to the undeformed state

${}^{A}\boldsymbol{f}$ surface traction's at point $\boldsymbol{x}$ with respect to the global reference frame

${}^{A}\bar{\boldsymbol{f}}$ surface traction's at point $\boldsymbol{x}$ with respect to the local reference frame

${}^{A}\hat{\bar{\boldsymbol{f}}}$ surface traction's at element nodes

12.0.8 Implementation notes

Many authors of textbooks about nonlinear Finite Element Methods spent a lot of effort to explain the consistent derivation of the tangent stiffness matrix, since this is crucial for any Newton-like nonlinear solver [5], [6]. In contradiction to that, the analytical derivation of a sparse tangent stiffness matrix is not required for MBDyn's solid elements, because they are exploiting a technique called "Automatic Differentiation" for Newton based- as well as Newton Krylov based solvers.

Chapter 13

Aerodynamic Elements

Aerodynamic elements apply aerodynamic forces to structural nodes.

This section is not intended to give details about the aerodynamic models adopted but mainly discuss the computation of the contributions to the Jacobian matrix of the aerodynamic elements.

Files. 2D aerodnamic elements are implemented in files
`mbdyn/aero/aeroelem.h`
`mbdyn/aero/aeroelem.cc`
2D aerodynamic models are implemented in files
`mbdyn/aero/aerodata.h`
`mbdyn/aero/aerodata.cc`
`mbdyn/aero/aerodata_impl.h`
`mbdyn/aero/aerodata_impl.cc`

13.1 Linearization of 2D Aerodynamic Forces and Moments

MBDyn's built-in 2D aerodynamics computes aerodynamic forces $\tilde{\boldsymbol{f}}_{a/\xi}$ and moments $\tilde{\boldsymbol{c}}_{a/\xi}$ per unit span, in a relative frame at station ξ, based on the instantaneous value of linear and angular velocity boundary conditions, respectively $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{\omega}}$, expressed in the same relative frame, namely

$$\tilde{\boldsymbol{f}}_{a/\xi} = \tilde{\boldsymbol{f}}_{a/\xi}\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \xi\right) \tag{13.1a}$$
$$\tilde{\boldsymbol{c}}_{a/\xi} = \tilde{\boldsymbol{c}}_{a/\xi}\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \xi\right). \tag{13.1b}$$

The boundary conditions are computed by projecting the *effective* linear and angular velocity at station ξ, respectively $\boldsymbol{v}\left(\xi\right)$ and $\boldsymbol{\omega}\left(\xi\right)$, in the reference frame of the aerodynamic forces, namely

$$\tilde{\boldsymbol{v}}\left(\xi\right) = \boldsymbol{R}^T\left(\xi\right)\boldsymbol{v}\left(\xi\right) \tag{13.2a}$$
$$\tilde{\boldsymbol{\omega}}\left(\xi\right) = \boldsymbol{R}^T\left(\xi\right)\boldsymbol{\omega}\left(\xi\right), \tag{13.2b}$$

where $\boldsymbol{R}\left(\xi\right)$ is the matrix that expresses the local orientation of the aerodynamic section at station ξ.

In detail, the effective linear velocity at an arbitrary station is the combination of the absolute velocity resulting from the kinematics of the model, of an airstream velocity that may depend on the absolute location of a reference point and on time, and of a contribution resulting from an inflow model, namely

$$\boldsymbol{v}\left(\xi\right) = \boldsymbol{v}_{\text{kin}}\left(\xi\right) + \boldsymbol{v}_{\infty}\left(\boldsymbol{x}\left(\xi\right), t\right) + \boldsymbol{v}_{\text{inflow}}\left(\boldsymbol{x}\left(\xi\right)\right). \tag{13.3}$$

It is assumed that the last two contributions do not depend on the state of the problem, or only depend on it in a loose manner, and thus do not directly participate in the linearization of the aerodynamic forces and moments.

Their linearization yields

$$\delta\tilde{\boldsymbol{v}} = \boldsymbol{R}^T(\xi)\left(\delta\boldsymbol{v}_{\text{kin}}(\xi) + \boldsymbol{v}(\xi) \times \boldsymbol{\theta}_\delta(\xi)\right) \tag{13.4a}$$

$$\delta\tilde{\boldsymbol{\omega}} = \boldsymbol{R}^T(\xi)\left(\delta\boldsymbol{\omega}(\xi) + \boldsymbol{\omega}(\xi) \times \boldsymbol{\theta}_\delta(\xi)\right), \tag{13.4b}$$

where

$$\boldsymbol{\theta}_\delta(\xi) = \operatorname{ax}\left(\delta\boldsymbol{R}(\xi)\,\boldsymbol{R}^T(\xi)\right). \tag{13.5}$$

By means of numerical integration, the force and moment per unit span are integrated into discrete contributions to the force and the moment applied to the appropriate node equilibrium. This is usually done by multiplying each force and moment per unit span contribution by an appropriate reference length coefficient. As a consequence, in the following force and moment contributions will be considered, namely

$$\tilde{\boldsymbol{f}}_a = \tilde{\boldsymbol{f}}_a(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \xi) \tag{13.6a}$$

$$\tilde{\boldsymbol{c}}_a = \tilde{\boldsymbol{c}}_a(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \xi). \tag{13.6b}$$

As already mentioned, each weighted sectional force and moment contribution is applied to the appropriate node after projection in the global reference frame. The contribution to the force and moment equilibrium of the n-th node is

$$\Delta\boldsymbol{f}_n = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_a \tag{13.7a}$$

$$\Delta\boldsymbol{c}_n = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_a + (\boldsymbol{x}(\xi) - \boldsymbol{x}_n) \times \Delta\boldsymbol{f}_n. \tag{13.7b}$$

Their linearization yields

$$\delta\Delta\boldsymbol{f}_n = -\Delta\boldsymbol{f}_n \times \boldsymbol{\theta}_\delta(\xi) + \boldsymbol{R}(\xi)\,\delta\tilde{\boldsymbol{f}}_a \tag{13.8a}$$

$$\begin{aligned} \delta\Delta\boldsymbol{c}_n = {} & -\Delta\boldsymbol{c}_n \times \boldsymbol{\theta}_\delta(\xi) + \boldsymbol{R}(\xi)\,\delta\tilde{\boldsymbol{c}}_a - \Delta\boldsymbol{f}_n \times (\delta\boldsymbol{x}(\xi) - \delta\boldsymbol{x}_n) \\ & + (\boldsymbol{x}(\xi) - \boldsymbol{x}_n) \times \left(-\Delta\boldsymbol{f}_n \times \boldsymbol{\theta}_\delta(\xi) + \boldsymbol{R}(\xi)\,\delta\tilde{\boldsymbol{f}}_a\right), \end{aligned} \tag{13.8b}$$

which can be summarized as

$$\begin{aligned} \left\{\begin{array}{c} \delta\Delta\boldsymbol{f}_n \\ \delta\Delta\boldsymbol{c}_n \end{array}\right\} = {} & \left[\begin{array}{c} -\Delta\boldsymbol{f}_n \times \\ -\Delta\boldsymbol{c}_n \times - (\boldsymbol{x}(\xi) - \boldsymbol{x}_n) \times \Delta\boldsymbol{f}_n \times \end{array}\right] \boldsymbol{\theta}_\delta(\xi) \\ & + \left[\begin{array}{c} \boldsymbol{0} \\ -\Delta\boldsymbol{f}_n \times \end{array}\right] (\delta\boldsymbol{x}(\xi) - \delta\boldsymbol{x}_n) \\ & + \left[\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{x}(\xi) - \boldsymbol{x}_n) \times & \boldsymbol{I} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{R}(\xi)\,\delta\tilde{\boldsymbol{f}}_a \\ \boldsymbol{R}(\xi)\,\delta\tilde{\boldsymbol{c}}_a \end{array}\right\}. \end{aligned} \tag{13.9}$$

It is assumed that the Jacobian matrix of the sectional force and moment with respect to the linear and angular velocity is either available or can be computed by numerical differentiation. The resulting force and moment perturbation is

$$\left\{\begin{array}{c} \delta\tilde{\boldsymbol{f}}_a \\ \delta\tilde{\boldsymbol{c}}_a \end{array}\right\} = \left[\begin{array}{cc} \tilde{\boldsymbol{f}}_{a/\tilde{v}} & \tilde{\boldsymbol{f}}_{a/\tilde{\omega}} \\ \tilde{\boldsymbol{c}}_{a/\tilde{v}} & \tilde{\boldsymbol{c}}_{a/\tilde{\omega}} \end{array}\right] \left\{\begin{array}{c} \delta\tilde{\boldsymbol{v}} \\ \delta\tilde{\boldsymbol{\omega}} \end{array}\right\}. \tag{13.10}$$

The computation of the matrix of Eq. (13.10) is delegated to the `AeroData` class.

Each type of element determines how the sectional force and moment contributions are applied to the nodes, and how the sectional boundary conditions at each section are computed from the kinematics of the nodes.

13.2 Numerical Linearization of Sectional Forces

Consider an arbitrary submatrix of the Jacobian matrix of Eq. (13.10), $\boldsymbol{J} = \boldsymbol{p}_{/\boldsymbol{q}}$. Its generic element, the c-th component of $\boldsymbol{p}$ derived by the r-th component of $\boldsymbol{q}$, is

$$J_{rc} = \frac{\partial \boldsymbol{p}_r}{\partial \boldsymbol{q}_c}. \tag{13.11}$$

A forward difference approach is used, namely

$$J_{rc} \cong \frac{\boldsymbol{p}_r\left(\boldsymbol{q} + \Delta q \boldsymbol{e}_c\right) - \boldsymbol{p}_r\left(\boldsymbol{q}\right)}{\Delta q}, \tag{13.12}$$

where $\boldsymbol{e}_c$ is the unit vector along the c-th component, and Δq is a suitably chosen perturbation. Alternatively, a centered difference approach can be used, namely

$$J_{rc} \cong \frac{\boldsymbol{p}_r\left(\boldsymbol{q} + \Delta q \boldsymbol{e}_c\right) - \boldsymbol{p}_r\left(\boldsymbol{q} - \Delta q \boldsymbol{e}_c\right)}{2\Delta q}. \tag{13.13}$$

The perturbation is

$$\Delta q = \varepsilon \left\|\boldsymbol{q}\right\| + \nu. \tag{13.14}$$

Since the boundary condition $\boldsymbol{q}$ is perturbed in order to determine an equivalent perturbation of angle of attack, a resolution of few tenth of degree is deemed sufficient. As a consequence, $\varepsilon > 0$ must be a "small" number that, in case $\boldsymbol{e}_c$ is orthogonal to $\boldsymbol{q}$, yields an angle of the order of a tenth of a degree. The default value is $\varepsilon = 10^{-3}$. However, in order to avoid divisions by too small numbers, the perturbation is corrected by another "small" parameter, $\nu > 0$. The default value is $\nu = 10^{-9}$.

13.3 Aerodynamic Forces with Internal States

Consider the case of a model of the aerodynamic forces that requires the use of internal states $\boldsymbol{q}$, namely

$$\tilde{\boldsymbol{f}}_a = \tilde{\boldsymbol{f}}_a\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{q}\right) \tag{13.15a}$$

$$\tilde{\boldsymbol{c}}_a = \tilde{\boldsymbol{c}}_a\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{q}\right) \tag{13.15b}$$

$$\boldsymbol{0} = \boldsymbol{g}\left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{q}, \dot{\boldsymbol{q}}\right). \tag{13.15c}$$

Usually, the dynamic model of the aerodynamics is differential, and thus its perturbation yields a linearized state-space system,

$$\delta \boldsymbol{g} = \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}\delta\tilde{\boldsymbol{v}} + \boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}\delta\tilde{\boldsymbol{\omega}} + \boldsymbol{g}_{/\boldsymbol{q}}\delta\boldsymbol{q} + \boldsymbol{g}_{/\dot{\boldsymbol{q}}}\delta\dot{\boldsymbol{q}} \tag{13.16a}$$

$$\delta\tilde{\boldsymbol{f}}_a = \tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}}\delta\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}}\delta\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{f}}_{a/\boldsymbol{q}}\delta\boldsymbol{q} \tag{13.16b}$$

$$\delta\tilde{\boldsymbol{c}}_a = \tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}}\delta\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}}\delta\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{c}}_{a/\boldsymbol{q}}\delta\boldsymbol{q}, \tag{13.16c}$$

where, with respect to $\delta\boldsymbol{g}$, the terms $\delta\tilde{\boldsymbol{v}}$ and $\delta\tilde{\boldsymbol{\omega}}$ play the role of the input, while $\delta\boldsymbol{q}$ plays the role of the state; $\delta\tilde{\boldsymbol{f}}_a$, $\delta\tilde{\boldsymbol{c}}_a$ play the role of the output.

In those cases, the underlying aerodynamic model has to deal with $\boldsymbol{g}$, Eq. (13.15c), and its perturbation $\delta\boldsymbol{g}$, Eq. (13.15c).

However, the aerodynamic elements have to:

1. provide the underlying aerodynamic model the Jacobian submatrices required to compute $\delta\tilde{\boldsymbol{v}}$ and $\delta\tilde{\boldsymbol{\omega}}$ from the perturbations of the nodal position, orientation, and linear and angular velocity that are needed to deal with $\delta\boldsymbol{g}$, Eq. (13.15c);
2. account for the contribution of the Jacobian matrices $\delta\tilde{\boldsymbol{f}}_{a/\boldsymbol{q}}$ and $\delta\tilde{\boldsymbol{c}}_{a/\boldsymbol{q}}$ respectively required by Eqs. (13.16b) and (13.16c).

13.4 Aerodynamic body

The `aerodynamic body` element applies aerodynamic forces to the structural node it is connected to, based on the relative velocity between an aerodynamic surface attached to the node and the airstream.

The boundary conditions are related to the rigid body motion of the node, so

$$\boldsymbol{R}(\xi) = \boldsymbol{R}_n \boldsymbol{R}_a \boldsymbol{R}_t(\xi) \tag{13.17a}$$

$$\boldsymbol{b}(\xi) = \boldsymbol{R}_n \left(\tilde{\boldsymbol{b}}_0 + \boldsymbol{R}_a \left(b(\xi)\, \boldsymbol{e}_1 + \xi \boldsymbol{e}_3\right)\right) \tag{13.17b}$$

$$\boldsymbol{\omega}(\xi) = \boldsymbol{\omega}_n \tag{13.17c}$$

$$\boldsymbol{v}_{\text{kin}}(\xi) = \boldsymbol{v}_n + \boldsymbol{\omega}_n \times \boldsymbol{b}(\xi), \tag{13.17d}$$

where $\boldsymbol{R}_n$ is the orientation of the node, $\boldsymbol{R}_a$ is the relative orientation of the aerodynamics with respect to the node, $\boldsymbol{R}_t$ is the pretwist matrix, $\boldsymbol{v}_n$ is the absolute velocity of the node, $\boldsymbol{\omega}_n$ is the absolute angular velocity of the node, $\tilde{\boldsymbol{b}}_0$ is an offset between the node and the reference location of the aerodynamic body, and $b(\xi)$ is the chordwise location of the point where the boundary conditions are evaluated.

Their linearization is straightforward:

$$\boldsymbol{\theta}_\delta(\xi) = \boldsymbol{\theta}_{n\delta} \tag{13.18a}$$

$$\delta\boldsymbol{b}(\xi) = -\boldsymbol{b}(\xi) \times \boldsymbol{\theta}_{n\delta} \tag{13.18b}$$

$$\delta\boldsymbol{\omega}(\xi) = \delta\boldsymbol{\omega}_n \tag{13.18c}$$

$$\delta\boldsymbol{v}_{\text{kin}}(\xi) = \delta\boldsymbol{v}_n - \boldsymbol{b}(\xi) \times \delta\boldsymbol{\omega}_n - \boldsymbol{\omega}_n \times \boldsymbol{b}(\xi) \times \boldsymbol{\theta}_{n\delta}. \tag{13.18d}$$

Eq. (13.7b) can be rewritten as

$$\Delta\boldsymbol{c}_n = \boldsymbol{R}(\xi) \left(\tilde{\boldsymbol{c}}_a + \tilde{\boldsymbol{o}}(\xi) \times \tilde{\boldsymbol{f}}_a\right), \tag{13.19}$$

where $\tilde{\boldsymbol{o}}(\xi) = \boldsymbol{R}^T(\xi)(\boldsymbol{x}(\xi) - \boldsymbol{x}_n)$, the offset between the point where the force is applied and the node, in the reference frame of the node, does not depend on the kinematics of the system, since the body is rigid. Its linearization yields

$$\delta\Delta\boldsymbol{c}_n = -\Delta\boldsymbol{c}_n \times \boldsymbol{\theta}_\delta(\xi) + \boldsymbol{R}(\xi) \left(\delta\tilde{\boldsymbol{c}}_a + \tilde{\boldsymbol{o}}(\xi) \times \delta\tilde{\boldsymbol{f}}_a\right), \tag{13.20}$$

since $\delta\tilde{\boldsymbol{o}}(\xi) \equiv 0$. So the linearized force and moment is

$$\left\{\begin{array}{c} \delta\Delta\boldsymbol{f}_n \\ \delta\Delta\boldsymbol{c}_n \end{array}\right\} = -\left[\begin{array}{c} \Delta\boldsymbol{f}_n \times \\ \Delta\boldsymbol{c}_n \times \end{array}\right] \boldsymbol{\theta}_{n\delta} + \left[\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{o}(\xi) \times & \boldsymbol{I} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{R}(\xi)\, \delta\tilde{\boldsymbol{f}}_a \\ \boldsymbol{R}(\xi)\, \delta\tilde{\boldsymbol{c}}_a \end{array}\right\}. \tag{13.21}$$

with $\boldsymbol{o}(\xi) = \boldsymbol{R}(\xi)\, \tilde{\boldsymbol{o}}(\xi)$.

The linearization of the boundary conditions yields

$$\begin{aligned} \left\{\begin{array}{c} \delta\tilde{\boldsymbol{v}}(\xi) \\ \delta\tilde{\boldsymbol{\omega}}(\xi) \end{array}\right\} = {} & \left[\begin{array}{cc} \boldsymbol{R}^T(\xi) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^T(\xi) \end{array}\right] \left(\left[\begin{array}{c} \boldsymbol{v}(\xi) \times \ -\boldsymbol{\omega}_n \times \boldsymbol{b}(\xi) \times \\ \boldsymbol{\omega}_n \times \end{array}\right] \boldsymbol{\theta}_{n\delta} \right. \\ & \left. + \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{b}(\xi) \times \\ \boldsymbol{0} & \boldsymbol{I} \end{array}\right] \left\{\begin{array}{c} \delta\boldsymbol{v}_n \\ \delta\boldsymbol{\omega}_n \end{array}\right\}\right) \end{aligned} \tag{13.22a}$$

$$\begin{aligned} \overset{\text{uu}}{=} {} & \left[\begin{array}{cc} \boldsymbol{R}^T(\xi) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^T(\xi) \end{array}\right] \left(\left[\begin{array}{c} (\boldsymbol{v}(\xi) + \boldsymbol{b}(\xi) \times \boldsymbol{\omega}_n) \times \\ \boldsymbol{0} \end{array}\right] \delta\boldsymbol{g}_n \right. \\ & \left. + \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{b}(\xi) \times \\ \boldsymbol{0} & \boldsymbol{I} \end{array}\right] \left\{\begin{array}{c} \delta\dot{\boldsymbol{x}}_n \\ \delta\dot{\boldsymbol{g}}_n \end{array}\right\}\right). \end{aligned} \tag{13.22b}$$

After defining

$$\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.23a}$$

$$\boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi) \tag{13.23b}$$

$$\boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.23c}$$

$$\boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi), \tag{13.23d}$$

the linearization becomes

$$\begin{aligned}
&\left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} \\ \boldsymbol{o}(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} \end{array}\right]\delta\dot{\boldsymbol{x}}_n \\
&+\left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} - \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{b}(\xi)\times \\ \boldsymbol{o}(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} - \left(\boldsymbol{o}(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\right)\boldsymbol{b}(\xi)\times \end{array}\right]\delta\dot{\boldsymbol{g}}_n \\
&+\left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\left(\boldsymbol{v}(\xi) + \boldsymbol{b}(\xi)\times\boldsymbol{\omega}_n\right)\times \\ \left(\boldsymbol{o}(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\right)\left(\boldsymbol{v}(\xi) + \boldsymbol{b}(\xi)\times\boldsymbol{\omega}_n\right)\times \end{array}\right]\delta\boldsymbol{g}_n \\
&-\left[\begin{array}{c} \Delta\boldsymbol{f}_n\times \\ \Delta\boldsymbol{c}_n\times \end{array}\right]\delta\boldsymbol{g}_n \stackrel{\text{uu}}{=} \left\{\begin{array}{c} \delta\Delta\boldsymbol{f}_n \\ \delta\Delta\boldsymbol{c}_n \end{array}\right\}.
\end{aligned} \tag{13.24}$$

13.5 Aerodynamic Beam (3 Nodes)

The `aerodynamic beam3` element applies aerodynamic forces to the nodes of a three node structural beam element.

The aerodynamic forces/moments acting on each node are computed based on the relative velocity of a set of locations along the beam and the airstream. The kinematic quantities of the beam are computed based on an interpolation of the kinematics of the three nodes.

Kinematics Interpolation. *Note: this part is common to all elements that use the three-node beam discretization and interpolation model.* The generic field variable $\boldsymbol{p}(x)$ is interpolated using parabolic functions related to the value of the field variable at three locations that in general can be offset from the nodes,

$$\boldsymbol{p}(\xi) = \sum_{i=1,2,3} N_i(\xi)\,\boldsymbol{p}_i, \tag{13.25}$$

with

$$N_1 = \frac{1}{2}\xi(\xi-1) \tag{13.26a}$$

$$N_2 = 1-\xi^2 \tag{13.26b}$$

$$N_3 = \frac{1}{2}\xi(\xi+1). \tag{13.26c}$$

Orientation at an arbitrary location ξ:

$$\boldsymbol{R}(\xi) = \boldsymbol{R}_{2_a}\boldsymbol{R}(\boldsymbol{\theta}(\xi)) \tag{13.27}$$

$$\boldsymbol{\theta}(\xi) = \sum_{i=1,3} N_i(\xi)\,\boldsymbol{\theta}_{2\leftarrow i} = N_1(\xi)\,\boldsymbol{\theta}_{2\leftarrow 1} + N_3(\xi)\,\boldsymbol{\theta}_{2\leftarrow 3} \tag{13.28}$$

$$\boldsymbol{\theta}_{2\leftarrow i} = \operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{2_a}^T\boldsymbol{R}_{i_a}\right)\right). \tag{13.29}$$

Orientation is dealt with specially, given its special nature. The orientation of the mid node is used as a reference, and the orientation parameters that express the relative orientation between each of the end nodes and the mid node are interpolated. The interpolated orientation parameters are used to compute the interpolated relative orientation matrix, which is then pre-multiplied by the orientation matrix of the mid node. Summation in this case occurs on $i = 1, 3$ only because by definition $\boldsymbol{\theta}_{2\leftarrow 2} = \mathrm{ax}(\exp^{-1}((\boldsymbol{I})) \equiv \mathbf{0}$.

Since the Euler-Rodrigues orientation parameters are used (the so-caller 'rotation vector'), the magnitude of the relative orientation between each end node and the mid node must be limited (formally, to π, but it should be less for accuracy).

Position at an arbitrary location ξ:

$$\boldsymbol{x}(\xi) = \sum_{i=1,2,3} N_i(\xi)(\boldsymbol{x}_i + \boldsymbol{o}_i), \tag{13.30}$$

with $\boldsymbol{o}_i = \boldsymbol{R}_i \tilde{\boldsymbol{o}}_i$.

Angular velocity at an arbitrary location ξ:

$$\boldsymbol{\omega}(\xi) = \sum_{i=1,2,3} N_i(\xi)\boldsymbol{\omega}_i. \tag{13.31}$$

Velocity at an arbitrary location ξ:

$$\boldsymbol{v}_{\mathrm{kin}}(\xi) = \sum_{i=1,2,3} N_i(\xi)(\boldsymbol{v}_i + \boldsymbol{\omega}_i \times \boldsymbol{o}_i). \tag{13.32}$$

Perturbation of Interpolated Kinematics. Orientation perturbation at an arbitrary location ξ:

$$\delta\boldsymbol{\theta}(\xi) = \sum_{i=1,3} N_i(\xi)\,\delta\boldsymbol{\theta}_{2\leftarrow i} \tag{13.33}$$

$$\delta\boldsymbol{\theta}_{2\leftarrow i} = \boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow i})\,\boldsymbol{\theta}_{(2\leftarrow i)\delta} \tag{13.34}$$

$$\boldsymbol{\theta}_{(2\leftarrow i)\delta} = \boldsymbol{R}_{2_a}^T(\boldsymbol{\theta}_{i\delta} - \boldsymbol{\theta}_{2\delta}) \tag{13.35}$$

$$\begin{aligned}
\boldsymbol{\theta}_\delta(\xi) &= \boldsymbol{\theta}_{2\delta} + \sum_{i=1,3} \boldsymbol{R}_{2_a}\boldsymbol{\Gamma}(\boldsymbol{\theta}(\xi))\,N_i(\xi)\,\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow i})\,\boldsymbol{R}_{2_a}^T(\boldsymbol{\theta}_{i\delta} - \boldsymbol{\theta}_{2\delta}) \\
&= \left(\boldsymbol{I} - \boldsymbol{R}_{2_a}\boldsymbol{\Gamma}(\boldsymbol{\theta}(\xi)) \sum_{i=1,3} N_i(\xi)\,\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow i})\,\boldsymbol{R}_{2_a}^T\right)\boldsymbol{\theta}_{2\delta} \\
&\quad + \boldsymbol{R}_{2_a}\boldsymbol{\Gamma}(\boldsymbol{\theta}(\xi))\,N_1\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow 1})\,\boldsymbol{R}_{2_a}^T\boldsymbol{\theta}_{1\delta} \\
&\quad + \boldsymbol{R}_{2_a}\boldsymbol{\Gamma}(\boldsymbol{\theta}(\xi))\,N_3\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow 3})\,\boldsymbol{R}_{2_a}^T\boldsymbol{\theta}_{3\delta} \\
&= \sum_{i=1,2,3} \boldsymbol{\Theta}_i(\xi)\,\boldsymbol{\theta}_{i\delta},
\end{aligned} \tag{13.36}$$

with

$$\boldsymbol{\Theta}_i(\xi) = \boldsymbol{R}_{2_a}\boldsymbol{\Gamma}(\boldsymbol{\theta}(\xi))\,N_i\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_{2\leftarrow i})\,\boldsymbol{R}_{2_a}^T \qquad i = 1 \text{ and } 3 \tag{13.37a}$$

$$\boldsymbol{\Theta}_2(\xi) = \boldsymbol{I} - \boldsymbol{\Theta}_1(\xi) - \boldsymbol{\Theta}_3(\xi) \tag{13.37b}$$

playing the role of shape functions.

In fact, note that, when $N_1 = 1$ and $N_3 = 0$, then $\boldsymbol{\Theta}_1 = \boldsymbol{I}$, $\boldsymbol{\Theta}_2 = \boldsymbol{\Theta}_3 = \mathbf{0}$ and $\boldsymbol{\theta}_\delta(\xi) = \boldsymbol{\theta}_{1\delta}$, while, when $N_1 = 0$ and $N_3 = 1$, then $\boldsymbol{\Theta}_1 = \boldsymbol{\Theta}_2 = \mathbf{0}$, $\boldsymbol{\Theta}_3 = \boldsymbol{I}$ and $\boldsymbol{\theta}_\delta(\xi) = \boldsymbol{\theta}_{3\delta}$. Finally, when $N_1 = N_3 = 0$, then $\boldsymbol{\Theta}_1 = \boldsymbol{\Theta}_3 = \mathbf{0}$, $\boldsymbol{\Theta}_2 = \boldsymbol{I}$ and $\boldsymbol{\theta}_\delta(\xi) = \boldsymbol{\theta}_{2\delta}$. Moreover, $\sum_{i=1,2,3} \boldsymbol{\Theta}_i(\xi) = \boldsymbol{I}\ \forall\xi$.

Position perturbation at an arbitrary location ξ:

$$\delta \boldsymbol{x}(\xi) = \sum_{i=1,2,3} N_i(\xi)\left(\delta \boldsymbol{x}_i + \boldsymbol{\theta}_{i\delta} \times \boldsymbol{o}_i\right). \tag{13.38}$$

Angular velocity perturbation at an arbitrary location ξ:

$$\begin{aligned}\delta \boldsymbol{\omega}(\xi) &= \sum_{i=1,2,3} N_i \delta \boldsymbol{\omega}_i \\ &\overset{\text{uu}}{=} \sum_{i=1,2,3} N_i\left(\delta \dot{\boldsymbol{g}}_i - \boldsymbol{\omega}_i \times \delta \boldsymbol{g}_i\right).\end{aligned} \tag{13.39}$$

Velocity perturbation at an arbitrary location ξ:

$$\begin{aligned}\delta \boldsymbol{v}_{\text{kin}}(\xi) &= \sum_{i=1,2,3} N_i(\xi)\left(\delta \boldsymbol{v}_i + \delta \boldsymbol{\omega}_i \times \boldsymbol{o}_i + \boldsymbol{\omega}_i \times \delta \boldsymbol{o}_i\right) \\ &= \sum_{i=1,2,3} N_i(\xi)\left(\delta \boldsymbol{v}_i - \boldsymbol{o}_i \times \delta \boldsymbol{\omega}_i - \boldsymbol{\omega}_i \times \boldsymbol{o}_i \times \boldsymbol{\theta}_{i\delta}\right) \\ &\overset{\text{uu}}{=} \sum_{i=1,2,3} N_i(\xi)\left(\delta \dot{\boldsymbol{x}}_i - \boldsymbol{o}_i \times \delta \dot{\boldsymbol{g}}_i - \left(\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \delta \boldsymbol{g}_i\right).\end{aligned} \tag{13.40}$$

Boundary Conditions Perturbation. Angular velocity perturbation:

$$\begin{aligned}\delta \tilde{\boldsymbol{\omega}}(\xi) &= \boldsymbol{R}^T(\xi) \sum_{i=1,2,3}\left(N_i(\xi)\, \delta \boldsymbol{\omega}_i + \boldsymbol{\omega}(\xi) \times \boldsymbol{\Theta}_i(\xi)\, \boldsymbol{\theta}_{i\delta}\right) \\ &\overset{\text{uu}}{=} \boldsymbol{R}^T(\xi)\left(N_i(\xi)\, \delta \dot{\boldsymbol{g}}_i + \left(\boldsymbol{\omega}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\, \boldsymbol{\omega}_i \times\right) \delta \boldsymbol{g}_i\right).\end{aligned} \tag{13.41}$$

Velocity perturbation:

$$\begin{aligned}\delta \tilde{\boldsymbol{v}}(\xi) &= \boldsymbol{R}^T(\xi)\big(N_i(\xi)\, \delta \boldsymbol{v}_i - N_i(\xi)\, \boldsymbol{o}_i \times \delta \boldsymbol{\omega}_i \\ &\qquad + \left(\boldsymbol{v}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\, \boldsymbol{\omega}_i \times \boldsymbol{o}_i \times\right) \boldsymbol{\theta}_{i\delta}\big) \\ &\overset{\text{uu}}{=} \boldsymbol{R}^T(\xi)\big(N_i(\xi)\, \delta \dot{\boldsymbol{x}}_i - N_i(\xi)\, \boldsymbol{o}_i \times \delta \dot{\boldsymbol{g}}_i \\ &\qquad + \left(\boldsymbol{v}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\left(\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times\right) \delta \boldsymbol{g}_i\big).\end{aligned} \tag{13.42}$$

They can be summarized as

$$\begin{aligned}\left\{\begin{array}{c}\delta \tilde{\boldsymbol{v}}(\xi) \\ \delta \tilde{\boldsymbol{\omega}}(\xi)\end{array}\right\} &= \left[\begin{array}{cc}\boldsymbol{R}^T(\xi) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}^T(\xi)\end{array}\right] \sum_{i=1,2,3}\Bigg(N_i(\xi)\left[\begin{array}{cc}\boldsymbol{I} & -\boldsymbol{o}_i \times \\ \mathbf{0} & \boldsymbol{I}\end{array}\right]\left\{\begin{array}{c}\delta \boldsymbol{v}_i \\ \delta \boldsymbol{\omega}_i\end{array}\right\} \\ &\qquad + \left[\begin{array}{c}\boldsymbol{v}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\, \boldsymbol{\omega}_i \times \boldsymbol{o}_i \times \\ \boldsymbol{\omega}(\xi) \times \boldsymbol{\Theta}_i(\xi)\end{array}\right] \boldsymbol{\theta}_{i\delta}\Bigg) \\ &\overset{\text{uu}}{=} \left[\begin{array}{cc}\boldsymbol{R}^T(\xi) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}^T(\xi)\end{array}\right] \sum_{i=1,2,3}\Bigg(N_i(\xi)\left[\begin{array}{cc}\boldsymbol{I} & -\boldsymbol{o}_i \times \\ \mathbf{0} & \boldsymbol{I}\end{array}\right]\left\{\begin{array}{c}\delta \dot{\boldsymbol{x}}_i \\ \delta \dot{\boldsymbol{g}}_i\end{array}\right\} \\ &\qquad + \left[\begin{array}{c}\boldsymbol{v}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\left(\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \\ \boldsymbol{\omega}(\xi) \times \boldsymbol{\Theta}_i(\xi) - N_i(\xi)\, \boldsymbol{\omega}_i \times\end{array}\right] \delta \boldsymbol{g}_i\Bigg).\end{aligned} \tag{13.43}$$

Contribution to Jacobian Matrix. After defining

$$\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.44a}$$

$$\boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi) \tag{13.44b}$$

$$\boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.44c}$$

$$\boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi) \tag{13.44d}$$

$$\boldsymbol{B}_{\tilde{\boldsymbol{v}}} = \boldsymbol{v}(\xi)\times\boldsymbol{\Theta}_i(\xi) - N_i(\xi)\,(\boldsymbol{\omega}_i\times\boldsymbol{o}_i)\,\times \tag{13.44e}$$

$$\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} = \boldsymbol{\omega}(\xi)\times\boldsymbol{\Theta}_i(\xi) - N_i(\xi)\,\boldsymbol{\omega}_i\,\times \tag{13.44f}$$

$$\boldsymbol{d}_n(\xi) = \boldsymbol{x}(\xi) - \boldsymbol{x}_n, \tag{13.44g}$$

the contribution of the i-th node's motion to the equilibrium of the n-th node ($i = 1,2,3$, $n = 1,2,3$) is

$$\begin{aligned}
&N_i(\xi)\begin{bmatrix} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} \\ \boldsymbol{d}_n(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} \end{bmatrix}\delta\dot{\boldsymbol{x}}_i \\
&+ N_i(\xi)\begin{bmatrix} \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} - \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{o}_i\,\times \\ \boldsymbol{d}_n(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} - \left(\boldsymbol{d}_n(\xi)\times\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\right)\boldsymbol{o}_i\,\times \end{bmatrix}\delta\dot{\boldsymbol{g}}_i \\
&+ \begin{bmatrix} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} \\ \boldsymbol{d}_n(\xi)\times\left(\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}}\right) + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} \end{bmatrix}\delta\boldsymbol{g}_i \\
&+ \begin{bmatrix} \boldsymbol{0} \\ -(N_i - \delta_{ni})\,\Delta\boldsymbol{f}_n\,\times \end{bmatrix}\delta\boldsymbol{x}_i \\
&+ \begin{bmatrix} -\Delta\boldsymbol{f}_n\times\boldsymbol{\Theta}_i(\xi) \\ -\left(\boldsymbol{d}_n(\xi)\times\Delta\boldsymbol{f}_n\,\times + \Delta\boldsymbol{c}_n\,\times\right)\boldsymbol{\Theta}_i(\xi) + \Delta\boldsymbol{f}_n\times N_i\boldsymbol{o}_i\,\times \end{bmatrix}\delta\boldsymbol{g}_i \\
&\overset{\text{uu}}{=} \begin{Bmatrix} \delta\Delta\boldsymbol{f}_n \\ \delta\Delta\boldsymbol{c}_n \end{Bmatrix},
\end{aligned} \tag{13.45}$$

where δ_{ni} is Dirac's delta, which is 1 when $i = n$, and 0 otherwise.

13.6 Aerodynamic Beam (2 Nodes)

The `aerodynamic beam2` element applies aerodynamic forces to the nodes of a two node structural beam element.

The aerodynamic forces/moments acting on each node are computed based on the relative velocity of a set of locations along the beam and the airstream. The kinematic quantities of the beam are computed based on an interpolation of the kinematics of the three nodes.

Kinematics Interpolation. *Note: this part is common to all elements that use the two-node beam discretization and interpolation model.* The generic field variable $\boldsymbol{p}(x)$ is interpolated using linear functions related to the value of the field variable at two locations that in general can be offset from the nodes,

$$\boldsymbol{p}(\xi) = \sum_{i=1,2} N_i(\xi)\,\boldsymbol{p}_i, \tag{13.46}$$

with

$$N_1 = \frac{1}{2}(1-\xi) \tag{13.47a}$$

$$N_2 = \frac{1}{2}(1+\xi)\,. \tag{13.47b}$$

Orientation at an arbitrary location ξ:

$$\boldsymbol{R}(\xi) = \boldsymbol{R}_{\text{mid}} \exp\left(\boldsymbol{\theta}(\xi) \times\right) \tag{13.48}$$

$$\boldsymbol{R}_{\text{mid}} = \boldsymbol{R}_{1_a} \exp\left(\overline{\boldsymbol{\theta}} \times\right) = \boldsymbol{R}_{2_a} \exp\left(\overline{\boldsymbol{\theta}} \times\right)^T \tag{13.49}$$

$$\overline{\boldsymbol{\theta}} = \frac{1}{2} \operatorname{ax}\left(\exp^{-1}\left(\boldsymbol{R}_{1_a}^T \boldsymbol{R}_{2_a}\right)\right) \tag{13.50}$$

$$\boldsymbol{\theta}(\xi) = \sum_{i=1,2} N_i(\xi)\, \boldsymbol{\theta}_{\text{mid}\leftarrow i} = N_1(\xi)\, \boldsymbol{\theta}_{\text{mid}\leftarrow 1} + N_2(\xi)\, \boldsymbol{\theta}_{\text{mid}\leftarrow 2} \tag{13.51}$$

$$\boldsymbol{\theta}_{\text{mid}\leftarrow 1} = -\overline{\boldsymbol{\theta}} \tag{13.52}$$

$$\boldsymbol{\theta}_{\text{mid}\leftarrow 2} = \overline{\boldsymbol{\theta}}. \tag{13.53}$$

Orientation is dealt with specially, given its special nature. The orientation of the mid point is used as a reference, and the orientation parameters that express the relative orientation between each of the end nodes and the mid point are interpolated. The interpolated orientation parameters are used to compute the interpolated relative orientation matrix, which is then pre-multiplied by the orientation matrix of the mid point.

Since the Euler-Rodrigues parametrization is used (the so-called 'rotation vector'), the magnitude of the relative orientation between each end node must be limited (formally, to π, but it should be less for accuracy).

Note that since the generic orientation $\boldsymbol{R}(\xi)$ is the result of a sequence of orientations about a common axis $\overline{\boldsymbol{\theta}}$, it can be conveniently rewritten as

$$\begin{aligned}\boldsymbol{R}(\xi) &= \boldsymbol{R}_{1_a} \exp\left(\overline{\boldsymbol{\theta}} \times\right) \exp\left(\left(\left(N_2(\xi) - N_1(\xi)\right) \overline{\boldsymbol{\theta}}\right) \times\right) \\ &= \boldsymbol{R}_{1_a} \exp\left(\left(\left(1 + N_2(\xi) - N_1(\xi)\right) \overline{\boldsymbol{\theta}}\right) \times\right).\end{aligned} \tag{13.54}$$

In fact, when $\xi = -1$, $N_1 = 1$ and $N_2 = 0$, then $\boldsymbol{R}(\xi) = \boldsymbol{R}_{1_a}$, while when $\xi = 1$, $N_1 = 0$ and $N_2 = 1$, then $\boldsymbol{R}(\xi) = \boldsymbol{R}_{2_a}$.

Position at an arbitrary location ξ:

$$\boldsymbol{x}(\xi) = \sum_{i=1,2} N_i(\xi) \left(\boldsymbol{x}_i + \boldsymbol{o}_i\right), \tag{13.55}$$

with $\boldsymbol{o}_i = \boldsymbol{R}_i \tilde{\boldsymbol{o}}_i$.

Angular velocity at an arbitrary location ξ:

$$\boldsymbol{\omega}(\xi) = \sum_{i=1,2} N_i(\xi)\, \boldsymbol{\omega}_i. \tag{13.56}$$

Velocity at an arbitrary location ξ:

$$\boldsymbol{v}_{\text{kin}}(\xi) = \sum_{i=1,2} N_i(\xi) \left(\boldsymbol{v}_i + \boldsymbol{\omega}_i \times \boldsymbol{o}_i\right). \tag{13.57}$$

Perturbation of Interpolated Kinematics. Orientation perturbation at an arbitrary location ξ:

$$\boldsymbol{\theta}_\delta(\xi) = \frac{1 + N_2(\xi) - N_1(\xi)}{2} \boldsymbol{\theta}_{2\delta} + \frac{1 + N_1(\xi) - N_2(\xi)}{2} \boldsymbol{\theta}_{1\delta} \tag{13.58}$$

$$= \sum_{i=1,2} \mathcal{N}_i(\xi)\, \boldsymbol{\theta}_{i\delta}, \tag{13.59}$$

with

$$\mathcal{N}_i(\xi) = \frac{1 + N_i(\xi) - N_{3-i}(\xi)}{2}. \tag{13.60a}$$

Note, however, that according to Eqs. (13.47), the shape functions of Eq. (13.60a) are $\mathcal{N}_i(\xi) = N_i(\xi)$.

Position perturbation at an arbitrary location ξ:

$$\delta \boldsymbol{x}(\xi) = \sum_{i=1,2} N_i(\xi) \left(\delta \boldsymbol{x}_i + \boldsymbol{\theta}_{i\delta} \times \boldsymbol{o}_i\right). \tag{13.61}$$

Angular velocity perturbation at an arbitrary location ξ:

$$\begin{aligned} \delta \boldsymbol{\omega}(\xi) &= \sum_{i=1,2} N_i \delta \boldsymbol{\omega}_i \\ &\stackrel{\text{uu}}{=} \sum_{i=1,2} N_i \left(\delta \dot{\boldsymbol{g}}_i - \boldsymbol{\omega}_i \times \delta \boldsymbol{g}_i\right). \end{aligned} \tag{13.62}$$

Velocity perturbation at an arbitrary location ξ:

$$\begin{aligned} \delta \boldsymbol{v}_{\text{kin}}(\xi) &= \sum_{i=1,2} N_i(\xi) \left(\delta \boldsymbol{v}_i + \delta \boldsymbol{\omega}_i \times \boldsymbol{o}_i + \boldsymbol{\omega}_i \times \delta \boldsymbol{o}_i\right) \\ &= \sum_{i=1,2} N_i(\xi) \left(\delta \boldsymbol{v}_i - \boldsymbol{o}_i \times \delta \boldsymbol{\omega}_i - \boldsymbol{\omega}_i \times \boldsymbol{o}_i \times \boldsymbol{\theta}_{i\delta}\right) \\ &\stackrel{\text{uu}}{=} \sum_{i=1,2} N_i(\xi) \left(\delta \dot{\boldsymbol{x}}_i - \boldsymbol{o}_i \times \delta \dot{\boldsymbol{g}}_i - \left(\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \delta \boldsymbol{g}_i\right). \end{aligned} \tag{13.63}$$

Boundary Conditions Perturbation. Angular velocity perturbation:

$$\begin{aligned} \delta \tilde{\boldsymbol{\omega}}(\xi) &= \boldsymbol{R}^T(\xi) \sum_{i=1,2} \left(N_i(\xi) \delta \boldsymbol{\omega}_i + \boldsymbol{\omega}(\xi) \times \mathcal{N}_i(\xi) \boldsymbol{\theta}_{i\delta}\right) \\ &\stackrel{\text{uu}}{=} \boldsymbol{R}^T(\xi) \sum_{i=1,2} \left(N_i(\xi) \delta \dot{\boldsymbol{g}}_i + \left(\mathcal{N}_i \boldsymbol{\omega}(\xi) - N_i(\xi) \boldsymbol{\omega}_i\right) \times \delta \boldsymbol{g}_i\right) \\ &= \boldsymbol{R}^T(\xi) \sum_{i=1,2} N_i(\xi) \left(\delta \dot{\boldsymbol{g}}_i + \left(\boldsymbol{\omega}(\xi) - \boldsymbol{\omega}_i\right) \times \delta \boldsymbol{g}_i\right). \end{aligned} \tag{13.64}$$

Velocity perturbation:

$$\begin{aligned} \delta \tilde{\boldsymbol{v}}(\xi) &= \boldsymbol{R}^T(\xi) \left(N_i(\xi) \delta \boldsymbol{v}_i - N_i(\xi) \boldsymbol{o}_i \times \delta \boldsymbol{\omega}_i \right. \\ &\qquad \left. + \left(\mathcal{N}_i(\xi) \boldsymbol{v}(\xi) \times \; - N_i(\xi) \boldsymbol{\omega}_i \times \boldsymbol{o}_i \times \right) \boldsymbol{\theta}_{i\delta}\right) \\ &\stackrel{\text{uu}}{=} \boldsymbol{R}^T(\xi) \left(N_i(\xi) \delta \dot{\boldsymbol{x}}_i - N_i(\xi) \boldsymbol{o}_i \times \delta \dot{\boldsymbol{g}}_i \right. \\ &\qquad \left. + \left(\mathcal{N}_i(\xi) \boldsymbol{v}(\xi) - N_i(\xi) \boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \delta \boldsymbol{g}_i\right) \\ &= \boldsymbol{R}^T(\xi) N_i(\xi) \left(\delta \dot{\boldsymbol{x}}_i - \boldsymbol{o}_i \times \delta \dot{\boldsymbol{g}}_i + \left(\boldsymbol{v}(\xi) - \boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \delta \boldsymbol{g}_i\right). \end{aligned} \tag{13.65}$$

They can be summarized as

$$\left\{ \begin{array}{c} \delta\tilde{\boldsymbol{v}}(\xi) \\ \delta\tilde{\boldsymbol{\omega}}(\xi) \end{array} \right\} = \left[\begin{array}{cc} \boldsymbol{R}^T(\xi) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^T(\xi) \end{array} \right] \sum_{i=1,2} \left(N_i(\xi) \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{o}_i \times \\ \boldsymbol{0} & \boldsymbol{I} \end{array} \right] \left\{ \begin{array}{c} \delta\boldsymbol{v}_i \\ \delta\boldsymbol{\omega}_i \end{array} \right\} \right.$$
$$\left. + \left[\begin{array}{c} \mathcal{N}_i(\xi)\,\boldsymbol{v}(\xi) \times \; - N_i(\xi)\,\boldsymbol{\omega}_i \times \boldsymbol{o}_i \times \\ \mathcal{N}_i(\xi)\,\boldsymbol{\omega}(\xi) \times \end{array} \right] \boldsymbol{\theta}_{i\delta} \right)$$
$$\stackrel{\text{uu}}{=} \left[\begin{array}{cc} \boldsymbol{R}^T(\xi) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^T(\xi) \end{array} \right] \sum_{i=1,2} \left(N_i(\xi) \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{o}_i \times \\ \boldsymbol{0} & \boldsymbol{I} \end{array} \right] \left\{ \begin{array}{c} \delta\dot{\boldsymbol{x}}_i \\ \delta\dot{\boldsymbol{g}}_i \end{array} \right\} \right.$$
$$\left. + \left[\begin{array}{c} \left(\mathcal{N}_i(\xi)\,\boldsymbol{v}(\xi) - N_i(\xi)\,\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \\ \left(\mathcal{N}_i(\xi)\,\boldsymbol{\omega}(\xi) - N_i(\xi)\,\boldsymbol{\omega}_i\right) \times \end{array} \right] \delta\boldsymbol{g}_i \right)$$
$$= \left[\begin{array}{cc} \boldsymbol{R}^T(\xi) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^T(\xi) \end{array} \right] \sum_{i=1,2} N_i(\xi) \left(\left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{o}_i \times \\ \boldsymbol{0} & \boldsymbol{I} \end{array} \right] \left\{ \begin{array}{c} \delta\dot{\boldsymbol{x}}_i \\ \delta\dot{\boldsymbol{g}}_i \end{array} \right\} \right.$$
$$\left. + \left[\begin{array}{c} \left(\boldsymbol{v}(\xi) - \boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \\ \left(\boldsymbol{\omega}(\xi) - \boldsymbol{\omega}_i\right) \times \end{array} \right] \delta\boldsymbol{g}_i \right). \tag{13.66}$$

i

Contribution to Jacobian Matrix. After defining

$$\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.67a}$$
$$\boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi) \tag{13.67b}$$
$$\boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{R}^T(\xi) \tag{13.67c}$$
$$\boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} = \boldsymbol{R}(\xi)\,\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{R}^T(\xi) \tag{13.67d}$$
$$\begin{aligned} \boldsymbol{B}_{\tilde{\boldsymbol{v}}} &= \left(\mathcal{N}_i(\xi)\,\boldsymbol{v}(\xi) - N_i(\xi)\,\boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \\ &= N_i(\xi)\left(\boldsymbol{v}(\xi) - \boldsymbol{\omega}_i \times \boldsymbol{o}_i\right) \times \end{aligned} \tag{13.67e}$$
$$\begin{aligned} \boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} &= \left(\mathcal{N}_i(\xi)\,\boldsymbol{\omega}(\xi) - N_i(\xi)\,\boldsymbol{\omega}_i\right) \times \\ &= N_i(\xi)\left(\boldsymbol{\omega}(\xi) - \boldsymbol{\omega}_i\right) \times \end{aligned} \tag{13.67f}$$
$$\boldsymbol{d}_n(\xi) = \boldsymbol{x}(\xi) - \boldsymbol{x}_n, \tag{13.67g}$$

the contribution of the i-th node's motion to the equilibrium of the n-th node ($i = 1, 2$, $n = 1, 2$) is

$$\begin{aligned} & N_i(\xi) \left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} \\ \boldsymbol{d}_n(\xi) \times \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}} \end{array} \right] \delta\dot{\boldsymbol{x}}_i \\ & + N_i(\xi) \left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} - \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{o}_i \times \\ \boldsymbol{d}_n(\xi) \times \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}} - \left(\boldsymbol{d}_n(\xi) \times \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\right) \boldsymbol{o}_i \times \end{array} \right] \delta\dot{\boldsymbol{g}}_i \\ & + \left[\begin{array}{c} \boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} \\ \boldsymbol{d}_n(\xi) \times \left(\boldsymbol{f}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{f}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}}\right) + \boldsymbol{c}_{a/\tilde{\boldsymbol{v}}}\boldsymbol{B}_{\tilde{\boldsymbol{v}}} + \boldsymbol{c}_{a/\tilde{\boldsymbol{\omega}}}\boldsymbol{B}_{\tilde{\boldsymbol{\omega}}} \end{array} \right] \delta\boldsymbol{g}_i \\ & + \left[\begin{array}{c} \boldsymbol{0} \\ -\left(N_i - \delta_{ni}\right) \Delta\boldsymbol{f}_n \times \end{array} \right] \delta\boldsymbol{x}_i \\ & + \left[\begin{array}{c} -\mathcal{N}_i(\xi)\,\Delta\boldsymbol{f}_n \times \\ -\mathcal{N}_i(\xi)\left(\boldsymbol{d}_n(\xi) \times \Delta\boldsymbol{f}_n \times \; + \Delta\boldsymbol{c}_n \times\right) + \Delta\boldsymbol{f}_n \times N_i\boldsymbol{o}_i \times \end{array} \right] \delta\boldsymbol{g}_i \\ & \stackrel{\text{uu}}{=} \left\{ \begin{array}{c} \delta\Delta\boldsymbol{f}_n \\ \delta\Delta\boldsymbol{c}_n \end{array} \right\}, \end{aligned} \tag{13.68}$$

where δ_{ni} is Dirac's delta, which is 1 when $i = n$, and 0 otherwise.

Table 13.1: Coefficients of the Wagner indicial response approximation of Theodorsen's function ([11, 12])

	approx. 1	approx. 2
A_1	0.165	0.165
A_2	0.335	0.335
b_1	0.0455	0.041
b_2	0.3	0.32

13.7 Unsteady aerodynamics model

(Author: Mattia Mattaboni)

The 2D unsteady aerodynamics loads are computed implementing the Theodorsen theory in state-space form using the Wagner approximation of the Theodorsen function.

$$\tilde{\boldsymbol{f}}_a = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\tilde{\boldsymbol{c}}_{f_a}\left(\boldsymbol{y}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q}\right),U_\infty\right) \tag{13.69a}$$

$$\tilde{\boldsymbol{c}}_a = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\tilde{\boldsymbol{c}}_{c_a}\left(\boldsymbol{y}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q}\right),U_\infty\right) \tag{13.69b}$$

$$\boldsymbol{0} = \boldsymbol{g}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q},\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}} - \boldsymbol{A}\left(U_\infty\right)\boldsymbol{q} - \boldsymbol{B}\left(U_\infty\right)\boldsymbol{u}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}}\right), \tag{13.69c}$$

where ρ is the air density, c the airfoil chord and $\boldsymbol{y}$ is

$$\boldsymbol{y} = \boldsymbol{C}\left(U_\infty\right)\boldsymbol{q} + \boldsymbol{D}\left(U_\infty\right)\boldsymbol{u}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}}\right). \tag{13.70}$$

Matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -b_1 b_2\left(\frac{2U_\infty}{c}\right)^2 & -\left(b_1+b_2\right)\left(\frac{2U_\infty}{c}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\omega_{\mathrm{PD}} & -\omega_{\mathrm{PD}}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\omega_{\mathrm{PD}} & -\omega_{\mathrm{PD}}^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{13.71}$$

where A_1, A_2, b_1 and b_2 are the coefficients of the Theodorsen function approximation (Table 13.1 from [11] and [12]), ω_{PD} is the frequency of the pseudo-derivative algorithm

$$\mathcal{L}\left(\dot{f}\right) = s\mathcal{L}\left(f\right) - f\left(0\right) \cong \frac{s}{\left(1+s/\omega_{\mathrm{PD}}\right)^2}\mathcal{L}\left(f\right) \tag{13.72}$$

(where $f\left(0\right)$ can be neglected under broad assumptions), and

$$U_\infty = \sqrt{\tilde{v}_x^2 + \tilde{v}_y^2}. \tag{13.73}$$

Matrix $\boldsymbol{B}$ is

$$\boldsymbol{B} = \tilde{\boldsymbol{B}}\boldsymbol{T}_u\left(U_\infty\right), \tag{13.74}$$

where

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13.75}$$

and

$$\boldsymbol{T}_u = \left[\begin{array}{c} 0 \quad 1 \\ \left[\begin{array}{cc} 1 & -\frac{d_{1/4}}{U_\infty} \\ 1 & -\frac{d_{3/4}}{U_\infty} \end{array} \right]^{-1} \end{array} \right] = \left[\begin{array}{c} 0 \quad 1 \\ \frac{1}{d_{3/4}-d_{1/4}} \left[\begin{array}{cc} d_{3/4} & -d_{1/4} \\ U_\infty & -U_\infty \end{array} \right] \end{array} \right]. \tag{13.76}$$

$$\boldsymbol{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \frac{1}{d_{3/4}-d_{1/4}} d_{3/4} & -\frac{1}{d_{3/4}-d_{1/4}} d_{1/4} \\ 0 & 0 \\ \frac{1}{d_{3/4}-d_{1/4}} U_\infty & -\frac{1}{d_{3/4}-d_{1/4}} U_\infty \\ 0 & 0 \end{array} \right] \tag{13.77}$$

$$\boldsymbol{u} = \left\{ \begin{array}{c} \tan^{-1}\left(\frac{-V_y-\omega_z d_{1/4}}{V_x}\right) \\ \tan^{-1}\left(\frac{-V_y-\omega_z d_{3/4}}{V_x}\right) \end{array} \right\} \tag{13.78}$$

Matrix $\boldsymbol{C}$ is

$$\boldsymbol{C} = \left[\begin{array}{cccccc} (A_1+A_2)\, b_1 b_2 \left(\frac{2U_\infty}{c}\right)^2 & (A_1 b_1 + A_2 b_2)\left(\frac{2U_\infty}{c}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_{\mathrm{PD}}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_{\mathrm{PD}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]. \tag{13.79}$$

Matrix $\boldsymbol{D}$ is

$$\boldsymbol{D} = \tilde{\boldsymbol{D}} \boldsymbol{T}_u \left(U_\infty\right), \tag{13.80}$$

where

$$\tilde{\boldsymbol{D}} = \left[\begin{array}{ccc} (1-A_1-A_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]. \tag{13.81}$$

$$\boldsymbol{D} = \left[\begin{array}{cc} 0 & (1-A_1-A_2) \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{d_{3/4}-d_{1/4}} U_\infty & -\frac{1}{d_{3/4}-d_{1/4}} U_\infty \end{array} \right]. \tag{13.82}$$

$$\tilde{\boldsymbol{c}}_{f_a} = \left\{ \begin{array}{c} -c_d \\ c_l \\ 0 \end{array} \right\} = \boldsymbol{c}_{f_a}^{\mathrm{lookup}}(\boldsymbol{y}) + \tilde{\boldsymbol{T}}_{f_a} \boldsymbol{y}. \tag{13.83}$$

$$\tilde{\boldsymbol{c}}_{c_a} = \left\{ \begin{array}{c} 0 \\ 0 \\ c_m \end{array} \right\} = \boldsymbol{c}_{c_a}^{\mathrm{lookup}}(\boldsymbol{y}) + \tilde{\boldsymbol{T}}_{c_a} \boldsymbol{y}. \tag{13.84}$$

The second part is the noncirculatory effect.

$$\left\{ \begin{array}{c} c_l^{NC} \\ c_m^{NC} \end{array} \right\} = \boldsymbol{T}_1 \boldsymbol{T}_2 \left(U_\infty\right) \boldsymbol{y} = \tilde{\boldsymbol{T}} \boldsymbol{y}, \tag{13.85}$$

where

$$\boldsymbol{T}_1 = \left[\begin{array}{cc} 1 & 0 \\ -\frac{a+\frac{1}{2}}{2} & 1 \end{array} \right], \tag{13.86}$$

and

$$\boldsymbol{T}_2 = \frac{C_{l_\alpha}}{2} \frac{c}{2} \left[\begin{array}{cccc} 0 & \frac{1}{U_\infty} & -\frac{ca}{2U_\infty^2} & 0 \\ 0 & \frac{a}{2U_\infty} & -\frac{c}{4U_\infty^2}\left(\frac{1}{8} + a^2\right) & -\frac{1}{4U_\infty} \end{array} \right]. \tag{13.87}$$

$$\tilde{\boldsymbol{T}} = \frac{C_{l_\alpha}}{2} \frac{c}{2} \left[\begin{array}{cccc} 0 & \frac{1}{U_\infty} & -\frac{ca}{2U_\infty^2} & 0 \\ 0 & -\frac{1}{4U_\infty} & \left(\frac{ca}{8U_\infty^2} - \frac{c}{32U_\infty^2}\right) & -\frac{1}{4U_\infty} \end{array} \right] \tag{13.88}$$

$$\tilde{\boldsymbol{T}}_{f_a} = \frac{C_{l_\alpha}}{2} \frac{c}{2} \left[\begin{array}{cccc} 0 & \frac{1}{U_\infty} & -\frac{ca}{2U_\infty^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \tag{13.89}$$

$$\tilde{\boldsymbol{T}}_{c_a} = \frac{C_{l_\alpha}}{2} \frac{c}{2} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4U_\infty} & \left(\frac{ca}{8U_\infty^2} - \frac{c}{32U_\infty^2}\right) & -\frac{1}{4U_\infty} \end{array} \right] \tag{13.90}$$

Finally,

$$\tilde{\boldsymbol{c}}_{f_a} = \boldsymbol{c}_{f_a}^{\text{lookup}} \left(y_1\right) + \left\{ \begin{array}{c} 0 \\ \frac{c_{l_\alpha}}{2} \frac{c}{2U_\infty} \left(y_2 - \frac{ca}{2U_\infty} y_3\right) \\ 0 \end{array} \right\}. \tag{13.91}$$

$$\tilde{\boldsymbol{c}}_{c_a} = \boldsymbol{c}_{c_a}^{\text{lookup}} \left(y_1\right) + \left\{ \begin{array}{c} 0 \\ 0 \\ \frac{c_{l_\alpha}}{2} \frac{c}{2U_\infty} \left(-\frac{1}{4} y_2 + \left(\frac{ca}{8U_\infty} - \frac{c}{32U_\infty}\right) y_3 - \frac{1}{4} y_4\right) \end{array} \right\}. \tag{13.92}$$

13.7.1 Perturbation of the Equations

Perturbation of $\boldsymbol{g}$ with respect to $\tilde{\boldsymbol{v}}$

The perturbation of $\boldsymbol{g}$ is

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}} = -\left(\boldsymbol{A}_{/U_\infty} \boldsymbol{q} + \boldsymbol{B}_{/U_\infty} \boldsymbol{u}\right) U_{\infty/\tilde{\boldsymbol{v}}} - \boldsymbol{B} \boldsymbol{u}_{/\tilde{\boldsymbol{v}}}. \tag{13.93}$$

where, according to the previous definition of the matrices, the derivatives are

$$\boldsymbol{A}_{/U_\infty} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -b_1 b_2 \frac{8}{c^2} U_\infty & -\left(b_1 + b_2\right) \frac{2}{c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \tag{13.94}$$

$$\boldsymbol{B}_{/U_\infty} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{d_{3/4}-d_{1/4}} & -\frac{1}{d_{3/4}-d_{1/4}} \\ 0 & 0 \end{bmatrix} \tag{13.95}$$

$$U_{\infty/\tilde{\boldsymbol{v}}} = \begin{bmatrix} \frac{\tilde{v}_x}{U_\infty} & \frac{\tilde{v}_y}{U_\infty} & 0 \end{bmatrix} \tag{13.96}$$

$$\boldsymbol{u}_{/\tilde{\boldsymbol{v}}} = \begin{bmatrix} \frac{\tilde{v}_y+\tilde{\omega}_z d_{1/4}}{\tilde{v}_x^2+\tilde{v}_y^2+\tilde{\omega}_z^2 d_{1/4}^2+2\tilde{v}_y\tilde{\omega}_z d_{1/4}} & \frac{-\tilde{v}_x}{\tilde{v}_x^2+\tilde{v}_y^2+\tilde{\omega}_z^2 d_{1/4}^2+2\tilde{v}_y\tilde{\omega}_z d_{1/4}} & 0 \\ \frac{\tilde{v}_y+\tilde{\omega}_z d_{3/4}}{\tilde{v}_x^2+\tilde{v}_y^2+\tilde{\omega}_z^2 d_{3/4}^2+2\tilde{v}_y\tilde{\omega}_z d_{3/4}} & \frac{-\tilde{v}_x}{\tilde{v}_x^2+\tilde{v}_y^2+\tilde{\omega}_z^2 d_{3/4}^2+2\tilde{v}_y\tilde{\omega}_z d_{3/4}} & 0 \end{bmatrix} \tag{13.97}$$

The explicit computation of each term of Eq. 13.93 yields

$$\left(\boldsymbol{A}_{U_\infty}\boldsymbol{q} + \boldsymbol{B}_{U_\infty}\boldsymbol{u}\right) U_{\infty/\tilde{v}} = \begin{Bmatrix} 0 \\ -b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2 \\ 0 \\ 0 \\ \frac{u_1-u_2}{d_{3/4}-d_{1/4}} \\ 0 \end{Bmatrix} \begin{Bmatrix} \frac{\tilde{v}_x}{U_\infty} & \frac{\tilde{v}_x}{U_\infty} & 0 \end{Bmatrix} \tag{13.98}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_x}{U_\infty} & \left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_y}{U_\infty} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \left(\frac{u_1-u_2}{d_{3/4}-d_{1/4}}\right)\frac{\tilde{v}_x}{U_\infty} & \left(\frac{u_1-u_2}{d_{3/4}-d_{1/4}}\right)\frac{\tilde{v}_y}{U_\infty} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.99}$$

for the first, while the second yields

$$\boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}} = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) & \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) & 0 \\ \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(3,1) & \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(3,2) & 0 \\ 0 & 0 & 0 \\ \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(5,1) & \boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(5,2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.100}$$

The non-null terms are

$$\boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) = \boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) \tag{13.101}$$

$$\boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) = \boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) \tag{13.102}$$

$$\boldsymbol{B}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(3,1) = \boldsymbol{B}(3,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,1) + \boldsymbol{B}(3,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) \tag{13.103}$$

$$\boldsymbol{Bu}_{/\tilde{\boldsymbol{v}}}(3,2) = \boldsymbol{B}(3,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,2) + \boldsymbol{B}(3,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) \tag{13.104}$$

$$\boldsymbol{Bu}_{/\tilde{\boldsymbol{v}}}(3,1) = \boldsymbol{B}(5,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,1) + \boldsymbol{B}(5,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) \tag{13.105}$$

$$\boldsymbol{Bu}_{/\tilde{\boldsymbol{v}}}(3,2) = \boldsymbol{B}(5,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,2) + \boldsymbol{B}(5,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) \tag{13.106}$$

So, the Jacobian matrix $\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}$ is

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}} = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,1) & \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,2) & 0 \\ \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(3,1) & \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(3,2) & 0 \\ 0 & 0 & 0 \\ \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(5,1) & \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(5,2) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13.107}$$

where the matrix elements are

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,1) = -\left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1 + b_2)\,\frac{2}{c} q_2\right) \frac{\tilde{v}_x}{U_\infty} - \boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1) \tag{13.108}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,2) = -\left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1 + b_2)\,\frac{2}{c} q_2\right) \frac{\tilde{v}_y}{U_\infty} - \boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2) \tag{13.109}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(3,1) = -\big(\boldsymbol{B}(3,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,1) + \boldsymbol{B}(3,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1)\big) \tag{13.110}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(3,2) = -\big(\boldsymbol{B}(3,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,2) + \boldsymbol{B}(3,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2)\big) \tag{13.111}$$

$$\boldsymbol{g}_{\tilde{\boldsymbol{v}}}(3,1) = -\left(\frac{u_1 - u_2}{d_{3/4} - d_{1/4}}\right) \frac{\tilde{v}_x}{U_\infty} - \big(\boldsymbol{B}(5,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,1) + \boldsymbol{B}(5,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,1)\big) \tag{13.112}$$

$$\boldsymbol{g}_{\tilde{\boldsymbol{v}}}(3,2) = -\left(\frac{u_1 - u_2}{d_{3/4} - d_{1/4}}\right) \frac{\tilde{v}_y}{U_\infty} - \big(\boldsymbol{B}(5,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(1,2) + \boldsymbol{B}(5,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{v}}}(2,2)\big) \tag{13.113}$$

Perturbation of $\boldsymbol{g}$ with respect to $\tilde{\boldsymbol{\omega}}$

The perturbation of $\boldsymbol{g}$ yields

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}} = -\boldsymbol{Bu}_{/\tilde{\boldsymbol{\omega}}}. \tag{13.114}$$

Where

$$\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}} = \begin{bmatrix} 0 & 0 & \frac{-\tilde{v}_x d_{1/4}}{\tilde{v}_x^2 + \tilde{v}_y^2 + \tilde{\omega}_z^2 d_{1/4}^2 + 2\tilde{v}_y \tilde{\omega}_z d_{1/4}} \\ 0 & 0 & \frac{-\tilde{v}_x d_{3/4}}{\tilde{v}_x^2 + \tilde{v}_y^2 + \tilde{\omega}_z^2 d_{3/4}^2 + 2\tilde{v}_y \tilde{\omega}_z d_{3/4}} \end{bmatrix} \tag{13.115}$$

Thus,

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(2,3) \\ 0 & 0 & \boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(3,3) \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(5,3) \\ 0 & 0 & 0 \end{bmatrix} \tag{13.116}$$

where

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(2,3) = -\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}}(2,3) \tag{13.117}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(5,3) = -\big(\boldsymbol{B}(3,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}}(1,3) + \boldsymbol{B}(3,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}}(2,3)\big) \tag{13.118}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}}(5,3) = -\big(\boldsymbol{B}(5,1)\,\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}}(1,3) + \boldsymbol{B}(5,2)\,\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}}(2,3)\big) \tag{13.119}$$

Perturbation of $\boldsymbol{g}$ with respect to $\boldsymbol{q}$

The perturbation of $\boldsymbol{g}$ yields

$$\boldsymbol{g}_{/\boldsymbol{q}} = -\boldsymbol{A} \tag{13.120}$$

Perturbation of $\boldsymbol{g}$ with respect to $\dot{\boldsymbol{q}}$

The perturbation of $\boldsymbol{g}$ yields

$$\boldsymbol{g}_{/\dot{\boldsymbol{q}}} = \boldsymbol{I} \tag{13.121}$$

13.7.2 Perturbation of the aerodynamic forces

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\tilde{v}$

$$\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}} = \rho c \tilde{\boldsymbol{c}}_{f_a} \left[\begin{array}{ccc} \tilde{v}_x & \tilde{v}_y & \tilde{v}_z \end{array}\right] + \frac{1}{2}\rho \tilde{\boldsymbol{v}}^T \tilde{\boldsymbol{v}} c \left(\tilde{\boldsymbol{c}}_{f_a/\boldsymbol{y}} \boldsymbol{y}_{/\tilde{\boldsymbol{v}}} + \tilde{\boldsymbol{c}}_{f_a/U_\infty} U_{\infty/\tilde{\boldsymbol{v}}}\right) \tag{13.122}$$

where

$$\boldsymbol{y}_{/\tilde{\boldsymbol{v}}} = \big(\boldsymbol{C}_{/U_\infty}\boldsymbol{q} + \boldsymbol{D}_{/U_\infty}\boldsymbol{u}\big)\, U_{\infty/\tilde{\boldsymbol{v}}} + \boldsymbol{D}\boldsymbol{u}_{/\tilde{\boldsymbol{v}}} \tag{13.123}$$

$$\boldsymbol{C}_{/U_\infty} = \begin{bmatrix} (A_1 + A_2)\, b_1 b_2 \frac{8}{c^2} U_\infty & (A_1 b_1 + A_2 b_2)\frac{2}{c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13.124}$$

$$\boldsymbol{D}_{/U_\infty} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{d_{3/4} - d_{1/4}} & -\frac{1}{d_{3/4} - d_{1/4}} \end{bmatrix} \tag{13.125}$$

$$\tilde{\boldsymbol{c}}_{f_a/U_\infty} = \left\{ \begin{array}{c} 0 \\ \frac{C_{l_\alpha}}{2}\frac{c}{2}\frac{1}{U_\infty^2}\left(-y_2 + \frac{ca}{U_\infty} y_3\right) \\ 0 \end{array} \right\} \tag{13.126}$$

$$\tilde{\boldsymbol{c}}_{f_a/\boldsymbol{y}} = \begin{bmatrix} \frac{dc_d^{\text{lookup}}}{d\alpha} & 0 & 0 & 0 \\ \frac{dc_l^{\text{lookup}}}{d\alpha} & \frac{C_{l_\alpha}}{2}\frac{c}{2U_\infty} & -\frac{c_{l_\alpha}}{2}\frac{c^2 a}{4U_\infty^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13.127}$$

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\tilde{\boldsymbol{\omega}}$

$$\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\left(\tilde{\boldsymbol{c}}_{f_{a/\boldsymbol{y}}}\boldsymbol{y}_{/\tilde{\boldsymbol{\omega}}}\right) \tag{13.128}$$

where

$$\boldsymbol{y}_{/\tilde{\boldsymbol{\omega}}} = \boldsymbol{D}\boldsymbol{u}_{/\tilde{\boldsymbol{\omega}}} \tag{13.129}$$

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\boldsymbol{q}$

$$\tilde{\boldsymbol{f}}_{a/\boldsymbol{q}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\left(\tilde{\boldsymbol{c}}_{f_{a/\boldsymbol{y}}}\boldsymbol{y}_{/\boldsymbol{q}}\right) \tag{13.130}$$

where

$$\boldsymbol{y}_{/\boldsymbol{q}} = \boldsymbol{C} \tag{13.131}$$

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\dot{\boldsymbol{q}}$

$$\tilde{\boldsymbol{f}}_{a/\dot{\boldsymbol{q}}} = \boldsymbol{0} \tag{13.132}$$

13.7.3 Perturbation of the aerodynamic moments

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\tilde{v}$

$$\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}} = \rho c^2\tilde{\boldsymbol{c}}_{c_a}\begin{bmatrix} \tilde{v}_x & \tilde{v}_y & \tilde{v}_z \end{bmatrix} + \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/\boldsymbol{y}}}\boldsymbol{y}_{/\tilde{\boldsymbol{v}}} + \tilde{\boldsymbol{c}}_{c_{a/U_\infty}}U_{\infty/\tilde{\boldsymbol{v}}}\right) \tag{13.133}$$

$$\tilde{\boldsymbol{c}}_{c_{a/U_\infty}} = \begin{Bmatrix} 0 \\ 0 \\ \frac{c_{l_\alpha}}{2}\frac{c}{2}\frac{1}{4U_\infty^2}\left(y_2 - \left(\frac{ca}{U_\infty} - \frac{c}{4U_\infty}\right)y_3 + y_4\right) \end{Bmatrix} \tag{13.134}$$

$$\tilde{\boldsymbol{c}}_{c_{a/\boldsymbol{y}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{dc_m^{\text{lookup}}}{d\alpha} & -\frac{1}{4}\frac{c_{l_\alpha}}{2}\frac{c}{2U_\infty} & \frac{c_{l_\alpha}}{2}\frac{c}{2U_\infty}\left(\frac{ca}{8U_\infty} - \frac{c}{32U_\infty}\right) & -\frac{1}{4}\frac{c_{l_\alpha}}{2}\frac{c}{2U_\infty} \end{bmatrix} \tag{13.135}$$

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\tilde{\boldsymbol{\omega}}$

$$\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/\boldsymbol{y}}}\boldsymbol{y}_{/\tilde{\boldsymbol{\omega}}}\right) \tag{13.136}$$

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\boldsymbol{q}$

$$\tilde{\boldsymbol{c}}_{a/\boldsymbol{q}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/\boldsymbol{y}}}\boldsymbol{y}_{/\boldsymbol{q}}\right) \tag{13.137}$$

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\dot{\boldsymbol{q}}$

$$\tilde{\boldsymbol{c}}_{a/\dot{\boldsymbol{q}}} = \boldsymbol{0} \tag{13.138}$$

13.7.4 Finite difference version

A reduced state-space model, with just 2 states instead of 6 states, can be used deleting the pseudo-derivative algorithm and computing the derivate using the backward finite diffenence.

The set of equations that descibes the unsteady aerodynamic loads is still the same:

$$\tilde{\boldsymbol{f}}_a = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\tilde{\boldsymbol{c}}_{f_a}\left(y\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q}\right),U_\infty\right) \tag{13.139a}$$

$$\tilde{\boldsymbol{c}}_a = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\tilde{\boldsymbol{c}}_{c_a}\left(y\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q}\right),U_\infty\right) \tag{13.139b}$$

$$\boldsymbol{0} = \boldsymbol{g}\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}},\boldsymbol{q},\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}} - \boldsymbol{A}\left(U_\infty\right)\boldsymbol{q} - \boldsymbol{B}u\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}}\right), \tag{13.139c}$$

where ρ is the air density, c the airfoil chord and $\boldsymbol{y}$ is defined as:

$$y = \boldsymbol{C}\left(U_\infty\right)\boldsymbol{q} + \boldsymbol{D}u\left(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{\omega}}\right). \tag{13.140}$$

In this case the matrix $\boldsymbol{A}$ is:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -b_1b_2\left(\frac{2U_\infty}{c}\right)^2 & -\left(b_1+b_2\right)\left(\frac{2U_\infty}{c}\right) \end{bmatrix}, \tag{13.141}$$

where A_1, A_2, b_1 and b_2 are the coefficients of the Theodorsen function approximation (Table 13.1 from [11] and [12]),

The matrix $\boldsymbol{B}$ is simply:

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{13.142}$$

Whereas the input of the system u is the angle of attack computed at the 3/4 chord point:

$$u = \tan^{-1}\left(\frac{-V_y - \omega_z d_{3/4}}{V_x}\right) \tag{13.143}$$

The $\boldsymbol{C}$ matrix is:

$$\boldsymbol{C} = \begin{bmatrix} \left(A_1+A_2\right)b_1b_2\left(\frac{2U_\infty}{c}\right)^2 & \left(A_1b_1+A_2b_2\right)\left(\frac{2U_\infty}{c}\right) \end{bmatrix}. \tag{13.144}$$

and the $\boldsymbol{D}$ matrix:

$$\boldsymbol{D} = \begin{bmatrix} \left(1-A_1-A_2\right) \end{bmatrix}. \tag{13.145}$$

Using the otuput $\boldsymbol{y}$ of this SISO state-space model as input for the lookup table the circulatory part of the unsteady loads is computed.

The non-circulatory part is computed starting from the angle of attack computed at 1/4 chord (u_1) and 3/4 chord (u_2):

$$\begin{Bmatrix} \alpha_{pivot} \\ \dot{\alpha} \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{d_{1/4}}{U_\infty} \\ 1 & -\frac{d_{3/4}}{U_\infty} \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{d_{3/4}-d_{1/4}} \begin{bmatrix} d_{3/4} & -d_{1/4} \\ U_\infty & -U_\infty \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{13.146}$$

where:

$$\boldsymbol{u} = \begin{Bmatrix} \tan^{-1}\left(\frac{-V_y-\omega_z d_{1/4}}{V_x}\right) \\ \tan^{-1}\left(\frac{-V_y-\omega_z d_{3/4}}{V_x}\right) \end{Bmatrix} \tag{13.147}$$

Thus, the aerodynamic coefficients result:

$$\tilde{\boldsymbol{c}}_{f_a} = \boldsymbol{c}_{f_a}^{\text{lookup}}(y) + \left\{ \begin{array}{c} 0 \\ \frac{c_{l_\alpha}}{2}\frac{c}{2U_\infty}\left(\dot{\alpha}_{pivot} - \frac{ca}{2U_\infty}\ddot{\alpha}\right) \\ 0 \end{array} \right\}. \tag{13.148}$$

$$\tilde{\boldsymbol{c}}_{c_a} = \boldsymbol{c}_{c_a}^{\text{lookup}}(y) + \left\{ \begin{array}{c} 0 \\ 0 \\ \frac{c_{l_\alpha}}{2}\frac{c}{2U_\infty}\left(-\frac{1}{4}\dot{\alpha}_{pivot} + \left(\frac{ca}{8U_\infty} - \frac{c}{32U_\infty}\right)\ddot{\alpha} - \frac{1}{4}\dot{\alpha}\right) \end{array} \right\}. \tag{13.149}$$

In order to compute the aerodynamic coefficients $\dot{\alpha}_{pivot}$ and $\ddot{\alpha}$ are necessary and they can be computed using the backward finite difference:

$$\left\{ \begin{array}{c} \dot{\alpha}_{pivot} \\ \ddot{\alpha} \end{array} \right\} = \frac{1}{\Delta t}\left(\left\{ \begin{array}{c} \alpha_{pivot} \\ \dot{\alpha} \end{array} \right\}_k - \left\{ \begin{array}{c} \alpha_{pivot} \\ \dot{\alpha} \end{array} \right\}_{k-1}\right) \tag{13.150}$$

13.7.5 Perturbation of the equations

Perturbation of $\boldsymbol{g}$ with respect to $\tilde{\boldsymbol{v}}$

The perturbation of $\boldsymbol{g}$ is:

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}} = -\boldsymbol{A}_{/U_\infty}\boldsymbol{q}U_{\infty/\tilde{\boldsymbol{v}}} - \boldsymbol{B}u_{/\tilde{\boldsymbol{v}}}. \tag{13.151}$$

where, accordingly with the previous definition of the matrices the derivatives are:

$$\boldsymbol{A}_{/U_\infty} = \left[\begin{array}{cc} 0 & 0 \\ -b_1 b_2 \frac{8}{c^2} U_\infty & -(b_1+b_2)\frac{2}{c} \end{array} \right] \tag{13.152}$$

$$U_{\infty/\tilde{\boldsymbol{v}}} = \left[\begin{array}{ccc} \frac{\tilde{v}_x}{U_\infty} & \frac{\tilde{v}_y}{U_\infty} & 0 \end{array} \right] \tag{13.153}$$

$$u_{/\tilde{\boldsymbol{v}}} = \left[\begin{array}{ccc} \frac{\tilde{v}_y + \tilde{\omega}_z d_{3/4}}{\tilde{v}_x^2 + \tilde{v}_y^2 + \tilde{\omega}_z^2 d_{3/4}^2 + 2\tilde{v}_y\tilde{\omega}_z d_{3/4}} & \frac{-\tilde{v}_x}{\tilde{v}_x^2 + \tilde{v}_y^2 + \tilde{\omega}_z^2 d_{3/4}^2 + 2\tilde{v}_y\tilde{\omega}_z d_{3/4}} & 0 \end{array} \right] \tag{13.154}$$

We can now explicitly perform the computation of each term of Eq. 13.151:

$$\boldsymbol{A}_{U_\infty}\boldsymbol{q}U_{\infty/\tilde{v}} = \left\{ \begin{array}{c} 0 \\ -b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2 \end{array} \right\} \left\{ \begin{array}{ccc} \frac{\tilde{v}_x}{U_\infty} & \frac{\tilde{v}_x}{U_\infty} & 0 \end{array} \right\} \tag{13.155}$$

$$= \left[\begin{array}{ccc} 0 & 0 & 0 \\ \left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_x}{U_\infty} & \left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_y}{U_\infty} & 0 \end{array} \right] \tag{13.156}$$

wheres the second is:

$$\boldsymbol{B}u_{/\tilde{\boldsymbol{v}}} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ u_{/\tilde{\boldsymbol{v}}}(1,1) & u_{/\tilde{\boldsymbol{v}}}(1,2) & 0 \end{array} \right] \tag{13.157}$$

So, the jacobian matrix $\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}$ results:

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,1) & \boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,2) & 0 \end{array} \right] \tag{13.158}$$

where the matrix elements are:

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,1) = -\left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_x}{U_\infty} - u_{/\tilde{\boldsymbol{v}}}(1,1) \tag{13.159}$$

$$\boldsymbol{g}_{/\tilde{\boldsymbol{v}}}(2,2) = -\left(-b_1 b_2 \frac{8}{c^2} U_\infty q_1 - (b_1+b_2)\frac{2}{c} q_2\right)\frac{\tilde{v}_y}{U_\infty} - u_{/\tilde{\boldsymbol{v}}}(1,2) \tag{13.160}$$

Perturbation of $\boldsymbol{g}$ with respect to $\tilde{\boldsymbol{\omega}}$

The perturbation of $\boldsymbol{g}$ is:

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}} = -\boldsymbol{B} u_{/\tilde{\boldsymbol{\omega}}}. \tag{13.161}$$

Where:

$$u_{/\tilde{\boldsymbol{\omega}}} = \left[\begin{array}{ccc} 0 & 0 & \frac{-\tilde{v}_x d_{3/4}}{\tilde{v}_x^2 + \tilde{v}_y^2 + \tilde{\omega}_z^2 d_{3/4}^2 + 2\tilde{v}_y \tilde{\omega}_z d_{3/4}} \end{array} \right] \tag{13.162}$$

Thus, it results:

$$\boldsymbol{g}_{/\tilde{\boldsymbol{\omega}}} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -u_{/\tilde{\boldsymbol{\omega}}}(1,3) \end{array} \right] \tag{13.163}$$

Perturbation of $\boldsymbol{g}$ with respect to $\boldsymbol{q}$

The perturbation of $\boldsymbol{g}$ is simply:

$$\boldsymbol{g}_{/\boldsymbol{q}} = -\boldsymbol{A} \tag{13.164}$$

Perturbation of $\boldsymbol{g}$ with respect to $\dot{\boldsymbol{q}}$

The perturbation of $\boldsymbol{g}$ is simply:

$$\boldsymbol{g}_{/\dot{\boldsymbol{q}}} = \boldsymbol{I} \tag{13.165}$$

13.7.6 Perturbation of the aerodynamic forces

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\tilde{v}$

$$\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{v}}} = \rho c \tilde{\boldsymbol{c}}_{f_a} \left[\begin{array}{ccc} \tilde{v}_x & \tilde{v}_y & \tilde{v}_z \end{array} \right] + \frac{1}{2}\rho \tilde{\boldsymbol{v}}^T \tilde{\boldsymbol{v}} c \left(\tilde{\boldsymbol{c}}_{f_{a/y}} y_{/\tilde{\boldsymbol{v}}} + \tilde{\boldsymbol{c}}_{f_{a/U_\infty}} U_{\infty/\tilde{\boldsymbol{v}}} \right) \tag{13.166}$$

where:

$$y_{/\tilde{\boldsymbol{v}}} = \boldsymbol{C}_{/U_\infty} \boldsymbol{q} U_{\infty/\tilde{\boldsymbol{v}}} + \boldsymbol{D} u_{/\tilde{\boldsymbol{v}}} \tag{13.167}$$

$$\boldsymbol{C}_{/U_\infty} = \left[\begin{array}{cc} (A_1 + A_2) b_1 b_2 \frac{8}{c^2} U_\infty & (A_1 b_1 + A_2 b_2) \frac{2}{c} \end{array} \right] \tag{13.168}$$

$$\tilde{\boldsymbol{c}}_{f_{a/U_\infty}} = \left\{ \begin{array}{c} 0 \\ \frac{C_{l\alpha}}{2} \frac{c}{2} \frac{1}{U_\infty^2} \left(-\dot{\alpha}_{pivot} + \frac{ca}{U_\infty} \ddot{\alpha} \right) \\ 0 \end{array} \right\} \tag{13.169}$$

$$\tilde{\boldsymbol{c}}_{f_{a/y}} = \left\{ \begin{array}{c} \frac{dc_d^{\text{lookup}}}{d\alpha} \\ \frac{dc_l^{\text{lookup}}}{d\alpha} \\ 0 \end{array} \right\} \tag{13.170}$$

where the dependence of $\dot{\alpha}_{pivot}$, $\dot{\alpha}$ and $\ddot{\alpha}$ is neglected.

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\tilde{\boldsymbol{\omega}}$

$$\tilde{\boldsymbol{f}}_{a/\tilde{\boldsymbol{\omega}}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\left(\tilde{\boldsymbol{c}}_{f_{a/\boldsymbol{y}}}y_{/\tilde{\boldsymbol{\omega}}}\right) \tag{13.171}$$

where:

$$y_{/\tilde{\boldsymbol{\omega}}} = \boldsymbol{D}u_{/\tilde{\boldsymbol{\omega}}} \tag{13.172}$$

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\boldsymbol{q}$

$$\tilde{\boldsymbol{f}}_{a/\boldsymbol{q}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c\left(\tilde{\boldsymbol{c}}_{f_{a/y}}\boldsymbol{y}_{/\boldsymbol{q}}\right) \tag{13.173}$$

where:

$$y_{/\boldsymbol{q}} = \boldsymbol{C} \tag{13.174}$$

Perturbation of $\tilde{\boldsymbol{f}}_a$ with respect to $\dot{\boldsymbol{q}}$

$$\tilde{\boldsymbol{f}}_{a/\dot{\boldsymbol{q}}} = \boldsymbol{0} \tag{13.175}$$

13.7.7 Perturbation of the aerodynamic moments

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\tilde{v}$

$$\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{v}}} = \rho c^2\tilde{\boldsymbol{c}}_{c_a}\left[\begin{array}{ccc}\tilde{v}_x & \tilde{v}_y & \tilde{v}_z\end{array}\right] + \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/y}}y_{/\tilde{\boldsymbol{v}}} + \tilde{\boldsymbol{c}}_{c_{a/U_\infty}}U_{\infty/\tilde{\boldsymbol{v}}}\right) \tag{13.176}$$

$$\tilde{\boldsymbol{c}}_{c_{a/U_\infty}} = \left\{\begin{array}{c} 0 \\ 0 \\ \frac{c_{l_\alpha}}{2}\frac{c}{2}\frac{1}{4U_\infty^2}\left(\dot{\alpha}_{pivot} - \left(\frac{ca}{U_\infty} - \frac{c}{4U_\infty}\right)\ddot{\alpha} + \dot{\alpha}\right)\end{array}\right\} \tag{13.177}$$

$$\tilde{\boldsymbol{c}}_{c_{a/y}} = \left[\begin{array}{c} 0 \\ 0 \\ \frac{dc_m^{\text{lookup}}}{d\alpha}\end{array}\right] \tag{13.178}$$

where again the dependence of $\dot{\alpha}_{pivot}$, $\dot{\alpha}$ and $\ddot{\alpha}$ is neglected.

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\tilde{\boldsymbol{\omega}}$

$$\tilde{\boldsymbol{c}}_{a/\tilde{\boldsymbol{\omega}}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/y}}y_{/\tilde{\boldsymbol{\omega}}}\right) \tag{13.179}$$

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\boldsymbol{q}$

$$\tilde{\boldsymbol{c}}_{a/\boldsymbol{q}} = \frac{1}{2}\rho\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{v}}c^2\left(\tilde{\boldsymbol{c}}_{c_{a/y}}y_{/\boldsymbol{q}}\right) \tag{13.180}$$

Perturbation of $\tilde{\boldsymbol{c}}_a$ with respect to $\dot{\boldsymbol{q}}$

$$\tilde{\boldsymbol{c}}_{a/\dot{\boldsymbol{q}}} = \boldsymbol{0} \tag{13.181}$$

13.8 Rotor

Definitions. Axis 3 is the rotor's axis. $\tilde{\boldsymbol{v}}$ is the composition of the velocity of the 'aircraft' node and of the airstream speed, if any, projected in the reference frame of the 'aircraft' node, namely

$$\tilde{\boldsymbol{v}} = \boldsymbol{R}_{\text{craft}}^T \left(-\boldsymbol{v}_{\text{craft}} + \boldsymbol{v}_\infty\right). \tag{13.182}$$

$$v_{12} = \sqrt{v_1^2 + v_2^2} \tag{13.183a}$$

$$v = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\tilde{\boldsymbol{v}}^T \tilde{\boldsymbol{v}}} \tag{13.183b}$$

$$\sin\alpha_d = -v_3/v \tag{13.183c}$$

$$\cos\alpha_d = v_{12}/v \tag{13.183d}$$

$$\psi_0 = \tan^{-1}\left(\frac{v_2}{v_1}\right) \tag{13.183e}$$

$$v_{\text{tip}} = \Omega R \tag{13.183f}$$

$$\mu = \cos\alpha_d \frac{v}{v_{\text{tip}}} \tag{13.183g}$$

Note: $v \geq 0$ and $v_{12} \geq 0$ by definition; as a consequence, $\cos\alpha_d \geq 0$, while the sign of $\sin\alpha_d$ depends on whether the flow related to the absolute motion of the rotor enters the disk from above (> 0) or from below (< 0). $v_{\text{tip}} > 0$ by construction ($\Omega = \|\boldsymbol{\omega}\|$, and no induced velocity is computed if Ω is below a threshold). As a consequence, $\mu \geq 0$.

Ground effect. If defined, according to [13],

$$u_{\text{IGE}} = k_{\text{GE}} u_{\text{OGE}} \tag{13.184a}$$

$$k = 1 - \frac{1}{z^2} \tag{13.184b}$$

$$z = \max\left(\frac{h}{R}, \frac{1}{4}\right) \tag{13.184c}$$

where h is the component along axis 3 of the 'ground' node of distance between the 'aircraft' and the 'ground' node, assuming the 'aircraft' node is located at the hub center.

Reference Induced Velocity. The reference induced velocity u is computed by solving the implicit problem

$$f = \lambda_u - \frac{C_t}{2\sqrt{\mu^2 + \lambda^2}} = 0, \tag{13.185}$$

with

$$C_t = \frac{T}{\rho \pi R^4 \Omega^2} \tag{13.186a}$$

$$\lambda_u = \frac{u}{v_{\text{tip}}} \tag{13.186b}$$

$$\lambda = \frac{v \sin\alpha_d + u}{v_{\text{tip}}} = \mu \tan\alpha_d + \lambda_u; \tag{13.186c}$$

T is the component of the aerodynamic force of the rotor along the shaft axis. The value of λ_u is initialized using the reference induced velocity u at the previous step/iteration. Only when $u = 0$ and $C_t \neq 0$, u is initialized using its nominal value in hover,

$$u = \operatorname{sign}(T) \sqrt{\frac{\|T\|}{2\rho A}}. \tag{13.187}$$

The problem is solved by means of a local Newton iteration. The Jacobian of the problem is

$$\frac{\partial f}{\partial \lambda_u} = 1 + \frac{C_t}{2\left(\mu^2 + \lambda^2\right)^{3/2}} \lambda. \tag{13.188}$$

The solution,

$$\Delta\lambda_u = -\left(\frac{\partial f}{\partial \lambda_u}\right)^{-1} f, \tag{13.189}$$

is added to λ_u as $\lambda_u += \eta\Delta\lambda_u$, where $0 < \eta \leq 1$ is an optional relaxation factor (by default, $\eta = 1$).

Corrections. The reference induced velocity is corrected by separately correcting the inflow and advance parameters, namely

$$\lambda^* = \frac{\lambda}{k_{\mathrm{H}}^2} \tag{13.190a}$$

$$\mu^* = \frac{\mu}{k_{\mathrm{FF}}} \tag{13.190b}$$

(by default, $k_{\mathrm{H}} = 1$ and $k_{\mathrm{FF}} = 1$). The reference induced velocity is then recomputed as

$$u^* = (1 - \rho)\, k_{\mathrm{GE}} v_{\mathrm{tip}} \frac{C_t}{2\sqrt{\mu^{*^2} + \lambda^{*^2}}} + \rho u^*_{\mathrm{prev}}, \tag{13.191}$$

where $0 \leq \rho < 1$ is a memory factor (by default, $\rho = 0$).

Note: in principle, multiple solutions for λ_u are possible. However, only one solution is physical. Currently, no strategy is put in place to ensure that only the physical solution is considered.

13.8.1 Uniform Inflow Model

The induced velocity is equal to its reference value, u^*, everywhere.

13.8.2 Glauert Model

In forward flight (when $\mu > 0.15$) the inflow over the rotor disk can be approximated by:

$$\lambda_i = \lambda_0 \left(1 + k_x \frac{x}{R} + k_y \frac{y}{R}\right) \tag{13.192}$$

$$= \lambda_0 \left(1 + k_x r \cos\psi + k_y r \sin\psi\right), \tag{13.193}$$

where the mean induced velocity λ_0 is computed as shown in the previous section, while $r = \sqrt{x^2 + y^2}/R$ is the nondimensional radius.

Table 13.2: Glauert inflow model (source: Leishman [12])

Author(s)	k_x	k_y
Coleman et al. (1945)	$\tan\frac{\chi}{2}$	0
Drees (1949)	$\frac{4}{3}\frac{(1-\cos\chi-1.8\mu^2)}{\sin\chi}$	-2μ
Payne (1959)	$\frac{4}{3}(\mu/\lambda/(1.2+\mu/\lambda))$	0
White & Blake (1979)	$\sqrt{2}\sin\chi$	0
Pitt & Peters (1981)	$\frac{15\pi}{23}\tan\frac{\chi}{2}$	0
Howlett (1981)	$\sin^2\chi$	0

In literature a lot of expressions for the k_x and k_y coefficients have been proposed by different authors, as summarized in table 13.2. Up to now in MBDyn the following expressions are implemented:

$$k_x = \frac{4}{3}\left(1 - 1.8\mu^2\right)\tan\frac{\chi}{2} \tag{13.194}$$

$$k_y = 0, \tag{13.195}$$

FIXME: è diversa dalle espressioni riportate sul Leishman!
where χ is the wake skew angle:

$$\chi = \tan^{-1}\left(\frac{\mu}{\lambda}\right). \tag{13.196}$$

TODO: check, dovrebbe essere equivalente all'espressione di Leishman ma sarebbe meglio verificare!!!

Note: the Glauert inflow model exactly matches the uniform inflow model when the advance ratio is null (in hover). In fact, when $\mu = 0$ then $\chi = k_x = 0$. The inflow is thus uniform over the rotor disk, and equal to λ_0.

Table 13.3: Glauert inflow model as implemented in MBDyn

Name	**Author(s)**	k_x	k_y
(default)	Glauert	$\frac{4}{3}(1-1.8\mu^2)\tan\left(\frac{\chi}{2}\right)$	0
`coleman`	Coleman et al. (1945)	$\tan\left(\frac{\chi}{2}\right)$	0
`drees`	Drees (1949)	$\frac{4}{3}\frac{(1-\cos\chi-1.8\mu^2)}{\sin\chi}$	-2μ
`payne`	Payne (1959)	$\frac{4}{3}\frac{\mu/\lambda}{1.2+\mu/\lambda}$	0
`white and blake`	White & Blake (1979)	$\sqrt{2}\sin\chi$	0
`pitt and peters`	Pitt & Peters (1981)	$\frac{15\pi}{23}\tan\left(\frac{\chi}{2}\right)$	0
`howlett`	Howlett (1981)	$\sin^2\chi$	0
`drees 2`	Drees (?)	$\frac{4}{3}(1-1.8\mu^2)\sqrt{1-\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2}$	-2μ

13.8.3 Mangler-Squire Model

The Mangler-Squire model is developed under the high speed assumption and it should be used only for advance ratio grater than 0.1.

In the original formulation the induced velocity is computed as:

$$\lambda_i = \left(\frac{2C_T}{\mu}\right)\left[\frac{c_0}{2} - \sum_{n=1}^{\infty} c_n(r,\alpha_d)\cos n\psi\right], \tag{13.197}$$

since the advance ratio μ appears in the denominator this expression is not valid in hover. Bramwell [14] proposed a different expression for the induced velocity:

$$\lambda_i = 4\lambda_0\left[\frac{c_0}{2} - \sum_{n=1}^{\infty} c_n(r,\alpha_d)\cos n\psi\right], \tag{13.198}$$

where λ_0 is the mean inflow computed as shown before. In this way the Mangler-Squire inflow model makes sense also in hover. MBDyn uses the latter version.

The expression of the c_n coefficients depends on the form of the rotor disk loading. Mangler and Squire developed the theory for two fundamental forms: Type I (elliptical loading) and Type III (a loading that vanishes at the edges and at the center of the disk). The total loading is finally obtained by a linear combination of Type I and Type III loadings (see [12]).

MBDyn uses just a Type III loading, and the resulting expressions for the c_n coefficients are:

$$c_0 = \frac{15}{8}\eta\left(1-\eta^2\right), \tag{13.199}$$

where $\eta = \sqrt{1-r^2}$, and $r = \sqrt{x^2+y^2}/R$ is the nondimensional radius.

$$c_1 = -\frac{15\pi}{256}\left(5-9\eta^2\right)\left[\left(1-\eta^2\right)\left(\frac{1-\sin\alpha_d}{1+\sin\alpha_d}\right)\right]^{\frac{1}{2}}, \tag{13.200}$$

$$c_3 = \frac{45\pi}{256}\left[\left(1-\eta^2\right)\left(\frac{1-\sin\alpha_d}{1+\sin\alpha_d}\right)\right]^{\frac{3}{2}}, \tag{13.201}$$

and $c_n = 0$ for odd values of $n \geq 0$.

For even values:

$$c_n = (-1)^{\frac{n}{2}-1}\frac{15}{8}\left[\frac{\eta+n}{n^2-1}\frac{9\eta^2+n^2-6}{n^2-9} + \frac{3\eta}{n^2-9}\right]\left[\left(\frac{1-\eta}{1+\eta}\right)\left(\frac{1-\sin\alpha_d}{1+\sin\alpha_d}\right)\right]^{\frac{n}{2}}, \tag{13.202}$$

Note: the version proposed by Bramwell [14] makes sense also in hover but gives different results with respect to the uniform inflow and the Glauert inflow models. Let assume $\alpha_d = \pi/2$, it follows that $c_n = 0$ for $n \geq 1$. Therefore the induced velocity does not depend on the azimuthal position ψ but only on the radial position r, so the inflow is not uniform, but the mean inflow is still λ_0.

13.8.4 Dynamic Inflow Model

The dynamic inflow model implemented in MBDyn has been developed by Pitt and Peters [15]. Here the model is just briefly described together with the MBDyn implementation peculiarities.

The inflow dynamics is represented by a simple first-order linear model:

$$\boldsymbol{M}\dot{\boldsymbol{\lambda}} + \boldsymbol{L}^{-1}\boldsymbol{\lambda} = \boldsymbol{c}, \tag{13.203}$$

where $\boldsymbol{\lambda}$ is a vector with 3 elements:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_0 \\ \lambda_s \\ \lambda_c \end{bmatrix}, \tag{13.204}$$

the induced velocity on the rotor disk is finally obtained as function of the azimuthal angle ψ and the non dimensional radial position $r = \frac{\sqrt{x^2+y^2}}{R}$ using the following equation:

$$u_{ind}(r, \psi) = \Omega R(\lambda_0 + \lambda_s r \sin\psi + \lambda_c r \cos\psi). \tag{13.205}$$

The right-hand term in equation 13.203 contains the thrust, roll moment and pitch moment coefficients:

$$\boldsymbol{c} = \begin{bmatrix} C_T \\ C_L \\ C_M \end{bmatrix} = \begin{bmatrix} \frac{T}{\rho A \Omega^2 R^2} \\ \frac{L}{\rho A \Omega^2 R^3} \\ \frac{M}{\rho A \Omega^2 R^3} \end{bmatrix}, \tag{13.206}$$

while $\boldsymbol{\lambda}$ is derived with respect a non-dimensional time Ωt:

$$\dot{\boldsymbol{\lambda}} = \frac{d\boldsymbol{\lambda}}{d(\Omega t)}. \tag{13.207}$$

Equation 13.203 could be rewritten as:

$$\boldsymbol{M}\dot{\boldsymbol{\lambda}} + \Omega \boldsymbol{L}^{-1}\boldsymbol{\lambda} = \Omega \boldsymbol{c}, \tag{13.208}$$

where now the dot represents the (dimensional) time derivative. The latter is the form implemented in MBDyn.

Matrix $\boldsymbol{L}$ is defined as:

$$\boldsymbol{L} = \tilde{\boldsymbol{L}} \cdot \boldsymbol{K} = \begin{bmatrix} \dfrac{1}{2} & 0 & \dfrac{15\pi}{64} \tan\dfrac{\chi}{2} \\ 0 & -\dfrac{4}{2\cos^2\dfrac{\chi}{2}} & 0 \\ \dfrac{15\pi}{64} \tan\dfrac{\chi}{2} & 0 & -\dfrac{4(2\cos^2\dfrac{\chi}{2} - 1)}{2\cos^2\dfrac{\chi}{2}} \end{bmatrix} \begin{bmatrix} \dfrac{1}{V_T} & & \\ & \dfrac{1}{V_m} & \\ & & \dfrac{1}{V_m} \end{bmatrix}, \tag{13.209}$$

where χ is wake skew angle defined as:

$$\chi = \tan^{-1}\frac{\mu}{\lambda}, \tag{13.210}$$

where

$$\mu = \frac{V_\infty \cos\alpha_d}{\Omega R}, \tag{13.211}$$

and

$$\lambda = \frac{V_\infty \sin\alpha_d}{\Omega R} + \frac{u^0_{ind}}{\Omega R}, \tag{13.212}$$

where u_{ind}^0 is the steady uniform induced velocity computed as described in the uniform inflow model section. The elements in matrix $\boldsymbol{K}$ matrix are respectively:

$$V_T = \sqrt{\lambda^2 + \mu^2}, \tag{13.213}$$

and

$$V_m = \frac{\mu^2 + \lambda \left(\lambda + \dfrac{u_{ind}^0}{\Omega R} \right)}{\sqrt{\lambda^2 + \mu^2}} \tag{13.214}$$

Note: matrix $\boldsymbol{L}$ is slightly different from matrix $\boldsymbol{L}$ in Pitt and Peters' paper [15] because here the first column elements are divided by V_T while the second and third columns elements by V_m, whereas in matrix $\boldsymbol{L}$ an unique value v is used for all the elements in the matrix. The V_m term corresponds to the *steady lift mass-flow parameter* defined in the Pitt and Peters paper, while the V_T corresponds to the *no lift mass-flow parameter*, because $\overline{\lambda} + \overline{\nu} = \lambda$ and $\overline{\nu} = u_{ind}^0/(\Omega R)$. Moreover in the Pitt and Peters work the elements are function of $\alpha = \tan^{-1} \frac{\lambda}{\mu}$, the relation between α and χ is:

$$\alpha = \frac{\pi}{2} - \chi. \tag{13.215}$$

So,

$$\sin\alpha = \cos\chi = \cos\left(\frac{\chi}{2} + \frac{\chi}{2}\right) = \cos^2\frac{\chi}{2} - \sin^2\frac{\chi}{2} = 2\cos^2\frac{\chi}{2} - 1, \tag{13.216}$$

$$\sqrt{\frac{1 - \sin\alpha}{1 + \sin\alpha}} = \sqrt{\frac{2 - 2\cos^2\frac{\chi}{2}}{2\cos^2\frac{\chi}{2}}} = \sqrt{\frac{2\sin^2\frac{\chi}{2}}{2\cos^2\frac{\chi}{2}}} = \tan\frac{\chi}{2}. \tag{13.217}$$

That means that the only difference in the MBDyn implementation is related to matrix $\boldsymbol{K}$.

In MBDyn the inversion of the $\boldsymbol{L}$ matrix is formulated analytically; matrix $\boldsymbol{L}^*$ is defined as:

$$\boldsymbol{L}^* = \Omega \boldsymbol{L}^{-1} = \Omega \begin{bmatrix} \dfrac{l_{33}}{l_{11}l_{33} - l_{13}l_{31}} & 0 & -\dfrac{l_{13}}{l_{11}l_{33} - l_{13}l_{31}} \\ 0 & \dfrac{1}{l_{22}} & 0 \\ -\dfrac{l_{31}}{l_{11}l_{33} - l_{13}l_{31}} & 0 & \dfrac{l_{11}}{l_{11}l_{33} - l_{13}l_{31}} \end{bmatrix}. \tag{13.218}$$

Finally, matrix $\boldsymbol{M}$ is defined as:

$$\boldsymbol{M} = \begin{bmatrix} \dfrac{128}{75\pi} & 0 & 0 \\ 0 & -\dfrac{16}{45\pi} & 0 \\ 0 & 0 & -\dfrac{16}{45\pi} \end{bmatrix}. \tag{13.219}$$

Note: this choice of matrix M corresponds to the mixed *uncorrected-corrected* (following the authors' nomenclature) M-matrix proposed by Pitt and Peters in their paper.

13.9 Aero Modal

13.9.1 Clamped

TODO

The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, of a state space model according to the representation

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{q} \\
\boldsymbol{f} &= q\left(\boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_0\boldsymbol{q} + \frac{c}{2V_\infty}\boldsymbol{D}_1\dot{\boldsymbol{q}} + \left(\frac{c}{2V_\infty}\right)^2 \boldsymbol{D}_2\ddot{\boldsymbol{q}}\right)
\end{aligned}$$

where $\boldsymbol{x}$ are the aerodynamic state variables, $\boldsymbol{q}$ are the modal variables that describe the structural motion as defined in the related modal joint, c is the reference length, V_∞ is the free airstream velocity, $q = \rho V_\infty^2/2$ is the dynamic pressure, and $\boldsymbol{f}$ are the unsteady aerodynamic forces applied to the structural dynamics equations.

13.9.2 Free

TODO

The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, of a state space model according to the representation

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{q} + \boldsymbol{B}_{q_x}\boldsymbol{q}_x + \boldsymbol{B}_{q_\theta}\boldsymbol{q}_\theta \\
\boldsymbol{f} &= q\left(\boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_0\boldsymbol{q} + \boldsymbol{D}_{0q_x}\boldsymbol{q}_x + \boldsymbol{D}_{0q_\theta}\boldsymbol{q}_\theta + \frac{c}{2V_\infty}\left(\boldsymbol{D}_1\dot{\boldsymbol{q}} + \boldsymbol{D}_{1q_x}\dot{\boldsymbol{q}}_x + \boldsymbol{D}_{1q_\theta}\dot{\boldsymbol{q}}_\theta\right) + \left(\frac{c}{2V_\infty}\right)^2 \left(\boldsymbol{D}_2\ddot{\boldsymbol{q}} + \boldsymbol{D}_{2q_x}\ddot{\boldsymbol{q}}_x + \boldsymbol{D}_{2q_\theta}\ddot{\boldsymbol{q}}_\theta\right)\right) \\
\boldsymbol{F} &= q\boldsymbol{R}\left(\boldsymbol{C}_F\boldsymbol{x} + \boldsymbol{D}_{0F}\boldsymbol{q} + \boldsymbol{D}_{0Fq_x}\boldsymbol{q}_x + \boldsymbol{D}_{0Fq_\theta}\boldsymbol{q}_\theta + \frac{c}{2V_\infty}\left(\boldsymbol{D}_{1F}\dot{\boldsymbol{q}} + \boldsymbol{D}_{1Fq_x}\dot{\boldsymbol{q}}_x + \boldsymbol{D}_{1Fq_\theta}\dot{\boldsymbol{q}}_\theta\right) + \left(\frac{c}{2V_\infty}\right)^2 (\boldsymbol{D}_{2F}\ddot{\boldsymbol{q}} + \boldsymbol{D}_{2Fq_x}\ddot{\boldsymbol{q}}_x \right. \\
\boldsymbol{M} &=
\end{aligned}$$

where $\boldsymbol{q}_x = \boldsymbol{R}^T(\boldsymbol{x} - \boldsymbol{x}_0)$, $\boldsymbol{q}_\theta = \boldsymbol{R}^T(\boldsymbol{\theta}_\Delta - \boldsymbol{\theta}_{\Delta 0})$, $\dot{\boldsymbol{q}}_x = \boldsymbol{R}^T\dot{\boldsymbol{x}}$, $\dot{\boldsymbol{q}}_\theta = \boldsymbol{R}^T\boldsymbol{\omega}$, ...

Chapter 14

Forces

14.1 Abstract Force

14.1.1 Abstract

14.1.2 Abstract Internal

14.2 Structural Forces

14.2.1 Force

Absolute

Follower

Absolute Internal

A structural internal force is a pair of internal forces, of equal magnitude and opposite direction, applied to two nodes. Consider two nodes, 1 and 2, and an internal force $\boldsymbol{f}$. The force is applied to node 1 with an offset, $\tilde{\boldsymbol{o}}_1$, in the node's reference frame.

The force and moment acting on the first node are

$$\boldsymbol{f}_1 = \boldsymbol{f} \tag{14.1a}$$

$$\boldsymbol{m}_1 = \boldsymbol{o}_1 \times \boldsymbol{f} \tag{14.1b}$$

with $\boldsymbol{o}_1 = \boldsymbol{R}_1\tilde{\boldsymbol{o}}_1$, where $\boldsymbol{R}_1$ is the orientation matrix of the first node.

The force acting on the second node is

$$\boldsymbol{f}_2 = -\boldsymbol{f} \tag{14.2}$$

Two options are available for the moment applied to the second node:

- Case 1: the opposite force is applied to node 2 with another offset, $\tilde{\boldsymbol{o}}_2$, in the node's reference frame;
- Case 2: the opposite force is applied to node 2 in the same point where it is applied to node 1. This case is not implemented yet.

In the first case, the moment acting on the second node is

$$\boldsymbol{m}_2 = -\boldsymbol{o}_2 \times \boldsymbol{f} \tag{14.3}$$

with $\boldsymbol{o}_2 = \boldsymbol{R}_2\tilde{\boldsymbol{o}}_2$, where $\boldsymbol{R}_2$ is the orientation matrix of the second node.

In the second case, the moment acting on the second node is

$$\boldsymbol{m}_2 = -\left(\boldsymbol{p}_1 + \boldsymbol{o}_1 - \boldsymbol{p}_2\right) \times \boldsymbol{f} \tag{14.4}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the positions of the two nodes.

When the force is *absolute*, its direction is fixed in the global reference frame. As such, $\boldsymbol{f}$ does not depend on the orientation of any node.

The linearization of the forces and moments is thus:

$$\Delta\boldsymbol{f}_1 = \boldsymbol{0} \tag{14.5a}$$
$$\Delta\boldsymbol{m}_1 = \left(\boldsymbol{\theta}_{1\Delta} \times \boldsymbol{o}_1\right) \times \boldsymbol{f} = \boldsymbol{f} \times \boldsymbol{o}_1 \times \boldsymbol{\theta}_{1\Delta} \tag{14.5b}$$
$$\Delta\boldsymbol{f}_2 = \boldsymbol{0} \tag{14.5c}$$

Case 1:

$$\Delta\boldsymbol{m}_2 = -\left(\boldsymbol{\theta}_{2\Delta} \times \boldsymbol{o}_2\right) \times \boldsymbol{f} = -\boldsymbol{f} \times \boldsymbol{o}_2 \times \boldsymbol{\theta}_{2\Delta} \tag{14.5d}$$

Case 2:

$$\Delta\boldsymbol{m}_2 = -\left(\Delta\boldsymbol{p}_1 + \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{o}_1 - \Delta\boldsymbol{p}_2\right) \times \boldsymbol{f} = \boldsymbol{f} \times \Delta\boldsymbol{p}_1 - \boldsymbol{f} \times \boldsymbol{o}_1 \times \boldsymbol{\theta}_{1\Delta} - \boldsymbol{f} \times \Delta\boldsymbol{p}_2 \tag{14.5e}$$

Follower Internal

When the force is *follower*, it can be expressed as

$$\boldsymbol{f} = \boldsymbol{R}_1\tilde{\boldsymbol{f}} \tag{14.6}$$

with $\tilde{\boldsymbol{f}}$ expressed in the frame of the first node, i.e., it is assumed to follow the orientation of the first node.

The linearization of the forces and moments is thus:

$$\Delta\boldsymbol{f}_1 = \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{f} = -\boldsymbol{f} \times \boldsymbol{\theta}_{1\Delta} \tag{14.7a}$$
$$\Delta\boldsymbol{m}_1 = \left(\boldsymbol{\theta}_{1\Delta} \times \boldsymbol{o}_1\right) \times \boldsymbol{f} + \boldsymbol{o}_1 \times \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{f} = \left(\boldsymbol{f} \times \boldsymbol{o}_1\right) \times \boldsymbol{\theta}_{1\Delta} \tag{14.7b}$$
$$\Delta\boldsymbol{f}_2 = -\boldsymbol{\theta}_{1\Delta} \times \boldsymbol{f} = \boldsymbol{f} \times \boldsymbol{\theta}_{1\Delta} \tag{14.7c}$$

Case 1:

$$\Delta\boldsymbol{m}_2 = -\left(\boldsymbol{\theta}_{2\Delta} \times \boldsymbol{o}_2\right) \times \boldsymbol{f} - \boldsymbol{o}_2 \times \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{f} = \boldsymbol{o}_2 \times \boldsymbol{f} \times \boldsymbol{\theta}_{1\Delta} - \boldsymbol{f} \times \boldsymbol{o}_2 \times \boldsymbol{\theta}_{2\Delta} \tag{14.7d}$$

Case 2:

$$\begin{aligned}\Delta\boldsymbol{m}_2 &= -\left(\Delta\boldsymbol{p}_1 + \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{o}_1 - \Delta\boldsymbol{p}_2\right) \times \boldsymbol{f} - \boldsymbol{o}_1 \times \boldsymbol{\theta}_{1\Delta} \times \boldsymbol{f} \\ &= \boldsymbol{f} \times \Delta\boldsymbol{p}_1 - \left(\boldsymbol{f} \times \boldsymbol{o}_1\right) \times \boldsymbol{\theta}_{1\Delta} - \boldsymbol{f} \times \Delta\boldsymbol{p}_2\end{aligned} \tag{14.7e}$$

14.2.2 Couple

Absolute

Follower

Absolute Internal

Follower Internal

14.3 Modal

14.4 External

The external structural force element allows MBDyn to cooperate with external solvers by defining a meta-element that applies forces and moments to a set of structural nodes. The forces and moments are provided by an external solver, called the *peer*. MBDyn provides the peer information about the motion of the nodes participating in the set.

Optionally, an additional layer of field mapping can be interposed. In this case, the motion of the nodes is transformed in the motion of a set of intermediate points, which is further mapped into the motion of the points known by the peer by means of a linear mapping.

The forces at the mapped points returned by the peer are mapped back into forces and moments for the structural nodes participating in the set of the element.

14.4.1 External Structural

The external structural and external structural mapping forces can be formulated directly in the absolute frame, or referred to a reference node. In the former case, operations are straightforward; in the latter one, the kinematics are first expressed in the reference frame of the reference node and then sent to the peer along with the motion of the reference node. The latter returns nodal forces and moments oriented according to the reference frame of the reference node. The additional operations performed by the mapping variant are discussed separately in a subsequent section. The orientation of the reference node is $\boldsymbol{R}_r$; the position is $\boldsymbol{x}_r$.

The orientation of the generic node i is

$$\boldsymbol{R}_i = \boldsymbol{R}_r \overline{\boldsymbol{R}}_i. \tag{14.8}$$

The relative orientation passed to the peer solver is

$$\overline{\boldsymbol{R}}_i = \boldsymbol{R}_r^T \boldsymbol{R}_i. \tag{14.9}$$

The position of the generic node i is

$$\boldsymbol{x}_i = \boldsymbol{x}_r + \boldsymbol{R}_r \overline{\boldsymbol{x}}_i. \tag{14.10}$$

The relative position passed to the peer solver is

$$\overline{\boldsymbol{x}}_i = \boldsymbol{R}_r^T \left(\boldsymbol{x}_i - \boldsymbol{x}_r \right). \tag{14.11}$$

The angular velocity of the generic node i is

$$\boldsymbol{\omega}_i \times = \dot{\boldsymbol{R}}_i \boldsymbol{R}_i^T = \dot{\boldsymbol{R}}_r \boldsymbol{R}_r^T + \boldsymbol{R}_r \dot{\overline{\boldsymbol{R}}}_i \overline{\boldsymbol{R}}_i^T \boldsymbol{R}_r^T = \boldsymbol{\omega}_r \times + \boldsymbol{R}_r \overline{\boldsymbol{\omega}}_i \times \boldsymbol{R}_r^T. \tag{14.12}$$

The relative angular velocity passed to the peer solver is

$$\overline{\boldsymbol{\omega}}_i = \boldsymbol{R}_r^T \left(\boldsymbol{\omega}_i - \boldsymbol{\omega}_r\right). \tag{14.13}$$

The velocity of the generic node i is

$$\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{x}}_r + \boldsymbol{\omega}_r \times \boldsymbol{R}_r \overline{\boldsymbol{x}}_i + \boldsymbol{R}_r \dot{\overline{\boldsymbol{x}}}_i. \tag{14.14}$$

The relative velocity passed to the peer solver is

$$\dot{\overline{\boldsymbol{x}}}_i = \boldsymbol{R}_r^T \left(\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_r\right) - \left(\boldsymbol{R}_r^T \boldsymbol{\omega}_r\right) \times \overline{\boldsymbol{x}}_i \tag{14.15}$$

TODO: accelerations

The virtual work done by the forces applied to the nodes is

$$\delta\mathcal{L} = \delta\boldsymbol{x}_r \cdot \boldsymbol{f}_r + \boldsymbol{\theta}_{r\delta} \cdot \boldsymbol{m}_r + \sum_i \left(\delta\overline{\boldsymbol{x}}_i \cdot \overline{\boldsymbol{f}}_i + \overline{\boldsymbol{\theta}}_{i\delta} \cdot \overline{\boldsymbol{m}}_i\right). \tag{14.16}$$

The virtual rotation of the generic node i is

$$\delta\boldsymbol{R}_i \boldsymbol{R}_i^T = \boldsymbol{\theta}_{i\delta} \times\ = \delta\boldsymbol{R}_r \boldsymbol{R}_r^T + \boldsymbol{R}_r \delta\overline{\boldsymbol{R}}_i \overline{\boldsymbol{R}}_i^T \boldsymbol{R}_r^T = \boldsymbol{\theta}_{r\delta} \times\ + \boldsymbol{R}_r \overline{\boldsymbol{\theta}}_{i\delta} \times \boldsymbol{R}_r^T. \tag{14.17}$$

The relative virtual rotation is thus

$$\overline{\boldsymbol{\theta}}_{i\delta} = \boldsymbol{R}_r^T \left(\boldsymbol{\theta}_{i\delta} - \boldsymbol{\theta}_{r\delta}\right). \tag{14.18}$$

The virtual displacement of the generic node i is

$$\delta\boldsymbol{x}_i = \delta\boldsymbol{x}_r + \boldsymbol{\theta}_{r\delta} \times \boldsymbol{R}_r \overline{\boldsymbol{x}}_i + \boldsymbol{R}_r \delta\overline{\boldsymbol{x}}_i. \tag{14.19}$$

The relative virtual displacement is thus

$$\delta\overline{\boldsymbol{x}}_i = \boldsymbol{R}_r^T \left(\delta\boldsymbol{x}_i - \delta\boldsymbol{x}_r\right) + \overline{\boldsymbol{x}}_i \times \left(\boldsymbol{R}_r^T \boldsymbol{\theta}_{r\delta}\right). \tag{14.20}$$

The virtual work becomes

$$\delta\mathcal{L} = \delta\boldsymbol{x}_r \cdot \boldsymbol{f}_r + \boldsymbol{\theta}_{r\delta} \cdot \boldsymbol{m}_r + \sum_i \left(\delta\boldsymbol{x}_i \cdot \boldsymbol{R}_r \overline{\boldsymbol{f}}_i + \boldsymbol{\theta}_{i\delta} \cdot \boldsymbol{R}_r \overline{\boldsymbol{m}}_i - \delta\boldsymbol{x}_r \cdot \boldsymbol{R}_r \overline{\boldsymbol{f}}_i - \boldsymbol{\theta}_{r\delta} \cdot \boldsymbol{R}_r \left(\overline{\boldsymbol{m}}_i + \overline{\boldsymbol{x}}_i \times \overline{\boldsymbol{f}}_i\right)\right). \tag{14.21}$$

The force and moment acting on the generic node i are

$$\boldsymbol{f}_i = \boldsymbol{R}_r \overline{\boldsymbol{f}}_i \tag{14.22}$$

$$\boldsymbol{m}_i = \boldsymbol{R}_r \overline{\boldsymbol{m}}_i. \tag{14.23}$$

The moment is always intrinsically referred to the current position of the node. The force and moment acting on the reference node are

$$\boldsymbol{f} = \boldsymbol{f}_r - \sum_i \boldsymbol{R}_r \overline{\boldsymbol{f}}_i \tag{14.24}$$

$$\boldsymbol{m} = \boldsymbol{m}_r - \sum_i \boldsymbol{R}_r \left(\overline{\boldsymbol{m}}_i + \overline{\boldsymbol{x}}_i \times \overline{\boldsymbol{f}}_i\right). \tag{14.25}$$

The moment is always intrinsically referred to the current position of the node.

In principle, $\boldsymbol{f}$ and $\boldsymbol{m}$ should be identically zero, unless the reference node is specifically loaded. This fact can be exploited by setting `use reference node forces` to `no`, which results in ignoring $\boldsymbol{f}$ and $\boldsymbol{m}$.

14.4.2 External Structural Mapping

The external structural mapping case differs from the previous one in the fact that two additional intermediate layers are added. The first layer computes the motion of a set of points rigidly connected to the nodes that participate in the set. Each node n can have an arbitrary number of associated points p, whose position in the reference frame of the node is $\tilde{\boldsymbol{o}}_{np}$. The position of point p associated with node n is

$$\boldsymbol{x}_{np} = \boldsymbol{x}_n + \boldsymbol{R}_n \tilde{\boldsymbol{o}}_{np}. \tag{14.26}$$

The position $\boldsymbol{x}_{np}$, instead of $\boldsymbol{x}_i$, must be used to compute the relative position. Orientations are not mapped.

As soon as the positions, velocities (and accelerations, if needed) of the points are computed and collected in vectors denoted by the subscript $(\cdot)_{\text{mbdyn}}$, they are mapped into the corresponding quantities of the peer, denoted by the subscript $(\cdot)_{\text{peer}}$, using a linear transformation $\boldsymbol{H}$, namely

$$\boldsymbol{x}_{\text{peer}} = \boldsymbol{H} \boldsymbol{x}_{\text{mbdyn}} \tag{14.27}$$

$$\dot{\boldsymbol{x}}_{\text{peer}} = \boldsymbol{H} \dot{\boldsymbol{x}}_{\text{mbdyn}} \tag{14.28}$$

$$\delta \boldsymbol{x}_{\text{peer}} = \boldsymbol{H} \delta \boldsymbol{x}_{\text{mbdyn}}. \tag{14.29}$$

The work done in the peer domain must be equal by the work done in MBDyn's; this implies

$$\delta \mathcal{L}_{\text{mbdyn}} = \delta \boldsymbol{x}_{\text{mbdyn}} \cdot \boldsymbol{f}_{\text{mbdyn}} = \delta \boldsymbol{x}_{\text{peer}} \cdot \boldsymbol{f}_{\text{peer}} = \delta \mathcal{L}_{\text{peer}}, \tag{14.30}$$

which yields

$$\delta \boldsymbol{x}_{\text{mbdyn}} \cdot \boldsymbol{f}_{\text{mbdyn}} = \delta \boldsymbol{x}_{\text{mbdyn}} \cdot \boldsymbol{H}^T \boldsymbol{f}_{\text{peer}}, \tag{14.31}$$

and thus, thanks to the arbitrariness of virtual displacements,

$$\boldsymbol{f}_{\text{mbdyn}} = \boldsymbol{H}^T \boldsymbol{f}_{\text{peer}}. \tag{14.32}$$

The forces $\boldsymbol{f}_{\text{mbdyn}}$ corrrespond to the points of the intermediate mapping layer. They are transformed into the corresponding nodal forces and moments considering the work done by the virtual perturbation of Eq. (14.26),

$$\delta \boldsymbol{x}_{np} = \delta \boldsymbol{x}_n - (\boldsymbol{R}_n \tilde{\boldsymbol{o}}_{np}) \times \boldsymbol{\theta}_{n\delta}, \tag{14.33}$$

namely

$$\delta \mathcal{L}_n = \sum_p \delta \boldsymbol{x}_{np} \cdot \boldsymbol{f}_{np} = \sum_p \left(\delta \boldsymbol{x}_n \cdot \boldsymbol{f}_{np} + \boldsymbol{\theta}_{n\delta} \cdot (\boldsymbol{R}_n \tilde{\boldsymbol{o}}_{np}) \times \boldsymbol{f}_{np} \right). \tag{14.34}$$

The corresponding nodal force and moment are

$$\boldsymbol{f}_n = \sum_p \boldsymbol{f}_{np} \tag{14.35}$$

$$\boldsymbol{m}_n = \sum_p (\boldsymbol{R}_n \tilde{\boldsymbol{o}}_{np}) \times \boldsymbol{f}_{np}. \tag{14.36}$$

The moment is always intrinsically referred to the current position of the node.

When a reference node is defined, all symbols in the expressions of the nodal force and moment must bear an overbar $\overline{(\cdot)}$, to indicate that they are relative and thus need to be further transformed as previously shown for the external structural force element.

14.4.3 External Modal

14.4.4 External Modal Mapping

14.4.5 Client Library

The peer side of the communication protocol has been implemented in a set of client libraries. At the core there is `libmbc`, a library written in C that performs the core operations. Declarations are provided in the header file `mbc.h`.

A high-level interface is provided in C++ in `libmbcxx`. Declarations are provided in the header file `mbcxx.h`.

Another interface is provided in Python, in module `_mbc_py.so`. The corresponding Python code is defined in `mbc_py_interface.py`.

Socket-Based Protocol

Each communication is prefixed by a `uint8_t` value that indicates the type of operation being performed. It is optionally followed by a message. Legal values are

- `ES_REGULAR_DATA`
- `ES_GOTO_NEXT_STEP`
- `ES_ABORT`
- `ES_REGULAR_DATA_AND_GOTO_NEXT_STEP`
- `ES_NEGOTIATION`
- `ES_OK`

The corresponding numerical value is defined in `mbc.h`.

Operations are:

- negotiation: the client tells the server what
- kinematics exchange: the master sends the motion
- forces exchange: the peer receives the loads

Chapter 15

Hydraulic Library

15.1 Hydraulic Fluids

15.2 Hydraulic Nodes

15.3 Hydraulic Elements

15.3.1 Accumulator

The accumulator defines two internal states x and v that represent the position and the velocity of the cap that, in a conventional gas device, separates the fluid and the gas. However, both the gravity effect and a linear spring effect can be considered as well, and any combination of reaction forces can be modeled by setting the appropriate parameters: g for a gravity device, p_{g0} for a gas device, and k for a linear spring device.

$$
\begin{aligned}
0 &= q \\
m\dot{v} + kx &= -mg + Ap\left(p - p_g\right) - f_0 - \frac{1}{2}\rho A c_e \left(\frac{A}{A_p}\right)^2 |v|\, v \\
&\quad - \operatorname{step}\left(x_{\min} - x\right)\left(c_1\left(x - x_{\min}\right) + c_2 v + c_3 \dot{v}\right) \\
&\quad - \operatorname{step}\left(x - x_{\max}\right)\left(c_1\left(x - x_{\max}\right) + c_2 v + c_3 \dot{v}\right) \\
\dot{x} &= v
\end{aligned}
$$

where $p_g = p_{g0}\left(\frac{l}{l-x}\right)^\gamma$ and $q = \rho A v$.

15.3.2 Actuator

The hydraulic actuator element couples the hydraulic library with the structural library. It connects the displacement of two structural nodes to the flow through two hydraulic nodes, and the pressure at two hydraulic nodes to the forces applied at two structural nodes. In the spirit of the multibody analysis philosophy, this element provides the essential connection between structural and hydraulic nodes; the constraints between the structural nodes, and other flow elements, e.g. leakages between the chambers, must be added by the user.

Definitions

In the following, $(\cdot)_{s1}$ and $(\cdot)_{s2}$ refer to structural nodes 1 and 2, and $(\cdot)_{h1}$ and $(\cdot)_{h2}$ refer to hydraulic nodes 1 and 2. The structural node labeled as 1 is assumed as the cylinder, and its orientation determines the axis of the actuator. The relative orientation of the actuator is defined by the unit vector $\tilde{\boldsymbol{v}}$, and the absolute orientation is $\boldsymbol{R}_{s1}\tilde{\boldsymbol{v}}$. It is assumed that appropriate kinematic constraints allow only a relative displacement of the structural nodes along the axis $\tilde{\boldsymbol{v}}$, and the only relative rotation, if any, is about the axis itself. This can be obtained by combining an inline joint with a revolute rotation or a prismatic joint.

Equations

$$
\begin{aligned}
0 &= -\boldsymbol{F} \\
0 &= -\boldsymbol{f}_{s1} \times \boldsymbol{F} \\
0 &= \boldsymbol{F} \\
0 &= \boldsymbol{f}_{s2} \times \boldsymbol{F} \\
0 &= q_{h1} \\
0 &= q_{h2} \\
p_{h1} &= P_{h1} \\
p_{h2} &= P_{h2}
\end{aligned}
$$

The first four equations apply the force resulting from the hydraulic pressure to the structural nodes; the fifth and the sixth apply the flow resulting from the actuator kinematics to the flow balance equations of the hydraulic nodes. The last two equations are required to associate two scalar differential unknowns to the hydraulic node pressures, because the flow definitions require the derivative of the pressure, while the hydraulic nodes are defined as scalar algebraic.

The force is defined as

$$\boldsymbol{F} = \left(A_{h1}p_{h1} - A_{h2}p_{h2}\right)\left(\boldsymbol{R}_{s1}\tilde{\boldsymbol{v}}\right) \tag{15.1}$$

The distance between the structural nodes, along the actuator axis, is

$$l = \left(\boldsymbol{R}_{s1}\tilde{\boldsymbol{v}}\right)^T\left(\boldsymbol{x}_{s2} + \boldsymbol{f}_{s2} - \boldsymbol{x}_{s1} - \boldsymbol{f}_{s1}\right) \tag{15.2}$$

The relative velocity of the structural nodes, along the actuator axis, is

$$\dot{l} = \left(\boldsymbol{R}_{s1}\tilde{\boldsymbol{v}}\right)^T\left(\dot{\boldsymbol{x}}_{s2} - \dot{\boldsymbol{x}}_{s1} + \boldsymbol{\omega}_{s1} \times \left(\boldsymbol{x}_{s2} - \boldsymbol{x}_{s1}\right) + \left(\boldsymbol{\omega}_{s2} - \boldsymbol{\omega}_{s1}\right) \times \boldsymbol{f}_{s2}\right) \tag{15.3}$$

The flow at the two hydraulic nodes is

$$
\begin{aligned}
q_{h1} &= A_{h1}\left(l\frac{\partial \rho_{h1}}{\partial p}\dot{P}_{h1} + \rho_{h1}\dot{d}\right) \\
q_{h2} &= A_{h2}\left(\left(L - l\right)\frac{\partial \rho_{h2}}{\partial p}\dot{P}_{h2} - \rho_{h2}\dot{d}\right)
\end{aligned}
$$

where ρ is the fluid density (different fluids in the chambers are allowed), and L is the total length of the actuator. In case stroke limitations must be enforced, the kinematic constraints must account for them.

15.3.3 Dynamic Pipe

Finite Volume dynamic pipe.

Definitions

The dynamic pipe is formulated according to the finite volume approach. The pipe is discretized by means of the pressures and the flow at the two ends, which are interpolated linearly. The mass and the momentum balance equations are written by cutting the pipe in two halves and adding the contribution of each resulting subvolume to the respective nodal equations.

Consider the mass conservation and the momentum balance equations for a one-dimensional flow:

$$\frac{D}{Dt}(dm) = 0, \tag{15.4}$$

$$\frac{D}{Dt}(dQ) = df. \tag{15.5}$$

When a rigid pipe is considered, the total derivative D/Dt of the test mass $dm = \rho A dx$ of Equation (15.4) yields

$$\frac{D}{Dt}(dm) = \frac{\partial}{\partial t}(dm) + v\frac{\partial}{\partial x}(dm), \tag{15.6}$$

which results in

$$q_{/x} + A\rho_{/t} = 0, \tag{15.7}$$

where $q = \rho A v$ is the mass flow. Consider now the momentum equation (15.5); the total derivative of the momentum $dQ = vdm$ yields

$$\frac{D}{Dt}(dQ) = \left(q_{/t} + (qv)_{/x}\right) dx, \tag{15.8}$$

while the pressure gradient and the viscous contributions can be isolated from the force per unit length on the right hand side:

$$df = -Adp + f_v dx + df^*, \tag{15.9}$$

so, by neglecting the deformability of the pipe and the extra forces df^* acting on the fluid, the momentum balance equation yields

$$q_{/t} + (qv + Ap)_{/x} = f_v, \tag{15.10}$$

which can be reduced to the pressure and flow unknowns simply by recalling the definition of the flow:

$$q_{/t} + \left(\frac{q^2}{\rho A} + Ap\right)_{/x} = f_v. \tag{15.11}$$

A flexible pipe has been considered as well; the formulation is not reported for simplicity, because such a level of detail is required only for very specialized problems, and a first approximation can be obtained by altering the bulk modulus of the fluid.

The pipe is discretized by considering a finite volume approach, based on the use of constant stepwise (*Heavyside*) test functions with arbitrary trial functions. In the present case, linear trial functions have been considered both for the flow and for the pressure:

$$q(x) = \left[\begin{array}{cc} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{array}\right] \left\{\begin{array}{c} q_1 \\ q_2 \end{array}\right\},$$

$$p(x) = \left[\begin{array}{cc} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{array}\right] \left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\},$$

with $\xi = \xi(x) \in [-1, 1]$ and $d\xi/dx = 2/(b-a)$. The discrete form of the pipe equations results in

$$
\begin{aligned}
q(b) - q(a) &= -\int_a^b \frac{\partial \rho}{\partial p} p_{/t}\, dx, \\
\frac{b-a}{2}\left(q(b)_{/t} + q(a)_{/t}\right) + \left(\frac{q(b)^2}{\rho(b) A} + Ap(b)\right) & \\
-\left(\frac{q(a)^2}{\rho(a) A} + Ap(a)\right) &= \int_a^b f_v\, dx;
\end{aligned}
$$

by dividing the pipe in two portions, and by considering the domains $[-1, 0]$ and $[0, 1]$ for ξ in each portion, the discrete equations of the finite volume pipe become

$$
\begin{aligned}
-\frac{1}{2}(q_1 + q_2) - \frac{\partial \rho(-1/2)}{\partial p}\frac{L}{8}(3\dot{p}_1 + \dot{p}_2) &= \phi_1, \\
\frac{1}{2}(q_1 + q_2) - \frac{\partial \rho(1/2)}{\partial p}\frac{L}{8}(\dot{p}_1 + 3\dot{p}_2) &= \phi_2, \\
\frac{L}{8}(3\dot{q}_1 + \dot{q}_2) + \frac{(q_1 + q_2)^2}{4\rho(0) A} - \frac{q_1^2}{\rho(-1) A} & \\
+\frac{A}{2}(p_2 - p_1) &= \frac{L}{2}\int_{-1}^{0} f_v\, d\xi, \\
\frac{L}{8}(\dot{q}_1 + 3\dot{q}_2) + \frac{q_2^2}{\rho(1) A} - \frac{(q_1 + q_2)^2}{4\rho(0) A} & \\
+\frac{A}{2}(p_2 - p_1) &= \frac{L}{2}\int_{0}^{1} f_v\, d\xi,
\end{aligned}
$$

where ϕ_1 and ϕ_2 are the contributions of the two portions of pipe to the respective nodal flow balance equations. The integral of the time derivative of the density is numerically computed. The integral of the viscous forces per unit length is numerically performed as well, accounting for the flow regime in the pipe as function of the *Reynolds* number. In fact, for the forces per unit length, the dependency on the flow is considered linear for $0 < Re < 2000$, and quadratic for $Re > 4000$, while a polynomial fitting of the transition behavior, accounting also for the rate of the *Reynolds* number, is modeled for $2000 < Re < 4000$.

Equations

$$
\begin{aligned}
0 &= \frac{1}{2}(q_1 + q_2) + AL\frac{\partial \rho}{\partial p}_1\left(\frac{3}{8}\dot{P}_1 + \frac{1}{8}\dot{P}_2\right) \\
0 &= -\frac{1}{2}(q_1 + q_2) + AL\frac{\partial \rho}{\partial p}_2\left(\frac{1}{8}\dot{P}_1 + \frac{3}{8}\dot{P}_2\right) \\
0 &= -L\left(\frac{3}{8}\dot{q}_1 + \frac{1}{8}\dot{q}_2\right) - \left(\frac{1}{2}(q_1 + q_2)\right)^2\frac{1}{\rho_m A} + q_1^2\frac{1}{\rho_1 A} - \frac{A}{2}(p_2 - p_1) - f_1 \\
0 &= -L\left(\frac{1}{8}\dot{q}_1 + \frac{3}{8}\dot{q}_2\right) - q_2^2\frac{1}{\rho_2 A} + \left(\frac{1}{2}(q_1 + q_2)\right)^2\frac{1}{\rho_m A} - \frac{A}{2}(p_2 - p_1) - f_2 \\
p_1 &= P_1 \\
p_2 &= P_2
\end{aligned}
$$

15.3.4 Pressure Flow Control Valve

$$R_1 = Q_{12} + Q_{13} \tag{15.12}$$
$$R_2 = -Q_{12} + Q_{24} \tag{15.13}$$
$$R_3 = -Q_{13} + Q_{34} \tag{15.14}$$
$$R_4 = -Q_{34} - Q_{24} \tag{15.15}$$
$$R_5 = A_v s; \tag{15.16}$$
$$R_6 = -A_v s \tag{15.17}$$
$$R_7 = -F + M\dot{v} + C\dot{s} + Ks + c_1 s + c_2\dot{s} + c_3\dot{v} + cf_1(s - s_{\max}) + cf_2\dot{s} + cf_3\dot{v} \tag{15.18}$$
$$R_8 = \dot{s} - v \tag{15.19}$$

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Appendix A

On the optimization of n-sub-step composite time integration methods

This appendix includes a verbatim copy of the paper “On the optimization of n-sub-step composite time integration methods”, written by Huimin Zhang, Runsen Zhang, Yufeng Xing and Pierangelo Masarati and published in Nonlinear Dynamics.

The paper reports some of MBDyn's integrators coefficients.

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Appendix B

Performance of implicit A-stable time integration methods for multibody system dynamics

This appendix includes a verbatim copy of the paper "Performance of implicit A-stable time integration methods for multibody system dynamics", written by Huimin Zhang, Runsen Zhang, Andrea Zanoni and Pierangelo Masarati and published in Multibody System Dynamics.

The paper gives the most recent description of MBDyn's different integrators properties.

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