Genetic paths to evolutionary rescue and the distribution of fitness effects along them

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Dependencies

Directories

```
SetDirectory[NotebookDirectory[]];
(*sets current directory to be location of this file*)
datadir = "data/";(*directory with simulation data*)
figdir = "../main/";(*directory to save figures in*)
supfigdir = "../supplementary/";(*directory to save supplementary figures in*)
```

Plot styles

Functions (derived below)

```
fs[s_, so_, \lambda_, n_] :=  \frac{2}{\lambda} PDF \Big[ Noncentral ChiSquare Distribution \Big[ n, \frac{2 \text{ so}}{\lambda} \Big], \frac{2}{\lambda} \text{ (so - s)} \Big] \text{ Heaviside The ta} [\text{so - s}] \\ fm[m_, mwt_, mmax_, \lambda_, n_] := \\ Simplify[fs[s, so, \lambda, n] /. s \to m - mwt /. so \to mmax - mwt, \{mmax > m, \lambda > 0\} \] prescue = 1 - <math>(1 - p0)^{N0};
```

```
prescueApp = 1 - e^{-N0 p0};
pest[m_] := (1 - Exp[-2 m]) HeavisideTheta[m]
prescuem[m\_, \Lambda\_] := 1 - e^{\left(1 - \sqrt{1 + \frac{2\Lambda}{Abs[m]^2}}\right)Abs[m]}
\Lambda1[m0_?NumericQ, mmax_, \lambda_, n_, U_] :=
   UNIntegrate [fm[m, m0, mmax, \lambda, n] pest[m], {m, 0, mmax}]
PR1[m0\_?NumericQ] := prescue /. p0 \rightarrow prescuem[m0, \Lambda1[m0, mmax, \lambda, n, U]]
PR1App[m0_?NumericQ] := prescueApp /. p0 \rightarrow prescuem[m0, \Lambda1[m0, mmax, \lambda, n, U]]
\mathbf{f}\psi = \frac{e^{-\frac{1}{4}\rho \max (\psi - \psi \text{wt})^2} \sqrt{\rho \text{max}} \left(\frac{2 - \psi}{2 - \psi \text{wt}}\right)^{-\frac{1}{2} + \theta}}{2\sqrt{\pi}};
AnciauxEqnA12 = U \frac{\left(1 - \frac{\psi wt}{2}\right)^{\frac{1}{2} - \theta}}{1 - \frac{\psi wt}{2}} \left(\frac{\text{Exp}[-\alpha]}{\sqrt{\pi \alpha}} - \text{Erfc}[\sqrt{\alpha}]\right);
\Lambda 2 [m0_?NumericQ, mmax_, \lambda_, n_, U_] := UNIntegrate[
         \texttt{fm}[\texttt{m}, \texttt{m0}, \texttt{mmax}, \lambda, \texttt{n}] \; (\texttt{1-pest}[\texttt{m}]) \; \texttt{prescuem}[\texttt{m}, \Lambda \texttt{1}[\texttt{m}, \texttt{mmax}, \lambda, \texttt{n}, \texttt{U}]] \; , \; \{\texttt{m}, -\infty, \, \texttt{mmax}\}]
\Lambda 0 \operatorname{approx} = \frac{2 \, \mathrm{U} \, \sqrt{\operatorname{mmax} \, \lambda}}{\sqrt{\pi}};
small \psi approx Rescue = - \left[ e^{-\frac{mmax}{\lambda} \left( 1 - \sqrt{1 - \frac{mvt}{mmax}} \right)^{2}} \left( 1 - \frac{mwt}{mmax} \right)^{\frac{1-n}{2}} U Log \left[ \frac{1 - \sqrt{1 + \frac{\sqrt{U\sqrt{mmax}\lambda}}{mmax} \pi^{1/4}}}{1 - \sqrt{1 - \frac{mwt}{mmax}}} \right] \right] 
            \left| \left| 1 + \frac{1}{2} \left| -1 + \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right| \right| \pi \right|;
verylarge \rho approx Rescue = - \left[ 2 e^{-\frac{mmax}{2\lambda} \left| \frac{1 - \sqrt{1 - \frac{mvt}{mmax}}}{2\lambda} \right|} \left( 1 - \frac{mwt}{mmax} \right)^{\frac{1-n}{2}} \sqrt{\frac{2}{\pi}} U \right] /
```

$$\left(\left(1-\sqrt{1-\frac{mwt}{mmax}}\right)^3\left(1+\frac{1}{2}\left(-1+\sqrt{1-\frac{mwt}{mmax}}\right)\right)\left(\frac{mmax}{\lambda}\right)^{3/2}\right);$$

$$small \psi approx Rescue Super = \left(e^{-\frac{mmax}{\lambda} \left(1 - \sqrt{1 - \frac{mwt}{max}}\right)^2} \left(1 - \frac{mwt}{mmax}\right)^{\frac{1-n}{2}} U\right)$$

$$\operatorname{Log}\left[\frac{1}{\sqrt{2}\sqrt{\frac{\operatorname{mmax}}{\lambda}}\left(1-\sqrt{1-\frac{\sqrt{\operatorname{u}\sqrt{\operatorname{mmax}\lambda}}}{\operatorname{mmax}\pi^{1/4}}}\right)}\right] \bigg| \left(\left(1+\frac{1}{2}\left(-1+\sqrt{1-\frac{\operatorname{mwt}}{\operatorname{mmax}}}\right)\right)\pi\right);$$

$$glm = \frac{U fm[m, mwt, mmax, \lambda, n] pest[m]}{\Lambda 1[mwt, mmax, \lambda, n, U]};$$

$$\texttt{g1mApp} = \frac{ e^{\alpha - \frac{1}{4} \rho \texttt{max} \; (\psi - \psi \texttt{wt})^{\; 2}} \; \sqrt{\alpha \; \rho \texttt{max}} \; \psi}{\sqrt{1 - \frac{\texttt{m}}{\texttt{mmax}}} \; \texttt{mmax} \; \psi \texttt{wt} \; \left(-1 + e^{\alpha} \; \sqrt{\pi} \; \sqrt{\alpha} \; \texttt{Erfc} \left[\sqrt{\alpha} \; \right] \right)} \; ;$$

g2numerator[m0_, m2_?NumericQ] := UNIntegrate[fm[m, m0, mmax, λ , n] (1 - pest[m]) prescuem[m, U fm[m2, m, mmax, λ , n] pest[m2]], {m, $-\infty$, mmax}]; g2denominator[m0_?NumericQ] := NIntegrate[Ufm[m, m0, mmax, λ , n] (1 - pest[m]) prescuem[m, U fm[m2, m, mmax, λ , n] pest[m2]], {m, $-\infty$, mmax}, {m2, 0, mmax}];

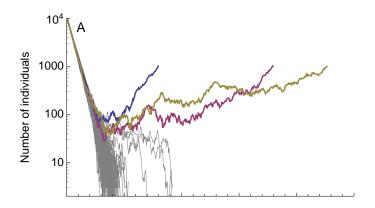
Example simulations

1-step rescue (figure 1)

Population size dynamics (panel A)

Total population size for many replicates

```
n0 = "10000";
n = "4";
U = "0.000100";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.10";
mutmax = "2";
nreps = 100;
data = Table[
    Import[datadir <> "alleles_N" <> n0 <> "_n" <>
        n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <> mmax <> "_mwt" <> mwt <> 
        "_mutmax" <> mutmax <> "_rep" <> ToString[rep] <> ".csv"][[All, 1]],
    {rep, 1, nreps}
   ];
rescued = Select[data, #[[-1]] > 1000 &];
extinct = Select[data, #[[-1]] < 1000 &];
genmax = 500;
nmax = n0;
fig1a = Show
  ListLogPlot[
    extinct,
    Joined → True,
    PlotRange \rightarrow {{0, genmax}, {1, nmax}},
    Frame → {True, True, False, False},
    FrameLabel → {"", "Number of individuals"},
    PlotStyle → Gray, LabelStyle → labelstyle,
    \texttt{FrameTicksStyle} \rightarrow \{\texttt{FontColor} \rightarrow \texttt{White}, \texttt{Automatic}, \texttt{Automatic}\},
    Epilog →
     \texttt{Text}[\texttt{Style}["A", 14, Bold, FontFamily} \rightarrow "\texttt{Helvetica"}], \texttt{Scaled@}\{0.05, 0.95\}]
  ListLogPlot[
    rescued,
    Joined → True,
    PlotStyle → Thickness[0.005]
  PlotRange \rightarrow \{\{0, \text{genmax}\}, \text{Log}@\{2, 10^4\}\}
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, nreps, genmax, nmax]
```



Mutation dynamics (panel B)

example of allele dynamics for one rescued replicate

```
n0 = "10000";
n = "4";
U = "0.000100";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.10";
mutmax = "2";
rep = "83";
genmax = 500;
nmax = 1000;
Import[datadir <> "alleles_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
alleles = Transpose[PadRight[%]];
Import[datadir <> "kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
allmuts = Transpose[PadRight[%]];
rescuemut = Ordering[alleles[[2;;]][[All, -1]], -1];
fig1b = Show
  ListLogPlot[
   alleles[[2;;]][[rescuemut]],
   Joined \rightarrow True,
   PlotRange \rightarrow \{\{0, \text{genmax}\}, \{1, 10^4\}\},
   Frame → {True, True, False, False},
   FrameLabel → {"Generation", "Number of copies of allele"},
   LabelStyle → labelstyle,
   Epilog →
    Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
   PlotStyle → Directive[Thickness[0.005], defaultcolors[[3]]]
  ],
  ListLogPlot[
   Drop[alleles[[2;;]], rescuemut],
   Joined → True,
   PlotStyle → Directive[Thickness[0.002], Black]
  ],
  ListLogPlot[
   allmuts[[1]],
   Joined \rightarrow True,
   PlotStyle → {Gray, Thickness[0.005]}
  ]
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, rep, genmax, nmax]
```

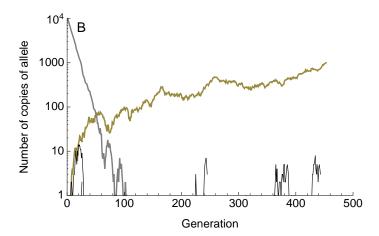
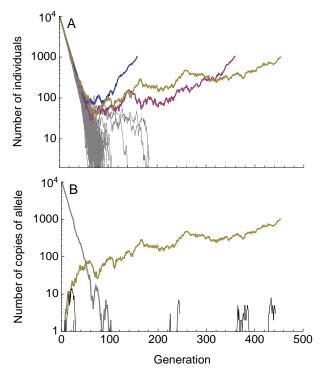


Figure 1

GraphicsColumn[$\{figla, figlb\}$, Spacings $\rightarrow -30$, ImageSize $\rightarrow 350$] Export[figdir <> "Figure1.eps", %];

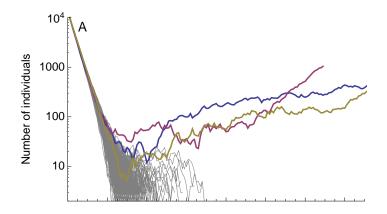


2-step rescue (figure 2)

Population size dynamics (panel A)

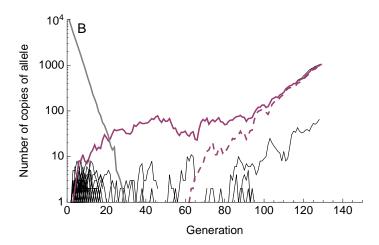
```
n0 = "10000";
n = "4";
U = "0.010000";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.30";
mutmax = "2";
intrep = 500;
endrep = 1000;
nrescued = 1000;
data = Table[
    Import[datadir <> "alleles_N" <> n0 <> "_n" <>
        n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <> mmax <> "_mwt" <> mwt <>
        "_mutmax" <> mutmax <> "_rep" <> ToString[rep] <> ".csv"][[All, 1]],
    {rep, intrep, endrep}
  ];
rescued = Select[data, #[[-1]] > 1000 &];
extinct = Select[data, #[[-1]] < 1000 &];
genmax = 150;
nmax = n0;
fig2a = Show
  ListLogPlot[
    extinct,
   Joined → True,
   PlotRange \rightarrow {{0, genmax}, {1, nmax}},
   Frame → {True, True, False, False},
   FrameLabel → {"", "Number of individuals"},
   PlotStyle → Gray,
   LabelStyle → labelstyle,
   \label{eq:fontColor} {\tt FrameTicksStyle} \rightarrow \{{\tt FontColor} \rightarrow {\tt White}, \, {\tt Automatic}, \, {\tt Automatic}, \, {\tt Automatic}\},
   Epilog →
     Text[Style["A", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]],
  ListLogPlot[rescued, Joined → True, PlotStyle → Thickness[0.005]
  PlotRange \rightarrow {{0, genmax}, Log@{2, 10<sup>4</sup>}}
```

Clear[n0, n, U, Es, mmax, mwt, r, mutmax, nreps, genmax, nmax, nrescued]



Mutation dynamics (panel B)

```
n0 = "10000";
n = "4";
U = "0.010000";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.30";
mutmax = "2";
rep = "675";
genmax = 150;
nmax = 1000;
ymax = 10^4;
Import[datadir <> "alleles_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
alleles = Transpose[PadRight[%]];
Import[datadir <> "kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
allmuts = Transpose[PadRight[%]];
rescuemuts = Ordering[alleles[[2;;]][[All, -1]], -2];
fig2b = Show[
  ListLogPlot[Drop[alleles[[2;;]], {rescuemuts[[1]]}, {rescuemuts[[2]]}],
   Joined → True, PlotStyle → Directive[Black, Thickness[0.002]],
   PlotRange \rightarrow {{0, genmax}, {1, ymax}},
   Frame → {True, True, False, False},
   FrameLabel → {"Generation", "Number of copies of allele"},
   LabelStyle → labelstyle,
   Epilog →
    Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]
  ],
  ListLogPlot[
   alleles[[2;;]][[rescuemuts]],
   Joined → True,
   PlotStyle → {Directive[Thickness[0.005], defaultcolors[[2]], Dashing[Medium]],
     Directive[Thickness[0.005], defaultcolors[[2]]]}
  ListLogPlot[allmuts[[1]], Joined → True, PlotStyle → {Gray, Thickness[0.005]}]
 ]
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, rep, genmax, nmax, ymax]
```



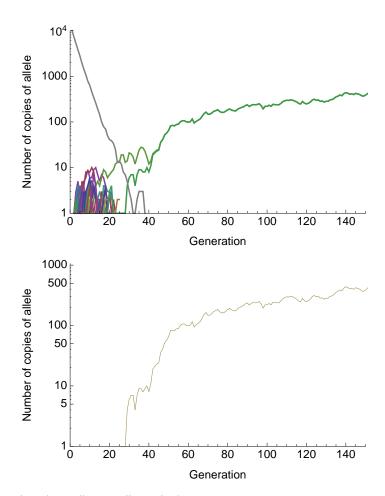
Note that the first mutation is subcritical, but the second mutation makes the double mutant supercritical (we can see this because we keep track of what is supercritical in the sims):

```
n0 = "10000";
n = "4";
U = "0.010000";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.30";
mutmax = "2";
rep = "675";
genmax = 150;
nmax = 1000;
ymax = 10^4;
Import[datadir <> "supercrit_kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <>
    "_mmax" <> mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
supermuts = Transpose[PadRight[%]];
ListLogPlot[supermuts, Joined → True,
 PlotRange \rightarrow \{\{0, genmax\}, \{0, nmax\}\}, Frame \rightarrow \{True, True, False, False\},
 FrameLabel → {"Generation", "Number of copies of allele"},
 LabelStyle → labelstyle]
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, rep, genmax, nmax, ymax]
   1000
    500
Number of copies of allele
    100
     50
     10
      5
                                80
                                      100
                                            120
                                                  140
                   40
                          60
```

The blue replicate is also 2-step rescue

Generation

```
n0 = "10000";
n = "4";
U = "0.010000";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.30";
mutmax = "2";
rep = "629";
genmax = 150;
nmax = 1000;
ymax = 10^4;
Import[datadir <> "alleles_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
alleles = Transpose[PadRight[%]];
Import[datadir <> "kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
    mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
allmuts = Transpose[PadRight[%]];
Show[
 ListLogPlot[alleles[[2;;]], Joined → True,
  \texttt{PlotRange} \rightarrow \{\{\texttt{0}, \texttt{genmax}\}, \{\texttt{1}, \texttt{ymax}\}\}, \texttt{Frame} \rightarrow \{\texttt{True}, \texttt{True}, \texttt{False}, \texttt{False}\},
  FrameLabel \rightarrow {"Generation", "Number of copies of allele"}, LabelStyle \rightarrow
   labelstyle, (*Epilog-Text[Style["B",14,Bold],Scaled@{0.05,0.95}],*)
  PlotStyle → Thickness[0.005]
 ١,
 ListLogPlot[allmuts[[1]], Joined → True, PlotStyle → {Gray, Thickness[0.005]}]
]
Import[datadir <> "supercrit_kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <>
    "_mmax" <> mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
supermuts = Transpose[PadRight[%]];
ListLogPlot[supermuts, Joined → True,
 PlotRange \rightarrow \{\{0, genmax\}, \{0, nmax\}\}, Frame \rightarrow \{True, True, False, False\},
 FrameLabel → {"Generation", "Number of copies of allele"},
 LabelStyle → labelstyle]
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, rep, genmax, nmax, ymax]
```



but the yellow replicate is 1-step

```
n0 = "10000";
n = "4";
U = "0.010000";
Es = "0.01000";
mmax = "0.50";
mwt = "-0.30";
mutmax = "2";
rep = "752";
genmax = 150;
nmax = 1000;
ymax = 10^4;
Import[datadir <> "alleles_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
   mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
alleles = Transpose[PadRight[%]];
Import[datadir <> "kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <> "_mmax" <>
    mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
allmuts = Transpose[PadRight[%]];
Show[
 ListLogPlot[alleles[[2;;]], Joined → True,
  \texttt{PlotRange} \rightarrow \{\{\texttt{0}, \texttt{genmax}\}, \{\texttt{1}, \texttt{ymax}\}\}, \texttt{Frame} \rightarrow \{\texttt{True}, \texttt{True}, \texttt{False}, \texttt{False}\},
  FrameLabel \rightarrow {"Generation", "Number of copies of allele"}, LabelStyle \rightarrow
   labelstyle, (*Epilog-Text[Style["B",14,Bold],Scaled@{0.05,0.95}],*)
  PlotStyle → Thickness[0.005]
 ١,
 ListLogPlot[allmuts[[1]], Joined → True, PlotStyle → {Gray, Thickness[0.005]}]
]
Import[datadir <> "supercrit_kmuts_N" <> n0 <> "_n" <> n <> "_U" <> U <> "_Es" <> Es <>
    "_mmax" <> mmax <> "_mwt" <> mwt <> "_mutmax" <> mutmax <> "_rep" <> rep <> ".csv"];
supermuts = Transpose[PadRight[%]];
ListLogPlot[supermuts, Joined → True,
 PlotRange \rightarrow \{\{0, genmax\}, \{0, nmax\}\}, Frame \rightarrow \{True, True, False, False\},
 FrameLabel → {"Generation", "Number of copies of allele"},
 LabelStyle → labelstyle]
Clear[n0, n, U, Es, mmax, mwt, r, mutmax, rep, genmax, nmax, ymax]
```

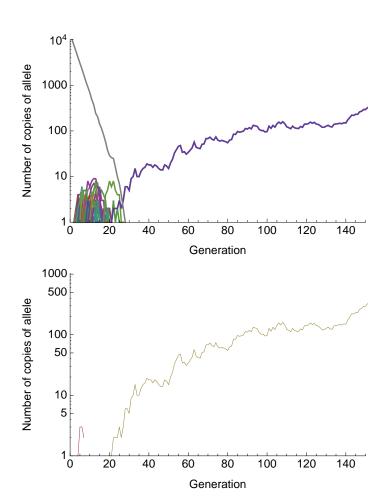
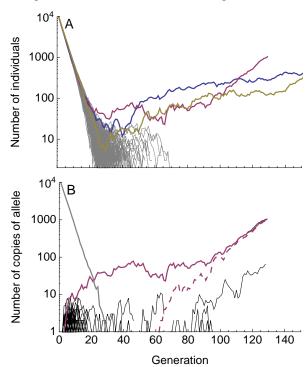


Figure 2

GraphicsColumn[{fig2a, fig2b}, Spacings → -30, ImageSize → 350] Export[figdir <> "Figure2.eps", %];



Background

Simulate some mutants

Distribution of mutant growth rates (equation 1)

In FGM, the PDF of selective effects of new mutations, $s = m - m_{wt}$, is (Martin & Lenormand 2015) Evolution)

$$\begin{split} &\text{fs[s_, so_, $\lambda_, n_] :=} \\ &\frac{2}{\lambda} \, \text{PDF} \Big[\text{NoncentralChiSquareDistribution} \Big[n, \, \frac{2 \, \text{so}}{\lambda} \Big] \,, \, \frac{2}{\lambda} \, \left(\text{so-s} \right) \Big] \, \text{HeavisideTheta[so-s];} \end{aligned}$$

where so = $m_{\text{max}} - m_{\text{wt}}$ is the selection coefficient of the optimum phenotype, λ is the mutational variance per trait, and n is the number of traits under selection.

We can translate this into the PDF of mutant growth rates (see also Anciaux et al 2018 Genetics)

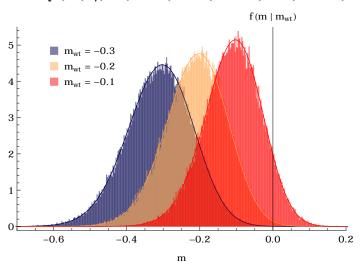
$$\begin{split} &\text{fm}\,[\text{m_, mwt_, mmax_, }\lambda_, \,\text{n_]} := \\ &\text{Simplify}[\text{fs}\,[\text{s, so, }\lambda, \,\text{n}] \,\, /. \,\, \text{s} \rightarrow \text{m-mwt} \,\, /. \,\, \text{so} \rightarrow \text{mmax-mwt, } \{\text{mmax} > \text{m, }\lambda > 0\}] \end{split}$$

```
n = 4; \eta = n / 2; (*phenotypic dimensions*)
Es = 0.01;(*mean mutational selective efffect*)
\lambda = 2 \text{ Es / n; (*mutational variance, i.e., scaled size*)}
uo = 0 UnitVector[n, 1];(*mean mutational size; set to all zeros*)
Idn = IdentityMatrix[n];(*matrix to make variance multidimensional*)
NT = 10<sup>5</sup>; (*number of mutants to create*)
{\tt dz = RandomReal[MultinormalDistribution[uo, \lambda Idn], NT];}
Clear[\lambda, n, Es, \eta, uo, Idn, NT]
Calculate selection coefficient given mutant vectors dz and vector to optimum zopt
ss[dz\_, zopt\_] := Table\Big[dz[[i]].zopt - \frac{dz[[i]].dz[[i]]}{2}, \{i, 1, Length[dz]\}\Big];
```

Check distributions with simulated mutants:

```
n = 4; \eta = n / 2; (*phenotypic dimensions*)
Es = 0.01;(*mean mutational selective efffect*)
\lambda = 2 \text{ Es / n; (*mutational variance per trait*)}
mmax = 0.5; (*max growth rate*)
mwts = \{-0.3, -0.2, -0.1\}; nmwts = Length[mwts]; (*scaled*)
zis = Table \left[\sqrt{2 \text{ (mmax - mwts[[i]])}} \text{ UnitVector[n, 1], {i, nmwts}}\right];
(*vector from wildtype from the optimum along trait 1 axis*)
color = {Darker[Blue, 0.7], Lighter[Orange], Red}; (*colors*)
styl = {FontFamily → "Times", FontSize → 10}; (*styles of axes and legend*)
legends = Table[StringForm["m_{wt} =  `", NumberForm[mwts[[i]], 3]], {i, nmwts}];
simuls = Histogram[Table[mwts[[i]] + ss[dz, zis[[i]]], {i, nmwts}], 500,
   "PDF", ChartStyle \rightarrow color, PlotRange \rightarrow {{-2, 0.5}, All}, Axes \rightarrow False,
  ChartLegends → Placed[legends, {0.2, 0.8}]]; (*simulation results*)
Table[fm[m, mwts[[i]], mmax, \lambda, n], {i, nmwts}];
theory = Plot[%, {m, -1, 0.5}, PlotRange \rightarrow {{-0.7, 0.2}, All},
   PlotStyle → color, Frame → {True, True, False, False}, FrameLabel → {"m"}];
Show[{theory, simuls, theory}, LabelStyle \rightarrow styl, AxesLabel \rightarrow {, "f(m \mid m_{wt})"}]
(*Export[imagedir<>"fmmwt.pdf",%];*)
```

Clear $[\lambda, n, \eta, Es, mmax, mwts, nmwts, zis, color, styl, legends, simuls, theory]$



Probability of rescue (equations 2 and 3)

Let p0 be the probability a wildtype individual has descendants that rescue the population. If we start with N0 wildtypes the probability of rescue is then

prescue =
$$1 - (1 - p0)^{N0}$$
;

which with small p0 and large N0 is approximately

```
prescueApp = Series \left[1 - (1 - p0)^{N0} /. p0 \rightarrow p0 \in /. N0 \rightarrow N0 / \epsilon, \{\epsilon, 0, 0\}\right] // Normal + (1 - p0)^{N0} /. p0 \rightarrow p0 \in /. N0 \rightarrow N0 / \epsilon, \{\epsilon, 0, 0\}
1 - e^{-N0 p0}
```

Let there be X[t,m] individuals in a lineage time t after it arose, given growth rate m. Then the total number of individuals in the lineage is $\int_0^{\tau} X[t, m] dt$, where τ is the time of extinction. The distribution of the random variable Y[m] = $\int_0^{\tau} X[t, m] dt$ is known (see Martin et al 2013 Phil Trans B sup mat); its MGF is

MomentGeneratingFunction[InverseGaussianDistribution[Abs[1/m],1],z]

$$\mathbb{P}\left(1-\sqrt{1-\frac{2z}{\mathrm{Abs}[\mathfrak{m}]^2}}\right)\mathrm{Abs}[\mathfrak{m}]$$

Let an individual with growth rate m produce rescue mutants at rate $\Lambda(m)$. The probability its lineage rescues the population is then $1 - \mathbb{E}_{Y} \left[e^{-Y (m) \wedge (m)} \right] = 1 - M_{Y} \left[- \wedge (m) \right]$ where M_{Y} is the MGF of Y. Thus evaluating our MGF above at $z = -\Lambda(m)$ we have the probability of rescue from a genotype with growth rate m

```
prescuem[m_, \Lambda_] :=
 1 - MomentGeneratingFunction[InverseGaussianDistribution[Abs[1/m], 1], -A]
```

Probability of establishment (equation 4)

Using the Feller diffusion approximation (see Martin et al 2013 Phil Trans B sup mat for details), the probability a mutant establishes is

```
1 - \text{Exp}[-2r/\sigma]
1 - e^{-\frac{2r}{\sigma}}
```

where r is the infinitesimal mean and σ is the infinitesimal variance in mutant growth rate.

```
In our discrete generation Poisson process, r is
Expectation[X - 1, X \( \) PoissonDistribution[Exp[m]]]
-1 + e^{m}
and \sigma is
Expectation [(X-1)^2, X \in PoissonDistribution[Exp[m]]] -
    Expectation [X - 1, X \acute{e} PoissonDistribution [Exp[m]]] ^2 // Simplify // Simplify
e^{m}
When m is small these are roughly
Series [Exp[m] - 1, \{m, 0, 1\}] // Normal
m
and
Series[Exp[m], {m, 0, 0}] // Normal
1
```

Then the probability of establishment for m>0 is roughly

1 - Exp[-2r/
$$\sigma$$
] /. r \rightarrow m /. $\sigma \rightarrow$ 1
1 - e^{-2m}

and this is zero when m<0.

```
pest[m_] := (1 - Exp[-2 m]) HeavisideTheta[m]
```

Note that this reduces to both Haldane's (1927 Mathematical Proceedings of the Cambridge Philosophical Society) constant population size result and Otto & Whitlock's (1997 Genetics) changing population size result when m is small

```
Normal[Series[1-Exp[-2m], {m, 0, 1}]] /. m \rightarrow m_{wt} + s /. m_{wt} \rightarrow 0
2 s
Normal[Series[1-Exp[-2m], {m, 0, 1}]] /. m \rightarrow m_{wt} + s
2 (s + m_{wt})
```

Mutant lineage dynamics

Probability generating function

Here we use a continuous time birth (λ) death (μ) process to approximate the dynamics of our discrete time Poisson process (nonoverlapping generations with expected offspring exp(m)). To align these two approaches we need m = λ - μ and, as discussed in Uecker & Hermisson 2016 Genetics and Uecker et al 2014 Am Nat, $\lambda + \mu = 1$. The latter requirement ensures both processes have the same amount of drift. We follow these studies in equally distributing m between λ and μ , such that $\lambda = (1 + m)/2$ and $\mu =$ (1 - m)/2. Note that this is only valid for |m|<1.

For a continuous-time birth death process starting from N0 individuals ($F[s, 0] = s^{N0}$) the PGF for the number of individuals at time t can be solved for explicitly

Probability of persistence

The probability of extinction at time t can be obtained from the PGF

pextt = F[s, t] /. fst /. s
$$\rightarrow$$
 0 // Factor // FullSimplify
$$\left(1 + \frac{-\lambda + \mu}{\lambda - e^{t} (-\lambda + \mu) \mu}\right)^{N0}$$

Evaluating at s=0 as t goes to infinity gives the probability the lineage ever goes extinct. When birth rate is greater than death rate, $\lambda > \mu$ ie m>0, we have

pextinction = Limit[F[s, t] /. fst /. s
$$\rightarrow$$
 0, t \rightarrow ∞ , Assumptions \rightarrow { $\lambda > \mu$, $\mu > 0$ }] $1 - \%$ /. $\lambda \rightarrow \frac{1+m}{2}$ /. $\mu \rightarrow \frac{1-m}{2}$ /. N0 \rightarrow 1; Series[%, {m, 0, 1}]

$$\left(\frac{\mu}{-}\right)^{N0}$$

$$2 m + O \lceil m \rceil^2$$

(giving Haldane's classic 2m approximation for the probability of establishment) and when death rate is greater than birth rate we have

$$\label{eq:limit} \begin{aligned} & \text{Limit}[\textbf{F}[\textbf{s},\,\textbf{t}] \ /. \ \textbf{fst} \ /. \ \textbf{s} \rightarrow \textbf{0} \,, \ \textbf{t} \rightarrow \infty \,, \ \text{Assumptions} \rightarrow \{\lambda < \mu \,, \ \mu > 0\}] \end{aligned}$$

The probability of persistence to time t is just one minus the probability of extinction by time t

probpersist = 1 - pextt

$$1 - \left(1 + \frac{-\lambda + \mu}{\lambda - e^{t(-\lambda + \mu)} \mu}\right)^{N0}$$

Distribution of extinction times (equation A4)

When |m| << 1/t << 1, ie |1/m| >> t >> 1 (that is, at late times but while the mutant is still effectively critical) the probability that a new mutant persists to time t is approximately

pestapp1 =

Normal[Series[probpersist/.
$$\lambda \rightarrow \frac{1+m}{2}$$
/. $\mu \rightarrow \frac{1-m}{2}$ /. $N0 \rightarrow 1$ /. $m \rightarrow m \epsilon^2$ /. $t \rightarrow t$ / ϵ , $\{\epsilon, 0, 1\}$]]/. $\epsilon \rightarrow 1$

2

And when -m t >>1, ie -m >> 1/t (that is, when the mutant is effectively subcritical) we have roughly

$$\begin{split} \text{pestapp2} &= \text{Simplify} \Big[\text{probpersist /. } \lambda \rightarrow \frac{1+m}{2} \text{ /. } \mu \rightarrow \frac{1-m}{2} \text{ /. } \text{NO} \rightarrow 1 \Big] \text{ /.} \\ &1 + e^{-m\,t} \, \left(-1 + m \right) + m \rightarrow e^{-m\,t} \, \left(-1 + m \right) \text{ /. } -1 + m \rightarrow -1 \\ &-2 \, e^{m\,t} \, m \end{split}$$

Visually check the critical (red) and subcritical (green) approximations of the true solution (blue)

Show LogPlot[probpersist /. $\lambda \rightarrow \frac{1+m}{2}$ /. $\mu \rightarrow \frac{1-m}{2}$ /. N0 \rightarrow 1 /. m \rightarrow -0.01, $\{t, 0, 1000\}, PlotRange \rightarrow \{10^{-4}, 1\}$, LogPlot[pestapp1 /. $m \rightarrow -0.01$, {t, 0, 1000}, PlotRange \rightarrow All, PlotStyle \rightarrow Red], $\texttt{LogPlot[pestapp2 /.m} \rightarrow \texttt{-0.01, \{t, 0, 1000\}, PlotRange} \rightarrow \texttt{All, PlotStyle} \rightarrow \texttt{Green]}$ 0.1 0.01 0.001 200 400 600 800 1000

Our equations for persistence times line-up with Weissman et al 2010 Genetics eq A2, but with an additional factor of 2 because we have b+d=2 while they have b+d=1. They point out that the distribution of persistence time has a long tail (like 1/t) before falling off exponentially when t > -1 / m (as can be seen in the plot above). Thus we can essentially say that a mutant lineage will not persist past t = -1 / m.

Distribution of lineage sizes

The probability that there is 1 individual at time t is

$$\begin{aligned} & \text{pn1} = \text{D[F[s,t] /. fst, \{s,n\}] / n! /. n \to 1 /. s \to 0 /. \lambda \to \frac{1+m}{2} /. \mu \to \frac{1-m}{2} /. N0 \to 1 // \\ & \text{Simplify} \\ & \frac{4 \, \text{e}^{\text{mt} \, \text{m}^2}}{\left(-1+\text{m}+\text{e}^{\text{mt} \, (1+\text{m})}\,\right)^2} \end{aligned}$$

and two individuals

$$\begin{aligned} & pn2 = D[F[s,t] \text{ /. fst, } \{s,n\}] \text{ / } n! \text{ /. } n \rightarrow 2 \text{ /. } s \rightarrow 0 \text{ /. } \lambda \rightarrow \frac{1+m}{2} \text{ /. } \mu \rightarrow \frac{1-m}{2} \text{ /. } N0 \rightarrow 1 \text{ //} \\ & \text{Simplify} \\ & \frac{4 \text{ e}^{\text{mt}} \left(-1 + \text{ e}^{\text{mt}}\right) \text{ m}^2 \left(1 + \text{m}\right)}{\left(-1 + \text{m} + \text{ e}^{\text{mt}} \left(1 + \text{m}\right)\right)^3} \end{aligned}$$

three

$$\begin{split} & \mathsf{D}[\mathsf{F}[\mathsf{s},\,\mathsf{t}] \; /. \; \mathsf{fst}, \; \{\mathsf{s},\,\mathsf{n}\}] \; / \, \mathsf{n} \; ! \; /. \; \mathsf{n} \to 3 \; /. \; \mathsf{s} \to 0 \; /. \; \lambda \to \frac{1+\mathsf{m}}{2} \; /. \; \mu \to \frac{1-\mathsf{m}}{2} \; /. \; \mathsf{N0} \to 1 \; // \; \mathsf{Simplify} \\ & \frac{4 \; e^{\mathsf{m} \, \mathsf{t}} \; \left(-1 + e^{\mathsf{m} \, \mathsf{t}} \right)^2 \, m^2 \; \left(1 + \mathsf{m}\right)^2}{\left(-1 + \mathsf{m} + e^{\mathsf{m} \, \mathsf{t}} \; \left(1 + \mathsf{m}\right)\right)^4} \end{split}$$

From this series we can see that we can write the probability of n individuals at time t like

$$\frac{\text{pnt = pn1 (pn2 / pn1)}^{n-1}}{4 e^{m t} m^{2} \left(\frac{\left(-1 + e^{m t}\right) (1 + m)}{-1 + m + e^{m t} (1 + m)}\right)^{-1 + n}}{\left(-1 + m + e^{m t} (1 + m)\right)^{2}}$$

Check for n=4:

pnt / (D[F[s,t] /. fst, {s,n}] / n!) /. n
$$\rightarrow$$
 4 /. s \rightarrow 0 /. $\lambda \rightarrow \frac{1+m}{2}$ /. $\mu \rightarrow \frac{1-m}{2}$ /. N0 \rightarrow 1 // Simplify

Distribution of lineage sizes given persistence (equation A5)

Dividing the probability of having n individuals at time t by the probability of survival to time t gives the conditional probability of having n individuals at time t given a mutant lineage lives to time t

pntgt = pnt / probpersist /.
$$\lambda \rightarrow \frac{1+m}{2}$$
 /. $\mu \rightarrow \frac{1-m}{2}$ /. N0 \rightarrow 1 // FullSimplify
$$\frac{2 \, m \, \left(1-\frac{2 \, m}{-1+m+e^{m\,t}\,\, (1+m)}\right)^n}{(-1+e^{m\,t})\,\, (1+m)}$$

When $t \ll |1/m|$ (effectively critical) this is approximately

pntgtapp1 = Normal[Series[pntgt /. m \rightarrow m ϵ , { ϵ , 0, 0}]] /. $\epsilon \rightarrow$ 1 // FullSimplify Simplify $\left[\% = 2 \left(\frac{1}{t} \right) \left(1 + \frac{2}{t} \right)^{-n}, \{t > 0, n > 0\} \right]$

 $2 t^{-1+n} (2 + t)^{-n}$

True

with expectation

Sum[npntgtapp1, $\{n, 0, \infty\}$]

$$\frac{2+t}{2}$$

Using this expectation, the cumulative number of individuals over T generations is roughly

$$\frac{1}{4}$$
 T (1 + T)

Or when -1/m << t (ie subcritical mutants) we have approximately

```
pntgtapp2 = pntgt /. -1 + m + e^{mt} (1 + m) \rightarrow -1 + m /. -1 + e^{mt} \rightarrow -1 // Simplify;
% /. 1 - m \rightarrow 1
-\,2 m (1 + m) ^{-1+n}
```

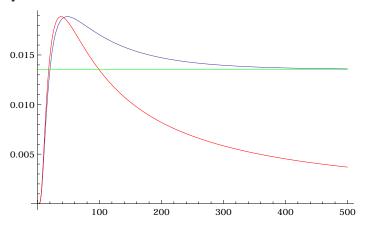
with expectation

 $Sum[npntgtapp2, {n, 0, \infty}]$

$$\frac{\text{-1} + \text{m}}{\text{2 m}}$$

Visually check critical (red) and subcritical (green) approximations against true solution (blue) for n=20 across time

```
Show[
 Plot[pntgt /. m \rightarrow -0.01 /. n \rightarrow 20, {t, 0, 500}, PlotRange \rightarrow All],
 Plot[pntgtapp1 /. m \rightarrow -0.01 /. n \rightarrow 20,
    \{t, 0, 500\}, PlotRange \rightarrow All, PlotStyle \rightarrow Red],
 \texttt{Plot[pntgtapp2 /. m \rightarrow -0.01 /. n \rightarrow 20, \{t, 0, 500\}, PlotRange \rightarrow \texttt{All, PlotStyle} \rightarrow \texttt{Green]}}
]
```



Check distribution of n at 3 different times

```
nmax = 100;
Show
 LogPlot[pntgt/.m\rightarrow-0.01/.t\rightarrow2,
   \{n, 0, nmax\}, PlotStyle \rightarrow Blue, PlotRange \rightarrow \{10^{-4}, 10\},
 LogPlot[pntgt/.m\rightarrow-0.01/.t\rightarrow20, {n, 0, nmax},
   PlotStyle → {Blue, Dashed}, PlotRange → All],
 LogPlot[pntgt /. m \rightarrow -0.01 /. t \rightarrow 200, {n, 0, nmax},
   PlotStyle → {Blue, Dotted, Thick}, PlotRange → All],
 LogPlot[pntgtapp1 /. m \rightarrow -0.01 /. t \rightarrow 2, {n, 0, nmax},
   PlotRange → All, PlotStyle → Red],
 LogPlot[pntgtapp1 /. m \rightarrow -0.01 /. t \rightarrow 20, {n, 0, nmax},
   PlotRange → All, PlotStyle → {Red, Dashed}],
 LogPlot[pntgtapp1 /. m \rightarrow -0.01 /. t \rightarrow 200, {n, 0, nmax},
  PlotRange → All, PlotStyle → {Red, Dotted, Thick}],
 LogPlot[pntgtapp2 /. m \rightarrow -0.01, {n, 0, nmax}, PlotRange \rightarrow All, PlotStyle \rightarrow Green]
  10
 0.01
0.001
```

These lineage sizes given persistence align with eq A3 in Weissman et al 2010. As they pointed out, the distribution of n given persistence to t and m is roughly geometric in both cases (t<<|1/m| and t>>-1/m), with p=1/t or p=-m respectively, which implies that the probability of n drops off exponentially when n>min(t, -1/m). Thus n will essentially never be greater than the minimum of t or -1/m, eg mutants with large -m will be restricted to small sizes.

Probability of rescue

1-step rescue (equation 5)

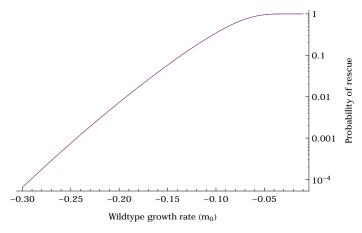
The probability that a mutant with growth rate in [m,m+dm] rescues the population is $f[m \mid m_{wt}]$ $p_{\rm est}[m]$ dm. Integrating over all m>0 gives the probability of 1-step rescue from an individual with growth rate m0

```
Clear[A1]
\Lambda1[m0_?NumericQ, mmax_, \lambda_, n_, U_] :=
 UNIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], {m, 0, mmax}]
```

```
The total probability of 1-step rescue is then
```

```
PR1[m0_?NumericQ] := prescue /. p0 \rightarrow prescuem[m0, \Lambda1[m0, mmax, \lambda, n, U]]
And we can approximate this as
PR1App[m0\_?NumericQ] := prescueApp /. p0 \rightarrow prescuem[m0, \Lambda1[m0, mmax, \lambda, n, U]]
Numerical example
N0 = 10^4;
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
LogPlot[
 {PR1[m0], PR1App[m0]},
 \{m0, -0.3, -0.01\},\
 Frame → {True, False, False, True},
 FrameTicks → {True, False, False, True},
 PerformanceGoal → "Speed"
1
```

Clear[N0, U, mmax, λ , Es, n, mwt]



2-step rescue (equation 6)

The rate of 2-step rescue from a single individual with growth rate m0 is the probability of a mutation to growth rate m, which does not establish, but then creates a second mutation that does

```
\Lambda 2 [m0_?NumericQ, mmax_, \lambda_, n_, U_] := UNIntegrate[
    fm[m, m0, mmax, \lambda, n] (1 - pest[m]) prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]], \{m, -\infty, mmax\}
Numerical comparison with probability of 1-step rescue
```

```
N0 = 10^4;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
Table[\{m0, prescue /. p0 \rightarrow prescuem[m0, \Lambda2[m0, mmax, \lambda, n, U]]\},
   \{m0, -0.3, -0.01, 0.1\}];
plot1 = ListLogPlot[%, Joined → True, PlotStyle → Red];
plot2 = LogPlot[PR1[m0], \{m0, -0.3, -0.01\}, Frame \rightarrow \{True, False, False, True\},
    FrameLabel \rightarrow {"Wildtype growth rate (m_0)", "", "", "Probability of rescue"},
    FrameTicks → {True, False, False, True}, PerformanceGoal → "Speed"];
Show[plot2, plot1]
Clear [mmax, \lambda, Es, n, N0, U]
                                                      0.1
                                                            Probability of rescue
                                                      0.01
                                                      0.001
-0.30
         -0.25
                  -0.20
                          -0.15
                                   -0.10
                                           -0.05
                  Wildtype growth rate (m<sub>0</sub>)
```

k-step rescue (equation 7)

One can carry on the logic used for 2-step rescue to get the probability of k-step rescue, e.g., 3-step is

```
\Lambda3[m0_?NumericQ, mmax_, \lambda_, n_, U_] := UNIntegrate[
      \texttt{fm}[\texttt{m}, \texttt{m0}, \texttt{mmax}, \lambda, \texttt{n}] \; (\texttt{1-pest}[\texttt{m}]) \; \texttt{prescuem}[\texttt{m}, \texttt{\Lambda2}[\texttt{m}, \texttt{mmax}, \lambda, \texttt{n}, \texttt{U}]] \; , \; \{\texttt{m}, -\infty, \texttt{mmax}\}]
and 4-step is
\Lambda4[m0_?NumericQ, mmax_, \lambda_, n_, U_] := UNIntegrate[
      fm[m, m0, mmax, \lambda, n] (1 - pest[m]) prescuem[m, \Lambda 3[m, mmax, \lambda, n, U]], \{m, -\infty, mmax\}
etc
```

Plot probability as a function of wildtype growth rate (figure 3)

Interpolation function

The probability of rescue by k steps involves k integrals, and so with even small k (e.g. k=3) they are very slow to compute in their exact form. We can, however, combine an approximation (don't integrate below mmin) with interpolation functions to greatly speed things up (with some loss of accuaracy)

```
mTempmin = -0.4;
mTempmax = -0.01;
step = 0.01;
mmin = mTempmin;
Clear[\lambdaInterpolated, \lambda2Interpolated, \lambda3Interpolated, \lambda4Interpolated]
\texttt{\Lambda1Interpolated[m0\_, mmax\_, } \lambda\_\texttt{, n\_, U\_] :=}
 \Lambda1Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n] pest[m],}
            {m, 0, mmax}, Method → {Automatic, "SymbolicProcessing" → 0}]},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
  ]
\Delta 2Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda2Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n]}
             (1 - pest[m]) prescuem[m, Λ1Interpolated[m, mmax, λ, n, U]],
            {m, mmin, mmax}, Method → {Automatic, "SymbolicProcessing" → 0}]},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
  ]
\Lambda3Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda3Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n]}
             (1-pest[m]) prescuem[m, Λ2Interpolated[m, mmax, λ, n, U]],
            \{m, mmin, mmax\}, Method \rightarrow \{Automatic, "SymbolicProcessing" \rightarrow 0\}]\}
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
  1
\Lambda4Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda4Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n]}
             (1 - pest[m]) prescuem[m, \Lambda 3Interpolated[m, mmax, \lambda, n, U]],
            {m, mmin, mmax}, Method → {Automatic, "SymbolicProcessing" → 0}]},
        {mTemp, mTempmin, mTempmax, step}]];
   f[m0]
  1
```

Numerical data

```
(*N0=10^4;
mmax=0.5;
\lambda = 2Es/n;
Es=0.01;
n=4;
U=2*10^{-3};
m0min=mTempmin;
m0max=mTempmax;
mstep=step;
rate4=Table[{m0, \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \(
rate3=Table[{m0,\deltaInterpolated[m0,mmax,\lambda,n,U]},{m0,m0min,m0max,mstep}];
rate2=Table[{m0, \Lambda2Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
rate1=Table[{m0, \lambda1Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
m0list=Table[m0,{m0,m0min,m0max,mstep}];
theory={
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate2[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate3[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate4[[i,2]]]},
          {i,Length[m0list]}]
   };
alltheory=
   Table[\{m0list[[i]],prescue/.p0\rightarrow prescuem[m0list[[i]],rate1[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+ra
                     rate3[[i,2]]+rate4[[i,2]]]},{i,Length[m0list]}];
Clear[N0,mmax,\lambda,Es,n,U,m0min,m0max]*)
theory = {{{-0.4}, 5.472398814099222\*^-7}, {-0.39\, 9.745688406281872\*^-7},
             {-0.38, 1.7238895513305152, 4^-6}, {-0.37, 3.0285772651961196, 4^-6},
             {-0.36000000000000000, 5.2841014319682245`*^-6}, {-0.35000000000000000,
                9.155420059081543`*^-6}, {-0.34`, 0.000015751880195202794`},
             {-0.33, 0.00002690955788409166}, {-0.32, 0.000045643083578017496},
             {-0.3100000000000005, 0.0000768622102641503},
             {-0.3000000000000004, 0.00012849810996373723},
             {-0.2900000000000004, 0.0002132574731799064},
             {-0.28`, 0.0003513291377449379`}, {-0.27`, 0.000574520974723769`},
             {-0.26`, 0.0009325213318174264`}, {-0.25`, 0.0015022797840921198`},
             {-0.24000000000000002, 0.0024019086800984857}},
             {-0.23, 0.0038110379258926574}, {-0.2200000000000003,
               0.00600020901304088<sup>*</sup>}, {-0.210000000000002<sup>*</sup>, 0.00937261491044139<sup>*</sup>},
             {-0.2`, 0.014522096798011974`}, {-0.1900000000000003`, 0.022311337733202485`},
             {-0.18000000000000002<sup>,</sup> 0.03397263148001661<sup>,</sup>},
             {-0.17`, 0.0512283883233019`}, {-0.160000000000003`, 0.07641581098977057`},
             {-0.15000000000000002`, 0.11257347217569635`},
             {-0.14`, 0.16339795924448708`}, {-0.13`, 0.23289952993154328`},
             {-0.12`, 0.3244893834485655`}, {-0.1100000000000004`, 0.43919411790398366`},
             {-0.10000000000000003, 0.5729341370203966},
             \{-0.09000000000000002^{, 0.7137213876566558^{}}\},
```

```
{-0.0800000000000002`, 0.8414602539868381`}, {-0.07`, 0.9344415307620567`},
 {-0.06`, 0.9829032755970813`}, {-0.049999999999999`, 0.9979187459679042`},
 {-0.04000000000000036`, 0.9999341480370483`},
 {-0.030000000000000027`, 0.9999998505645212`},
  \{-0.0200000000000018`,\, 0.999999999991431`\},\, \{-0.010000000000000000°,\, 1.`\}\},
{{-0.4`, 7.1817194564616216`*^-6}, {-0.39`, 0.000010382868435687698`},
{-0.38, 0.000015004899198478583, {-0.37, 0.000021674932411452907, },
 {-0.3600000000000004, 0.000031295722740964926},
 {-0.3500000000000003, 0.00004516336475557381},
 {-0.34, 0.00006513527103491068}, {-0.33, 0.00009386674884392932},
 {-0.32`, 0.00013514257904490012`}, {-0.31000000000000005`,
  0.0001943394640103735^{}), \{-0.300000000000004^{}, 0.00027906144855771586^{}\},
 {-0.29000000000000004, 0.00040002244449899305},
 {-0.28, 0.0005722318522486392, {-0.27, 0.0008166042269467377,},
 {-0.24000000000000002, 0.002330394186973561},
 {-0.23<sup>^</sup>, 0.0032813120522319217<sup>^</sup>}, {-0.2200000000000003<sup>^</sup>,
 0.0046003825714453495`}, {-0.210000000000002`, 0.006419557828723499`},
  \{-0.2\ ,\ 0.008912948912746632\ \}\ ,\ \{-0.190000000000003\ ,\ 0.012308013245967753\ \}\ ,
 {-0.18000000000000002`, 0.01689876878901897`}, {-0.17`, 0.02306102686962941`},
 {-0.16000000000000003, 0.031269462419949656},
 {-0.15000000000000002`, 0.042116115688766054`}, {-0.14`, 0.05632965853148175`},
 {-0.13<sup>,</sup> 0.07479449506069946<sup>,</sup> , {-0.12<sup>,</sup> 0.09856854888921651<sup>,</sup> },
 {-0.1100000000000004, 0.12889844527072303}, {-0.10000000000000003,
  \texttt{0.1672306176682965`} \texttt{, \{-0.09000000000000002`, 0.21521607762716344`\},} 
 {-0.08000000000000002`, 0.2747032194313106`}, {-0.07`, 0.3477011299288638`},
 {-0.06`, 0.4362577296013713`}, {-0.049999999999999`, 0.5420788502184701`},
 {-0.04000000000000036`, 0.6653477290582192`},
 {-0.030000000000000027<sup>,</sup>, 0.8010818232849237<sup>,</sup>}, {-0.020000000000000018<sup>,</sup>,
  0.9286233811012167`}, {-0.0100000000000000°, 0.9962496646879948`}},
{{-0.4`, 2.672518644497579`*^-6}, {-0.39`, 3.6200066483305093`*^-6},
 {-0.38, 4.9024286813414975, *^-6}, {-0.37, 6.639033992561494, *^-6},
 {-0.36000000000000004, 8.99229424278225**^-6}, {-0.35000000000000003,
  0.000012183830842893428`}, {-0.34`, 0.000016516674993138913`},
 {-0.33, 0.00002240514549411987, {-0.32, 0.000030416485379602953,
 {-0.3100000000000005, 0.000041327352556463914},
 {-0.3000000000000004, 0.00005620069935863281},
 {-0.2900000000000004, 0.00007648968361839703},
 {-0.28`, 0.0001041772194614321`}, {-0.27`, 0.00014196192976223543`},
 {-0.26`, 0.00019350379587734778`}, {-0.25`, 0.0002637455775598152`},
 {-0.2400000000000002, 0.0003593290092440693},
 {-0.23<sup>,</sup> 0.0004891277868906974<sup>,</sup>},
 {-0.220000000000003, 0.0006649222935298793}, {-0.2100000000000002,
 0.0009022438378168962`}, {-0.2`, 0.0012214190541605108`},
 {-0.1900000000000003, 0.0016488487049626244}, {-0.1800000000000002,
  0.0022185608705778126^{}, \{-0.17^{}, 0.0029740895246452848^{},
 {-0.16000000000000003<sup>^</sup>, 0.003970751527472172<sup>^</sup>}, {-0.15000000000000000<sup>^</sup>,
   0.0052784386334274735`\} \text{, } \{-0.14`, 0.006985125056896413`\} \text{,} \\
 {-0.13`, 0.009201449505447323`}, {-0.12`, 0.012067026227695399`},
 {-0.1100000000000000<sup>1</sup>, 0.015759693621743676<sup>1</sup>}, {-0.10000000000000000<sup>3</sup>,
 0.020509966391728862`}, {-0.0900000000000002`, 0.02662504581244618`},
 {-0.0800000000000002`, 0.03453108060938248`}, {-0.07`, 0.04485200223554198`},
 {-0.06`, 0.0585665300648599`}, {-0.04999999999999°, 0.0773474840786541`},
 {-0.04000000000000036`, 0.10438028623999529`},
 {-0.030000000000000027, 0.14667096966766024},
```

```
{-0.02000000000000018, 0.2233073280524227},
{-0.010000000000000000, 0.4098900908454398`}},
{{-0.4`, 3.665400664099039`*^-7}, {-0.39`, 4.882948413387211`*^-7},
 {-0.38, 6.507825516255394, 4-7}, {-0.37, 8.679175866888045, 4-7},
 \{-0.3600000000000004, 1.1585392625246271*^-6\}, \{-0.35000000000000000,
  1.548213704460899`*^-6}, {-0.34`, 2.071738417153135`*^-6},
 {-0.33`, 2.776581790708832`*^-6}, {-0.32`, 3.727611290194055`*^-6},
 {-0.3100000000000005, 5.013589215119296**^-6},
 {-0.30000000000000004, 6.756030736299046**^-6},
 {-0.2900000000000004, 9.12124881535803**^-6},
 {-0.28`, 0.00001233658455390696`}, {-0.27`, 0.000016712145389896627`},
 {-0.26`, 0.00002266963795427568`}, {-0.25`, 0.000030780242171069006`},
 {-0.2400000000000002, 0.00004181381972367948},
 {-0.23, 0.000056802136495281985},
 {-0.2200000000000003, 0.00007711911056884624}, {-0.2100000000000002,
 0.00010458146610448082`}, {-0.2`, 0.0001415735100674942`},
 {-0.1900000000000003`,0.00019120021529694942`},{-0.18000000000000002`,
  0.0002574735404686779^{}, \{-0.17^{}, 0.00034553832838446485^{}\},
 {-0.16000000000000003, 0.0004619470399811476}, {-0.1500000000000002,
  0.0006149981944676419`}, {-0.14`, 0.0008151642974213447`},
 {-0.13<sup>^</sup>, 0.0010756555359718867<sup>^</sup>}, {-0.12<sup>^</sup>, 0.0014132038091301258<sup>^</sup>},
 {-0.11000000000000004`, 0.0018492239194836113`},
 \{-0.10000000000000003^{,}0.002411647781310755^{,}\}
 {-0.09000000000000002, 0.003138005818001832},
 {-0.08000000000000002, 0.004080917925158634},
 {-0.07`, 0.005318491045821605`}, {-0.06`, 0.006975434272220693`},
 {-0.049999999999999999999, 0.009269923451119766`},
 {-0.04000000000000036, 0.012631023980203326},
 \{-0.030000000000000027^{, 0.01804961542287531^{}},
 \{-0.02000000000000018^{\circ}, 0.0284671089107692^{\circ}\},
```

```
alltheory = {{-0.4}, 0.000010767989638504893}, {-0.39}, 0.000015465680252679803},
   {-0.38`, 0.000022281877958607588`}, {-0.37`, 0.00003221020658283891`},
   {-0.3600000000000004`, 0.00004673011156652862`}, {-0.350000000000000003`,
    0.00006804965109896699`}, {-0.34`, 0.00009947299961354794`},
   {-0.33`, 0.0001459524027258574`}, {-0.32`, 0.00021491729975098295`},
   {-0.3100000000000005, 0.00031751489398534893},
   {-0.3000000000000004, 0.0004704543728014654},
   \{-0.2900000000000004, 0.0006987523028609832},
   {-0.28, 0.0010397647754660788, {-0.27, 0.0015491068347829318},
   {-0.26`, 0.0023092555158806816`}, {-0.25`, 0.0034419561736643844`},
   {-0.24000000000000002, 0.005125934145003619},
   {-0.23`, 0.007621874910163862`}, {-0.220000000000003`, 0.011307117362613917`},
   {-0.21000000000000002`, 0.01672286182842886`},
   {-0.2`, 0.024636589665609887`}, {-0.190000000000003`, 0.03612111665412221`},
   {-0.18000000000000002, 0.0526478810505413}, {-0.17, 0.07618321541025108},
   {-0.16000000000000003, 0.10925842311524103},
   {-0.15000000000000002`,0.154951959045035`},{-0.14`,0.21667029213726385`},
   {-0.13, 0.29754827077524637}, {-0.12, 0.39924782631433087},
   {-0.1100000000000004`, 0.5200288215021587`},
   {-0.10000000000000003, 0.6524246688522214},
   {-0.09000000000000002, 0.7819169338181124}},
   {-0.0800000000000002`, 0.8893420776854711`}, {-0.07`, 0.9592948833994817`},
   {-0.06`, 0.9909501188644341`}, {-0.04999999999999`, 0.9991194054587093`},
   {-0.04000000000000036`, 0.9999799413255674`},
   {-0.03000000000000027, 0.999999972757273},
   {-0.02000000000000018`, 0.99999999999342`}, {-0.01000000000000000°, 1.`}};
Figure 3
N0 = 10^4;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
U = 2 * 10^{-3};
dat1 = Import[datadir <>
    "prescue_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mutmax10_nreps1000.csv"];
dat2 = Import[datadir <>
    "prescue_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mutmax10_nreps10000.csv"];
dat3 = Import[datadir <>
    "prescue_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mutmax10_nreps100000.csv"];
alldat = Flatten[{dat1, dat2, dat3}, {1, 2}];
alldat =
  Select[Select[Select[alldat, \#[[1]] \neq -0.1 \&], \#[[1]] \neq -0.2 \&], \#[[1]] \neq -0.3 \&];
dat = Import[datadir <>
    "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.10_mutmax10
       _nreps100000.csv"];
nreps = 100000;
onestep = Length[Select[dat, #[[2]] == 1 &]] / nreps;
twostep = Length[Select[dat, #[[2]] == 2 &]] / nreps;
threestep = Length[Select[dat, #[[2]] == 3 &]] / nreps;
fourstep = Length[Select[dat, #[[2]] == 4 &]] / nreps;
```

```
total = onestep + twostep + threestep + fourstep;
k1 = {onestep, twostep, threestep, fourstep, total} // N;
Table[{{-0.1, k1[[i]]}}, {i, Length[k1]}];
k1plot = ListLogPlot[%, PlotMarkers → {
     Graphics[{Circle[], defaultcolors[[1]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[2]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[3]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[4]]}, ImageSize → 10],
     Graphics[{Black, Disk[]}, ImageSize → 8]
    }];
dat = Import[datadir <>
    "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.20_mutmax10
       _nreps100000.csv"];
nreps = 100 000;
onestep = Length[Select[dat, #[[2]] == 1 &]] / nreps;
twostep = Length[Select[dat, #[[2]] == 2 &]] / nreps;
threestep = Length[Select[dat, #[[2]] == 3 &]] / nreps;
fourstep = Length[Select[dat, #[[2]] == 4 &]] / nreps;
total = onestep + twostep + threestep + fourstep;
k2 = {onestep, twostep, threestep, fourstep, total} // N;
Table[{{-0.2, k2[[i]]}}, {i, Length[k2]}];
k2plot = ListLogPlot[%, PlotMarkers → {
     Graphics[{Circle[], defaultcolors[[1]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[2]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[3]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[4]]}, ImageSize → 10],
     Graphics[{Black, Disk[]}, ImageSize → 8]
    }];
dat = Import[datadir <>
    "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.30_mutmax10
       _nreps1000000.csv"];
nreps = 1000000;
onestep = Length[Select[dat, #[[2]] == 1 &]] / nreps;
twostep = Length[Select[dat, #[[2]] == 2 &]] / nreps;
threestep = Length[Select[dat, #[[2]] == 3 &]] / nreps;
fourstep = Length[Select[dat, #[[2]] == 4 &]] / nreps;
total = onestep + twostep + threestep + fourstep;
k3 = {onestep, twostep, threestep, fourstep, total} // N;
Table[{{-0.3, k3[[i]]}}, {i, Length[k3]}];
k3plot =
  ListLogPlot[%,
   PlotMarkers → {
      Graphics[{Circle[], defaultcolors[[1]]}, ImageSize → 10],
      Graphics[{Circle[], defaultcolors[[2]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[3]]}, ImageSize → 10],
     Graphics[{Circle[], defaultcolors[[4]]}, ImageSize → 10],
     Graphics[{Black, Disk[]}, ImageSize → 8]
  ];
Show
 ListLogPlot[
```

```
theory,
  Joined → True,
  PlotStyle → Thick,
  PlotRange \rightarrow \{\{-0.4, 0\}, \{5 * 10^{-7}, 1\}\},\
  Frame → {True, False, False, True},
  FrameLabel → {"Wildtype growth rate", , , "Probability of rescue"},
  FrameTicks → {True, False, False, True},
  LabelStyle \rightarrow labelstyle,
  PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
        {"1-step", "2-step", "3-step", "4-step"}], Scaled@{0.85, 0.23}]
 ],
 \texttt{ListLogPlot[alltheory, Joined} \rightarrow \texttt{True, PlotStyle} \rightarrow \{\texttt{Black, Thick}\},
  PlotLegends → Placed[LineLegend[
      Style [#, 12, FontFamily \rightarrow "Helvetica"] & /@ {"total"}], Scaled@ {0.56, 0.23}]
 ListLogPlot[alldat, PlotMarkers → Graphics[{Black, Disk[]}, ImageSize → 8]],
 k1plot, k2plot, k3plot,
 Epilog ->
  braceLabel[\{\{-0.125, Log@10^{-6}\}, \{-0.125, Log@(2*10^{-4})\}\}, Style["", Larger]],
 ImageSize → 350
Export[figdir <> "Figure3.eps", %];
Clear[N0, mmax, \lambda, Es, n, U, m0min, m0max]
                                              0.1
                                              0.01
                                              0.001
                          total
                                      3-step
                                              10^{-6}
           -0.3
                      -0.2
                                 -0.1
                                             0.0
               Wildtype growth rate
```

Plot probability as a function of mutation rate (figure S1)

Interpolation functions

We again use interpolation functions to speed things up, this time across U instead of m0

```
Umin = -5;
Umax = 1;
Ustep = 0.1;
mmin = -0.4;
Clear[\lambdaInterpolatedU, \lambda2InterpolatedU, \lambda3InterpolatedU, \lambda4InterpolatedU]
\Lambda1InterpolatedU[m0_, mmax_, \lambda_, n_, x_] :=
   \Lambda1InterpolatedU[m0, mmax, \lambda, n, x] = Block[{},
           f = Interpolation Table
                     \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], \{m, 0, mmax\}, Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrate[fm[m, m0, mmax, \lambda, n] pest[m], Method \rightarrow \{xtemp, 10^{xtemp} NIntegrat
                                   {Automatic, "SymbolicProcessing" \rightarrow 0}]}, {xtemp, Umin, Umax, Ustep}]];
          f[x]
\Lambda2InterpolatedU[m0_, mmax_, \lambda_, n_, x_] :=
   \Lambda2InterpolatedU[m0, mmax, \lambda, n, x] = Block[{},
           f = Interpolation[
                 Table [\{xtemp, 10^{xtemp} NIntegrate | fm[m, m0, mmax, \lambda, n] (1 - pest[m]) prescuem ]
                                      m, \Lambda1Interpolated[m, mmax, \lambda, n, 10^{\text{xtemp}}]], {m, mmin, mmax}, Method \rightarrow
                                   {Automatic, "SymbolicProcessing" \rightarrow 0}]}, {xtemp, Umin, Umax, Ustep}]];
          f[x]
\Lambda3InterpolatedU[m0_, mmax_, \lambda_, n_, x_] :=
   \Lambda3InterpolatedU[m0, mmax, \lambda, n, x] = Block[{},
           f = Interpolation
                 Table [\{xtemp, 10^{xtemp} NIntegrate | fm[m, m0, mmax, \lambda, n] (1-pest[m]) prescuem [
                                      m, \Lambda 2Interpolated[m, mmax, \lambda, n, 10^{\text{xtemp}}]], {m, mmin, mmax}, Method \rightarrow
                                   {Automatic, "SymbolicProcessing" → 0}]}, {xtemp, Umin, Umax, Ustep}]];
          f[x]
\Lambda4InterpolatedU[m0_, mmax_, \lambda_, n_, x_] :=
   \Lambda4InterpolatedU[m0, mmax, \lambda, n, x] = Block[{},
           f = Interpolation[
                  Table [\{xtemp, 10^{xtemp} NIntegrate | fm[m, m0, mmax, \lambda, n] (1-pest[m]) prescuem [
                                      m, \Lambda3Interpolated[m, mmax, \lambda, n, 10^{\text{xtemp}}]], {m, mmin, mmax}, Method \rightarrow
                                   {Automatic, "SymbolicProcessing" \rightarrow 0}]}, {xtemp, Umin, Umax, Ustep}]];
          f[x]
```

Numerical data for panel A

```
(*N0=10^4;
m0 = -0.1;
mmax=0.5;
\lambda = 2Es/n;
Es=0.01;
n=4;
rate4=Table [Λ4InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate3=Table[Λ3InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate2=Table[Λ2InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate1=Table[Λ1InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
Ulist=Table[10*, {x,Umin,Umax,Ustep}];
theory={
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,ratel[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate2[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate3[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0→prescuem[m0,rate4[[i]]]},{i,Length[Ulist]}]
alltheory=Table[{Ulist[[i]],
   prescue/.p0-prescuem[m0,rate1[[i]]+rate2[[i]]+rate3[[i]]+rate4[[i]]]},{i,
   Length[Ulist] }];
Clear [mmax, \lambda, Es, n, m0, N0, U] *)
theory = {{{0.00001}, 0.004246842378678117}},
    {0.000012589254117941661, 0.0053435122405096624},
    {0.00001584893192461114, 0.006722418845097056}, {0.000019952623149688786,
     0.008455638902157614`}, {0.000025118864315095822`, 0.010633329634670008`},
    {0.000031622776601683795, 0.013368073668290736},
    {0.000039810717055349695, 0.01680015577769267},
    {0.00005011872336272725, 0.021103903488885356},
    {0.00006309573444801929, 0.026495197360892786},
    {0.00007943282347242822`, 0.03324019208311246`},
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    {0.00019952623149688788, 0.08140749078007126},
    {0.0002511886431509582, 0.10138225267144507},
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    {0.00039810717055349735, 0.15584223469333813},
    {0.0005011872336272725, 0.19206791297501513},
    {0.000630957344480193, 0.2354690490355157},
    {0.0007943282347242822, 0.2868051576697973},
    {0.001, 0.3465563408388336, {0.0012589254117941675, 0.4147065687699266, },
    {0.001584893192461114, 0.49048019295427137},
    {0.0019952623149688807, 0.5720729042194002},
    {0.002511886431509582, 0.6564634113805501},
    {0.0031622776601683794, 0.7394414586022069},
    {0.003981071705534973, 0.8160091713130636},
    {0.005011872336272725, 0.8812494467922933},
    {0.006309573444801936, 0.9315527837690738},
    {0.00794328234724282`, 0.9657778626534119`}, {0.01`, 0.9856911478435505`},
    {0.012589254117941675, 0.9952212549506977},
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{0.07943282347242822, 0.99999999999963}, {0.1, 1.},
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{7.943282347242821,
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 0.7642508154485257 ^{\texttt{`}} \} \text{, } \{ 0.00794328234724282 ^{\texttt{`}} , \, 0.8862494786994978 ^{\texttt{`}} \} \text{,} \\
{0.01`, 0.961720591184484`}, {0.012589254117941675`, 0.9924489494028248`},
{0.01584893192461114`, 0.9993232933992733`},
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\{\{0.00001^{\dagger}, 8.752998326144734^{\star}^{-9} + 0.^{\dagger}\},
\{0.000012589254117941661^{1.7090773130057357^{-*}-8+0.^{1}},
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\{0.000019952623149688786^{, 6.4872549887518^{*^-8+0.^{i}}\},
\{0.000025118864315095822^{, 1.260103055233941^{*^-7} + 0.^{i}\},
\{0.000031622776601683795^{,}2.440958251082037^{*^-7} + 0.^{i}\},
\{0.000039810717055349695^4.712729129163762^*^-7+0.^{i}\}
\{0.00005011872336272725^{,}9.064345221077019^{*^-7}+0.^{i}\},
\{0.00006309573444801929^{,}1.7363417931237635^{*}-6+0.^{\dot{1}}\},
\{0.00007943282347242822^{,}3.3125780688481044^{*^-6}+0.^{i}\},
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\{0.01995262314968881^{\circ}, 0.9987489284470218^{\circ} + 0.^{\circ}i\},
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\{0.07943282347242822^{,}1.^{+}0.^{\pm}\}, \{0.1^{,}1.^{+}0.^{\pm}\},
\{0.12589254117941687^{+}, 1.^{+} + 2.764473032650923^{*} - 174 i\},
{0.15848931924611143`, 1.`+6.954318481178101`*^-245 i}, {0.1995262314968879`,
6357191551341868~355.29367778426564+
2.087234795303845338146796740467790237 `15.954589770191005* ^ - 349
 i), {0.25118864315095824`,
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2.681074505449572837767527147147354826589274784238`15.954589770191005*^-491 i), {0.31622776601683794`,

- 99994157040602949059406001228781806385930601112`684.8869268966961+
- 2.53989668964218087585074425191485894`15.653559774527023*^-677 il, {0.39810717055349776,

351~498.62612764362206+

- 99999999999999987633434402601386607801705650442636671~922.561310611644.
 - 3.2670811791520309978005568966896573057580040633735`15.653559774527023*^-9 14 i), {0.5011872336272725`,

99999999999999999766297047968852068872401546227681929637429681257~1220% .2848954657975 +

- 1.1891297176213295084513677641885012334805`15.653559774527023*^-1211 i), {0.6309573444801942`,
- 99999999999999997581373265499837237011155134731883880215539706`1587.968 960597958 +
- 2.463992640505014090339631605843690926322`15.35252972355796*^-1579 i), {0.7943282347242822`,

9999999999999999999999967436863321838278054299691194879721229`2037.8306: 551720596 +

- 5.095797644003562223499538532687619528851876246054813`15.352529695680404*^ -2029 i}, {1.`,
- 2423610028474382014695691126329~2582.315817553162+
 - 1.52711565691458167661226239381689454233188347141`15.051499516560282*^-2573

- i), {1.2589254117941688`,
- ~3236.4387974342512+
- 1.816437040670734990355761197170370558877529238132`15.051499362228064*^-32: 27 i}, {1.584893192461114`,

99999654729493821296276072320367513851868270182102~4015.5133378966484+

- 2.185224002438526808295169872783051684272`15.051499181929639*^-4006
- il, {1.9952623149688828,

297314325691~4936.096123882469+

- 7.853917392523956250555846410873893534560484`15.051498956311525*^-4927 i}, {2.5118864315095824`,

99999994918622061318685440250894871271231765363822102111 `6015.044478259875 + 1.25177057306061402873847238443669667040511525`14.750466862129239*^-6005i},

{3.1622776601683795,

99997282153603657727460796384545424989523095468653~7270.316229003585+

1.0597518678248573247051189957963363520496`14.750465157092915*^-7260 i}, {3.9810717055349776`,

590997088441568`8718.754157176232+

- 5.3737234781142323672526454757651104327`14.75046335135372*^-8709 i), {5.011872336272725`,

99937377758758979775719623276320164474839753`10376.953663472077+

- 1.1952145546961707192695407173501453833`14.750447122126301*^-10366 i), {6.309573444801943`,

4577004~12260.803672504146+

- 2.73106070260253711853020160907243257795446`14.750433117738908*^-12250 i), {7.943282347242821,

99999999999998642703943106485956402588673177592482734~14385.617639841981+ 5.17917942107663023483942194603421606`14.750424473100997*^-14375i}, {10.`,

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58~16766.66867403286+
 1.82897815340040880960126122582955262534531088`14.75028960875842*^-16755
  i)}, {{0.00001`, 7.771561172376096`*^-12+0.`i},
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\{0.00001584893192461114^{,}, 4.440892098500626^{*^-11} + 0.^{i}\},
\{0.000019952623149688786^{,}1.0880185641326534^{*}-10+0.^{\dot{1}}\},
\{0.000025118864315095822^{, 2.653433028854124^{*^-10 + 0.^{i}}\}
\{0.000031622776601683795^{, 6.472600233564663^{*^-10 + 0.^{i}}\}
\{0.000039810717055349695^{,}1.574296248918472^{*^-9}+0.^{i}\},
\{0.00005011872336272725^{,}3.810285420513537^{*^-9} + 0.^{i}\},
\{0.00006309573444801929^{,}9.183764859699295^{*}-9+0.^{i}\},
\{0.00007943282347242822^{,}2.203459625871318^{*^-8+0.^{\dot{1}}}\}
\{0.0001^{\circ}, 5.263456226423102^{*}-8+0.^{i}\},
\{0.00012589254117941674^{,} 1.2518318948107066^{*^-7} + 0.^{i}\},
\{0.00015848931924611142^{,} 2.9639508802858927^{*^-7} + 0.^{\dot{1}}\},
\{0.00019952623149688788^{\circ}, 6.982079188144752^{*}-7+0.^{\circ}i\},
\{0.0002511886431509582^{, 1.634581147769154^{*^-6+0.^{i}}\},
\{0.00031622776601683794^{,}3.7975450881688744^{*^-6+0.^{i}}\}
\{0.00039810717055349735^{,} 8.744046156916596^{*^-6} + 0.^{i}\},
\{0.0005011872336272725^{\circ}, 0.000019943908636133045^{\circ} + 0.^{\circ} \pm \},
\{0.000630957344480193^{,}0.00004508176259132579^{+}0.^{i}\},
\{0.0007943282347242822^{,}0.0001010974426889355^{+}0.^{\dot{1}}\}
\{0.001^{\circ}, 0.0002251406424497171^{\circ} + 0.^{\circ}i\},
\{0.0012589254117941675^{,}0.0004981729268668245^{+}0.^{\dot{1}}\},
\{0.001584893192461114^{\circ}, 0.001095349877802243^{\circ} + 0.^{\circ} i\},
\{0.0019952623149688807^{,}0.0023924079609550786^{+}0.^{i}\},
\{0.002511886431509582^{,}0.005186972207966822^{+}0.^{i}\},
\{0.0031622776601683794^{,}0.011148757660240483^{+}0.^{i}\},
\{0.003981071705534973^{,0.023701036601979975^{+0.^{i}}\},
\{0.005011872336272725^{,}0.04962209139705409^{+}0.^{i}\},
\{0.006309573444801936^{\circ}, 0.10149876716581652^{\circ} + 0.^{\circ} i\},
\{0.00794328234724282^{,}0.19978722710392505^{,}+0.^{i}\},
\{0.01^{\circ}, 0.3682276758319597^{\circ} + 0.^{\circ} i\},
\{0.012589254117941675^{\circ}, 0.6075523467641892^{\circ} + 0.^{\circ} \pm \},
\{0.01584893192461114^{\circ}, 0.8476300474296083^{\circ} + 0.^{\circ}i\},
\{0.01995262314968881^{,}0.9760902792835732^{+}0.^{i}\},
\{0.025118864315095822^{,}0.9993244941897982^{+}0.^{i}\},
\{0.03162277660168379^{,}0.9999992050363701^{+}0.^{i}\},
\{0.039810717055349734^{\circ}, 0.99999999999970253^{\circ} + 0.^{\circ}i\},
{0.05011872336272725`, 1.` + 0.` i}, {0.06309573444801936`, 1.` + 0.` i},
{0.07943282347242822`, 1.` + 7.326670348383186`*^-88 i},
\{0.1^{\cdot}, 1.^{\cdot} + 3.757868417519997^{\cdot} *^{-123} i\},
\{0.12589254117941687^{+}, 1.^{+} + 2.470147808875862^{*} - 193 i\},
{0.15848931924611143`, 1.` + 1.5888192023257797`*^-299 i}, {0.1995262314968879`,
```

99868238785224842888139389040468112399`461.8348021800242+

- 4.16299245260377231854517234848457891036548515492123`15.954589770191005*^-449 i), {0.25118864315095824,
- 7703295786176929601409179007`663.1462139993372+
- 1.7620966616132219996790516276012661075265709`15.653559774527023*^-650 i), {0.31622776601683794,
- 999999999999999999997933917010054400727695895082484429843206132826716791 28 925.3369395921532 +
- 1.398602133116689581135611072747850237324010890356658`15.653559774527023*^; -912 i}, {0.39810717055349776`,
- 66267142007479155024657431741608~1257.563822116834+
- 4.188928035928510250214495276920232208795278215`15.352157496235277*^-1245 i), {0.5011872336272725`,

99999999999999999999999997500510063048460231820273665452382301444~1670. 9528337751806 +

- 1.4151742940332911171838054637897385191`15.35222209138885*^-1658 i), {0.6309573444801942`,
- 99999999999999895515513875497345790985924138710936~2176.331590326622+
 - 6.053724638248544987237103719408129141423329919497289`15.352214907844614*^: -2164 i), {0.7943282347242822`,

- 9~2784.772334698187+
- 7.919865117599821828930925817610741554589`15.051102565111776*^-2773 i), {1.`,

9999999999999821573010909763593295415254397580801764933588132927`3507.9 79055991208 +

- 3.538944447851822797228938758036404918417389864768`15.051214408253964*^-34: 96 i), {1.2589254117941688,

695896267573754686133211194~4355.888692230206+

- 4.658426174444860314296747655673362400361070760841`15.051194639595089*^-43: 44 il, {1.584893192461114,

74762460433795022256138240876918616129571980529`5337.0112463587275+

- 1.64759940500436302120657794219120508598033969440549`14.750112055889216*^-5325 i), {1.9952623149688828,

999999337535587169903331302596897883332955806743404297~6458.92841028019+

- 1.459564375052413705131345625397015100208183279`14.750208121320368*^-6447
- i), {2.5118864315095824`,

9999999999999978082586849401326483560918795607048116130906368553484`7726: .408287227432+

- 7.5013494480701199642981231909532200459166598882883`14.750063286932834*^-7 715 i}, {3.1622776601683795`,

99999999999999999999999466404779966324679093685209326952530229165~9143: .021869852722+

- 1.819359825442730443852663385781787295727~14.750064822519922*^-9131 i), {3.9810717055349776`,

69`10711.832994179247 -

- 1.954380865861466795319476937008471962008280567742`14.75046698089892*^-107 02 i), {5.011872336272725`,

9139903723102083397952631214860704~12436.954441844173+

- 3.3408868958785375385913202427814342387434`14.7498318637798*^-12425 i), {6.309573444801943`,

999999999999999999999999998321395513766091820032199988665919928980735`1: 4325.522340752208+

- 1.3118212737013177059126475736832209257922`14.749541417211237*^-14313 i), {7.943282347242821`,

99999999999999999998843002767591683777559835733668430728466`16389.684162 456175 +

8.457065655566641259065390831705081119`14.749601481265914*^-16378 i}, {10.`,

```
006416389136843081752~18648.24727789297 +
           3.017873804981505379977654690647825621571677669637647`14.749334185622937*^:
              -18636 i\}\};
alltheory = {{0.00001, 0.004255901923682859+0. i},
    \{0.000012589254117941661^{\circ}, 0.0053575492358338694^{\circ} + 0.^{\circ}i\},
    {0.00001584893192461114, 0.0067441516854744865+0. i},
    \{0.000019952623149688786^{\circ}, 0.008489259139161986^{\circ} + 0.^{\circ} i\},
    \{0.000025118864315095822^{,0.01068529164012788^{+0.^{i}}\},
    \{0.000031622776601683795^{,}0.013448300625872878^{+}0.^{i}\},
    \{0.000039810717055349695^{\circ}, 0.016923875613529704^{\circ} + 0.^{\circ} i\},
    \{0.00005011872336272725^{,}0.02129443271835585^{+}0.^{i}\},
    \{0.00006309573444801929^{,}0.026788141996554327^{+}0.^{i}\},
    \{0.00007943282347242822^{,}0.03368974767574706^{+}0.^{i}\},
    \{0.0001^{\circ}, 0.04235348825450003^{\circ} + 0.^{\circ} \dot{\mathbf{1}}\},
    \{0.00012589254117941674^{,}0.05321818691335278^{+}0.^{i}\},
    \{0.00015848931924611142^{\circ}, 0.06682428147338215^{\circ} + 0.^{\circ} i\},
    \{0.00019952623149688788^{\circ}, 0.08383199232399552^{\circ} + 0.^{\circ}i\},
    \{0.0002511886431509582^{\circ}, 0.10503876670250001^{\circ} + 0.^{\circ}i\},
    \{0.00031622776601683794^{\circ}, 0.13139230800576218^{\circ} + 0.^{\circ} i\},
    \{0.00039810717055349735^{,}0.16399243239687444^{+}0.^{i}\},
    \{0.0005011872336272725^{,}0.204070097507436^{+}0.^{i}\},
    \{0.000630957344480193^{,}0.25292457924668266^{+}0.^{\dot{1}}\}
    \{0.0007943282347242822^{,}0.3117896993698642^{+}0.^{i}\},
    \{0.001^{\circ}, 0.38158864596192277^{\circ} + 0.^{\circ} i\},
    \{0.0012589254117941675^{,}0.4625305052352049^{+}0.^{i}\},
    \{0.001584893192461114^{\circ}, 0.553516711619294^{\circ} + 0.^{\circ} \pm \},
    \{0.0019952623149688807^{,}0.6513985086181926^{+}0.^{i}\},
    \{0.002511886431509582^{,}0.7503139738505484^{+}0.^{i}\},
    \{0.0031622776601683794^{,}0.8416742198469033^{+}0.^{i}\},
    \{0.003981071705534973^{,}0.9157168838785625^{+}0.^{i}\},
    \{0.005011872336272725^{,}0.9652521927879277^{+}0.^{\dot{1}}\},
    \{0.006309573444801936^{\circ}, 0.9902055896921497^{\circ} + 0.^{\circ} i\},
    \{0.00794328234724282^{\circ}, 0.9984478578862828^{\circ} + 0.^{\circ} i\},
    \{0.01^{\circ}, 0.9998986899380937^{\circ} + 0.^{\circ}i\},
    {0.012589254117941675`, 0.9999983479084642` + 0.` i},
    {0.01584893192461114`, 0.9999999969990214`+0.`i}, {0.01995262314968881`,
     0.99999999998354`+0.`i}, {0.025118864315095822`,1.`+0.`i},
    {0.03162277660168379`, 1.` + 0.` i}, {0.039810717055349734`, 1.` + 0.` i},
    {0.05011872336272725, 1. + 0. i}, {0.06309573444801936, 1. + 0. i},
    \{0.07943282347242822^{+}, 1.^{+} + 7.925116675345696^{+} - 192 i\}
    \{0.1^{\circ}, 1.^{\circ} + 1.325850814458533^{\circ} *^{-265} \pm \}, \{0.12589254117941687^{\circ}, \{0.12589254117947^{\circ}, \{0.1258925417^{\circ}, \{0.1
     5544370494843266675760495623965870191877083~393.677105332175+
       8.8096426117104020809430720304899022027527~15.954589770191005*^-383 i},
```

{0.15848931924611143,

- 99999997877005711393764140654024117679236351`556.6276409443859+
- 1.42456216451978839720619080292861272279620504619044725`15.954589770191005*^: -544 i}, {0.1995262314968879,
- 4599412591575342911014428675970685902875689672122`771.3170219790189+
- 5.06622582867291220787243987525325846434080102793786`15.653559774527023*^-759 i}, {0.25118864315095824`,
- 37255131511401~1048.671910856781+
 - $3.874942183534860300293149810957302624599942`15.653559774527023*^-1036\, \mathtt{i}\}\,,$

{0.31622776601683794,

9999999999999961021482178828457408511857231603379732283`1399.7600747452423+ 1.94985210959028942979835550752567166033`15.352257962982229*^-1387 i},

{0.39810717055349776,

- 71988026125436800140541527002296005823898784~1838.3833189626719+
- 4.720650251756940774803143408061445574166883`15.352253369029476*^-1826 id },

{0.5011872336272725,

999999775582709217306226405103856605227581669775113~2379.000095638583+

- 9.49137833905216965933816749798400528804549`15.352300000353498*^-2367id,
- {0.6309573444801942,
- 83120250759970027924518~3037.5127270842336+
- 1.49851811410178856716322936681565168454`15.051086320049887*^-3025i},

{0.7943282347242822,

- 999999999999998652325854559539534291182874245498177~3831.9207175839215+
- 4.24652358220754704821391492650732171104736002863901`15.051200350726608*^-38 20 i), {1.`,

1433154106122235811488107659~4780.151905554485+

- 1.797815037828110042906587192153850528496055536492`15.051284092775497*^-4768 i), {1.2589254117941688`,

8823 5900.540202074338 +

- 3.93234164021937870852681223736397749117115875`14.750181342080543*^-5889 il, {1.584893192461114\.

37197358213158018375518~7212.31240601915+

- 6.226244265760924188390656661815403747135483`14.75019953039571*^-7201 i},
- {1.9952623149688828,

1851407626570899180725691~8733.47218975716+

- 3.1472105355544011740227005944598306897286243¹⁴.750272546166403*^-8722¹/₂},
- {2.5118864315095824,

8107330585783784115971165772729~10481.933514092443+

- 1.6833590270750803174403280129878951189755339094447~14.750164321868958*^-104% 70 i), {3.1622776601683795`,

336454334134902831113313812058375404244 12475.502031373631 +

4.510584285245385250000329710564164841807~14.750166728605452*^-12464 i),

{3.9810717055349776`,

4793594659065`14732.57956703659+

- 2.2713229781720432365974409052144436`14.750467967729048*^-14723 i), {5.011872336272725`,

99999970658919999043780302902185468125272822432`17273.281357680135+

- 1.179210322339889937309859350455571054299`14.74999243642257*^-17261
- i}, {6.309573444801943`,

288064093055 20120.87577685515 +

- 4.3231858597076347397834452233263834847780033`14.749780771825648*^-20109 i), {7.943282347242821`,

23303.308809008122+

- 1.4901979990699957961887232577575697087163`14.749826303043864*^-23291 i}, {10.`,

```
9999999999873285559433971429478101252767874326`26854.644534775794+
9.6791864110205398321730458104448<sup>14</sup>.749562375722904*^-26843 i}
\{0.00001, 0.0042559 + 0.i\}, \{0.0000125893, 0.00535755 + 0.i\},
\{0.0000158489, 0.00674415 + 0.i\}, \{0.0000199526, 0.00848926 + 0.i\},
\{0.0000251189, 0.0106853 + 0.i\}, \{0.0000316228, 0.0134483 + 0.i\},
\{0.0000398107, 0.0169239 + 0.i\}, \{0.0000501187, 0.0212944 + 0.i\},
```

```
\{0.0000630957, 0.0267881 + 0.i\}, \{0.0000794328, 0.0336897 + 0.i\},
\{0.0001, 0.0423535 + 0.i\}, \{0.000125893, 0.0532182 + 0.i\},
\{0.000158489, 0.0668243 + 0.i\}, \{0.000199526, 0.083832 + 0.i\},
\{0.000251189, 0.105039 + 0.i\}, \{0.000316228, 0.131392 + 0.i\},
\{0.000398107, 0.163992 + 0.i\}, \{0.000501187, 0.20407 + 0.i\},
\{0.000630957, 0.252925 + 0.i\}, \{0.000794328, 0.31179 + 0.i\},
\{0.001, 0.381589 + 0.i\}, \{0.00125893, 0.462531 + 0.i\},
\{0.00158489, 0.553517 + 0.i\}, \{0.00199526, 0.651399 + 0.i\},
\{0.00251189, 0.750314 + 0.i\}, \{0.00316228, 0.841674 + 0.i\},
\{0.00398107, 0.915717 + 0.i\}, \{0.00501187, 0.965252 + 0.i\},
\{0.00630957, 0.990206 + 0.i\}, \{0.00794328, 0.998448 + 0.i\},
\{0.01, 0.999899 + 0.i\}, \{0.0125893, 0.999998 + 0.i\}, \{0.0158489, 1. + 0.i\},
\{0.0199526, 1.+0.i\}, \{0.0251189, 1.+0.i\}, \{0.0316228, 1.+0.i\},
\{0.0398107, 1.+0.i\}, \{0.0501187, 1.+0.i\}, \{0.0630957, 1.+0.i\},
\{0.0794328, 1. + 7.92512 \times 10^{-192} i\}, \{0.1, 1. + 1.32585 \times 10^{-265} i\}, \{0.125893, 1. + 1.32585 \times 10^{-265} i\}
48433 + 8.809642611710402 \times 10^{-383} i, \{0.158489,
057113937641 + 1.424562164519788 \times 10^{-544} i, \left\{0.199526, \right\}
5.06622582867291 \times 10^{-759} i, \{0.251189,
```

9999999999999999990433160262793884 + 3.874942183534860 \times 10⁻¹⁰³⁶ i }, $\left\{0.316228, 0.31628, 0.31622$

- - 1.949852109590289 \times 10⁻¹³⁸⁷ $\dot{\text{1}}$ }, $\left\{0.398107,\right.$

{0.501187,

9999999999999999999977558270921730623 + 9.49137833905217 \times 10⁻²³⁶⁷ $\dot{\text{i}}$ },

{0.630957,

- $1.49851811410179 \times 10^{-3025} i$, $\{0.794328,$

 $25854559540 + 4.24652358220755 \times 10^{-3820} i$, {1.,

999999992079881325200526 + 1.79781503782811 \times 10⁻⁴⁷⁶⁸ $\dot{\text{1}}$ }, $\{1.25893,$

9999999999999999999999999999999998380880196490324 +

- $3.93234164021938 \times 10^{-5889} \ \text{i} \, \big\} \, , \, \, \big\{ 1.58489 \, , \,$

999999999999999999997263865941313625 + $6.2262442657609 \times 10^{-7201} \ i \ \}$, $\left\{1.99526 \right\}$

 $3.14721053555440 \times 10^{-8722} i$, {2.51189,

99999999999999345501524780310 + 1.68335902707508 \times 10⁻¹⁰⁴⁷⁰ $\dot{\text{1}}$ }, $\left\{3.16228, \frac{1}{3}\right\}$

- $4.51058428524539\times 10^{-12\,464}~\text{i}\,\big\}\,,~\big\{3.98107\,,$

- 2.27132297817204 \times 10⁻¹⁴⁷²³ $\dot{\text{1}}$ }, $\left\{5.01187,\right.$

 $9999999999999999999997065891999904378 + 1.17921032233989 \times 10^{-17\,261}\,\,\dot{\mathtt{1}}\,\big\}\,,\,\,\big\{6.30957\,,\,\,\,0.30957\,,\,0.30957\,,\,0.30957\,,\,0.30957\,,$

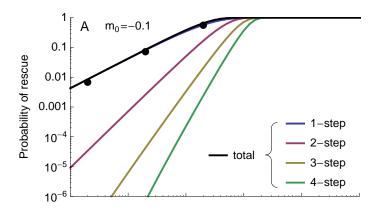
- $4.32318585970763\times 10^{-20\,109}~\text{i}\,\big\}\,,~\big\{7.94328\,,$

- $1.49019799907000 \times 10^{-23291} i$, {10.,

999999998732855594339714 + 9.6791864110205 \times 10⁻²⁶⁸⁴³ i}

Plot panel A

```
N0 = 10^4;
m0 = -0.1;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
letter = "A";
dat = Import[datadir <> ToString[StringForm[
        "prescue_poisson_N``_n``_Es``_mmax``_mwt``_mutmax10_nreps10000.csv",
       NumberForm[N0, 1], n, Es, NumberForm[mmax, {3, 2}], NumberForm[m0, {3, 2}]]]];
dat[[All, \{1\}]] = dat[[All, \{1\}]] * Uc /. Uc \rightarrow n^2 \lambda / 4 /. \lambda \rightarrow 2 Es / n;
figSla = Show
  ListLogLogPlot [Re[theory], Joined → True, PlotStyle → Thick,
    PlotRange \rightarrow \{\{10^{-5}, 1\}, \{10^{-6}, 1\}\},\
    Frame → {True, True, False, False},
    FrameLabel → {, "Probability of rescue", ,},
    FrameTicks → {True, True, False, False},
   LabelStyle → labelstyle,
    FrameTicksStyle → {FontColor → White, Automatic, Automatic},
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
         {"1-step", "2-step", "3-step", "4-step"}], Scaled@{0.85, 0.23}]
  ],
  ListLogLogPlot[Re[alltheory], Joined → True, PlotStyle → {Thick, Black},
   PlotLegends → Placed[LineLegend[
        Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@{"total"}], Scaled@{0.56, 0.23}]
  ],
  ListLogLogPlot[dat, PlotMarkers → {Automatic, Medium}, PlotStyle → Black],
  Epilog → {
     Text[Style[letter, 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
     Text[Style[StringForm["m_0 =  ", m0], 12, FontFamily \rightarrow "Helvetica"],
      Scaled@\{2/10, 9.5/10\}], braceLabel[\{\{\text{Log@}(3*10^{-2}), \text{Log@}(2*10^{-6})\},
        \left\{ \text{Log@}\left(3*10^{-2}\right), \text{Log@}\left(3*10^{-4}\right) \right\} \right\}, Style["", Larger]
    }
Clear[mmax, \(\lambda\), Es, n, m0, N0, U, letter]
```



Numerical data for panel B

```
(*N0=10^4;
m0 = -0.2;
mmax=0.5;
\lambda = 2Es/n;
Es=0.01;
n=4:
rate4=Table[Λ4InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate3=Table[Λ3InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate2=Table [Λ2InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate1=Table [Λ1InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
Ulist=Table[10x, {x,Umin,Umax,Ustep}];
theory={
  Table[{Ulist[[i]],prescue/.p0→prescuem[m0,rate1[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate2[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate3[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0→prescuem[m0,rate4[[i]]]},{i,Length[Ulist]}]
 } ;
alltheory=Table[{Ulist[[i]],
   prescue/.p0-prescuem[m0,rate1[[i]]+rate2[[i]]+rate3[[i]]+rate4[[i]]]},{i,
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Clear [mmax, \lambda, Es, n, m0, N0, U] *)
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- 8.986172143414903153655593302490237426`15.051499781605775*^-4568 i), {3.9810717055349776`,

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- 7.4873109916051305142017576876050609260987232350397`15.352527582846808*^-2 185 i}, {1.2589254117941688`,
- 7520679 2935.0482693955632 +
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 - i), {1.584893192461114`,

99999999999999999999999999999999999995564567509754848206382418180217: 929321~3831.4045408879015+

- 2.675434749888148644209993736691143338221036070264297`15.051494052692103*^\ -3821 i}, {1.9952623149688828`,

6780486946992456095260535857978771246324471745291~4896.599351441554+

- 1.8759138331963642812714943882323456503968775598`15.051493490679299*^-4886 i), {2.5118864315095824,

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- 1.33208963181718317641150472533886721`14.750455501487203*^-6130
- i), {3.1622776601683795`,

99999999868508013930708412236923590398162452777681302658729927~7571.63151% 4537781 +

- 1.8074268800907733536206570674617707399949541143781~14.750453464284291*^-7 561 i}, {3.9810717055349776`,

99954213240177996767483084468588905421069782392876538`9193.08970303599+

3.018398517165474216474583577295165136590228813~14.750461959001617*^-9183

i), {5.011872336272725`,

36602310626496`11006.338025543002+

- 5.9691905565223974077208168773216453176877991`14.750442358384815*^-10996 i), {6.309573444801943`,

78344677660243079715~13011.343546785583+

- 7.892198515857649194892648679010697027783209`14.750433055105809*^-13001 il, {7.943282347242821,

99999999999999999999999630445139427130226012027926397038893972640273`152: 09.18266447023 +

- 1.14728690415930658197824494003733690663782`14.75043292019074*^-15198 i}, {10.`,

```
17749758440470909~17605.111037298033+
      1.6548757005928404625130224896339642<sup>14</sup>.75042469157325*^-17594 i}};
alltheory = \{\{0.00001^{\circ}, 0.00007353868931081475^{\circ} + 0.^{\circ} i\},
   {0.000012589254117941661, 0.00009269683280133112+0. i},
   {0.00001584893192461114, 0.00011688016643474075+0. i},
   {0.000019952623149688786, 0.00014742545042500943+0. i},
   \{0.000025118864315095822^{\circ}, 0.00018603514306725355^{\circ} + 0.^{\circ} i\},
   \{0.000031622776601683795^{,}0.00023488281227146413^{+}0.^{\dot{1}}\},
   {0.000039810717055349695, 0.00029675175164667333+0. i},
   \{0.00005011872336272725^{,}0.00037521859908717925^{+}0.^{i}\},
   \{0.00006309573444801929^{,}, 0.00047489861384109044^{+}, 0.^{i}\}
   \{0.00007943282347242822^{\circ}, 0.0006017764515722224^{\circ} + 0.^{\circ}i\},
   \{0.0001^{\circ}, 0.0007636566304430881^{\circ} + 0.^{\circ}1\},
   {0.00012589254117941674, 0.0009707838076260522 + 0. i},
   {0.00015848931924611142, 0.0012367065256327914+0. i},
   \{0.00019952623149688788^{\circ}, 0.0015794943486258095^{\circ} + 0.^{\circ} i\},
   {0.0002511886431509582, 0.0020234738347264525+0.i},
   \{0.00031622776601683794^{,}0.002601735063939792^{+}0.^{\dot{1}}\},
   \{0.00039810717055349735^{,}0.003359795530018528^{+}0.^{\dot{1}}\},
   \{0.0005011872336272725^{\circ}, 0.00436102179747655^{\circ} + 0.^{\circ} i\},
   {0.000630957344480193, 0.005694750129110915+0. i},
   \{0.0007943282347242822^{\dagger}, 0.0074885938616378045^{\dagger} + 0.^{\dagger} \dot{n}\},
   \{0.001^{\circ}, 0.009927309374558435^{\circ} + 0.^{\circ} i\},
   \{0.0012589254117941675^{,}0.013282014848632229^{+}0.^{i}\},
   \{0.001584893192461114^{\circ}, 0.017955846125511266^{\circ} + 0.^{\circ} i\},
   \{0.0019952623149688807^{,}0.024555720893033373^{+}0.^{i}\},
   \{0.002511886431509582^{,}0.03400527987740665^{+}0.^{\pm}\},
   \{0.0031622776601683794^{,}0.04772128169536172^{+}0.^{i}\},
   \{0.003981071705534973^{,}0.06788259385275097^{+}0.^{i}\},
   \{0.005011872336272725^{\circ}, 0.09781776056513802^{\circ} + 0.^{\circ}i\},
   {0.006309573444801936`, 0.14249197168917993`+0.`i}, {0.00794328234724282`,
    0.20889963138070355 + 0. i}, {0.01, 0.30567121904306294 + 0. i},
   \{0.012589254117941675^, 0.44013821788519225^+ 0.^i\},
   {0.01584893192461114`, 0.6099676829216958` + 0.` i},
   {0.01995262314968881, 0.7894961882297732+0. i},
   \{0.025118864315095822^{,}, 0.9275635037742979^{+}0.^{i}\},
   \{0.03162277660168379^{\circ}, 0.9887901345810385^{\circ} + 0.^{\circ} i\},
   \{0.039810717055349734^{,}, 0.999582611323144^{+}, 0.^{i}\},
   \{0.05011872336272725^{\circ}, 0.9999987763242779^{\circ} + 0.^{\circ}i\},
   \{0.06309573444801936^{\circ}, 0.999999999999609833^{\circ} + 0.^{\circ}i\},
   \{0.07943282347242822^{+}, 1.^{+} + 2.3688088639289025^{*} - 44 i\}
   \{0.1^{\prime}, 1.^{\prime} + 2.4618446884222435^{\prime} *^{\prime} - 44 i\},
   \{0.12589254117941687^{+}, 1.^{+}1.6926801174727181^{*}^{-}65i\},
   \{0.15848931924611143^{,} 1.^{+} 2.9170369302038582^{*}^{-}105i\},
   \{0.1995262314968879^{+}, 1.^{+} + 2.581892315725188^{+} - 172 i\},
```

- {0.25118864315095824`, 1.` + 9.713609093068933`*^-281 id}, {0.31622776601683794`,
- 90420303511738164667323983289738~455.78311261283443+
- 1.979445076079181588747079480675390037\15.954589770191005*^-445i},
- {0.39810717055349776`,
- 9999552278342387371582441427060756954572570740724947174386`694.002548004204: 1 +
- 7.56139689980818569403409064163738630047276131587`15.653559774527023*^-684 i), {0.5011872336272725`,
- 4080507524~1024.0061553823589+
- 7.624616097793328219876523574920157456011702012~15.653559774527023*^-1014 i), {0.6309573444801942`,

99999999999999999999867311756007885163710763033654347791`1463.2296895288 482 +

- 3.178843357332366043073089931788506614634633737902`15.352528478295362*^-1453 i), {0.7943282347242822`,
- 2336812079457436062579153841518289402 2030 964986575233 +
 - 5.8779844100207923403962984135080408418725`15.352528471520623*^-2021 i}, {1.`,

999999999999999999999999999994889372493776486042996911920674610400963832: ~2746.3430125381806+

- 1.75061804809622743868245294048842513443`15.051496528949595*^-2736 i), {1.2589254117941688`,

999999999999999803649256735711005472486587916980286779655394~3630.75845062 19 + 8.583317438446112601955328928333813004918`15.051495630249555*^-3621 i), {1.584893192461114`,

291572499367852~4705.392621374156+

- 2.31761143172762584505339722039110822334849046`15.05149495354723*^-4695 i}, {1.9952623149688828`,

50039403377`5991.516549667056+

- 9.534151258495109078002545419437881647355669215271`14.750463180173256*^-5982 i), {2.5118864315095824`,

074805353695105144147000796685784~7510.768191437132+

- 9.657008022279147956809002045266549460352031924525179`14.75045784078847*^-7: 501 i), {3.1622776601683795`,

562126~9282.89546513887+

- 8.181817787601263182096203053636934357905681900864512~14.750456218892575*^---9273 i}, {3.9810717055349776`,

76146496714845670560774~11326.925205782489+

- 3.6369527826289916083594838378824861001458117861275~14.750463328921363*^-11: 317 i}, {5.011872336272725`,

4833290189538751123347717364827231998500365819455`13661.437545868213+

- 3.87982199781847496886439850430715578262177109133`14.750447368805625*^-13651 i), {6.309573444801943`,

99999999999999992068709616278486504097397505718753611175521489`16305.85102 4116953 +

- 1.983471160429126499078553923091341318219995`14.750440089374996*^-16295 i}, {7.943282347242821`,

152605008585862528483138037945`19282.34086740881+

- 6.363805293415704697114641558089888108693988037904476`14.750440352613316*^-19272 i}, {10.\`,

999999999999999999999999999716556370266807752120831953837863`22618.11855 5049143+

1.2801817559858338633552424569094621615600684739077~14.750434295811687*^-22 607 i}};

Plot panel B

```
N0 = 10^4;
m0 = -0.2;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
letter = "B";
dat = Import[datadir <> ToString[StringForm[
        "prescue_poisson_N``_n``_Es``_mmax``_mwt``_mutmax10_nreps10000.csv",
        NumberForm[N0, 1], n, Es, NumberForm[mmax, {3, 2}], NumberForm[m0, {3, 2}]]]];
dat[[All, {1}]] = dat[[All, {1}]] * Uc /. Uc \rightarrow n<sup>2</sup> \lambda / 4 /. \lambda \rightarrow 2 Es / n;
figS1b = Show
  \texttt{ListLogLogPlot} \Big[ \texttt{Re} \big[ \texttt{theory} \big] \text{, Joined} \rightarrow \texttt{True} \text{, PlotStyle} \rightarrow \texttt{Thick},
    PlotRange \rightarrow \{\{10^{-5}, 1\}, \{10^{-6}, 1\}\},\
    Frame → {True, True, False, False},
    FrameLabel → {, "Probability of rescue", ,},
    FrameTicks → {True, True, False, False},
    LabelStyle → labelstyle,
    Epilog →
     {Text[Style[letter, 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
       Text[Style[StringForm["m_0 =  \ ", m0], 12, FontFamily \rightarrow "Helvetica"],
        Scaled@{2/10,9.5/10}]},
    FrameTicksStyle → {FontColor → White, Automatic, Automatic}
  ListLogLogPlot[Re[alltheory], Joined → True, PlotStyle → {Thick, Black}],
  ListLogLogPlot[dat, PlotMarkers → {Automatic, Medium}, PlotStyle → Black]
Clear [mmax, \lambda, Es, n, m0, N0, U, letter]
     0.1
Probability of rescue
    0.01
   0.001
```

 10^{-4}

 10^{-5}

 10^{-6}

Numerical data for panel C

```
(*N0=10^4;
m0 = -0.3;
mmax=0.5;
\lambda = 2Es/n;
Es=0.01;
n=4;
rate4=Table [Λ4InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate3=Table[A3InterpolatedU[m0,mmax,\lambda,n,x],{x,Umin,Umax,Ustep}];
rate2=Table[Λ2InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
rate1=Table[Λ1InterpolatedU[m0,mmax,λ,n,x],{x,Umin,Umax,Ustep}];
Ulist=Table[10*, {x,Umin,Umax,Ustep}];
theory={
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate1[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate2[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0-prescuem[m0,rate3[[i]]]},{i,Length[Ulist]}],
  Table[{Ulist[[i]],prescue/.p0→prescuem[m0,rate4[[i]]]},{i,Length[Ulist]}]
alltheory=Table[{Ulist[[i]],
   prescue/.p0-prescuem[m0,rate1[[i]]+rate2[[i]]+rate3[[i]]+rate4[[i]]]},{i,
   Length[Ulist] }];
Clear [mmax, \lambda, Es, n, m0, N0, U] *)
theory = {{{0.00001}, 6.425313789915066`*^-7},
    {0.000012589254117941661, 8.088992932142602, *^-7}, {0.00001584893192461114,
     1.018343781855613`*^-6}, {0.000019952623149688786`, 1.2820181188866187`*^-6},
    {0.000025118864315095822, 1.6139654472890896**^-6},
    {0.000031622776601683795, 2.031861507045818**^-6},
    {0.000039810717055349695, 2.557961690463628**^-6},
    {0.00005011872336272725, 3.220282207117009**^-6},
    {0.00006309573444801929, 4.054093288852201,*^-6},
    {0.00007943282347242822, 5.103798812799987, *^-6},
    {0.0001, 6.425297459533574**^-6}, {0.00012589254117941674,
     8.08896460335884`*^-6}, {0.00015848931924611142`, 0.000010183391189189095`},
    {0.00019952623149688788, 0.000012820113925116061,
    {0.0002511886431509582`, 0.00001613954059820344`},
    {0.00031622776601683794, 0.00002031843485172935},
    {0.00039810717055349735, 0.00002557932584146272},
    {0.0005011872336272725, 0.00003220235756073997},
    {0.000630957344480193, 0.00004054019664556652},
    {0.0007943282347242822`, 0.00005103681593787712`},
    {0.001, 0.00006425111905661307}, {0.0012589254117941675,
     0.00008088669342876109`}, {0.001584893192461114`, 0.00010182924751034328`},
    {0.0019952623149688807`, 0.0001281937380451792`},
    {0.002511886431509582, 0.00016138367485374339},
    {0.0031622776601683794, 0.00020316576204426617},
    {0.003981071705534973, 0.0002557638106833737},
    {0.005011872336272725, 0.0003219768956345126},
    {0.006309573444801936, 0.00040532799164916256},
    {0.00794328234724282`, 0.000510250921635591`}, {0.01`, 0.000642325398099941`},
    {0.012589254117941675`, 0.0008085724878367229`},
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 {0.03162277660168379, 0.0020298005758730975},
 {0.039810717055349734`, 0.0025546954574532377`},
 \{0.05011872336272725^{\circ}, 0.0032151062945346975^{\circ}\},
 {0.06309573444801936`, 0.00404589236538766`},
 {0.07943282347242822`, 0.005090805230013262`},
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 {0.15848931924611143, 0.010131750671247608},
 {0.1995262314968879`, 0.012738340082092292`},
 {0.25118864315095824, 0.016010080931823767}, {0.31622776601683794,
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 {0.5011872336272725`, 0.03168971719900204`}, {0.6309573444801942`,
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 {1.`, 0.06223181354444862`}, {1.2589254117941688`, 0.07770382531038755`},
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 {3.9810717055349776`, 0.22569206110805295`}, {5.011872336272725`,
  0.27531012647733377`}, {6.309573444801943`, 0.33327845755254115`},
{7.943282347242821`, 0.39970722185453755`}, {10.`, 0.47400472671919136`}},
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 {0.000019952623149688786, 3.719247076983123**^-8},
 {0.000025118864315095822, 5.82034407337062,*^-8},
 {0.000031622776601683795, 9.106826037719173**^-8},
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 \{0.00005011872336272725^{,} 2.2282284772856542^{*^-7}\},
 {0.00006309573444801929`, 3.484489891070197`*^-7},
 {0.00007943282347242822, 5.448007234232222**^-7},
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 {0.0031622776601683794, 0.0006739423615106377},
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 {0.005011872336272725, 0.0016324246574039858},
 {0.006309573444801936`, 0.0025386974323072797`},
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 {0.01584893192461114, 0.014723640491375867},
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 {0.025118864315095822`, 0.035093089130379784`},
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{0.05011872336272725, 0.12442366513963876},
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{0.15848931924611143, 0.688372124798042},
{0.1995262314968879, 0.8337357597192929}, {0.25118864315095824,
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{0.39810717055349776`, 0.9984684299695721`}, {0.5011872336272725`,
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{5.011872336272725`, 1.`}, {6.309573444801943`,
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{7.943282347242821`,
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\{\{0.00001^{\dagger}, 1.554312234475219^{\star} -11 + 0.^{\dagger}\},
{0.000012589254117941661`, 3.108624468950438`*^-11+0.` i},
\{0.00001584893192461114^{,}, 5.995204332975845^{*}-11+0.^{i}\},
\{0.000019952623149688786^{,}1.1657341758564144^{*}-10+0.^{i}\},
\{0.000025118864315095822^{, 2.275957200481571^*^-10+0.^{i}},
\{0.000031622776601683795^{,}4.418687638008123^{*}-10+0.^{\dot{1}}\},
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\{0.00005011872336272725^{\dagger}, 1.6686652060116103^{*}-9+0.^{\dagger}\},
\{0.00006309573444801929^{,}3.2318592246838307^{*^-9} + 0.^{i}\}
\{0.00007943282347242822^{,} 6.247224959565756^{*}-9+0.^{i}\}
\{0.0001^{\dagger}, 1.2060352716503075^{\star}-8+0.^{\dagger}\},
\{0.00012589254117941674^{,} 2.324695980160385^{*}-8+0.^{i}\},
\{0.00015848931924611142^{,}4.475419956850857^{*^-8+0.^{i}},
\{0.00019952623149688788^{,}8.605560386509836^{*^-8+0.^{i}},
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- 583842139435116765709 \ 953.1194818377461 +
- 4.280676039726981980747762940737726538068790344170356`15.653559774527023*^: -950 i}, {2.5118864315095824`,
- 99999999999999999999999999999944127121170068059426508292114614751247 9~1382.6053287300592+
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 - i), {3.9810717055349776`,
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- {5.011872336272725,

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- 6.957492505255906295461578818412256217009408734~15.051499783195817*^-3596 i), {6.309573444801943`,

81261256588216789900403641403817111602957~4739.7222892764585+

- 1.856969125620337932138928134888490586415379`15.051499783190762*^-4735 i), {7.943282347242821,

761955960710872333509686`6133.399938013075+

2.618001215353933015878995671400061376`14.750469787521325*^-6129 il}, {10.`,

```
818.4750128539235 +
2.2844175812487870054514634264022168465500438392`14.750469787520727*^-7814
 i)}, {{0.00001, 0. + 0. i}, {0.000012589254117941661, 0. + 0. i},
{0.00001584893192461114`, 0.`+0.`i}, {0.000019952623149688786`, 0.`+0.`i},
\{0.000025118864315095822^{1.1102230246251565^{4}-12+0.^{1}},
\{0.000031622776601683795^{,}1.1102230246251565^{*^-12}+0.^{i}\},
\{0.000039810717055349695^{,}3.3306690738754696^{*^-12}+0.^{i}\},
\{0.00005011872336272725^{,}7.771561172376096^{*}-12+0.^{i}\},
\{0.00006309573444801929^{,}1.887379141862766^{*}-11+0.^{i}\},
{0.00007943282347242822, 4.551914400963142**-11+0. i},
\{0.0001^{\dagger}, 1.0769163338864018^{\star} -10 + 0.^{\dagger}\},
\{0.00012589254117941674^{,} 2.575717417130363^{*}-10+0.^{i}\},
\{0.00015848931924611142^{,}6.139533326177116^{*}-10+0.^{i}\},
\{0.00019952623149688788^{, 1.454392162258955^{*^-9} + 0.^{i}\},
\{0.0002511886431509582^{, 3.43836070726411^{*^-9} + 0.^{i}\},
\{0.00031622776601683794^{,}8.094636072542016^{*^-9}+0.^{i}\},
\{0.00039810717055349735^{1.898370338704325^{*^-8} + 0.^{i}\}
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\{0.000630957344480193^{,}1.031596985612282^{*^-7} + 0.^{i}\}
\{0.0007943282347242822^{, 2.39186419892512^{*^-7} + 0.^{i}\},
\{0.001^{\circ}, 5.528032063217836^{*}-7+0.^{\circ}i\},
\{0.0012589254117941675^{, 1.2737014484232034^{*^-6+0.^{i}}\}
\{0.001584893192461114^{,} 2.9256610027950103^{*}-6+0.^{i}\},
```

```
\{0.0019952623149688807^{,}6.698744719368932^{*}-6+0.^{i}\},
\{0.002511886431509582^{,}0.000015286385312895945^{+}0.^{\dot{1}}\},
\{0.0031622776601683794^{\circ}, 0.00003475909339045735^{\circ} + 0.^{\circ} i\},
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\{0.006309573444801936^{\circ}, 0.0003989857795272389^{\circ} + 0.^{\circ} i\},
\{0.00794328234724282^{\dagger}, 0.0008919245374604534^{\dagger} + 0.^{\dagger}i\},
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\{0.012589254117941675^{,0.004384909537185067^{,+0.^{i}}\}
\{0.01584893192461114^{\circ}, 0.009625975972992307^{\circ} + 0.^{\circ} i\},
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{0.03162277660168379, 0.09460326538138408+0. i},
\{0.039810717055349734^{\circ}, 0.19194325088236586^{\circ} + 0.^{\circ}i\},
\{0.05011872336272725^{\circ}, 0.3646632823623164^{\circ} + 0.^{\circ}i\},
\{0.06309573444801936^{\circ}, 0.616211017620879^{\circ} + 0.^{\circ}i\},
{0.07943282347242822`, 0.8652565489948346` + 6.86480542158121`*^-33 i},
\{0.1^{\circ}, 0.9843254695918204^{\circ} + 4.944427652253176^{\circ} *^{-}18 i\},
{0.12589254117941687, 0.9998018994576795+8.70662587436642**^-16 i},
{0.15848931924611143`, 0.9999999690628478` + 1.0609054519606643`*^-17 i},
{0.1995262314968879`, 0.9999999999999`+1.3295337783362737`*^-24i},
\{0.25118864315095824^{,} 1.^{+} 3.433760447134135^{*}-38i\},
\{0.31622776601683794^{,}1.^{+}2.5727910003219743^{*}-65i\},
\{0.39810717055349776^{+}, 1.^{+} + 2.8676653039367497^{*} - 115 i\},
\{0.5011872336272725^{\circ}, 1.^{\circ} + 4.680366506440045^{\circ} *^{-}204 \text{ i}\}, \{0.6309573444801942^{\circ}, 1.6444801942^{\circ}\}
8313131556438317 \ 361.8553283626163 +
 6.549461433460595882670450894037620584676104037270159`15.954589770191005*^\
  -353 i}, {0.7943282347242822`,
86159626729728223192836100592996684839711406`597.348078771817+
 2.55650230634637484115856407536728569331871756964`15.954589770191005*^-588
 i}, {1.`,
```

975537067382287418299`946.3907448908188+

- 1.80007298843587418487723087386396319738356961`15.653559774527023*^-937 i), {1.2589254117941688`,
- 99999999999946347343426762092996710108174246018602275494`1435.226551757 1977 +
 - 1.74860563894107988844684741210950969828`15.352529707838459*^-1426 i), {1.584893192461114`,

918305353575855049794201131256374035426687474144~2087.440335678966+

- 1.26012185934870972033141381751472515287731`15.352529695126812*^-2078 i), {1.9952623149688828`,
- 999999999999999999999375995068168981436996654750539899491615`2923.256311. 113066 +
- 1.0640467374363521400017100729332260674449073655424466`15.051499621202774* ^-2914 i}, {2.5118864315095824`,

99999999999999999999999745905844402632297555060090246729587757121~3961.6% 465039101327 +

- 8.00433793948543151599752605819826901`15.051499483928996*^-3953
- i), {3.1622776601683795`,

9999999999999999999956355462900224192393130338966417411245`5217.1105384 41931 +

- 1.558550846791585706498162846653020227237935456028252`14.750469363469087*^\ -5208 i), {3.9810717055349776`,

9999999999999999999999999400603708962153531828190357039303051`6700.7682 247323455 +

- 2.674110302342741777141672303855951841605995307081359`14.750469456725629*^\ -6692 i), {5.011872336272725`,

629`8417.12063107332+

- 2.6530696166418445051747103355565664049053388445941`14.75046904868732*^-84 08 i), {6.309573444801943`,

766080576765282258887659`10364.987685395936+

- 4.41740708961220419545068823849221496644646878261`14.75046888174273*^-10356 i), {7.943282347242821`,

999999999999999970820992370299676327359137902009176404`12538.28539611174 5 +

- 2.263381438061784040509916930251108308352519698480295`14.750468866386253*^\ -12529 i, {10.\`,

```
149 14928 691454512516 +
   1.10036598677555690205800295780627328777961313571916`14.750468646843236*^-
    14919 i}};
alltheory = {{0.00001, 6.522424942367877*^-7+0. i},
 \{0.000012589254117941661^{3}, 8.241093366612162^{4}-7+0.^{1},
 \{0.00001584893192461114^{,} 1.0421669225380015^{*^-6+0.^{i}}\}
 {0.000019952623149688786`, 1.3193282266676931`*^-6+0.`i},
 \{0.000025118864315095822^{, 1.6723974951560194^{*,-6+0.^{i}}\}
 {0.000031622776601683795, 2.1233736705594808**^-6+0. i},
 \{0.000039810717055349695^{,}2.701287778905659^{*^-6}+0.^{i}\},
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 \{0.00006309573444801929^{\circ}, 4.405791574679796^{\star} - 6 + 0.^{\circ} \pm \},
 \{0.00007943282347242822^{,}, 5.654889480677383^{*^-6} + 0.^{\dot{1}}\}
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 \{0.001^{\circ}, 0.0001462411717603196^{\circ} + 0.^{\circ}i\},
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 \{0.001584893192461114^{\circ}, 0.00031237150299834315^{\circ} + 0.^{\circ}i\},
 \{0.0019952623149688807^{,}0.0004684516151626239^{+}0.^{\dot{1}}\}
```

```
\{0.002511886431509582^{,}0.000715191584378716^{+}0.^{i}\},
{0.0031622776601683794, 0.0011120790901624211+0. i},
\{0.003981071705534973^{\circ}, 0.0017618873994811324^{\circ} + 0.^{\circ} i\},
\{0.005011872336272725^{\circ}, 0.002845094326627051^{\circ} + 0.^{\circ} i\},
\{0.006309573444801936^{\circ}, 0.004683574129253243^{\circ} + 0.^{\circ}i\},
{0.00794328234724282, 0.007859450549812408+0. i},
\{0.01^{\circ}, 0.013437158034683927^{\circ} + 0.^{\circ}i\},
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\{0.01584893192461114^{,}, 0.041263601475378^{+}, 1.564893192461114^{,}, 0.041263601475378^{+}, 1.564893192461114^{,}, 0.041263601475378^{,}
\{0.01995262314968881^{\circ}, 0.07356939955763908^{\circ} + 0.^{\circ}i\},
\{0.025118864315095822^{,}0.1313616570542161^{+}0.^{i}\},
\{0.03162277660168379^{,}0.23144690353800978^{+}0.^{i}\},
{0.039810717055349734, 0.39226297632515994+0. i},
\{0.05011872336272725^{\circ}, 0.613743271143804^{\circ} + 0.^{\circ}i\},
\{0.06309573444801936^{\circ}, 0.8395897950578348^{\circ} + 0.^{\circ}i\},
{0.07943282347242822`, 0.9708993564874809`+1.4818406183380653`*^-33i},
{0.1, 0.9989412696807748 + 3.3367019574748103 * ^ - 19 i},
{0.12589254117941687, 0.9999982616792098+7.628006216915802**-18 i},
{0.15848931924611143`, 0.999999999923229` + 2.6254259232651074`*^-21 i,},
\{0.1995262314968879^{+}, 1.^{+} + 6.818559425001438^{+} - 31 i\},
\{0.25118864315095824^{,}1.^{+}4.578605834220335^{*}-49i\},
\{0.31622776601683794^{+}, 1.^{+} + 7.46949882281936^{+} - 84 i\},
\{0.39810717055349776^{+}, 1.^{+} 3.5246244259058307^{*}-146 i\},
{0.5011872336272725`, 1.` + 5.4228587885864995`*^-254 i}, {0.6309573444801942`,
4304430373694~439.2736496325759+
 2.387571618452056003444834975123947293958717904639768`15.954589770191005*^-
  430 i), {0.7943282347242822`,
99999999999999999999918596307308820306487416889476564434`711.7429155389339+
 4.854194514604713184312418336516634239446371747682`15.653559774527023*^-703
 i}, {1.`,
```

560291862332~1109.118669016324+

- 3.139246795877328143671639671107141653`15.653559774527023*^-1100 i), {1.2589254117941688`,
- 3355620640090943107475217140354486845234041566`1658.198840510377+
- 1.7227380330462907506212930133350062024892008775`15.35252971321454*^-1649 i}, {1.584893192461114`,

413448674~2385.5898049738853+

- 8.20688226074579045991425928376617509648155142834813`15.352529701923736*^-2: 377 i}, {1.9952623149688828,

98397221724089707404499612391842700875846139526026274~3315.8466257431082+ 2.499974904863851284591997273710314416721139125`15.051499635017365*^-3307

- i), {2.5118864315095824`,

234126398 4474.010150123129 +

- 3.1567069122030296606167988239206745419523542`15.051499510541094*^-4465 i), {3.1622776601683795`,

999999999999999985422323433211021990782883760026580668968108732`5882.5867 80158169 +

- 4.7225915396612605147387320322274309232708~14.750469402824116*^-5874 i}, {3.9810717055349776`,

1105000457166247~7563.002642827339+

- 1.406856143760017301032239131893529944`14.75046948895129*^-7554 i), {5.011872336272725`,

7636632813167587930456264922442~9532.85106442276+

- 4.426463600901113479617753980451128432089813340210718`14.750469124866694*^-: 9524 i), {6.309573444801943,

9999999999999997505737058769073452266815526728837352`11807.35352478981+ 1.69297888204994600951838797083851028160700839831653~14.75046898150409*^-11:

- 798 i), {7.943282347242821`,

5272753266883530893727184939994~14400.587145903299+

9.95712643614615469567320877195902353814~14.75046897572326*^-14392i}, {10.~,

5341367315845292178648152944073249`17328.376667979843+

1.978767758281649622793477614434503003587`14.750468793872267*^-17319 il};

Plot panel C

 10^{-4}

 10^{-5}

 10^{-6}

 10^{-5}

0.001

Mutation probability

 10^{-4}

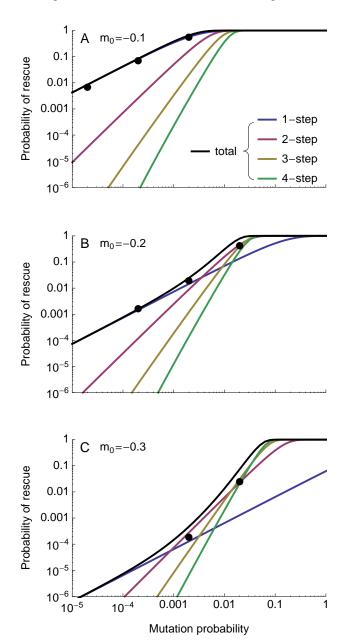
0.01

```
N0 = 10^4;
m0 = -0.3;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
letter = "C";
dat = Import[datadir <> ToString[StringForm[
        "prescue_poisson_N``_n``_Es``_mmax``_mwt``_mutmax10_nreps10000.csv",
       NumberForm[N0, 1], n, Es, NumberForm[mmax, {3, 2}], NumberForm[m0, {3, 2}]]]];
dat[[All, \{1\}]] = dat[[All, \{1\}]] * Uc /. Uc \rightarrow n^2 \lambda / 4 /. \lambda \rightarrow 2 Es / n;
figS1c = Show
  ListLogLogPlot [Re[theory], Joined → True, PlotStyle → Thick,
    PlotRange \rightarrow \{\{10^{-5}, 1\}, \{10^{-6}, 1\}\},\
    Frame → {True, True, False, False},
   FrameLabel → {"Mutation probability", "Probability of rescue", ,},
   FrameTicks → {True, True, False, False},
   LabelStyle → labelstyle,
   Epilog →
     {Text[Style[letter, 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
      \texttt{Text[Style[StringForm["$m_0$=``", m0], 12, FontFamily} \rightarrow "\texttt{Helvetica"],}
       Scaled@{2/10,9.5/10}]}
  ],
  ListLogLogPlot[Re[alltheory], Joined → True, PlotStyle → {Thick, Black}],
  ListLogLogPlot[dat, PlotMarkers → {Automatic, Medium}, PlotStyle → Black]
Clear[mmax, \(\lambda\), Es, n, m0, N0, U, letter]
              m_0 = -0.3
     0.1
Probability of rescue
    0.01
   0.001
```

0.1

Figure S1

 $\texttt{GraphicsColumn[\{figS1a, figS1b, figS1c\}, Spacings} \rightarrow -30, \texttt{ImageSize} \rightarrow 350]$ Export[supfigdir <> "FigureS1.eps", %];



Approximating the probability of 1-step rescue

Replicating Anciaux et al 2018, Genetics

Change of variables

We first introduce the variables y = m/mmax, ywt = mwt/mmax, $\rho max = mmax/\lambda$, $\theta = n/2$. The distribution of y among new mutations is then

```
fy = e^{-(2-y-ywt) \rho max} \rho max^{\theta} (1-y)^{\theta-1}
      Hypergeometric0F1Regularized[\theta, (1 - y) (1 - ywt) \rhomax<sup>2</sup>];
Simplify \left[ fy == \frac{fm \left[ y \, mmax, \, ywt \, mmax, \, mmax, \, mmax \, / \, \rho max, \, 2 \, \theta \right]}{D \left[ \frac{m}{mmax}, \, m \right]}, \, \left\{ \rho max > 0 \, , \, 0 < y < 1 \right\} \right]
fy /. y \rightarrow m / mmax /. ywt \rightarrow mwt / mmax /. \rhomax \rightarrow mmax / \lambda /. \theta \rightarrow n / 2;
Simplify[%D[m / mmax, m] == fm[m, mwt, mmax, \lambda, n], {mmax > 0, \lambda > 0}]
True
True
```

Approximate hypergeometric function

We next want to approximate the hypergeometric function in fy.

Note first that Hypergeometric0F1Regularized[θ ,z] is defined in *Mathematica* as Hypergeomet $ric0F1[\theta,z]/Gamma[\theta]$:

```
Hypergeometric0F1Regularized[\theta, z];
Hypergeometric0F1[\theta, z] / Gamma[\theta];
FullSimplify[% / %%]
1
```

Next note that Hypergeometric0F1[θ ,z] can be written as $\frac{\text{Gamma}\left[\theta\right]}{\left(\sqrt{z}\right)^{\theta-1}}$ BesselI $\left[\theta-1,\ 2\sqrt{z}\right]$ using

9.6.47 of Abramowitz and Stegun (1964):

```
Hypergeometric0F1[\theta, z];
 \frac{1}{\left(\sqrt{z}\right)^{\theta-1}} Bessell \left[\theta-1, 2\sqrt{z}\right];
FullSimplify[%/%%]
```

Thus, the Hypergeometric0F1Regularized[θ ,z] can be written as $\frac{1}{\left(\sqrt{z}\right)^{\theta-1}}$ Bessell[θ -1, 2 \sqrt{z}]:

Hypergeometric0F1Regularized[θ , z];

$$\frac{1}{\left(\sqrt{z}\right)^{\theta-1}} \text{ Bessell} \left[\theta-1, 2\sqrt{z}\right];$$

FullSimplify[%/%%]

1

Finally, 9.7.1 of Abramowitz and Stegun (1964) gives an asymptotic expansion for Bessell that holds for large |z|:

 $\texttt{BesselI}[\theta - 1, \ 2 \ \texttt{Sqrt}[\mathbf{z}]] \ /. \ \texttt{BesselI} \rightarrow \texttt{Function}\Big[\{\mathbf{v}, \ \mathbf{x}\}\,,$

$$\frac{e^{x}}{\sqrt{2\pi x}} \left(1 - \frac{\mu - 1}{8x} + \frac{(\mu - 1)(\mu - 9)}{2!(8x)^{2}} - \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3!(8x)^{3}} + \text{added} \right) / \cdot \mu \rightarrow 4 \text{ v}^{2} \right];$$

where the k^{th} term added to "1" is obtained by taking the previous term and multiplying by $-\frac{\left(\mu-\left(2\ k-1\right)^2\right)}{k\left(8\ z\right)}$. For large z, the Bessel function will be dominated by the leading term:

$$\text{BesselI}[\theta-1, 2 \, \text{Sqrt}[z]] \ /. \ \text{BesselI} \rightarrow \text{Function}\Big[\{v, x\}, \frac{e^x}{\sqrt{2 \, \pi \, x}}\Big]$$

$$\frac{e^{2\sqrt{z}}}{2\sqrt{\pi}z^{1/4}}$$

This allows us to conclude that

 $\label{eq:hypergeometricOF1Regularized} \text{HypergeometricOF1Regularized} \left[\theta \text{, z}\right] = \frac{1}{\left(\sqrt{\mathtt{z}}\right)^{\theta - 1}} \text{ BesselI}\left[\theta - 1 \text{, 2}\sqrt{\mathtt{z}}\right] \text{ can be approximately approxima$

$$\text{mated for z large as } \frac{1}{\left(\sqrt{z}\right)^{\theta-1}} \; \frac{e^{2\sqrt{z}}}{2\sqrt{\pi} \; z^{1/4}}, \text{ which upon rearranging gives } \quad \frac{e^{2\sqrt{z}} \; z^{\frac{1}{4} \, (1-2\,\theta)}}{2\,\sqrt{\pi}}.$$

We can therefore use the following approximation as ρ max = mmax/ λ goes to infinity, i.e., mutant growth rates are much less than maximal

$$\begin{split} &\text{fya} = \text{FullSimplify} \Big[\\ &\text{fy /. HypergeometricOFlRegularized} \rightarrow \text{Function} \Big[\left\{ \theta \text{, z} \right\}, \, \frac{1}{2 \sqrt{\pi}} \, z^{\frac{1-2\theta}{4}} \, e^{2 \sqrt{z}} \, \Big] \, / / \\ &\text{PowerExpand, 0 < y < 1 && \rho \text{max} > 0 && \text{ywt} < 0 && \theta \ge 1 \, / \, 2} \Big] \\ &\text{FullSimplify} \Big[\% = \frac{e^{-v[y]} \, \left(1 - y \right)^{\frac{\theta}{2} - \frac{3}{4}} \, \left(1 - y \text{wt} \right)^{\frac{1}{4} - \frac{\theta}{2}} \, \sqrt{\rho \text{max}}}{2 \, \sqrt{\pi}} \, / . \\ &v[y] \rightarrow \left(2 - y \text{wt} - 2 \, \sqrt{\left(1 - y \text{wt} \right) \, \left(1 - y \right)} \, - y \right) \, \rho \text{max}, \\ &\left\{ \rho \text{max} > 0 \, , \, y \text{wt} < 0 \, , \, 0 < y < 1 \, , \, \theta \ge 1 \, / \, 2 \right\} \, \Big[\, \left(* \text{compare to eqn A9*} \right) \\ &- \left(e^{\left(-2 + y + 2 \, \sqrt{\left(-1 + y \right) \, \left(-1 + y \text{wt} \right)} \, + y \text{wt} \right) \, \rho \text{max}} \, \left(\frac{-1 + y}{-1 + y \text{wt}} \right)^{\theta / 2} \, \left(-1 + y \text{wt} \right) \, \sqrt{\rho \text{max}} \, \right) / \\ &\left(2 \, \sqrt{\pi} \, \left(\left(-1 + y \right) \, \left(-1 + y \text{wt} \right) \right)^{3/4} \right) \end{split}$$

True

This does great for large ρ max, especially at the tails

```
\theta = 2; ywt = -0.2; \rhomax = {3, 10, 20}; fy;
{\tt Plot[\%, \{y, -3, 1\}, PlotRange \rightarrow All, PlotLegends \rightarrow LineLegend[}
       \texttt{Table[StringForm["$\rho_{\max}$ = ``", Round[$\rho$max[[i]]]], \{i, Length[$\rho$max]\}]]]"};
fya;
\texttt{Plot}[\$, \{y, -3, 1\}, \texttt{PlotRange} \rightarrow \texttt{All}, \texttt{PlotStyle} \rightarrow \{\{\texttt{Dashed}, \texttt{Black}\}\}];
Show[%%%, %, AxesLabel \rightarrow {y, "f<sub>y</sub>[y]"}]
fy /. y \rightarrow 0
Plot[%, {y, 0, 1}, PlotRange \rightarrow All, PlotLegends \rightarrow LineLegend[
       \texttt{Table[StringForm["$\rho_{max} = ``", Round[$\rho$max[[i]]]], \{i, Length[$\rho$max]$\}]]];}
\frac{}{\text{fya} /. y \rightarrow 0};
Plot[%, {y, 0, 1}, PlotRange \rightarrow All, PlotStyle \rightarrow {\{Dashed, Black\}}];
Show \left[ \%\%, \%, AxesLabel \rightarrow \left\{ y, "\frac{f_y[y]}{f_y[0]} " \right\} \right]
\theta = .; ywt = .; \rhomax = .;
                                                        f_y[y]
                                                      8,0
                                                                                          \rho_{\text{max}} = 3
                                                                                          \rho_{\text{max}} = 10
                                                      0.6
                                                                                          \rho_{\rm max} = 20
                                                      0.4
                                                      0.2
-3
                                      -1
  f_y[y]
  f_y[0]
1.0
0.8
                                                                                          -\rho_{\rm max} = 3
0.6
                                                                                          -\rho_{\rm max} = 10
                                                                                         -\rho_{\rm max} = 20
0.4
0.2
                 0.2
                               0.4
                                              0.6
                                                             8.0
```

Change of variables

We now consider the variable $\psi = 2\left(1 - \sqrt{1 - y}\right)$, which varies between 0 and 2 as y varies between 0 and 1, and is $2\frac{A}{B}$, where A is the phenotypic distance to the closest critical phenotype and B is the distance from a critical phenotype to the optimal

$$2\left(\sqrt{1-y}-1\right)/.\ y \to \frac{m}{mmax} \text{ // Expand}$$

$$FullSimplify \left[2\frac{\mathbf{x}_0 - \mathbf{x}_c}{\mathbf{x}_c} == \% \text{ /. Flatten} \left[Solve \left[\left\{ \sqrt{2 \text{ (mmax - m)}} =: \mathbf{x}_0 \text{ , } \sqrt{2 \text{ mmax}} =: \mathbf{x}_c \right\} \text{ , } \left\{ mmax, m \right\} \right] \right] \text{ // }$$

$$Expand, \ \mathbf{x}_0 > 0 \&\& \ \mathbf{x}_c > 0 \right]$$

$$-2 + 2\sqrt{1 - \frac{m}{mmax}}$$

True

The pdf of ψ is then approximately

$$\begin{split} & \text{Solve} \Big[\Big\{ 2 \left(1 - \sqrt{1 - y} \right) == \psi, \ 2 \left(1 - \sqrt{1 - ywt} \right) == \psi wt \Big\}, \ \{y, ywt\} \Big] \ // \ \text{Flatten} \\ & \text{Simplify} [\text{Abs}[\text{D}[y / \cdot \%, \psi]] \ \text{fya} \ / \cdot \%, \ \{0 < \psi < 2, \psi wt < 0, \rho \text{max} > 0, \theta \ge 1 \ / 2\} \Big] \\ & = \frac{e^{-\frac{1}{4} \rho \text{max} \left(\psi - \psi wt \right)^2} \sqrt{\rho \text{max}} \ \left(\frac{2 - \psi}{2 - \psi wt} \right)^{-\frac{1}{2} + \theta}}{2 \sqrt{\pi}}, \ \{0 < \psi < 2, \psi wt < 0, \rho \text{max} > 0, \theta \ge 1 \ / 2\} \Big] \\ & \{\psi = \frac{e^{-\frac{1}{4} \rho \text{max} \left(\psi - \psi wt \right)^2} \sqrt{\rho \text{max}} \ \left(\frac{2 - \psi}{2 - \psi wt} \right)^{-\frac{1}{2} + \theta}}{2 \sqrt{\pi}}; \\ & \left\{ y \to \frac{1}{4} \left(4 \ \psi - \psi^2 \right), \ ywt \to \frac{1}{4} \left(4 \ \psi \text{wt} - \psi \text{wt}^2 \right) \right\} \\ & - \frac{e^{-\frac{1}{4} \rho \text{max} \left(\psi - \psi \text{wt} \right)^2} \sqrt{\rho \text{max}} \ \left(\frac{-2 + \psi}{-2 + \psi \text{wt}} \right)^{1 + \theta} \left(-2 + \psi \text{wt} \right)^3}{2 \sqrt{\pi}} \end{aligned}$$

True

Note that this is a normal distribution, with mean ψ wt and variance $\frac{2}{\rho \text{max}}$, as $\left(\frac{-2+\psi}{-2+\psi \text{wt}}\right)^{-\frac{1}{2}+\Theta} \to 1$ (which is reasonable when the absolute values of ψ and ψ wt are much less than 2, i.e., growth rates small relative to max, or when θ is 1/2, i.e., when dimensionality is low).

$$\begin{split} & \text{f} \psi \, / \, \cdot \, \left(\frac{2 - \psi}{2 - \psi \text{wt}} \right)^{-\frac{1}{2} + \theta} \to 1; \\ & \text{Simplify} \Big[\% == & \text{Simplify} \Big[\text{PDF} \Big[\text{NormalDistribution} \Big[\psi \text{wt} \, , \, \sqrt{\frac{2}{\rho \text{max}}} \, \Big] \, , \, \psi \Big] \, , \, \rho \text{max} > 0 \Big] \, , \\ & \left\{ 0 < \psi < 2 \, , \, \psi \text{wt} < 0 \, , \, \rho \text{max} > 0 \, , \, \theta > 1 \, / \, 2 \right\} \Big] \end{split}$$
 True

Laplace approximation and compact form (equations 19 and 20)

Anciaux et al's equation A5 is the integral of the following over y from 0 to 1

$$\begin{split} &\frac{\textbf{U}}{-\textbf{mwt}} \, \textbf{pest[m] fy /. m} \rightarrow \textbf{mmax y} \\ &-\frac{1}{\textbf{mwt}} \, \textbf{e}^{(-2+y+y\text{wt}) \, \rho \text{max}} \, \left(1 - \textbf{e}^{-2 \, \text{mmax y}}\right) \, \textbf{U} \, \left(1 - \textbf{y}\right)^{-1+\theta} \, \rho \text{max}^{\theta} \\ &\quad \textbf{HeavisideTheta[mmax y] HypergeometricOF1Regularized[θ, (1-y) (1-y\text{wt}) ρmax^2]} \end{split}$$

which, using the approximation in equation A9, is nearly

$$\frac{\mathtt{U}}{-\mathtt{mwt}} \, \mathtt{pest[m] \, fya \, /. \, m \to mmax \, y} \\ \left(e^{\left(-2 + \mathtt{y} + 2 \, \sqrt{\, (-1 + \mathtt{y}) \, (-1 + \mathtt{ywt}) \,} \, + \mathtt{ywt} \right) \, \rho \max} \, \left(1 - e^{-2 \, mmax \, \mathtt{y}} \right) \, \mathtt{U} \left(\frac{-1 + \mathtt{y}}{-1 + \mathtt{ywt}} \right)^{\theta/2} \, \left(-1 + \mathtt{ywt} \right) \\ \sqrt{\rho max} \, \, \mathtt{HeavisideTheta[mmax \, \mathtt{y}]} \, \middle/ \, \left(2 \, \mathtt{mwt} \, \sqrt{\pi} \, \left(\, (-1 + \mathtt{y}) \, (-1 + \mathtt{ywt}) \, \right)^{3/4} \right) \\ \end{array}$$

which is roughly the integral of the following over ψ from 0 to 2 (A10)

$$\frac{\text{U}}{-\text{mwt}} \frac{\text{e}^{-\frac{1}{4}\rho \text{max} (\psi - \psi \text{wt})^2} \sqrt{\rho \text{max}} \left(\frac{2-\psi}{2-\psi \text{wt}}\right)^{-\frac{1}{2}+\theta}}{2 \sqrt{\pi}}$$

$$= \frac{\text{e}^{-\frac{1}{4}\rho \text{max} (\psi - \psi \text{wt})^2} \text{U} \sqrt{\rho \text{max}} \left(\frac{2-\psi}{2-\psi \text{wt}}\right)^{-\frac{1}{2}+\theta}}{2 \text{mwt} \sqrt{\pi}}$$

Anciuaux et al 2018 say we can write this as (A11)

$$\begin{split} & h[\psi_{-}] := \left(\frac{1 - \psi / 2}{1 - \psi wt / 2}\right)^{\theta - \frac{1}{2}} \left(1 - e^{-2 \max y}\right) / \cdot y \to \psi (1 - \psi / 4) \\ & q[\psi_{-}] := \frac{1}{4} (\psi - \psi wt)^{2} \\ & \frac{U}{-mwt} \frac{\sqrt{\rho max}}{2 \sqrt{\pi}} h[\psi] \exp[-\rho max \, q[\psi]] \\ & - \frac{1}{2 \, mwt \, \sqrt{\pi}} e^{-\frac{1}{4} \, \rho max \, (\psi - \psi wt)^{2}} \left(1 - e^{-2 \, mmax \, \left(1 - \frac{\psi}{4}\right) \, \psi}\right) \, U \, \sqrt{\rho max} \, \left(\frac{1 - \frac{\psi}{2}}{1 - \frac{\psi wt}{2}}\right)^{-\frac{1}{2} + \theta} \end{split}$$

which appears to be true only when we ignore this second term they have in h (but this will drop out anyway when we take the leading order in the next step)

Simplify
$$\left[\left(-\frac{1}{2 \text{ mwt } \sqrt{\pi}} e^{-\frac{1}{4} \rho \text{max } (\psi - \psi \text{wt})^2} \left(1 - e^{-2 \text{ mmax } \left(1 - \frac{\psi}{4} \right) \psi} \right) \text{ U } \sqrt{\rho \text{max}} \left(\frac{1 - \frac{\psi}{2}}{1 - \frac{\psi \text{wt}}{2}} \right)^{-\frac{1}{2} + \theta} \right) \right]$$

$$\left(\frac{\text{U}}{-\text{mwt}} \frac{e^{-\frac{1}{4} \rho \text{max } (\psi - \psi \text{wt})^2} \sqrt{\rho \text{max}} \left(\frac{2 - \psi}{2 - \psi \text{wt}} \right)^{-\frac{1}{2} + \theta}}{2 \sqrt{\pi}} \right) \right]$$

As ρ max=mmax/ λ goes to ∞ the Exp term dominates and we take the leading order of h

$$\texttt{h0 = Normal[Series[h[}\psi\texttt{], }\{\psi\texttt{, 0, 1}\}\texttt{]]}$$

$$2 \operatorname{mmax} \psi \left(\frac{1}{1 - \frac{\psi \operatorname{wt}}{2}} \right)^{-\frac{1}{2} + \Theta}$$

Now taking the integral over ψ from 0 to ∞ we get A12

Integrate
$$\left[\frac{U}{-mwt} \frac{\sqrt{\rho max}}{2\sqrt{\pi}} \text{ h0 Exp}[-\rho max q[\psi]], \{\psi, 0, \infty\},\right]$$

Assumptions $\rightarrow \{\rho \max > 0, \psi \text{wt} < 0\}$ /. $\text{mwt} \rightarrow \psi \text{wt} (1 - \psi \text{wt} / 4) \text{ mmax}$;

$$\text{FullSimplify} \Big[\% == U \, \frac{\left(1 - \frac{\psi \text{wt}}{2}\right)^{\frac{1}{2} - \theta}}{1 - \frac{\psi \text{wt}}{4}} \left(\frac{\text{Exp}[-\alpha]}{\sqrt{\pi \, \alpha}} - \text{Erfc} \Big[\sqrt{\alpha} \, \Big] \right) / \cdot \alpha \rightarrow \frac{\rho \text{max } \psi \text{wt}^2}{4} ,$$

$$\{\psi wt < 0, \rho max > 0, U > 0\}$$

True

AnciauxEqnA12 = U
$$\frac{\left(1 - \frac{\psi \text{wt}}{2}\right)^{\frac{1}{2} - \theta}}{1 - \frac{\psi \text{wt}}{\epsilon}} \left(\frac{\text{Exp}[-\alpha]}{\sqrt{\pi \alpha}} - \text{Erfc}[\sqrt{\alpha}]\right);$$

As m0 and thus ψ wt gets small, the rate of 1-step rescue becomes

Simplify [
$$Series \left[m0 \text{ AnciauxEqnA12 /. } m0 \rightarrow mmax \psi wt \left(1 - \psi wt / 4 \right) /. \alpha \rightarrow \frac{\rho max \psi wt^2}{4}, \left\{ \psi wt, 0, 0 \right\} \right] /.$$

$$\rho max \rightarrow mmax / \lambda / / \text{ Normal, } \left\{ mmax > 0, \lambda > 0 \right\} \right]$$

$$\frac{2 \text{ U } \sqrt{mmax \lambda}}{\sqrt{\pi}}$$

Approximating the probability of 2-step rescue

Defining "sufficiently critical" and "sufficiently non-critical" regimes (equation 8)

The probability of rescue from a new lineage with growth rate m and rescue rate Λ is

prescuem[m, A]

$$1 - e^{\left(1 - \sqrt{1 + \frac{2 \Lambda}{\text{Abs}[\mathfrak{m}]^2}}\right)} \text{Abs}[\mathfrak{m}]$$

Let's now consider single mutants with growth rates far from 0, such that $\Lambda(m) << m^2$. We can then approximate this by

$$Normal \left[Series \left[prescuem[m, \Lambda] \ /. \ \Lambda \rightarrow \varepsilon \ Abs[m]^2, \ \left\{ \varepsilon, \ 0, \ 1 \right\} \right] \right] \ /. \ \varepsilon \rightarrow \frac{\Lambda}{Abs[m]^2}$$

$$\frac{\wedge}{\mathsf{Abs}[\mathfrak{m}]}$$

Alternatively, consider single mutants with growth rates sufficiently near 0, such that $\Lambda(m) >> m^2$. We then have approximately

$$\texttt{Limit[prescuem[m, \Lambda], m} \rightarrow 0]$$

Normal [Series [% /.
$$\Lambda \rightarrow \mathbf{x}^2$$
, {x, 0, 1}]] /. $\mathbf{x} \rightarrow \sqrt{\Lambda}$

$$1-e^{-\sqrt{2}\sqrt{\Lambda}}$$

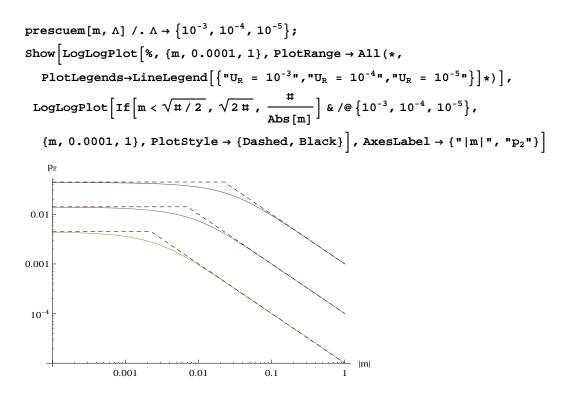
$$\sqrt{2} \sqrt{\wedge}$$

So we should transition from one approx to the other at

Solve
$$\left[\sqrt{2 \Lambda} = \Lambda / m, m \right] / flatten$$

$$\left\{ m \to \frac{\sqrt{\Lambda}}{\sqrt{2}} \right\}$$

Check approximations and transition point (for given Λ)



Approximate probability of rescue: sufficiently critical single mutants

House of Cards approximation (equation 9)

When $m \ll \sqrt{\Lambda/2}$ the probability of 2-step rescue from this single mutant lineage, as calculated above, is $\sqrt{2} \Lambda$. In this circumstance we can further approximate, since $\Lambda[m] \sim \Lambda[0]$, $f(m|m0) \sim f(0|m0)$, and pest(m)~0. So the rate of 2-step "sufficiently critical" rescue from single mutants is roughly

U Integrate
$$\left[\text{fm}[0, m0, mmax, \lambda, n] \sqrt{2 \Lambda 0}, \left\{m, -\sqrt{\Lambda 0/2}, \sqrt{\Lambda 0/2}\right\}\right]$$
; % == 2 U fm[0, m0, mmax, λ , n] $\Lambda 0$

with $\Lambda 0 = \Lambda 1[0, mmax, \lambda, n, U]$.

Check integration bounds:

```
U = 2 * 10^{-5};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{mneg} = \text{FindRoot}[2 m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, -0.1\}],
 mpos = FindRoot[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, 0.1\}]
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
\texttt{Plot}\Big[\Big\{\texttt{Abs}[\texttt{m}]\,,\,\texttt{pr0}\,,\,\sqrt{\texttt{\Lambda1}[\texttt{m},\,\texttt{mmax},\,\lambda,\,\texttt{n}\,,\,\texttt{U}]\,/\,2}\,\Big\}\,,\,\,\{\texttt{m}\,,\,\,2\,\texttt{m}\,/\,.\,\,\texttt{mneg}\,,\,\,2\,\texttt{m}\,/\,.\,\,\texttt{mpos}\}\,,
  PlotRange → {0, 1.2 pr0}, PerformanceGoal → "Speed"
Clear[mmax, \lambda, Es, n, n0, U]
 \{ \{ m \rightarrow -0.000655285 \}, \{ m \rightarrow 0.00066309 \} \}
                                    0.0007
                                    0.0006
                                    0.0005
                                    0.0004
                                    0.0003
                                    0.0002
                                    0.0001
        -0.0010
                        -0.0005
                                                         0.0005
                                                                         0.0010
```

Now check the contribution of single mutant growth rates to 2-step sufficiently critical rescue

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
m0 = -0.3;
\{\text{mneg = FindRoot}[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, -0.1\}],
  mpos = FindRoot[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   fm[m, m0, mmax, \lambda, n] (1-pest[m]) prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]],
   fm[m, m0, mmax, \lambda, n] (1-pest[m]) \sqrt{2 \Lambda 1[m, mmax, \lambda, n, U]}
    HeavisideTheta[(m - (m /. mneg)) ((m /. mpos) - m)],
   fm[0, m0, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m+pr0) (pr0-m)],
   fm[0, m0, mmax, \lambda, n] (1-pest[m]) prescuem[m, \Lambda1[m, mmax, \lambda, n, U]]
Plot[%, {m, 10 m/. mneg, 10 m/. mpos}, PlotRange \rightarrow {0, All}, Filling \rightarrow Bottom,
 PlotLegends → LineLegend[{"exact", "critical", "critical approx.", "HoC"}],
 PerformanceGoal → "Speed"]
Clear [mmax, \lambda, Es, n, U, m0]
                       0.00004
                       0.00003

    exact

    critical

                       0,00002
                                                                  critical approx.
                                                                    HoC
                       0.00001
  -0.04
               -0.02
                                         0.02
                                                      0.04
```

Finally, let's check the total rates of rescue across all m

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es/n;
Es = 0.01;
n = 4;
\{\text{mneg = FindRoot}[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, -0.1\}],
   mpos = FindRoot \left[2 m^2 - \Lambda 1 [m, mmax, \lambda, n, U], \{m, 0.1\}\right];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   Table\Big[\Big\{m0\,,\, NIntegrate\Big[fm\,[m,\,m0\,,\,mmax\,,\,\lambda\,,\,n]\,\,(1-pest\,[m]\,)\,\,\sqrt{2\,\Lambda 1\,[m,\,mmax\,,\,\lambda\,,\,n\,,\,U]}\,\,,
        \{m, m/. mneg, m/. mpos\}, \{m0, -0.8 mmax, -0.1 mmax, 0.1 mmax\}
  };
Show[
 ListLogPlot[%],
 LogPlot[2 fm[0, m0, mmax, \lambda, n] \Lambda1[0, mmax, \lambda, n, U],
    {m0, -0.8 mmax, -0.1 mmax}, PerformanceGoal → "Speed"]
]
Clear[mmax, \lambda, Es, n, U, n0]
  0.1
0.001
 10^{-5}
 10^{-7}
                    -0.3
                                    -0.2
                                                   -0.1
                                                                    0.0
```

Closed form approximation (equations 10 and 11)

OK, so a good approximation for sufficiently critical rescue from a mutant is

```
2 \Lambda 0 \text{ fm}[0, m0, mmax, \lambda, n];
with \Lambda 0 = \Lambda 1[0, mmax, \lambda, n, U]
```

but note that, while this has reduced a lot of complexity, $\Lambda 0 = U \int f(m \mid 0) p_{est}(m) dm$ is still an integral we have to compute. Fortunately however, we can just use the approximations Anciaux et al used (and we replicated above), so that when we take m=mmax $\psi(1-\psi/4)$ to zero we have $\Lambda 0$ as roughly

(*multiply by m because anciaux A12 accounts for number of individuals in lineage, which they estimate as 1/m*) m AnciauxEqnA12 /. ψ wt $\rightarrow \psi$ /. m \rightarrow mmax ψ (1 - ψ / 4) /. $\alpha \rightarrow \frac{\rho$ max $\psi^2}{4}$; Limit[%, $\psi \rightarrow 0$]; Simplify[% /. ρ max \rightarrow mmax / λ , {mmax > 0, λ > 0}] 2 U $\sqrt{\text{mmax }\lambda}$ $\sqrt{\pi}$

so that when we include mutation rate from the wildtype we have

$$4 \text{ U}^2 \text{ fm}[0, \text{m0}, \text{mmax}, \lambda, \text{n}] \sqrt{\text{mmax} \lambda / \pi};$$

We can further approximate fm to give a simpler analytic form. Using the approximation over ψ above (and incorporating the change in scale as we have integrated over fm above) we get

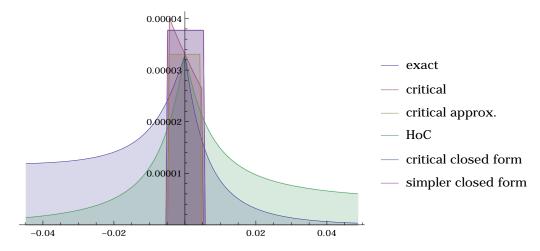
$$\begin{split} & \text{Simplify} \Big[\\ & 4 \, \text{U}^{\,2} \, \sqrt{\text{mmax} \, \lambda \, / \, \pi} \, \left(\text{D} \Big[\, 2 \, \left(1 - \sqrt{1 - \text{m} \, / \, \text{mmax}} \, \right) \, , \, \text{m} \Big] \, \text{f} \psi \, / \, . \, \psi \to 0 \right) \, / \, . \, \, \text{mmax} \to \rho \text{max} \, \lambda \, / \, . \, \, \text{m} \to 0 \, , \\ & \left\{ \lambda > 0 \, , \, \theta > 1 \, / \, 2 \, , \, \psi \text{wt} < 0 \right\} \Big] \, / \, . \, \, - \frac{\rho \text{max} \, \psi \text{wt}^2}{4} \to - \alpha \left(* / \, . \, \theta \to \text{n} / \, 2 \star \right) \, / \, . \, \, \psi \text{wt} \to \psi_0 \, / / \, \, \text{Simplify} \\ & \text{FullSimplify} \Big[\text{U}^2 \, \left(1 - \frac{\psi_0}{2} \right)^{\frac{1}{2} - \theta} \, \text{e}^{-\alpha} \, \frac{2}{\pi} = \% \Big] \\ & \frac{2^{\frac{1}{2} + \theta} \, \text{e}^{-\alpha} \, \text{U}^2 \, \left(2 - \psi_0 \right)^{\frac{1}{2} - \theta}}{\pi} \end{split}$$

True

Check contributions across m

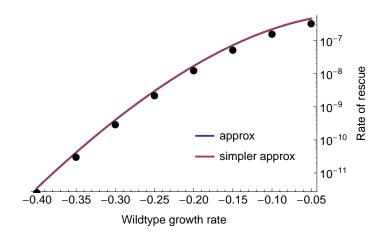
Clear[m0]

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
m0 = -0.3;
\{\text{mneg} = \text{FindRoot}[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, -0.1\}],
    mpos = FindRoot[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U]/2};
    fm[m, m0, mmax, \lambda, n] (1 - pest[m]) prescuem[m, \Lambda1[m, mmax, \lambda, n, U]],
    fm[m, m0, mmax, \lambda, n] (1-pest[m]) \sqrt{2 \Lambda 1[m, mmax, \lambda, n, U]}
      HeavisideTheta[(m - (m /. mneg)) ((m /. mpos) - m)],
     fm[0, m0, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m+pr0) (pr0-m)],
    fm[0, m0, mmax, \lambda, n] (1-pest[m]) prescuem[m, \Lambda1[m, mmax, \lambda, n, U]],
    fm[0, m0, mmax, \lambda, n] \sqrt{2 \frac{2 \, \text{U} \, \sqrt{\text{mmax} \, \lambda}}{\sqrt{}}}
     \text{HeavisideTheta} \left[ \left( m + \sqrt{\frac{\text{U}\sqrt{\text{mmax }\lambda}}{\sqrt{\pi}}} \right) \left( \sqrt{\frac{\text{U}\sqrt{\text{mmax }\lambda}}{\sqrt{\pi}}} - m \right) \right],
    \left(D\left[2\left(1-\sqrt{1-m\,/\,\mathrm{mmax}}\right),\,\mathrm{m}\right]\,\mathrm{f}\psi\,/\,.\,\,\psi\to0\,\,/\,.\,\,\psi\mathrm{wt}\to2\,\left(1-\sqrt{1-m0\,/\,\mathrm{mmax}}\right)\,/\,.
             \rho \max \rightarrow \max / \lambda / . \theta \rightarrow n / 2 / . m \rightarrow 0 \bigg) \sqrt{2 \frac{2 U \sqrt{\max \lambda}}{\sqrt{\pi}}}
     HeavisideTheta \left[ \left( m + \sqrt{\frac{U\sqrt{mmax \lambda}}{\sqrt{\pi}}} \right) \left( \sqrt{\frac{U\sqrt{mmax \lambda}}{\sqrt{\pi}}} - m \right) \right]
  };
Plot[%, {m, 10 m/. mneg, 10 m/. mpos}, PlotRange \rightarrow {0, All}, Filling \rightarrow Bottom,
  PlotLegends → LineLegend[{"exact", "critical", "critical approx.", "HoC",
        "critical closed form", "simpler closed form"}], PerformanceGoal → "Speed"]
Clear [mmax, \lambda, Es, n, U, m0]
```



And check total rate

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{mneg = FindRoot} [2 m^2 - \Lambda 1 [m, mmax, \lambda, n, U], \{m, -0.1\}],
   mpos = FindRoot \left[2 m^2 - \Lambda 1 [m, mmax, \lambda, n, U], \{m, 0.1\}\right];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   Table \left[ \left\{ m0, NIntegrate \left[ U fm[m, m0, mmax, \lambda, n] (1 - pest[m]) \sqrt{2 \Lambda 1[m, mmax, \lambda, n, U]} \right. \right. \right] \right]
        \{m, m/. mneg, m/. mpos\}, \{m0, -0.8 mmax, -0.1 mmax, 0.1 mmax\}
 };
Show
 LogPlot
   \{U \text{ 2 $\Lambda$0approx fm[0, m0, mmax, $\lambda$, n],}
    U^{2}\left(1-\frac{\psi_{0}}{2}\right)^{\frac{1}{2}-\theta}e^{-\alpha}\frac{2}{\pi}/.\alpha\rightarrow\rho\max\psi_{0}{}^{2}\left/4/.\psi_{0}\rightarrow2\left(1-\sqrt{1-m0/mmax}\right)/.\rho\max\rightarrow\min\lambda/\lambda/.
      \theta \to n/2, {m0, -0.8 mmax, -0.1 mmax},
   PlotStyle → Thick,
   Frame → {True, False, False, True},
   FrameLabel → {"Wildtype growth rate", , , "Rate of rescue"},
   FrameTicks → {True, False, False, True},
   LabelStyle → labelstyle,
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
          {"approx", "simpler approx"}], Scaled@{3/4,1/4}],
   PerformanceGoal → "Speed"
 ListLogPlot[%, PlotMarkers → {Automatic, Medium}, PlotStyle → Black]
Clear [mmax, \lambda, Es, n, U, mwt]
```



Approximate probability of rescue: sufficiently non-critical single mutants

Approximation 1

When $m>>\sqrt{\Lambda/2}$ the probability of 2-step rescue from this single mutant lineage, as calculated above, is roughly $\frac{\Lambda}{\mathsf{Abs}[m]}$.

We have seen above that the solutions to $m^2 = \Lambda[m]/2$ can be approximated well by $m = \pm \sqrt{\Lambda[0]/2}$. Check the contribution of single mutant growth rates to 2-step sufficiently critical rescue

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.3;
\{mneg = FindRoot[2m1^2 - \Lambda 1[m1, mmax, \lambda, n, U], \{m1, -0.1\}], \}
   mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}]\};
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   fm[m1, mwt, mmax, \lambda, n] (1-pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]],
   fm[m1, mwt, mmax, \lambda, n] (1-pest[m1])
     \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{\Lambda 1[m1, mmax, \lambda, n, U]} \text{ HeavisideTheta[(-pr0-m1) (pr0-m1)]}
              Abs[m1]
 };
Plot[%, \{m1, 50 m1 /. mneg, 10 m1 /. mpos\}, PlotRange \rightarrow \{0, All\}, Filling \rightarrow Bottom,
 PlotLegends → LineLegend[{"exact", "non-critical"}], PerformanceGoal → "Speed"]
Clear[mmax, \lambda, Es, n, U, mwt]
                                              0.0000
                                             0.00002
                                              0.00002
                                                                          exact
                                             0.000015

    non–critical

                                              0.00001
                                              5.×10
                                       -0.05
                                                               0.05
                            -0.10
```

And let's check the total rates of rescue across all m1

```
n0 = 10^4;
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{mneg = FindRoot} [2 \text{ m1}^2 - \Lambda 1 [\text{m1, mmax}, \lambda, n, U], \{\text{m1, -0.1}\}],
   mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   Table[{mwt, NIntegrate[
         fm[m1, mwt, mmax, \lambda, n] (1-pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]],
         {m1, mwt, mmax}]}, {mwt, -0.8 mmax, -0.1 mmax, 0.1 mmax}],
   Table \left[\left\{\text{mwt, NIntegrate}\left[\text{fm}\left[\text{m1, mwt, mmax, }\lambda,\text{n}\right]\right.\right]\right]
          , {m1, mwt, -pr0}] +
        NIntegrate \left[\text{fm}[\text{m1, mwt, mmax, } \lambda, \text{n}] \text{ } (1-\text{pest}[\text{m1}]) \right. \frac{\text{Al}[\text{m1, mmax, } \lambda, \text{n, U}]}{\text{Abs}[\text{m1}]}
          , {m1, pr0, mmax} ] }, {mwt, -0.8 mmax, -0.1 mmax, 0.1 mmax} ]
  };
Show[
 ListLogPlot[%, PlotLegends → LineLegend[{"exact", "non-critical"}]]
]
Clear [mmax, \lambda, Es, n, U, n0]
  0.1
0.001
                                                                                 exact
                                                                                 non-critical
 10^{-5}
                    -0.3
                                    -0.2
                                                     -0.1
                                                                     0.0
```

Approximation 2

We can next borrow the approximation of $\Lambda 1$ from Anciaux et al. (their eqn A12 without the 1/mwt term)

Abs[mwt] AnciauxEqnA12

$$\frac{\text{U}\left(1-\frac{\psi\text{wt}}{2}\right)^{\frac{1}{2}-\Theta}\,\text{Abs}\left[\text{mwt}\right]\,\left(\frac{e^{-\alpha}}{\sqrt{\pi}\,\,\sqrt{\alpha}}-\text{Erfc}\left[\sqrt{\alpha}\,\,\right]\right)}{1-\frac{\psi\text{wt}}{4}}$$

Check contributions of m1

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.3;
\{\text{mneg} = \text{FindRoot}[2 \text{ ml}^2 - \Lambda 1 [\text{ml}, \text{mmax}, \lambda, n, U], \{\text{ml}, -0.1\}],
   \texttt{mpos} = \texttt{FindRoot} \left[ 2 \, \texttt{m1}^2 - \texttt{\Lambda1} \left[ \texttt{m1}, \, \texttt{mmax}, \, \lambda, \, \texttt{n}, \, \texttt{U} \right], \, \left\{ \texttt{m1}, \, 0.1 \right\} \, \right] \right\};
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
    fm[m1, mwt, mmax, \lambda, n] (1-pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]],
    fm[m1, mwt, mmax, \lambda, n] (1-pest[m1])
       \texttt{fm[ml, mwt, mmax, } \lambda, \, \texttt{n] (1-pest[ml])} \; \frac{\texttt{Abs[ml] AnciauxEqnA12}}{\texttt{Abs[ml]}} \; \textit{/. } \psi \texttt{wt} \rightarrow \psi \; \textit{/.}
                   m \rightarrow mmax \psi (1 - \psi / 4) /. \alpha \rightarrow \frac{\rho max \psi^2}{4} /. \psi \rightarrow 2 \left(1 - \sqrt{1 - \frac{m}{mmax}}\right) /.
             \rhomax \rightarrow mmax / \lambda /. \theta \rightarrow n / 2 /. m \rightarrow ml HeavisideTheta[(-pr0 - m1) (pr0 - m1)]
  };
Plot[%, \{m1, 50 m1 /. mneg, 10 m1 /. mpos\}, PlotRange \rightarrow \{0, All\}, Filling \rightarrow Bottom,
  PlotLegends → LineLegend[{"exact", "non-critical", "non-critical anciaux"}],
  PerformanceGoal → "Speed"]
Clear [mmax, \lambda, Es, n, U, mwt]
                                                        0.00004
                                                        0.00003

    exact

    non–critical

                                                        0.00002
                                                                                          non-critical anciaux
                                                        0.0000
     -0.20
                    -0.15
                                  -0.10
                                                -0.05
                                                                            0.05
```

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
\{\text{mneg = FindRoot} [2 \text{ m1}^2 - \Lambda 1 [\text{m1, mmax}, \lambda, n, U], \{\text{m1, -0.1}\}],
    \texttt{mpos} = \texttt{FindRoot} \left[ 2 \, \texttt{m1}^2 - \texttt{\Lambda1} \left[ \texttt{m1}, \, \texttt{mmax}, \, \lambda, \, \texttt{n}, \, \texttt{U} \right], \, \left\{ \texttt{m1}, \, 0.1 \right\} \, \right] \right\};
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
     fm[m1, mwt, mmax, \lambda, n] (1-pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]],
     fm[m1, mwt, mmax, \lambda, n] (1-pest[m1])
       \frac{\text{A1[m1, mmax, } \lambda, n, U]}{\text{Abs[m1]}} \text{ HeavisideTheta[(-pr0-m1) (pr0-m1)],}
       \texttt{fm[m1, mwt, mmax, } \lambda, \texttt{n] (1-pest[m1])} \; \frac{\texttt{Abs[m1] AnciauxEqnA12}}{\texttt{Abs[m1]}} \; \textit{/. } \psi \texttt{wt} \rightarrow \psi \; \textit{/.}
                     m \rightarrow mmax \psi (1 - \psi / 4) /. \alpha \rightarrow \frac{\rho max \psi^2}{4} /. \psi \rightarrow 2 \left[1 - \sqrt{1 - \frac{m}{mmax}}\right] /.
               \rho \texttt{max} \rightarrow \texttt{mmax} \, / \, \lambda \, / \, . \, \theta \rightarrow \texttt{n} \, / \, 2 \, / \, . \, \texttt{m} \rightarrow \texttt{ml} \, \bigg| \, \, \texttt{HeavisideTheta[(-pr0-ml) (pr0-ml))} \, \bigg|
  };
Plot[%, \{m1, 50 m1 /. mneg, 10 m1 /. mpos\}, PlotRange \rightarrow \{0, All\}, Filling \rightarrow Bottom,
  PlotLegends → LineLegend[{"exact", "non-critical", "non-critical anciaux"}],
  PerformanceGoal → "Speed"]
Clear [mmax, \lambda, Es, n, U, mwt]
                                                                 0.001
                                                                                                     exact
                                                                0.001
                                                                                                      non-critical
                                                                                                      non-critical anciaux
                                                                 0.0005
      -0.20
                      -0.15
                                      -0.10
                                                      -0.05
                                                                                      0.05
```

Closed form approximation - subcriticals (equations 12 and 13)

For the subcriticals our approximation for $\Lambda 2$ is now the integral of

 $fm[m, mwt, mmax, \lambda, n]$ AnciauxEqnA12

$$-\left(e^{\frac{m-2\,\,\text{mmax}+\text{mwt}}{\lambda}}\,\text{U}\left(\frac{-\,\text{m}+\text{mmax}}{\lambda}\right)^{n/2}\left(1-\frac{\psi\text{wt}}{2}\right)^{\frac{1}{2}-\theta}\left(\frac{e^{-\alpha}}{\sqrt{\pi}\,\,\sqrt{\alpha}}-\text{Erfc}\!\left[\sqrt{\alpha}\,\,\right]\right)$$

$$\text{HypergeometricOFlRegularized}\!\left[\frac{n}{2}\,,\,\,\frac{\left(-\,\text{m}+\text{mmax}\right)\,\left(\text{mmax}-\text{mwt}\right)}{\lambda^2}\right]\right/\left(\left(\text{m}-\text{mmax}\right)\,\left(1-\frac{\psi\text{wt}}{4}\right)\right)$$

over m from $-\infty$ to $-m^*$, with α dependent on m.

As before we can approximately write fm in the ψ scale, leaving us with the integral of

$$\begin{split} & \text{f} \psi \, \text{AnciauxEqnA12} \, / \cdot \, \alpha \to \frac{\rho \text{max} \, \psi^2}{4} \\ & \text{constant} = \frac{\text{U} \, \sqrt{\rho \text{max}}}{2 \, \sqrt{\pi} \, \left(1 - \psi \text{wt} \, / \, 4 \right) \, \left(1 - \psi \text{wt} \, / \, 2 \right)^{-1 + 2 \, \theta}}; \\ & \psi \text{term} = \text{e}^{-\frac{1}{4} \, \rho \text{max} \, \left(\psi - \psi \text{wt} \right)^2} \, \left(1 - \psi \, / \, 2 \right)^{-\frac{1}{2} + \theta} \, \left(\frac{2 \, \text{e}^{-\frac{\rho \text{max} \, \psi^2}{4}}}{\sqrt{\pi} \, \sqrt{\rho \text{max} \, \psi^2}} - \text{Erfc} \left[\frac{\sqrt{\rho \text{max} \, \psi^2}}{2} \right] \right); \\ & \text{Simplify} \left[\\ & \left(\text{e}^{-\frac{1}{4} \, \rho \text{max} \, \left(\psi - \psi \text{wt} \right)^2} \, \text{U} \, \sqrt{\rho \text{max}} \, \left(\frac{2 - \psi}{2 - \psi \text{wt}} \right)^{-\frac{1}{2} + \theta} \, \left(1 - \frac{\psi \text{wt}}{2} \right)^{\frac{1}{2} - \theta} \, \left(\frac{2 \, \text{e}^{-\frac{\rho \text{max} \, \psi^2}{4}}}{\sqrt{\pi} \, \sqrt{\rho \text{max} \, \psi^2}} - \text{Erfc} \left[\frac{\sqrt{\rho \text{max} \, \psi^2}}{2} \right] \right) \right) / \\ & \left(2 \, \sqrt{\pi} \, \left(1 - \frac{\psi \text{wt}}{4} \right) \right) = \text{constant} \, \psi \text{term}, \, \left\{ \psi \text{wt} < 0 \right\} \right] \\ & \left(e^{-\frac{1}{4} \, \rho \text{max} \, \left(\psi - \psi \text{wt} \right)^2} \, \text{U} \, \sqrt{\rho \text{max}} \, \left(\frac{2 - \psi}{2 - \psi \text{wt}} \right)^{-\frac{1}{2} + \theta}} \, \left(1 - \frac{\psi \text{wt}}{2} \right)^{\frac{1}{2} - \theta} \, \left(\frac{2 \, \text{e}^{-\frac{\rho \text{max} \, \psi^2}{4}}}{\sqrt{\pi} \, \sqrt{\rho \text{max} \, \psi^2}} - \text{Erfc} \left[\frac{\sqrt{\rho \text{max} \, \psi^2}}{2} \right] \right) \right) / \\ & \left(2 \, \sqrt{\pi} \, \left(1 - \frac{\psi \text{wt}}{4} \right) \right) \end{split}$$

True

over ψ between $-\infty$ and ψ^*

$$\begin{aligned} & \texttt{Simplify} \bigg[2 \left(1 - \sqrt{1 - \frac{m}{mmax}} \right) / . \ m \rightarrow \{ -\infty, -m^* \}, \ mmax > 0 \bigg] \\ & \bigg\{ -\infty, \ 2 - 2 \sqrt{\frac{mmax + m^*}{mmax}} \bigg\} \end{aligned}$$

We make 2 different approximations. When ψ - ψ wt and ψ ρ max = ψ $\frac{\text{mmax}*n}{\lambda} = \psi$ $\frac{\text{mmax}*n}{2 \text{ Es}}$ are really small we

have

$$\begin{array}{l} \psi \text{term /.} \; \psi - \psi \text{wt} \to \psi \text{wt} + \text{d} \psi \in \; /. \; \psi \to \psi \; \epsilon; \\ \text{Simplify[Normal[Series[\%, \{\varepsilon, \, 0 \, , \, -1\}]] /. } \; \varepsilon \to 1 \, , \; \{\psi < \, 0\}] \end{array}$$

$$-\frac{2 e^{-\frac{\rho \max \psi wt^2}{4}}}{\sqrt{\pi} \sqrt{\rho \max \psi}}$$

And when ψ is small but ρ max is really large we have

$$\psi$$
term /. ρ max -> ρ max / ϵ ;

$$\text{Simplify[Normal[Series[\%, \{\varepsilon,\, 0\,,\, 2\}]] /. } \varepsilon \to 1\,, \; \{\psi < 0\}] \;/. \; \left(1 - \frac{\psi}{2}\right)^{\theta} \to 1\;/.$$

$$2\pi - \pi \psi \rightarrow 2\pi$$

$$-\,\frac{4\;\mathrm{e}^{-\frac{1}{4}\;\rho\mathrm{max}\,\left(\psi^2+\,\left(\psi-\psi\mathrm{wt}\right)^{\,2}\right)}}{\sqrt{\pi}\;\rho\mathrm{max}^{3/2}\,\psi^3}$$

Compare numerically over m (from ψ wt to ψ^*)

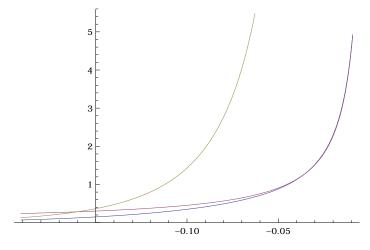
U =
$$10^{-3}$$
;
mmax = 0.5;
 λ = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
m* = $\sqrt{\Lambda 1[0, mmax, \lambda, n, U]/2}$;

$$\left\{\psi\text{term,} - \frac{2\,e^{-\frac{\rho\text{max}\,\psi\text{vrt}^2}{4}}}{\sqrt{\pi}\,\,\sqrt{\rho\text{max}}\,\,\psi}\,, - \frac{4\,e^{-\frac{1}{4}\,\rho\text{max}\,\left(\psi^2+(\psi-\psi\text{wt})^2\right)}}{\sqrt{\pi}\,\,\rho\text{max}^{3/2}\,\psi^3}\right\}\,/\,.\,\,\rho\text{max} \to \text{mmax}\,/\,\lambda\,/\,.\,\,\theta \to \text{n}\,/\,2\,/\,.$$

$$\psi\text{wt} \to 2\,\left(1-\sqrt{1-\frac{\text{mwt}}{\text{mmax}}}\right);$$

$$Plot\left[\%, \left\{\psi, 2\left(1 - \sqrt{1 - \frac{mwt}{mmax}}\right), 2 - 2\sqrt{\frac{mmax + m^*}{mmax}}\right\}, PlotRange \rightarrow \{0, Automatic\}\right]$$

Clear $[mmax, \lambda, Es, n, U, mwt]$



And compare integrals across range of mwt. We see the nice transition from the small ψ approx to the large ρ max approx as we increase ρ max:

For ρ max = 10

U =
$$10^{-3}$$
;
mmax = 0.5;
 λ = 2 Es / n;
Es = 0.1;
n = 4;
m* = $\sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2}$;
mmax / λ

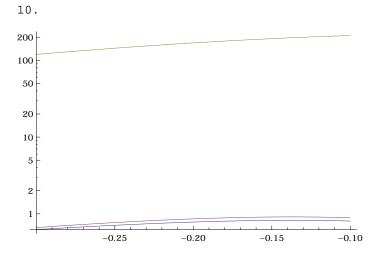
$$\left\{\psi\text{term,} - \frac{2\,e^{-\frac{\rho\text{max}\,\psi\text{wt}^2}{4}}}{\sqrt{\pi}\,\,\sqrt{\rho\text{max}}\,\,\psi}\,, - \frac{4\,e^{-\frac{1}{4}\,\rho\text{max}\,\left(\psi^2+\left(\psi-\psi\text{wt}\right)^2\right)}}{\sqrt{\pi}\,\,\rho\text{max}^{3/2}\,\psi^3}\right\}\,/\,.\,\,\rho\text{max} \to \text{mmax}\,/\,\lambda\,/\,.\,\,\theta \to \text{n}\,/\,2\,/\,.$$

$$\psi\text{wt} \to 2\,\left(1-\sqrt{1-\frac{\text{mwt}}{\text{mmax}}}\right);$$

$$\text{Table} \Big[\text{Table} \Big[\Big\{ \text{mwt, NIntegrate} \Big[\% \text{[[i]], } \Big\{ \psi \text{, 2} \left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right) \text{, 2 - 2} \sqrt{\frac{\text{mmax} + \text{m}^*}{\text{mmax}}} \right\} \Big] \Big\} \text{,}$$

{mwt, -0.3, -0.1, 0.01}], {i, Length[%]}]; ListLogPlot[%, Joined → True]

Clear $[mmax, \lambda, Es, n, U, mwt]$



For ρ max=100

U =
$$10^{-3}$$
;
mmax = 0.5;
 λ = 2 Es / n;
Es = 0.01;
n = 4;
m* = $\sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2}$;
mmax / λ

$$\left\{ \psi \text{term,} - \frac{2 \, \mathrm{e}^{-\frac{\rho \mathrm{max} \, \psi \mathrm{wt}^2}{4}}}{\sqrt{\pi} \, \sqrt{\rho \mathrm{max}} \, \psi} \,, - \frac{4 \, \mathrm{e}^{-\frac{1}{4} \, \rho \mathrm{max} \, \left(\psi^2 + \left(\psi - \psi \mathrm{wt}\right)^2\right)}}{\sqrt{\pi} \, \rho \mathrm{max}^{3/2} \, \psi^3} \right\} \,/ \,. \, \, \rho \mathrm{max} \to \mathrm{mmax} \,/ \, \lambda \,/ \,. \, \theta \to \mathrm{n} \,/ \, 2 \,/ \,.$$

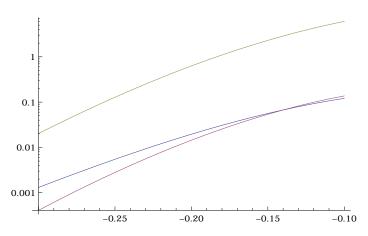
$$\psi \mathrm{wt} \to 2 \, \left(1 - \sqrt{1 - \frac{\mathrm{mwt}}{\mathrm{mmax}}} \,\right);$$

$$\text{Table} \Big[\text{Table} \Big[\Big\{ \text{mwt, NIntegrate} \Big[\% \text{[[i]], } \Big\{ \psi \text{, 2} \left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right) \text{, 2 - 2} \sqrt{\frac{\text{mmax} + \text{m}^*}{\text{mmax}}} \right\} \Big] \Big\} \text{,}$$

{mwt, -0.3, -0.1, 0.01}], {i, Length[%]}]; ListLogPlot[%, Joined \rightarrow True]

Clear[mmax, λ, Es, n, U, mwt]

100.



for ρ max=1000

U =
$$10^{-3}$$
;
mmax = 0.5;
 λ = 2 Es / n;
Es = 0.001;
n = 4;
m* = $\sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2}$;
mmax / λ

$$\left\{\psi\text{term,} - \frac{2\,e^{-\frac{\rho\text{max}\,\psi\text{wt}^2}{4}}}{\sqrt{\pi}\,\,\sqrt{\rho\text{max}}\,\,\psi}, - \frac{4\,e^{-\frac{1}{4}\,\rho\text{max}\,\left(\psi^2+\left(\psi-\psi\text{wt}\right)^2\right)}}{\sqrt{\pi}\,\,\rho\text{max}^{3/2}\,\psi^3}\right\} / \cdot \rho\text{max} \to \text{mmax}\,/\,\lambda\,/\,\cdot\,\theta \to \text{n}\,/\,2\,/\,\cdot\,$$

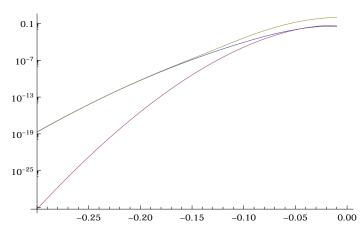
$$\psi\text{wt} \to 2\left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}}\right);$$

$$\text{Table} \Big[\text{Table} \Big[\Big\{ \text{mwt, NIntegrate} \Big[\% \text{[[i]], } \Big\{ \psi \text{, 2} \left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right) \text{, 2 - 2} \sqrt{\frac{\text{mmax} + \text{m}^*}{\text{mmax}}} \right\} \Big] \Big\},$$

{mwt, -0.3, -0.01, 0.01}], {i, Length[%]}]; ListLogPlot[%, Joined → True, PlotRange → All]

Clear[mmax, λ , Es, n, U, mwt]

1000.



and for ρ max=10000

U =
$$10^{-3}$$
;
mmax = 0.5;
 λ = 2 Es / n;
Es = 0.0001;
n = 4;
m* = $\sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2}$;
mmax / λ

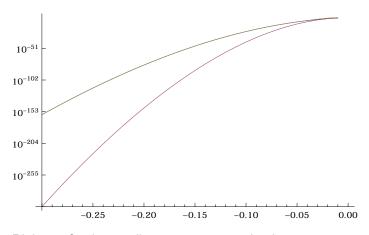
$$\left\{\psi\text{term,} - \frac{2\,e^{-\frac{\rho\text{max}\,\psi\text{vt}^2}{4}}}{\sqrt{\pi}\,\,\sqrt{\rho\text{max}}\,\,\psi}\,, - \frac{4\,e^{-\frac{1}{4}\,\rho\text{max}\,\left(\psi^2+\left(\psi-\psi\text{wt}\right)^2\right)}}{\sqrt{\pi}\,\,\rho\text{max}^{3/2}\,\psi^3}\right\}\,/\,.\,\,\rho\text{max} \to \text{mmax}\,/\,\lambda\,/\,.\,\,\theta \to \text{n}\,/\,2\,/\,.$$

$$\psi\text{wt} \to 2\,\left(1-\sqrt{1-\frac{\text{mwt}}{\text{mmax}}}\right);$$

$$\text{Table} \Big[\text{Table} \Big[\Big\{ \text{mwt, NIntegrate} \Big[\% \text{[[i]], } \Big\{ \psi \text{, 2} \left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right) \text{, 2 - 2} \sqrt{\frac{\text{mmax} + \text{m}^*}{\text{mmax}}} \right\} \Big] \Big\},$$

{mwt, -0.3, -0.01, 0.01}], {i, Length[%]}]; ListLogPlot[%, Joined \rightarrow True, PlotRange \rightarrow All]

Clear[mmax, λ , Es, n, U, mwt] 10000.



Right, so for the small ψ approx we need to integrate

$$small \psi approx = -\frac{2 e^{-\frac{cmax}{4}\psi wt^{2}}}{\sqrt{\pi} \sqrt{cmax} \psi}$$

which has a simple expression if we do not integrate all the way back to $-\infty$

 $\label{eq:continuous_prox} \texttt{Integrate[small} \psi \texttt{approx, } \{\psi \texttt{, a, b} \texttt{, Assumptions} \rightarrow \{\texttt{a} < \texttt{b} < \texttt{0} \} \texttt{]}$

$$-\frac{2 e^{-\frac{\rho \max \psi w t^2}{4}} Log\left[\frac{b}{a}\right]}{\sqrt{\pi} \sqrt{\rho max}}$$

And for the large ρ max we need to integrate

$$large \rho approx = -\frac{4 e^{-\frac{1}{4} \rho max \left(\psi^2 + (\psi - \psi wt)^2\right)}}{\sqrt{\pi} \rho max^{3/2} \psi^3};$$

here we can again use the Laplace approx with

$$q[\psi_{-}] := -\frac{1}{4} \rho \max (\psi^{2} + (\psi - \psi wt)^{2})$$

$$h[\psi_{-}] := -\frac{4}{\sqrt{\pi} \rho \max^{3/2} \psi^{3}}$$

and we know that q is peaked at

$$\texttt{Solve}[\mathtt{D}[\mathtt{q}[\psi]\,,\,\psi] = \mathtt{0}\,,\,\psi]$$

$$\left\{\left\{\psi \to \frac{\psi wt}{2}\right\}\right\}$$

where h equals

h[\psi\wt/2]

$$-\frac{32}{\sqrt{\pi} \rho \text{max}^{3/2} \psi \text{wt}^3}$$

giving

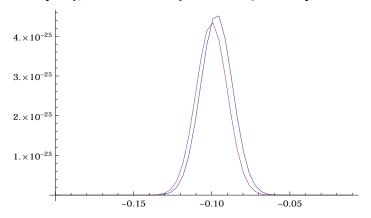
 $h[\psi wt / 2] Exp[q[\psi]]$

verylarge
$$\rho$$
approx = $-\frac{32 e^{-\frac{1}{4} \rho \max \left(\psi^2 + (\psi - \psi \text{wt})^2\right)}}{\sqrt{\pi} \rho \max^{3/2} \psi \text{wt}^3}$;

$$-\frac{32 \ e^{-\frac{1}{4} \, \rho \text{max} \, \left(\psi^2 + \, \left(\psi - \psi \text{wt}\right)^{\, 2}\right)}}{\sqrt{\pi} \, \, \rho \text{max}^{3/2} \, \, \psi \text{wt}^{\, 3}}$$

Numerical check

{large ρ approx, very large ρ approx} /. ρ max \rightarrow 10000 /. ψ wt \rightarrow -0.2; Plot[%, $\{\psi, -0.2, -0.01\}$, PlotRange \rightarrow All]



which we can integrate over all ψ with little error (because exponential tails)

Integrate[verylarge ρ approx, { ψ , $-\infty$, ∞ }, Assumptions \rightarrow { ρ max > 0, ψ wt < 0}]

$$-\frac{32\sqrt{2}}{\rho \text{max}^2} e^{-\frac{\rho \text{max} \psi \text{wt}^2}{8}}$$

OK, so our small ρ max approx for the probability of 2-step subcritical rescue is

 $small \psi approx Rescue \psi =$

constant Integrate [small ψ approx, { ψ , a, b}, Assumptions \rightarrow {a < b < 0}] small#approxRescue =

% /. a
$$\rightarrow$$
 2 $\left(1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}}\right)$ /. b \rightarrow 2 $\left(1 - \sqrt{1 + \frac{\text{mstar}}{\text{mmax}}}\right)$ /. mstar $\rightarrow \sqrt{\Lambda 0 \text{approx}/2}$ /.

$$\rho$$
max \rightarrow mmax $/ \lambda /. \theta \rightarrow$ n $/ 2 /. \psi$ wt \rightarrow 2 $\left(1 - \sqrt{1 - \frac{mwt}{mmax}}\right)$

$$-\frac{\mathrm{e}^{-\frac{\rho max\,\psi wt^2}{4}}\,U\,\left(1-\frac{\psi wt}{2}\right)^{1-2\,\Theta}\,Log\left[\frac{b}{a}\right]}{\pi\,\left(1-\frac{\psi wt}{4}\right)}$$

$$-\left(e^{-\frac{mmax}{\lambda}\left(1-\sqrt{1-\frac{mwt}{mmax}}\right)^{2}}\left(1-\frac{mwt}{mmax}\right)^{\frac{1-n}{2}}ULog\left[\frac{1-\sqrt{1+\frac{\sqrt{U\sqrt{mmax}\lambda}}{mmax\pi^{1/4}}}}{1-\sqrt{1-\frac{mwt}{mmax}}}\right]\right)\bigg/\left(\left(1+\frac{1}{2}\left(-1+\sqrt{1-\frac{mwt}{mmax}}\right)\right)\pi\right)$$

and our very large ρ max approx for the probability of 2-step subcritical rescue is

verylargeρapproxRescueψ =

 $\texttt{constant Integrate[verylarge} \rho \texttt{approx, } \{\psi \texttt{, -}\infty, \infty\} \texttt{, Assumptions} \rightarrow \{\rho \texttt{max} \texttt{> 0} \texttt{, } \psi \texttt{wt} \texttt{< 0} \}]$

verylargeρapproxRescue = % /. ρmax \rightarrow mmax / λ /. $\theta \rightarrow$ n / 2 /. ψ wt \rightarrow 2 $1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}}$

$$-\frac{16 e^{-\frac{\rho \max \psi \mathsf{wt}^2}{8}} \sqrt{\frac{2}{\pi}} \mathsf{U} \left(1 - \frac{\psi \mathsf{wt}}{2}\right)^{1 - 2\theta}}{\rho \mathsf{max}^{3/2} \left(1 - \frac{\psi \mathsf{wt}}{4}\right) \psi \mathsf{wt}^3}$$

$$-\left(2 e^{-\frac{mmax\left(1-\sqrt{1-\frac{mvt}{max}}\right)^2}{2\lambda}} \left(1-\frac{mwt}{mmax}\right)^{\frac{1-n}{2}}\sqrt{\frac{2}{\pi}} U\right) \middle/$$

$$\left(\left(1 - \sqrt{1 - \frac{mwt}{mmax}} \right)^3 \left(1 + \frac{1}{2} \left(-1 + \sqrt{1 - \frac{mwt}{mmax}} \right) \right) \left(\frac{mmax}{\lambda} \right)^{3/2} \right)$$

Check formula for text

Simplify

$$small \psi approx Rescue \psi = U \frac{\left(1 - \frac{\psi wt}{2}\right)^{1-2\theta}}{\left(1 - \frac{\psi wt}{4}\right)} e^{-\alpha} \frac{Log\left[\frac{a}{b}\right]}{\pi} / . \alpha \rightarrow \rho max \psi wt^{2} / 4, \{a < b < 0\}$$

Simplify verylargeρapproxRescueψ ==

$$\left(-U \frac{\left(1-\frac{\psi wt}{2}\right)^{1-2\theta}}{\left(1-\frac{\psi wt}{4}\right)} \left(e^{-\alpha} \frac{1}{\left(\alpha/2\right)^{3}\pi}\right)^{1/2}/. \alpha \rightarrow \rho \max \psi wt^{2}/4\right), \{\rho \max > 0, \psi \text{wt} > 0\}\right]$$

True

True

Compare to better approximation

```
n0 = 10^4;
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{mneg} = \text{FindRoot}[2 \text{ ml}^2 - \Lambda 1 [\text{ml}, \text{mmax}, \lambda, n, U], \{\text{ml}, -0.1\}],
   mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
   Table \Big[ \Big\{ mwt, \, U \, NIntegrate \Big[ fm[m1, \, mwt, \, mmax, \, \lambda, \, n] \, \, \frac{ \Lambda 1 \, [m1, \, mmax, \, \lambda, \, n, \, U] }{ Abs \, [m1] } \Big] \Big] \Big] \\
         , {m1, mwt, -pr0} }, {mwt, -0.8 mmax, -0.1 mmax, 0.1 mmax}
 };
Show[
 LogPlot[
   {U small ψapproxRescue, U verylargeρapproxRescue}, {mwt, -0.8 mmax, -0.1 mmax},
   PlotStyle → Thick,
   Frame → {True, False, False, True},
   FrameLabel → {"Wildtype growth rate", , , "Rate of rescue"},
   FrameTicks → {True, False, False, True},
   LabelStyle → labelstyle,
   PlotLegends → Placed[
      LineLegend[Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@ {"small \psi", "large \rho_{\text{max}}"}],
      Scaled@{3/4,1/4}]
 ListLogPlot[%, PlotMarkers → {Automatic, Medium}, PlotStyle → Black]
]
Clear [mmax, \lambda, Es, n, U, n0]
                                                        10^{-6}
                                      - small ψ
                                       · large 
ho_{\sf max}
-0.40 -0.35 -0.30 -0.25 -0.20 -0.15 -0.10 -0.05
                   Wildtype growth rate
```

Closed form approximation - supercriticals (equation 14)

For the supercriticals our approximation for $\Lambda 2$ is now the integral of

 $fm[m, mwt, mmax, \lambda, n]$ (1 - pest[m]) AnciauxEqnA12

$$\begin{split} &-\frac{1}{\left(\text{m-mmax}\right) \left(1-\frac{\psi \text{wt}}{4}\right)} \, e^{\frac{m-2 \, \text{mmax+mwt}}{\lambda}} \, \text{U} \left(\frac{-\, \text{m} + \text{mmax}}{\lambda}\right)^{n/2} \\ & \left(1-\frac{\psi \text{wt}}{2}\right)^{\frac{1}{2}-\Theta} \left(\frac{e^{-\alpha}}{\sqrt{\pi} \, \sqrt{\alpha}} - \text{Erfc} \left[\sqrt{\alpha} \, \right]\right) \, \left(1-\left(1-e^{-2\, \text{m}}\right) \, \text{HeavisideTheta[m]} \right) \\ & \text{HypergeometricOF1Regularized} \left[\frac{n}{2} \, , \, \frac{\left(-\, \text{m} + \text{mmax}\right) \, \left(\text{mmax} - \text{mwt}\right)}{\lambda^2}\right] \end{split}$$

over m from m^* to m_{max} , with α dependent on m.

As before we can approximately write fm in the ψ scale, leaving us with the integral of

$$\begin{split} & \text{f} \psi \; (\text{1-pest[m]}) \; \text{AnciauxEqnA12} \; /. \; \alpha \rightarrow \frac{\rho \text{max} \; \psi^2}{4} \; /. \; \text{m} \rightarrow \; \text{mmax} \; (\text{1-}\psi / \, 4) \; \psi \\ & \text{constant} = \frac{\text{U} \; \sqrt{\rho \text{max}}}{2 \; \sqrt{\pi} \; \left(1 - \psi \text{wt} \; / \; 4 \right) \; \left(1 - \psi \text{wt} \; / \; 2 \right)^{-1 + 2 \; \theta}}; \\ & \psi \text{term} = e^{-2 \; \text{mmax} \left(1 - \frac{\psi}{4} \right) \; \psi - \frac{1}{4} \; \rho \text{max} \; \left(\psi - \psi \text{wt} \right)^2} \; \left(1 - \psi \; / \; 2 \right)^{-\frac{1}{2} + \theta} \left(\frac{2 \; e^{-\frac{\rho \text{max} \; \psi^2}{4}}}{\sqrt{\pi} \; \sqrt{\rho \text{max} \; \psi^2}} - \text{Erfc} \left[\frac{\sqrt{\rho \text{max} \; \psi^2}}{2} \right] \right); \\ & \text{Simplify} \left[\left(\text{f} \psi \; (\text{1-pest[m]}) \; \text{AnciauxEqnA12} \; /. \; \alpha \rightarrow \frac{\rho \text{max} \; \psi^2}{4} \; /. \; \text{m} \rightarrow \; \text{mmax} \; (\text{1-}\psi / \; 4) \; \psi \right] = \\ & \text{constant} \; \psi \text{term} \; , \; \{\psi \text{wt} < 0 \; , \; 2 > \psi > 0 \; , \; \text{mmax} > 0 \} \right] \\ & \frac{1}{2 \; \sqrt{\pi} \; \left(1 - \frac{\psi \text{wt}}{4} \right)} \end{split}$$

$$e^{-\frac{1}{4}\rho\max\left(\psi-\psi\mathrm{wt}\right)^{2}}\,\mathrm{U}\,\sqrt{\rho\mathrm{max}}\,\left(\frac{2-\psi}{2-\psi\mathrm{wt}}\right)^{-\frac{1}{2}+\theta}\left(1-\frac{\psi\mathrm{wt}}{2}\right)^{\frac{1}{2}-\theta}\left(\frac{2\,\mathrm{e}^{-\frac{\rho\mathrm{max}\,\psi^{2}}{4}}}{\sqrt{\pi}\,\sqrt{\rho\mathrm{max}\,\psi^{2}}}-\mathrm{Erfc}\!\left[\frac{\sqrt{\rho\mathrm{max}\,\psi^{2}}}{2}\right]\right) \\ \left(1-\left(1-\mathrm{e}^{-2\,\mathrm{mmax}\,\left(1-\frac{\psi}{4}\right)\,\psi}\right)\,\mathrm{HeavisideTheta}\!\left[\mathrm{mmax}\,\left(1-\frac{\psi}{4}\right)\,\psi\right]\right)$$

True

over ψ between ψ^* and 2

Clear[m]
$$Simplify\left[2\left(1-\sqrt{1-\frac{m}{mmax}}\right)/.m \rightarrow \{mstar, mmax\}, mmax > 0\right]$$

$$\left\{2-2\sqrt{1-\frac{mstar}{mmax}}, 2\right\}$$

Here we only need one approximation as large m single mutants will establish themselves and are unlikely to rescue. So we just look at when ψ - ψ wt and ψ ρ max = ψ $\frac{\text{mmax}}{\lambda} = \psi$ $\frac{\text{mmax}*n}{2 \, \text{Es}}$ are really small, giving

$$\frac{2 e^{-\frac{\pi}{4}}}{\sqrt{\pi} \sqrt{\rho \max} \psi}$$

Right, so for the small ψ approx we need to integrate

$$small \psi approx = \frac{2 e^{-\frac{\rho max \psi wt^2}{4}}}{\sqrt{\pi} \sqrt{\rho max} \psi};$$

which has a simple expression

Integrate[small ψ approx, { ψ , a, b}, Assumptions \rightarrow {0 < a < b < 2}]

$$\frac{2 e^{-\frac{\rho \text{max} \psi \text{wt}^2}{4}} \text{Log}\left[\frac{b}{a}\right]}{\sqrt{\pi} \sqrt{\rho \text{max}}}$$

OK, so our small ρ max approx for the probability of 2-step supercritical rescue is

 $small \psi approx Rescue Super \psi =$

constant Integrate[small ψ approx, { ψ , a, b}, Assumptions \rightarrow {0 < a < b}] small ψ approxRescueSuper =

% /. a
$$\rightarrow$$
 2 $\left(1 - \sqrt{1 - \frac{\text{mstar}}{\text{mmax}}}\right)$ /. b $\rightarrow \frac{\sqrt{2}}{\sqrt{\rho \text{max}}}$ /. mstar $\rightarrow \sqrt{\Lambda 0 \text{approx}/2}$ /. $\rho \text{max} \rightarrow \text{mmax}/\lambda$ /.

$$\theta \rightarrow n / 2 / . \psi wt \rightarrow 2 \left(1 - \sqrt{1 - \frac{mwt}{mmax}} \right)$$

$$\frac{e^{-\frac{\rho max \, \psi wt^2}{4}} \, U \, \left(1 - \frac{\psi wt}{2}\right)^{1-2 \, \theta} \, Log\left[\frac{b}{a}\right]}{\pi \, \left(1 - \frac{\psi wt}{4}\right)}$$

$$e^{-\frac{m \max \left(1 - \sqrt{1 - \frac{m v t}{m \max }}\right)^2}{\lambda}} \left(1 - \frac{m w t}{m m a x}\right)^{\frac{1 - n}{2}} U \text{ Log}\left[\frac{1}{\sqrt{2} \sqrt{\frac{m m a x}{\lambda}}} \left(1 - \sqrt{1 - \frac{\sqrt{U \sqrt{m m a x} \lambda}}{m m a x} \pi^{1/4}}\right)\right] / \frac{1}{\sqrt{2} \sqrt{\frac{m m a x}{\lambda}}} \sqrt{\frac{1 - \sqrt{1 - \frac{\sqrt{U \sqrt{m m a x} \lambda}}{m m a x} \pi^{1/4}}}{\frac{1 - \sqrt{1 - \frac{\sqrt{U \sqrt{m m a x} \lambda}}{m m a x} \pi^{1/4}}}}\right]}$$

$$\left(\left(1 + \frac{1}{2} \left(-1 + \sqrt{1 - \frac{\mathsf{mwt}}{\mathsf{mmax}}} \right) \right) \pi \right)$$

where we've used our rough cut-off of $\sqrt{2/\rho max}$, after which the rate of rescue declines very very quickly and therefore is negligible.

Check formula in text

$$\text{Simplify} \bigg[\frac{\text{U} \left(1 - \frac{\psi \text{wt}}{2} \right)^{1-2\theta}}{\left(1 - \frac{\psi \text{wt}}{4} \right)} \, \text{e}^{-\alpha} \, \text{Log} \bigg[\frac{b}{a} \bigg] \bigg/ \, \pi = = \, \text{small} \psi \text{approxRescueSuper} \psi \, / \, \cdot \, \alpha \to \frac{\rho \text{max } \psi \text{wt}^2}{4} \bigg]$$

True

Compare to better approximation

```
n0 = 10^4;
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{mneg} = \text{FindRoot}[2 \text{ ml}^2 - \Lambda 1 [\text{ml}, \text{mmax}, \lambda, n, U], \{\text{ml}, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
  , {m1, pr0, mmax} }, {mwt, -0.8 mmax, -0.1 mmax, 0.1 mmax}
 };
Show[
 LogPlot[
   {U small \u03c4approxRescueSuper}, {mwt, -0.8 mmax, -0.1 mmax},
  PlotStyle → Thick,
  Frame → {True, False, False, True},
  FrameLabel → {"Wildtype growth rate", , , "Rate of rescue"},
  FrameTicks → {True, False, False, True},
  LabelStyle → labelstyle,
  PlotLegends → Placed[LineLegend[
      \texttt{Style[\#, 12, FontFamily} \rightarrow \texttt{"Helvetica"] \& /@ \{"approx"\}], Scaled@\{3/4,1/4\}]
 ListLogPlot[%, PlotMarkers → {Automatic, Medium}, PlotStyle → Black]
Clear [mmax, \lambda, Es, n, U, n0]
                                               10^{-7}
                                 approx
                                               10^{-11}
-0.40 -0.35 -0.30 -0.25 -0.20 -0.15 -0.10 -0.05
               Wildtype growth rate
```

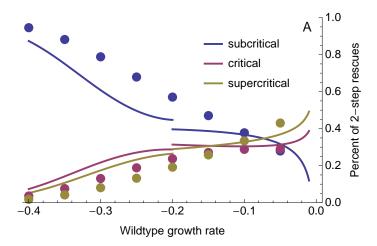
Plot approximations (figure 5)

Panel A data

```
(*n0=10^4;
U=10^{-3};
mmax=0.5:
\lambda = 2Es/n;
Es=0.01;
n=4;
\{\text{mneg=FindRoot}[2m1^2-\Lambda 1[m1,mmax,\lambda,n,U],\{m1,-0.1\}],
 mpos=FindRoot[2m1^2-\Lambda1[m1,mmax,\lambda,n,U],\{m1,0.1\}];
pr0=\sqrt{\Lambda 1[0,mmax,\lambda,n,U]/2};
xs=Table[mwt,{mwt,-0.8mmax,-0.1mmax,0.1mmax}];
temptab={
  Table NIntegrate \left[ \text{fm}[\text{m1,mwt,mmax},\lambda,\text{n}] \frac{\Lambda 1[\text{m1,mmax},\lambda,\text{n},\text{U}]}{2 \text{he}[\text{m1}]} \right]
      ,{m1,mwt,-pr0}],{mwt,-0.8mmax,-0.1mmax,0.1mmax}],
  Table [NIntegrate fm[m1,mwt,mmax,\lambda,n] (1-pest[m1]) \sqrt{2\Lambda 1} [m1,mmax,\lambda,n,U],
      \{m1,m1/.mneg,m1/.mpos\}],\{mwt,-0.8mmax,-0.1mmax,0.1mmax\}],
   \texttt{Table} \Big[ \texttt{NIntegrate} \Big[ \texttt{fm} [\texttt{m1,mwt,mmax,} \lambda, \texttt{n}] \, (\texttt{1-pest} [\texttt{m1}]) \, \tfrac{\Lambda 1 \, [\texttt{m1,mmax}, \lambda, \texttt{n}, \texttt{U}]}{\texttt{abe} \, [\texttt{m1}]} \\
      ,{m1,pr0,mmax} |,{mwt,-0.8mmax,-0.1mmax,0.1mmax}|
 };
Transpose[temptab];
Table[%[[i]]/Total[%[[i]]],{i,Length[%]}];
test=Transpose[%];
data=Table[Transpose[Join[{xs},{%[[i]]},1]],{i,Length[%]}];
Clear [mmax, \lambda, Es, n, U, n0] *)
data = {{{-0.4}, 0.9456638573208591}}, {-0.350000000000003}, 0.8818894810731134}},
      {-0.3000000000000004`, 0.7882778363153661`}, {-0.25`, 0.6791421962065166`},
     {-0.2`, 0.5706393639682957`}, {-0.1500000000000002`, 0.4705197731917115`},
      {-0.0999999999999998`, 0.3776908923323233`},
     {-0.049999999999999999, 0.2793108104800298`}},
    {{-0.4`, 0.03623085604695413`}, {-0.35000000000000003`, 0.07601020185743679`},
      {-0.30000000000000004`, 0.13070865247937516`}, {-0.25`, 0.1885039970593472`},
      {-0.2`, 0.23742270239875216`}, {-0.1500000000000002`, 0.2712516380984508`},
     {-0.0999999999999998, 0.2883094661397107},
      {-0.049999999999999999, 0.2901562484218649`}},
    {{-0.4`, 0.018105286632186777`}, {-0.3500000000000003`, 0.042100317069449986`},
      \{-0.300000000000004^{\text{`}}, 0.08101351120525872^{\text{`}}\}, \{-0.25^{\text{`}}, 0.13235380673413624^{\text{`}}\}, \} 
      {-0.2`, 0.1919379336329521`}, {-0.1500000000000002`, 0.2582285887098376`},
      {-0.0999999999999998, 0.3339996415279661},
      {-0.04999999999999999, 0.4305329410981054`}}};
```

Panel A

```
n0 = 10^4;
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
\{\text{small}\ \psi \text{approxRescue}, 2\ \Lambda 0 \text{approx}\ \text{fm}[0, \text{mwt}, \text{mmax}, \lambda, \text{n}], \text{small}\ \psi \text{approxRescueSuper}\};
Total[%]
\{\text{verylarge} \rho \text{approxRescue}, 2 \Lambda 0 \text{approx} \text{fm}[0, \text{mwt}, \text{mmax}, \lambda, \text{n}], \text{small} \psi \text{approxRescueSuper}\};
Total[%]
fig5a = Show[
  Plot[
    %, \{mwt, -0.4, -0.2\},
    PlotRange \rightarrow \{0, 1\},
    PlotStyle → Thick,
    Frame → {True, False, False, True},
    FrameLabel → {"Wildtype growth rate", , , "Percent of 2-step rescues"},
    FrameTicks → {True, False, False, All},
    LabelStyle → labelstyle,
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
          {"subcritical", "critical", "supercritical"}], Scaled@{3/4,3/4}],
    Epilog → Text[Style["A", 14, Bold, FontFamily → "Helvetica"],
       Scaled@{0.95, 0.95}]
   ],
  Plot[
    %%%, {mwt, -0.2, -0.01},
    PlotRange \rightarrow \{0, 1\},
    PlotStyle → Thick
  ListPlot[data,
    PlotMarkers → {Graphics[{defaultcolors[[1]], Disk[]}, ImageSize → 10],
       Graphics[{defaultcolors[[2]], Disk[]}, ImageSize \rightarrow 10],
       Graphics[{defaultcolors[[3]], Disk[]}, ImageSize \rightarrow 10]}, PlotRange \rightarrow {0, 1}],
  PlotRange \rightarrow \{\{-0.4, 0\}, \{0, 1\}\}
 ]
Clear [mmax, \lambda, Es, n, U, n0]
```



Panel B data

```
(*n0=10^4;
mwt = -0.1;
mmax=0.5;
\lambda=2Es/n;
Es=0.01;
n=4;
U=10x;
xs=Table[U, \{x, -6, -1, 1\}];
temptab2=Table
    mneg=FindRoot \left[2m1^2-\Lambda 1\left[m1,mmax,\lambda,n,U\right],\left\{m1,-0.1\right\}\right];
    mpos=FindRoot[2m1^2-\Lambda1[m1,mmax,\lambda,n,U],\{m1,0.1\}];
    pr0=\sqrt{\Lambda 1[0,mmax,\lambda,n,U]/2};
        \text{NIntegrate} \left[ \text{fm} \left[ \text{m1,mwt,mmax,} \lambda, \text{n} \right] \xrightarrow{\Delta \text{l} \left[ \text{m1,mmax,} \lambda, \text{n}, \text{U} \right]}, \left\{ \text{m1,mwt,-pr0} \right\} \right], 
       NIntegrate [
         \texttt{fm} [\texttt{m1}, \texttt{mwt}, \texttt{mmax}, \lambda, \texttt{n}] \ (1-\texttt{pest}[\texttt{m1}]) \ \sqrt{2 \land 1 [\texttt{m1}, \texttt{mmax}, \lambda, \texttt{n}, \texttt{U}]} \ , \\ \{\texttt{m1}, \texttt{m1}/.\texttt{mneg}, \texttt{m1}/.\texttt{mpos}\} \ ] \ ,
       \label{eq:normax} \texttt{NIntegrate} \bigg[ \texttt{fm} \big[ \texttt{m1,mwt,mmax,} \lambda, \texttt{n} \big] \, (1 - \texttt{pest} \big[ \texttt{m1} \big] \big) \, \frac{ \Lambda 1 \, [\texttt{m1,mmax,} \lambda, \texttt{n}, \texttt{U}] }{ \texttt{Abs} \, [\texttt{m1}] } \, , \\ \big\{ \texttt{m1,pr0,mmax} \big\} \, \bigg] \, \bigg] \, \bigg] \, .
     ,\{x,-6,-1,1\};
temptab2;
Table[%[[i]]/Total[%[[i]]],{i,Length[%]}];
test=Transpose[%];
data=Table[Transpose[Join[{xs},{%[[i]]},1]],{i,Length[%]}];
Clear[mmax, \lambda, Es, n, mwt, n0, U] *)
```

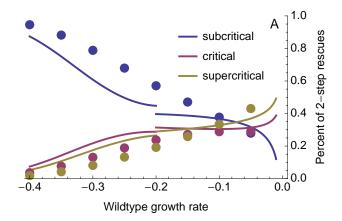
$$\begin{aligned} & \text{data} = \left\{ \left\{ \left\{ \frac{1}{1000000}, \, 0.4394221756678203^{\circ} \right\}, \, \left\{ \frac{1}{100000}, \, 0.4272522411954631^{\circ} \right\}, \\ & \left\{ \frac{1}{10000}, \, 0.40885542554335036^{\circ} \right\}, \, \left\{ \frac{1}{1000}, \, 0.3776908923323242^{\circ} \right\}, \\ & \left\{ \frac{1}{100}, \, 0.3146639788436848^{\circ} \right\}, \, \left\{ \frac{1}{10}, \, 0.16762056652824764^{\circ} \right\} \right\}, \\ & \left\{ \left\{ \frac{1}{1000000}, \, 0.14473575990819615^{\circ} \right\}, \, \left\{ \frac{1}{100000}, \, 0.17364469110334504^{\circ} \right\}, \\ & \left\{ \frac{1}{10000}, \, 0.2168952147581818^{\circ} \right\}, \, \left\{ \frac{1}{1000}, \, 0.2883094661397102^{\circ} \right\}, \\ & \left\{ \frac{1}{100}, \, 0.4246353608625372^{\circ} \right\}, \, \left\{ \frac{1}{10}, \, 0.7093264193395161^{\circ} \right\} \right\}, \\ & \left\{ \left\{ \frac{1}{1000000}, \, 0.4158420644239835^{\circ} \right\}, \, \left\{ \frac{1}{100000}, \, 0.39910306770119186^{\circ} \right\}, \\ & \left\{ \frac{1}{10000}, \, 0.37424935969846795^{\circ} \right\}, \, \left\{ \frac{1}{1000}, \, 0.33339996415279655^{\circ} \right\}, \\ & \left\{ \frac{1}{100}, \, 0.26070066029377786^{\circ} \right\}, \, \left\{ \frac{1}{10}, \, 0.1230530141322361^{\circ} \right\} \right\} \right\}, \end{aligned}$$

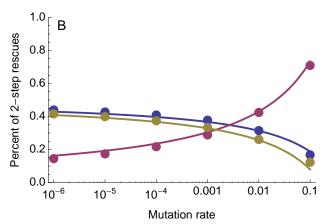
Panel B

```
n0 = 10^4;
mwt = -0.1;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
{small\psi approxRescue(*verylarge\rhoapproxRescue*),
   2 \Lambda0approx fm[0, mwt, mmax, \lambda, n], small\psiapproxRescueSuper};
Total[%]
fig5b = Show
  LogLinearPlot
    %, \{U, 10^{-6}, 10^{-1}\},
    PlotRange \rightarrow \{0, 1\},
    PlotStyle → Thick,
    Frame → {True, True, False, False},
    FrameLabel → {"Mutation rate", "Percent of 2-step rescues", ,},
    FrameTicks → {True, True, False, False},
    LabelStyle \rightarrow labelstyle,
    Epilog →
     Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]
  ListLogLinearPlot[data,
    {\tt PlotMarkers} \rightarrow \{{\tt Graphics[\{defaultcolors[[1]], Disk[]\}, ImageSize} \rightarrow 10],
       Graphics[{defaultcolors[[2]], Disk[]}, ImageSize \rightarrow 10],
       Graphics[{defaultcolors[[3]], Disk[]}, ImageSize \rightarrow 10]}, PlotRange \rightarrow {0, 1}]
Clear [mmax, \lambda, Es, n, mwt, n0, U]
    1.0
Percent of 2-step rescues
   8.0
   0.6
   0.4
   0.2
                                    0.001
                                              0.01
                                                         0.1
      10^{-6}
                10^{-5}
                          10^{-4}
                           Mutation rate
```

Figure 5

GraphicsColumn[$\{fig5a, fig5b\}$, Spacings $\rightarrow -30$, ImageSize $\rightarrow 350$] Export[figdir <> "Figure5.eps", %];





Distribution of growth rates given rescue

Distribution of growth rates among 1-step rescue mutants (equations 15 and 16)

The distribution of growth rates among the rescue mutants in 1-step rescue is simply

$$glm = \frac{U fm[m, mwt, mmax, \lambda, n] pest[m]}{\Lambda 1[mwt, mmax, \lambda, n, U]};$$

We can approximate this using our Laplacian approach. In this case we have

$$\begin{split} \mathbf{h} \left[\psi_{-} \right] &:= \left(\frac{1 - \psi \, / \, 2}{1 - \psi \mathrm{wt} \, / \, 2} \right)^{\theta - \frac{1}{2}} \left(1 - \mathrm{e}^{-2 \, \mathrm{mmax} \, y} \right) \, / \cdot \, y \rightarrow \psi \, \left(1 - \psi \, / \, 4 \right) \\ \mathbf{q} \left[\psi_{-} \right] &:= \frac{1}{4} \, \left(\psi - \psi \mathrm{wt} \right)^{2} \\ \mathbf{h} \mathbf{0} &= \mathrm{Normal} \left[\mathrm{Series} \left[\mathbf{h} \left[\psi \right] \, , \, \left\{ \psi \, , \, 0 \, , \, 1 \right\} \right] \right] ; \\ \mathbf{g} \\ \mathrm{Imapprox} &= \mathrm{Simplify} \left[\\ & \frac{\mathbf{u}}{-\mathrm{mwt}} \, \frac{\sqrt{\rho \mathrm{max}}}{2 \, \sqrt{\pi}} \, \mathbf{h} \mathbf{0} \, \mathrm{Exp} \left[-\rho \mathrm{max} \, \mathbf{q} \left[\psi \right] \right] }{2 \, \sqrt{\pi}} \, / \cdot \, \mathrm{mwt} \rightarrow \mathrm{mmax} \, \left(4 - \psi \mathrm{wt} \right) \, \psi \mathrm{wt} \, / \, 4 \, , \, \left\{ \psi \mathrm{wt} < 0 \, , \, \alpha > 0 \, , \, \rho \mathrm{max} > 0 \right\} \right] \\ & \frac{\mathrm{e}^{\alpha - \frac{1}{4} \, \rho \mathrm{max} \, \left(\psi - \psi \mathrm{wt} \right)^{2} \, \sqrt{\alpha \, \rho \mathrm{max}} \, \psi}}{\psi \mathrm{wt} \, \left(-1 + \mathrm{e}^{\alpha} \, \sqrt{\pi} \, \sqrt{\alpha} \, \, \mathrm{Erfc} \left[\sqrt{\alpha} \, \right] \right)} \end{split}$$

And if we want to plot on an m scale (and still want a pdf) we need to scale by

scaleglapprox =
$$D\left[2\left(1-\sqrt{1-\frac{m}{mmax}}\right), m\right]$$

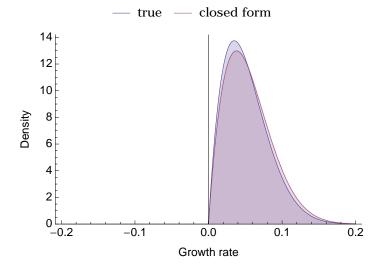
giving

glmApp = scaleglapprox glmapprox;

Numerically compare

```
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
{
            glm,
            g1mApp
           } /. \alpha \rightarrow \psi \text{wt}^2 \rho \text{max} / 4 /. \rho \text{max} \rightarrow \text{mmax} / \lambda /. \theta \rightarrow \text{n} / 2 /. \psi \text{wt} \rightarrow 2 \left[ 1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}} \right] /.
   \psi \to 2 \left( 1 - \sqrt{1 - \frac{m}{mmax}} \right);
Plot[%, \{m, -0.2, 0.2\}, PlotRange \rightarrow \{0, All\},
 Frame → {True, True, False, False}, FrameLabel → {"Growth rate", "Density"},
 PlotLegends → Placed[{"true", "closed form"}, Top],
 Filling → Bottom, LabelStyle → labelstyle]
```

Clear $[mmax, \lambda, Es, n, mwt]$



Distribution of growth rates among 2-step rescue genotypes (equation 17)

The rate of 2-step rescue through double mutants with growth rate m2 is

```
g2numerator[m0_, m2_?NumericQ] := UNIntegrate[fm[m, m0, mmax, <math>\lambda, n]
       (1-pest[m]) prescuem[m, U fm[m2, m, mmax, \lambda, n] pest[m2]], {m, -\infty, mmax}];
Integrating this over m2 then provides the correct normalization,
g2denominator[m0_?NumericQ] := NIntegrate[U fm[m, m0, mmax, λ, n] (1 - pest[m])
     prescuem[m, Ufm[m2, m, mmax, \lambda, n] pest[m2]], \{m, -\infty, mmax\}, \{m2, 0, mmax\}];
```

Plot growth rates of rescue genotypes (figure 6)

Panels

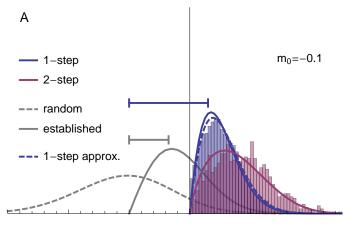
```
xmin = -0.3;
xmax = 0.25;
ymax = 28;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
   (*random*)
   Table[\{m2, fm[m2, mwt, mmax, \lambda, n]\}, \{m2, xmin, xmax, 0.01\}],
   (*established*)
   Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, 0.01\}]
 };
oldtheory = ListPlot[%, PlotRange → All, Joined → True,
    PlotStyle → {Directive[Thick, Gray, Dashed], Directive[Thick, Gray]},
    PerformanceGoal → "Speed",
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
          {"random", "established"}], Scaled@{1.5 / 8, 1 / 2}]];
   (*1 step*)
   Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, 0.005\}],
   (*2 step*)
   total = Re[g2denominator[mwt]];
   Table \left[\left\{m2, \frac{1}{total} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax}, 0.005\right\}\right]
 };
theory = ListPlot[%, PlotRange → All,
    Joined → True, PlotStyle → Thick, PerformanceGoal → "Speed",
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
          {"1-step", "2-step"}], Scaled@{1.5/8,1/2}]];
glmApp /. \alpha \rightarrow \psi wt^2 \rho max / 4 /. \rho max \rightarrow mmax / \lambda /. \theta \rightarrow n / 2 /. \psi wt \rightarrow 2 \left[ 1 - \sqrt{1 - \frac{mwt}{mmax}} \right] /.
  \psi \to 2 \left(1 - \sqrt{1 - \frac{m}{mmax}}\right);
onestepapp = Plot[%, {m, 0, xmax}, PlotStyle → {Thick, Dashed},
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
          {"1-step approx."}], Scaled@{1.5/8,1/2}]];
```

```
dat = Import[datadir <>
    "dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.10_mutmax10_nreps10000.
onestep = Select[dat, #[[2]] == 1 &][[All, 1]];
twostep = Select[dat, #[[2]] == 2 &][[All, 1]];
data = Histogram[{onestep, twostep}, 50, "PDF", AxesOrigin \rightarrow {0, 0}];
fig6a = Show[oldtheory, data, theory, onestepapp,
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  LabelStyle → labelstyle,
  FrameTicksStyle → {FontColor → White, Automatic, Automatic},
  Epilog \rightarrow {
    Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily → "Helvetica"], Scaled@{7/8,3/4}],
    Text[Style["A", 14, Bold, FontFamily \rightarrow "Helvetica"], Scaled@{0.05, 0.95}],
     {Directive[Gray, Thick], Line[{{mwt, 10}, {-0.035, 10}}]},
     {Directive[Gray, Thick], Line[{{mwt, 9.5}, {mwt, 10.5}}]},
     {Directive[Gray, Thick], Line[{{-0.035, 9.5}, {-0.035, 10.5}}]},
     {Directive[defaultcolors[[1]], Thick], Line[{{mwt, 15}, {0.03, 15}}]},
     {Directive[defaultcolors[[1]], Thick], Line[{{mwt, 14.5}, {mwt, 15.5}}]},
    {Directive[defaultcolors[[1]], Thick], Line[{{0.03, 14.5}, {0.03, 15.5}}]}
   }
 ]
```

NIntegrate::izero:

Clear $[mmax, \lambda, Es, n, U, mwt]$

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. \gg



```
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
   (*random*)
```

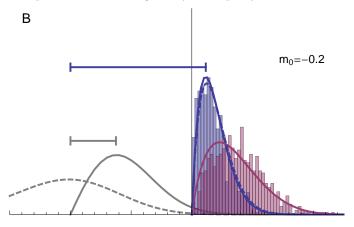
```
Table [\{m2, fm[m2, mwt, mmax, \lambda, n]\}, \{m2, xmin, xmax, 0.01\}],
   (*established*)
  Table [\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, 0.01\}]
oldtheory = ListPlot[%, PlotRange → All, Joined → True,
    PlotStyle → {Directive[Thick, Gray, Dashed], Directive[Thick, Gray]},
    PerformanceGoal → "Speed"];
   (*1 step*)
  Table [\{m2, g1m / . m \rightarrow m2\}, \{m2, 0, xmax, 0.005\}],
   (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax}, 0.005\right\}\right]
theory = ListPlot[%, PlotRange → All,
    Joined → True, PlotStyle → Thick, PerformanceGoal → "Speed"];
glmApp /. \alpha \rightarrow \psiwt<sup>2</sup> \rhomax / 4 /. \rhomax \rightarrow mmax / \lambda /. \theta \rightarrow n / 2 /. \psiwt \rightarrow 2 \left| 1 - \sqrt{1 - \frac{mwt}{mmax}} \right| /.
  \psi \to 2 \left( 1 - \sqrt{1 - \frac{m}{mmax}} \right);
onestepapp = Plot[%, {m, 0, xmax}, PlotStyle → {Thick, Dashed}];
dat = Import[datadir <>
     "dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.20_mutmax10_nreps100000
        .csv"];
onestep = Select[dat, #[[2]] == 1 &][[All, 1]];
twostep = Select[dat, #[[2]] == 2 &][[All, 1]];
data = Histogram[{onestep, twostep}, 50, "PDF", AxesOrigin \rightarrow {0, 0}];
fig6b = Show[oldtheory, data, theory, onestepapp,
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  LabelStyle → labelstyle,
  FrameTicksStyle → {FontColor → White, Automatic, Automatic}, Automatic},
  Epilog → {
     Text[Style["m0=" <> ToString[mwt],
        12, FontFamily → "Helvetica"], Scaled@{7/8,3/4}],
     Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
     {Directive[Gray, Thick], Line[{{mwt, 10}, {-0.125, 10}}]},
     {Directive[Gray, Thick], Line[{{mwt, 9.5}, {mwt, 10.5}}]},
     {Directive[Gray, Thick], Line[{{-0.125, 9.5}, {-0.125, 10.5}}]},
     {Directive[defaultcolors[[1]], Thick], Line[{{mwt, 20}, {0.023, 20}}]},
     {Directive [defaultcolors [[1]], Thick], Line [{{mwt, 20 - 0.5}}, {mwt, 20 + 0.5}}]},
     {Directive[defaultcolors[[1]], Thick],
      Line[\{(0.023, 20-0.5), \{0.023, 20+0.5\}\}\}]
```

```
}
]
```

Clear[mmax, λ , Es, n, U, mwt]

NIntegrate::izero:

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>



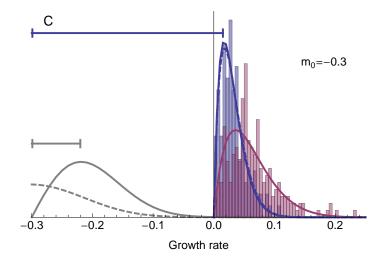
```
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.3;
   (*random*)
  Table[\{m2, fm[m2, mwt, mmax, \lambda, n]\}, \{m2, xmin, xmax, 0.01\}],
   (*established*)
  Table [\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2 - mwt]) / NIntegrate[
        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, 0.01\}
oldtheory = ListPlot[%, PlotRange → All, Joined → True,
    PlotStyle → {Directive[Thick, Gray, Dashed], Directive[Thick, Gray]},
    PerformanceGoal → "Speed"];
   (*1 step*)
  Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, 0.005\}],
   (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax, 0.005}\right\}\right]
theory = ListPlot[%, PlotRange → All,
    Joined → True, PlotStyle → Thick, PerformanceGoal → "Speed"];
```

```
glmApp /. \alpha \rightarrow \psi \text{wt}^2 \rho \text{max} / 4 /. \rho \text{max} \rightarrow \text{mmax} / \lambda /. \theta \rightarrow \text{n} / 2 /. \psi \text{wt} \rightarrow 2 \left[1 - \sqrt{1 - \frac{\text{mwt}}{\text{mmax}}}\right] /.
  \psi \rightarrow 2 \left[1 - \sqrt{1 - \frac{m}{mmax}}\right];
onestepapp = Plot[%, {m, 0, xmax}, PlotStyle → {Thick, Dashed}];
dat = Import[datadir <>
     "dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.30_mutmax10
        _nreps1000000.csv"];
onestep = Select[dat, #[[2]] == 1 &][[All, 1]];
twostep = Select[dat, #[[2]] == 2 &][[All, 1]];
data = Histogram[{onestep, twostep}, 50, "PDF", AxesOrigin \rightarrow {0, 0}];
fig6c = Show[oldtheory, data, theory, onestepapp,
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  FrameLabel → {"Growth rate"},
  LabelStyle → labelstyle, Epilog → {
     Text[Style["m0=" <> ToString[mwt],
        12, FontFamily → "Helvetica"], Scaled@{7/8,3/4}],
     Text[Style["C", 14, Bold, FontFamily \rightarrow "Helvetica"], Scaled@{0.05, 0.95}],
     {Directive[Gray, Thick], Line[{{mwt, 10}, {-0.22, 10}}]},
     {Directive[Gray, Thick], Line[{{mwt + 0.001, 9.5}, {mwt + 0.001, 10.5}}]},
     {Directive[Gray, Thick], Line[{{-0.22, 9.5}, {-0.22, 10.5}}]},
     {Directive[defaultcolors[[1]], Thick], Line[{{mwt, 25}, {0.015, 25}}]},
     {Directive[defaultcolors[[1]], Thick],
       Line[\{\{\text{mwt} + 0.001, 25 - 0.5\}, \{\text{mwt} + 0.001, 25 + 0.5\}\}\}\},
     {Directive[defaultcolors[[1]], Thick],
       Line[\{\{0.015, 25-0.5\}, \{0.015, 25+0.5\}\}\}]
    }
 ]
```

NIntegrate::izero:

Clear $[mmax, \lambda, Es, n, U, mwt]$

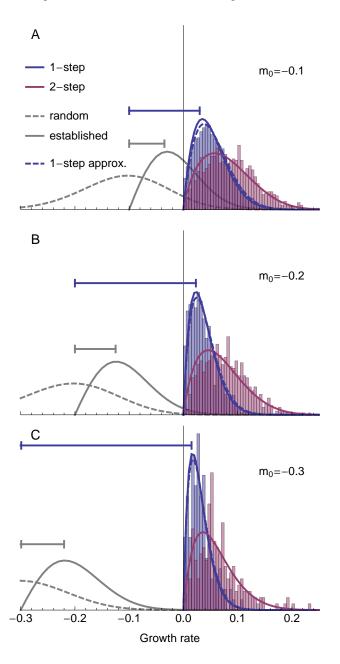
Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>



Clear[ymax, xmin, xmax]

Figure 6

 $GraphicsColumn[\{fig6a, fig6b, fig6c\}, Spacings \rightarrow -30, ImageSize \rightarrow 350]$ Export[figdir <> "Figure6.pdf", %];



Distribution of first-step growth rates in 2-step rescue genotypes (equation 18)

As, in the 1-step case, we can use the $\Lambda 2$ term to write the distribution of growth rates of rescue genotypes

h2[m_?NumericQ, m0_] :=
$$(U fm[m, m0, mmax, \lambda, n] (1-pest[m]) prescuem[m, \Lambda1[m, mmax, \lambda, n, U]]) / \Lambda2[m0, mmax, \lambda, n, U]$$

We can also use our approximations above to get a closed form approximation for this.

For sufficiently subcritical m near -m* we have

small
$$\psi$$
approx = $-\frac{2 e^{-\frac{\rho \max \psi v^2}{4}}}{\sqrt{\pi} \sqrt{\rho \max} \psi}$

Integrate[small ψ approx, { ψ , a, b}, Assumptions \rightarrow {a < b < 0}]; $small \psi approx$

$$D\left[2\left(1-\sqrt{1-\frac{m}{mmax}}\right), m\right] % /. a \rightarrow 2\left(1-\sqrt{1-\frac{mwt}{mmax}}\right) /. b \rightarrow 2\left(1-\sqrt{1+\frac{mstar}{mmax}}\right) /.$$

$$mstar \rightarrow \sqrt{\Lambda 0approx/2}$$
 /. $\rho max \rightarrow mmax/\lambda$ /. $\theta \rightarrow n/2$ /.

$$\psi$$
wt \rightarrow 2 $\left(1 - \sqrt{1 - \frac{mwt}{mmax}}\right)$ /. $\psi \rightarrow$ 2 $\left(1 - \sqrt{1 - \frac{m}{mmax}}\right)$ // Simplify

$$\frac{1}{\psi \operatorname{Log}\left[\frac{b}{a}\right]}$$

$$-1 \bigg/ \left[2 \left(-1 + \sqrt{1 - \frac{m}{mmax}} \right) \sqrt{1 - \frac{m}{mmax}} \right. \\ \left. mmax \, Log \left[\frac{1 - \sqrt{1 + \frac{\sqrt{U \sqrt{mmax} \, \lambda}}{mmax \, \pi^{1/4}}}}{1 - \sqrt{1 - \frac{mwt}{mmax}}} \right] \right]$$

Compare distribution of subcriticals

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
total = UNIntegrate[
        fm[m, mwt, mmax, \lambda, n] (1-pest[m]) prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]], {m, -\infty, 0}];
Show
  Plot \Big[ \frac{1}{total} U \, fm[m, mwt, mmax, \lambda, n] \, (1 - pest[m]) \, prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]], \\
    \{m, -0.2, 0\}, PerformanceGoal \rightarrow "Speed", PlotRange \rightarrow \{0, All\},
\operatorname{Plot}\left[\left\{,-1\right/\left(2\left(-1+\sqrt{1-\frac{m}{mmax}}\right)\sqrt{1-\frac{m}{mmax}}\right)\operatorname{mmax}\operatorname{Log}\left[\frac{1-\sqrt{1+\frac{\sqrt{U\sqrt{mmax}\lambda}}{mmax}\pi^{1/4}}}{1-\sqrt{1-\frac{mwt}{mmax}}}\right]\right]\right\},
    \{m, -0.2, 0\}, PlotRange \rightarrow \{0, 70\}, PlotLegends \rightarrow \{"true", "approx"\}
Clear [mmax, \lambda, Es, n, U, mwt]
                                                                                              true
                                                                                              approx
                                                                              20
```

10

and for large ρ max we have

-0.15

-0.10

-0.20

-0.05

$$\begin{aligned} & \text{verylarge} \rho approx = -\frac{32 \, e^{-\frac{1}{4} \, \rho max} \, (\theta^2 + (\psi - \psi wt)^2)}{\sqrt{\pi} \, \rho max^{3/2} \, \psi wt^3}; \\ & \text{Integrate} [\text{verylarge} \rho approx, \{\psi, -\infty, 0\}, \text{ Assumptions} \rightarrow \{\rho max > 0, \psi wt < 0\}]; \\ & \text{verylarge} \rho approx, \\ & \text{%} \\ & \text{Simplify} \\ & \text{D} \Big[2 \, \Big(1 - \sqrt{1 - \frac{m}{mmax}} \Big), \, m \Big] \, \% \, /. \, a \rightarrow 2 \, \left(1 - \sqrt{1 - \frac{mwt}{mmax}} \right) \, /. \, b \rightarrow 2 \, \left(1 - \sqrt{1 + \frac{mstar}{mmax}} \right) \, /. \, mstar \rightarrow \\ & \sqrt{\Lambda 0 approx / 2} \, /. \, \rho max \rightarrow mmax \, / \, \lambda \, /. \, \theta \rightarrow n \, / \, 2 \, /. \, \psi wt \rightarrow 2 \, \left(1 - \sqrt{1 - \frac{mwt}{mmax}} \right) \, /. \\ & \psi \rightarrow 2 \, \left(1 - \sqrt{1 - \frac{m}{mmax}} \right), \, \{m < 0, \, mwt < 0, \, mmax > 0, \, \lambda > 0\} \right] \\ & e^{-\frac{1}{4} \, \rho max} \, (\psi^2 + (\psi - \psi wt)^2) + \frac{\rho max}{6} \, \psi dt} \\ & Erfc \Big[\frac{\sqrt{\rho max} \, \psi wt}{2 \, \sqrt{2}} \Big] \\ & e^{\frac{1}{4} \, \rho max} \, (\psi^2 + (\psi - \psi wt)^2) + \frac{\rho max}{2} \, \psi dt} \\ & - \frac{1}{2} \, \sqrt{\rho max} \, \psi dt} \\ & - \frac{1}{2} \, \sqrt{\rho max} \, \frac{\rho max}{2} \, \sqrt{\frac{2}{\pi}} \, \sqrt{\rho max} \, \frac{\rho max}{\lambda} \\ & \sqrt{-m + mmax} \, /. \, \Delta \, Erfc} \Big[\frac{1 - \sqrt{1 - \frac{mwt}{mmax}} \, \sqrt{\frac{mmax}{\lambda}}}{\sqrt{2}} \Big] \end{aligned}$$

Compare distribution of subcriticals

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.4;
total = UNIntegrate[
           \texttt{fm}[\texttt{m},\texttt{mwt},\texttt{mmax},\lambda,\texttt{n}] \; (\texttt{1-pest}[\texttt{m}]) \; \texttt{prescuem}[\texttt{m},\Lambda\texttt{1}[\texttt{m},\texttt{mmax},\lambda,\texttt{n},\texttt{U}]] \; , \; \{\texttt{m},-\infty,0\}] \; ; \\
Show
   Plot \Big[ \frac{1}{total} U \, fm[m, mwt, mmax, \lambda, n] \, (1 - pest[m]) \, prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]], \\
     \{m, -0.4, 0\}, PerformanceGoal \rightarrow "Speed", PlotRange \rightarrow \{0, All\},
  \text{Plot} \Big[ \Big\{, \frac{1}{\sqrt{1 - \frac{m}{m \text{max}}}} \, \text{mmax} \Big] e^{\frac{4 \, \text{m+mwt+mmax} \Big[ - 6 + 4 \, \sqrt{1 - \frac{m}{m \text{max}}} \, - 2 \, \sqrt{1 - \frac{m \text{ot}}{m \text{max}}} \, + 4 \, \sqrt{1 - \frac{m}{m \text{max}}} \, \sqrt{1 - \frac{m \text{ot}}{m \text{max}}} \Big]} \, \sqrt{\frac{2}{\pi}} \, \sqrt{\frac{m \text{max}}{\lambda}} \Big] \Big/ 
             \operatorname{Erfc}\left[\frac{\left(1-\sqrt{1-\frac{\operatorname{mwt}}{\operatorname{mmax}}}\right)\sqrt{\frac{\operatorname{mmax}}{\lambda}}}{\sqrt{2}}\right], \{m, -0.4, 0\},
     PlotRange → {0, 70}, PlotLegends → {"true", "approx"}
Clear [mmax, \lambda, Es, n, U, mwt]
                                                                                                                                   true
                                                                                                                                   approx
```

Our sufficiently critical approximation is

-0.2

-0.3

-0.4

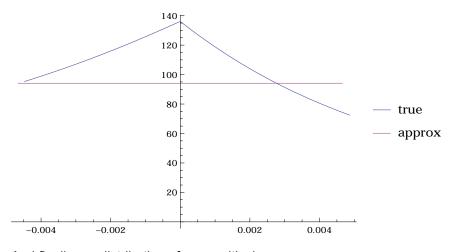
-0.1

$$\begin{split} &\text{fm} [\texttt{0, m0, mmax, } \lambda, \texttt{n}] \, \sqrt{2 \, \frac{2 \, \text{U} \, \sqrt{\text{mmax} \, \lambda}}{\sqrt{\pi}}} \; ; \\ &\text{Integrate} \Big[\$, \, \Big\{ \texttt{m, -} \sqrt{\frac{\, \text{U} \, \sqrt{\text{mmax} \, \lambda}}{\sqrt{\pi}}} \; , \, \sqrt{\frac{\, \text{U} \, \sqrt{\text{mmax} \, \lambda}}{\sqrt{\pi}}} \; \Big\} \Big] ; \\ &\text{Simplify} [\$\$ / \$, \, \{ \text{mmax} > \texttt{0, } \lambda > \texttt{0} \}] \\ &\frac{\pi^{1/4}}{2 \, \sqrt{\text{U}} \, \left(\text{mmax} \, \lambda \right)^{1/4}} \end{aligned}$$

Compare distributions of criticals

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
\{\text{mneg = FindRoot}[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, -0.1\}],
   mpos = FindRoot[2m^2 - \Lambda 1[m, mmax, \lambda, n, U], \{m, 0.1\}];
pr0 = \sqrt{\Lambda 1[0, mmax, \lambda, n, U] / 2};
total = U NIntegrate[fm[m, mwt, mmax, \lambda, n] (1 - pest[m])
         prescuem[m, \Lambda1[m, mmax, \lambda, n, U]], {m, m/. mneg, m/. mpos}];
Show
 Plot \left[\frac{1}{\text{total}} \text{U fm}[\text{m, mwt, mmax}, \lambda, \text{n}] (1-\text{pest}[\text{m}]) \text{ prescuem}[\text{m, } \Lambda 1[\text{m, mmax}, \lambda, \text{n, } U]], \right]
   \{m, m/. mneg, m/. mpos\}, PerformanceGoal \rightarrow "Speed", PlotRange \rightarrow \{0, All\},
 Plot\left[\left\{, \frac{\pi^{1/4}}{2\sqrt{U} \left(\max \lambda\right)^{1/4}}\right\}, \{m, -pr0, pr0\}, PlotRange \rightarrow \{0, All\},\right]
   PlotLegends → {"true", "approx"}],
  PlotRange → All
```

Clear $[mmax, \lambda, Es, n, U, mwt]$



And finally our distribution of supercriticals

$$small \psi approx = \frac{2 e^{-\frac{\rho max \psi v t^2}{4}}}{\sqrt{\pi} \sqrt{\rho max} \psi};$$

Integrate[small ψ approx, { ψ , a, b}, Assumptions \rightarrow {0 < a < b}]; small#approx

$$D\left[2\left(1-\sqrt{1-\frac{m}{mmax}}\right), m\right] % /. a \rightarrow 2\left(1-\sqrt{1-\frac{mstar}{mmax}}\right) /. b \rightarrow \frac{\sqrt{2}}{\sqrt{\rho max}} /.$$

 $mstar \rightarrow \sqrt{\Lambda 0 approx / 2} /. \rho max \rightarrow mmax / \lambda /. \theta \rightarrow n / 2 /.$

$$\psi$$
wt $\rightarrow 2 \left(1 - \sqrt{1 - \frac{mwt}{mmax}}\right) / \cdot \psi \rightarrow 2 \left(1 - \sqrt{1 - \frac{m}{mmax}}\right)$

$$\frac{1}{\psi \operatorname{Log}\left[\frac{b}{a}\right]}$$

$$1 \bigg/ \left(2 \left(1 - \sqrt{1 - \frac{m}{mmax}} \right) \sqrt{1 - \frac{m}{mmax}} \, \operatorname{mmax} \, \operatorname{Log} \left[\frac{1}{\sqrt{2} \, \sqrt{\frac{mmax}{\lambda}}} \left(1 - \sqrt{1 - \frac{\sqrt{U\sqrt{mmax}\,\lambda}}{mmax\,\pi^{1/4}}} \right) \right] \right)$$

Note this is good only to

$$\begin{aligned} & \mathtt{Solve} \Big[\frac{\sqrt{2}}{\sqrt{\rho \mathtt{max}}} = 2 \left(1 - \sqrt{1 - \frac{\mathtt{m}}{\mathtt{mmax}}} \right), \, \mathtt{m} \Big] \, / \, . \, \, \rho \mathtt{max} \to \mathtt{mmax} \, / \, \lambda \, / / \, \, \mathtt{Simplify} \\ & \Big\{ \Big\{ \mathtt{m} \to \left(-\frac{1}{2} + \sqrt{2} \, \sqrt{\frac{\mathtt{mmax}}{\lambda}} \, \right) \, \lambda \Big\} \Big\} \end{aligned}$$

Compare distn of supercriticals

```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
total = U NIntegrate [fm[m, mwt, mmax, \lambda, n]
          (1 - pest[m]) prescuem[m, \Lambda 1[m, mmax, \lambda, n, U]], {m, 0, mmax}];
Show
 Plot \left[\frac{1}{1+\alpha+1}U \text{ fm}[m, mwt, mmax, \lambda, n] (1-pest[m]) \text{ prescuem}[m, \Lambda 1[m, mmax, \lambda, n, U]], \right]
    \{m, 0, 0.2\}, PerformanceGoal \rightarrow "Speed", PlotRange \rightarrow \{0, All\},
Plot\left[\left\{, 1 \middle/ 2 \left(1 - \sqrt{1 - \frac{m}{mmax}}\right) \sqrt{1 - \frac{m}{mmax}} \right] \right]
           \log\left[\frac{1}{\sqrt{2}\sqrt{\frac{mmax}{\lambda}}\left[1-\sqrt{1-\frac{\sqrt{U\sqrt{mmax}\lambda}}{mmax}\pi^{1/4}}\right]}\right], \left\{m, 0, \left(-\frac{1}{2}+\sqrt{2}\sqrt{\frac{mmax}{\lambda}}\right)\lambda\right\},
    PlotRange → {0, 150}, PlotLegends → {"true", "approx"}
Clear [mmax, \lambda, Es, n, U, mwt]
140
120
100
                                                                                          - true
 80
                                                                                            approx
 60
 40
 20
```

0.20

0.15

0.05

0.10

Plot growth rates of rescue intermediates (figures 7 and S2)

As function of wildtype growth rate (figure 7)

```
xmin = -0.3;
xmax = 0.15;
ymax = 30;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
\{\text{mneg} = \text{FindRoot}[2 \text{ m1}^2 - \Lambda 1[\text{m1}, \text{mmax}, \lambda, n, U], \{\text{m1}, -0.1\}],
   mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
   fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
   \texttt{fm}[\texttt{m1, mwt, mmax, } \lambda, \texttt{n}] \; \frac{\texttt{\Lambda1}[\texttt{m1, mmax, } \lambda, \texttt{n, U}]}{\texttt{Abs}[\texttt{m1}]} \; \texttt{HeavisideTheta[(-m1-pr0)];}
supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
     \frac{\text{A1}[\text{m1}, \text{mmax}, \lambda, \text{n}, \text{U}]}{\text{Abs}[\text{m1}]} \text{HeavisideTheta}[(\text{m1} - \text{pr0})];
allexact = NIntegrate[exact, {m1, mwt, 0.1}];
   fm[m1, mwt, mmax, \lambda, n],
   (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
    NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}]
 };
oldtheory = Plot[
     %,
     {m1, xmin, xmax},
     PlotRange \rightarrow \{0, All\},
     (*Filling→Bottom,*)
     PlotStyle → {Directive[Gray, Thick, Dashed], Directive[Gray, Thick]},
    PerformanceGoal → "Speed",
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
           {"random", "established"}], Scaled@{1/8, 1.25/2}]
   ];
```

```
subcritical critical supercritical
   allexact
               allexact
                             allexact
theory = Plot[
   %,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   Filling → Bottom,
   PerformanceGoal → "Speed",
   PlotLegends → Placed[SwatchLegend[{Directive[defaultcolors[[1]], Opacity[0.5]],
        Directive[defaultcolors[[2]], Opacity[0.5]], Directive[defaultcolors[[3]],
         Opacity[0.5]]}, Style[#, 12, FontFamily → "Helvetica"] & /@
        {"subcritical", "critical", "supercritical"}], Scaled@{1/8,0.65/2}]
  ];
exactplot = Plot[
   exact / allexact, {m1, xmin, xmax},
   PlotStyle → Directive[Thick, Black], PerformanceGoal → "Speed", PlotRange → All,
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
        {"first-step"}], Scaled@{1/8, 1.25/2}]
  ];
dat = Import[datadir <>
     "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.10_mutmax10
       _nreps100000.csv"];
Select[dat, #[[2]] = 2 &][[All, 3]];
data = Histogram[%, 50, "PDF",
   AxesOrigin \rightarrow {0, 0}, ChartStyle \rightarrow Directive[Gray, Opacity[0.25]]];
fig7a = Show[
  oldtheory, data, theory, exactplot,
  Frame → {True, False, False, False},
  FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  LabelStyle → labelstyle,
  Epilog \rightarrow {Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily → "Helvetica"], Scaled@{7/8,3/4}],
    Text[Style["A", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]}
 ]
Clear [mmax, \lambda, Es, n, U, mwt]
```

```
Α
 --- random
                                                  m_0 = -0.1
 — established
 first-step
 subcritical
 critical
 supercritical
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es/n;
Es = 0.01;
n = 4;
mwt = -0.2;
\{\text{mneg = FindRoot}[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, -0.1\}],
   mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
   fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
   fm[m1, mwt, mmax, \lambda, n] \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{\Pi 1[m1, mmax, \lambda, n]} HeavisideTheta[(-m1-pr0)];
supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
     \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{} HeavisideTheta[(m1-pr0)];
             Abs[m1]
allexact = NIntegrate[exact, {m1, mwt, 0.1}];
{
   fm[m1, mwt, mmax, \lambda, n],
   (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
    NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}]
oldtheory = Plot[
    %,
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},\
    (*Filling→Bottom,*)
    PlotStyle → {Directive[Gray, Thick, Dashed], Directive[Gray, Thick]},
```

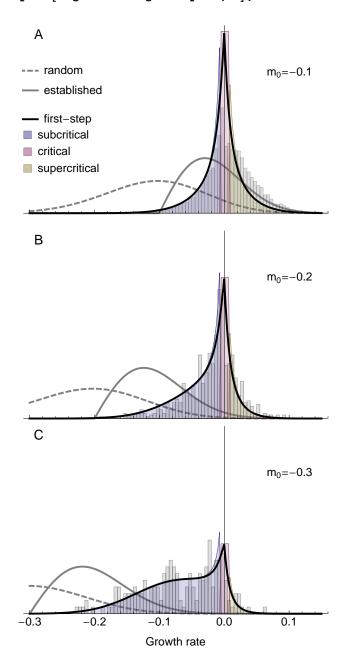
```
PerformanceGoal → "Speed"
  ];
 subcritical
                critical
                            supercritical
   allexact
                allexact
                               allexact
theory = Plot[
   %,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   Filling → Bottom,
   PerformanceGoal → "Speed"
  ];
exactplot = Plot[
   exact/allexact, {m1, xmin, xmax},
   PlotStyle → Directive [Thick, Black], PerformanceGoal → "Speed", PlotRange → All
  ];
dat = Import[datadir <>
     "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.20_mutmax10
       _nreps100000.csv"];
Select[dat, #[[2]] = 2 &][[All, 3]];
data = Histogram[%, 50, "PDF",
   AxesOrigin \rightarrow {0, 0}, ChartStyle \rightarrow Directive[Gray, Opacity[0.25]]];
fig7b = Show[
  oldtheory, data, theory, exactplot,
  Frame → {True, False, False, False},
  FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  LabelStyle → labelstyle,
  Epilog \rightarrow {Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily \rightarrow "Helvetica"], Scaled@\{7/8, 3/4\}],
     Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]}
 1
Clear [mmax, \lambda, Es, n, U, mwt]
  В
                                          m_0 = -0.2
U = 2 * 10^{-3};
mmax = 0.5;
```

```
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.3;
\{mneg = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
  fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
  supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
    \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{} HeavisideTheta[(m1-pr0)];
allexact = NIntegrate[exact, {m1, mwt, 0.1}];
  fm[m1, mwt, mmax, \lambda, n],
   (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
   NIntegrate [fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}]
 } ;
oldtheory = Plot[
    %,
    {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
    (*Filling→Bottom,*)
    PlotStyle → {Directive[Gray, Thick, Dashed], Directive[Gray, Thick]},
   PerformanceGoal → "Speed"
  ];
 subcritical allexact, supercritical allexact ;
theory = Plot[
    %,
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},
   Filling → Bottom,
   PerformanceGoal → "Speed"
  ];
exactplot = Plot[
    exact/allexact, {m1, xmin, xmax},
   PlotStyle → Directive [Thick, Black], PerformanceGoal → "Speed", PlotRange → All
  ];
```

```
dat = Import[datadir <>
     "int_dfe_poisson_N10000_n4_U0.00200_Es0.01_mmax0.50_mwt-0.30_mutmax10
       _nreps1000000.csv"];
Select[dat, #[[2]] == 2 &][[All, 3]];
data = Histogram[%, 50, "PDF",
   AxesOrigin \rightarrow {0, 0}, ChartStyle \rightarrow Directive[Gray, Opacity[0.25]]];
fig7c = Show[
  oldtheory, data, theory, exactplot,
  Frame → {True, False, False, False},
  FrameLabel \rightarrow {"Growth rate"},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  LabelStyle → labelstyle,
  Epilog → {Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily \rightarrow "Helvetica"], Scaled@\{7/8, 3/4\}],
     Text[Style["C", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]}
 ]
Clear [mmax, \lambda, Es, n, U, mwt]
  С
                                           m_0 = -0.3
                      -0.1
                                  0.0
                                              0.1
                      Growth rate
```

Clear[ymax, xmin, xmax]

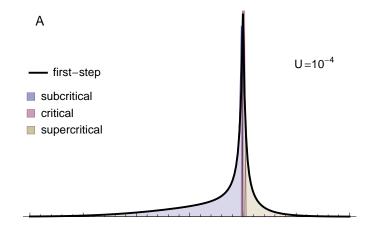
 $\texttt{GraphicsColumn[\{fig7a, fig7b, fig7c\}, Spacings} \rightarrow \texttt{-30, ImageSize} \rightarrow \texttt{350]}$ Export[figdir <> "Figure7.pdf", %];



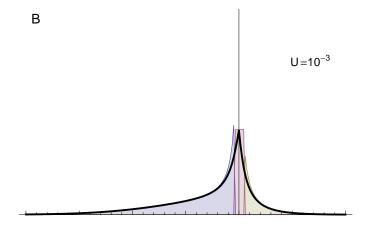
As function of mutation (figure S2)

```
xmin = mwt;
xmax = -mwt / 2;
ymax = 70;
U = 10^{-4};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
```

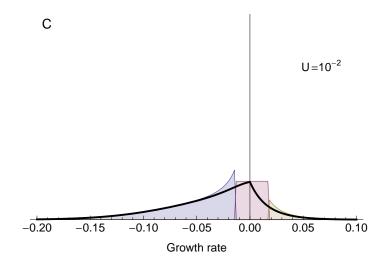
```
n = 4;
mwt = -0.2;
\{\text{mneg} = \text{FindRoot}[2 \text{ m1}^2 - \Lambda 1[\text{m1}, \text{mmax}, \lambda, n, U], \{\text{m1}, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
  fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1} [0, mmax, \lambda, n, U] HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
  supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
    \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{} HeavisideTheta[(m1-pr0)];
total = NIntegrate[exact, {m1, mwt, 0.1}];
figS2a = Show
  Plot
    {subcritical / total, critical / total, supercritical / total},
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},
    (*PlotStyle→Thick,*)
    Filling → Bottom,
    Frame → {True, False, False, False},
    FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
    PlotLegends →
     Placed[SwatchLegend[{Directive[defaultcolors[[1]], Opacity[0.5]],
         Directive[defaultcolors[[2]], Opacity[0.5]], Directive[defaultcolors[[3]],
          Opacity[0.5]]}, Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@
         {"subcritical", "critical", "supercritical"}], Scaled@{1/8, 0.5}],
    LabelStyle → labelstyle,
    Epilog →
     \{\text{Text}[\text{Style}["U=10^{-4}", 12, \text{FontFamily} \rightarrow "\text{Helvetica"}], \text{Scaled}@\{7/8, 3/4\}],
      Text[Style["A", 14, Bold, FontFamily \rightarrow "Helvetica"], Scaled@\{0.05, 0.95\}],
   PerformanceGoal → "Speed"
  Plot[exact / total, {m1, xmin, xmax},
    PlotStyle → Directive[Thick, Black], PerformanceGoal → "Speed", PlotRange → All,
   PlotLegends → Placed[Style[#, 12, FontFamily → "Helvetica"] & /@ {"first-step"},
      Scaled@{1/8,0.7}]
   ],
  PlotRange → {{xmin, xmax}, {0, ymax}}
Clear [mmax, \lambda, Es, n, U, mwt]
```



```
U = 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
\{\text{mneg} = \text{FindRoot}[2 \text{ m1}^2 - \Lambda 1[\text{m1}, \text{mmax}, \lambda, n, U], \{\text{m1}, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1 [m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
   fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
   supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
     \frac{\text{A1}[\text{m1, mmax, } \lambda, \text{n, U}]}{\text{Abs}[\text{m1}]} \text{ HeavisideTheta[(m1-pr0)];}
total = NIntegrate[exact, {m1, mwt, 0.1}];
figS2b = Show
   Plot[
    {subcritical / total, critical / total, supercritical / total},
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},\
    (*PlotStyle→Thick,*)
    Filling → Bottom,
    Frame → {True, False, False, False},
    FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
    LabelStyle → labelstyle,
    Epilog →
     \{\text{Text}[\text{Style}["U=10^{-3}", 12, \text{FontFamily} \rightarrow "\text{Helvetica"}], \text{Scaled@}\{7/8, 3/4\}],
       Text[Style["B", 14, Bold, FontFamily \rightarrow "Helvetica"], Scaled@\{0.05, 0.95\}],
    PerformanceGoal → "Speed"
   Plot[{exact / total}, {m1, xmin, xmax}, PlotStyle → Directive[Thick, Black],
    PerformanceGoal → "Speed", PlotRange → All(*,Filling→Bottom*)],
   PlotRange → {{xmin, xmax}, {0, ymax}}
Clear [mmax, \lambda, Es, n, U, mwt]
```

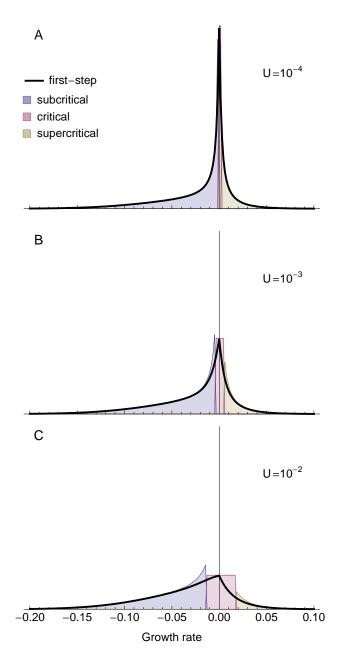


```
U = 10^{-2};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
\{\text{mneg} = \text{FindRoot}[2 \text{ m1}^2 - \Lambda 1[\text{m1}, \text{mmax}, \lambda, n, U], \{\text{m1}, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1 [m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
   fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
   supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
     \frac{\text{A1}[\text{m1, mmax, } \lambda, \text{n, U}]}{\text{Abs}[\text{m1}]} \text{ HeavisideTheta[(m1-pr0)];}
total = NIntegrate[exact, {m1, mwt, 0.1}];
figS2c = Show
  Plot[
    {subcritical / total, critical / total, supercritical / total},
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},\
    Filling → Bottom,
    Frame → {True, False, False, False},
    FrameLabel → {"Growth rate", ""},
    LabelStyle → labelstyle,
    Epilog →
     \{\text{Text}[\text{Style}["U=10^{-2}", 12, \text{FontFamily} \rightarrow "\text{Helvetica"}], \text{Scaled@}\{7/8, 3/4\}], \}
       Text[Style["C", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]},
    PerformanceGoal → "Speed"
   ],
  Plot[{exact/total}, {m1, xmin, xmax},
    PlotStyle → Directive[Thick, Black], PerformanceGoal → "Speed", PlotRange → All],
   PlotRange → {{xmin, xmax}, {0, ymax}}
Clear [mmax, \lambda, Es, n, U, mwt]
```



Clear[ymax, xmin, xmax]

 $\texttt{GraphicsColumn[\{figS2a, figS2b, figS2c\}, Spacings} \rightarrow -30, \texttt{ImageSize} \rightarrow 350]$ Export[supfigdir <> "FigureS2.eps", %];



Kurtosis

Mutational distributions (figure S3A)

Let's compare our results when we take mutational distributions of phenotypic changes with the same mean (0), variance (λ), and skewness (0), but different kurtosis.

```
n = 4; (*phenotypic dimensions*)
Es = 0.01;(*mean mutational selective effect*)
\lambda = 2 Es / n; (*mutational variance per trait*)
styl = {FontFamily → "Times", FontSize → 10}; (*styles of axes and legend*)
\left\{ \texttt{PDF} \middle[ \texttt{WignerSemicircleDistribution} \middle[ \sqrt{4 \lambda} \middle], z \middle], \right.
    \texttt{PDF} \Big[ \texttt{NormalDistribution} \Big[ \texttt{0} \,,\, \sqrt{\lambda} \, \Big] \,,\, \texttt{z} \Big] \,,\, \texttt{PDF} \Big[ \texttt{LaplaceDistribution} \Big[ \texttt{0} \,,\, \sqrt{\lambda/2} \, \Big] \,,\, \texttt{z} \Big] \Big\} ; 
figS3a = Plot \left[\%, \left\{z, -4\sqrt{\lambda}, 4\sqrt{\lambda}\right\},\right]
   PlotStyle → {Directive[Thick, Dashing[Medium], Black], Directive[Thick, Black],
      Directive[Thick, Dotted, Black]}, PlotRange → {0, All},
   Filling → Bottom, FillingStyle → Directive[Black, Opacity[0.1]],
   PlotLegends → Placed[LineLegend[Style[#, labelstyle] & /@
         {"platykurtic", "normal", "leptokurtic"}], Scaled@{0.8, 0.8}],
   Frame → {True, False, False, False},
   FrameLabel \rightarrow \{ \Delta z_i \},
   FrameStyle → labelstyle,
   PlotRangePadding → None,
   Epilog →
     {Text[Style["A", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]}
Clear[n, Es, \lambda, styl, platy, meso, lepto]
  Α
                                                - platykurtic

    normal

                                              ---- leptokurtic
        -0.2
                                0.0
                                            0.1
                                \Delta z_i
```

Probability of rescue (figure S3B,C)

Interpolation function

The probability of rescue by k steps involves k integrals, and so with even small k (e.g., k=3) they are very slow to compute in their exact form. We can, however, combine an approximation (don't integrate below mmin) with interpolation functions to greatly speed things up (with some loss of accuaracy)

```
mTempmin = -0.4;
mTempmax = -0.01;
step = 0.01;
mmin = mTempmin;
Clear[\lambdaInterpolated, \lambda2Interpolated, \lambda3Interpolated, \lambda4Interpolated]
\Lambda1Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda1Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    \texttt{f} = \texttt{Interpolation}[\texttt{Table}[\{\texttt{mTemp}, \texttt{UNIntegrate}[\texttt{fm}[\texttt{m}, \texttt{mTemp}, \texttt{mmax}, \lambda, \texttt{n}] \texttt{ pest}[\texttt{m}],\\
            \{m, 0, mmax\}, Method \rightarrow \{Automatic, "SymbolicProcessing" \rightarrow 0\}]\},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
   1
\Lambda2Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda2Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n]}
              (1 - pest[m]) prescuem[m, Λ1Interpolated[m, mmax, λ, n, U]],
            {m, mmin, mmax}, Method → {Automatic, "SymbolicProcessing" → 0}]},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
   1
\Lambda3Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda3Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    (1 - pest[m]) prescuem[m, \Lambda 2Interpolated[m, mmax, \lambda, n, U]],
            \{m, mmin, mmax\}, Method \rightarrow \{Automatic, "SymbolicProcessing" \rightarrow 0\}]\},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
   1
\Lambda4Interpolated[m0_, mmax_, \lambda_, n_, U_] :=
 \Lambda4Interpolated[m0, mmax, \lambda, n, U] = Block[{},
    f = Interpolation[Table[{mTemp, UNIntegrate[fm[m, mTemp, mmax, \lambda, n]}
              (1-pest[m]) prescuem[m, Λ3Interpolated[m, mmax, λ, n, U]],
            {m, mmin, mmax}, Method → {Automatic, "SymbolicProcessing" → 0}]},
        {mTemp, mTempmin, mTempmax, step}]];
    f[m0]
   1
```

Numerical data (for normal distribution)

```
(*N0=10^4;
mmax=0.5;
\lambda = 2Es/n;
Es=0.01;
n=4;
U=2*10^{-3};
m0min=mTempmin;
m0max=mTempmax;
mstep=step;
rate4=Table[{m0, \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \(
rate3=Table[{m0,\deltaInterpolated[m0,mmax,\lambda,n,U]},{m0,m0min,m0max,mstep}];
rate2=Table[{m0, \Lambda2Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
rate1=Table[{m0, \lambda1Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
m0list=Table[m0,{m0,m0min,m0max,mstep}];
theory={
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate2[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate3[[i,2]]]},
          {i,Length[m0list]}],
      Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate4[[i,2]]]},
          {i,Length[m0list]}]
   };
alltheory=
   Table[\{m0list[[i]],prescue/.p0\rightarrow prescuem[m0list[[i]],rate1[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+rate2[[i,2]]+ra
                     rate3[[i,2]]+rate4[[i,2]]]},{i,Length[m0list]}];
Clear[N0,mmax,\lambda,Es,n,U,m0min,m0max]*)
theory = {{{-0.4}, 5.472398814099222\*^-7}, {-0.39\, 9.745688406281872\*^-7},
             {-0.38, 1.7238895513305152, 4^-6}, {-0.37, 3.0285772651961196, 4^-6},
             {-0.36000000000000000, 5.2841014319682245`*^-6}, {-0.35000000000000000, 
                9.155420059081543`*^-6}, {-0.34`, 0.000015751880195202794`},
             {-0.33, 0.00002690955788409166, {-0.32, 0.000045643083578017496},
             {-0.3100000000000005, 0.0000768622102641503},
             {-0.3000000000000004, 0.00012849810996373723},
             {-0.2900000000000004, 0.0002132574731799064},
             {-0.28`, 0.0003513291377449379`}, {-0.27`, 0.000574520974723769`},
             {-0.26`, 0.0009325213318174264`}, {-0.25`, 0.0015022797840921198`},
             {-0.24000000000000002, 0.0024019086800984857}},
             {-0.23, 0.0038110379258926574}, {-0.2200000000000003,
               0.00600020901304088^{}, {-0.2100000000000002^{}, 0.00937261491044139^{}},
             {-0.2`, 0.014522096798011974`}, {-0.1900000000000003`, 0.022311337733202485`},
             {-0.18000000000000002<sup>,</sup> 0.03397263148001661<sup>,</sup>},
             {-0.17`, 0.0512283883233019`}, {-0.160000000000003`, 0.07641581098977057`},
             {-0.15000000000000002`, 0.11257347217569635`},
             {-0.14`, 0.16339795924448708`}, {-0.13`, 0.23289952993154328`},
             {-0.12`, 0.3244893834485655`}, {-0.1100000000000004`, 0.43919411790398366`},
             {-0.10000000000000003, 0.5729341370203966},
             \{-0.09000000000000002^{, 0.7137213876566558^{}}\},
```

```
{-0.0800000000000002`, 0.8414602539868381`}, {-0.07`, 0.9344415307620567`},
 {-0.06`, 0.9829032755970813`}, {-0.049999999999999`, 0.9979187459679042`},
 {-0.04000000000000036`, 0.9999341480370483`},
 {-0.030000000000000027`, 0.9999998505645212`},
 {-0.02000000000000018`, 0.99999999991431`}, {-0.01000000000000000°, 1.`}},
{{-0.4`, 7.1817194564616216`*^-6}, {-0.39`, 0.000010382868435687698`},
{-0.38, 0.000015004899198478583, {-0.37, 0.000021674932411452907, },
 {-0.3600000000000004, 0.000031295722740964926},
 {-0.3500000000000003, 0.00004516336475557381},
 {-0.32`, 0.00013514257904490012`}, {-0.31000000000000005`,
  0.0001943394640103735^{}), \{-0.300000000000004^{}, 0.00027906144855771586^{}\},
 {-0.29000000000000004, 0.00040002244449899305},
 {-0.28, 0.0005722318522486392, {-0.27, 0.0008166042269467377,},
 {-0.24000000000000002, 0.002330394186973561},
 {-0.23<sup>^</sup>, 0.0032813120522319217<sup>^</sup>}, {-0.2200000000000003<sup>^</sup>,
 0.0046003825714453495`}, {-0.210000000000002`, 0.006419557828723499`},
 {-0.2, 0.008912948912746632}, {-0.190000000000003, 0.012308013245967753},
 {-0.18000000000000002`, 0.01689876878901897`}, {-0.17`, 0.02306102686962941`},
 {-0.16000000000000003, 0.031269462419949656},
 {-0.15000000000000002`, 0.042116115688766054`}, {-0.14`, 0.05632965853148175`},
 {-0.13<sup>,</sup> 0.07479449506069946<sup>,</sup> , {-0.12<sup>,</sup> 0.09856854888921651<sup>,</sup> },
 {-0.1100000000000004, 0.12889844527072303, -0.1000000000000000,
  \texttt{0.1672306176682965`} \texttt{, \{-0.09000000000000002`, 0.21521607762716344`\},} 
 {-0.08000000000000002`, 0.2747032194313106`}, {-0.07`, 0.3477011299288638`},
 {-0.06`, 0.4362577296013713`}, {-0.049999999999999`, 0.5420788502184701`},
 {-0.04000000000000036`, 0.6653477290582192`},
 {-0.030000000000000027<sup>,</sup>, 0.8010818232849237<sup>,</sup>}, {-0.020000000000000018<sup>,</sup>,
  0.9286233811012167`}, {-0.0100000000000000°, 0.9962496646879948`}},
{{-0.4`, 2.672518644497579`*^-6}, {-0.39`, 3.6200066483305093`*^-6},
 {-0.38, 4.9024286813414975, *^-6}, {-0.37, 6.639033992561494, *^-6},
 {-0.36000000000000004, 8.99229424278225**^-6}, {-0.35000000000000003,
  0.000012183830842893428`}, {-0.34`, 0.000016516674993138913`},
 {-0.33, 0.00002240514549411987, {-0.32, 0.000030416485379602953,
 {-0.3100000000000005, 0.000041327352556463914},
 {-0.3000000000000004, 0.00005620069935863281},
 {-0.2900000000000004, 0.00007648968361839703},
 {-0.28`, 0.0001041772194614321`}, {-0.27`, 0.00014196192976223543`},
 {-0.26`, 0.00019350379587734778`}, {-0.25`, 0.0002637455775598152`},
 {-0.2400000000000002, 0.0003593290092440693},
 {-0.23<sup>,</sup> 0.0004891277868906974<sup>,</sup>},
 {-0.220000000000003, 0.0006649222935298793}, {-0.2100000000000002,
 0.0009022438378168962`}, {-0.2`, 0.0012214190541605108`},
 {-0.1900000000000003, 0.0016488487049626244}, {-0.1800000000000002,
  0.0022185608705778126^{}, \{-0.17^{}, 0.0029740895246452848^{},
 {-0.16000000000000003<sup>^</sup>, 0.003970751527472172<sup>^</sup>}, {-0.15000000000000000<sup>^</sup>,
   0.0052784386334274735`\} \text{, } \{-0.14`, 0.006985125056896413`\} \text{,} \\
 {-0.13`, 0.009201449505447323`}, {-0.12`, 0.012067026227695399`},
 {-0.1100000000000000<sup>1</sup>, 0.015759693621743676<sup>1</sup>}, {-0.10000000000000000<sup>3</sup>,
 0.020509966391728862`}, {-0.0900000000000002`, 0.02662504581244618`},
 {-0.0800000000000002`, 0.03453108060938248`}, {-0.07`, 0.04485200223554198`},
 {-0.06`, 0.0585665300648599`}, {-0.04999999999999°, 0.0773474840786541`},
 {-0.04000000000000036`, 0.10438028623999529`},
 {-0.030000000000000027, 0.14667096966766024},
```

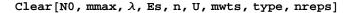
```
{-0.02000000000000018, 0.2233073280524227},
{-0.010000000000000000, 0.4098900908454398`}},
{{-0.4`, 3.665400664099039`*^-7}, {-0.39`, 4.882948413387211`*^-7},
 {-0.38, 6.507825516255394, 4-7}, {-0.37, 8.679175866888045, 4-7},
 \{-0.3600000000000004, 1.1585392625246271*^-6\}, \{-0.35000000000000000,
  1.548213704460899`*^-6}, {-0.34`, 2.071738417153135`*^-6},
 {-0.33`, 2.776581790708832`*^-6}, {-0.32`, 3.727611290194055`*^-6},
 {-0.3100000000000005, 5.013589215119296**^-6},
 {-0.3000000000000004, 6.756030736299046**^-6},
 {-0.2900000000000004, 9.12124881535803**^-6},
 {-0.28`, 0.00001233658455390696`}, {-0.27`, 0.000016712145389896627`},
 {-0.26`, 0.00002266963795427568`}, {-0.25`, 0.000030780242171069006`},
 {-0.2400000000000002, 0.00004181381972367948},
 {-0.23, 0.000056802136495281985},
 {-0.2200000000000003, 0.00007711911056884624}, {-0.2100000000000002,
 0.00010458146610448082`}, {-0.2`, 0.0001415735100674942`},
 {-0.1900000000000003`, 0.00019120021529694942`}, {-0.18000000000000002`,
  0.0002574735404686779`}, {-0.17`, 0.00034553832838446485`},
 {-0.16000000000000003, 0.0004619470399811476}, {-0.1500000000000002,
  0.0006149981944676419`}, {-0.14`, 0.0008151642974213447`},
 {-0.13<sup>^</sup>, 0.0010756555359718867<sup>^</sup>}, {-0.12<sup>^</sup>, 0.0014132038091301258<sup>^</sup>},
 {-0.11000000000000004`, 0.0018492239194836113`},
 \{-0.10000000000000003^{,}0.002411647781310755^{,}\}
 {-0.09000000000000002, 0.003138005818001832},
 \{-0.08000000000000002^{, 0.004080917925158634^{}},
 {-0.07`, 0.005318491045821605`}, {-0.06`, 0.006975434272220693`},
 {-0.049999999999999999999, 0.009269923451119766`},
 {-0.04000000000000036, 0.012631023980203326},
 \{-0.030000000000000027^{, 0.01804961542287531^{}},
 \{-0.02000000000000018^{\circ}, 0.0284671089107692^{\circ}\},
```

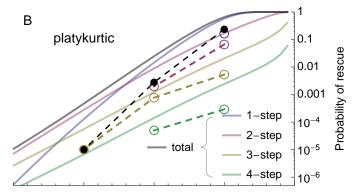
```
alltheory = {{-0.4}, 0.000010767989638504893}, {-0.39}, 0.000015465680252679803},
        {-0.38`, 0.000022281877958607588`}, {-0.37`, 0.00003221020658283891`},
        {-0.3600000000000004`, 0.00004673011156652862`}, {-0.350000000000000003`,
         0.00006804965109896699`}, {-0.34`, 0.00009947299961354794`},
        {-0.33`, 0.0001459524027258574`}, {-0.32`, 0.00021491729975098295`},
        {-0.3100000000000005, 0.00031751489398534893},
        {-0.3000000000000004, 0.0004704543728014654},
        \{-0.2900000000000004, 0.0006987523028609832},
        {-0.28, 0.0010397647754660788, {-0.27, 0.0015491068347829318},
        {-0.26`, 0.0023092555158806816`}, {-0.25`, 0.0034419561736643844`},
        {-0.24000000000000002, 0.005125934145003619},
        {-0.23`, 0.007621874910163862`}, {-0.220000000000003`, 0.011307117362613917`},
        {-0.2100000000000002`, 0.01672286182842886`},
       {-0.2`, 0.024636589665609887`}, {-0.190000000000003`, 0.03612111665412221`},
        {-0.18000000000000002`, 0.0526478810505413`}, {-0.17`, 0.07618321541025108`},
        {-0.16000000000000003, 0.10925842311524103},
        {-0.15000000000000002`,0.154951959045035`},{-0.14`,0.21667029213726385`},
        {-0.13, 0.29754827077524637}, {-0.12, 0.39924782631433087},
        {-0.1100000000000004`, 0.5200288215021587`},
        {-0.10000000000000003, 0.6524246688522214},
        {-0.09000000000000002, 0.7819169338181124}},
        {-0.08000000000000002`, 0.8893420776854711`}, {-0.07`, 0.9592948833994817`},
       {-0.06`, 0.9909501188644341`}, {-0.04999999999999`, 0.9991194054587093`},
        {-0.04000000000000036`, 0.9999799413255674`},
        {-0.03000000000000027, 0.999999972757273},
        {-0.02000000000000018<sup>\colon, 0.999999999999342<sup>\colon, 1.\colon, </sup></sup>
```

Platykurtic (semicircle, panel B)

```
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwts = \{-0.1, -0.2, -0.3\};
type = "platy";
nreps = 100 000;
data = Table[
   file = StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps``.csv",
      type, N0, n, NumberForm[U // N, {6, 5}], NumberForm[Es, {3, 2}],
      NumberForm[mmax, {3, 2}], mwts[[i]], nreps];
   dat = Import[ToString[file]];
   onestep = Length[Select[dat, #[[2]] == 1 &]] / nreps;
   twostep = Length[Select[dat, #[[2]] == 2 &]] / nreps;
   threestep = Length[Select[dat, #[[2]] == 3 &]] / nreps;
   fourstep = Length[Select[dat, #[[2]] == 4 &]] / nreps;
   total = onestep + twostep + threestep + fourstep;
   {onestep, twostep, threestep, fourstep, total} // N,
   {i, Length[mwts]}
Table[Table[{mwts[[i]], data[[i, j]]}, {i, Length[mwts]}], {j, Length[data[[1]]]}};
```

```
dataplot = ListLogPlot[%,
   PlotRange \rightarrow \{\{-0.4, 0\}, \{10^{-6}, 1\}\},\
   Frame → {True, False, False, True},
   Joined → True,
   PlotStyle \rightarrow {
     Directive[defaultcolors[[1]], Thick, Dashing[Medium]],
     Directive[defaultcolors[[2]], Thick, Dashing[Medium]],
     Directive[defaultcolors[[3]], Thick, Dashing[Medium]],
     Directive[defaultcolors[[4]], Thick, Dashing[Medium]],
     Directive[Black, Thickness[0.005], Dashing[Medium]]
   PlotMarkers → {
     Graphics[{Dashing[None], Thickness[Medium],
        defaultcolors[[1]], Circle[]}, ImageSize → 10],
     Graphics[{Dashing[None], Thickness[Medium], Circle[], defaultcolors[[2]]},
       ImageSize → 10], Graphics[{Dashing[None], Thickness[Medium],
        Circle[], defaultcolors[[3]]}, ImageSize → 10],
     Graphics[{Dashing[None], Thickness[Medium], Circle[],
        defaultcolors[[4]]}, ImageSize \rightarrow 10],
     Graphics[{Black, Disk[]}, ImageSize → 8]
    }
  ];
figS3b = Show
  ListLogPlot
   theory,
   Joined → True,
   PlotStyle → Directive[Thick, Opacity[0.5]],
   PlotRange \rightarrow \{\{-0.4, 0\}, \{5*10^{-7}, 1\}\},\
   Frame → {True, False, False, True},
   FrameLabel → {(*"Wildtype growth rate"*), , , "Probability of rescue"},
   FrameTicks → {True, False, False, True},
   FrameTicksStyle → {FontColor → White, Automatic, Automatic},
   LabelStyle → labelstyle,
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
        {"1-step", "2-step", "3-step", "4-step"}], Scaled@{0.85, 0.22}]
  ],
  ListLogPlot[alltheory, Joined → True, PlotStyle → {Black, Thick, Opacity[0.5]},
   PlotLegends → Placed[LineLegend[
       Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@ {"total"}], Scaled@ {0.56, 0.22}]
  ],
  dataplot,
  PlotRangePadding \rightarrow 0,
  (*ImagePadding→probpadding,*)
  Epilog \rightarrow \{
    Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
    Text[Style["platykurtic", 14, FontFamily → "Helvetica"], Scaled@{0.25, 0.85}],
    {Opacity[0.5], braceLabel[
       \{\{-0.125, Log@10^{-6}\}, \{-0.125, Log@(1.5*10^{-4})\}\}, Style["", Larger]]\}
```





Leptokurtic (Laplace, panel C)

```
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwts = \{-0.1, -0.2, -0.3\};
type = "lepto";
nreps = 100 000;
data = Table[
   file = StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps``.csv",
      type, N0, n, NumberForm[U // N, {6, 5}], NumberForm[Es, {3, 2}],
      NumberForm[mmax, {3, 2}], mwts[[i]], nreps];
   dat = Import[ToString[file]];
   onestep = Length[Select[dat, #[[2]] == 1 &]] / nreps;
   twostep = Length[Select[dat, #[[2]] == 2 &]] / nreps;
   threestep = Length[Select[dat, #[[2]] == 3 &]] / nreps;
   fourstep = Length[Select[dat, #[[2]] == 4 &]] / nreps;
   total = onestep + twostep + threestep + fourstep;
   {onestep, twostep, threestep, fourstep, total} // N,
   {i, Length[mwts]}
  ];
Table[Table[{mwts[[i]], data[[i, j]]}, {i, Length[mwts]}], {j, Length[data[[1]]]}};
dataplot = ListLogPlot[%,
   PlotRange \rightarrow \{\{-0.4, 0\}, \{10^{-6}, 1\}\},\
   Frame → {True, False, False, True},
   Joined → True,
   PlotStyle → {
      Directive [defaultcolors[[1]], Thick, Dotted],
      Directive[defaultcolors[[2]], Thick, Dotted],
      Directive[defaultcolors[[3]], Thick, Dotted],
      Directive [defaultcolors [[4]], Thick, Dotted],
```

```
Directive[Black, Thickness[0.005], Dotted]
    },
   PlotMarkers → {
      Graphics[{Dashing[None], Thickness[Medium],
        defaultcolors[[1]], Circle[]}, ImageSize → 10],
      Graphics[{Dashing[None], Thickness[Medium], Circle[], defaultcolors[[2]]},
       ImageSize \rightarrow 10], Graphics[{Dashing[None], Thickness[Medium],}]
        Circle[], defaultcolors[[3]]}, ImageSize → 10],
      Graphics[{Dashing[None], Thickness[Medium], Circle[],
        defaultcolors[[4]]}, ImageSize → 10],
      Graphics[{Black, Disk[]}, ImageSize → 8]
    }
  ];
figS3c = Show
  ListLogPlot |
   theory,
   Joined → True,
   PlotStyle → Directive[Thick, Opacity[0.5]],
   PlotRange \rightarrow \{\{-0.4, 0\}, \{5*10^{-7}, 1\}\},\
   Frame → {True, False, False, True},
   FrameLabel → {"Wildtype growth rate", , , "Probability of rescue"},
   FrameTicks → {True, False, False, True},
   LabelStyle → labelstyle(*,
   PlotLegends→Placed[LineLegend[Style[#,12,FontFamily→"Helvetica"]&/@
        {"1-step","2-step","3-step","4-step"}],Scaled@{7/8,1/4}]*)
  ],
  ListLogPlot[alltheory, Joined → True, PlotStyle → {Black, Thick, Opacity[0.5]}(*,
   PlotLegends→Placed[LineLegend[Style[#,12,FontFamily→"Helvetica"]&/@
        {"1-, 2-, 3-, or 4-step"}],Scaled@{1/4,7/8}]*)
  ],
  dataplot,
  PlotRangePadding \rightarrow 0,
  (*ImagePadding→probpadding,*)
  \texttt{Epilog} \rightarrow \{
    Text[Style["C", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
    Text[Style["leptokurtic", 14, FontFamily → "Helvetica"], Scaled@{0.25, 0.85}]
   }
Clear [NO, mmax, \lambda, Es, n, U, mwts, type, nreps]
```

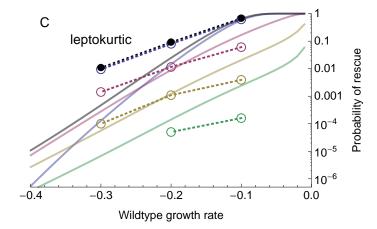
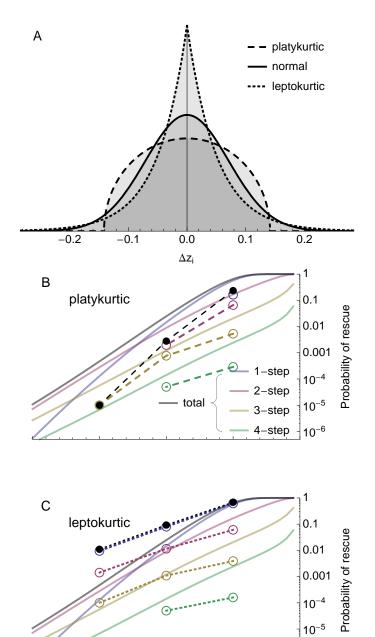


Figure S3

 $\label{eq:graphicsColumn} $$ GraphicsColumn[{figS3a, figS3b, figS3c}, Spacings $\to -70$, ImageSize $\to 375$] $$ Export[supfigdir <> "FigureS3.eps", %];$



-0.3

-0.2

Wildtype growth rate

-0.1

-0.4

10-6

0.0

Rescue genotype growth rate (figure S4)

Platykurtic (semicircle, panel A)

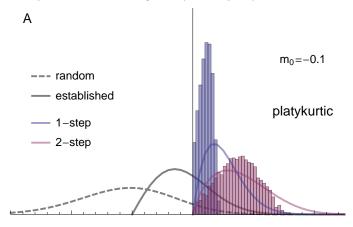
```
xmin = -0.3;
xmax = 0.25;
ymax = 40;
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
xint = 0.01;
type = "platy";
   (*random*)
  Table [\{m2, fm[m2, mwt, mmax, \lambda, n]\}, \{m2, xmin, xmax, xint\}],
  (*established*)
  Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}]\}, \{m2, mwt, xmax, xint\}]
 };
oldtheory = ListPlot[%, PlotRange → All, Joined → True,
    PlotStyle → {Directive[Thick, Gray, Dashed], Directive[Thick, Gray]},
   PerformanceGoal → "Speed",
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
         {"random", "established"}], Scaled@{1.5 / 8, 1 / 2}]];
   (*1 step*)
  Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
  (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax, xint}\right\}\right]
theory = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → Directive[Thick, Opacity[0.5]], PerformanceGoal → "Speed",
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
         {"1-step", "2-step"}], Scaled@{1.5 / 8, 1 / 2}]];
StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps100000.csv",
  type, N0, n, NumberForm [U // N, \{6, 5\}],
  NumberForm[Es, {3, 2}], NumberForm[mmax, {3, 2}], mwt];
dat = Import[ToString[%]];
onestep = Select[dat, #[[2]] == 1 &][[All, 1]];
twostep = Select[dat, #[[2]] == 2 &][[All, 1]];
data = Histogram[{onestep, twostep}, Automatic, "PDF", AxesOrigin → {0, 0}];
```

```
figS4a = Show[
   {oldtheory, data, theory},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  LabelStyle → labelstyle,
  \texttt{FrameTicksStyle} \rightarrow \{\texttt{FontColor} \rightarrow \texttt{White}, \texttt{Automatic}, \texttt{Automatic}\},
  Epilog \rightarrow {
     Text[Style["m<sub>0</sub>=" <> ToString[mwt],
        12, FontFamily \rightarrow "Helvetica"], Scaled@\{7/8, 3/4\}],
     Text[Style["A", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
     \texttt{Text[Style["platykurtic", 14, FontFamily} \rightarrow \texttt{"Helvetica"], Scaled@{7/8, 1/2}]}
    }
 ]
```

Clear $[mmax, \lambda, Es, n, U, mwt]$

NIntegrate::izero:

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. \gg



Leptokurtic (Laplace, panel B)

```
xmin = -0.3;
xmax = 0.25;
ymax = 40;
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.1;
xint = 0.01;
type = "lepto";
   (*random*)
  Table[\{m2, fm[m2, mwt, mmax, \lambda, n]\}, \{m2, xmin, xmax, xint\}],
```

```
(*established*)
  Table[{m2, (fm[m2, mwt, mmax, λ, n] pest[m2-mwt]) / NIntegrate[
       fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}]
 };
oldtheory = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → {Directive[Thick, Gray, Dashed], Directive[Thick, Gray]},
   PerformanceGoal → "Speed" (*,
   PlotLegends→Placed[LineLegend[Style[#,12,FontFamily→"Helvetica"]&/@
         {"random", "established"}], Scaled@{1.5/8,1/2}]*)];
   (*1 step*)
  Table [\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
  (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{total} g2numerator[mwt, m2]\right\}, \left\{m2, 0, xmax, xint\right\}\right]
 };
theory = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → Directive [Thick, Opacity [0.5]], PerformanceGoal → "Speed" (*,
   PlotLegends→Placed[LineLegend[Style[#,12,FontFamily→"Helvetica"]&/@
         {"1-step","2-step"}],Scaled@{1.5/8,1/2}]*)];
StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps100000.csv",
  type, N0, n, NumberForm [U // N, \{6, 5\}],
  NumberForm[Es, {3, 2}], NumberForm[mmax, {3, 2}], mwt];
dat = Import[ToString[%]];
onestep = Select[dat, #[[2]] == 1 &][[All, 1]];
twostep = Select[dat, #[[2]] == 2 &][[All, 1]];
data = Histogram[{onestep, twostep}, Automatic, "PDF", AxesOrigin → {0, 0}];
figS4b = Show[
  {oldtheory, data, theory},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  LabelStyle \rightarrow labelstyle,
  FrameTicksStyle → {Automatic, Automatic, Automatic},
  FrameLabel → {"Growth rate"},
  Epilog \rightarrow {
     (*Text[
      Style["m_0="<>ToString[mwt],12,FontFamily\rightarrow"Helvetica"],Scaled@\{7/8,3/4\}],\star)
     Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
    Text[Style["leptokurtic", 14, FontFamily → "Helvetica"], Scaled@{7/8,1/2}]
   }
 1
Clear [mmax, \lambda, Es, n, U, mwt]
NIntegrate::izero:
```

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>>

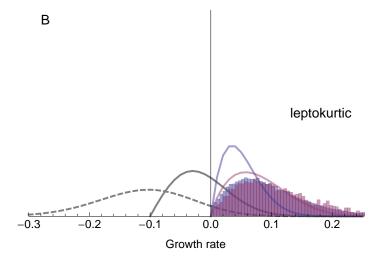
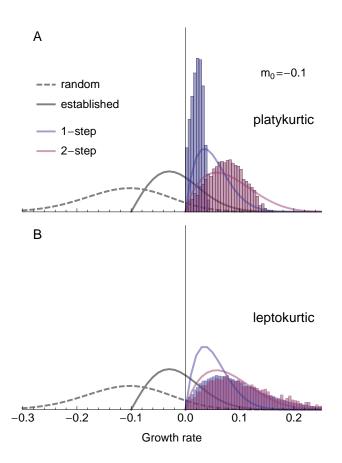


Figure S4

GraphicsColumn[$\{figS4a, figS4b\}$, Spacings $\rightarrow -50$, ImageSize $\rightarrow 350$] Export[supfigdir <> "FigureS4.pdf", %];

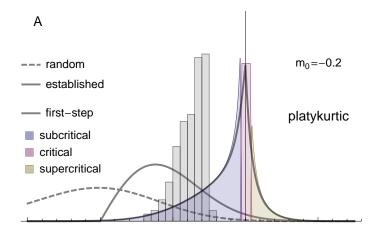


2-step intermediate growth rate (figure S5)

Platykurtic (semicircle, panel A)

```
xmin = -0.3;
xmax = 0.15;
ymax = 30;
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es/n;
Es = 0.01;
n = 4;
mwt = -0.2;
type = "platy";
\{\text{mneg = FindRoot}[2m1^2 - \Lambda 1[m1, mmax, \lambda, n, U], \{m1, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
  fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
  supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
    \frac{\Lambda 1[m1, mmax, \lambda, n, U]}{} HeavisideTheta[(m1-pr0)];
allexact = NIntegrate[exact, {m1, mwt, 0.1}];
{
  fm[m1, mwt, mmax, \lambda, n],
   (fm[m1, mwt, mmax, λ, n] pest[m1 - mwt] HeavisideTheta[m1 - mwt]) /
   NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}]
oldtheory = Plot[
    %,
    {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
    (*Filling→Bottom,*)
   PlotStyle → {Directive[Gray, Thick, Dashed], Directive[Gray, Thick]},
    PerformanceGoal → "Speed",
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
         {"random", "established"}], Scaled@{1/8, 1.25/2}]
  ];
```

```
subcritical
              critical
                          supercritical.
   allexact
               allexact
                             allexact
theory = Plot[
   %,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},
   Filling → Bottom,
   PerformanceGoal → "Speed",
   PlotLegends → Placed[SwatchLegend[{Directive[defaultcolors[[1]], Opacity[0.5]],
        Directive[defaultcolors[[2]], Opacity[0.5]], Directive[defaultcolors[[3]],
         Opacity[0.5]]}, Style[#, 12, FontFamily → "Helvetica"] & /@
        {"subcritical", "critical", "supercritical"}], Scaled@{1/8,0.65/2}]
  ];
exactplot = Plot[
   exact / allexact, {m1, xmin, xmax},
   PlotStyle → Directive[Thick, Black, Opacity[0.5]],
   PerformanceGoal → "Speed", PlotRange → All,
   PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
        {"first-step"}], Scaled@{1/8, 1.25/2}]
  ];
StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps100000.csv",
  type, N0, n, NumberForm [U // N, \{6, 5\}],
  NumberForm[Es, {3, 2}], NumberForm[mmax, {3, 2}], mwt];
dat = Import[ToString[%]];
Select[dat, #[[2]] = 2 &][[All, 3]];
data = Histogram[%, Automatic, "PDF",
   AxesOrigin → {0, 0}, ChartStyle → Directive[Gray, Opacity[0.25]]];
figS5a = Show[
  oldtheory, data, theory, exactplot,
  Frame → {True, False, False, False},
  FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
  PlotRange \rightarrow \{\{xmin, xmax\}, \{0, ymax\}\},\
  LabelStyle → labelstyle,
  Epilog → {
    Text[Style["mo=" <> ToString[mwt],
       12, FontFamily \rightarrow "Helvetica"], Scaled@\{7/8, 3/4\}],
    Text[Style["A", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}],
    Text[Style["platykurtic", 14, FontFamily → "Helvetica"], Scaled@{7/8,1/2}]
   }
 ]
Clear [mmax, \lambda, Es, n, U, mwt]
```



Leptokurtic (Laplace, panel B)

```
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
type = "lepto";
\{\text{mneg = FindRoot}[2m1^2 - \Lambda 1[m1, mmax, \lambda, n, U], \{m1, -0.1\}],
  mpos = FindRoot[2m1^2 - \Lambda1[m1, mmax, \lambda, n, U], \{m1, 0.1\}];
pr0 = \sqrt{\Lambda 1[m1, mmax, \lambda, n, U] / 2};
exact = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
critical =
   fm[0, mwt, mmax, \lambda, n] \sqrt{2 \Lambda 1[0, mmax, \lambda, n, U]} HeavisideTheta[(m1+pr0) (pr0-m1)];
subcritical =
   supercritical = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1])
     \frac{\text{A1}[\text{m1}, \text{mmax}, \lambda, \text{n}, \text{U}]}{\text{Abs}[\text{m1}]} \text{ HeavisideTheta[(m1-pr0)];}
allexact = NIntegrate[exact, {m1, mwt, 0.1}];
   fm[m1, mwt, mmax, \lambda, n],
   (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
    NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], \{m1, mwt, mmax\}]
 };
oldtheory = Plot[
```

```
%,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   (*Filling→Bottom,*)
   PlotStyle → {Directive[Gray, Thick, Dashed], Directive[Gray, Thick]},
   PerformanceGoal → "Speed" (*,
   PlotLegends→Placed[LineLegend[Style[#,12,FontFamily→"Helvetica"]&/@
        {"random", "established"}], Scaled@{1/8,1.25/2}]*)
  ];
 subcritical
              critical
   allexact
               allexact
theory = Plot[
   %,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   Filling → Bottom,
   PerformanceGoal → "Speed" (*,
   PlotLegends→Placed[SwatchLegend[{Directive[defaultcolors[[1]],Opacity[0.5]],
        Directive[defaultcolors[[2]],Opacity[0.5]],Directive[defaultcolors[[3]],
         Opacity[0.5]]},Style[#,12,FontFamily→"Helvetica"]&/@
        {"subcritical","critical","supercritical"}],Scaled@{1/8,0.65/2}]*)
  ];
exactplot = Plot[
   exact / allexact, {m1, xmin, xmax},
   PlotStyle → Directive[Thick, Black, Opacity[0.5]],
   PerformanceGoal → "Speed", PlotRange → All(*,
   PlotLegends-Placed[LineLegend[Style[#,12,FontFamily-"Helvetica"]&/@
        {"first-step"}],Scaled@{1/8,1.25/2}]*)
  ];
StringForm[datadir <> "dfe_``_N``_n``_U``_Es``_mmax``_mwt``_nreps100000.csv",
  type, N0, n, NumberForm [U // N, \{6, 5\}],
  NumberForm[Es, {3, 2}], NumberForm[mmax, {3, 2}], mwt];
dat = Import[ToString[%]];
Select[dat, #[[2]] = 2 &][[All, 3]];
data = Histogram[%, Automatic, "PDF",
   AxesOrigin → {0, 0}, ChartStyle → Directive[Gray, Opacity[0.25]]];
figS5b = Show[
  oldtheory, data, theory, exactplot,
  Frame → {True, False, False, False},
  FrameTicksStyle → {Automatic, Automatic, Automatic},
  FrameLabel → {"Growth rate"},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  LabelStyle \rightarrow labelstyle,
  Epilog \rightarrow {
     (*Text[
      Style["m_0="<>ToString[mwt],12,FontFamily\rightarrow"Helvetica"],Scaled@\{7/8,3/4\}],\star)
    Text[Style["B", 14, Bold, FontFamily \rightarrow "Helvetica"], Scaled@\{0.05, 0.95\}],
    Text[Style["leptokurtic", 14, FontFamily → "Helvetica"], Scaled@{7/8,1/2}]
   }
 ]
```

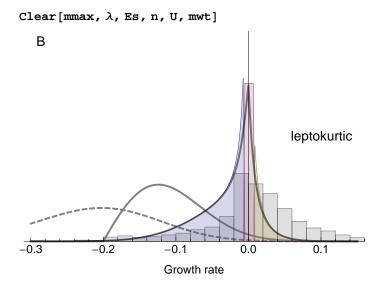
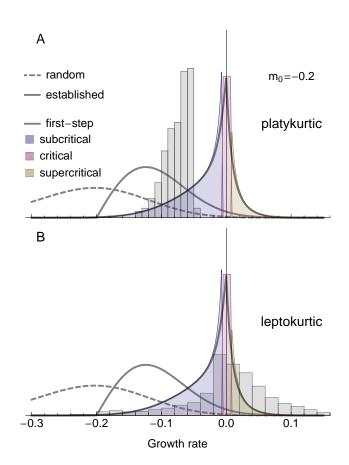


Figure S5

 $\label{eq:graphicsColumn} $$ GraphicsColumn[\{figS5a, figS5b\}, Spacings \to -50, ImageSize \to 350] $$ Export[supfigdir <> "FigureS5.pdf", %];$



Dimensionality, n

Probability of rescue (figure S6A)

Numerical data

```
(*N0=10^4;
mmax=0.5;
\lambda = 2Es/4;
Es=0.01;
U=2*10^{-3};
m0min=mTempmin;
m0max=mTempmax;
mstep=step;
m0list=Table[m0,{m0,m0min,m0max,mstep}];
n=2;
rate4=Table[{m0, \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \(
rate3=Table[{m0,Λ3Interpolated[m0,mmax,λ,n,U]},{m0,m0min,m0max,mstep}];
rate2=Table[{m0, \(\Lambda\)2Interpolated[m0, \(\max\), \(\max\), \(\max\), \(\max\)]}, \(\max\), \(\max\)
rate1=Table[{m0, \lambda1Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
theory2={
               Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]]},
                      {i,Length[m0list]}],
               Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate2[[i,2]]]},
                      {i,Length[m0list]}],
              Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate3[[i,2]]]},
                      {i,Length[m0list]}],
              Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate4[[i,2]]]},
                      {i,Length[m0list]}]
       };
 alltheory2=
       Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]+rate2[[i,2]]+
                                                rate3[[i,2]]+rate4[[i,2]]]},{i,Length[m0list]}];
n=20;
rate4=Table[{m0, \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \(
rate3=Table[{m0, \Lambda3Interpolated[m0, \Lambdamax, \lambda, \Lambda, \Lambda]; {m0, \Lambda0min, \Lam
rate2=Table[{m0, \( \lambda \)2Interpolated[m0, \( \max \), \( \lambda \), \( \lambda \), \( \max \), \( \max
rate1=Table[{m0, \lambda1Interpolated[m0, mmax, \lambda, n, U]}, {m0, m0min, m0max, mstep}];
               Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]]},
                       {i,Length[m0list]}],
               Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate2[[i,2]]]},
                      {i,Length[m0list]}],
              Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate3[[i,2]]]},
                      {i,Length[m0list]}],
               Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate4[[i,2]]]}},
                      {i,Length[m0list]}]
 alltheory20=
        Table[{m0list[[i]],prescue/.p0→prescuem[m0list[[i]],rate1[[i,2]]+rate2[[i,2]]+
                                                rate3[[i,2]]+rate4[[i,2]]]},{i,Length[m0list]}];
Clear[N0,mmax, \lambda, Es, n, U, m0min, m0max, mstep] *)
 theory2 = {{ -0.4, 7.550146599033525, *^-7}, {-0.39, 1.3377479120002533, *^-6},
                             {-0.38`, 2.3541607676591525`*^-6}, {-0.37`, 4.114440329328062`*^-6},
```

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 {-0.03000000000000027, 0.16254620716433155},
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\{-0.010000000000000000^{\circ}, 0.43935473836510974^{\circ}\}\},
{{-0.4}, 5.338328372683776}*^-7}, {-0.39}, 7.072562302479213`*^-7},
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 {-0.3600000000000004, 1.649783406221239<sup>**</sup>-6}, {-0.35000000000000003,
  2.1918565153988823`*^-6\} \,, \, \{-0.34`, \, 2.915686786786331`*^-6\} \,, \\
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alltheory2 = {{-0.4}, 0.000015350202181019235}}, {-0.39}, 0.000021926220291734744}},
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theory20 = {{{-0.4}, 3.6721736207390165\*^-8}, {-0.39\, 6.809108610283943\*^-8},
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        0.0008832005339582949`}, {-0.210000000000002`, 0.0014480669951962843`},
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        0.0038113391574159383`}, {-0.1800000000000002`, 0.006118324069079195`},
       {-0.17, 0.009752194976848538}, {-0.16000000000000000,
         0.015432390043889188^{\circ})\,,\, \{-0.1500000000000000^{\circ},\, 0.02423916555868677^{\circ}\}\,,
       {-0.14, 0.037771223298540546}, {-0.13, 0.05834834323446625},
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       {-0.1000000000000003, 0.20069983428605798},
       {-0.09000000000000002, 0.2926736492678442},
       {-0.08000000000000002`, 0.4150076889285138`}, {-0.07`, 0.5658189827878293`},
       {-0.06`, 0.7303782065372255`}, {-0.04999999999999°, 0.8768257341172561`},
       {-0.04000000000000036`, 0.9680381655938813`},
       {-0.03000000000000027, 0.997401209946258},
       {-0.02000000000000018, 0.9999877420421525},
       {-0.010000000000000000, 0.9999999999959232`}},
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```

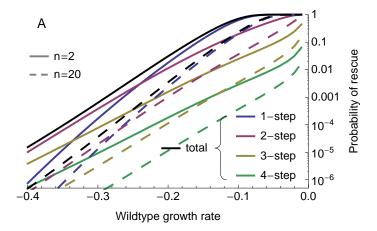
```
{-0.38, 7.625996985005656, *^-7}, {-0.37, 1.1436489691929452, *^-6},
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 2.573369322966812`*^-6}, {-0.34`, 3.860955236412167`*^-6},
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 \{-0.02000000000000018, 0.5790997599763441},
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 \{-0.3600000000000004, 3.83096816936046*^-7\}, \{-0.35000000000000000,
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 {-0.3100000000000005, 2.19020228642286**^-6},
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 {-0.0499999999999999, 0.018796767828342165`},
 {-0.04000000000000036`, 0.02750298571925336`},
 \{-0.0300000000000000027^{, 0.04220002528993849^{}},
 {-0.02000000000000018, 0.07125431410015437},
 \{-0.010000000000000000^{\circ}, 0.15330403861172748^{\circ}\}\},
```

```
{{-0.4}, 1.0911271886016038**^-8}, {-0.39}, 1.5165646405357336**^-8},
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  {-0.36000000000000000, 4.1023850316790345`*^-8}, {-0.35000000000000003`,
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  \{-0.2900000000000004^{, 4.50536176255234^{*^-7}},
  \{-0.28^{\circ}, 6.412768283059478^{\circ}*^{-7}\}, \{-0.27^{\circ}, 9.15076491470046^{\circ}*^{-7}\}, \{-0.28^{\circ}, 6.412768283059478^{\circ}*^{-7}\}, \{-0.27^{\circ}, 9.15076491470046^{\circ}*^{-7}\}, \{-0.28^{\circ}, 6.412768283059478^{\circ}*^{-7}\}, \{-0.27^{\circ}, 9.15076491470046^{\circ}*^{-7}\}, \{-0.27^{\circ}, 9.15076491470046^{\circ}*^{-7}\}\}
  {-0.26, 1.3086989646415859, 4-6}, {-0.25, 1.8751116518389566, *^-6},
  {-0.2400000000000002, 2.6904020712370524**^-6},
  \{-0.23^{,}3.863469882126047^{*}-6\},
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   7.968629597288057`*^-6}, {-0.2`, 0.000011431327242572564`},
  {-0.1900000000000003, 0.00001637322207959091}, {-0.1800000000000002,
   0.000023402016377183088^{}, \{-0.17^{}, 0.00003336083728600059^{}\},
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  {-0.1100000000000004`, 0.00026106844205453505`},
  {-0.1000000000000003, 0.0003639201714409346},
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  {-0.08000000000000002, 0.0007058861356389334},
  {-0.07`, 0.0009867178388376363`}, {-0.06`, 0.0013899202861142879`},
  {-0.04999999999999999, 0.0019868453399276564`},
  {-0.04000000000000036, 0.0029171473631195433},
  {-0.03000000000000027, 0.0045017862748640836},
  {-0.02000000000000018, 0.0076938147012420854},
```

```
alltheory20 = {{-0.4}, 4.834442790535576\**-7}, {-0.39}, 7.280806675336038\**-7},
   {-0.38`, 1.1013550670346817`*^-6}, {-0.37`, 1.6739418127187378`*^-6},
   {-0.36000000000000004, 2.5571145922942407**^-6}, {-0.35000000000000003,
    3.926711245827441`*^-6}, {-0.34`, 6.0616106480892995`*^-6},
   {-0.33`, 9.405275660845369`*^-6}, {-0.32`, 0.00001466408464945168`},
   {-0.3100000000000005, 0.00002296397914403059},
   {-0.3000000000000004, 0.000036099947758549966},
   {-0.2900000000000004, 0.000056929773823877206},
   {-0.28, 0.00008999608544957294, {-0.27, 0.00014250026368112234},
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    0.0014147467761264076`}, {-0.2100000000000002`, 0.0022270230475007757`},
   {-0.2`, 0.003494639623003337`}, {-0.1900000000000003`, 0.005463884823775689`},
   {-0.1800000000000002`, 0.008507921962300657`},
   {-0.15000000000000002`, 0.031189923818135568`},
   {-0.14<sup>^</sup>, 0.04752275577813281<sup>^</sup>}, {-0.13<sup>^</sup>, 0.07186039463277039<sup>^</sup>},
   {-0.12`, 0.10766009539364907`}, {-0.1100000000000004`, 0.15941014043993607`},
   {-0.10000000000000003, 0.23240769609962075},
   {-0.09000000000000002, 0.33176964127173436},
   {-0.08000000000000002`, 0.4599591492094549`}, {-0.07`, 0.6121606584397329`},
   {-0.06`, 0.7703936794609902`}, {-0.04999999999999`, 0.9023396525261022`},
   {-0.04000000000000036, 0.9773439970496514},
   {-0.03000000000000027, 0.9984758621156562},
   {-0.02000000000000018, 0.9999949973318434},
```

Plot

```
N0 = 10^4;
mmax = 0.5;
\lambda = 2 Es / 4;
Es = 0.01;
U = 2 * 10^{-3};
letter = "A";
figS6a = Show
      ListLogPlot theory2,
         Joined → True,
         PlotStyle \rightarrow Thick,
         PlotRange \rightarrow \{\{-0.4, 0\}, \{5*10^{-7}, 1\}\},\
         Frame → {True, False, False, True},
         \label{limits} {\tt FrameLabel} \rightarrow \{"{\tt Wildtype growth rate", ,, "Probability of rescue"}\},
         FrameTicks → {True, False, False, True},
         LabelStyle → labelstyle,
         {\tt PlotLegends} \rightarrow {\tt Placed[LineLegend[Style[\#, 12, FontFamily \rightarrow "Helvetica"] \& /@ Content of the property o
                       {"1-step", "2-step", "3-step", "4-step"}], Scaled@{0.85, 0.23}]
       ],
      ListLogPlot[alltheory2, Joined → True, PlotStyle → {Black, Thick},
         PlotLegends → Placed[LineLegend[
                   Style[\#, 12, FontFamily \rightarrow "Helvetica"] & /@ {"total"}], Scaled@ {0.56, 0.23}]
      ],
      ListLogPlot[theory20,
         Joined → True,
         PlotStyle → Directive[Thick, Dashing[Large]]
      ListLogPlot[alltheory20,
         Joined → True, PlotStyle → {Black, Thick, Dashing[Large]},
         PlotLegends → Placed[LineLegend[
                    {Directive[Thick, Gray], Directive[Thick, Gray, Dashing[Medium]]},
                   Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@\{"n=2", "n=20"\}],
                Scaled@{1/10,7/10}]
      ],
      Epilog → {
            Text[Style[letter, Bold, 14, FontFamily > "Helvetica"], Scaled@{0.05, 0.95}],
            braceLabel \{\{-0.125, \log@10^{-6}\}, \{-0.125, \log@(2*10^{-4})\}\}, Style["", Larger]
```



Rescue genotype growth rate (figure S6B)

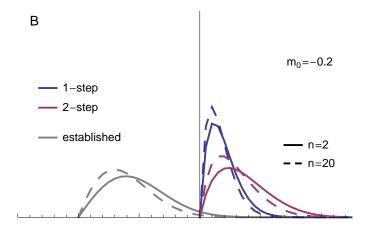
```
xmin = -0.3;
xmax = 0.25;
ymax = 40;
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / 4;
Es = 0.01;
mwt = -0.2;
xint = 0.01;
n = 2;
{
   (*established*)
  Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}
oldtheory2 = ListPlot[%, PlotRange \rightarrow All, Joined \rightarrow True,
    PlotStyle → Directive [Thick, Gray], PerformanceGoal → "Speed", PlotLegends →
     Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"established"}],
      Scaled@{1.5/8,1/2}]];
   (*1 step*)
  Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
   (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax, xint}\right\}\right]
theory2 = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → Directive[Thick], PerformanceGoal → "Speed",
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
```

```
{"1-step", "2-step"}], Scaled@{1.5/8,1/2}]];
n = 20;
   (*established*)
  Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
       fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}
 };
oldtheory4 = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → Directive [Thick, Gray, Dashing [Large]], PerformanceGoal → "Speed"];
  (*1 step*)
  Table [\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
  (*2 step*)
  total = Re[g2denominator[mwt]];
  Table \left[\left\{m2, \frac{1}{total} g2numerator[mwt, m2]\right\}, \left\{m2, 0, xmax, xint\right\}\right]
 };
theory4 = ListPlot[%, PlotRange → All,
   Joined → True, PlotStyle → Directive[Thick, Dashing[Large]],
   PerformanceGoal → "Speed", PlotLegends → Placed[LineLegend[
       {Directive[Thick, Black], Directive[Thick, Black, Dashing[Medium]]}, Style[#,
           12, FontFamily \rightarrow "Helvetica"] & /@ {"n=2", "n=20"}], Scaled@ {7 / 8, 3 / 10}]];
figS6b = Show[
  {oldtheory2, theory2, oldtheory4, theory4},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  Frame → {True, False, False, False},
  LabelStyle → labelstyle,
  FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
  Epilog → {
     Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily → "Helvetica"], Scaled@{7/8,3/4}],
     Text[Style["B", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]
    }
 ]
Clear [mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, xint]
NIntegrate::izero:
```

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>

NIntegrate::izero:

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>



2-step intermediate growth rate (figure S6C)

```
xmin = -0.3;
xmax = 0.25;
ymax = 30;
N0 = 10000;
U = 2 * 10^{-3};
mmax = 0.5;
\lambda = 2 Es / 4;
Es = 0.01;
mwt = -0.2;
n = 2;
\texttt{exact2} = \texttt{fm} [\texttt{m1}, \texttt{mwt}, \texttt{mmax}, \lambda, \texttt{n}] \; (\texttt{1-pest}[\texttt{m1}]) \; \texttt{prescuem} [\texttt{m1}, \Lambda \texttt{1}[\texttt{m1}, \texttt{mmax}, \lambda, \texttt{n}, U]] \; ;
allexact2 = NIntegrate[exact2, {m1, mwt, 0.1}];
oldtheory2 = (fm[m1, mwt, mmax, \lambda, n] pest[m1 - mwt] HeavisideTheta[m1 - mwt]) /
    NIntegrate[fm[m1, mwt, mmax, \(\lambda\), n] pest[m1-mwt], \(\lambda\), mmax\);
n = 20;
exact4 = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
allexact4 = NIntegrate[exact4, {m1, mwt, 0.1}];
\verb|oldtheory4 = (fm[m1, mwt, mmax, \lambda, n]| pest[m1-mwt]| HeavisideTheta[m1-mwt]) / \\
    NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}];
figS6c = Show[
  Plot[
    {oldtheory2, oldtheory4},
    {m1, xmin, xmax},
    PlotRange \rightarrow \{0, All\},\
    PlotStyle → {Directive[Gray, Thick], Directive[Gray, Thick, Dashing[Large]]},
    PerformanceGoal → "Speed",
    PlotLegends →
     Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"established"}],
       Scaled@{1/8, 1.25/2}]
   ],
  Plot[
```

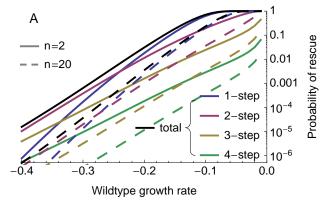
```
exact2 / allexact2,
    {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   PlotStyle → Directive[Thick],
   PerformanceGoal → "Speed",
   PlotLegends →
    Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"first-step"}],
      Scaled@{1/8, 1.25/2}]
  ],
  Plot[
   exact4/allexact4,
   {m1, xmin, xmax},
   PlotRange \rightarrow \{0, All\},\
   PlotStyle → Directive[Thick, Dashing[Large]],
   PerformanceGoal → "Speed",
   PlotLegends → Placed[LineLegend[
       {Directive[Thick, Black], Directive[Thick, Black, Dashing[Medium]]},
       Style[#, 12, FontFamily → "Helvetica"] & /@
        {"n=2", "n=20"}], Scaled@{7/8,3/10}]
  ],
  Frame → {True, False, False, False},
  FrameTicksStyle → {Automatic, Automatic, Automatic},
  FrameLabel → {"Growth rate"},
  PlotRange → {{xmin, xmax}, {0, ymax}},
  LabelStyle → labelstyle,
  Epilog \rightarrow {
     Text[Style["m<sub>0</sub>=" <> ToString[mwt],
       12, FontFamily \rightarrow "Helvetica"], Scaled@\{7/8, 3/4\}],
     Text[Style["C", 14, Bold, FontFamily → "Helvetica"], Scaled@{0.05, 0.95}]
   }
 ]
Clear [mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, N0]
  С
                                          m_0 = -0.2
 first-step

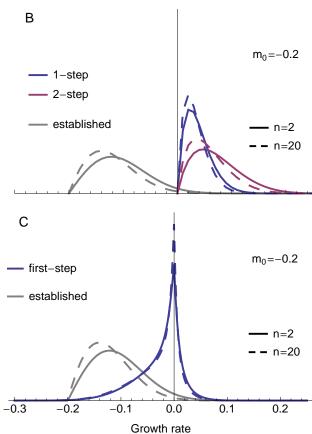
    established

                                            - n=2
                                           - n=20
-0.3
         -0.2
                                    0.1
                                             0.2
                  -0.1
                           0.0
                      Growth rate
```

Figure S6

 $\texttt{GraphicsColumn[\{figS6a, figS6b, figS6c\}, Spacings} \rightarrow -110, \texttt{ImageSize} \rightarrow 350]$ Export[supfigdir <> "FigureS6.eps", %];





Mutation rate U vs mutational variance λ

Rate. U

Probability of rescue

```
N0 = 10^4;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
m0min = mTempmin;
m0max = mTempmax;
mstep = step;
m0list = Table[m0, {m0, m0min, m0max, mstep}];
U = 10^{-3};
rate4 = Table[{m0, Λ4Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate3 = Table[{m0, Λ3Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate2 = Table[{m0, Λ2Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate1 = Table[{m0, Λ1Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
theory2 = {
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]]]\},
           {i, Length[m0list]}],
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate2[[i, 2]]]\},
           {i, Length[m0list]}],
        Table[{m0list[[i]], prescue /. p0 → prescuem[m0list[[i]], rate3[[i, 2]]]},
           {i, Length[m0list]}],
        Table[{m0list[[i]], prescue /. p0 → prescuem[m0list[[i]], rate4[[i, 2]]]},
          {i, Length[m0list]}]
     };
alltheory2 = Table[
        \{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]] + rate2[[i, 2]] + rate2[[i
                    rate3[[i, 2]] + rate4[[i, 2]]]}, {i, Length[m0list]}];
U = 10^{-4};
rate4 = Table[{m0, A4Interpolated[m0, mmax, \(\lambda\), n, U]}, {m0, m0min, m0max, mstep}];
rate3 = Table[{m0, Λ3Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate2 = Table[{m0, Λ2Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate1 = Table[{m0, Λ1Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
theory4 = {
        Table[{m0list[[i]], prescue /. p0 → prescuem[m0list[[i]], rate1[[i, 2]]]},
           {i, Length[m0list]}],
        Table[{m0list[[i]], prescue /. p0 → prescuem[m0list[[i]], rate2[[i, 2]]]},
           {i, Length[m0list]}],
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate3[[i, 2]]]\},
          {i, Length[m0list]}],
```

```
Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate4[[i, 2]]]\},
          {i, Length[m0list]}]
     };
alltheory4 = Table[
        \{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]] + rate2[[i, 2]] + rate2[[i
                    rate3[[i, 2]] + rate4[[i, 2]]]}, {i, Length[m0list]}];
Show
  ListLogPlot theory2,
     Joined → True,
     PlotStyle → Thick,
     PlotRange \rightarrow \{\{-0.4, 0\}, \{5*10^{-7}, 1\}\},\
     Frame → {True, False, False, True},
    FrameLabel → {"Wildtype growth rate", , , "Probability of rescue"},
     FrameTicks → {True, False, False, True},
    LabelStyle → labelstyle,
    PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
               {"1-step", "2-step", "3-step", "4-step"}], Scaled@{3/4,1/4}]
   ],
  ListLogPlot[alltheory2, Joined → True, PlotStyle → {Black, Thick},
     PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
               {"1-, 2-, 3-, or 4-step"}], Scaled@{1/4,8/10}]
  ],
  ListLogPlot[theory4,
    Joined → True,
     PlotStyle → Directive[Thick, Dashing[Large]]
  ListLogPlot [alltheory4, Joined → True, PlotStyle → {Black, Thick, Dashing[Large]},
    PlotLegends → Placed
         LineLegend[{Directive[Thick, Gray], Directive[Thick, Gray, Dashing[Medium]]},
            Style[#, 12, FontFamily \rightarrow "Helvetica"] & /@
               {"U=10^{-3}", "U=10^{-4}"}, Scaled@{1/10, 6/10}
  Epilog \rightarrow Text[Style["A", Bold, 14], Scaled@{0.05, 0.95}]
Clear [NO, mmax, \lambda, Es, n, U, m0min, m0max, mstep]
NIntegrate::ncvb:
   Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near \{m\} = \{0.000767263\}.
         NIntegrate obtained 7.279705329608732`*^-8 and
         8.761530263711027*^-13 for the integral and error estimates. \gg
NIntegrate::ncvb:
   Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near \{m\} = \{0.000767263\}.
         NIntegrate obtained 1.0286191497588282 *^-7 and
          1.124360034010241`*^-12 for the integral and error estimates. \gg
NIntegrate::ncvb:
   NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near \{m\} = \{0.000767263\}.
         NIntegrate obtained 1.4522794837384644`*^-7 and
         2.2232504361922817`*^-12 for the integral and error estimates. \gg
General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>
```

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.4807240285122168`*^-8 and

7.012035920314079`*^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.961272248188249`*^-8 and

1.2926522573741938*^-12 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.5960369823117813`*^-8 and

2.0146265215810807*^-12 for the integral and error estimates. \gg

 $General::stop: Further \ output \ of \ NIntegrate::ncvb \ will \ be \ suppressed \ during \ this \ calculation. \gg$

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 7.279705329608732`*^-8 and

 $8.761530263711027\ ^*^-13$ for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.0286191497588282^{*^-} and

1.124360034010241`*^-12 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.4522794837384644`*^-7 and

 $2.2232504361922817\ensuremath{\,^{\circ}}\ensuremat$

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near {m} = {0.000767263}. NIntegrate obtained 7.594589776123759`*^-9 and

4.002636289199166`*^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.0811191772406686`*^-8 and

6.179494725282621 *^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.539096864690094`*^-8 and

1.0371883643828802 *^- 12 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.9750026573164113`*^- 10 and

1.9587973588764628*^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.6362598419413893`*^- 10 and

3.2561917484902713*^-13 for the integral and error estimates. \gg

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 3.518449596971985`*^- 10 and

5.369851615725374 *^- 13 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 7.594589776123759`*^-9 and

4.002636289199166`*^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.0811191772406686`*^-8 and

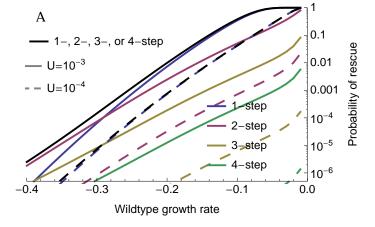
6.179494725282621*^-13 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.539096864690094`*^-8 and

1.0371883643828802 *^- 12 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>



Rescue genotype growth rate

```
xmin = -0.3;
xmax = 0.25;
ymax = 40;
xint = 0.01;
N0 = 10000;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
n = 4;
mwt = -0.2;
U = 10^{-3};
   (*established*)
```

```
Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
                     fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}
   };
oldtheory2 = ListPlot[%, PlotRange → All, Joined → True,
          {\tt PlotStyle} \rightarrow {\tt Directive[Thick, Gray]} \; , \; {\tt PerformanceGoal} \rightarrow {\tt "Speed"} \; , \; {\tt PlotLegends} \rightarrow {\tt Thick} \; , \; {\tt PlotLegends} \rightarrow {\tt PlotLegends}
              Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"established"}],
                  Scaled@{1.5/8,1/2}]];
       Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
        (*2 step*)
       total = Re[g2denominator[mwt]];
       Table \left[\left\{m2, \frac{1}{total} g2numerator[mwt, m2]\right\}, \left\{m2, 0, xmax, xint\right\}\right]
   };
theory2 = ListPlot[%, PlotRange → All, Joined → True,
          PlotStyle → Directive[Thick], PerformanceGoal → "Speed",
          PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
                         {"1-step", "2-step"}], Scaled@{1.5 / 8, 1 / 2}]];
U = 10^{-4};
        (*established*)
       Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
                     fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}]
oldtheory4 = ListPlot[%, PlotRange → All, Joined → True,
          PlotStyle → Directive[Thick, Gray, Dashing[Large]], PerformanceGoal → "Speed"];
        (*1 step*)
       Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
        (*2 step*)
       total = Re[g2denominator[mwt]];
       Table \left[\left\{m2, \frac{1}{total} g2numerator[mwt, m2]\right\}, \left\{m2, 0, xmax, xint\right\}\right]
theory4 = ListPlot [%, PlotRange → All,
          Joined → True, PlotStyle → Directive[Thick, Dashing[Large]],
           PerformanceGoal → "Speed", PlotLegends → Placed[LineLegend[
                     {Directive[Thick, Black], Directive[Thick, Black, Dashing[Medium]]},
                     Style[#, 12, FontFamily \rightarrow "Helvetica"] & / @ \{ "U=10^{-3} ", "U=10^{-4} " \} ],
                  Scaled@{7/8,3/10}]];
Show[
    {oldtheory2, theory2, oldtheory4, theory4},
   PlotRange → {{xmin, xmax}, {0, ymax}},
   Frame → {True, False, False, False},
```

```
LabelStyle → labelstyle,
 FrameTicksStyle → {FontColor → White, Automatic, Automatic, Automatic},
 Epilog \rightarrow {
   Text[Style["m<sub>0</sub>=" <> ToString[mwt], 12, FontFamily → "Helvetica"],
    Scaled@{7/8,3/4}],
   Text[Style["B", 14, Bold], Scaled@{0.05, 0.95}]
  }
]
```

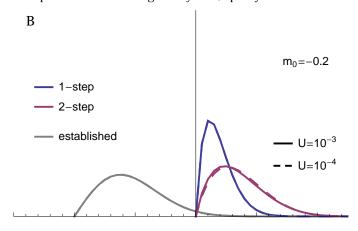
Clear $[mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, xint]$

NIntegrate::izero:

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>

NIntegrate::izero:

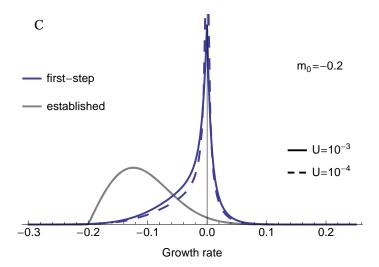
Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>



2-step intermediate growth rate

```
xmin = -0.3;
xmax = 0.25;
ymax = 30;
N0 = 10000;
mmax = 0.5;
\lambda = 2 Es / n;
Es = 0.01;
mwt = -0.2;
n = 4;
U = 10^{-3};
exact2 = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
allexact2 = NIntegrate[exact2, {m1, mwt, 0.1}];
oldtheory2 = (fm[m1, mwt, mmax, λ, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
   NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}];
U = 10^{-4};
```

```
exact4 = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
allexact4 = NIntegrate[exact4, {m1, mwt, 0.1}];
oldtheory4 = (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] HeavisideTheta[m1-mwt]) /
   NIntegrate[fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}];
Show
 Plot[
  {oldtheory2, oldtheory4},
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},\
  PlotStyle → {Directive[Gray, Thick], Directive[Gray, Thick, Dashing[Large]]},
  PerformanceGoal → "Speed",
  PlotLegends →
   Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ { "established" }],
     Scaled@{1/8, 1.25/2}]
 ],
 Plot[
  exact2/allexact2,
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},
  PlotStyle → Directive[Thick],
  PerformanceGoal → "Speed",
  PlotLegends →
   Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"first-step"}],
     Scaled@{1/8, 1.25/2}]
 ],
 Plot
  exact4 / allexact4,
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},
  PlotStyle → Directive[Thick, Dashing[Large]],
  PerformanceGoal → "Speed",
  PlotLegends → Placed
    LineLegend [Directive[Thick, Black], Directive[Thick, Black, Dashing[Medium]]},
      Style[\#, 12, FontFamily \rightarrow "Helvetica"] & /@
       {"U=10^{-3}", "U=10^{-4}"}, Scaled@{7/8, 3/10}
 |,
 Frame → {True, False, False, False},
 FrameTicksStyle → {Automatic, Automatic, Automatic},
 FrameLabel → {"Growth rate"},
 PlotRange → {{xmin, xmax}, {0, ymax}},
 LabelStyle → labelstyle,
 Epilog \rightarrow \{
   Text[Style["m_0=" <> ToString[mwt], 12, FontFamily \rightarrow "Helvetica"],
    Scaled@\{7/8, 3/4\}],
   Text[Style["C", 14, Bold], Scaled@{0.05, 0.95}]
  }
Clear [mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, N0]
```



Variance, λ

Probability of rescue

```
N0 = 10^4;
mmax = 0.5;
Es = 0.01;
n = 4;
U = 2 * 10^{-3};
m0min = mTempmin;
m0max = mTempmax;
mstep = step;
m0list = Table[m0, {m0, m0min, m0max, mstep}];
\lambda = 0.005;
rate4 = Table[{m0, Λ4Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate3 = Table[\{m0, \Lambda 3Interpolated[m0, mmax, \lambda, n, U]\}, \{m0, m0min, m0max, mstep\}];
rate2 = Table[{m0, Λ2Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate1 = Table[{m0, Λ1Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
theory2 = {
            Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], ratel[[i, 2]]]\}, \\
              {i, Length[m0list]}],
           Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate2[[i, 2]]]\},
              {i, Length[m0list]}],
          Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate3[[i, 2]]]\},
              {i, Length[m0list]}],
            Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate4[[i, 2]]]\}, \\
              {i, Length[m0list]}]
       };
alltheory2 = Table[
           \{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]] + rate2[[i, 2]] + rate2[[i
                           rate3[[i, 2]] + rate4[[i, 2]]]}, {i, Length[m0list]}];
```

```
\lambda = \frac{0.005}{2};
rate4 = Table[\{m0, \Lambda 4Interpolated[m0, mmax, \lambda, n, U]\}, \{m0, m0min, m0max, mstep\}];
rate3 = Table[{m0, Λ3Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
rate2 = Table[{m0, Λ2Interpolated[m0, mmax, λ, n, U]}, {m0, m0min, m0max, mstep}];
\texttt{rate1} = \texttt{Table}[\{\texttt{m0}, \texttt{\Lambda1Interpolated}[\texttt{m0}, \texttt{mmax}, \lambda, \texttt{n}, \texttt{U}]\}, \{\texttt{m0}, \texttt{m0min}, \texttt{m0max}, \texttt{mstep}\}];
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]]]\},
           {i, Length[m0list]}],
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate2[[i, 2]]]\},
           {i, Length[m0list]}],
        Table[{m0list[[i]], prescue /. p0 → prescuem[m0list[[i]], rate3[[i, 2]]]},
           {i, Length[m0list]}],
        Table[\{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate4[[i, 2]]]\},
          {i, Length[m0list]}]
      };
alltheory4 = Table[
        \{m0list[[i]], prescue /. p0 \rightarrow prescuem[m0list[[i]], rate1[[i, 2]] + rate2[[i, 2]] + rate2[[i
                    rate3[[i, 2]] + rate4[[i, 2]]]}, {i, Length[m0list]}];
 Show
   ListLogPlot[
      theory2,
     Joined → True,
     PlotStyle → Thick,
     PlotRange \rightarrow \{\{-0.4, 0\}, \{5*10^{-7}, 1\}\},\
     Frame → {True, False, False, True},
     FrameLabel → {"Wildtype growth rate", , , "Probability of rescue"},
     FrameTicks → {True, False, False, True},
     LabelStyle → labelstyle,
     PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
                {"1-step", "2-step", "3-step", "4-step"}], Scaled@{3/4,1/4}]
   ListLogPlot[alltheory2, Joined → True, PlotStyle → {Black, Thick},
      PlotLegends \rightarrow Placed[LineLegend[Style[\#, 12, FontFamily \rightarrow "Helvetica"] \& /@ \\
                {"1-, 2-, 3-, or 4-step"}], Scaled@{1/4,8/10}]
   ],
   ListLogPlot[theory4,
     Joined \rightarrow True,
     PlotStyle → Directive[Thick, Dashing[Large]]
   ListLogPlot[alltheory4, Joined → True, PlotStyle → {Black, Thick, Dashing[Large]},
     PlotLegends → Placed[
          LineLegend[{Directive[Thick, Gray], Directive[Thick, Gray, Dashing[Medium]]},
             Style[#, 12, FontFamily → "Helvetica"] & /@
                {"\lambda=0.005", "\lambda=0.0025"}], Scaled@{1/10, 6/10}]
   Epilog \rightarrow Text[Style["A", Bold, 14], Scaled@{0.05, 0.95}]
Clear [NO, mmax, \lambda, Es, n, U, m0min, m0max, mstep]
```

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.4363490176861695`*^-7 and

1.034782441470326`*^-12 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.0246698484863734`*^-7 and

1.0921054693023465`*^-12 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.850952330867838`*^-7 and

2.321744002394699*^-12 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 5.3450453526009294`*^-8 and $5.951525075263353`*^-13$ for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 7.059025621526559`*^-8 and

 $8.588848983658455\ensuremath{\,^{\circ}}\ensuremath{\,^{\circ}}\ensuremath{^{\circ}}\e$

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 9.314638149255934`*^-8 and

1.711619241364099`*^-12 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 1.4363490176861695`*^-7 and

 $1.034782441470326`*^-12$ for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.0246698484863734`*^-7 and

1.0921054693023465`*^-12 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.850952330867838`*^-7 and

2.321744002394699`*^-12 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 1.371611099561599`*^- 10 and 3.2533774948230864^{*} -16 for the integral and error estimates. \gg

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 2.459129528372128`*^-10 and

5.379711113614932`*^-16 for the integral and error estimates. \gg

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 4.382809860895368`*^- 10 and

 $8.119644899145538`*^-16$ for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. ≫

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 2.830813641888082`*^-9 and

3.4578366587790035`*^-15 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 4.427814198985973`*^-9 and

8.978439335615778*^-15 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{0.000767263\}$. NIntegrate obtained 6.8986954726390565`*^-9 and

2.174871671811652`*^-14 for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. ≫

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 1.371611099561599`*^- 10 and 3.2533774948230864*^-16 for the integral and error estimates. \gg

NIntegrate::ncvb:

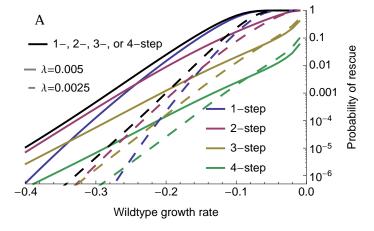
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 2.459129528372128`*^-10 and 5.379711113614932*^-16 for the integral and error estimates. \gg

NIntegrate::ncvb:

Nintegrate failed to converge to prescribed accuracy after 9 recursive bisections in m near $\{m\} = \{-0.199623\}$. NIntegrate obtained 4.382809860895368`*^- 10 and

 $8.119644899145538`*^-16$ for the integral and error estimates. \gg

General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation. >>



Rescue genotype growth rate

xmin = -0.3;xmax = 0.25;

```
ymax = 40;
xint = 0.01;
N0 = 10000;
mmax = 0.5;
Es = 0.01;
n = 4;
mwt = -0.2;
U = 2 * 10^{-3};
\lambda = 0.005;
         (*established*)
        Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
                        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}], \{m2, mwt, xmax, xint\}
oldtheory2 = ListPlot[%, PlotRange → All, Joined → True,
            {\tt PlotStyle} \rightarrow {\tt Directive[Thick, Gray]} \; , \; {\tt PerformanceGoal} \rightarrow {\tt "Speed"} \; , \; {\tt PlotLegends} \rightarrow {\tt Thick} \; , \; {\tt PlotLegends} \rightarrow {\tt PlotLegends}
                Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"established"}],
                    Scaled@{1.5/8,1/2}]];
         (*1 step*)
        Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
         (*2 step*)
        total = Re[g2denominator[mwt]];
        Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax, xint}\right\}\right]
theory2 = ListPlot[%, PlotRange → All, Joined → True,
            PlotStyle → Directive[Thick], PerformanceGoal → "Speed",
            PlotLegends → Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@
                             {"1-step", "2-step"}], Scaled@{1.5/8,1/2}]];
\lambda = 0.005 / 2;
         (*established*)
        Table[\{m2, (fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt]) / NIntegrate[
                        fm[m2, mwt, mmax, \lambda, n] pest[m2-mwt], \{m2, mwt, mmax\}]\}, \{m2, mwt, xmax, xint\}]
oldtheory4 = ListPlot[%, PlotRange → All, Joined → True,
            PlotStyle → Directive[Thick, Gray, Dashing[Large]], PerformanceGoal → "Speed"];
         (*1 step*)
        Table[\{m2, g1m /. m \rightarrow m2\}, \{m2, 0, xmax, xint\}],
         (*2 step*)
        total = Re[g2denominator[mwt]];
       Table \left[\left\{m2, \frac{1}{\text{total}} \text{ g2numerator}[\text{mwt, m2}]\right\}, \left\{m2, 0, \text{xmax, xint}\right\}\right]
```

```
};
theory4 = ListPlot[%, PlotRange → All, Joined → True,
   PlotStyle → Directive[Thick, Dashing[Large]], PerformanceGoal → "Speed",
   PlotLegends → Placed[LineLegend[{Directive[Thick, Black], Directive[Thick,
         Black, Dashing [Medium]]}, Style [#, 12, FontFamily → "Helvetica"] & /@
        {"\lambda=0.005", "\lambda=0.0025"}], Scaled@{7/8, 3/10}]];
Show[
 {oldtheory2, theory2, oldtheory4, theory4},
 PlotRange → {{xmin, xmax}, {0, ymax}},
 Frame → {True, False, False, False},
 LabelStyle → labelstyle,
 FrameTicksStyle → {FontColor → White, Automatic, Automatic},
 Epilog \rightarrow {
   Text[Style["m_0=" <> ToString[mwt], 12, FontFamily \rightarrow "Helvetica"],
    Scaled@{7/8,3/4}],
   Text[Style["B", 14, Bold], Scaled@{0.05, 0.95}]
  }
]
Clear [mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, xint]
```

NIntegrate::izero:

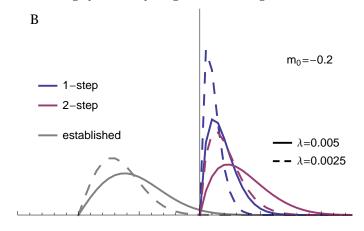
Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>

NIntegrate::izero:

Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option. >>

NIntegrate::slwcon:

Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or Working Precision too small. \gg



2-step intermediate growth rate

```
xmin = -0.3;
xmax = 0.25;
ymax = 30;
```

```
N0 = 10000;
mmax = 0.5;
Es = 0.01;
mwt = -0.2;
n = 4;
U = 2 * 10^{-3};
\lambda = 0.005;
exact2 = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
allexact2 = NIntegrate[exact2, {m1, mwt, 0.1}];
\verb|oldtheory2| = (fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt] | HeavisideTheta[m1-mwt]) / 
   NIntegrate [fm[m1, mwt, mmax, \lambda, n] pest[m1-mwt], {m1, mwt, mmax}];
\lambda = 0.005 / 2;
exact4 = fm[m1, mwt, mmax, \lambda, n] (1 - pest[m1]) prescuem[m1, \Lambda1[m1, mmax, \lambda, n, U]];
allexact4 = NIntegrate[exact4, {m1, mwt, 0.1}];
\verb|oldtheory4 = (fm[m1, mwt, mmax, \lambda, n]| pest[m1-mwt]| HeavisideTheta[m1-mwt]) / \\
   NIntegrate[fm[m1, mwt, mmax, \(\lambda\), n] pest[m1-mwt], {m1, mwt, mmax}];
Show[
 Plot[
  {oldtheory2, oldtheory4},
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},\
  PlotStyle → {Directive[Gray, Thick], Directive[Gray, Thick, Dashing[Large]]},
  PerformanceGoal → "Speed",
  PlotLegends →
   Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"established"}],
     Scaled@{1/8,1.25/2}]
 ],
 Plot[
  exact2/allexact2,
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},
  PlotStyle → Directive[Thick],
  PerformanceGoal → "Speed",
  PlotLegends →
   Placed[LineLegend[Style[#, 12, FontFamily → "Helvetica"] & /@ {"first-step"}],
     Scaled@{1/8, 1.25/2}]
 ],
 Plot[
  exact4/allexact4,
  {m1, xmin, xmax},
  PlotRange \rightarrow \{0, All\},\
  PlotStyle → Directive[Thick, Dashing[Large]],
  PerformanceGoal → "Speed",
  PlotLegends → Placed[
     LineLegend[{Directive[Thick, Black], Directive[Thick, Black, Dashing[Medium]]},
      Style[#, 12, FontFamily → "Helvetica"] & /@
       \{ \lambda=0.005, \lambda=0.0025 \} , Scaled@\{7/8, 3/10\} 
 ],
 Frame → {True, False, False, False},
 FrameTicksStyle → {Automatic, Automatic, Automatic},
```

```
FrameLabel → {"Growth rate"},
 PlotRange → {{xmin, xmax}, {0, ymax}},
 LabelStyle \rightarrow labelstyle,
 Epilog \rightarrow {
    \texttt{Text}[\texttt{Style}["m_0 = " <> \texttt{ToString}[mwt] \,,\, 12 \,,\, \texttt{FontFamily} \rightarrow "\texttt{Helvetica"}] \,,
      Scaled@{7/8,3/4}],
    Text[Style["C", 14, Bold], Scaled@{0.05, 0.95}]
]
Clear[mmax, \lambda, Es, n, U, mwt, xmin, xmax, ymax, N0]
   \mathsf{C}
                                                       m_0 \! = \! -0.2
   first-step

established

                                                        \lambda = 0.005
                                                      - \lambda=0.0025
-0.3
            -0.2
                       -0.1
                                   0.0
                                               0.1
                                                           0.2
                            Growth rate
```