Higher-order Cheeger inequalities for graphons

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The spectrum of a graph

- Consider a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ for $n \geq 2$.
- Assume that $\deg(v) > 0$ for all $v \in V$.
- The Laplacian of G:

$$\Delta_G := I_n - D^{-1/2} A D^{-1/2},$$

  where $A :=$ the adjacency matrix of G, and
  $D := \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n))$.
- Δ_G is symmetric.
- The eigenvalues of Δ_G:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Higher-order Cheeger inequalities for graphs

Let k be a positive integer with $k \leq n$.

- $h_G(k) :=$ the k-way expansion constant of the graph G.
- The higher-order Cheeger inequality[1]:

$$\frac{\lambda_k}{2} \leq h_G(k) \leq O(k^2)\sqrt{\lambda_k}.$$

[1] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan, *Multiway spectral partitioning and higher-order Cheeger inequalities*, J. ACM **61** (2014), no. 6, Art. 37, 30.

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- $k = 2$: the discrete Cheeger–Buser inequality.

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Graphons

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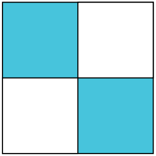
- The associated graphon W_G of the graph G:

Let $P_k := \left[\frac{k-1}{n}, \frac{k}{n}\right)$ for $1 \leq k < n$, and $P_n := \left[\frac{n-1}{n}, 1\right]$.

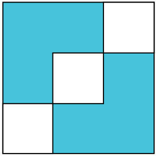
For any $1 \leq i, j \leq n$ and $(x, y) \in P_i \times P_j$, define

$$W_G(x, y) = \begin{cases} 1 & \text{if there is an edge between } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

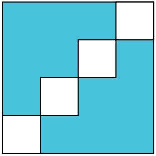
Examples



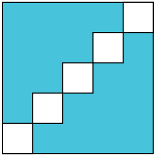
W_{K_2}



W_{K_3}



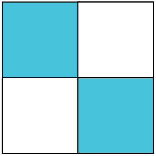
W_{K_4}



W_{K_5}

$\cdots$

Examples



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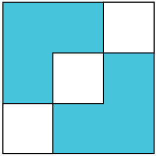

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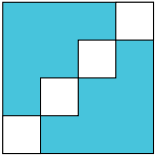

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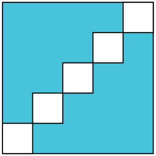

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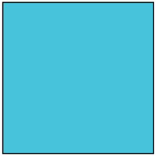

$W = 1$

Preliminaries

- Assume that a graphon W is connected, that is, $\int_{A\times A^c} W > 0$, for every measurable $A \subseteq I$ with $0 < \mu_L(A) < 1$.
- The degree function $d_W : I \to \mathbb{R}$ of W:

$$d_W(x) := \int_I W(x,y)\,\mathrm{d}y \quad \text{for all } x \in I.$$

- For any measurable $A \subseteq I$,

$$\nu(A) := \int_{A\times I} W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then $L^2(I,\nu)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_v = \int_{I^2} f(x)g(x)W(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Preliminaries

- For any measurable $S \subseteq I^2$,

$$\eta(S) := \int_S W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- $E := \{(x,y) \in I^2 : y > x\}$.
- The inner product on the Hilbert space $L^2(E,\eta)$:

$$\langle f, g\rangle_e = \int_0^1 \int_x^1 f(x,y)g(x,y)W(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

The Laplacian of a graphon

- For any $f\colon I \to \mathbb{R}$ and $(x, y) \in I^2$,

$$df(x, y) \coloneqq f(y) - f(x).$$

- The map $d\colon L^2(I, \nu) \to L^2(E, \eta)$ is a bounded linear operator.
- The Laplacian of W:

$$\Delta_W \coloneqq d^* d\colon L^2(I, \nu) \to L^2(I, \nu).$$

- Δ_W is a self-adjoint, positive semidefinite linear operator.
- The Rayleigh quotient of $f \in L^2(I, \nu) \setminus \{0\}$ with respect to Δ_W:

$$R_{\Delta_W}(f) \coloneqq \frac{\langle \Delta_W f, f \rangle_v}{\langle f, f \rangle_v}.$$

Analogs of eigenvalues and expansion constants

Let k be a positive integer.

- $\mathcal{S}_k :=$ the set of all k-dimensional subspaces of $L^2(I, \nu)$.

$$\lambda_k := \inf_{S \in \mathcal{S}_k} \max_{f \in S \setminus \{0\}} R_{\Delta_W}(f).$$

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- $\mathcal{P}_k :=$ the set of all k-tuples $(A_1, \ldots, A_k)$ of measurable subsets of I with $\nu(A_i) > 0$ for all i, and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

The *k-way expansion constant* $h_W(k)$ of a connected graphon W:

$$h_W(k) := \inf_{(A_1, \ldots, A_k) \in \mathcal{P}_k} \max_{1 \le i \le k} h_W(A_i),$$

where $h_W(A_i) := \dfrac{\eta(A_i \times A_i^c)}{\nu(A_i)}$ for $1 \le i \le k$.

Higher-order Cheeger inequalities for graphons

Theorem 1

Given a connected graphon W, the inequality

$$\frac{\lambda_k}{2} \leq h_W(k) \leq O(k^{3.5})\sqrt{\lambda_k}$$

holds, for every positive integer k.

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We prove Theorem 1 using ideas in Trevisan's lecture notes[2].

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The above inequality is proved by Khetan and Mj[3] for $k = 2$.

[2] Luca Trevisan, *Lecture Notes on Graph Partitioning, Expanders and Spectral Methods*, 2017.

[3] Abhishek Khetan and Mahan Mj, *Cheeger inequalities for graph limits*, Ann. Inst. Fourier (Grenoble) **74** (2024), no. 1, 257–305.

Outline of the proof of the upper bound on $h_W(k)$

Lemma 2

Given any orthonormal subset $\{f_1, \ldots, f_k\}$ of $L^2(I, \nu)$, there exist disjointly supported functions $g_1, \ldots, g_k \in L^2(I, \nu) \setminus \{0\}$ such that for all $1 \leq i \leq k$, the inequality

$$R_{\Delta_W}(g_i) \leq O(k^7) \max_{1 \leq j \leq k} R_{\Delta_W}(f_j)$$

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Lemma 3

Let g be an element of $L^2(I, \nu) \setminus \{0\}$. Then there exists a measurable subset A of I with $A \subseteq \mathrm{supp}(g)$ and $\nu(A) > 0$, satisfying the inequality

$$h_W(A) \le \sqrt{2 R_{\Delta_W}(g)}.$$

Constructing disjointly supported functions having small Rayleigh quotients

- Let $\{f_1, \ldots, f_k\}$ be an orthonormal subset of $L^2(I, \nu)$.
- Define $F \colon I \to \mathbb{R}^k$ as $F(x) = (f_1(x), \ldots, f_k(x))$ for all $x \in I$.

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- To "localize" F on k disjoint subsets of I.

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  - Construct k measurable subsets of I each having reasonably large "mass", and which are "well-separated".

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  - Consider "smoothened" indicator functions of these sets, scaled by $\|F\|$.

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  - Construct k measurable subsets of I each having reasonably large "mass", and which are "well-separated".
  - Consider "smoothened" indicator functions of these sets, scaled by $\|F\|$.
- To construct "such" sets:
  Construct measurable "well-separated" subsets of I each having small "mass", but large total "mass", and then "merge" them.

Radial projection distance on I and mass of subsets of I

- Define $\overline{F} \colon F^{-1}(\mathbb{R}^k \setminus \{\mathbf{0}\}) \to \mathbb{R}^k$ by

$$\overline{F}(x) = \frac{1}{\|F(x)\|} F(x),$$

  for any $x \in I$ with $F(x) \neq \mathbf{0}$.

- Define $d_F \colon I \times I \to [0, \infty]$ as follows. For any $x, y \in I$,

$$d_F(x, y) = \begin{cases} \left\|\overline{F}(x) - \overline{F}(y)\right\| & \text{if } F(x), F(y) \neq \mathbf{0}, \\ 0 & \text{if } F(x) = F(y) = \mathbf{0}, \\ \infty & \text{otherwise.} \end{cases}$$

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- The F-mass of a measurable subset A of I:

$$\mathrm{mass}_F(A) := \int_A \|F(x)\|^2 d_W(x)\,\mathrm{d}x.$$

Well-separated small sets with large total mass

Lemma 4

Let $\{f_1, \ldots, f_k\}$ be an orthonormal subset of $L^2(I, \nu)$. Then for some $m \geq 1$, there exist pairwise disjoint measurable subsets $T_1, \ldots, T_m$ of I such that the following conditions hold.

1. $\sum_{i=1}^{m} \mathrm{mass}_F(T_i) \geq k - \frac{1}{4}$.
2. *For any $1 \leq i, j \leq k$ with $i \neq j$, if $x \in T_i$ and $y \in T_j$, then $d_F(x, y) \geq \frac{1}{4\sqrt{5}k^3}$.*
3. $\mathrm{mass}_F(T_i) \leq 1 + \frac{1}{4k}$ *for all $1 \leq i \leq m$.*

Well-separated sets with reasonably large mass

Lemma 5

Let $\{f_1, \ldots, f_k\}$ be an orthonormal subset of $L^2(I, \nu)$. Then there exist pairwise disjoint measurable subsets $A_1, \ldots, A_k$ of I such that the following conditions hold.

1. *For any $1 \leq i \leq k$, $\mathrm{mass}_F(A_i) \geq \frac{1}{2}$.*
2. *For any $1 \leq i, j \leq k$ with $i \neq j$, if $x \in A_i$ and $y \in A_j$, then $d_F(x, y) \geq \frac{1}{4\sqrt{5}k^3}$.*

Constructing disjointly supported functions with small Rayleigh quotients

- Choose subsets $A_1, \ldots, A_k$ of I as guaranteed by the previous lemma.
- $\delta := \frac{1}{4\sqrt{5}k^3}$.
- For any $1 \leq i \leq k$, define $g_i \colon I \to \mathbb{R}$ by

$$g_i(x) = \begin{cases} \left(1 - \frac{2}{\delta} d_F(x, A_i)\right) \|F(x)\| & \text{if } d_F(x, A_i) \leq \frac{\delta}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

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- $g_1, \ldots, g_k$ are disjointly supported.
- $\|g_i\|_v \geq \frac{1}{2}$ for all i.
- For all $x, y \in I$,

$$|g_i(x) - g_i(y)| \leq \|F(x) - F(y)\| \left(1 + \frac{4}{\delta}\right).$$

Disjointly supported functions with small Rayleigh quotients

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Small diameter implies small mass

Lemma 6

Any nonempty Borel measurable subset R of the unit sphere in $\mathbb{R}^k$, with $\operatorname{diam}(R) < \sqrt{2}$*, satisfies the inequality*

$$\operatorname{mass}_F(\overline{F}^{-1}(R)) \leq \left(1 - \frac{1}{2}\operatorname{diam}(R)^2\right)^{-2}.$$

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- For any $R \subseteq \mathbb{S}^{k-1}$, $V(R) \coloneqq \overline{F}^{-1}(R)$.
- To find subsets $R_1, \ldots, R_m$ of $\mathbb{S}^{k-1}$ such that the subsets $V(R_1), \ldots, V(R_m)$ of I are as desired.

Constructing well-separated small sets with large total mass

- $s := \frac{1}{\sqrt{5}k}$.
- For any $\mathbf{n} = (n_1, \ldots, n_k)$ of $\mathbb{Z}^k$,

$$C_{\mathbf{n}} := \prod_{i=1}^{k} [n_i s, n_i s + s),$$

$$\widetilde{C}_{\mathbf{n}} := \prod_{i=1}^{k} \left[n_i s + \frac{s}{8k^2}, n_i s + s - \frac{s}{8k^2} \right].$$

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- The distance between any two "cores" $\geq \frac{s}{4k^2} = \frac{1}{4\sqrt{5}k^3}$.
- The diameter of each "core" $\leq s\sqrt{k} = \frac{1}{\sqrt{5k}}$.

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- The diameter of each "core" $\leq s\sqrt{k} = \frac{1}{\sqrt{5}k}$.
- There is a shift of the partition such that the intersections of the shifted cores with $\mathbb{S}^{k-1}$ have large total F-mass.

Thank you