Fractals in Pascal's Triangle

Math 496 Spring 2022

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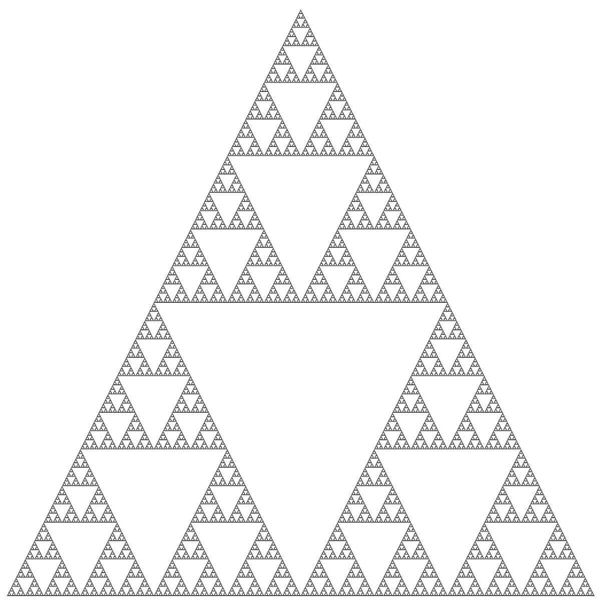


May 13, 2022

The Problem

Pascal's triangle is a commonplace mathematical construction which exhibits a multitude of deep and complex behaviors. Here n denotes row number, which is indexed at 0.

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Rows 0 and 1 of the triangle are always given. A new row is constructed and added to the bottom of the triangle by placing 1 at each of the edges, and for each pair of digits in the bottom row, taking their sum and writing it in the place centered below them (so that the 6 in row $n = 4$ is the sum of 3 and 3 from the row above).

Rows n and row indices k in Pascal's triangle describe a location in the triangle, and it is known that the binomial coefficient n choose k is the value in the location (n, k):

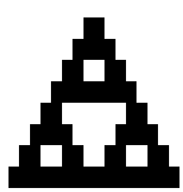
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n and k are indexed at 0, so that row n has k ranging from 0 to n, and row n has $n + 1$-many terms.

It is interesting to choose a number p, called the modulus, and for all values in the triangle (n, k), to replace them with $(n, k)(\text{mod } p)$. For $p = 2$, the triangle is generated:

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 0 & & 1 & & \\
n=3: & & 1 & & 1 & & 1 & & 1 & \\
n=4: & 1 & & 0 & & 0 & & 0 & & 1
\end{array}$$

From now on, the triangle will sometimes be presented as an array of black and transparent square blocks representing the values in the triangle. In this representation, a black block indicates that the value in its location is nonzero, and a transparent block indicates that the value is zero. An example follows:



This is the representation of the first 8 rows of Pascal's triangle, modulo 2. The triangle, represented in this way, appears to show some self-similarity. This was the observation that prompted the research project; I happened across this while playing with the triangle, and found it very interesting. Because I could only feasibly draw so many rows by hand, I wrote a computer program to generate the triangle.

It was apparent at this point that the triangle was continuing to exhibit self similarity mod 2, at least when expanded to any finite number of rows, so I began testing different moduli. My hypothesis was that any prime modulus would produce a fractal, including when taken to infinitely many rows. Importantly, fractals do not have a commonly accepted definition. I will provide an appropriate one here, from Mandelbrot [1]:

Definition (Fractal)**.** *A* ***fractal*** *is a geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole.*

It seems visually intuitive that this figure is a fractal, but it is not obviously the case that this behavior continues for arbitrarily many rows of Pascal's triangle, with $p = 2$ or otherwise. The goal of this project was to prove that these figures are fractals for arbitrarily many rows, and to classify the fractals based on properties of the modulus p.

Before moving on to the next section, I should mention that my main result, namely, that Pascal's triangle mod p (a prime) is always a fractal, is known. I developed an intuition for this fact while playing with the triangle on paper one day, and decided to prove the result using only my limited background in elementary number theory, which I supplemented by collecting troves of empirical data. This was helpful because it gave me a sense of how certain structures *usually* formed; though this doesn't amount to a mathematical proof, it certainly helped me figure out which things were worth trying to prove.

All images and data in this project were generated by a computer program I wrote for this purpose. This paper is about a new method for proving an old fact. The method, I think, is simple and intuitive, and it can likely be modified to fit other numerical constructions.

Results

Let $p \in \mathbb{N}$. There are four relevant cases for the type of number the modulus p can be: prime, square-free, prime powers, and otherwise composite numbers. Correspondingly, the fractals generated will be referred to as prime fractals, etc. for linguistic convenience. The first result will ease the transition from prime to square-free fractals.

Definition (Square-Free Integer)**.** *Let an integer x be called* ***square-free*** *if there are no repeated factors in its prime factorization.*

By this definition, prime numbers are square-free. We are going to prove a method by which prime fractals construct square-free fractals, which is trivially valid for prime fractals themselves.

Lemma (Union-Intersection)**.** *A fractal generated by a square-free modulus p is, equivalently, the union of the nonzero values, or the intersection of the zero values, in all of the fractals in the corresponding locations generated by the factors of p.*

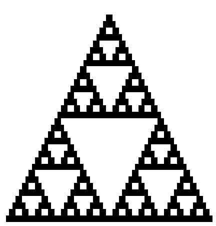
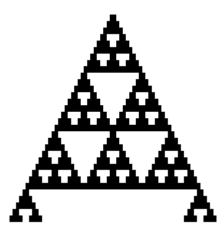
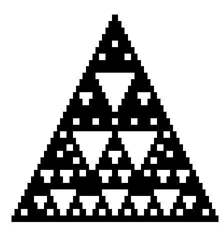
Proof. For each prime p, let $\Delta_p = \{(n,k) : \binom{n}{k} \not\equiv 0 (\text{mod } p)\}$ be the set of all locations for nonzero values in the fractal generated by p. In our diagrams, these are the black boxes. Let $c = p_1 \cdot p_2 \cdot \ldots \cdot p_j$ be a square-free integer, so that all of its prime factors p_i are distinct. Then the set of nonzero locations for c is the union of all Δ_{p_i} in its prime factorization:

$$\Delta_c = \bigcup_{1 \leq i \leq j} \Delta_{p_i}$$

The reason is intuitive; if the value at some location (n, k) is not divisible by a factor of c, then it cannot be divisible by c itself.

Conversely, for any prime p, we can define a set of all locations: $\{(n,k) : n \in \mathbb{N}, k \leq n\}$. If we subtract Δ_p from this set, we yield the set of locations of all zero values. If we have a similar c and list of factors, the intersection of these sets is the set of all locations of zero values in c. $\square$

This proof is simple, but it gives an interesting insight: factors of the modulus directly correspond to 'factors' of the fractal it generates, insofar as the union or intersection of the factors fractals completely describe the fractal generated by the modulus. A visual demonstration follows; since $2 * 3 = 6$, we can observe the following relationship between the fractals those numbers generate, taken to 32 rows:

[p=2]  [p=3]  [p=6] 

Since $p = 6$ is the union of black spaces in $p = 2$ and $p = 3$, and since every space in the bottom row of $p = 2$ is black, the bottom row of $p = 6$ must also be completely black. We now define an operation that is essential to the next proof [2]:

Definition (Discrete Logarithm)**.** *The discrete logarithm of n base p, written $\nu_p(n)$, is the integer a such that $p^a \mid n$ while $p^{a+1} \nmid n$.*

So if $\nu_p(n) = a$, then a is the greatest integer power of p which divides n. It is easy to see that, in case $\nu_p(n) = 0$, then $n \not\equiv 0 (\text{mod } p)$, so $p \nmid n$. These are three ways of representing the same fact: p does not divide n. In terms of our fractals, a value n is represented with a black block when it is not divisible by the modulus p.

Importantly, we have that $\nu_p(nm) = \nu_p(n) + \nu_p(m)$. This means that:

$$\nu_p(n!) = \nu_p(n) + \nu_p(n+1) + \ldots + \nu_p(1) = \sum_{j=1}^{n} \nu_p(j)$$

This is enough preparation for the next result:

Lemma (Full Rows)**.** *Let a row of Pascal's triangle be called **full** if none of its values are divisible by the modulus. For a prime modulus p, rows of the form* $p^n - 1$ *are full, where* $n \in \mathbb{N}$.

Proof. Every value in a row $p^n - 1$ is represented by the binomial coefficient $\binom{p^n-1}{k}$, for a row index $k \in \{0, \ldots, p^n - 1\}$. The row is full if every value of this form is not divisible by p. This can be represented:

$$\nu_p\left(\frac{(p^n-1)!}{k!(p^n-1-k)!}\right) = 0$$

We show this using properties of the discrete logarithm:

$$\begin{aligned}
& \nu_p\left(\frac{(p^n-1)!}{k!(p^n-1-k)!}\right) = \nu_p((p^n-1)!) - (\nu_p(k!(p^n-1-k)!)) \\
= \quad & \nu_p((p^n-1)!) - (\nu_p(k!) + \nu_p((p^n-1-k)!)) \\
= \quad & \sum_{j=1}^{p^n-1} \nu_p(j) - \left(\sum_{j=1}^{k} \nu_p(j) + \sum_{j=1}^{p^n-1-k} \nu_p(j)\right) \\
= \quad & \sum_{j=1}^{k} \nu_p(j) + \sum_{j=k+1}^{p^n-1-k} \nu_p(j) + \sum_{j=p^n-k}^{p^n-1} \nu_p(j) - \left(\sum_{j=1}^{k} \nu_p(j) + \sum_{j=1}^{k} \nu_p(j) + \sum_{j=k+1}^{p^n-1-k} \nu_p(j)\right) \\
= \quad & \sum_{j=1}^{k} \nu_p(j) + \sum_{j=k+1}^{p^n-1-k} \nu_p(j) + \sum_{j=p^n-k}^{p^n-1} \nu_p(j) - \left(2\sum_{j=1}^{k} \nu_p(j) + \sum_{j=k+1}^{p^n-1-k} \nu_p(j)\right) \\
= \quad & \sum_{j=p^n-k}^{p^n-1} \nu_p(j) - \sum_{j=1}^{k} \nu_p(j)
\end{aligned}$$

Notice that these sums both have k-many terms. We will rewrite the sums, so that they are facing opposite directions, keeping in mind that we subtract the bottom row and take the discrete logarithm base p of every term:

$$\nu_p\left(\begin{matrix} p^n-1 \\ -1 \end{matrix} + \begin{matrix} p^n-2 \\ -2 \end{matrix} + \quad \cdots \quad + \begin{matrix} p^n-k \\ -k \end{matrix}\right)$$

Where we reverse the conventional order of the top sum in order to form pairs. We can see that these cancel term-for-term: before the pth pair, that is, $\nu_p(p^n - p) - \nu_p(p)$, we see that the terms are too small to be divisible by p, so they are all equal to 0. In the case of the pth pair: $p^n - p = p(p^n - 1)$, and $\nu_p(p(p^n - 1)) = 1$. The corresponding value is p, and obviously $\nu_p(p) = 1$. So, these terms cancel. This relationship holds for all terms which are factors of p as well. Since all of terms cancel, we have that:

$$\nu_p\left(\frac{(p^n-1)!}{k!(p^n-1-k)!}\right) = 0$$

□

So, we know that for a prime modulus p, rows of the form $p^n - 1$ are full. It is important to note that, for prime moduli p, rows of this form only form a complete list of full rows for $p = 2$; when $p > 2$, there are additional full rows:

Lemma (Strong Full Rows)**.** *For a prime modulus p, rows of the form $ap^n - 1$ are full, where $n \in \mathbb{N}, a \in \{1, \ldots, p-1\}$.*

Proof. We follow essentially the same calculations as above, substituting $p^n - 1$ with $ap^n - 1$. This yields:

$$\begin{aligned}
& \nu_p\left(\frac{(ap^n-1)!}{k!(ap^n-1-k)!}\right) = \nu_p((ap^n-1)!) - (\nu_p(k!(ap^n-1-k)!)) \\
= \quad & \nu_p((ap^n-1)!) - (\nu_p(k!) + \nu_p((ap^n-1-k)!)) \\
= \quad & \sum_{j=ap^n-k}^{ap^n-1} \nu_p(j) - \sum_{j=1}^{k} \nu_p(j)
\end{aligned}$$

Which gives us:

$$\nu_p\left(\begin{matrix} ap^n-1 \\ -1 \end{matrix} + \begin{matrix} ap^n-2 \\ -2 \end{matrix} + \quad \cdots \quad + \begin{matrix} ap^n-k \\ -k \end{matrix}\right)$$

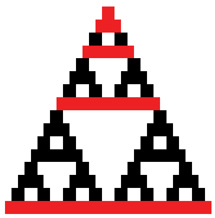
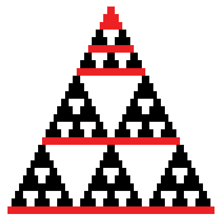
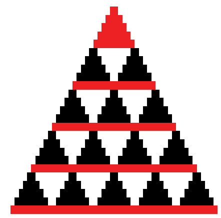
Since $a < p$, the corresponding terms satisfy the same relationship as above. We yield that:

$$\nu_p\left(\frac{(ap^n-1)!}{k!(ap^n-1-k)!}\right) = 0$$

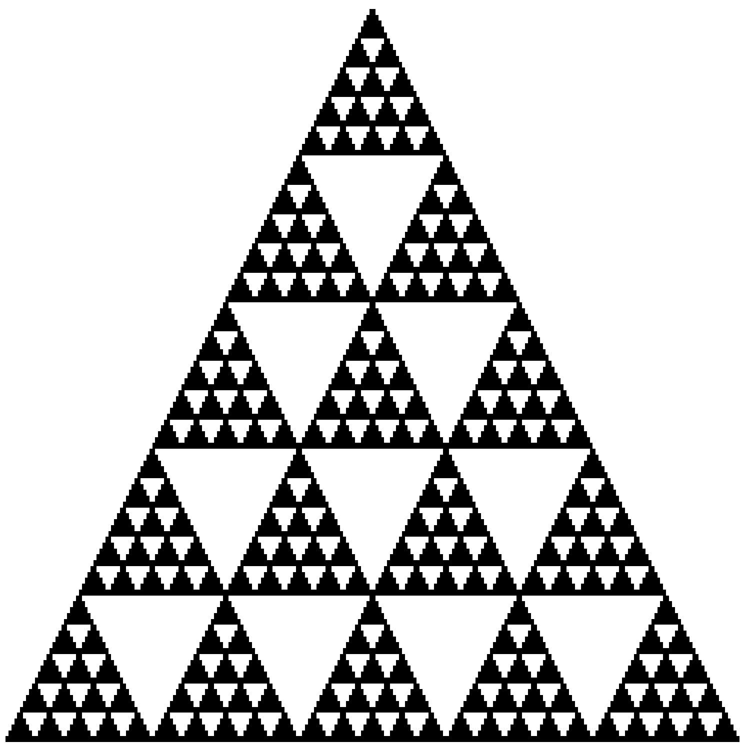
□

Importantly, when $p = 2$, we have that $a \in \{1\}$. This case is equivalent to the original full row lemma, when accounting for the fact that row 0 is full in any case trivially. In cases where p is a prime greater than 2, we now have a sense of the structure of these fractals.

This structure is demonstrated by the following images, where the full rows in $p = 2, 3, 5$ are highlighted in red:

[p=2]  [p=3]  [p=5] 

In order to prove that these constructions really are fractals, it is not enough to know where the full rows are. A large image of $p = 5$, taken to the first 125 rows, is included below for ease of exposition:



To be precise, we use *triangle* here to mean a white section of the figure which approximates the geometric shape. In the figure above, there is a clear pattern of downward-pointed triangles: simply put, we see a row of one, two, ..., $p - 1$ triangles, before they increase in size and the pattern continues. Since $p = 5$, there are 4 rows of triangles of any size before the pattern restarts, with the triangles increasing in size. This is no accident; we will prove why this happens momentarily.

When a downward-pointed triangle is the first (reading top-to-bottom) of its size, call it **major**. Every row n has 1 in the 0 and n places, by construction of Pascal's triangle. We will prove that these triangles occur at regular intervals.

Lemma (Placement of Major Triangles)**.** *Major triangles begin at rows of the form p^n, and are $p^n - 1$ rows tall.*

Proof. Rows of the form p^n have values of the form $\binom{p^n}{k}$ for $k \in \{0, \ldots, p^n\}$. For $k = 0$ and $k = p^n$, it is trivial that $\binom{p^n}{0} = \binom{p^n}{p^n} = 1$. These are the first and last row indices, which is consistent with the construction rules for Pascal's triangle. Otherwise, entries in the row must be 0:

$$\binom{p^n}{k} \equiv 0 \pmod{p} \text{ for } k \in \{1, \ldots, p^n - 1\}$$

This is easy to see:

$$\binom{p^n}{k} = \frac{(p^n)!}{k!(p^n - k)!} = p^n \left(\frac{(p^n - 1)!}{(k!)(p^n - k)!} \right)$$

Since $k \in \{1, \ldots, p^n - 1\}$, we know that

$$\frac{(p^n - 1)!}{(k!)(p^n - k)!} \in \mathbb{Z}$$

Call that value m. Then $\binom{p^n}{k} = mp^n \equiv 0 \pmod{p}$, as desired. So, these rows have values of 1 at the edges, and 0 elsewhere.

By construction of Pascal's triangle, the following row must be one entry longer with a value of 1 at both edges. Since the rest of the values are calculated by taking the sum of the two values above them, we will yield one fewer 0 entry. This is best understood with an example. Consider Pascal's triangle with $p = 5$, starting with row $p^1 = 5$:

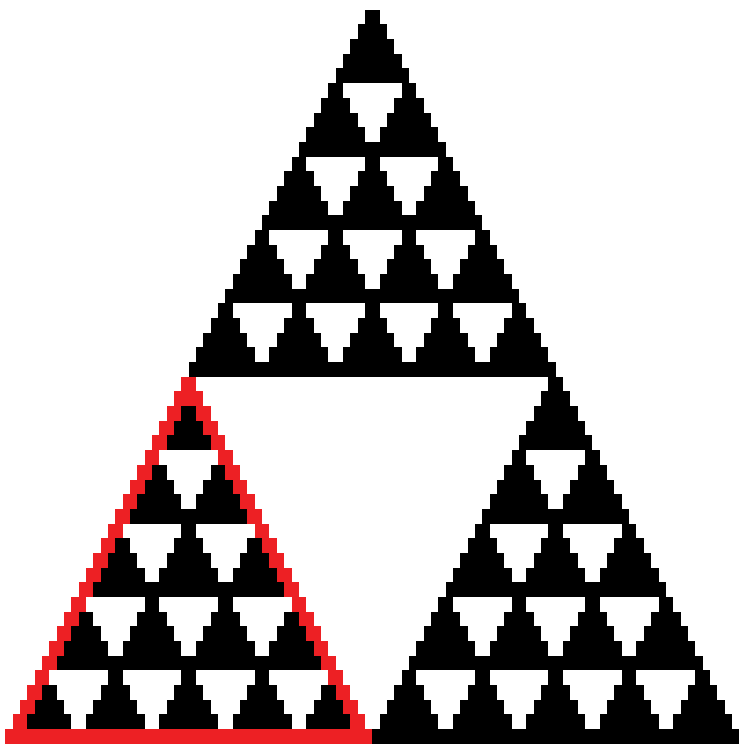
$$\begin{array}{ccccccccccccccccccc}
 & & & & 1 & & 0 & & 0 & & 0 & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 0 & & 0 & & 0 & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & 0 & & 0 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & & 0 & & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 1 & & 4 & & 1 & & 1 & & 4 & & 1 & & 4 & & 1
\end{array}$$

We see that the triangle decreases in width by one value per row, by construction rules for rows of Pascal's triangle. Since a row p^n has $p^n + 1$-many values, we have that $p^n - 1$-many central values in the row are zeroes. Therefore, the triangle has a height of $p^n - 1$ rows. $\square$

Notably, it would have been easy to say that row 9 is where the triangle disappeared without actually writing it out; this is guaranteed by the strong full row lemma, since $9 = 2 * 5^1 - 1$.

We have seen that, since rows must begin and end with a 1, smaller copies of Pascal's triangle always form on the sides of any major triangle. Because these smaller copies are contained in Pascal's triangle mod p to begin with, they must form exactly the same way. This means that the copies have the same exact set of full rows as the original, relative to their first rows. If the first row of a major triangle is p^n, then these copies must have full rows at $ap^n - 1 + p^n = (a+1)p^n$, where all p, n, a are relative to the original construction.

Most importantly, this means that those copies must also produce major triangles. It is not possible to demonstrate this with text, because all examples are too large to place on a page. As such, a copy having its own major triangle is outlined in red here, for $p = 5$ taken to 49 rows:



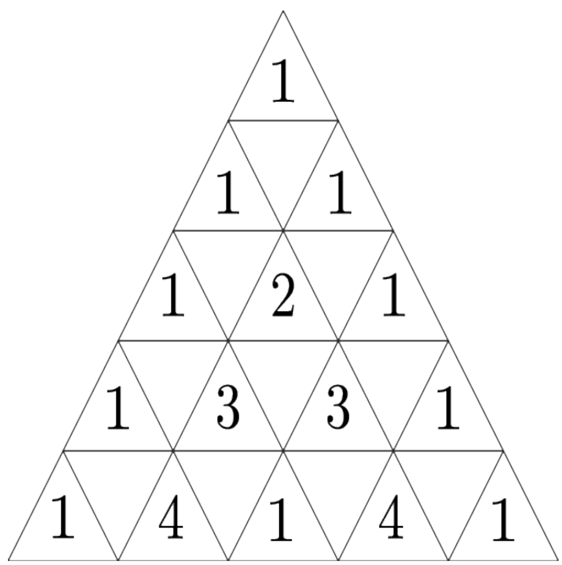
We know that row 49 is full for $p = 5$. It is important that, relative to the outlined copy beginning at row 25, this is row 24; then it produces its own major triangle directly below (see p. 6). This is where things come together:

- On the immediate next row, the outlined copy will produce its second major triangle.

- The construction is symmetric, so the copy on the other side will follow suit.
- Both of these downward-pointed triangles will be accompanied by a new copy of Pascal's triangle on either side.

The new copies of Pascal's triangle will be duplicated in the middle. This is necessarily additive; since the copies must have values of 1 on the edges, the middle copy on the next segment is like our usual Pascal's triangle mod 5, but every value is doubled. Since $2 < 5$, we retain all features of the fractal, and it is still an exact copy from the perspective of zero versus nonzero values.

Let the numbers inside of the triangles represent the factor by which they are multiplied in this construction (so that a 4 is a copy of Pascal's triangle mod 5, with every value multiplied by 4). This yields a familiar pattern:



In this diagram, horizontal lines are representative of full rows. It is tempting to wonder what happens on the next row, since $1 + 4 = 5$, and we are working this example mod 5. Fortunately, the bottom line of the diagram represents row 124; $124 = 5^3 - 1$. Then the answer must be that a major triangle appears, and its associated copies are the diagram above!

It is now pertinent to make a clarification. We have that, by the strong full row lemma, rows of the form $ap^n - 1$ are full. We have not yet mentioned that these are *all* of the full rows for some prime modulus p. The idea is simple: at least one downward-pointed triangle exists between any two consecutive rows we have proven to be full (except in the first $p - 1$ rows of the triangle, which are of the form $ap^0 - 1$; the lemma accounts for these already).

Claim. *For a prime modulus p, $n \in \mathbb{N}$ and $a \in \{1, \ldots, p-1\}$, every full row is of the form $ap^n - 1$.*

Proof. The proof is by induction on n. Suppose that there are no exceptional full rows before row $p^n - 1$ for some $n \in \mathbb{N}$. We will show that there are no exceptional rows before $p^{n+1} - 1$.

Base Case: Let $n = 1$. We want to show that there are no exceptional rows before $p - 1$. This is trivial; every row above $p - 1$ is of the form $ap^0 - 1$, meaning that exceptions cannot exist.

Inductive Step: Suppose there are no exceptional rows before $p^n - 1$. The next row, p^n, generates a major triangle of width $p^n - 1$. So, the next $p^n - 1$ rows cannot be full. The next candidate is:

$$p^n - 1 + p^n - 1 + 1 = 2p^n - 1$$

The lemma tells us that this row is full, so it is not an exception. When a major triangle is generated, it produces copies of Pascal's triangle mod p on either side. These copies produce their own major triangles; so, the next candidate row is another $p^n - 1$ rows down. This row, too, is not exceptional. Continue in this way until row $(p - 1)p^n - 1$. The next candidate row is:

$$(p - 1)p^n - 1 + p^n - 1 + 1 = p^{n+1} - p^n - 1 + p^n - 1 + 1 = p^{n+1} - 1$$

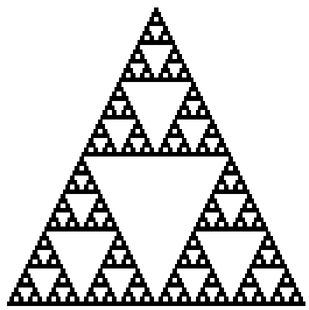
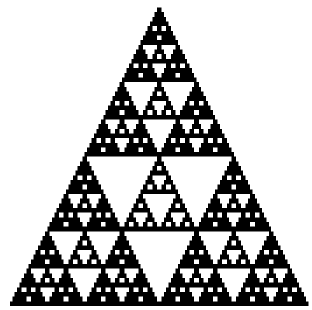
This row is full by the lemma. Then for all $n \in \mathbb{N}$, no exceptions occur before row $p^n - 1$. Therefore, there are no exceptions. $\square$

So, we have determined the locations of all full rows. This, together with the placement of major triangles and the observation that major triangles produce copies of the entire construction above them, is enough to ensure that for any prime p, Pascal's triangle mod p is a fractal.

Further Work

During this project, I was able to prove that taking the modulus of Pascal's triangle by p a prime produces a fractal. I also showed that, by the Union-Intersection method demonstrated at the beginning of the previous section, these prime fractals can be intersected in order to produce the fractals generated by any square-free integer modulus as well.

There were four cases for the kind of number that p could be; two of those cases are now settled. In case the modulus is a power of a prime number, some interesting behaviors occur. Compare $p = 2$ with $p = 2^2 = 4$, taken to 64 rows:

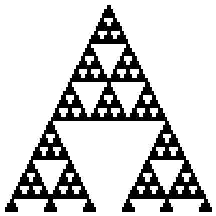
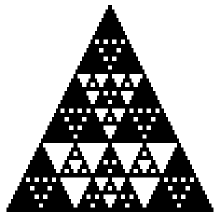
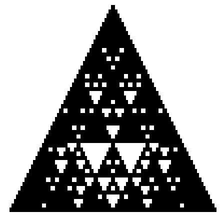
[p=2]  [p=4] 

The triangle mod 4 is obviously more dense; this is simply a function of the fact that more numbers are divisible by 2 than are by 4. The interesting feature here is that, for each major triangle in $p = 2$, $p = 4$ places a copy of Pascal's triangle mod 2 inside. This has happened in all tested cases, but I have not yet proved that it is always the case for prime powers. These triangles will necessarily have more full rows. There is a modified full row lemma for prime powers: Rows of the form $bp^n - 1$ are full, where $n \in \mathbb{N}, b \in \{x : p \nmid x, p < x < p^t\}$ and p^t is the modulus. An interesting trend is that full row proofs follow essentially the same two steps:

1. Reduce $\nu_p(\binom{n}{k})$ to a difference of sums.
2. Reverse the order of the top sum to produce pairs that cancel.

As of writing, the full rows are not enough to completely describe the behavior, especially with regards to placing copies of $p = 2$ inside of $p = 4$, where $p = 2$ would have a major triangle. This also shows us why union-intersection doesn't work for prime powers; the factors are all the same, so we produce an identical fractal (this doesn't give us the small copies inside of the major triangles).

The last case, I called *otherwise composite*. This term does not need a formal definition; I only use it to mean, 'numbers which are composite but not square-free or powers of primes.' A modified union-intersection method will work for these, where the intersection is by powers of primes, not by prime factors themselves. As an example: $24 = 2^3 * 3^1$. 24 is composite, but it is neither square-free nor a power of a prime number. The relevant fractals, then, are 2^3 and 3^1. A visual is on the next page:

[p=3]  [p=8]  [p=24] 

This is an interesting consequence of modular arithmetic, but until the fractals generated by prime powers have a more rigorously defined set of behaviors, it remains somewhat informal.

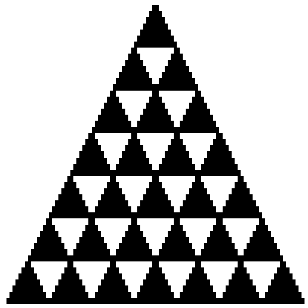
Outside of finishing these proofs, there are a number of directions this project could go in the future. These constructions provide a framework within which many interesting phenomena can be studied in their own right; as an example, quantum walks, which describe the movement of a quantum particle through a system, are studied in the context of these fractals [3].

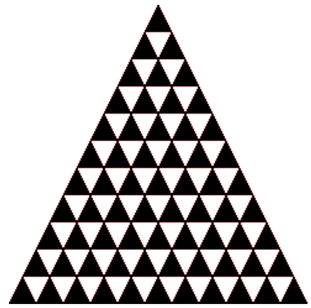
As a final note, I think it would be interesting to see what kinds of fractals Pascal's triangle, or something like it, produces under different row-building rules. As an example, we can always give the first three rows of the usual triangle, but substituting the 2 for any value c we like. Then, if the values in the next row are formed by multiplying, not adding, the two overhead values, we can take the triangles modulo some other value p. In this case, it might be interesting to assign different shades of grey based on the actual values mod p, as opposed to only using black and white for nonzero or zero, respectively. This would be the beginning of such a structure:

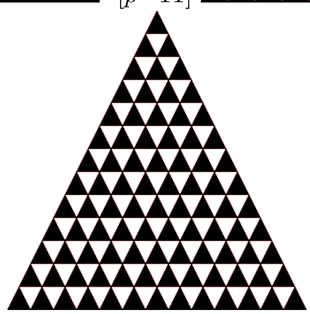
$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & c & & 1 & & \\
n=3: & & 1 & & c & & c & & 1 & \\
n=4: & 1 & & c & & c^2 & & c & & 1
\end{array}$$

Gallery

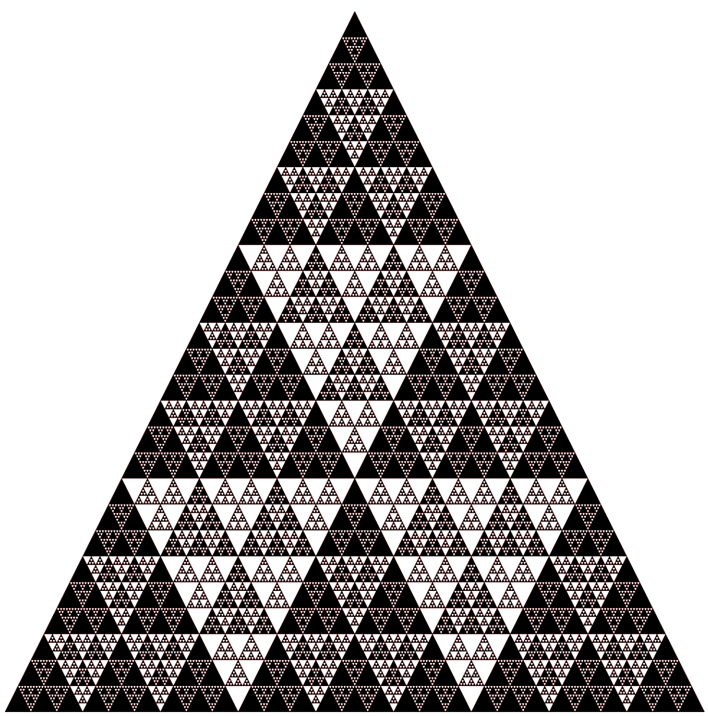
Because this project has such a heavy visual component, it seems fitting to include some of the more interesting fractals I generated. Primes always follow the same pattern: we have sections of 1, 2, ..., $p-1$-many downward-pointed triangles, and then the size increases:

[p=7] 

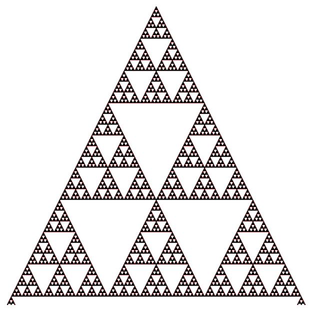
[p=11] 

[p=13] 

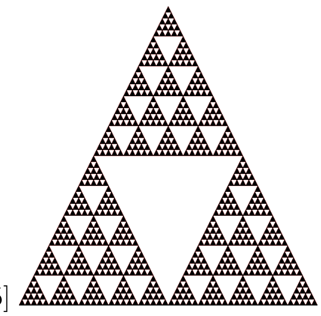
Primes are interesting for their uniformity, but prime powers produce more intricate patterns. Below is $p = 3^3 = 27$, taken to 729 rows:



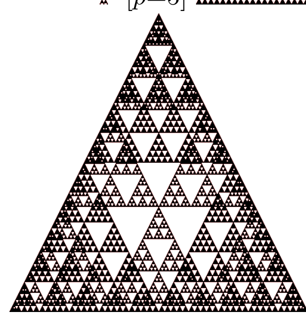
The behavior is interesting, but still observably uniform. This is not usually the case for square-free integers, though it is easy in some small cases to determine which parts of a square-free fractal are inherited from which factors. We will use a new example:



[p=3]

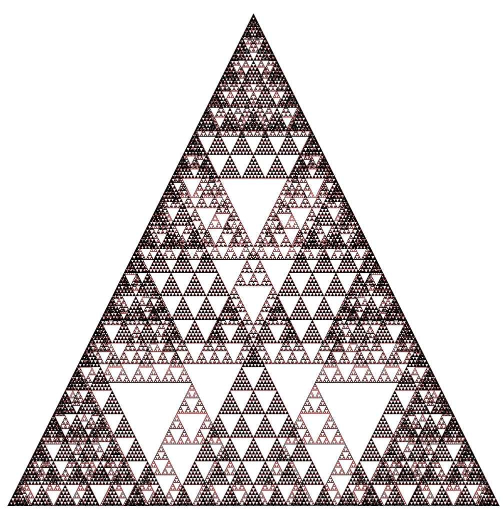


[p=5]



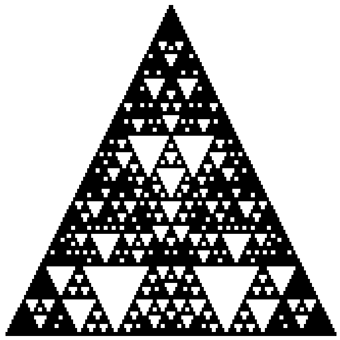
[p=15]

15 is a relatively small square-free integer; we will look at $42 = 2 * 3 * 7$:

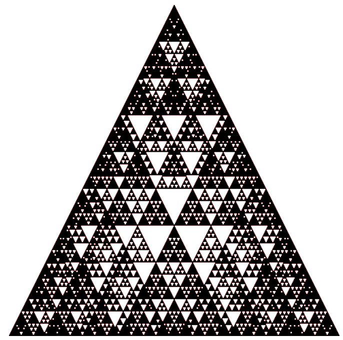


At this size, it becomes very clear that the factors act as 'layers' of the triangle; they are a little bit like pieces of transparency film.

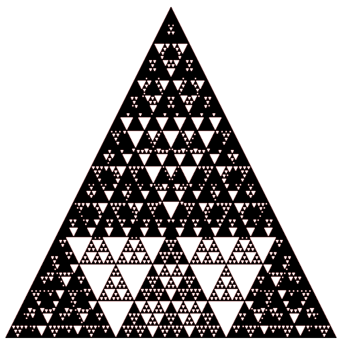
Finally, a page of typical composite numbers:



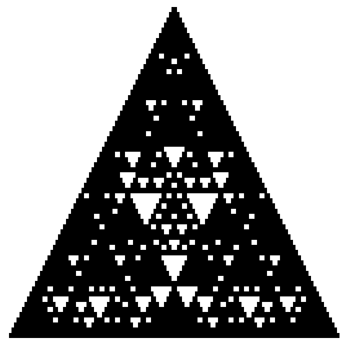
[p=12]



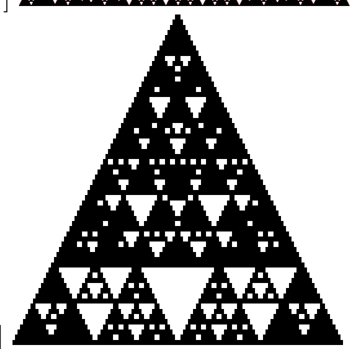
[p=45]



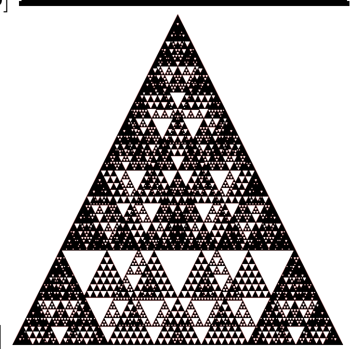
[p=117]



[p=36]



[p=28]



[p=63]

References

[1] B. Mandelbrot, W. H. Freeman, and Company, *The Fractal Geometry of Nature*. Einaudi paperbacks, Henry Holt and Company, 1983.

[2] B. Reznick, “Math 453 bonus notes 1-3,” Jan 2013.

[3] T. Bannink and H. Buhrman, “Quantum pascal’s triangle and sierpinski’s carpet,” 2017.