In [27]:

```
## HW02 Code

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#You will complete the following notebook, as described in the PDF for Homework
#1. This notebook file, along with your COLLABORATORS.txt file, to the Gradescop
#2. A PDF of this notebook and all of its output, once it is completed, to the G


#Please report any questions to the [class Piazza page](https://piazza.com/tufts

# import libraries as needed
import operator
import numpy as np
import pandas as pd
import math

from sklearn import linear_model
from sklearn.metrics import mean_squared_error
from sklearn.preprocessing import PolynomialFeatures

from matplotlib import pyplot as plt
import seaborn as sns
%matplotlib inline
plt.style.use('seaborn') # pretty matplotlib plots
```

Plotting function

Do not modify the following: it takes in a list of polynomial (integer) values, along with associated lists consisting of the predictions made for the associated model, and the resulting error, and plots the results in a grid.

In [28]:

```
def plot_predictions(polynomials=list(), prediction_list=list(), error_list=list
    '''Plot predicted results for a number of polynomial regression models

    Args
    ----
    polynomials : list of positive integer values
        Each value is the degree of a polynomial regression model.
    prediction_list: list of arrays ((# polynomial models) x (# input data))
        Each array contains the predicted y-values for input data.
    error_list: list of error values ((# polynomial models) x 1)
        Each value is the mean squared error (MSE) of the model with
        the associated polynomial degree.

        Note: it is expected that all lists are of the same length, and
            that this length be some perfect square (for grid-plotting).
    '''
    length = len(prediction_list)
    grid_size = int(math.sqrt(length))
    if not (length == len(polynomials) and length == len(error_list)):
        raise ValueError("Input lists must be of same length")
    if not length == (grid_size * grid_size):
        raise ValueError("Need a square number of list items (%d given)" % (leng

    fig, axs = plt.subplots(grid_size, grid_size, figsize =(14,14), sharey=True)
```

```
    for subplot_id, prediction in enumerate(prediction_list):
        # order data for display
        data_frame = pd.DataFrame(data=[x[:, 0], prediction]).T
        data_frame = data_frame.sort_values(by=0)
        x_sorted = data_frame.iloc[:, :-1].values
        prediction_sorted = data_frame.iloc[:, 1].values

        ax = axs.flat[subplot_id]
        ax.set_title('degree = %d; MSE = %.3f' % (polynomials[subplot_id], error
        ax.plot(x, y, 'r.')
        ax.plot(x_sorted, prediction_sorted, color='blue')

    plt.show()
```

Load the dataset

In [29]:
```
data = pd.read_csv('data.csv')
data
```

Out[29]:

	x_i	y_i
0	1.590909	2.846988
1	1.803030	2.959811
2	4.984848	13.041394
3	1.696970	3.971889
4	1.272727	2.454520
...	...	...
95	5.090909	11.537465
96	10.500000	10.381492
97	1.484848	2.683212
98	0.636364	1.437600
99	0.848485	0.990251

100 rows × 2 columns

In [30]:
```
x = data.iloc[:, :-1].values
y = data.iloc[:, 1].values
```

1. Test a range of polynomial functions fit to the data

Fit models to data of polynomial degree $d \in \{1, 2, 3, 4, 5, 6, 10, 11, 12\}$. For each such model, we will record its predictions on the input data, along with the mean squared error (MSE) that it makes. These results are then plotted for comparison.

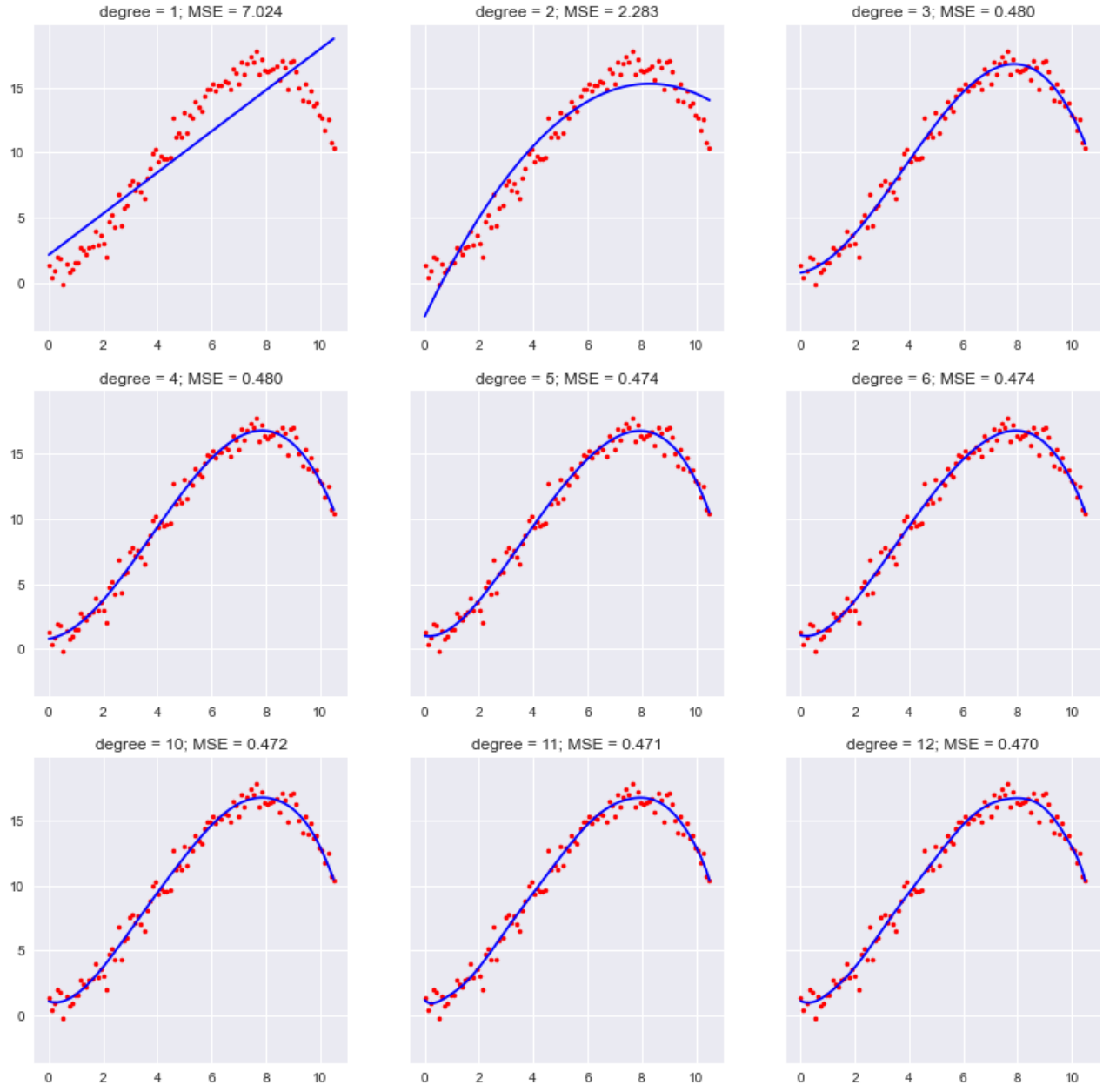
1.1 Create function to generate models, make predictions, measure error.

In [31]:
```
def test_polynomials(polynomials=list()):
```

```
    '''Generates a series of polynomial regression models on input data.
       Each model is fit to the data, then used to predict values of that
       input data.  Predictions and mean squared error are collected and
       returned as two lists.

    Args
    ----
    polynomials : list of positive integer values
        Each value is the degree of a polynomial regression model, to be built.

    Returns
    -------
    prediction_list: list of arrays ((# polynomial models) x (# input data))
        Each array contains the predicted y-values for input data.
    error_list: list of error values ((# polynomial models) x 1)
        Each value is the mean squared error (MSE) of the model with
        the associated polynomial degree.
    '''
    prediction_list = list()
    error_list = list()

    for degree in polynomials:
        poly = PolynomialFeatures(degree)
        xtrain = poly.fit_transform(x)
        linearreg = linear_model.LinearRegression()
        linearreg.fit(xtrain, y)
        y_train_pred = linearreg.predict(xtrain)
        prediction_list.append(y_train_pred)
        error_list.append(mean_squared_error(y, y_train_pred))

    # COMPLETED-TODO: fill in this function to generate the required set of mode
    #       returning the predictions and the errors for each.

    return prediction_list, error_list
```

In [32]:

```
# COMPLETED-TODO: generate the sequence of degrees, call test_polynomials to cre
#       use plot_predictions to show the results
degree1 = [1, 2, 3, 4, 5, 6, 10, 11, 12]
prediction_list, error_list = test_polynomials(degree1)
plot_predictions(degree1, prediction_list, error_list)
```



1.2 Discuss the results seen in the plots above

Discussion: The plotted results above show that degree 12 is the best model out of the nine given degrees. The polynomial degrees of 1 & 2 are the worest since they don't represent our data correctly compare to other degrees. Afterwards, the degrees 3-12 represent the better MSE values (low error). However, we may want to stop increasing degrees since we run a risk of overfitting. When looking at MSE values on all degrees charts above, we can see that degree 3-12 are within a small range of values (0.48 to 0.47), compare to degrees 1 & 2. Thus, we may want to stop at degree 3 here.

2. k-fold cross-validation

For each of the polynomial degrees, 5-fold cross-validation is performed. Data is divided into 5 equal parts, and 5 separate models are trained and tested. Results are averaged over the 5 runs and plotted (in a single plot), comparing training and test error for each of the polynomial degrees. Error values are also shown in a tabular form.

2.1 Creating the k folds

A function that generates the distinct, non-overlapping folds of the data.

In [33]:
```
def make_folds(num_folds=1):
    '''Splits data into num_folds separate folds for cross-validation.
       Each fold should consist of M consecutive items from the
       original data; each fold should be the same size (we will assume
       that  the data divides evenly by num_folds).  Every data item should
       appear in exactly one fold.

       Args
       ----
       num_folds : some positive integer value
           Number of folds to divide data into.

        Returns
        -------
        x_folds : list of sub-sequences of original x-data
            There will be num_folds such sequences; each will
            consist of 1/num_folds of the original data, in
            the original order.
        y_folds : list of sub-sequences of original y data
            There will be num_folds such sequences; each will
            consist of 1/num_folds of the original data, in
            the original order.
       '''
    x_folds = list()
    y_folds = list()

    step = int(x.size / num_folds)

    for i in range(0, x.size, step):
        x_folds.append(x[i:i+step])
        y_folds.append(y[i:i+step])

    # COMPLETED-TODO: Complete method to generate partition into folds.

    return x_folds, y_folds
```

In [34]:
```
# Print out start/end of each fold for sanity check.  Should see 5 folds,
# with the (x,y) pairs at the start/end of each.  (Can be manually verified
# by looking at original input file.)
#
# DO NOT MODIFY THIS CODE.  Its output will be used to check your work.
k = 5
x_folds, y_folds = make_folds(k)
for i in range(k):
    print("Fold %d: (%.3f, %.3f) ... (%.3f, %.3f)"
          % (i, x_folds[i][0], y_folds[i][0], x_folds[i][-1], y_folds[i][-1]))
```

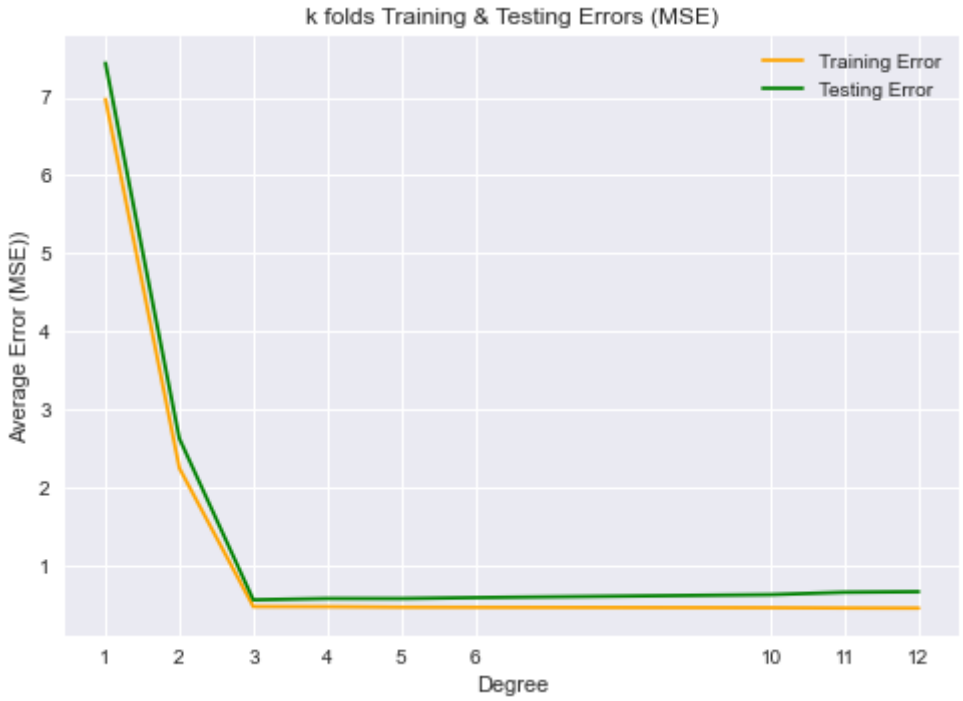
```
Fold 0: (1.591, 2.847) ... (10.394, 10.739)
Fold 1: (6.788, 16.408) ... (2.227, 4.722)
Fold 2: (9.545, 13.897) ... (3.924, 10.229)
Fold 3: (2.864, 5.929) ... (7.212, 16.030)
Fold 4: (7.530, 16.982) ... (0.848, 0.990)
```

2.2 Perform cross-validation

For each of the polynomial degrees already considered, k-fold cross-validation is performed. Average training error (MSE) and test error (MSE) are reported, both in the form of a plot and a tabular print of the values.

In [35]:

```
# COMPLETED-TODO: Perform 5-fold cross-validation for each polynomial degree.

final = list()
for degree in degree1:
    train_errors = list()
    test_errors = list()
    for i in range(k):
        x_test = x_folds[0]
        x_folds.remove(x_test)
        x_train = np.concatenate(x_folds, axis=0)

        y_test = y_folds[0]
        y_folds.remove(y_test)
        y_train = np.concatenate(y_folds)

        poly = PolynomialFeatures(degree)
        xs_train = poly.fit_transform(x_train)
        xs_test = poly.fit_transform(x_test)
        linreg = linear_model.LinearRegression()
        linreg.fit(xs_train, y_train)

        y_trained = linreg.predict(xs_train)
        y_pred = linreg.predict(xs_test)

        train_errors.append(mean_squared_error(y_train, y_trained))
        test_errors.append(mean_squared_error(y_test, y_pred))
        x_folds.append(x_test)
        y_folds.append(y_test)

    error_tr = np.mean(train_errors)
    error_te = np.mean(test_errors)
    final.append([degree, error_tr, error_te])

# Keep track of average training/test error for each degree

df_final = pd.DataFrame(final, columns = ['Degree', 'Training Error (MSE)', 'Tes
df_final.set_index('Degree',inplace=True)

#Plot results in a single table, properly labeled, and also
plt.plot(degree1, df_final['Training Error (MSE)'], color='orange')
plt.plot(degree1, df_final['Testing Error (MSE)'], color='green')
plt.xlabel('Degree')
plt.ylabel('Average Error (MSE))')
plt.title('k folds Training & Testing Errors (MSE)')
plt.legend(['Training Error','Testing Error'])
plt.xticks(degree1)

plt.show()

#print out the results in some clear tabular format.
print(df_final)
```



Degree	Training Error (MSE)	Testing Error (MSE)
1	6.977641	7.441157
2	2.245920	2.625608
3	0.471621	0.558083
4	0.469909	0.574623
5	0.463663	0.574580
6	0.462069	0.586606
10	0.457199	0.624748
11	0.453294	0.654943
12	0.451891	0.662556

2.3 Discuss the results seen in the plots above

Discussion: The above results prove our intuition about training MSE vs testing MSE and the concept of overfitting. As Training data MSE is getting smaller but after the first two degrees, it is steady and not much has changed. For Testing data, the MSE increases back after the first three degrees, meaning degree 3 may be our best degree model to stop at.

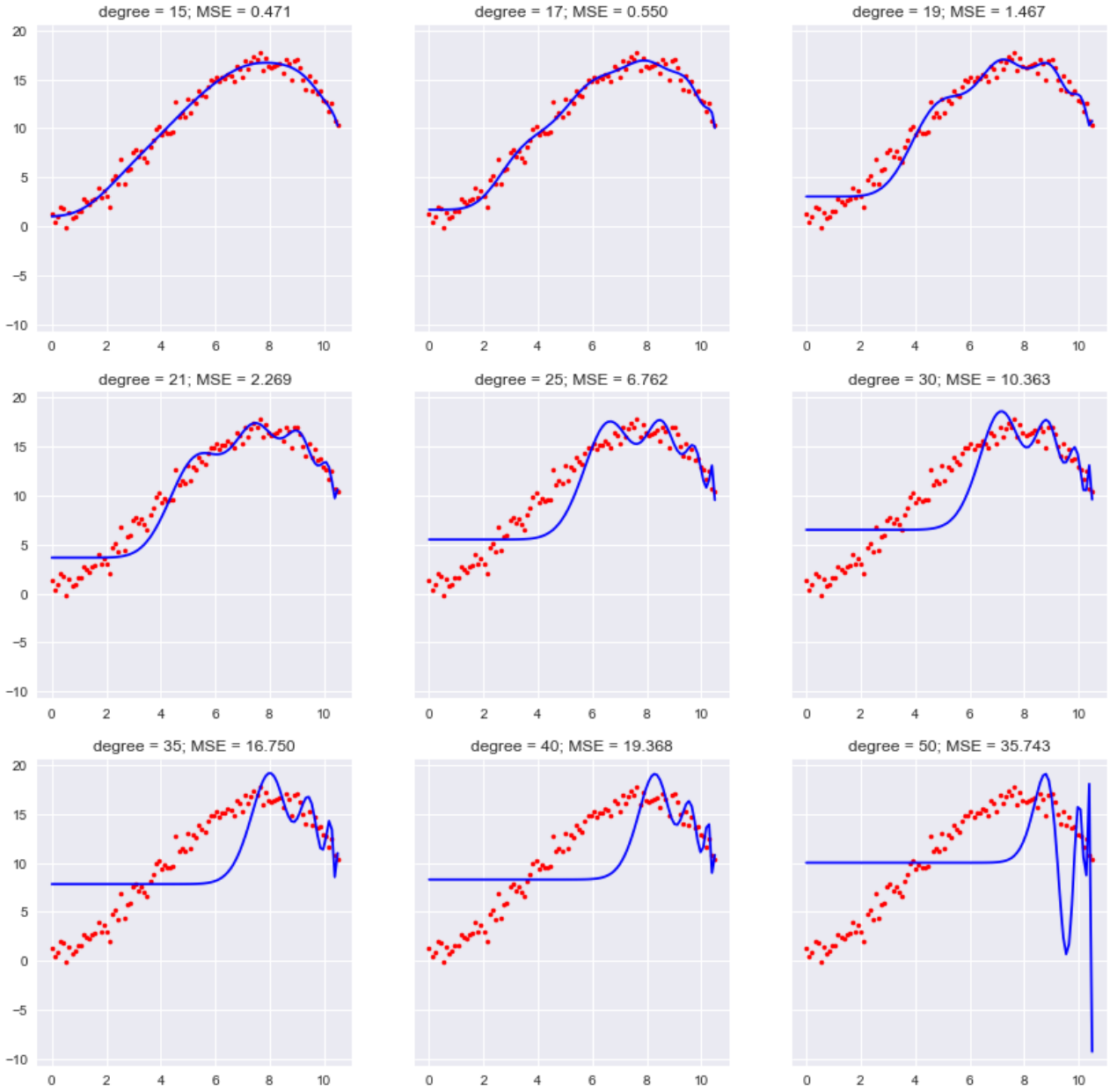
3. Higher-order polynomials

Results are generated and plotted (as for part 1), for the higher polynomial degrees $d = \{15, 17, 19, 21, 25, 30, 35, 40, 50\}$.

3.1 Plot a grid of prediction results/errors for the higher-order polynomials

In [36]:

```
# COMPLETED-TODO: generate and plot 9 more models, for the higher-degree polynom
degree2 = [15, 17, 19, 21, 25, 30, 35, 40, 50]
prediction_list, error_list = test_polynomials(degree2)
plot_predictions(degree2, prediction_list, error_list)
```



3.2 Discuss the results seen in the plots above

Discussion: The above plotted results show that as we increase the degree of the polynomial, the overfitting is a problem that doesn't help repersent or define our original data correctly. The high MSE is also a problem with increased degrees since its our goal to reduce MSE as much as possible. Thus, the high degrees of polynomila doesn't convey the model will be correct or accurate and we would have to stop somehwere in the middle.

4. Regularized (ridge) regression

Ridge regularization is a process whereby the loss function that is minimized combines the usual measure (error on the training data) with a penalty that is applied to the magnitude of individual coefficients. This latter penalty discourages models that overly emphasize any single feature, and can often prevent over-fitting.

Here, a set of 50 different `sklearn.linear_model.Ridge` models are generated, each using a single polynomial degree (the one that was determined to be best for the data-set in earlier tests), and using a range of different regularization penalties, chosen from a logarithmic series: $s \in [0.01, 100]$. 5-fold cross-validation is again used to examine how robust these models are.

4.1 Cross-validation for each regularization strength value

In [37]:

```
# TODO: Generate a sequence of 50 ridge models, varying the regularization stren
#       from 0.01 (10^-2) to 100 (10^2).  Each model is 5-fold cross-validated a
#       the resulting average training/test errors are tracked.  Errors are then
#       plotted (on a logarithmic scale) and printed in some legible tabular for

alphas = np.logspace(-2, 2, base=10, num=50)
models = list()

pol = PolynomialFeatures(degree=3)

for alpha in np.nditer(alphas):
    ridge = linear_model.Ridge(alpha)

    train_errs = list()
    test_errs = list()

    for i in range(k):
        x_test = x_folds[0]
        x_folds.remove(x_test)
        x_train = np.concatenate(x_folds, axis=0)

        y_test = y_folds[0]
        y_folds.remove(y_test)
        y_train = np.concatenate(y_folds)

        xs_train = pol.fit_transform(x_train)
        xs_test = pol.fit_transform(x_test)

        ridge.fit(xs_train, y_train)
        y_trained = ridge.predict(xs_train)
        y_pred = ridge.predict(xs_test)

        train_errs.append(mean_squared_error(y_train, y_trained))
        test_errs.append(mean_squared_error(y_test, y_pred))
        x_folds.append(x_test)
        y_folds.append(y_test)

    error_tr = np.mean(train_errs)
    error_te = np.mean(test_errs)

    models.append([alpha, error_tr, error_te])

models_df = pd.DataFrame(models, columns=['Alpha', 'Training Error (MSE)', 'Test

plt.plot(alphas, models_df['Training Error (MSE)'], color='orange')
plt.plot(alphas, models_df['Testing Error (MSE)'], color='green')
plt.xlabel('Regularization Strength (Alpha)')
plt.ylabel('Average Error (MSE)')
plt.title('Ridge Regression')
plt.legend(['Training Error (MSE)','Testing Error (MSE)'])
```

```
plt.xscale('log')

plt.show()

print(models_df)
```

	Alpha	Training Error (MSE)	Testing Error (MSE)
0	0.01	0.471621	0.558036
1	0.012067926406393288	0.471621	0.558026
2	0.014563484775012436	0.471621	0.558015
3	0.017575106248547922	0.471621	0.558001
4	0.021209508879201904	0.471621	0.557984
5	0.025595479226995357	0.471621	0.557963
6	0.030888435964774818	0.471621	0.557939
7	0.0372759372031494	0.471621	0.557909
8	0.04498432668969444	0.471621	0.557873
9	0.054286754393238594	0.471621	0.557831
10	0.0655128556859551	0.471621	0.557779
11	0.07906043210907697	0.471622	0.557717
12	0.09540954763499938	0.471622	0.557643
13	0.1151395399326447	0.471623	0.557554
14	0.13894954943731375	0.471623	0.557447
15	0.16768329368110074	0.471625	0.557320
16	0.20235896477251566	0.471627	0.557168
17	0.2442053094548651	0.471629	0.556987
18	0.29470517025518095	0.471633	0.556773
19	0.35564803062231287	0.471638	0.556520
20	0.42919342601287763	0.471646	0.556221
21	0.517947467923121	0.471656	0.555871
22	0.6250551925273969	0.471671	0.555462
23	0.7543120063354615	0.471692	0.554989
24	0.9102981779915218	0.471721	0.554445
25	1.0985411419875584	0.471760	0.553824
26	1.325711365590108	0.471813	0.553124
27	1.5998587196060574	0.471884	0.552341
28	1.9306977288832496	0.471978	0.551479
29	2.329951810515372	0.472099	0.550540
30	2.811768697974228	0.472253	0.549535
31	3.3932217718953264	0.472445	0.548476
32	4.094915062380423	0.472681	0.547379

33	4.941713361323833	0.472965	0.546263
34	5.963623316594643	0.473300	0.545149
35	7.196856730011514	0.473687	0.544059
36	8.68511373751352	0.474128	0.543014
37	10.481131341546853	0.474625	0.542034
38	12.648552168552959	0.475181	0.541138
39	15.264179671752318	0.475808	0.540348
40	18.420699693267146	0.476521	0.539689
41	22.229964825261934	0.477353	0.539195
42	26.826957952797247	0.478354	0.538914
43	32.374575428176435	0.479600	0.538920
44	39.06939937054613	0.481209	0.539322
45	47.1486636345739	0.483350	0.540285
46	56.89866029018293	0.486267	0.542053
47	68.66488450042998	0.490310	0.544979
48	82.86427728546842	0.495975	0.549575
49	100.0	0.503957	0.556572

4.2 Discuss the results seen in the plots above

Discussion: The above plotted results show that we have a range of point (10^1 to 10^2) with lowest point of MSE. This point can be the perfect regularization strength (alpha) that can help us avoid overfitting. It seems that the Training MSE is not affected since it increases slowly but Testing MSE is minimized at a point where Training MSE is one of the lowest if not the lowest.