Tufts

Class #05: Linear Classification with the Perceptron

Machine Learning (COMP 135)

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Review: The General Learning Problem

- We want to learn functions from inputs to outputs, where each input has n features:

Inputs $\langle x_1, x_2, \ldots, x_n \rangle$, with each feature x_i from domain X_i.
Outputs y from domain Y.

Function to learn: $f : X_1 \times X_2 \times \cdots \times X_n \rightarrow Y$

- The type of learning problem we are solving really depends upon the type of the output domain, Y

1. If $Y \subseteq \mathcal{R}$ (i.e., is real-valued), this is regression
2. If Y is a finite discrete set, this is classification

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Decisions to Make

- When collecting our training example pairs, $(x, f(x))$, we still have some decisions to make
- ***Example***: Medical Informatics
  - We have some genetic information about patients
  - Some get sick with a disease and some don't
  - Patients live for a number of years (sick or not)
- ***Question***: what do we want to learn from this data?
- Depending upon what we decide, we may use:
  - Different models of the data
  - Different machine learning approaches
  - Different measurements of successful learning

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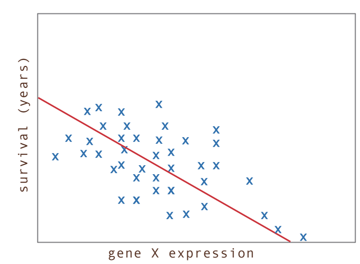
One Approach: Regression

- We decide that we want to try to learn to predict how long patients will live
- We base this upon information about the degree to which they express a specific gene
- A regression problem: the function we learn is the "best (linear) fit" to the data we have



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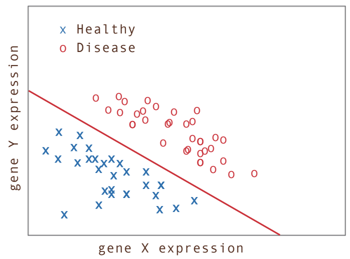
Another Approach: Classification

- We decide instead that we simply want to decide whether a patient will get the disease or not
- We base this upon information about expression of two genes
- A classification problem: function separates data into two different groups (binary classes)



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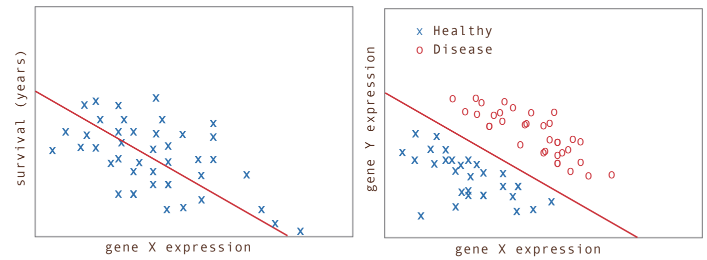
Which is the Correct Approach?



- The approach we use depends upon what we want to achieve, and what works best based upon the data we have
- Much machine learning involves investigating different approaches

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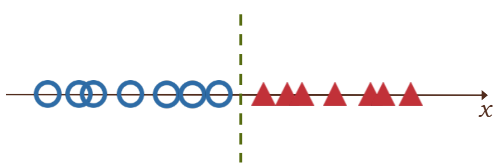
From Regression to Classification



- Suppose we have two classes of data, defined by a single attribute x
- We seek a decision boundary that splits the data in two
- When such a boundary can be defined using a linear function, it is called a linear separator

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Threshold Functions

1. We have data-points with n features:
$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$
2. We have a linear function defined by $n+1$ weights:
$$\mathbf{w} = (w_0, w_1, w_2, \ldots, w_n)$$
3. We can write this linear function as:
$$\mathbf{w} \cdot \mathbf{x}$$
4. We can then find the linear boundary, where:
$$\mathbf{w} \cdot \mathbf{x} = 0$$
5. And use it to define our threshold between classes:
$$h_{\mathbf{w}} = \begin{cases} 1 & \mathbf{w} \cdot \mathbf{x} \geq 0 \\ 0 & \mathbf{w} \cdot \mathbf{x} < 0 \end{cases}$$

Outputs 1 and 0 here are ***arbitrary labels*** for one of two possible classes

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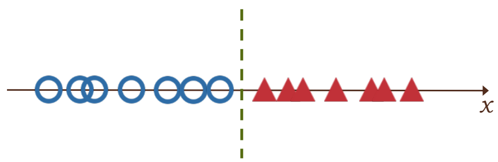
From Regression to Classification



- Data is linearly separable if it can be divided into classes using a linear boundary:

$$\mathbf{w} \cdot \mathbf{x} = 0$$

- Such a boundary, in 1-dimensional space, is a ***threshold value***

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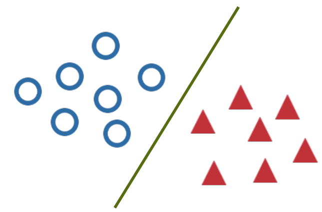
From Regression to Classification



- Data is linearly separable if it can be divided into classes using a linear boundary:

$$\mathbf{w} \cdot \mathbf{x} = 0$$

- Such a boundary, in 2-dimensional space, is a ***line***

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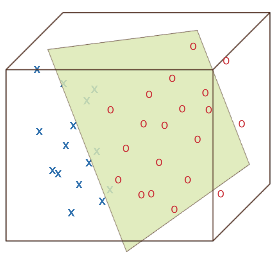
From Regression to Classification



- Data is linearly separable if it can be divided into classes using a linear boundary:

$$\mathbf{w} \cdot \mathbf{x} = 0$$

- Such a boundary, in 3-dimensional space, is a ***plane***
  - In higher dimensions, it is a ***hyper-plane***

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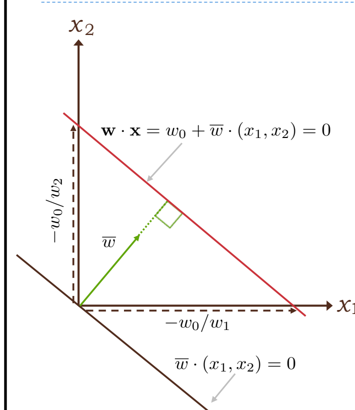
The Geometry of Linear Boundaries



- Suppose we have 2-dimensional inputs $\mathbf{x} = (x_1, x_2)$
- The "real" weights $\overline{w} = (w_1, w_2)$ define a vector
- The boundary where our linear function is zero,

$$\mathbf{w} \cdot \mathbf{x} = w_0 + \overline{w} \cdot (x_1, x_2) = 0$$

  is an orthogonal line, parallel to $\overline{w} \cdot (x_1, x_2) = 0$
- Its offset from origin is determined by w_0 (which is called the bias weight)

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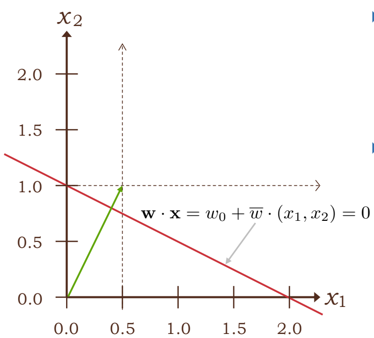
The Geometry of Linear Boundaries



- For example, with "real" weights:
  $$\overline{w} = (w_1, w_2) = (0.5, 1.0)$$
  we get the vector shown as a green arrow
- Then, for a bias weight
  $$w_0 = -1.0$$
  the boundary where our linear function is zero,
  $$\mathbf{w} \cdot \mathbf{x} = w_0 + \overline{w} \cdot (x_1, x_2) = 0$$
  is the line shown in red, crossing origin at (2,0) & (0,1)

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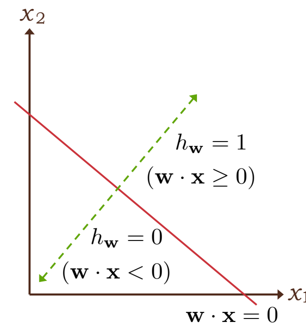
The Geometry of Linear Boundaries



- Once we have our linear boundary, data points are classified according to our threshold function

$$h_{\mathbf{w}} = \begin{cases} 1 & \mathbf{w} \cdot \mathbf{x} \geq 0 \\ 0 & \mathbf{w} \cdot \mathbf{x} < 0 \end{cases}$$

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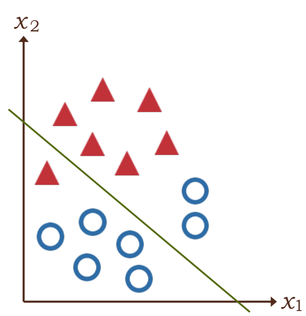
Zero-One Loss



- For a training set made up of input/output pairs,
  $$\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_k, y_k)\}$$
  we could define the zero/one loss
  $$L(h_{\mathbf{w}}(\mathbf{x}_i), y_i) = \begin{cases} 0 & \text{if } h_{\mathbf{w}}(\mathbf{x}_i) = y_i \\ 1 & \text{if } h_{\mathbf{w}}(\mathbf{x}_i) \neq y_i \end{cases}$$
- Summed for the entire set, this is simply the ***count*** of examples that we get wrong
- In this example, if data-points marked ○ should be in class 0 (below the line) and those marked ▲ should be in class 1 (above the line) the loss would be equal to 3

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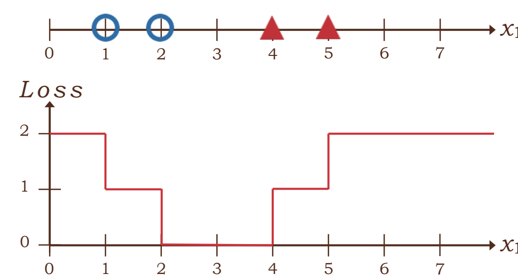
Minimizing Zero/One Loss

- Sadly, it is not easy to compute weights that minimize zero/one loss
  - It is a piece-wise constant function of weights
  - It is not continuous, however, and gradient descent won't work
- E.g., for the following one-dimensional data, we get loss shown below:



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Perceptron Loss

- Instead, we define the perceptron loss on a training item:

$$\mathbf{x}_i = (x_{i,1},\, x_{i,2},\, \ldots,\, x_{i,n})$$

$$L_\pi(h_\mathbf{w}(\mathbf{x}_i), y_i) = \sum_{j=0}^{n} (y_i - h_\mathbf{w}(\mathbf{x_i})) \times x_{i,j}$$

- For example, suppose we have a 2-dimensional element in our training set for which the correct output is 0, but our threshold function says 1:

$$\mathbf{x}_i = (0.5,\, 0.4) \qquad y_i = 1 \qquad h_\mathbf{w}(\mathbf{x}_i) = 0$$

$$L_\pi(h_\mathbf{w}(\mathbf{x}_i), y_i) \;=\; \boxed{(1-0)}\boxed{(1+0.5+0.4)} \;=\; 1.9$$

The difference between what output ***should*** be, and what our weights make it

Sum of input attributes (1 is the "dummy" attribute that is multiplied by bias weight w_0)

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Perceptron Learning

- To minimize perceptron loss we can start from initial weights—perhaps chosen uniformly from interval $[-1, 1]$—and then:

1. Choose an input $\mathbf{x}_i$ from our data set that is wrongly classified.
2. Update vector of weights, $\mathbf{w} = (w_0,\, w_1,\, w_2,\, \ldots,\, w_n)$, as follows:

$$w_j \;\leftarrow\; w_j + \alpha(y_i - h_\mathbf{w}(\mathbf{x}_i)) \times x_{i,j}$$

3. Repeat until no classification errors remain.

- The update equation means that:

1. If correct output should be *below* the boundary ($y_i = 0$) but our threshold has placed it *above* ($h_\mathbf{w}(\mathbf{x}_i) = 1$) then we *subtract* each feature ($x_{i,j}$) from the corresponding weight (w_i)
2. If correct output should be *above* the boundary ($y_i = 1$) but our threshold has placed it *below* ($h_\mathbf{w}(\mathbf{x}_i) = 0$) then we *add* each feature ($x_{i,j}$) to the corresponding weight (w_i)

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Perceptron Updates

$$w_j \;\leftarrow\; w_j + \alpha(y - h_\mathbf{w}(\mathbf{x}_i)) \times x_{i,j}$$

- The perceptron update rule shifts the weight vector positively or negatively, trying to get all data on the right side of the linear decision boundary
- Again, supposing we have an error as before, with weights as given below:

$$\mathbf{x}_i = (0.5,\, 0.4) \qquad \mathbf{w} = (0.2,\, -2.5,\, 0.6) \qquad y_i = 1$$

$$\mathbf{w} \cdot \mathbf{x}_i = 0.2 + (-2.5 \times 0.5) + (0.6 \times 0.4) = -0.81 \qquad h_\mathbf{w}(\mathbf{x}_i) = 0$$

- This means we add the value of each attribute to its matching weight (assuming again that "dummy" $x_{i,0} = 1$, and that parameter $\alpha = 1$):

$$w_0 \;\leftarrow\; (w_0 + x_{i,0}) = (0.2 + 1) = 1.2$$

$$w_1 \;\leftarrow\; (w_1 + x_{i,1}) = (-2.5 + 0.5) = -2.0$$

$$w_2 \;\leftarrow\; (w_2 + x_{i,2}) = (0.6 + 0.4) = 1.0$$

$$\mathbf{w} \cdot \mathbf{x}_i = 1.2 + (-2.0 \times 0.5) + (1.0 \times 0.4) = 0.6 \qquad \boxed{h_\mathbf{w}(\mathbf{x}_i) = 1}$$

After adjusting weights, our function is now correct on this input

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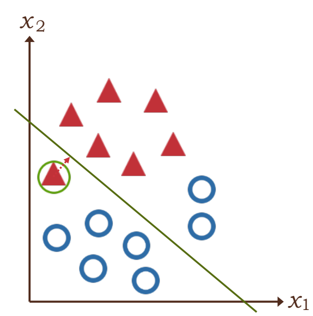
Progress of Perceptron Learning



- For an example like this, we:

1. Choose a mis-classified item (marked in green)
2. Compute the weight updates, based on the "distance" away from the boundary (so weights shift more based upon errors in boundary placement that are more extreme)

- Here, this ***adds*** to each weight, changing the decision boundary

- In this example, data-points marked ○ should be in class 0 (below the line) and those marked ▲ should be in class 1 (above the line)

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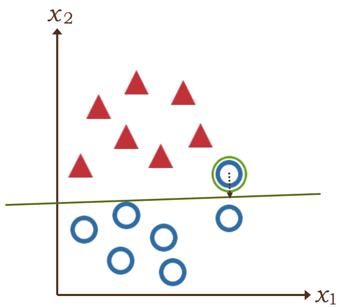
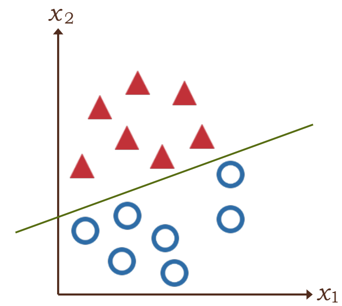
Progress of Perceptron Learning



- Once we get a new boundary, we repeat the process
  1. Choose a mis-classified item (marked in green)
  2. Compute the weight updates, based on the "distance" away from the boundary (so weights shift more based upon errors in boundary placement that are more extreme)
- Here, this ***subtracts*** from each weight, changing the decision boundary in the other direction

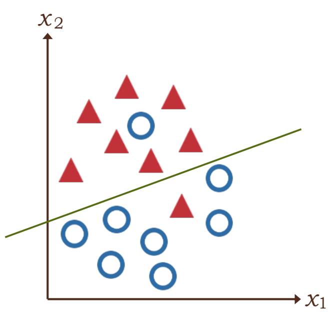
- In this example, data-points marked should be in class 0 (below the line) and those marked should be in class 1 (above the line)

Linear Separability



- The process of adjusting weights stops when there is no classification error left
- A data-set is linearly separable if a linear separator exists for which there will be no error
- It is possible that there are ***multiple*** linear boundaries that achieve this
- It is also possible that there is ***no such*** boundary!

Linearly Inseparable Data



- Some data ***can't*** be separated using a linear classifier
  - Any line drawn will always leave some error
- The perceptron update method is guaranteed to eventually converge to an error-free boundary ***if*** such a boundary really exists
  - If it ***doesn't*** exist, then the most basic version of the algorithm will never terminate

Linearly Inseparable Data

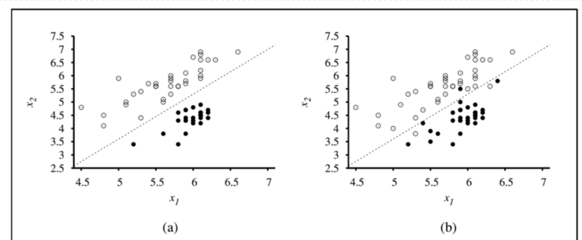


Figure 18.15 (a) Plot of two seismic data parameters, body wave magnitude x_1 and surface wave magnitude x_2, for earthquakes (white circles) and nuclear explosions (black circles) occurring between 1982 and 1990 in Asia and the Middle East (Kebeasy *et al.*, 1998). Also shown is a decision boundary between the classes. (b) The same domain with more data points. The earthquakes and explosions are no longer linearly separable.

Image source: Russel & Norvig, *AI: A Modern Approach* (Prentice Hal, 2010)

- Unfortunately, data that can't be separated linearly is very common…

Modifying Perceptron Learning

- To minimize error, we can modify the algorithm slightly:

1. Choose an input $\mathbf{x}_i$ from our data set that is wrongly classified.
2. Update vector of weights, $\mathbf{w} = (w_0, w_1, w_2, \ldots, w_n)$, as follows:

$$w_j \leftarrow w_j + \alpha(y_i - h_{\mathbf{w}}(\mathbf{x}_i)) \times x_{i,j}$$

~~3. Repeat until no classification errors remain.~~

3. Repeat until weights no longer change; modify learning parameter α over time to guarantee this.

- If we make α smaller and smaller over time, then as $\alpha \rightarrow 0$, the weights will quit changing, and the algorithm converges
- To get down to a least-error possible final separator, we do this slowly, e.g., setting $\alpha(t) = 1000/(1000 + t)$, where t is the current iteration of the update algorithm

Modifying Perceptron Learning

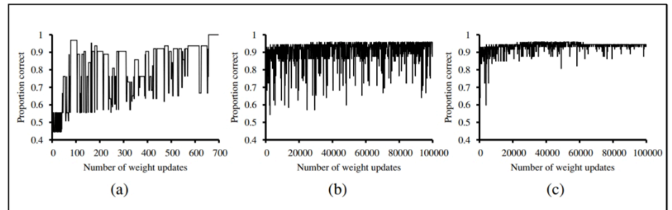


Figure 18.16 (a) Plot of total training-set accuracy vs. number of iterations through the training set for the perceptron learning rule, given the earthquake/explosion data in Figure 18.15(a). (b) The same plot for the noisy, non-separable data in Figure 18.15(b); note the change in scale of the x-axis. (c) The same plot as in (b), with a learning rate schedule $\alpha(t) = 1000/(1000 + t)$.

Image source: Russel & Norvig, *AI: A Modern Approach* (Prentice Hal, 2010)