

4.7 Rank and Nullity

In this section, we look at relationships between the row space, column space, null space of a matrix and its transpose. We will derive fundamental results which in turn will give us deeper insight into solving linear systems.

4.7.1 Rank and Nullity

The first important result, one which follows immediately from the previous section, is that the row space and the column space of a matrix have the same dimension. It is easy to see. For the row space, we use the corresponding matrix in row-echelon form. The number of leading ones is the number of row vectors in the basis of the row space, hence its dimension. But the same number of leading ones also gives us the number of vectors in the basis of the column space, hence also its dimension. We give this result as a theorem.

Theorem 378 *If A is any matrix, then its row space and column space have the same dimension.*

Definition 379 *Let A be a matrix.*

1. *The dimension of its row space (or column space) is called the **rank** of A . It is denoted $\text{rank}(A)$.*
2. *The dimension of its null space is called the **nullity** of A . It is denoted $\text{nullity}(A)$.*

Example 380 Find $\text{rank}(A)$ and $\text{nullity}(A)$ for $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$.

- *$\text{rank}(A)$. It is enough to put A in row-echelon form and count the number of leading ones. The reader will verify that a row-echelon form of*

A is $\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are three leading ones, therefore $\text{rank}(A) = 3$.

- *$\text{nullity}(A)$. For this, we need to find a basis for the solution set of*

$Ax = 0$. Its reduced row-echelon form is: $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The

corresponding system is $\begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 - 2x_3 + 3x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases}$. The leading variables

are x_1 , x_2 , and x_4 . Hence, the free variables are x_3 and x_5 . We can

write the solution in parametric form as follows:
$$\begin{cases} x_1 = -s - t \\ x_2 = 2s - 3t \\ x_3 = s \\ x_4 = 5t \\ x_5 = t \end{cases} \quad \text{or}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}. \quad \text{Thus, } \text{nullity}(A) = 2.$$

Remark 381 In the previous example, if we had had to find a basis for the row space and column space, we would have used the row-echelon form of A . For the column space, we simply take the row vectors of the row-echelon form with a leading one. Thus, we see that a vector for the row space is $\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, 0, 0)\}$. For the column space, we must use the column vectors from the original matrix corresponding to the columns of the row-echelon form with leading ones. The column having leading ones are 1, 2, and 4. Hence, a basis for the column space

of A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$

Remark 382 In the above example, A was 4×5 . We see that $\text{rank}(A) + \text{nullity}(A) = 5$, the number of columns of A . This is not an accident. We will state it as a theorem below.

Remark 383 Since the rows of A are the columns of A^T , we see that the row space of A is the column space of A^T and vice-versa. So, the dimension of the row space of A is the same as that of the column space of A^T . We state this as a theorem.

Theorem 384 $\text{rank}(A) = \text{rank}(A^T).$

Now, we state an important result regarding the relationship between $\text{rank}(A)$ and $\text{nullity}(A)$.

Theorem 385 If A is $m \times n$, then $\text{rank}(A) + \text{nullity}(A) = n$. In other words, $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A$.

Proof. Variables in a system can be separated in two categories. The leading variables, the ones corresponding to the leading 1's and the free variables, the ones to which we usually assign a parameter. We have $\left[\begin{array}{c} \text{number of} \\ \text{leading variables} \end{array} \right] + \left[\begin{array}{c} \text{number of} \\ \text{free variables} \end{array} \right] = n$. But the number of leading variables, being the same as the number of leading 1's, so it is $\text{rank}(A)$. Similarly, the number of free variables is the number of parameters in the solution of the homogeneous system, hence it is the dimension of the null space, that is $\text{nullity}(A)$. ■

Let us state as a corollary two important results used in the proof.

Corollary 386 *If A is $m \times n$, then:*

- ✓ 1. $\text{rank}(A) = \text{the number of leading variables in the solution of } Ax = 0.$
- ✓ 2. $\text{nullity}(A) = \text{the number of parameters in the solution of } Ax = 0.$

Remark 387 *One important consequence of the theorem is that once we know the rank of a matrix, we also know its nullity and vice-versa. We illustrate it with an example.*

Example 388 Find the rank and nullity of $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$

Not this matrix has 5 columns. The row-echelon form is $\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

We see that $\text{rank}(A) = 2$ (2 leading 1's). Therefore $\text{nullity}(A) = 5 - 2 = 3.$

Proposition 389 *Let A be an $m \times n$ matrix. Then $\text{rank}(A) \leq \min(m, n).$*

Proof. Let us note that $\min(m, n)$ is the smallest values between m and n . The column vectors of A are in \mathbb{R}^m , hence the dimension of the column space is at most m . Similarly, the row vectors of A are in \mathbb{R}^n . Hence, the dimension of the row space is at most n . Since the column space and the row space have the same dimension, which is called $\text{rank}(A)$, we see that $\text{rank}(A)$ is at most m and at most n . Hence, we must have $\text{rank}(A) \leq \min(m, n).$ ■

We now see how this applies to linear systems.

4.7.2 Applications to Linear Systems

We look at conditions which must be satisfied in order for a system of m linear equations in n variables to have no solutions, a unique solution, or infinitely many solutions. Let us begin with some definitions.

Definition 390 *A linear system with more equations than unknowns is called **overdetermined**. If it has fewer equations than unknowns, it is called **underdetermined**.*

We already know certain results, which we have developed so far. We will recall these results, then add new results. We first consider the case $m = n$. Then, we will look at what happens if $m \neq n$.

Linear Systems Having m Equations and n Unknowns, $m = n$

Theorem 391 *If A is an $n \times n$ matrix, then the following statements are equivalent:*

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ column matrix \mathbf{b} .
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row equivalent to I_n .
5. A can be written as the product of elementary matrices.
6. $|A| \neq 0$.
7. The column vectors of A are linearly independent.
8. The row vectors of A are linearly independent.
9. The column vectors of A span \mathbb{R}^n .
10. The row vectors of A span \mathbb{R}^n .
11. The column vectors of A form a basis for \mathbb{R}^n .
12. The row vectors of A form a basis for \mathbb{R}^n .
13. $\text{rank}(A) = n$.
14. $\text{nullity}(A) = 0$.

The statements added are 7 – 14. It is easy to see why they are true. We look briefly why they are true. Number 7 and 8 follow from properties of determinants and previous theorems. If say the rows of A were dependent, the one row would be a linear combination of others. But in this case, the determinant of A would be 0. The same applies for the columns. 9, 10, 11, and 12 follow immediately because we have n independent vectors in an space of dimension n . hence, they must also span and form a basis. Since $\text{rank}(A)$ is the dimension of the row space and the row vectors are a basis for \mathbb{R}^n , it follows that $\text{rank}(A) = n$ and hence $\text{nullity}(A) = 0$ since we must have $\text{rank}(A) + \text{nullity}(A) = n$.

Linear Systems Having m Equations and n Unknowns, $m \neq n$

This case is a little bit more difficult. We already know some results. In the case of a homogeneous system, if the system has more unknowns than equations, then it will have infinitely many solutions. We derive more similar results.

First, we look at conditions under which if A is an $m \times n$ matrix the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix \mathbf{b} . We already know that this

system has a solution if \mathbf{b} is in the column space of A . Since \mathbf{b} is an element of \mathbb{R}^m , for the system to have a solution for every \mathbf{b} , the column space of A must span \mathbb{R}^m . But the column space of A is also a subspace of \mathbb{R}^m . This means that the column space of A must be equal to \mathbb{R}^m . This implies in particular that $\text{rank}(A) = m$. We state these as a theorem.

Theorem 392 *If A is an $m \times n$ matrix, then the following statements are equivalent:*

1. *the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix \mathbf{b} .*
2. *The column space of A spans \mathbb{R}^m .*
3. *$\text{rank}(A) = m$.*

This has important consequences. Recall, we saw earlier that if A is an $m \times n$ matrix, then $\text{rank}(A) \leq \min(m, n)$. So, if $m > n$ (more equations than unknowns or the system is overdetermined), then $\text{rank}(A) \leq n$, hence we cannot have $\text{rank}(A) = m$, so the system cannot be consistent for every $m \times 1$ matrix \mathbf{b} .

Next, we look at the homogeneous system $A\mathbf{x} = \mathbf{0}$. We know that such a system always has the trivial solution. We also know the conditions which guaranty uniqueness of solution when $m = n$. What about when $m \neq n$? It is easy to see that the system only has the trivial solution if and only if the columns of A are linearly independent. It is so because $A\mathbf{x}$ is a linear combination of the columns of A . Saying that $A\mathbf{x} = \mathbf{0}$ only has the trivial solution is therefore the same as saying that the only linear combination of the columns of A equal to the zero vector is the one for which all the coefficients are 0. This is the definition of linear independence. It also turns out that in this case, the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ either has no solution, or a unique solution. We state this as a theorem.

Theorem 393 *If A is an $m \times n$ matrix, then the following statements are equivalent:*

1. *$A\mathbf{x} = \mathbf{0}$ only has the trivial solution.*
2. *The columns of A are linearly independent.*
3. *$A\mathbf{x} = \mathbf{b}$ has at most one solution (1 or 0) for every $m \times 1$ matrix \mathbf{b} .*

4.7.3 Concepts Review

- Know the definition of rank and nullity.
- Be able to find the rank and nullity of any matrix A .
- Know the relationship between rank and nullity.
- Know the various conditions under which a system of m equations in n unknowns is consistent.