complex variables:

complex Number:

standard form, $\boxed{z=(x+iy)}$

$iy \rightarrow$ imaginary part.

$x \rightarrow$ real part.

i = imaginary unit.

$$\boxed{\begin{array}{l} ***\ i=\sqrt{-1} \\ ***\ i^2=-1 \end{array}}$$

x = Real part object (z).

i = Imaginary part object. (z).

** If, $z=(x+iy)$ is a complex number than the conjugate of z is denoted by $\bar{z}$ and defined by

$\bar{z}=(x-iy)$.

*** The modulus of complex number is denoted by $|z|$ and defined by,

$$\boxed{\begin{array}{l} |z|=|x+iy|=\sqrt{x^2+y^2} \\ |\bar{z}|=|x-iy|=\sqrt{x^2+y^2} \end{array}}$$

*** $\boxed{\text{Properties:}}$

(i) $|z| = |\bar{z}|$

(ii) $\bar{\bar{z}} = z$

(iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iv) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(v) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

(vi) $z + \bar{z} = 2 . Re(z)$

(vii) $z . \bar{z} = |z|^2 = x^2 + y^2$

x

Operations on complex Numbers:

If z_1, z_2, z_3 be three complex numbers, then,

① $(z_1+z_2)=(z_2+z_1) \rightarrow$ cumulative law of addition.

$z_1 \cdot z_2 = z_2 \cdot z_1 \rightarrow$ cumulative law of multiplication.

② $(z_1+z_2)+z_3 = z_1+(z_2+z_3) \rightarrow$ Associative law of addition.

$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3 \rightarrow$ Associative law of multiplication.

③ $z_1 \cdot (z_2+z_3) = (z_1 \cdot z_2 + z_1 \cdot z_3) \rightarrow$ Distributive law.

④ $z_1+0 = 0+z_1 = z_1$; "0" is "additive identity".

⑤ $(z_1+z_2)=0$; where "z_2" is the "additive inverse" of z_1.

⑥ $z_1 \times 1 = 1 \times z_1 = z_1$; "1" is "multiplicative identity".

$z_1 \times z_2 = z_2 \times z_1 = 1$; "z_2" is the "multiplicative inverse" of z_1.

$\therefore \boxed{z_2 = z_1^{-1}}$ ***

$\therefore \boxed{z_2 = \frac{1}{z_1}}$ ***

⑦ If (z_1 and z_2) $\in C$,

then $(z_1+z_2) \in C \rightarrow$ closure law of addition.

$(z_1 \cdot z_2) \in C \rightarrow$ closure law of multiplication.

Evaluate:

① $\left\{\dfrac{4}{1-i} + \dfrac{2-i}{1+i}\right\} \cdot (2i-1)^2$

$= \left\{\dfrac{4+4i+2-2i-i+i^2}{(1-i)\cdot(1+i)}\right\} \cdot (2i-1)^2$

$= \left(\dfrac{6+i-1}{1-i^2}\right) \cdot (2i-1)^2$

$= \left(\dfrac{5+i}{1-(-1)}\right) \cdot (4i^2-4i+1)$

$= \left(\dfrac{5+i}{2}\right) \cdot (-4-4i+1)$

$= \left(\dfrac{5+i}{2}\right) \cdot (-3-4i)$

$= \dfrac{-15-20i-3i-4i^2}{2}$

$= \dfrac{-15-23i+4}{2}$

$= \dfrac{-11-23i}{2} = -\left(\dfrac{11}{2}\right) - \left(\dfrac{23}{2}\right) i$ (ans)

(11) If, $z_1 = (1-i)$.

$z_2 = (-2+4i)$

$z_3 = (\sqrt{3}-2i)$.

then evaluate, $|z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1}|$

Now, $|z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1}|$

$= |(1-i) \cdot \overline{(-2+4i)} + (-2+4i) \cdot \overline{(1-i)}|$

$= |(1-i) \cdot (-2-4i) + (-2+4i) \cdot (1+i)|$

$= |(-2-4i+2i+4i^2) + (-2-2i+4i+4i^2)|$

$= |-2-4i+2i+4i^2-2-2i+4i+4i^2|$

$= |-4+4i^2+4i^2|$

$= |-4+8i^2|$

$= |-4-8|$

$= |-12|$

$= 12$ (Ans)

Now, if, $z=(x+iy)$.

$\therefore |z|=\sqrt{x^2+y^2}$

$\therefore |z|^2=(x^2+y^2)$.

$\therefore |z| = \{Re(z)\}^2+\{Im(z)\}^2$

$\therefore |z|^2 \geqslant \{Re(z)\}^2$ | $\therefore |z|^2 \geqslant \{Im(z)\}^2$

$\therefore \boxed{|z| \geqslant Re(z)}$ | $\therefore \boxed{|z| \geqslant Im(z)}$

Triangular Inequality:

Prove that:

(i) $|z_1 \pm z_2| \leq |z_1|+|z_2|$

(ii) $|z_1 \pm z_2| \geqslant |z_1|-|z_2|$

(i) Now, $|z_1+z_2|^2 = (z_1+z_2).(\overline{z_1+z_2})$ $[\because |z|^2 = z.\bar{z}$

$=(z_1+z_2).(\bar{z}_1+\bar{z}_2)$

$= z_1.\bar{z}_1 + z_1.\bar{z}_2 + z_2.\bar{z}_1 + z_2.\bar{z}_2$

$=|z_1|^2 + z_1.\bar{z}_2 + z_2.\bar{z}_1 + |z_2|^2$

$$= |z_1|^2 + z_1 \cdot \bar{z}_2 + \overline{\bar{z}_2 \cdot z_1} + |z_2|^2$$

$$= |z_1|^2 + 2 \cdot Re(z_1 \cdot \bar{z}_2) + |z_2|^2 \quad [\because z + \bar{z} = 2 Re(z)]$$

$$\therefore |z_1 + z_2|^2 \leq |z_1|^2 + 2 \cdot |z_1 \cdot \bar{z}_2| + |z_2|^2$$

$$\leq |z_1|^2 + 2 \cdot |z_1| \cdot |\bar{z}_2| + |z_2|^2$$

$$\leq |z_1|^2 + 2 \cdot |z_1| \cdot |z_2| + |z_2|^2$$

$$\therefore |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$$

Replacing z_2 by $(-z_2)$, we get;

$$|z_1 - z_2| \leq |z_1| + |-z_2|$$

$$\therefore |z_1 - z_2| \leq |z_1| + |z_2|$$

$$\therefore |z_1 \pm z_2| \leq |z_1| + |z_2|$$

[Proved]

(iii) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

Now,

$$|z_1 - z_2|^2 = (z_1 - z_2) \cdot (\overline{z_1 - z_2}) \quad [\because |z|^2 = z \cdot \bar{z}]$$

$$= (z_1 - z_2) \cdot (\bar{z}_1 - \bar{z}_2)$$

$$= z_1 \cdot \bar{z}_1 - \bar{z}_1 \cdot z_2 - z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_2$$

$$= z_1 \cdot \bar{z}_1 - z_1 \cdot \bar{z}_2 - \bar{z}_1 \cdot z_2 + z_2 \cdot \bar{z}_2$$

$$= |z_1|^2 - z_1 \cdot \bar{z}_2 - \overline{z_1 \cdot \bar{z}_2} + |z_2|^2$$

$$= |z_1|^2 - 2 \cdot Re(z_1 \cdot \bar{z}_2) + |z_2|^2 \quad [\because z + \bar{z}$$

$$\therefore |z_1 - z_2|^2 \geq |z_1|^2 - 2 \cdot |z_1 \cdot \bar{z}_2| + |z_2|^2$$

$$\geq |z_1|^2 - 2 \cdot |z_1| \cdot |\bar{z}_2| + |z_2|^2$$

$$\geq |z_1|^2 - 2 \cdot |z_1| \cdot |z_2| + |z_2|^2$$

$$\therefore |z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2$$

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

Replacing z_2 by $(-z_2)$ we get,

$$|z_1 + z_2| \geq |z_1| - |-z_2|$$

$$\therefore |z_1 + z_2| \geq |z_1| - |z_2|$$

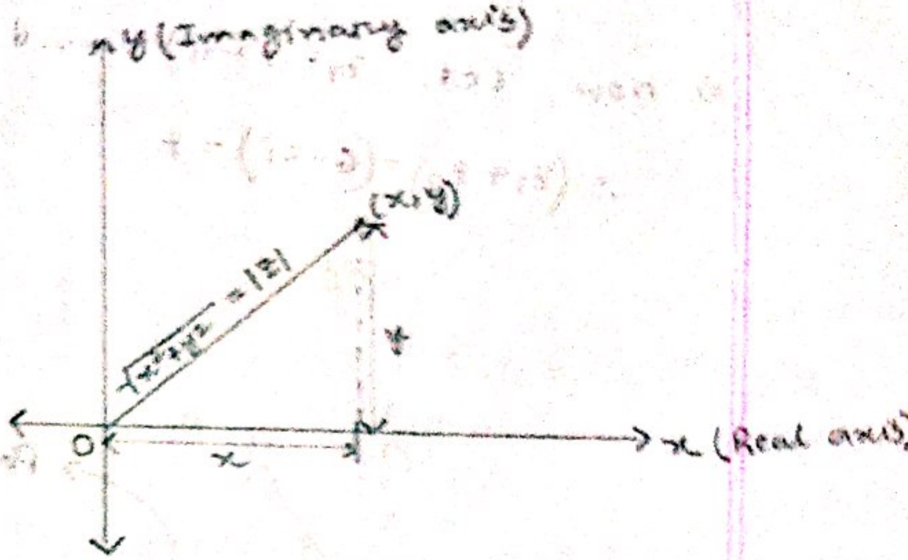
$\therefore |z_1 \pm z_2| \geq |z_1| -$

[Pro

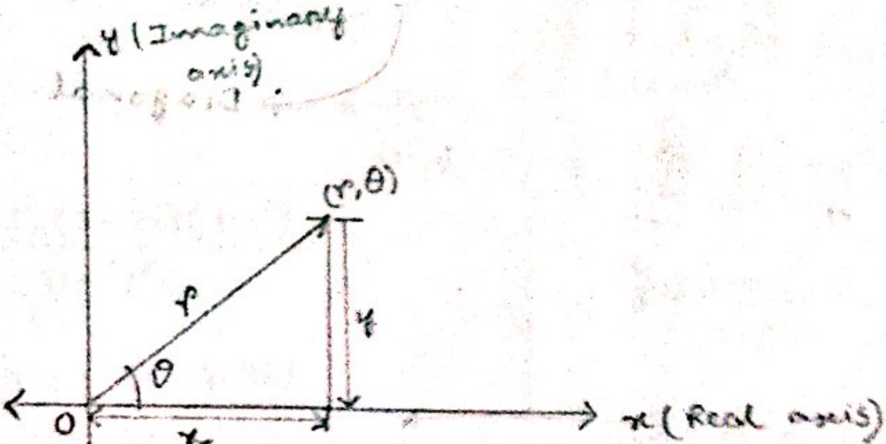
Geometrical Representation of complex number:

Cartesian:

$z = (x + iy)$



Polar:

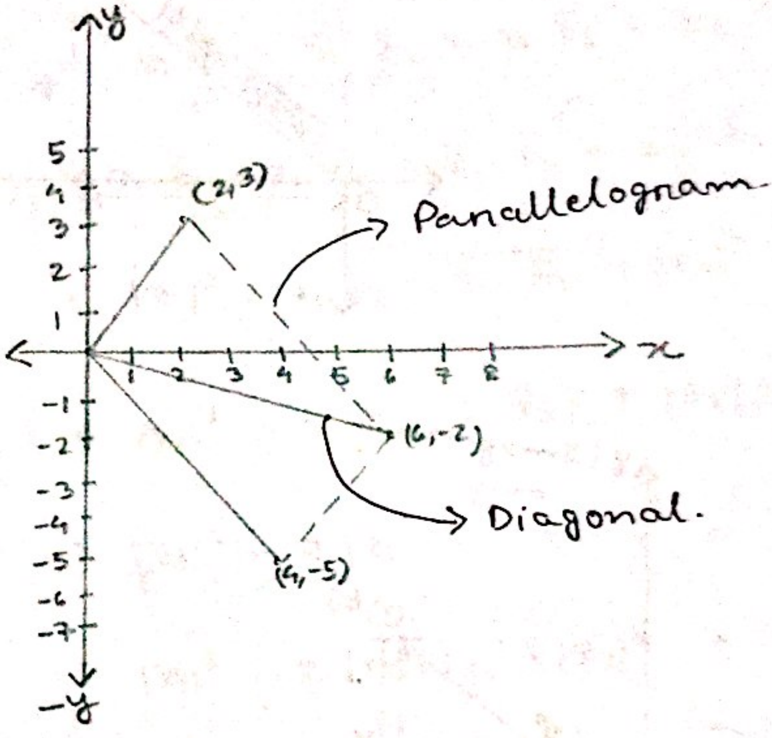


$$\boxed{x = r.\cos\theta}$$
$$\boxed{y = r.\sin\theta}$$

① Perform $(2+3i)+(4-5i)$ analytically and graphically.

$\Rightarrow$ Now, Let, $z_1=(2+3i)$ and $z_2=(4-5i)$

$\therefore (z_1+z_2)=(6-2i)=z$



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Part-A:

① Perform each of the indicated operations: (iii)

(i) $(i-2)\cdot\{2\cdot(1+i)-3(i-1)\}$

$= (i-2)\cdot\{2+2i-3i+3\}$

$= (i-2)\cdot\{5-i\}$

$= 5i-i^2-10+2i$

$= 7i+1-10$

$= (-9+7i)$ (ans)

(ii) $\dfrac{(2+i)(3-2i)(1-i)}{(1-i)^2}$

$= \dfrac{(2+i)\cdot(1-i)\cdot(3-2i)}{(1-i)^2}$

$= \dfrac{(2-2i+i-i^2)\cdot(3-2i)}{(1-i)^2}$

$= \dfrac{(2+1-i)\cdot(3-2i)}{(1-i)^2}$

$= \dfrac{(3-i)(3-2i)}{(1-i)^2}$

$= \dfrac{9-6i-3i+2i^2}{(1-i)^2}$

$= \dfrac{9-2-9i}{1-2i+i^2}$

$= \dfrac{7-9i}{1-2i-1}$

$= \dfrac{7-9i}{-2i}$

$= \dfrac{7i-9i^2}{-2i^2}$

$= \dfrac{9+7i}{2}$

$= \left(\dfrac{9}{2}\right)+\left(\dfrac{7}{2}\right)\cdot i$ (ans)

(iii) $(2i-1)^2 \cdot \left\{ \frac{4}{1-i} + \frac{2-i}{1+i} \right\}$

$= (2i-1)^2 \cdot \left\{ \frac{4+4i+(2-i).(1-i)}{(1-i)(1+i)} \right\}$

$= (2i-1)^2 \cdot \left\{ \frac{4+4i+2-2i-i+i^2}{1-i^2} \right\}$

$= (2i-1)^2 \cdot \left\{ \frac{6-1+i}{1+1} \right\}$

$= (4i^2-4i+1) \cdot \left(\frac{5+i}{2} \right)$

$= \frac{(-3-4i).(5+i)}{2}$

$= \frac{-15-3i-20i-4i^2}{2}$

$= \frac{-15+4-23i}{2}$

$= \frac{-11-23i}{2}$

$= -\left(\frac{11}{2}\right) - \left(\frac{23}{2}\right)i$ (Ans)

*** (IV) $3\cdot\left(\frac{1+i}{1-i}\right)^2 - 2\cdot\left(\frac{1-i}{1+i}\right)^3$ (IV)

$= 3\cdot\left(\frac{1+2i+i^2}{1-2i+i^2}\right) - 2\cdot\frac{(1-i)^2\cdot(1-i)}{(1+i)^2\cdot(1+i)}$

$= 3\cdot\left(\frac{1+2i-1}{1-2i-1}\right) - 2\cdot\frac{(1-2i+i^2)\cdot(1-i)}{(1+2i+i^2)\cdot(1+i)}$

$= 3\cdot\left(\frac{2i}{-2i}\right) - 2\cdot\frac{-2i\cdot(1-i)}{+2i\cdot(1+i)}$

$= -3 + 2\cdot\frac{1-i}{1+i}$

$= \frac{-3(1+i)+2(1-i)}{1+i}$

$= \frac{-3-3i+2-2i}{1+i}$

$= \frac{-1-5i}{(1+i)}$

$= \frac{(-1-5i)\cdot(1-i)}{(1+i)\cdot(1-i)}$

$= \frac{-1+i-5i+5i^2}{1-i^2}$

$= \frac{-1-5-4i}{1+1}$

$= \frac{-6-4i}{2}$

$= (-3-2i)$

(Ans)

(v) $\dfrac{3i^{10} - i^{19}}{2i - 1}$

$= \dfrac{3(i^2)^5 - i.i^{18}}{2i-1}$

$= \dfrac{3(-1)^5 - i.(i^2)^9}{2i-1}$

$= \dfrac{-3 - i.(-1)^9}{2i-1}$

$= \dfrac{-3+i}{2i-1}$

$= \dfrac{(-3+i).(2i+1)}{(2i-1)(2i+1)}$

$= \dfrac{-6i-3+2i^2+i}{4i^2-1}$

$= \dfrac{-3-2-5i}{-4-1}$

$= \dfrac{-5-5i}{-5}$

$= \dfrac{-5(1+i)}{-5}$

$= (1+i)$ (Ans)

(vi) $\dfrac{i^4 - i^7 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

$= \dfrac{(i^2)^2 - i.i^8 + (i^2)^8}{2 - i.i^4 + (i^2)^5 - i.i^{14}}$

$= \dfrac{(-1)^2 - i.(i^2)^4 + (-1)^8}{2 - i.(i^2)^2 + (-1)^5 - i.(i^2)^7}$

$= \dfrac{1 - i.(-1)^4 + 1}{2 - i.(-1)^2 - 1 - i.(-1)^7}$

$= \dfrac{2-i}{2-i-1+i}$

$= (2-i)$ (Ans)

(2) (i) $(5+3i) + \{(-1+2i) + (7-5i)\}$

$= 5+3i + \{6-3i\}$

$= 11$

(ii) $\{(5+3i) + (-1+2i)\} + (7-5i)$

$= \{4+5i\} + (7-5i)$

$= 11$ $\quad \therefore$ (i) = (ii)

$\therefore$ (i) and (ii) illustrate the associative law of addition.

③ $z_1 = (1-i)$

$z_2 = (-2+4i)$

$z_3 = (\sqrt{3}-2i)$

ⓘ $|2z_2 - 3z_1|^2$

$= |2.(-2+4i) - 3(1-i)|^2$

$= |-4+8i-3+3i|^2$

$= |-7+11i|^2$

$= \left(\sqrt{(-7)^2+11^2}\right)^2$

$= 170$ (Ans)

ⓘⓘ $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$

$= \left|\frac{(1-i)+(-2+4i)+1}{(1-i)-(-2+4i)+i}\right|$

$= \left|\frac{1-i-2+4i+1}{1-i+2-4i+i}\right|$

$= \left|\frac{3i}{3-4i}\right|$

$= \left|\frac{3i.(3+4i)}{3^2-(4i)^2}\right|$

$= \left|\frac{9i+12i^2}{9-16i^2}\right|$

$= \left|\frac{-12+9i}{9+16}\right|$

$= \left|-\left(\frac{12}{25}\right)+\left(\frac{9}{25}\right)i\right|$

$= \sqrt{\left(-\frac{12}{25}\right)^2 + \left(\frac{9}{25}\right)^2}$

$= 3/5$ (Ans)

(iii) $\overline{\overline{(z_2+z_3)}.(z_1-z_3)}$

$= (\overline{\overline{z_2+z_3}}).(\overline{z_1-z_3})$

$= (\overline{\overline{z_2}} + \overline{\overline{z_3}}).(\overline{z_1} - \overline{z_3})$

$= \left\{(\overline{-2+4i}) + (\overline{\sqrt{3}-2i})\right\}.\left\{(\overline{1-i}) - (\overline{\sqrt{3}-2i})\right\}$

$= \left\{-2-4i + \sqrt{3}+2i\right\}.\left\{1+i-(\sqrt{3}+2i)\right\}$

$= (-2+\sqrt{3}-2i).(1+i-\sqrt{3}-2i)$

$= (-2+\sqrt{3}-2i).(1-\sqrt{3}-i)$

$= -2+2\sqrt{3}+2i+\sqrt{3}-3-i\sqrt{3}-2i+2\sqrt{3}i+2i^2$

$= -2-2+3\sqrt{3}-3+\sqrt{3}i$

$= (3\sqrt{3}-7)+\sqrt{3}i$ (Ans)

*** (iv) $Re\{2z_1^3 + 3z_2^3 - 5z_3^2\}$

$= 2z_1^3 + 3z_2^3 - 5.z_3^2$

$= 2.(1-i)^3 + 3.(-2+4i)^3 - 5.(\sqrt{3}-2i)^2$

$= 2.(1-3.i+3.i^2-i^3) + 3.(-8+3.(-2)^2.4i+3.(-2).(4i)^2 + (4i)^3) - 5.\{3-4\sqrt{3}i+4i^2\}$

$= 2.(1-3-3i+i) + 3.(-8+48i+96-64i) - 5.(3-4\sqrt{3}i-4)$

$= 2.(-2-2i) + 3.(88-16i) - 5.(-1-4\sqrt{3}i)$

$= -4-4i+264-48i+5+20\sqrt{3}i$

$= 265 - 52i + 20\sqrt{3}i$

$= 265 + (-52+20\sqrt{3})i$

$\therefore Re\{2z_1^3 + 3z_2^3 - 5z_3^2\} = 265$ (Ans)

*** (V) $\text{Im}.\left\{\frac{z_1 . z_2}{z_3}\right\}$

$$= \frac{z_1 . z_2}{z_3}$$

$$= \frac{(1-i).(-2+4i)}{(\sqrt{3}-2i)}$$

$$= \frac{-2+4i+2i-4i^2}{\sqrt{3}-2i}$$

$$= \frac{-2+4+6i}{\sqrt{3}-2i}$$

$$= \frac{(2+6i).(\sqrt{3}+2i)}{(\sqrt{3}-2i).(\sqrt{3}+2i)}$$

$$= \frac{2\sqrt{3}+4i+6\sqrt{3}i+12i^2}{(\sqrt{3})^2-4i^2}$$

$$= \frac{2\sqrt{3}-12+(4+6\sqrt{3})i}{3+4}$$

$$= \left(\frac{2\sqrt{3}-12}{7}\right)+\left(\frac{4+6\sqrt{3}}{7}\right).i$$

$$\therefore \text{Im}.\left\{\frac{z_1 . z_2}{z_3}\right\} = \left(\frac{4+6\sqrt{3}}{7}\right)$$

(ans)

(VI) $z_1^2+2z_1-3$

$$= (1-i)^2+2.(1-i)-3$$

$$= 1-2i+i^2+2-2i-3$$

$$= 1-1-1-4i$$

$$= (-1-4i)$$

(ans)

*** (vii) $\left| z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1} \right|$

$= \left| (1-i) \cdot (\overline{-2+4i}) + (-2+4i) \cdot (\overline{1-i}) \right|$

$= \left| (1-i) \cdot (-2-4i) + (-2+4i) \cdot (1+i) \right|$

$= \left| (-2 - \cancel{4i} + \cancel{2i} + 4i^2) + (-2 - \cancel{2i} + \cancel{4i} + 4i^2) \right|$

$= \left| -2 - 4 - 2 - 4 \right|$

$= |-12|$

$= 12$ (Ans)

***** VIII**

$$\frac{1}{2}\cdot\left(\frac{z_3}{\overline{z_3}}+\frac{\overline{z_3}}{z_3}\right)$$

$$=\frac{1}{2}\cdot\left\{\frac{\sqrt{3}-2i}{\overline{\sqrt{3}-2i}}+\frac{\overline{\sqrt{3}-2i}}{\sqrt{3}-2i}\right\}$$

$$=\frac{1}{2}\cdot\left\{\frac{\sqrt{3}-2i}{\sqrt{3}+2i}+\frac{\sqrt{3}+2i}{\sqrt{3}-2i}\right\}$$

$$=\frac{1}{2}\cdot\left\{\frac{(\sqrt{3}-2i)^2+(\sqrt{3}+2i)^2}{(\sqrt{3})^2-(2i)^2}\right\}$$

$$=\frac{1}{2}\cdot\left\{\frac{3-4\sqrt{3}i+4i^2+3+4\sqrt{3}i+4i^2}{3-4i^2}\right\}$$

$$=\frac{1}{2}\cdot\left\{\frac{6-4-4}{3+4}\right\}$$

$$=\frac{1}{2}\cdot\left(\frac{-2}{7}\right)$$

$$=\frac{-1}{7}$$

(IX) $(z_3 - \overline{z_3})^5$

$= \left\{(\sqrt{3}-2i) - (\overline{\sqrt{3}-2i})\right\}^5$

$= \left\{\sqrt{3}-2i - (\sqrt{3}+2i)\right\}^5$

$= \left\{\sqrt{3}-2i - \sqrt{3} - 2i\right\}^5$

$= (-4i)^5$

$= -1024 . i^5$

$= -1024 . i . i^4$

$= -1024 . i . (i^2)^2$

$= -1024i$ (Ans)

⊛⊛⊛ Polar form:

$z = (x + iy)$

$x = r.\cos\theta$

$y = r.\sin\theta$

$\therefore z = r.\cos\theta + i.r.\sin\theta$

$= r.(\cos\theta + i.\sin\theta)$

$= r.\text{cis}\theta$

$r \rightarrow$ modulus of z.

$\theta \rightarrow$ argument or amplitude of z.

$\therefore \boxed{z = r.\text{cis}(\arg z + 2n\pi)}$

$\arg z \rightarrow$ principal argument.

*** Euler's Formula:

$$\boxed{(\cos\theta + i.\sin\theta) = e^{i\theta}}$$

$$\therefore \boxed{z = r.e^{i\theta}}$$

*** (4) Express in polar form and show graph

(i) $2+2\sqrt{3}i$

$\Rightarrow$ Here, $x=2$
$y=2\sqrt{3}$

$\therefore r=\sqrt{2^2+(2\sqrt{3})^2}$

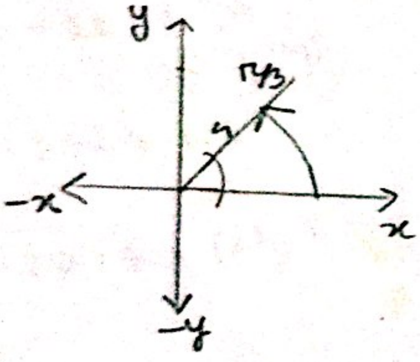
$\therefore r=4$

$\therefore \theta=\tan^{-1}(y/x)$

$=\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$=\tan^{-1}(\sqrt{3})$

$=\tan^{-1}(\tan \pi/3)$



$\therefore \theta=\pi/3$

$\therefore (2+2\sqrt{3}i)=r.\text{cis}.(\arg. z+2n\pi)$

$=4.\text{cis}.(\pi/3+2n\pi)$

$=4.e^{(\pi/3).i}$

(ans)

(ii) $2\sqrt{2}+2\sqrt{2}i$

$\Rightarrow$ Here, $x=2\sqrt{2}$

$y=2\sqrt{2}$

$\therefore r=\sqrt{(2\sqrt{2})^2+(2\sqrt{2})^2}$

$\therefore r=4.$

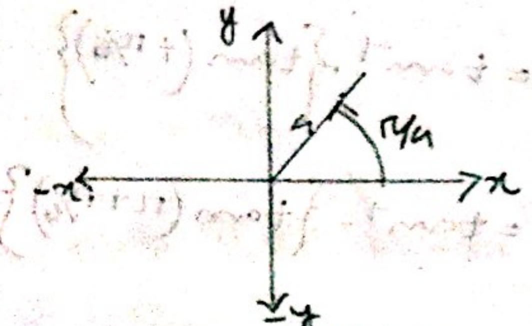
$\therefore \theta=\tan^{-1}\left(\frac{y}{x}\right)$

$=\tan^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right)$

$=\tan^{-1}(1)$

$=\tan^{-1}.\left(\tan \pi/4\right)$

$\therefore \theta=\pi/4$



$\therefore (2\sqrt{2}+2\sqrt{2}i)=r.\text{cis}(\text{arg}.\ z+2n\pi)$

$=4.\text{cis}(\pi/4+2n\pi)$

$=4.e^{(\pi/4).i}$ (ans)

(iii) $-2\sqrt{3}-2i$

$\Rightarrow$ Here, $x = -2\sqrt{3}$

$y = -2$

$\therefore r = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$\therefore r = 4$

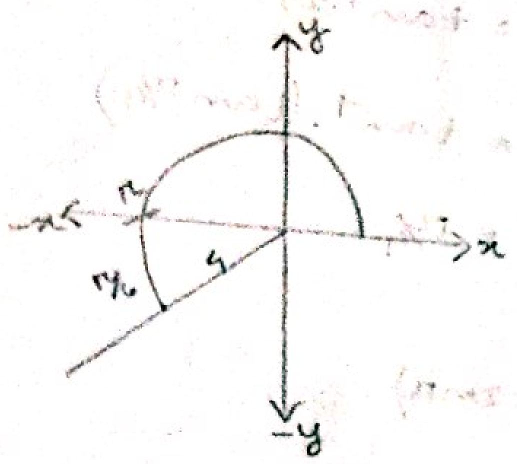
$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{+1}{\sqrt{3}}\right)$

$= \tan^{-1}\left\{\tan\left(+\pi/6\right)\right\}$

$= \tan^{-1}\left\{\tan\left(\pi + \pi/6\right)\right\}$



$= \pi + \pi/6$

$\therefore \theta = \frac{7\pi}{6}$

$\therefore (-2\sqrt{3}-2i) = r.\text{cis}(\arg.z + 2n\pi)$

$= 4.\text{cis}.\left(\frac{7\pi}{6} + 2n\pi\right)$

$= 4.e^{\left(\frac{7\pi}{6}\right).i}$

(ans)

(iv) $-1+\sqrt{3}i$

$\Rightarrow$ Here, $x=-1$

$y=\sqrt{3}$

$\therefore r=\sqrt{(-1)^2+(\sqrt{3})^2}$

$\therefore r=2$

$\therefore \theta=\tan^{-1}(y/x)$

$\therefore \quad =\tan^{-1}(\sqrt{3}/-1)$

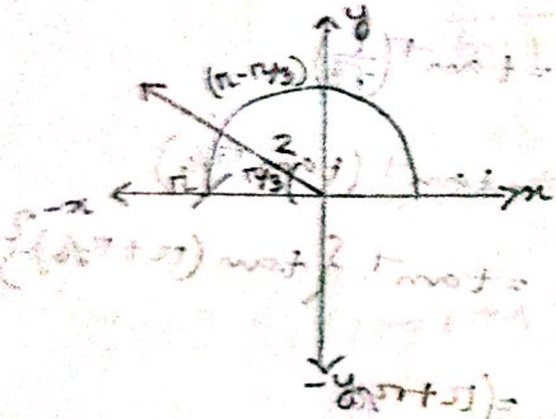
$=\tan^{-1}(-\sqrt{3})$

$=\tan^{-1}.\{\tan(-\pi/3)\}$

$=\tan^{-1}.\{\tan(\pi-\pi/3)\}$

$=\pi-\pi/3$

$\therefore \theta = 2\pi/3$



$\therefore (-1+\sqrt{3}i) = r.cis\theta$

$=r.cis(\arg z + 2n\pi)$

$=2.cis.\left(\frac{2\pi}{3}+2n\pi\right)$

$=2.e^{\left(\frac{2\pi}{3}\right).i}$

$\left(-\text{ans}\right)$

(5) Prove that:

***(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$\Rightarrow$ Here, Let, $z_1 = (x_1 + iy_1)$

$z_2 = (x_2 + iy_2)$

Now, $\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1 + x_2) + (y_1 + y_2) \cdot i}$

$= (x_1 + x_2) - i \cdot (y_1 + y_2)$ $\left[\because \overline{a+ib} = (a - ib)\right]$

$= x_1 + x_2 - iy_1 - i \cdot y_2$

$= (x_1 - i \cdot y_1) + (x_2 - i \cdot y_2)$

$= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)}$

$\therefore \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

[Proved]

*** (11) $|z_1 \cdot z_2| = |z_1| + |z_2|$

$\Rightarrow$ Here, Let, $z_1 = (x_1 + i \cdot y_1)$

$z_2 = (x_2 + i \cdot y_2)$

Now,

$$|z_1 \cdot z_2| = |(x_1 + iy_1) \cdot (x_2 + iy_2)|$$

$$= |x_1 x_2 + i \cdot x_1 \cdot y_2 + iy_1 x_2 + i^2 \cdot y_1 \cdot y_2|$$

$$= |(x_1 \cdot x_2 - y_1 \cdot y_2) + (x_1 \cdot y_2 + x_2 \cdot y_1) i|$$

$$= \sqrt{(x_1 \cdot x_2 - y_1 \cdot y_2)^2 + (x_1 \cdot y_2 + x_2 \cdot y_1)^2}$$

$$= \sqrt{x_1^2 \cdot x_2^2 \cancel{- 2 x_1 x_2 \cdot y_1 \cdot y_2} + y_1^2 \cdot y_2^2 + x_1^2 \cdot y_2^2 \cancel{+ 2 \cdot x_1 \cdot x_2 \cdot y_1 \cdot y_2} + x_2^2 \cdot y_1^2}$$

$$= \sqrt{x_1^2 \cdot (x_2^2 + y_2^2) + y_1^2 \cdot (x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$$

$\therefore |z_1 \cdot z_2| = |z_1| \cdot |z_2|$ [Proved]

*** (iii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

⇒ Now,

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ $\left[\because |z|^2 = z\bar{z}\right]$

$= (z_1 + z_2)\cdot(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + z_1\bar{z}_2 + \overline{\bar{z}_2 z_1} + |z_2|^2$

$= |z_1|^2 + 2\,Re\{z_1\bar{z}_2\} + |z_2|^2$ $\left[\because (z + \bar{z}) = 2\,Re(z)\right]$

$\therefore |z_1 + z_2|^2 \leq |z_1|^2 + 2\cdot|z_1\cdot\bar{z}_2| + |z_2|^2$

$\leq |z_1|^2 + 2\cdot|z_1|\cdot|\bar{z}_2| + |z_2|^2$

$\leq |z_1|^2 + 2\cdot|z_1|\cdot|z_2| + |z_2|^2$

$\therefore |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$

By replacing z_2 by $(-z_2)$ we have,

$|z_1 - z_2| \leq |z_1| + |-z_2|$

$\therefore |z_1 - z_2| \leq |z_1| + |z_2|$

$\therefore |z_1 \pm z_2| \leq |z_1| + |z_2|$ [Proved]

*** (IV) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

$\Rightarrow$ Now,

$$|z_1 - z_2|^2 = (z_1 - z_2) \cdot (\overline{z_1 - z_2}) \quad [\because |z|^2 = z \cdot \bar{z}]$$

$$= (z_1 - z_2) \cdot (\bar{z}_1 - \bar{z}_2)$$

$$= z_1 \cdot \bar{z}_1 - z_1 \cdot \bar{z}_2 - z_2 \cdot \bar{z}_1 + z_2 \bar{z}_2$$

$$= |z_1|^2 - z_1 \cdot \bar{z}_2 - \overline{\bar{z}_2 \cdot z_1} + |z_2|^2$$

$$= |z_1|^2 - 2 \cdot Re\{z_1 \cdot \bar{z}_2\} + |z_2|^2$$

$$\therefore |z_1 - z_2|^2 \geq |z_1|^2 - 2 \cdot |z_1 \cdot \bar{z}_2| + |z_2|^2$$

$$\geq |z_1|^2 - 2 \cdot |z_1| \cdot |\bar{z}_2| + |z_2|^2$$

$$\geq |z_1|^2 - 2 \cdot |z_1| \cdot |z_2| + |z_2|^2$$

$$\therefore |z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2$$

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

Now, replacing z_2 by $(-z_2)$ we have,

$$|z_1 + z_2| \geq |z_1| - |-z_2|$$

$$\therefore |z_1 + z_2| \geq |z_1| - |z_2|$$

$$\therefore |z_1 \pm z_2| \geq |z_1| - |z_2| \quad \text{[Proved]}$$

***⑥ State and Prove De Moivre's Theorem:

⇒ **Statement:**

$$(\cos\theta + i.\sin\theta)^n = \{\cos(n\theta) + i.\sin(n\theta)\} \; ; \; n = 0, 1, 2, 3 \ldots$$

is known as the D' Moiver's Theorem.

Prove:

Here, we will ~~use~~ the principal of mathematical induction.

Step 1: we take the given statement is true for $n=1$.

$\therefore$ we have, $(\cos\theta + i.\sin\theta)^1 = (\cos\theta + i.\sin\theta)$

Step 2: we show that if the statement is true for $n=k$, then it is also true for, $n=(k+1)$.

i.e $(\cos\theta + i.\sin\theta)^k = \{\cos(k\theta) + i.\sin(k\theta)\}$

multiplied by $(\cos\theta + i.\sin\theta)$, we get,

$$(\cos\theta + i.\sin\theta)^k.(\cos\theta + i.\sin\theta) = \{\cos(k\theta) + i.\sin(k\theta)\}.(\cos\theta + i.\sin$$

$$\Rightarrow (\cos\theta + i.\sin\theta)^{k+1} = \cos\theta.\cos(k\theta) + i.\sin\theta.\cos(k\theta) + i.\sin(k\theta).\cos$$
$$- \sin\theta.\sin(k\theta)$$

$$= \cos\theta \cdot \cos(k\theta) - \sin\theta \cdot \sin(k\theta) + i \cdot (\sin k\theta \cdot \cos\theta + \cos k\theta \cdot \sin\theta)$$

$$= \cos(k+1) \cdot \theta + i \cdot \sin(k+1) \cdot \theta$$

$$\left[\begin{array}{l} \because \cos A \cdot \cos B - \sin A \cdot \sin B = \cos(A+B) \\ \because \sin A \cdot \cos B + \cos A \cdot \sin B = \sin(A+B) \end{array} \right]$$

$$\therefore (\cos\theta + i \cdot \sin\theta)^{k+1} = \cos(k+1) \cdot \theta + i \cdot \sin(k+1) \cdot \theta$$

$\therefore$ By mathematical induction, this statement is true for all n, where, $n = 0, 1, 2, 3 \cdots \cdots$

[Proved]

*** $z = r \cdot e^{i\theta}$

$= r \cdot (\cos\theta + i \cdot \sin\theta)$

$\therefore z^2 = r^2 \cdot (\cos\theta + i \cdot \sin\theta)^2$

$= r^2 (\cos^2\theta - \sin^2\theta + 2 \cdot i \cdot \cos\theta \cdot \sin\theta)$

$= r^2 \cdot (\cos 2\theta + i \cdot \sin 2\theta)$

$\therefore z^2 = r^2 \cdot e^{2i\theta} \quad [\because (\cos\theta + i \cdot \sin\theta) = e^{i\theta}]$

$\therefore \boxed{z^n = r^n \cdot e^{ni\theta}}$

In De Moivene's Theorem the radius vector is fixed and that is 1.

***Law:**

$z_1 = r_1 \cdot e^{i\theta_1}, \quad z_2 = r_2 \cdot e^{i\theta_2}$

① $(z_1 + z_2) = (r_1 \cdot e^{i\theta_1} + r_2 \cdot e^{i\theta_2})$

② $(z_1 \cdot z_2) = r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)}$

③ $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) \cdot e^{i(\theta_1 - \theta_2)}$

*** (7) **Evaluate by De moivre's Theorem:**

***(i)
$$\frac{\left(3e^{\frac{\pi i}{6}}\right)\cdot\left(2e^{\frac{-5\pi i}{4}}\right)\left(6\cdot e^{\frac{5\pi i}{3}}\right)}{\left(4\cdot e^{\frac{2\pi i}{3}}\right)^2}$$

$$= \frac{36\cdot e^{\frac{\pi i}{6}-\frac{5\pi i}{4}+\frac{5\pi i}{3}}}{16\cdot e^{\frac{4\pi i}{3}}}$$

$$= \frac{36}{16}\cdot\left(e^{\frac{\pi i}{6}+\frac{5\pi i}{3}-\frac{5\pi i}{4}-\frac{4\pi i}{3}}\right)$$

$$= \frac{9}{4}\cdot\left(e^{\frac{2\pi i+20\pi i-15\pi i-16\pi i}{12}}\right)$$

$$= \frac{9}{4}\cdot e^{-\frac{9\pi i}{12}}$$

$$= \frac{9}{4}\cdot e^{-\left(\frac{3}{4}\right)\pi i}$$

$$= \frac{9}{4}\cdot\left\{\cos\left(\frac{-3\pi}{4}\right)+i\cdot\sin\left(\frac{-3\pi}{4}\right)\right\}$$

$$= \frac{9}{4}\cdot\left\{\cos\left(\frac{3\pi}{4}\right)-i\ \sin\left(\frac{3\pi}{4}\right)\right\}$$

$$= \frac{9}{4}\cdot\left(\frac{-1}{\sqrt{2}}-i\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{-9}{4\sqrt{2}}-\frac{9}{4\sqrt{2}}\cdot i$$

$$= -\left(\frac{9\sqrt{2}}{8}\right)-\left(\frac{9\sqrt{2}}{8}\right)i \quad \text{(Ans)}$$

(11)

$$\frac{(8\,\text{cis}\,40)^3}{(2.\text{cis}\,60)^4}$$

$$= \frac{512.\text{cis}\,3\times40}{16.\text{cis}\,4\times60}$$

$$= 32.\text{cis}\,(120-240)$$

$$= 32.\text{cis}\,(-120)$$

$$= 32.\{\cos(-120) + i.\sin(-120)\}$$

$$= 32.\{\cos 120 - i.\sin(120)\}$$

$$= 32.\left(\frac{-1}{2} - i.\frac{\sqrt{3}}{2}\right)$$

$$= (-16 - 16\sqrt{3}i)$$

(Ans)

(iii) $(5 \operatorname{cis} 20^\circ)(3 \operatorname{cis} 40^\circ)$

$= 15 \cdot \operatorname{cis}(20+40)$

$= 15 \operatorname{cis} 60^\circ$

$= 15 \cdot (\cos 60^\circ + i \cdot \sin 60^\circ)$

$= 15\left(\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right)$

$= \left(\frac{15}{2}\right) + \left(\frac{15\sqrt{3}}{2}\right) i$

(Ans)

*** (iv) $(2 \cdot \operatorname{cis} 50^\circ)^6$

$= 2^6 \cdot \operatorname{cis} 6 \times 50$

$= 2^6 \cdot \operatorname{cis} 300$

$= 2^6 \cdot (\cos 300 + i \cdot \sin 300)$

$= 2^6 \cdot \left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$

(Ans)

=

$$\{3.(\cos 40 + i \sin 40)\}.(4 \text{ cis} 80)$$

$$= 12 . \text{cis} 40 . \text{cis} 80$$

$$= 12 . \text{cis} 120$$

$$= 12 (\cos 120 + i \sin 120)$$

$$= 12 \left(-\frac{1}{2} + i . \frac{\sqrt{3}}{2}\right)$$

$$= (-6 + 6\sqrt{3} i)$$

(Ans)

$$\frac{(2 \text{ cis} 15^\circ)^7}{(4 . \text{cis} 45^\circ)^3}$$

$$= \frac{2^7 . \text{cis}(15\times 7)}{4^3 . \text{cis}(45\times 3)}$$

$$= \frac{128 . \text{cis} 105^\circ}{64 . \text{cis} 135^\circ}$$

$$= 2 . \text{cis}(105 - 135)$$

$$= 2 . \text{cis}(-30)$$

$$= 2 . \{\cos(-30) + i . \sin(-30)\}$$

$$= 2 . \left\{\frac{\sqrt{3}}{2} - i . \frac{1}{2}\right\}$$

$$= (\sqrt{3} - i)$$

(Ans)

*** (VI) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

Now for nominator

$r = \sqrt{1^2+(\sqrt{3})^2} = 2$

$\therefore \theta = \tan^{-1}(\sqrt{3})$

$= \tan^{-1}(\tan \frac{\pi}{3})$

$\therefore \theta = \pi/3 = 60^\circ$

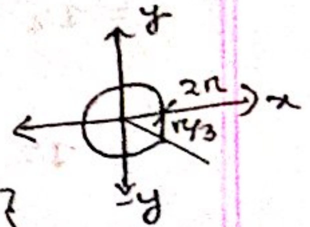
$\therefore (1+\sqrt{3}i) = 2.cis60^\circ$

for denominator,

$r = \sqrt{1^2+(-\sqrt{3})^2} = 2$

$\therefore \theta = \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\{\tan(-\pi/3)\}$



$= \tan^{-1}\{\tan(2\pi-\pi/3)\}$

$= \frac{5\pi}{3}$

$\therefore \theta = 300^\circ$

$\therefore (1-\sqrt{3}i) = 2.cis300^\circ$

(IV)

$$\therefore \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{2.cis60^\circ}{2.cis300}\right)^{10}$$

$$= \{cis(60-300)\}^{10}$$

$$= cis(-240\times10)$$

$$= \cos(-2400) + i.\sin(-2400)$$

$$= \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)$$

(ans)

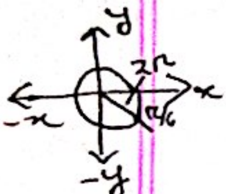
***VIII $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 \cdot \left(\frac{1+i}{1-i}\right)^5$

$\Rightarrow$ Now, Part 1,

Nominator,

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\therefore \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$



$= \tan^{-1}\{\tan(-\pi/6)\}$

$= \tan^{-1}.\{\tan(2\pi - \pi/6)\}$

$= 2\pi - \pi/6$

$= \frac{11\pi}{6}$

$\therefore \theta = 330^\circ$

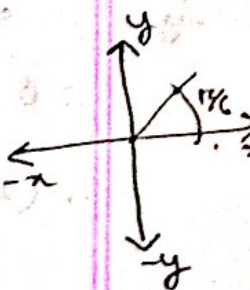
$\therefore (\sqrt{3}-i) = 2.\text{cis}\,330^\circ$

Denominator,

$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\therefore \theta = \tan^{-1}(1/\sqrt{3})$

$= \tan^{-1}.(\tan \pi/6)$



$= \tan^{-1}(\tan \pi/6)$

$= \pi/6$

$\therefore \theta = 30^\circ$

$\therefore (\sqrt{3}+i) = 2.\text{cis}\,30^\circ$

$\therefore \left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 = \left(\frac{2.\text{cis}\,330^\circ}{2.\text{cis}\,30^\circ}\right)^4$

$= \frac{\text{cis}(330\times4)}{\text{cis}(30\times4)}$

$= \text{cis}(1320 - 120)$

$= \text{cis}\,1200$

Part-2:

Nominator,

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \theta = \tan^{-1}(1)$

$= \tan^{-1}(\tan \pi/4)$

$= \pi/4$

$\therefore \theta = 45^\circ$

$\therefore (1+i) = \sqrt{2}.\text{cis}\,45^\circ$

Denominator,

$r = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\therefore \theta = \tan^{-1}(-1)$

$= \tan^{-1}\{\tan(-\pi/4)\}$

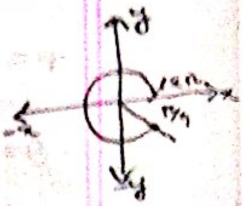
$= \tan^{-1}\{\tan(2\pi-\pi/4)\}$

$= 2\pi - \pi/4$

$= \frac{7\pi}{4}$

$\therefore \theta = 315^\circ$

$\therefore (1-i) = \sqrt{2}.\text{cis}\,315^\circ$



$$\therefore \left(\frac{1+i}{1-i}\right)^5 = \left(\frac{\sqrt{2}.\text{cis}45^\circ}{\sqrt{2}.\text{cis}315}\right)^5$$

$$= \frac{\text{cis}(45\times5)}{\text{cis}(315\times5)}$$

$$= \text{cis}(225-1575)$$

$$= \text{cis}(-1350)$$

$\therefore \left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 \cdot \left(\frac{1+i}{1-i}\right)^5$

$= cis(1200) . cis(-1350)$

$= cis(1200-1350)$

$= cis(-150)$

$= cos(-150) + i . sin(-150)$

$= -\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)i$

(Ans)

If, $z_1 = (2+i)$

$z_2 = (3-2i)$

$z_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

*** (i) $\left(\overline{z_3}\right)^4$

$= \left(\overline{-\frac{1}{2} + \frac{\sqrt{3}}{2} i}\right)^4$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i\right)^4$

$= \left\{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i\right)^2\right\}^2$

$= \left\{\frac{1}{4} + 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot i + \left(\frac{\sqrt{3}}{2} \cdot i\right)^2\right\}^2$

$= \left(\frac{1}{4} + \frac{\sqrt{3}}{2} \cdot i - \frac{3}{4}\right)^2$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2$

$= \frac{1}{4} - 2 \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{2} \cdot i + \left(\frac{\sqrt{3}}{2} \cdot i\right)^2$

$= \frac{1}{4} - \frac{\sqrt{3}}{4} \cdot i - \frac{3}{4}$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i\right)$

(ans)

(ii) $\left|\dfrac{2z_2+z_1-5-i}{2z_1-z_2+3-i}\right|^2$

$= \left|\dfrac{2.(3-2i)+(2+i)-5-i}{2.(2+i)-(3-2i)+3-i}\right|$

$= \left|\dfrac{6-4i+2+i-5-i}{4+2i-3+2i+3-i}\right|$

$= \left|\dfrac{3-4i}{4+3i}\right|$

$= \dfrac{\sqrt{3^2+(-4)^2}}{\sqrt{4^2+3^2}}$

$= 1$

(Ans)

(ii) If, $z_1 = (1-i)$

$z_2 = (-2+4i)$

$z_3 = (\sqrt{3}-2i)$

(i) $|z_1^2 + \overline{z_2}^2|^2 + |\overline{z_3}^2 - z_2^2|^2$

$= |(1-i)^2 + (\overline{-2+4i})^2|^2 + |(\overline{\sqrt{3}-2i})^2 - (-2+4i)^2|^2$

$= |1-2i+i^2 + (-2-4i)^2|^2 + |(\sqrt{3}+2i)^2 - (4-16i+16i^2)|^2$

$= |-2i + (4+16i+16i^2)|^2 + |3+4\sqrt{3}i + 4i^2 - 4 + 16i - 16i^2|^2$

$= |-2i+4+16i+16i^2|^2 + |3+4\sqrt{3}i - 4 - 4 + 16i + 16|^2$

$= |-2i+4+16i-16|^2 + |3-8+4\sqrt{3}i + 16i + 16|^2$

$= |-12+14i|^2 + |11 + (4\sqrt{3}+16)i|^2$

$= (-12)^2 + (14)^2 + 11^2 + (4\sqrt{3}+16)^2$

$= (765 + 128\sqrt{3})$ (Ans)

Express in Polar form:

1. $-5+5i$

$\Rightarrow$ Here, $x = -5$

$y = 5$

$\therefore r = \sqrt{(-5)^2 + 5^2} = 5\sqrt{2}$

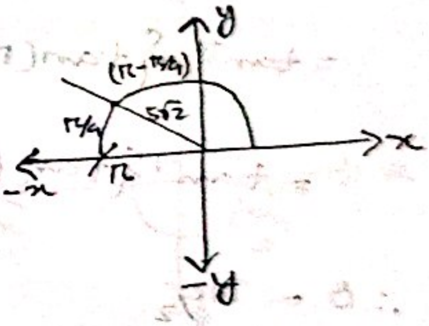
$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(5/-5)$

$= \tan^{-1}(-1)$

$= \tan^{-1}.\{\tan(-\pi/4)\}$

$= \tan^{-1}\{\tan(\pi - \pi/4)\}$



$\therefore \theta = \frac{3\pi}{4}$

$\therefore (-5+5i) = r.\text{cis}(\arg z + 2n\pi)$

$= 5\sqrt{2}.\text{cis}\left(\frac{3\pi}{4} + 2n\pi\right)$

$= 5\sqrt{2}.e^{\left(\frac{3\pi}{4}\right).i}$

(Ans)

(ii) $-3i$

Here, $x = 0$

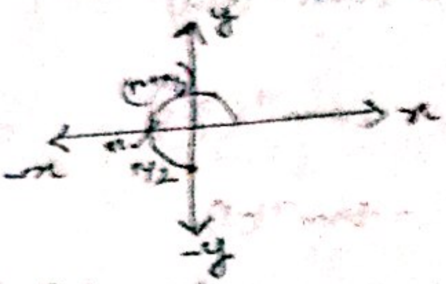
$y = -3$

$\therefore r = \sqrt{0^2 + (-3)^2} = 3$

$\therefore \theta = \tan^{-1}(-3/0)$

$= \tan^{-1}(\alpha)$

$= \tan^{-1}\{\tan(\pi/2)\}$



$= \tan^{-1}\{\tan(\pi + \pi/2)\}$

$= \tan^{-1}\{\tan(3\pi/2)\}$

$\therefore \theta = 3\pi/2$

$\therefore -3i = r.\operatorname{cis}(\arg z + 2n\pi)$

$= 3.\operatorname{cis}\left(\frac{3\pi}{2} + 2n\pi\right)$

$= 3.e^{\left(\frac{3\pi}{2}\right).i}$

(Ans)

(iii) $(2-2i)$

⇒ Here, $x = 2$

$y = -2$

$\therefore r = \sqrt{2^2 + (-2)^2}$

$\therefore r = 2\sqrt{2}$

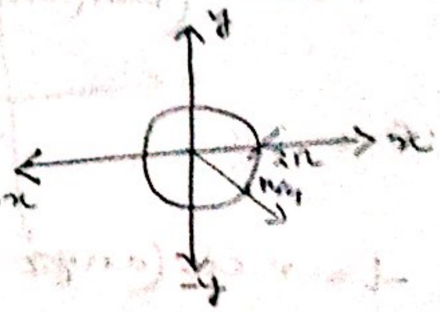
$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(-2/2)$

$= \tan^{-1}(-1)$

$= \tan^{-1}.\{\tan(-\pi/4)\}$

$= \tan^{-1}.\{\tan(2\pi - \pi/4)\}$



$\therefore \theta = \frac{7\pi}{4}$

$\therefore (2-2i) = 2\sqrt{2}.\operatorname{cis}\left(\frac{7\pi}{4} + 2n\pi\right)$

$= 2\sqrt{2}.\,e^{\left(\frac{7\pi}{4}\right).i}$

(Ans)

(ii) $-i$

$\Rightarrow$ Here, $x = 0$

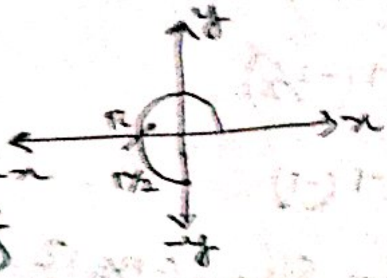
$y = -1$

$\therefore r = \sqrt{0^2 + (-1)^2} = 1$

$\therefore \theta = \tan^{-1}(-1/0)$

$= \tan^{-1}(\infty)$

$= \tan^{-1} \cdot \tan(\pi/2)$



$= \tan^{-1} \{\tan(\pi + \pi/2)\}$

$\therefore \theta = \frac{3\pi}{2}$

$\therefore -i = r \cdot cis(\arg z + 2n\pi)$

$= cis(\frac{3\pi}{2} + 2n\pi)$

$= e^{(\frac{3\pi}{2})i}$

$(-2n\pi)$

(v) -4

$\Rightarrow$ Here, $x = -4$

$y = 0$

$\therefore r = \sqrt{(-4)^2 + 0}$

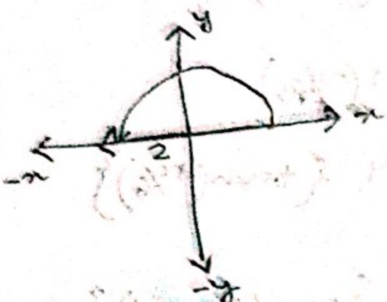
$\therefore r = 4$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-4)$

$= \tan^{-1}(0)$

$= \tan^{-1}.\{\tan(\pi)\}$



$\therefore \theta = \pi$

$\therefore -4 = r.\text{cis}(\arg z + 2n\pi)$

$= 4.\text{cis}(\pi + 2n\pi)$

$= 4.e^{i\pi}$

(ans)

(VI) $2\sqrt{3}-2i$

$\Rightarrow$ Here, $x = 2\sqrt{3}$

$y = -2$

$\therefore r = \sqrt{(2\sqrt{3})^2 + (-2)^2}$

$\therefore r = 4$

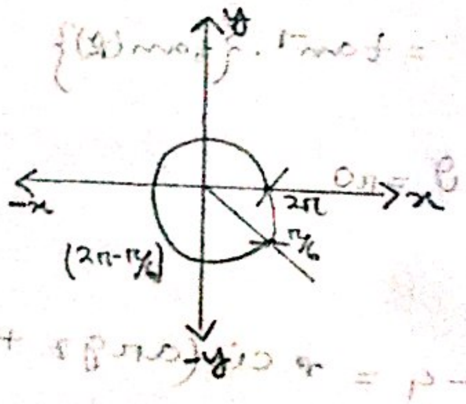
$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}\left(-2/2\sqrt{3}\right)$

$= \tan^{-1}(-1/\sqrt{3})$

$= \tan^{-1}.\{\tan(-\pi/6)\}$

$= \tan^{-1}.\{\tan(2\pi - \pi/6)\}$



$= \frac{11\pi}{6}$

$\therefore \theta = \frac{11\pi}{6}$

$\therefore (2\sqrt{3}-2i) = r.\text{cis}(\arg z + 2n\pi)$

$= 4.\text{cis}\left(\frac{11\pi}{6} + 2n\pi\right)$

$= 4e^{\left(\frac{11\pi}{6}\right).i}$

(Ans)

** (XII) $\sqrt{2}i$

$\Rightarrow$ Here,

$x = 0$

$y = \sqrt{2}$

$\therefore r = \sqrt{0^2 + (\sqrt{2})^2}$

$\therefore r = \sqrt{2}$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(\sqrt{2}/0)$

$= \tan^{-1}(\infty)$

$= \tan^{-1}\{\tan(\pi/2)\}$

$\therefore \theta = \pi/2$

$\therefore \sqrt{2}i = \sqrt{2}.\,cis(\pi/2 + 2n\pi)$

$= \sqrt{2}.e^{(\frac{\pi}{2}).i}$

(ans)

(VIII) $\frac{\sqrt{3}}{2} - \left(\frac{3}{2}\right)i$

$\Rightarrow$ Here, $x = \frac{\sqrt{3}}{2}$

$y = -\frac{3}{2}$

$\therefore r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{3}{2}\right)^2}$

$\therefore r = \sqrt{3}$

$\therefore \theta = \tan^{-1}(y/x)$

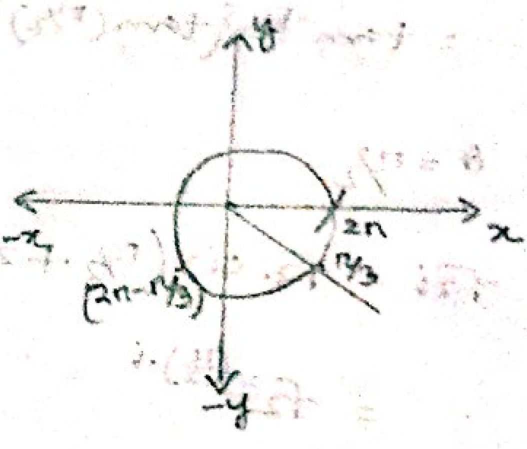
$= \tan^{-1}\left(\frac{-3/2}{+\sqrt{3}/2}\right)$

$= \tan^{-1}\left(-\frac{3}{2} \times \frac{2}{\sqrt{3}}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\{\tan(-\pi/3)\}$

$= \tan^{-1}\{\tan(2\pi - \pi/3)\}$



$= 2\pi - \pi/3$

$\therefore \theta = 5\pi/3$

$\therefore \frac{\sqrt{3}}{2} - \left(\frac{3}{2}\right) \cdot i = \sqrt{3} \cdot \text{cis} \cdot \left(\frac{5\pi}{3} + 2n\pi\right)$

$= \sqrt{3} \cdot e^{\left(\frac{5\pi}{3}\right) \cdot i}$

(Ans)

Show that:

$$(2+i) = \sqrt{5}\, e^{i \tan^{-1}(1/2)}$$

$\Rightarrow$ Now, $(2+i)$

Here, $x = 2$
$y = 1$

$\therefore r = \sqrt{2^2 + 1^2}$

$\therefore r = \sqrt{5}$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(1/2)$

$\therefore \theta = \tan^{-1}(1/2)$

$\therefore (2+i) = \sqrt{5} . \operatorname{cis}\left(\tan^{-1}(1/2) + 2n\pi\right)$

$\therefore (2+i) = \sqrt{5} . e^{i . \tan^{-1}(1/2)}$

[Shown]

Express in Polar form

(a) $-3-4i$

$\Rightarrow$ Here, $x=-3$

$y=-4$

$\therefore r=\sqrt{(-3)^2+(-4)^2}$

$\therefore r=5$

$\therefore \theta=\tan^{-1}(y/x)$

$=\tan^{-1}(-4/-3)$

$\therefore \theta=\tan^{-1}(4/3)$

$\therefore (-3-4i)=5.\text{cis}\{\tan^{-1}(4/3)+(2n\pi)\}$

$=5.e^{i.\tan^{-1}(4/3)}$

(Ans)

(b) $1-2i$

$\Rightarrow$ Here, $x=1$

$y=-2$

$\therefore r=\sqrt{1^2+(-2)^2}$

$\therefore r=\sqrt{5}$

$\therefore \theta=\tan^{-1}(-2/1)$

$=\tan^{-1}(-2)$

$\therefore \theta=\tan^{-1}(-2)$

$\therefore (1-2i)=\sqrt{5}.\text{cis}\{\tan^{-1}(-2)+2n\pi\}$

$=\sqrt{5}.e^{i.\tan^{-1}(-2)}$

(Ans)

⑧ Find each of the indicated roots:

ⓘ $z^5 = (-4+4i)$

$\Rightarrow z = (-4+4i)^{1/5}$

Here, $x = -4$

$y = 4$

$\therefore r = \sqrt{(-4)^2 + 4^2}$

$= \sqrt{32}$

$\therefore r = 4\sqrt{2}$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(4/-4)$

$= \tan^{-1}(-1)$

$= \tan^{-1}.\{\tan(-\pi/4)\}$

$= \tan^{-1}.\{\tan(\pi - \pi/4)\}$

$= \pi - \pi/4$

$\therefore \theta = \frac{3\pi}{4}$

$\therefore (-4+4i) = r.\text{cis}(\arg z \pm 2n\pi)$

$= 4\sqrt{2}.\text{cis}(\frac{3\pi}{4} + 2n\pi)$

$\therefore (-4+4i)^{1/5} = (4\sqrt{2})^{1/5}.\text{cis}.(\frac{3\pi}{4} + 2n\pi)^{1/5}$

$= (4\sqrt{2})^{1/5}.\text{cis}\{(\frac{3\pi}{4} + 2n\pi).\frac{1}{5}\}$

$\therefore (-4+4i)^{1/5} = (4\sqrt{2})^{1/5}.\text{cis}.(\frac{3\pi}{20} + \frac{2n\pi}{5})$; $n = 0, 1, 2, 3, 4.$

$\therefore$ If, $n = 0$,

$z_1 = (4\sqrt{2})^{1/5}.\text{cis}(\frac{3\pi}{20})$

$\therefore$ If, $n = 1$,

$z_2 = (4\sqrt{2})^{1/5}.\text{cis}(\frac{3\pi}{20} + \frac{2\pi}{5}) = (4\sqrt{2})^{1/5}.\text{cis}(\frac{11\pi}{20})$

$\therefore$ If, $n=2$,

$\therefore z_3 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{3\pi}{20} + \frac{4\pi}{5}\right)$

$\therefore z_3 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{19\pi}{20}\right)$

If, $n=3$,

$\therefore z_4 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{3\pi}{20} + \frac{6\pi}{5}\right)$

$\therefore z_4 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{27\pi}{20}\right)$

If, $n=4$,

$\therefore z_5 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{3\pi}{20} + \frac{8\pi}{5}\right)$

$\therefore z_5 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{35\pi}{20}\right)$,

$\therefore z_5 = (4\sqrt{2})^{1/5} \cdot cis\left(\frac{7\pi}{4}\right)$

(ii) $z^4 = -16i$

$\therefore z = (-16i)^{1/4}$

Here, $x = 0$

$y = -16$

$\therefore r = \sqrt{0^2 + (-16)^2}$

$\therefore r = 16$

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{-16}{0}\right)$

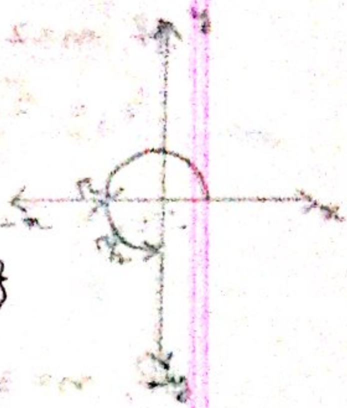
$= \tan^{-1}(\infty)$

$= \tan^{-1}(\tan \pi/2)$

$= \tan^{-1}\{\tan(\pi + \pi/2)\}$

$= \pi + \pi/2$

$\therefore \theta = \frac{3\pi}{2}$



$\therefore (-16i) = r.cis(\arg z + 2n\pi)$

$= 16.cis\left(\frac{3\pi}{2} + 2n\pi\right)$

$\therefore (-16i)^{1/4} = (16)^{1/4}.cis\left(\frac{3\pi}{2} + 2n\pi\right)^{1/4}$

$= 2.cis.\left\{\left(\frac{3\pi}{2} + 2n\pi\right).\frac{1}{4}\right\}$

$\therefore (-16i)^{1/4} = 2.cis.\left\{\frac{3\pi}{8} + \frac{n\pi}{2}\right\}$; $n = 0, 1, 2, 3$

$\therefore$ If, $n = 0$

$\therefore z_1 = 2.cis\left(\frac{3\pi}{8}\right)$

If, $n=1$,

$\therefore z_2 = 2.\text{cis}\left(\frac{3\pi}{8} + \frac{1\pi}{2}\right)$

$\therefore z_2 = 2.\text{cis}\left(\frac{7\pi}{8}\right)$

If, $n=2$,

$\therefore z_3 = 2.\text{cis}\left(\frac{3\pi}{8} + \frac{2\pi}{2}\right)$

$\therefore z_3 = 2.\text{cis}\left(\frac{11\pi}{8}\right)$

If, $n=3$,

$\therefore z_4 = 2.\text{cis}\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right)$

$\therefore z_4 = 2.\text{cis}\left(\frac{15\pi}{8}\right)$

(~~Ans~~)

(iii) $(-1+i)^{1/3}$

Here, $x = -1$

$y = 1$

$\therefore r = \sqrt{(-1)^2 + 1^2}$

$\therefore r = \sqrt{2}$

$\therefore \theta = \tan^{-1}(y/x)$

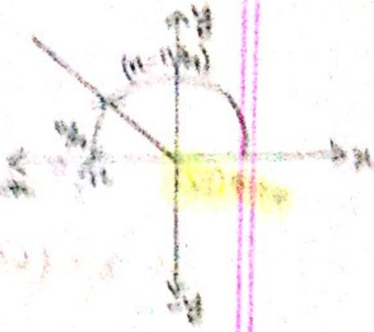
$= \tan^{-1}(-1)$

$= \tan^{-1}\cdot\{\tan(\pi/4)\}$

$= \tan^{-1}\cdot\{\tan(\pi - \pi/4)\}$

$= \pi - \pi/4$

$\therefore \theta = \frac{3\pi}{4}$



$\therefore (-1+i) = r.\text{cis}(\arg z + 2n\pi)$

$= \sqrt{2}.\text{cis}\left(\frac{3\pi}{4} + 2n\pi\right)$

$\therefore (-1+i)^{1/3} = (\sqrt{2})^{1/3}.\text{cis}\left(\frac{3\pi}{4} + 2n\pi\right)^{1/3}$

$= (\sqrt{2})^{1/3}.\text{cis}.\left\{\left(\frac{3\pi}{4} + 2n\pi\right).\frac{1}{3}\right\}$

$\therefore (-1+i)^{1/3} = (\sqrt{2})^{1/3}.\text{cis}\left(\frac{\pi}{4} + \frac{2n\pi}{3}\right)$; $n = 0, 1, 2$

Now,

If, $n = 0$,

$\therefore z_1 = (\sqrt{2})^{1/3}.\text{cis}\left(\frac{\pi}{4}\right)$

If, $n = 1$,

$\therefore z_2 = (\sqrt{2})^{1/3}.\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = (\sqrt{2})^{1/3}.\text{cis}\left(\frac{11\pi}{12}\right)$

If, $n=2$,

$\therefore z_3 = (\sqrt{2})^{1/3} \cdot cis\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)$

$\therefore z_3 = (\sqrt{2})^{1/3} \cdot cis\left(\frac{19\pi}{12}\right)$

IV $\quad z^6 = 64$

$\therefore z = (64)^{1/6}$

Here, $x = 64$

$y = 0$

$\therefore r = \sqrt{(64)^2 + 0}$

$\therefore r = 64$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/64)$

$= \tan^{-1}(0)$

$\therefore \theta = 0$

$\therefore 64 = r \cdot cis(\arg z + 2n\pi)$

$= 64 \cdot cis(0 + 2n\pi)$

$= 64 \cdot cis(2n\pi)$

$\therefore (64)^{1/6} = (64)^{1/6} \cdot cis(2n\pi)^{1/6}$

$= 2 \cdot cis\left(2n\pi \cdot \frac{1}{6}\right)$

$\therefore (64)^{1/6} = 2 \cdot cis\left(\frac{n\pi}{3}\right)$; $n = 0, 1, 2, 3, 4, 5$

if, $n=0$,

$\therefore z_1 = 2.\text{cis}\,0$

if, $n=1$,

$\therefore z_2 = 2.\text{cis}\left(\frac{\pi}{3}\right)$

if, $n=2$,

$\therefore z_3 = 2.\text{cis}\left(\frac{2\pi}{3}\right)$

if, $n=3$,

$\therefore z_4 = 2.\text{cis}\,\pi$.

if, $n=4$,

$\therefore z_5 = 2.\text{cis}\left(\frac{4\pi}{3}\right)$

if, $n=5$,

$\therefore z_6 = 2.\text{cis}\left(\frac{5\pi}{3}\right)$

(V) $(-2\sqrt{3}-2i)^{1/4}$

$\Rightarrow$ Here, $x = -2\sqrt{3}$
$y = -2$

$\therefore r = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$\therefore r = 4$

$\therefore \theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right)$

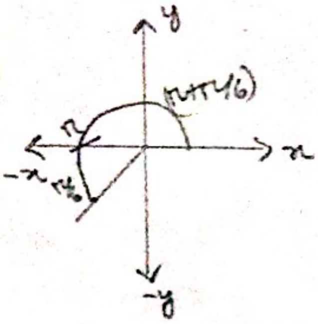
$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \tan^{-1}(\tan \pi/6)$

$= \tan^{-1}\{\tan(\pi + \pi/6)\}$

$= \pi + \pi/6$

$\therefore \theta = \frac{7\pi}{6}$



$\therefore (-2\sqrt{3}-2i) = r.\text{cis}(\arg z + 2n\pi)$

$= 4.\text{cis}\left(\frac{7\pi}{6} + 2n\pi\right)$

$\therefore (-2\sqrt{3}-2i)^{1/4} = (4)^{1/4}.\text{cis}\left(\frac{7\pi}{6} + 2n\pi\right)^{1/4}$

$= (4)^{1/4}.\text{cis}\left\{\left(\frac{7\pi}{6} + 2n\pi\right).\frac{1}{4}\right\}.$

$\therefore (-2\sqrt{3}-2i)^{1/4} = (4)^{1/4}.\text{cis}\left(\frac{7\pi}{24} + \frac{n\pi}{2}\right)$; $n = 0, 1, 2, 3.$

if, $n = 0$,
$\therefore z_1 = (4)^{1/4}.\text{cis}\left(\frac{7\pi}{24}\right)$

if, $n = 1$,
$z_2 = (4)^{1/4}.\text{cis}\left(\frac{7\pi}{24} + \frac{\pi}{2}\right) = (4)^{1/4}.\text{cis}\left(\frac{19\pi}{24}\right)$

if, $n=2$,

$\therefore z_3 = (4)^{1/4} \cdot cis\left(\frac{7\pi}{24} + \pi\right)$

$\therefore z_3 = (4)^{1/4} \cdot cis\left(\frac{31\pi}{24}\right)$

if, $n=3$,

$\therefore z_4 = (4)^{1/4} \cdot cis\left(\frac{7\pi}{24} + \frac{3\pi}{2}\right)$

$\therefore z_4 = (4)^{1/4} \cdot cis\left(\frac{43\pi}{24}\right)$

*** (VI) $(z^4+z^2+1)=0$

Let, $z^2 = x$

$\therefore x^2+x+1=0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1^2-4.1.1}}{2.1}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$\therefore x = \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore z^2 = \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore z = \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^{1/2}$

Now, Part 1:

$\left(\frac{-1+\sqrt{3}i}{2}\right)^{1/2} = \left\{\left(\frac{-1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)i\right\}^{1/2}$

Here, $x = -1/2$

$y = \sqrt{3}/2$

$\therefore r = \sqrt{(-1/2)^2 + (\sqrt{3}/2)^2}$

$\therefore r = 1$

Now, $\theta = \tan^{-1}(y/x)$

$= \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\{\tan(-\pi/3)\}$

$$= \tan^{-1}.\{\tan(\pi - \pi/3)\}$$

$$= \pi - \pi/3$$

$$\therefore \theta = \frac{2\pi}{3}$$

$$\therefore \left(\frac{-1+\sqrt{3}i}{2}\right) = r.cis(\arg z + 2n\pi)$$

$$= cis\left(\frac{2\pi}{3} + 2n\pi\right)$$

$$\therefore \left(\frac{-1+\sqrt{3}i}{2}\right)^{1/2} = cis.\left(\frac{2\pi}{3} + 2n\pi\right)^{1/2}$$

$$= cis\left\{\left(\frac{2\pi}{3} + 2n\pi\right).\frac{1}{2}\right\}$$

$$\therefore \left(\frac{-1+\sqrt{3}i}{2}\right)^{1/2} = cis.\left(\frac{\pi}{3} + n\pi\right) ; n = 0, 1$$

$\therefore$ if, $n = 0$,

$$z_1 = cis\left(\frac{\pi}{3}\right)$$

$\therefore$ if, $n = 1$,

$$z_2 = cis.\left(\frac{\pi}{3} + \pi\right) = cis\left(\frac{4\pi}{3}\right)$$

Part-2:

$$\left(\frac{-1-\sqrt{3}i}{2}\right)^{1/2} = \left\{\left(-1/2\right) - \left(\frac{\sqrt{3}}{2}\right)i\right\}^{1/2}$$

Here, $x = (-1/2)$

$y = \left(\frac{-\sqrt{3}}{2}\right)$ $\therefore r = \sqrt{(-1/2)^2 + (-\sqrt{3}/2)^2} = 1$

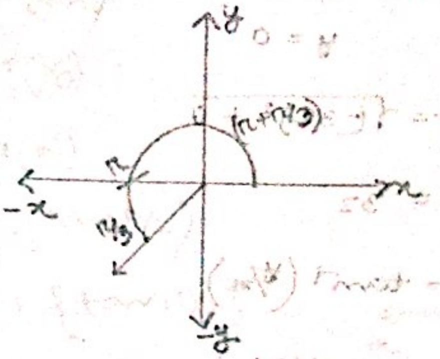
$$\therefore \theta = \tan^{-1}(y/x)$$

$$= \tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$

$$= \tan^{-1}.(\sqrt{3})$$

$$= \tan^{-1}.(\tan \pi/3)$$

$$= \tan^{-1}.\{\tan(\pi + \pi/3)\}$$



$$= \pi + \pi/3$$

$$\therefore \theta = \frac{4\pi}{3}$$

$$\therefore \left(\frac{-1-\sqrt{3}i}{2}\right) = r.\text{cis}.(\arg z + 2n\pi)$$

$$= \text{cis}\left(\frac{4\pi}{3} + 2n\pi\right)$$

$$\therefore \left(\frac{-1-\sqrt{3}i}{2}\right)^{1/2} = \text{cis}.\left(\frac{4\pi}{3} + 2n\pi\right)^{1/2}$$

$$= \text{cis}.\left(\frac{4\pi}{3} + 2n\pi\right).\frac{1}{2}$$

$$\therefore \left(\frac{-1-\sqrt{3}i}{2}\right)^{1/2} = \text{cis}.\left(\frac{2\pi}{3} + n\pi\right) ; \; n = 0, 1$$

$\therefore$ if, $n = 0$,

$$z_3 = \text{cis}\left(\frac{2\pi}{3}\right)$$

$\therefore$ if, $n = 1$,

$$z_4 = \text{cis}.\left(\frac{2\pi}{3} + \pi\right) = \text{cis}\left(\frac{5\pi}{3}\right)$$

(Ans)

(XII) $z^5 = -32$

$\therefore z = (-32)^{1/5}$

Here, $x = -32$

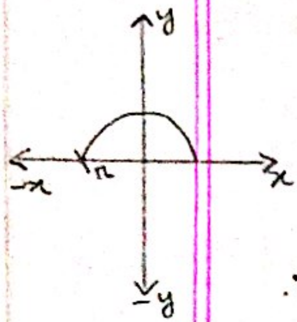
$y = 0$

$\therefore r = \sqrt{(-32)^2 + 0}$

$\therefore r = 32$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-32)$



$= \tan^{-1}(0)$

$= \tan^{-1}.\tan(\pi)$

$\therefore \theta = \pi$

$\therefore (-32) = r.\text{cis}(\arg z + 2n\pi)$

$= 32.\text{cis}(\pi + 2n\pi)$

$\therefore (-32)^{1/5} = (32)^{1/5}.\text{cis}(\pi + 2n\pi)^{1/5}$

$= 2.\text{cis}.\{(\pi + 2n\pi).\frac{1}{5}\}$

$\therefore (-32)^{1/5} = 2.\text{cis}\left(\frac{\pi}{5} + \frac{2n\pi}{5}\right); n = 0, 1, 2, 3, 4$

$\therefore$ If, $n = 0$,

$\therefore z_1 = 2.\text{cis}\left(\frac{\pi}{5}\right)$

If, $n = 1$,

$z_2 = 2.\text{cis}\left(\frac{\pi}{5} + \frac{2\pi}{5}\right)$

$\therefore z_2 = 2.\text{cis}\left(\frac{3\pi}{5}\right)$

If, $n = 2$,

$z_3 = 2.\text{cis}\left(\frac{\pi}{5} + \frac{4\pi}{5}\right)$

$\therefore z_3 = 2.\text{cis}(\pi)$

If, $n = 3$,

$z_4 = 2.\text{cis}\left(\frac{\pi}{5} + \frac{6\pi}{5}\right)$

$\therefore z_4 = 2.\text{cis}\left(\frac{7\pi}{5}\right)$

If, $n = 4$,

$z_5 = 2.\text{cis}.\left(\frac{\pi}{5} + \frac{8\pi}{5}\right)$

$\therefore z_5 = 2.\text{cis}\left(\frac{9\pi}{5}\right)$

(Ans)

***xvii $(i)^{2/3}$

Here, $x = 0$

$y = 1$

$\therefore r = \sqrt{0^2 + 1^2}$

$\therefore r = 1$

$\therefore \theta = \tan^{-1}(y/x)$

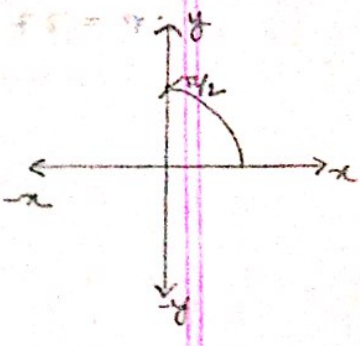
$= \tan^{-1}(1/0)$

$= \tan^{-1}(\infty)$

$= \tan^{-1}.(\tan \pi/2)$

$= \tan^{-1}.\{\tan \pi/2\}$

$\therefore \theta = \pi/2$



$\therefore (i) = r\ cis\,(\arg z + 2n\pi)$

$= cis\,(\pi/2 + 2n\pi)$

$\therefore (i)^{2/3} = cis\,(\pi/2 + 2n\pi)^{2/3}$

$= cis\,\left(\frac{\pi}{2} + 2n\pi\right).\frac{2}{3}$

$\therefore (i)^{2/3} = cis\left(\frac{\pi}{3} + \frac{4n\pi}{3}\right)$; $n = 0, 1, 2$

$\therefore$ If, $n = 0$,

$z_1 = cis\left(\frac{\pi}{3}\right)$

$\therefore$ If, $n = 1$

$z_2 = cis\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) = cis\left(\frac{5\pi}{3}\right)$

$\therefore$ If, $n = 2$,

$z_3 = cis\left(\frac{\pi}{3} + \frac{8\pi}{3}\right) = cis(3\pi)$

(Ans)

*** IX $(-27i)^{1/6}$

Here, $x = 0$

$y = -27$

$\therefore r = \sqrt{0^2 + (-27)^2}$

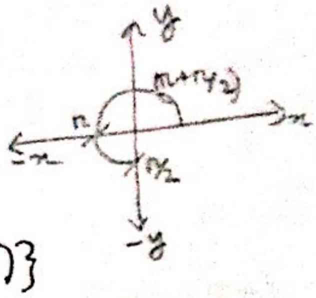
$\therefore r = 27$

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{-27}{0}\right)$

$= \tan^{-1}(\alpha)$

$= \tan^{-1}.\left(\tan \pi/2\right)$



$= \tan^{-1}.\{\tan(\pi+\pi/2)\}$

$= (\pi+\pi/2)$

$\therefore \theta = \frac{3\pi}{2}$

$\therefore (-27i) = r.\operatorname{cis}(\arg z + 2n\pi)$

$= 27.\operatorname{cis}\left(\frac{3\pi}{2} + 2n\pi\right)$

$\therefore (-27i)^{1/6} = (27)^{1/6}.\operatorname{cis}.\left(\frac{3\pi}{2} + 2n\pi\right)^{1/6}$

$= \sqrt{3}.\operatorname{cis}.\left(\frac{3\pi}{2}. + 2n\pi\right).\frac{1}{6}$

$\therefore (-27i)^{1/6} = \sqrt{3}.\operatorname{cis}.\left(\frac{\pi}{4} + \frac{n\pi}{3}\right); \; n = 0, 1, 2, 3, 4, 5$

[illegible], $n=0$,

$\therefore z_0 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}\right)$

[illegible], $n=1$,

$\therefore z_1 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$\therefore z_1 = \sqrt{3}\cdot cis\left(\frac{7\pi}{12}\right)$

[illegible], $n=2$,

$\therefore z_2 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$

$\therefore z_2 = \sqrt{3}\cdot cis\left(\frac{11\pi}{12}\right)$

[illegible], $n=3$,

$\therefore z_3 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}+\pi\right)$

$\therefore z_3 = \sqrt{3}\cdot cis\left(\frac{5\pi}{4}\right)$

[illegible], $n=4$,

$\therefore z_4 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)$

$\therefore z_4 = \sqrt{3}\cdot cis\left(\frac{19\pi}{12}\right)$

[illegible], $n=5$,

$\therefore z_5 = \sqrt{3}\cdot cis\left(\frac{\pi}{4}+\frac{5\pi}{3}\right)$

$\therefore z_5 = \sqrt{3}\cdot cis\left(\frac{23\pi}{12}\right)$

(Ans)

***8 Find square roots of:

(a) $5-12i$ (b) $8+4\sqrt{5}\,i$

(a) $5-12i$

Let, $(p+qi)$, where p and q are real, represent the required square roots,

$\therefore (p+qi)^2 = 5-12i$

$\Rightarrow p^2+2pqi+q^2i^2 = 5-12i$

$\therefore (p^2-q^2)+2pqi = 5-12i$

$\therefore p^2-q^2=5$ ——— (i)

$\therefore 2pq=-12$ ——— (ii)

$\therefore$ (ii) $\Rightarrow$ $pq=-6$

$\therefore q=\left(\frac{-6}{p}\right)$ ——— (iii)

Now, (i) $\Rightarrow$

$$p^2-\left(\frac{-6}{p}\right)^2=5$$

$\Rightarrow p^2-\frac{36}{p^2}=5$

$\Rightarrow p^4-36=5p^2$

$\Rightarrow p^4-5p^2-36=0$

$\Rightarrow p^4-9p^2+4p^2-36=0$

$\Rightarrow p^2\cdot(p^2-9)+4\cdot(p^2-9)=0$

$\Rightarrow (p^2-9)\cdot(p^2+4)=0$

$\therefore p^2=9$

$\therefore p=\pm 3$

$\therefore p^2=-4$

$\therefore p=\sqrt{-4}$, but p is a real number. so th won't be allowed.

$\therefore$ when, $p=3$

(iii) $\Rightarrow$ $q=-2$

$\therefore$ when, $p=-3$

(iii) $\Rightarrow$ $q=2$

$\therefore$ Two roots are, $(3-2i)$, $(-3+2i)$

(Ans)

---

(4) $8+4\sqrt{5}i$

Let, $(p+iq)$, where p and q are real, represent the required square roots,

$\therefore (p+iq)^2 = 8+4\sqrt{5}i$

$\Rightarrow p^2+2pq.i+q^2.i^2 = 8+4\sqrt{5}i$

$\Rightarrow (p^2-q^2)+2pq.i = 8+4\sqrt{5}i$

$\therefore (p^2-q^2) = 8$ —— (i)

$\therefore 2pq = 4\sqrt{5}$

$\therefore pq = 2\sqrt{5}$

$\therefore q = \left(\frac{2\sqrt{5}}{p}\right)$ —— (ii)

$\therefore$ (i) $\Rightarrow$ $p^2-\left(\frac{2\sqrt{5}}{p}\right)^2 = 8$

$\Rightarrow p^2-\frac{20}{p^2} = 8$

$\Rightarrow p^4-20 = 8p^2$

$\Rightarrow p^4-8p^2-20=0$

$\Rightarrow p^4-10p^2+2p^2-20=0$

$\Rightarrow p^2(p^2-10)+2.(p^2-10)=0$

$\Rightarrow (p^2-10).(p^2+2)=0$

$\therefore p^2=10$

$\therefore p=\pm\sqrt{10}$

$\therefore p^2=-2$

$\therefore p=\sqrt{-2}$;

but p is a real number

$\therefore$ this won't be allowed.

$\therefore$ when, $p=\sqrt{10}$

(ii) $\Rightarrow$ $q=\left(\frac{2\sqrt{5}}{\sqrt{10}}\right) = \frac{2\sqrt{5}}{\sqrt{5}.\sqrt{2}} = \sqrt{2}$

$\therefore$ when, $p=-5$

(ii) $\Rightarrow$ $q=\left(\frac{2\sqrt{5}}{-\sqrt{10}}\right) = -\sqrt{2}$

$\therefore$ Two roots are,

$(\sqrt{10}+\sqrt{2}i)$, $(-\sqrt{10}-\sqrt{2}i)$

(Ans)

***(XI) Solve the equations:

***(a) $z^4 + 81 = 0$

$\Rightarrow z^4 = -81$

$\therefore z = (-81)^{1/4}$

Here, $x = -81$

$y = 0$

$\therefore r = \sqrt{(-81)^2 + 0}$

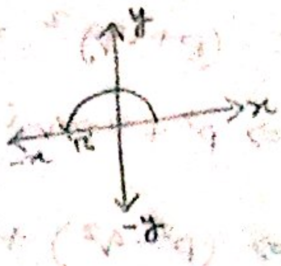
$\therefore r = 81$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-81)$

$= \tan^{-1}(\tan\pi)$

$\therefore \theta = \pi$



$\therefore (-81) = r.\text{cis}.(\arg z + 2n\pi)$

$= 81.\text{cis}(\pi + 2n\pi)$

$\therefore (-81)^{1/4} = (81)^{1/4}.\text{cis}(\pi + 2n\pi)^{1/4}$

$= 3.\text{cis}.(\pi + 2n\pi).\frac{1}{4}$

$\therefore (-81)^{1/4} = 3.\text{cis}.\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$; $n = 0, 1, 2, 3$

If, $n = 0$,

$z_1 = 3.\text{cis}\left(\frac{\pi}{4}\right)$

If, $n = 1$,

$z_2 = 3.\text{cis}.\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = 3.\text{cis}\left(\frac{3\pi}{4}\right)$

If, $n=2$,

$\therefore z_3 = 3 \cdot \text{cis}\left(\frac{\pi}{4}+\pi\right)$

$\therefore z_3 = 3 \cdot \text{cis}\left(\frac{5\pi}{4}\right)$

If, $n=3$,

$\therefore z_4 = 3 \cdot \text{cis} \cdot \left(\frac{\pi}{4}+\frac{3\pi}{2}\right)$

$\therefore z_4 = 3 \cdot \text{cis}\left(\frac{7\pi}{4}\right)$

(Ans)

(b) $z^6+1=\sqrt{3}i$

$\Rightarrow z^6 = -1+\sqrt{3}i$

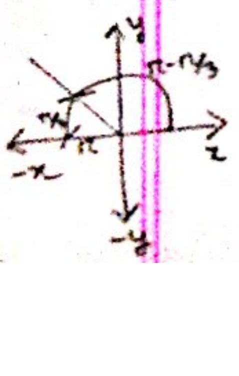
$\therefore z = (-1+\sqrt{3}i)^{1/6}$

Here, $x=-1$
$y=\sqrt{3}$

$\therefore r = \sqrt{1^2+(\sqrt{3})^2}$

$\therefore r = 2$

$\therefore \theta = \tan^{-1}(y/x)$



$= \tan^{-1}(\sqrt{3})$

$= \tan^{-1}(\tan(\pi/3))$

$= \tan^{-1}\cdot\{\tan(\pi-\pi/3)\}$

$= \pi - \pi/3$

$\therefore \theta = \frac{2\pi}{3}$

$\therefore (-1+\sqrt{3}i) = r\,\text{cis}(\arg z + 2n\pi)$

$= 2\,\text{cis}\left(\frac{2\pi}{3}+2n\pi\right)$

$\therefore (-1+\sqrt{3}i)^{1/6} = (2)^{1/6} \cdot \text{cis} \cdot \left(\frac{2\pi}{3}+2n\pi\right)^{1/6}$

$= (2)^{1/6} \cdot \text{cis}\left(\frac{2\pi}{3}+2n\pi\right) \cdot \frac{1}{6}$

$\therefore (-1+\sqrt{3}i)^{1/6} = (2)^{1/6} \text{cis}\left(\frac{\pi}{9}+\frac{n\pi}{3}\right)$; $n = 0,1,2,$ $4,5$

$\therefore$ if, $n=0$,

$\therefore z_1 = 2^{1/6} \cdot cis\left(\frac{\pi}{9}\right)$

if, $n=1$,

$\therefore z_2 = 2^{1/6} \cdot cis\left(\frac{\pi}{9} + \frac{\pi}{3}\right)$

$\therefore z_2 = 2^{1/6} \cdot cis\left(\frac{4\pi}{9}\right)$

if, $n=2$,

$\therefore z_3 = 2^{1/6} \cdot cis\left(\frac{\pi}{9} + \frac{2\pi}{3}\right)$

$\therefore z_3 = 2^{1/6} \cdot cis\left(\frac{7\pi}{9}\right)$

if, $n=3$,

$\therefore z_4 = 2^{1/6} \cdot cis\left(\frac{\pi}{9} + \pi\right)$

$\therefore z_4 = 2^{1/6} \cdot cis\left(\frac{10\pi}{9}\right)$

if, $n=4$,

$\therefore z_5 = 2^{1/6} \cdot cis\left(\frac{\pi}{9} + \frac{4\pi}{3}\right)$

$\therefore z_5 = 2^{1/6} \cdot cis\left(\frac{13\pi}{9}\right)$

if, $n=5$,

$\therefore z_6 = 2^{1/6} \cdot cis\left(\frac{\pi}{9} + \frac{5\pi}{3}\right)$

$\therefore z_6 = 2^{1/6} \cdot cis\left(\frac{16\pi}{9}\right)$

(Ans)

Equation of circle:

Let, $z = (x + iy)$ and $z_0 = (x_0 + iy_0)$

$\therefore (z - z_0) = (x - x_0) + i(y - y_0)$

$\therefore |z - z_0| = |(x - x_0) + i.(y - y_0)|$

$= \sqrt{(x - x_0)^2 + (y - y_0)^2}$

Let,
$|z - z_0| = R$

$\Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = R$

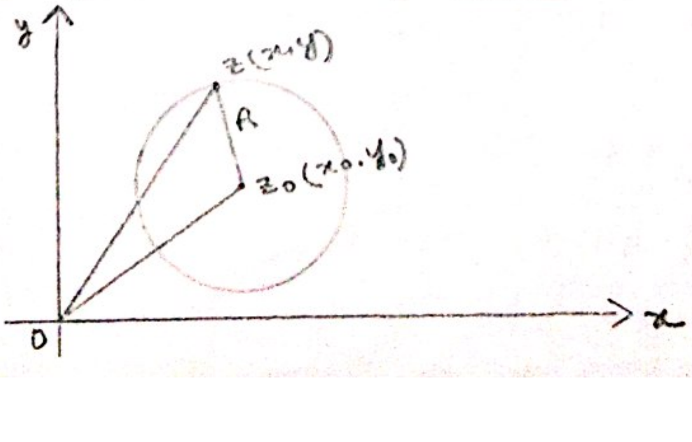
$\therefore R^2 = (x - x_0)^2 + (y - y_0)^2$

$\therefore$ $\boxed{|z - z_0| = R}$ is an equation of circle in complex form with centre at z_0.

with centre origin, $|z| = R$

If the circle has centre z_0, then the parametric representation of circle equation is, $\boxed{z = R.e^{i\theta}}$ $[0 \le \theta \le 2\pi]$

If the circle has centre z_0, then $\boxed{z = z_0 + R.e^{i\theta}}$



Part-B

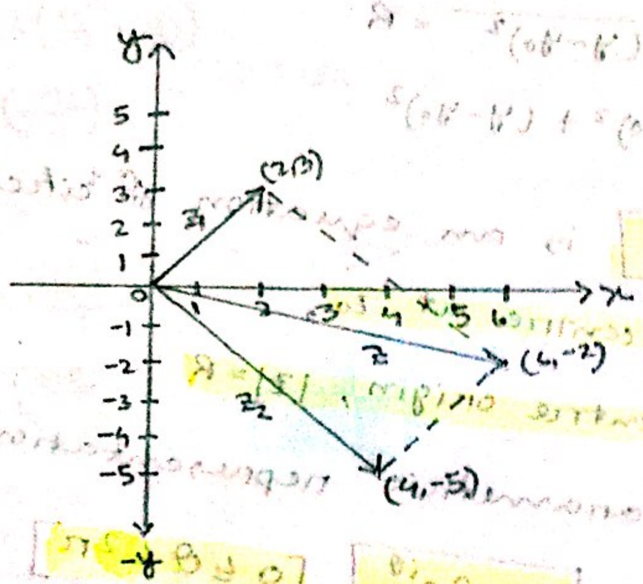
① Perform the indicated operations analytically and graphically:

ⓐ $(2+3i)+(4-5i)$

Let, $z_1=(2+3i)$

$z_2=(4-5i)$

$\therefore (z_1+z_2) = 2+3i+4-5i = (6-2i) = z$



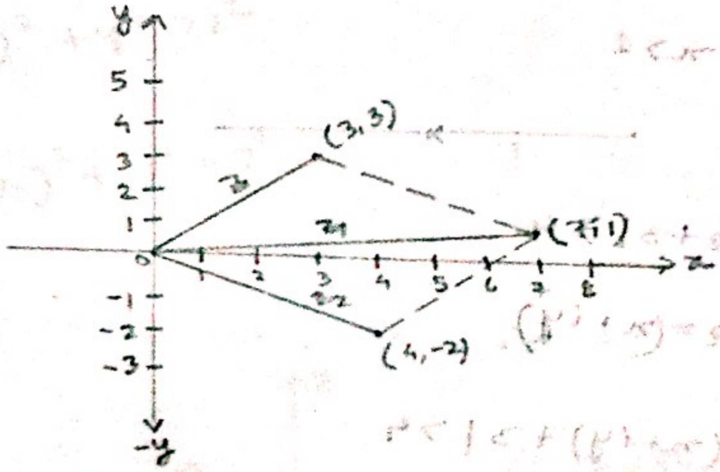
(b) $(7+i)-(4-2i)$

$\Rightarrow$ Let, $z_1 = (7+i)$

$z_2 = (4-2i)$

$\therefore z = (z_1 - z_2) = 7+i-4+2i$

$\therefore z = (3+3i)$

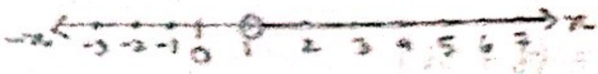


② Describe the conditions geometrically:

ⓐ $Re(z) > 1$

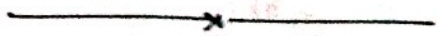
Now, $z = (x+iy)$

$\therefore Re(z) = x$



$\therefore Re(z) > 1$

$\Rightarrow x > 1$



**ⓑ $|2z+3| > 4$

Here, $z = (x+iy)$

$\therefore |2(x+iy)+3| > 4$

$\Rightarrow |2x+2iy+3| > 4$

$\Rightarrow |2x+3+2iy| > 4$

$\Rightarrow \sqrt{(2x+3)^2 + (2y)^2} > 4$

$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$

$\Rightarrow 4x^2 + 12x + 4y^2 > 16 - 9$

$\Rightarrow x^2 + 3x + y^2 > 7/4$

$\Rightarrow \quad x^2 + 2 \cdot \frac{1}{2} \cdot 3x + \left(\frac{3}{2}\right)^2 + y^2 > 7/4 + (3/2)^2$

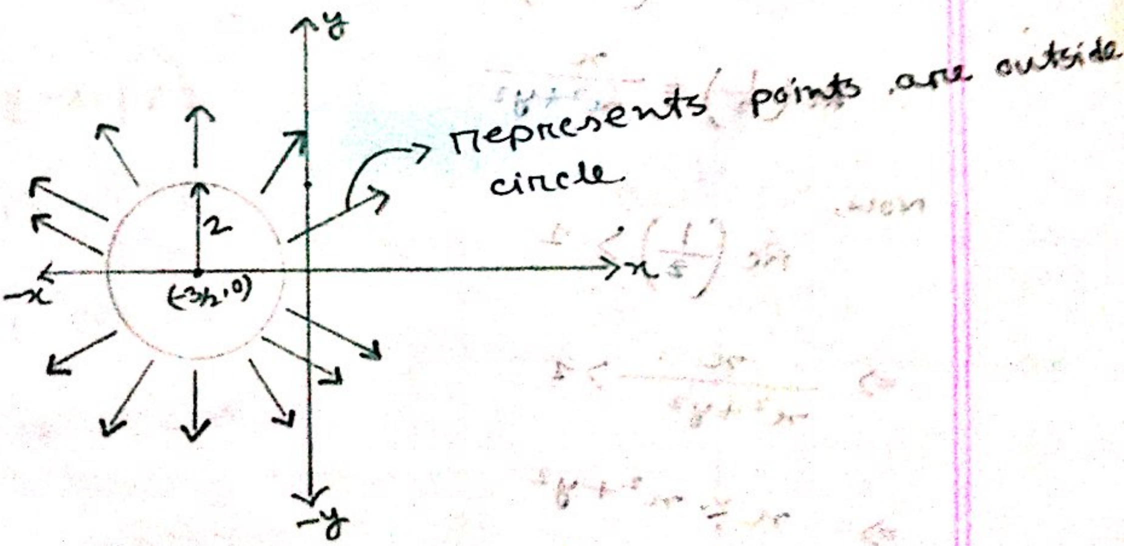
$\Rightarrow \quad (x + 3/2)^2 + y^2 > 7/4 + 9/4$

$\Rightarrow \quad (x + 3/2)^2 + y^2 > \frac{16}{4}$

$\Rightarrow \quad (x + 3/2)^2 + y^2 > 4$

$\Rightarrow \quad (x + 3/2)^2 + y^2 > 2^2$

Now, $(x + 3/2)^2 + y^2 = 2^2$



*** © $Re\left(\frac{1}{z}\right) > 1$

Now,

$$z = x + iy$$

$$\Rightarrow \frac{1}{z} = \frac{1}{x+iy}$$

$$\Rightarrow \frac{1}{z} = \frac{x-iy}{(x+iy)\cdot(x-iy)}$$

$$= \frac{x-iy}{x^2 - i^2 y^2}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$\therefore \frac{1}{z} = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$$

$$\therefore Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$

Now,

$$Re\left(\frac{1}{z}\right) > 1$$

$$\Rightarrow \frac{x}{x^2+y^2} > 1$$

$$\Rightarrow x > x^2+y^2$$

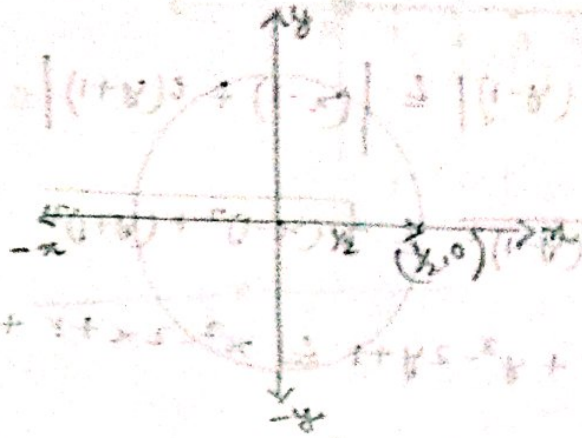
$$\Rightarrow x^2+y^2 - x < 1$$

$$\Rightarrow x^2 - 2\cdot\frac{1}{2}\cdot x + \left(\frac{1}{2}\right)^2 + y^2 < \frac{1}{4}$$

$$\Rightarrow (x - 1/2)^2 + y^2 < (1/2)^2$$

Now,

$$(x-\tfrac{1}{2})^2+y^2=(\tfrac{1}{2})^2$$



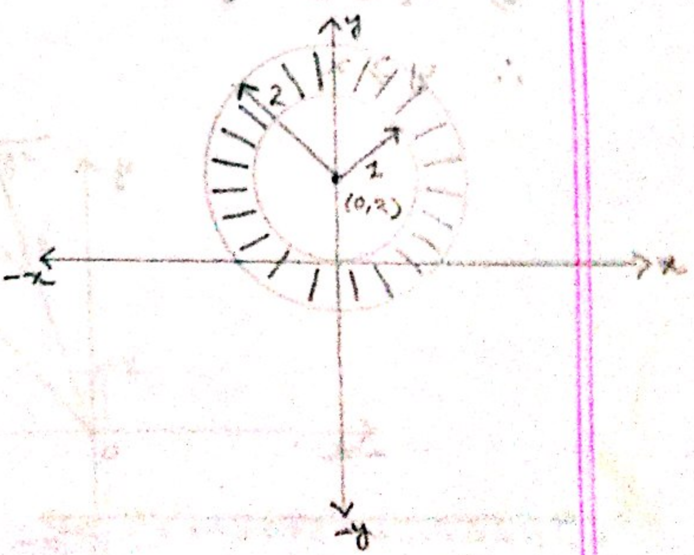
(d) $1<|z-2i|<2$

$\Rightarrow 1<|x+iy-2i|<2$

$\Rightarrow 1<|x+i(y-2)|<2$

$\Rightarrow 1<\sqrt{x^2+(y-2)^2}<2$

$\Rightarrow 1^2<x^2+(y-2)^2<2^2$



Now,

$1^2=x^2+(y-2)^2$

and

$2^2=x^2+(y-2)^2$

(Ans)

*** ⓒ $|z+1-i| \le |z-1+i|$

$\Rightarrow |x+iy+1-i| \le |x+iy-1+i|$

$\Rightarrow |(x+1)+i(y-1)| \le |(x-1)+i(y+1)|$

$\Rightarrow \sqrt{(x+1)^2+(y-1)^2} \le \sqrt{(x-1)^2+(y+1)^2}$

$\Rightarrow \cancel{x^2}+2x+\cancel{1}+\cancel{y^2}-2y+\cancel{1} \le \cancel{x^2}-2x+\cancel{1}+\cancel{y^2}+2y+\cancel{1}$

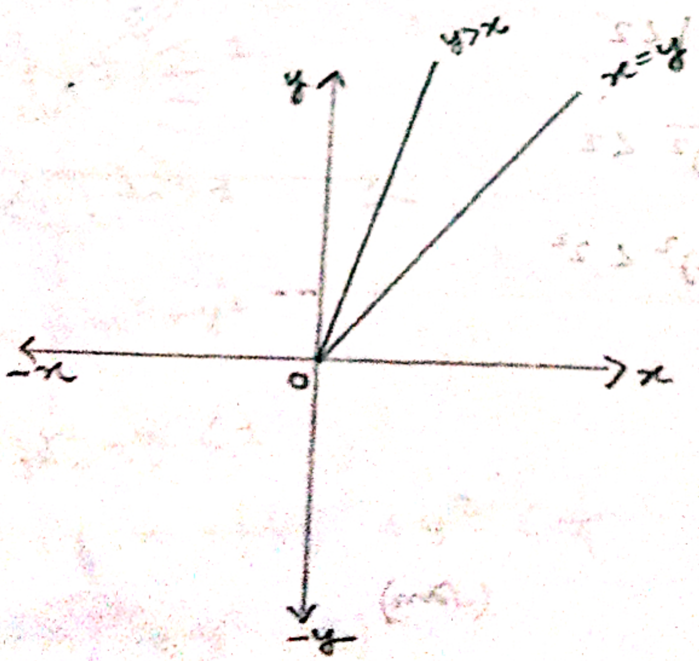
$\Rightarrow 2(x-y) \le 2(y-x)$

$\Rightarrow x-y \le y-x$

$\Rightarrow x+x \le y+y$

$\Rightarrow 2x \le 2y$

$\therefore y \geqslant x$



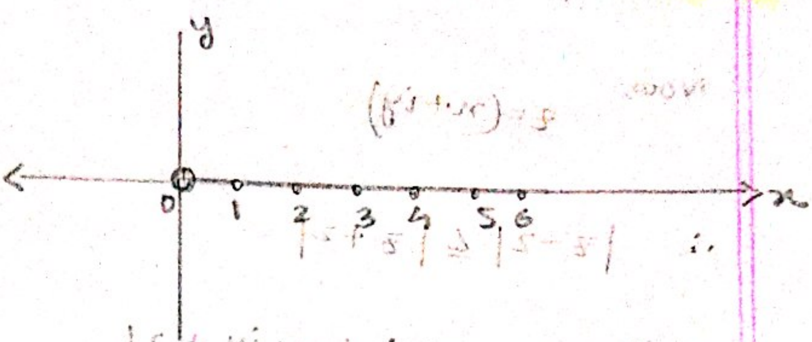
(f) $R(z) \geq 0$

Now, $z = x + iy$

$\therefore Re(z) = x$

$\therefore Re(z) \geq 0$

$\therefore x \geq 0$



***(g) $|z-4| \geq |z|$

Now, $z = x + iy$

$\therefore |z-4| \geq |z|$

$\Rightarrow |x+iy-4| \geq |x+iy|$

$\Rightarrow |x-4+iy| \geq |x+iy|$

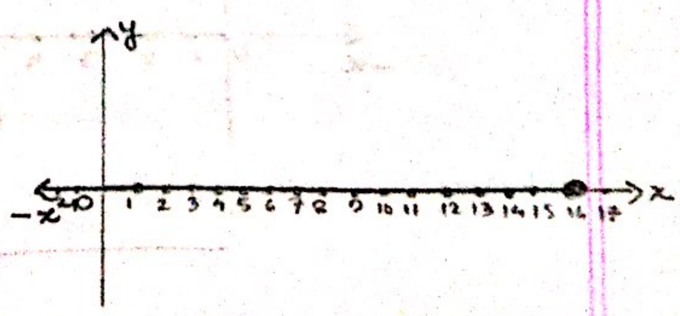
$\Rightarrow \sqrt{(x-4)^2+(y)^2} \geq \sqrt{x^2+y^2}$

$\Rightarrow x^2-8x+16+y^2 \geq x^2+y^2$

$\Rightarrow -8x+16 \geq 0$

$\Rightarrow 8x \leq 16$

$\therefore x \leq 2$



***(2) $|z-2| \le |z+2|$

Now $z=(x+iy)$

$\therefore \quad |z-2| \le |z+2|$

$\Rightarrow |x+iy-2| \le |x+iy+2|$

$\Rightarrow |(x-2)+iy| \le |(x+2)+iy|$

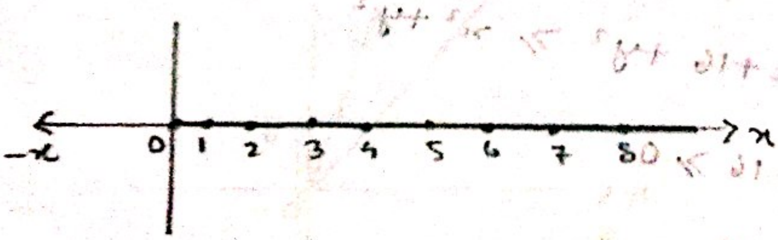
$\Rightarrow \sqrt{(x-2)^2+y^2} \le \sqrt{(x+2)^2+y^2}$

$\Rightarrow x^2-4x+4+y^2 \le x^2+4x+4+y^2$

$\Rightarrow -4x \le 4x$

$\Rightarrow 8x \ge 0$

$\therefore x \ge 0$



***(i) $Re\left(\frac{1}{z}\right) \leq (1/2)$

$\Rightarrow$ Here, $z = (x+iy)$

Now, $\frac{1}{z} = \frac{1}{x+iy}$

$= \frac{x-iy}{(x+iy).(x-iy)}$

$= \frac{x-iy}{x^2-i^2y^2}$

$= \frac{x-iy}{x^2+y^2}$

$\therefore \frac{1}{z} = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$

$\therefore Re\left(\frac{1}{z}\right) = \left(\frac{x}{x^2+y^2}\right)$

$\therefore Re\left(\frac{1}{z}\right) \leq 1/2$

$\Rightarrow \frac{x}{x^2+y^2} \leq 1/2$

$\Rightarrow x \leq 1/2 . x^2 + 1/2 y^2$

$\Rightarrow \frac{1}{2}.x^2 - x + 1/2 . y^2 \geq 0$

$\Rightarrow \frac{1}{2}.(x^2-2x+y^2) \geq 0$

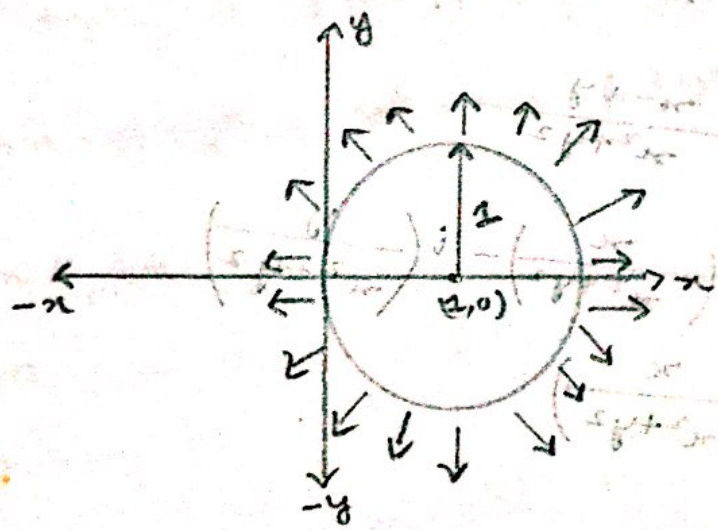
$$\Rightarrow \quad x^2 - 2x + y^2 \geq 0$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 \geq 0 + 1$$

$$\Rightarrow \quad (x-1)^2 + y^2 \geq 1^2$$

$$\therefore \quad (x-1)^2 + y^2 \geq 1^2$$

Now, $(x-1)^2 + y^2 = 1^2$



(j) $\pi/2 < \arg z < \frac{3\pi}{2}$, $|z| > 2$

$\Rightarrow$ Here,

$z = (x+iy)$

$\therefore$ $|z| > 2$

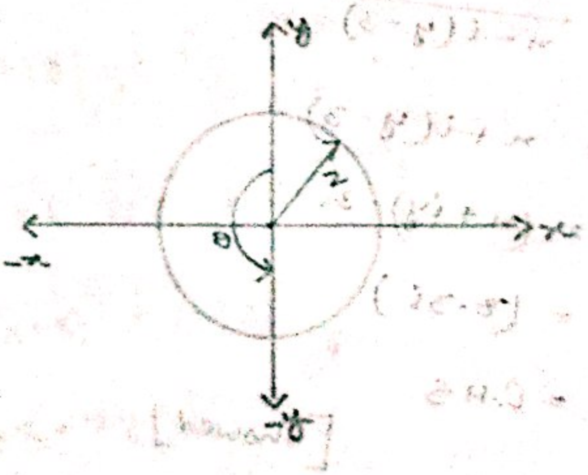
$\Rightarrow$ $|x+iy| > 2$

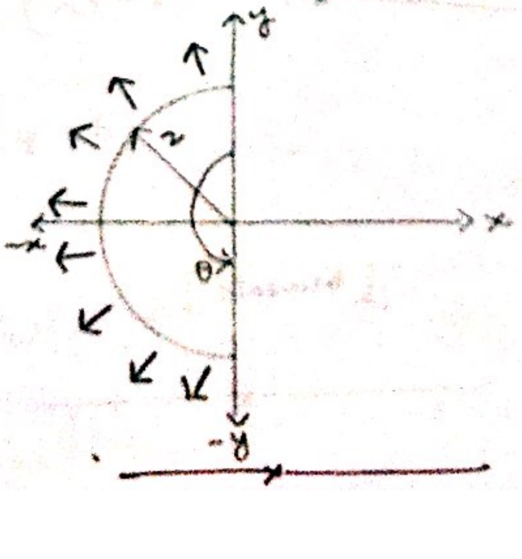
$\Rightarrow$ $\sqrt{x^2+y^2} > 2$

$\therefore$ $(x^2+y^2) > 2^2$

$\pi/2 < \arg z < \frac{3\pi}{2}$

$\therefore$ $90^\circ < \arg z < 270^\circ$





***③ ***ⓘ $\overline{\bar{z}+3i} = (z-3i)$

⇒ Here,

$z = (x+iy)$

$\therefore$ L.H.S $= \overline{\bar{z}+3i}$

$= \bar{\bar{z}} + \overline{3i}$

$= \overline{\overline{x+iy} + 3i}$

$= \overline{x-iy+3i}$

$= \overline{x-i(y-3)}$

$= x+i(y-3)$

$= (x+iy)-3i$

$= (z-3i)$

$=$ R.H.S

[showed]

***(ii) $\left|(2\bar{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}.\left|2z+5\right|$

$\Rightarrow$ Here, $z=(x+iy)$

$\therefore$ L.H.S $= \left|\{2.\overline{(x+iy)}+5\}(\sqrt{2}-i)\right|$

$= \left|\{2(x-iy)+5\}.(\sqrt{2}-i)\right|$

$= \left|(2x-2iy+5).(\sqrt{2}-i)\right|$

$= \left|(2x+5)-2iy\right|.\left|\sqrt{2}-i\right|$

$= \sqrt{(2x+5)^2+(-2y)^2}.\sqrt{(\sqrt{2})^2+1^2}$

$= \sqrt{3}.\sqrt{(2x+5)^2+(+2y)^2}$

$= \sqrt{3}.\left|2x+5+2y.i\right|$

$= \sqrt{3}.\left|2.(x+iy)+5\right|$

$= \sqrt{3}.\left|2.z+5\right|$

$=$ R.H.S [showed]

***(III) $|2z+3\bar{z}| \leq 4.|Re(z).| + |z|$

$\Rightarrow$ Here, $z=(x+iy)$

Now,

L.H.S $= |2z+3\bar{z}|$

$= \left|2.(x+iy) + 3.\overline{(x+iy)}\right|$

$= \left|2x+2iy + 3.(x-iy)\right|$

$= |2x+2iy+3x-3iy|$

$= |5x-iy|$

$= |4x+x-iy|$

$\leq |4x| + |x-iy|$ $[\because |z_1+z_2| \leq |z_1|+|z_2|]$

$\leq 4|x| + |x-iy|$

$\leq 4.|Re(z)| + |\bar{z}|$

$\leq 4.|Re(z)| + |z|$

$\leq$ R.H.S

[Showed]

***④ Find modulus and argument:

***ⓘ $\dfrac{2-i}{2+i}$

Now, Numerator,

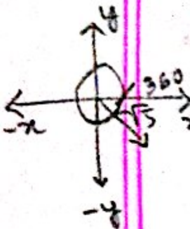
$x=2,\ y=-1$

$r=\sqrt{(2)^2+(-1)^2}$

$\therefore r=\sqrt{5}$

$\therefore \theta=\tan^{-1}(-1/2)$

$=\tan^{-1}\{\tan(-26.57)\}$



$=\tan^{-1}\{\tan(360°-26.57°)\}$

$\therefore \theta = 333.43°$

$\therefore (2-i)=\sqrt{5}.cis(333.43°)$

For denominator,

$x=2,\ y=1$

$\therefore r=\sqrt{2^2+1^2}=\sqrt{5}$

$\therefore \theta=\tan^{-1}(1/2)$

$=\tan^{-1}(\tan 26.57°)$

$\therefore \theta = 26.57°$

$\therefore (2+i)=\sqrt{5}.cis(26.57°)$

$\therefore \left(\dfrac{2-i}{2+i}\right)=\dfrac{\sqrt{5}.cis(333.43°)}{\sqrt{5}.cis(26.57°)}$

$=cis(333.43-26.57)$

$=1.cis(307°)$

$\therefore$ modulus $=1$

$\therefore$ argument $=307°$ (Ans)

***(II) $\dfrac{\sqrt{3}+i}{\sqrt{3}-i}$

$\Rightarrow$ Here,

For Numerator

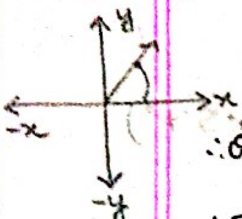
$x = \sqrt{3}$, $y = 1$

$\therefore r = \sqrt{(\sqrt{3})^2 + 1^2}$

$\therefore r = 2$

$\therefore \theta = \tan^{-1}(1/\sqrt{3})$

$= \tan^{-1}(\tan \pi/6)$



$\therefore \theta = (\pi/6) = 30^\circ$

$\therefore (\sqrt{3}+i) = 2.\text{cis}\,30^\circ$

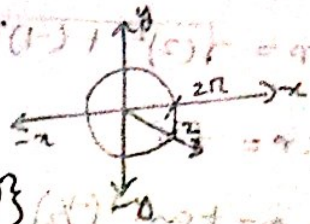
For, denominator,

$x = \sqrt{3}$, $y = -1$

$\therefore r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\therefore \theta = \tan^{-1}(-1/\sqrt{3})$

$= \tan^{-1}\{\tan(-\pi/3)\}$



$= \tan^{-1}\{\tan(2\pi - \pi/3)\}$

$= (2\pi - \pi/3)$

$= \left(\dfrac{6\pi - \pi}{3}\right)$

$\therefore \theta = \left(\dfrac{5\pi}{3}\right) = 300^\circ = (-\pi/3) = -30^\circ$

$\therefore (\sqrt{3}-i) = 2.\text{cis}(-30^\circ)$

$\therefore \left(\dfrac{\sqrt{3}+i}{\sqrt{3}-i}\right) = \dfrac{2.\text{cis}30^\circ}{2.\text{cis}(-30^\circ)}$

$= 1.\text{cis}(30^\circ + 30^\circ)$

$= 1.\text{cis}60^\circ$

$\therefore$ modulus $= 1$

$\therefore$ argument $= 60^\circ$ (Ans)

(III) $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^2$

$\Rightarrow$ Here, for Numerator,

$x=1,\ y=\sqrt{3}$

$\therefore r=\sqrt{1^2+(\sqrt{3})^2}$

$\therefore r=2$

$\therefore \theta=\tan^{-1}(\sqrt{3})$

$=\tan^{-1}(\tan \pi/3)$

$=\pi/3$

$\therefore \theta=60^\circ$

$\therefore (1+\sqrt{3}i)=2.cis(60^\circ)$

for de-nominator,

$x=1,\ y=-\sqrt{3}$

$\therefore r=\sqrt{1^2+(-\sqrt{3})^2}=2$

$\therefore \theta=\tan^{-1}(-\sqrt{3})$

$=\tan^{-1}\{\tan(-\pi/3)\}$

$=-\pi/3$

$\therefore \theta=-60^\circ$

$\therefore (1-\sqrt{3}i)=2.cis(-60^\circ)$

$$\therefore \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^2=\left\{\frac{2.cis(60^\circ)}{2.cis(-60^\circ)}\right\}^2$$

$$=\{cis(60+60)\}^2$$

$$=(cis 120)^2$$

$$=cis(120\times 2)$$

$$=1cis(240^\circ)$$

$\therefore$ modulus $=1$

$\therefore$ argument $=240^\circ$ (Ans)

(5) Prove that $|z-i| = |z+i|$ represent a straight line: (III)

$\Rightarrow$ Here, $z = (x+iy)$

Now,

$\therefore |x+iy-i| = |x+iy+i|$

$\Rightarrow |x+i\cdot(y-1)| = |x+i\cdot(y+1)|$

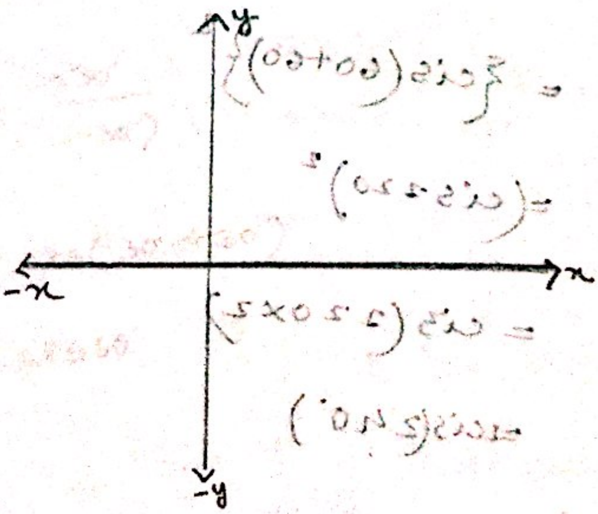
$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$

$\Rightarrow \cancel{x^2} + (y-1)^2 = \cancel{x^2} + (y+1)^2$

$\Rightarrow \cancel{y^2} - 2y + \cancel{1} = \cancel{y^2} + 2y + \cancel{1}$

$\Rightarrow 4y = 0$

$\therefore y = 0$



$\therefore y=0$ or x-axis is a straight line.

[Proved]

***⑥ find an equation of a circle at (2,3) with radius 3.

$\Rightarrow$ Here, we know,

$$|z-z_0| = R$$

$R = 3$ | $x = 2$

$z_0 = (2+3i)$ | $y = 3$

$$\therefore \quad |z - z_0| = 3$$

$$\Rightarrow \quad |z - 2 - 3i| = 3$$

$$\therefore \quad |z-2-3i| = 3$$

(ans)

*** (6) Prove that $|z+2i|+|z-2i|=6$ represents an ellipse.

$\Rightarrow$ Now, $z=(x+iy)$

$|z+2i|+|z-2i|=6$

$\Rightarrow |x+iy+2i|+|x+iy-2i|=6$

$\Rightarrow |x+i.(y+2).|+|x+i.(y-2)|=6$

$\Rightarrow \sqrt{x^2+(y+2)^2}+\sqrt{x^2+(y-2)^2}=6$

$\Rightarrow \left(\sqrt{x^2+(y+2)^2}\right)^2=\left\{6-\sqrt{x^2+(y-2)^2}\right\}^2$

$\Rightarrow \cancel{x^2}+(y+2)^2=36-12.\sqrt{x^2+(y-2)^2}+\cancel{x^2}+(y-2)^2$

$\Rightarrow \cancel{y^2}+4y+4=36-12.\sqrt{x^2+(y-2)^2}+\cancel{y^2}-4y+4$

$\Rightarrow 8y=36-12.\sqrt{x^2+(y-2)^2}$

$\Rightarrow 2y=9-3.\sqrt{x^2+(y-2)^2}$

$\Rightarrow \left\{3.\sqrt{x^2+(y-2)^2}\right\}^2=(9-2y)^2$

$\Rightarrow 9.(x^2+y^2-4y+4)=81-36y+4y^2$

$\Rightarrow 9x^2+9y^2-3\cancel{6}y+36=81-3\cancel{6}y+4y^2$

$\Rightarrow 9x^2 + 5y^2 = 45$

$\therefore \left(\frac{x^2}{5} + \frac{y^2}{9}\right) = 1$

This is an equation of ellipse

[Proved]

***(8) Sketch the region:**

***(i)** $Re(\bar{z}-1) = 2$ $\quad\Big|\quad z = (x+iy)$

Now,

$(\overline{x+iy} - 1)$

$= (x - iy - 1)$

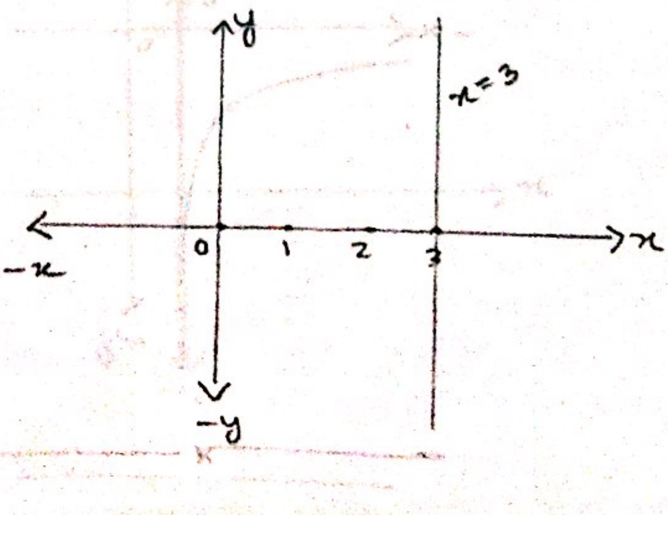
$= (x-1) - iy$

$\therefore Re(\bar{z}-1) = (x-1)$

$\therefore Re(\bar{z}-1) = 2$

$\Rightarrow x - 1 = 2$

$\therefore x = 3$ (Ans)



***(ii) $Im(z^2) = 4$

now,

$z = x + iy$

$\Rightarrow z^2 = (x+iy)^2$

$\Rightarrow z^2 = x^2 + 2ixy + y^2 i^2$

$\therefore z^2 = (x^2 - y^2) + 2xy.i$

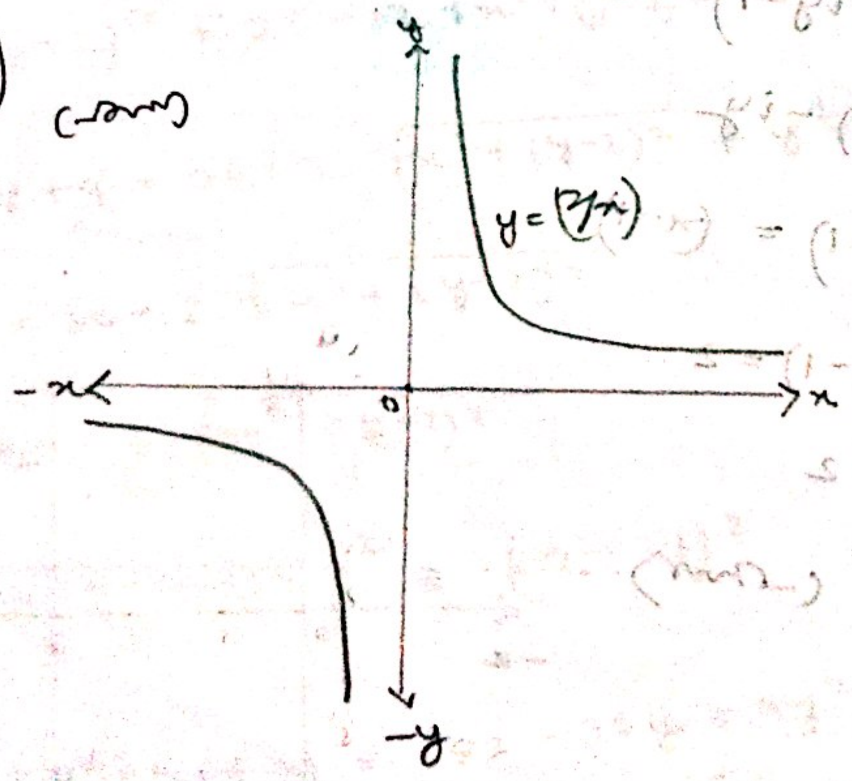
$\therefore Im(z^2) = 2xy$

$\therefore Im(z^2) = 4$

$\Rightarrow 2xy = 4$

$\Rightarrow xy = 2$

$\therefore y = \left(\frac{2}{x}\right)$ (Ans)



***(iii) $\left|\frac{2z-3}{2z+3}\right| = 1$

Here, $z = (x+iy)$

$$\therefore \left|\frac{2(x+iy)-3}{2.(x+iy)+3}\right| = 1$$

$$\Rightarrow \left|\frac{(2x-3)+iy}{(2x+3)+iy}\right| = 1$$

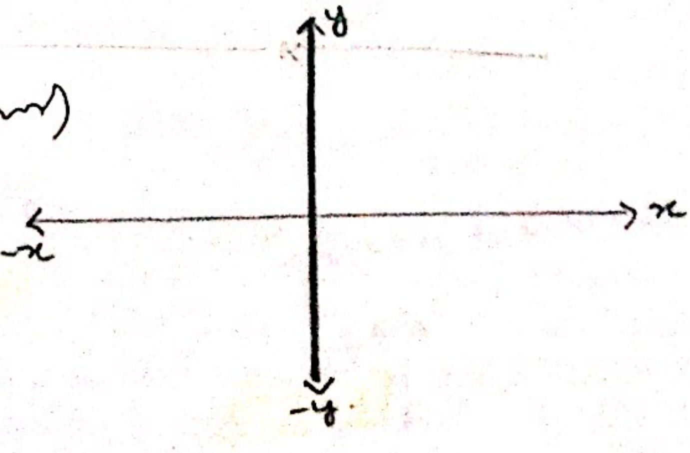
$$\Rightarrow \frac{\sqrt{(2x-3)^2+y^2}}{\sqrt{(2x+3)^2+y^2}} = 1$$

$$\Rightarrow (2x+3)^2 + \cancel{y^2} = (2x-3)^2 + \cancel{y^2}$$

$$\Rightarrow \cancel{4x^2} + 12x + \cancel{9} = \cancel{4x^2} - 12x + \cancel{9}$$

$$\Rightarrow 24x = 0$$

$\therefore x = 0$ (ans)



(iv) $Re(z) + Im(z) = 0$

Now,
$$z = x + iy$$

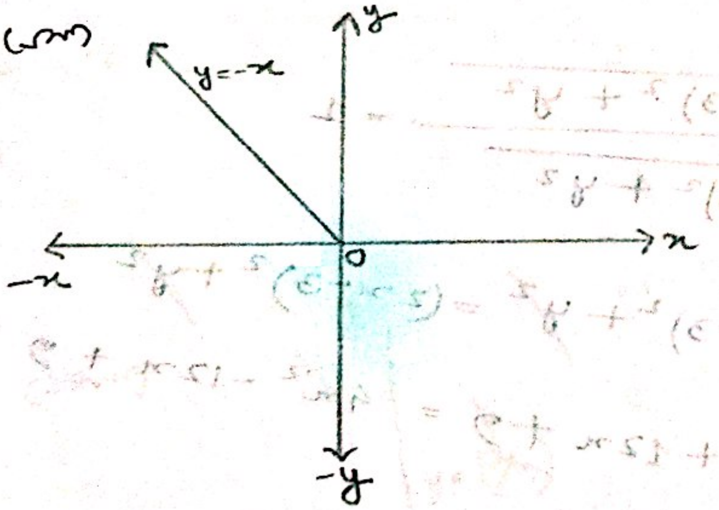
$\therefore Re(z) = x$

$\therefore Im(z) = y$

$\therefore Re(z) + Im(z) = 0$

$\Rightarrow (x+y) = 0$

$\therefore y = -x$ (ans)



P.S → 13

★★★ Exponential functions of complex variable:

$f(z) = e^z = e^{(x+iy)}$

$= e^x \cdot e^{iy}$

$= e^x \cdot (\cos y + i \cdot \sin y)$ $\left[\because (\cos\theta + i.\sin\theta) = e^{i\theta}\right]$

$\therefore |f(z)| = |e^z| = |e^x (\cos y + i \cdot \sin y)|$

$= |e^x| \cdot |\cos y + i \sin y|$

$= e^x \cdot \sqrt{\cos^2 y + \sin^2 y}$

$= e^x \cdot 1$

$\therefore |f(z)| = e^x$

$\therefore |f(z)| = e^x > 0$ for all $x \in \mathbb{R}$.

① $\left(\frac{1}{e^z}\right) = e^{-z}$

② $(e^z)^n = e^{nz}$

★★★③ $e^{z+2\pi i} = e^z \cdot e^{2\pi i}$

$\therefore \boxed{e^{(z+2\pi i)} = e^z}$ $\left[\because e^{2\pi i} = \cos 2\pi + i \cdot \sin 2\pi\right.$

$= 1 + 0$

$\left.\therefore \boxed{e^{2\pi i} = 1}\right]$

***① Show that:

***(i) $e^{(2\pm 3\pi i)} = -e^2$

Now, $e^{2+3\pi i} = e^2 \cdot e^{3\pi i}$

$= e^2 \cdot (\cos 3\pi + i \cdot \sin 3\pi)$

$= e^2 \cdot (-1+0)$

$\therefore e^{(2+3\pi i)} = -e^2$

Also, $e^{(2-3\pi i)} = -e^2$

$\therefore e^{(2\pm 3\pi i)} = -e^2$ [Showed]

***(ii) $e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{\frac{e}{2}} \cdot (1+i)$

Now, $e^{\left(\frac{2+\pi i}{4}\right)} = e^{\left(2/4 + \frac{\pi}{4} \cdot i\right)}$

$= e^{1/2} \cdot e^{(\pi/4)i}$

$= e^{1/2} \cdot (\cos \pi/4 + i \cdot \sin \pi/4)$

$= e^{1/2} \cdot \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{e}}{\sqrt{2}} \cdot (1+i)$

$\therefore e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{\frac{e}{2}} \cdot (1+i)$ [Showed]

***(iii) $e^{(z+\pi i)} = -e^{z}$

Now,

$$e^{z+\pi i} = e^{z}.e^{\pi i}$$

$$= e^{z}.(\cos\pi + i\cdot\sin\pi) \quad [\because e^{i\theta} = \cos\theta + i\cdot\sin\theta]$$

$$= e^{z}.(-1+0)$$

$$\therefore e^{(z+\pi i)} = -e^{z}$$

[Showed]

***

$$z = r.e^{i\theta}$$

$$\therefore \boxed{z = r.e^{i(\theta+2n\pi)}}$$

***(2) find all values of z such that:

***(i) $e^{z} = (1+\sqrt{3}i)$

Now, $x = 1$

$y = \sqrt{3}$

$\therefore r = \sqrt{1^2+(\sqrt{3})^2}$

$\therefore r = 2$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(\sqrt{3})$

$= \tan^{-1}.(\tan\pi/3)$

$\therefore \theta = (\pi/3)$

$\therefore (1+\sqrt{3}i) = r.e^{i(\theta+2n\pi)}$

$= 2.e^{i(\pi/3+2n\pi)}$

$\Rightarrow e^{z} = 2.e^{i(\pi/3+2n\pi)}$

$\Rightarrow \ln.e^{z} = \ln\{2.e^{i(\pi/3+2n\pi)}\}$

$\Rightarrow z = \ln 2 + \ln.e^{i(\pi/3+2n\pi)}$

$\therefore z = \ln 2 + i(\pi/3+2n\pi)$; $n = 0, \pm1, \pm2, \ldots$

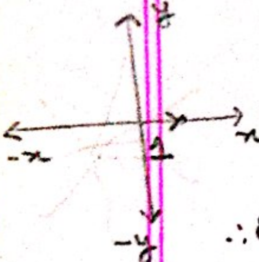
(Ans)

*** (ii) $e^{2z-1} = 1$

$\Rightarrow$ Here, $x = 1$
$y = 0$

$\therefore r = \sqrt{1^2 + 0^2} = 1$

$\therefore \theta = \tan^{-1}(y/x)$



$= \tan^{-1}(0/1)$

$= \tan^{-1}(0)$

$= \tan^{-1}(\tan 0)$

$\therefore \theta = 0$

Now, $1 = r.e^{i(\theta + 2n\pi)}$

$\Rightarrow e^{2z-1} = 1.e^{i(0+2n\pi)}$

$\Rightarrow \ln.e^{2z-1} = \ln.e^{2n\pi.i}$

$\Rightarrow 2z - 1 = 2n\pi i$

$\Rightarrow 2z = 2n\pi i + 1$

$\Rightarrow z = n\pi i + 1/2$

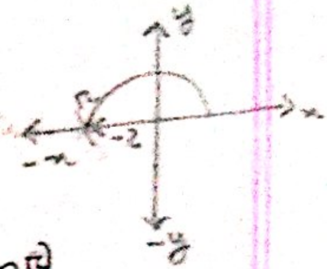
$\therefore z = \left(\frac{1}{2} + in\pi\right)$ (Ans)

*** (iii) $e^{z} = -2$

Here, $x = -2$
$y = 0$

$\therefore r = \sqrt{(-2)^2 + 0^2} = 2$

$\therefore \theta = \tan^{-1}(y/x)$



$= \tan^{-1}(0/-2)$

$= \tan^{-1}(0)$

$= \tan^{-1}.(\tan\pi)$

$\therefore \theta = \pi$

Now, $e^{z} = -2$

$\therefore -2 = r.e^{i(\theta + 2n\pi)}$

$\Rightarrow e^{z} = 2.e^{i(2n\pi + \pi)}$

$\Rightarrow \ln e^{z} = \ln\{2.e^{i(2n\pi+\pi)}\}$

$\Rightarrow z = \ln 2 + \ln e^{(2n\pi+\pi)i}$

$\Rightarrow z = \ln 2 + (2n\pi + \pi)i$

$\therefore z = \ln 2 + (2n+1).\pi i.$

(Ans)

***(iv) $e^z = -1$

$\Rightarrow$ Here, $x = -1$

$y = 0$

$\therefore r = \sqrt{(-1)^2 + 0^2} = 1$

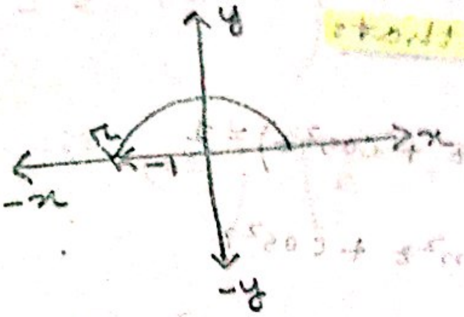
$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-1)$

$= \tan^{-1}(0)$

$= \tan^{-1}(\tan\pi)$

$\therefore \theta = \pi$



Now,

$-1 = r.e^{i(\theta + 2n\pi)}$

$\Rightarrow e^z = 1.e^{i(\pi+2n\pi)}$

$\Rightarrow \ln.e^z = \ln e^{i(\pi+2n\pi)}$

$\Rightarrow z = i(\pi + 2n\pi)$

$\therefore z = (2n+1).\pi i.$

(Ans)

*** Trigonometric Functions:

Here, $e^{iz} = (\cos z + i\sin z)$; $e^{-iz} = (\cos z - i\cdot\sin z)$

$\therefore (e^{iz} + e^{-iz}) = 2\cos z$

$\therefore \boxed{\cos z = \frac{1}{2}\cdot(e^{iz} + e^{-iz})}$

$\therefore (e^{iz} - e^{-iz}) = (2i\cdot\sin z)$

$\therefore \boxed{\sin z = \frac{1}{2i}\cdot(e^{iz} - e^{-iz})}$

*** Prove that:

**① $(\sin^2 z + \cos^2 z) = 1$

$L.H.S = \sin^2 z + \cos^2 z$

$$= \left\{\frac{1}{2i}\cdot(e^{iz} - e^{-iz})\right\}^2 + \left\{\frac{1}{2}\cdot(e^{iz} + e^{-iz})\right\}^2$$

$$= \frac{1}{4i^2}\left(e^{2iz} - 2\cdot e^{iz}\cdot e^{-iz} + e^{-2iz}\right) + \frac{1}{4}\left(e^{2iz} + 2e^{iz}\cdot e^{-iz} + e^{-2iz}\right)$$

$$= \frac{1}{-4}\cdot\left(e^{2iz} - 2 + e^{-2iz}\right) + \frac{1}{4}\cdot\left(e^{2iz} + 2 + e^{-2iz}\right)$$

$$= \frac{1}{4}\cdot\left(e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}\right)$$

$$= \frac{1}{4}\cdot 4$$

$= 1 = R.H.S$ —— [Proved]

*** (ii) $\sin(z+2\pi) = \sin z$

Now,

$$L.H.S = \sin(z+2\pi)$$

$$= \frac{1}{2i} \cdot \left\{ e^{i(z+2\pi)} - e^{-i(z+2\pi)} \right\} \quad \left[\because \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \right]$$

$$= \frac{1}{2i} \cdot \left(e^{iz} \cdot e^{2\pi i} - e^{-iz} \cdot e^{-2\pi i} \right)$$

$$= \frac{1}{2i} \cdot \left(e^{iz} \cdot 1 - e^{-iz} \cdot (+1) \right) \quad \left[\because e^{\pm 2\pi i} = 1 \right]$$

$$= \frac{1}{2i} \cdot \left(e^{iz} - e^{-iz} \right)$$

$$= \sin z$$

$$= R.H.S$$

[Proved]

*** (iii) $\cos(z+2\pi) = \cos z$

Now,
LHS =
$\cos(z+2\pi)$

$= \frac{1}{2}\cdot\left\{e^{i(z+2\pi)} + e^{-i(z+2\pi)}\right\}$

$= \frac{1}{2}\cdot\left(e^{iz}\cdot e^{2\pi i} + e^{-iz}\cdot e^{-2\pi i}\right)$

$= \frac{1}{2}\cdot\left(e^{iz}\cdot 1 + e^{-iz}(1)\right)$ $\left[\because e^{2\pi i} = 1\right]$

$= \frac{1}{2}\cdot\left(e^{iz} + e^{-iz}\right)$

$= \cos z$

$=$ R.H.S

[Proved]

*** Hyperbolic Functions:

① $\boxed{\cosh z = \frac{1}{2}\cdot\left(e^{z}+e^{-z}\right)}$

② $\boxed{\sinh z = \frac{1}{2}\cdot\left(e^{z}-e^{-z}\right)}$

③ $\boxed{\cosh^2 z - \sinh^2 z = 1}$

④ $\frac{d}{dz}\cdot(\sinh z) = \cosh z$

⑤ $\frac{d}{dz}\cdot(\cosh z) = \sinh z$

*** Prove that:

***(i) $\sin z = (\sin x\cdot\cosh y + i\cos x\cdot\sinh y)$

***(ii) $\cos z = (\cos x\cdot\cosh y - i\cdot\sin x\cdot\sinh y)$

Now, $\sin z = \frac{1}{2i}\cdot\left(e^{iz}-e^{-iz}\right)$ —— (I)

$\sinh z = \frac{1}{2}\cdot\left(e^{z}-e^{-z}\right)$ —— (II)

$\cos z = \frac{1}{2}\cdot\left(e^{iz}+e^{-iz}\right)$ —— (III)

$\cosh z = \frac{1}{2}\cdot\left(e^{z}+e^{-z}\right)$ —— (IV)

From (I),
$$\sin iz = \frac{1}{2i}\cdot\left(e^{i.iz}-e^{-i.iz}\right)$$
$$= \frac{+1}{2i}\cdot\left(e^{i^2z}-e^{-i^2z}\right)$$
$$= \frac{1}{2i}\cdot\left(e^{-z}-e^{z}\right)$$
$$= \frac{-1}{2i}\cdot\left(e^{z}-e^{-z}\right)$$

From (III),
$$\cos iz = \frac{1}{2}\cdot\left(e^{iiz}+e^{-i.iz}\right)$$
$$= \frac{1}{2}\cdot\left(e^{i^2z}+e^{-i^2.z}\right)$$
$$= \frac{1}{2}\cdot\left(e^{-z}+e^{+z}\right)$$
$$= \frac{1}{2}\cdot\left(e^{z}+e^{-z}\right)$$

$$= \frac{-i}{2i^2}\cdot(e^z - e^{-z})$$

$$= \frac{-i}{-2}\cdot(e^z - e^{-z})$$

$$= i\cdot\frac{1}{2}\cdot(e^z - e^{-z})$$

$\therefore \boxed{\sin iz = i\cdot\sinh z}$

$\therefore \boxed{\cos iz = \cosh z}$

Now,

(i) $\Rightarrow$ $L.H.S = \sin z$

$= \sin(x+iy)$

$= \sin x\cdot\cos iy + \cos x\cdot\sin iy$

$= \sin x\cdot\cosh y + \cos x\cdot i\cdot\sinh y$ $\left[\begin{array}{l}\cos iz = \cosh z\\ \sin iz = i\cdot\sinh z\end{array}\right]$

$= \sin x\cdot\cosh y + i\cdot\cos x\cdot\sinh y$

$= R.H.S$

[Proved]

(ii) $\Rightarrow$ $L.H.S = \cos z$

$= \cos(x+iy)$

$= \cos x\cdot\cos iy - \sin x\cdot\sin iy$

$= \cos x\cdot\cosh y - \sin x\cdot i\cdot\sinh y$

$= \cos x\cdot\cosh y - i\cdot\sin x\cdot\sinh y$

$= R.H.S$ [Proved]

Prove that:

***(i) $\sinh z = \sinh x.\cos y + i.\cosh x.\sin y$.

Now, L.H.S = $\sinh z$

$= -i\sin iz$

$= -i.\sin i(x+iy)$

$= -i.\sin(ix-y)$

$= -i.(\sin ix\cos y - \cos ix\sin y)$

$= -i.\sin ix.\cos y + i.\cos ix.\sin y$

$= \sinh x.\cos y + i.\cosh x.\sin y$

= R.H.S [Proved]

$i\sinh z = \sin iz$

$\Rightarrow i^2\sinh z = .i.\sin iz$

$\Rightarrow -\sinh z = i.\sin iz$

$\therefore \sinh z = -i.\sin iz$

***(ii) $\cosh z = \cosh x.\cos y + i.\sinh x.\sin y$

Now,

L.H.S = $\cosh z$

$= \cos iz$

$= \cos i(x+iy)$

$= \cos(ix-y)$

$= \cos ix.\cos y + \sin ix.\sin y$

$= \cosh x.\cos y + i.\sinh x.\sin y$

= R.H.S [Proved]

*** (iii) $|\sinh z|^2 = (\sinh^2 x + \sin^2 y)$

Now,

$$i.\sinh z = \sin iz$$

$$\Rightarrow |i.\sinh z|^2 = |\sin iz|^2$$

$$\Rightarrow |i|^2.|\sinh z|^2 = |\sin i(x+iy)|^2$$

$$\Rightarrow |i^2|.|\sinh z|^2 = |\sin(ix-y)|^2$$

$$\Rightarrow |\sinh z|^2 = |\sin ix.\cos y - \cos ix.\sin y|^2$$

$$= |i.(\sinh x.\cos y) - \cosh x.\sin y|^2$$

$$= \left(\sqrt{(\sinh x.\cos y)^2 + (-\cosh x.\sin y)^2}\right)^2$$

$$= \sinh^2 x.\cos^2 y + \cosh^2 x.\sin^2 y$$

$$= \sinh^2 x.\cos^2 y + (1+\sinh^2 x).\sin^2 y$$

$$[\because \cosh^2 x - \sinh^2 x = 1]$$

$$= \sinh^2 x.\cos^2 y + \sin^2 y.\sinh^2 x + \sin^2 y$$

$$= \sinh^2 x.(\cos^2 y + \sin^2 y) + \sin^2 y$$

$$= (\sinh^2 x + \sin^2 y)$$

$= R.H.S$ [Proved]

(iv) $|\cosh z|^2 = (\sinh^2 x + \cos^2 y)$

Now,

L.H.S $= |\cosh z|^2$

$= |\cos iz|^2$

$= |\cos i(x+iy)|^2$

$= |\cos(ix-y)|^2$

$= |\cos ix . \cos y + \sin ix . \sin y|^2$

$= |(\cosh x . \cos y) + i.(\sinh x . \sin y)|^2$

$= \sqrt{(\cosh x . \cos y)^2 + (\sinh x . \sin y)^2}$

$= \cosh^2 x \cos^2 y + \sinh^2 x . \sin^2 y$

$= (1+\sinh^2 x) . \cos^2 y + \sinh^2 x . \sin^2 y$ $[\because \cosh^2 x - \sinh^2 x = 1]$

$= \cos^2 y + \sinh^2 x . \cos^2 y + \sinh^2 x . \sin^2 y$

$= \cos^2 y + (\cos^2 y + \sin^2 y) ; \sinh^2 x$

$= (\sinh^2 x + \cos^2 y)$

$=$ R.H.S [Proved]

*** Inverse Trigonometric function:

** Prove that:

***(i) $\sin^{-1} z = -i \ln\left[iz \pm (1-z^2)^{1/2}\right]$

Now,

Let, $u = \sin^{-1} z$

$\Rightarrow z = \sin u$

$\Rightarrow z = \frac{1}{2i} \cdot \left(e^{iu} - e^{-iu}\right)$

$\Rightarrow 2iz = e^{iu} - e^{-iu}$

$\Rightarrow 2iz \cdot e^{iu} = e^{2iu} - 1$ [multiply by e^{iu}]

$\Rightarrow e^{2iu} - 1 - 2iz \cdot e^{iu} = 0$

$\Rightarrow (e^{iu})^2 - 2iz \cdot e^{iu} - 1 = 0$

$\Rightarrow e^{iu} = \frac{-(-2iz) \pm \sqrt{(-2iz)^2 - 4.1.(-1)}}{2.1}$

$= \frac{2iz \pm \sqrt{4i^2z^2 + 4}}{2}$

$= \frac{2\left(iz \pm \sqrt{-z^2+1}\right)}{2}$

$= iz \pm \sqrt{1-z^2}$

$$\Rightarrow e^{iu} = iz \pm \sqrt{1-z^2}$$

$$\Rightarrow \ln . e^{iu} = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow iu = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow u = \frac{1}{i} . \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow u = \frac{-i^2}{i} . \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow u = -i . \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\therefore \sin^{-1} z = -i . \ln\left(iz \pm \sqrt{1-z^2}\right)$$

[Proved]

***(ii) $\cos^{-1}z = -i.\ln\left[z \pm i(1-z^2)^{1/2}\right]$

Let,

$u = \cos^{-1}z$

$\Rightarrow z = \cos u$

$\Rightarrow z = \frac{1}{2}.(e^{iu} + e^{-iu})$

$\Rightarrow 2z = e^{iu} + e^{-iu}$

$\Rightarrow 2.z.e^{iu} = (e^{iu})^2 + 1$ [multiply by e^{iu}]

$\Rightarrow (e^{iu})^2 + 1 - 2z.e^{iu} = 0$

$\Rightarrow (e^{iu})^2 - 2z.e^{iu} + 1 = 0$

$\Rightarrow e^{iu} = \frac{-(-2z) \pm \sqrt{(-2z)^2 - 4.1.1}}{2.1}$

$\Rightarrow e^{iu} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$

$\Rightarrow e^{iu} = \frac{2(z \pm \sqrt{z^2-1})}{2}$

$\Rightarrow e^{iu} = z \pm \sqrt{-i^2.z^2 + i^2}$

$\Rightarrow e^{iu} = z \pm i.\sqrt{-z^2+1}$

$\Rightarrow \ln.e^{iu} = \ln(z \pm i.\sqrt{1-z^2})$

$\Rightarrow iu = \ln(z \pm i.\sqrt{1-z^2})$

$\Rightarrow u = \frac{1}{i} \cdot \ln\left(z \pm i \cdot \sqrt{1-z^2}\right)$

$\Rightarrow u = \frac{-i^2}{i} \cdot \ln\left(z \pm i \cdot \sqrt{1-z^2}\right)$

$\Rightarrow \cos^{-1} z = -i \cdot \ln\left(z \pm i \cdot \sqrt{1-z^2}\right)$

$\therefore \cos^{-1} z = -i \cdot \ln\left(z \pm i\sqrt{1-z^2}\right)$

[Proved]

*** Solve the following equations:

***(i) $\sinh z = i$

Now, $\sinh z = i$

$\Rightarrow \frac{1}{2} \cdot \left(e^{z} - e^{-z}\right) = i$

$\Rightarrow e^{z} - e^{-z} = 2i$

$\Rightarrow (e^{z})^2 - 1 = 2i \cdot e^{z}$ [multiply by e^{z}]

$\Rightarrow (e^{z})^2 - 2i \cdot e^{z} - 1 = 0$

$\Rightarrow e^{z} = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$

$\Rightarrow e^{z} = \frac{2i \pm \sqrt{4i^2 + 4}}{2}$

$$\Rightarrow e^z = \frac{2(i \pm \sqrt{i^2+1})}{2}$$

$$\Rightarrow e^z = i \pm \sqrt{-1+1}$$

$$\therefore e^z = i$$

Here, $x = 0, y = 1$

$$\therefore r = \sqrt{0^2+1^2} = 1$$

$$\therefore r = 1$$

$$\therefore \theta = \tan^{-1}(1/0)$$

$$= \tan^{-1} \infty$$

$$= \tan^{-1}(\tan \pi/2)$$

$$\therefore \theta = \pi/2$$

$$\therefore i = r \cdot e^{i(\theta + 2n\pi)}$$

$$\Rightarrow e^z = 1 \cdot e^{i(\pi/2 + 2n\pi)}$$

$$\Rightarrow z = i(\pi/2 + 2n\pi)$$

$$\therefore z = i \cdot (\pi/2 + 2n\pi);\ n = 0, \pm 1, \pm 2, \ldots\ldots$$

***(ii) $\cosh z = 1/2$

Now,

$$\cosh z = 1/2$$

$$\Rightarrow \frac{1}{2}\cdot(e^z + e^{-z}) = 1/2$$

$$\Rightarrow e^z + e^{-z} = 1$$

$$\Rightarrow (e^z)^2 + 1 = e^z \quad [\text{multiply by } e^z]$$

$$\Rightarrow (e^z)^2 - e^z + 1 = 0$$

$$\Rightarrow e^z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.1}}{2.1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= (1/2) \pm \frac{\sqrt{-3}}{2}$$

$$\therefore e^z = \left(\frac{1}{2}\right) \pm \left(\frac{\sqrt{3}}{2}\right)i$$

Now, $e^z = \left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)i$

$\therefore x = 1/2, \; y = \sqrt{3}/2$

$$\therefore r = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2}$$

$$\therefore r = 1$$

$$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \tan^{-1}(\tan \pi/3)$$

$$\therefore \theta = \pi/3$$

$$\therefore \left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)i = r.e^{i(\theta + 2n\pi)}$$

$$\Rightarrow e^z = 1.e^{i(\pi/3 + 2n\pi)}$$

$$\Rightarrow z = i(\pi/3 + 2n\pi)$$

$$\therefore z = i(\pi/3 + 2n\pi); \; n = 0, \pm 1, \pm 2, \ldots$$

P.T.O

Again,

$$e^z = \left(\frac{1}{2}\right) - \left(\frac{-\sqrt{3}}{2}\right) i$$

Here, $x = (1/2)$, $y = \left(\frac{-\sqrt{3}}{2}\right)$

$\therefore r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2}$

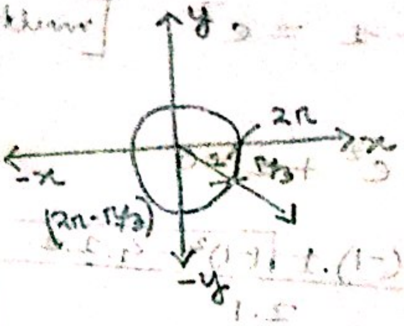
$\therefore r = 1$

$\therefore \theta = \tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\{\tan(-\pi/3)\}$

$= \tan^{-1}\{\tan(2\pi - \pi/3)\}$



$\therefore \theta = \frac{5\pi}{3}$

$\therefore \left(\frac{1}{2}\right) - \left(\frac{-\sqrt{3}}{2}\right) i = r \cdot e^{i\left(\frac{5\pi}{3} + 2n\pi\right)}$

$\Rightarrow e^z = 1 \cdot e^{i\left(\frac{5\pi}{3} + 2n\pi\right)}$

$\therefore z = i \cdot \left(\frac{5\pi}{3} + 2n\pi\right)$; $n = 0, \pm 1, \pm 2, \cdots$

(Answer)

O.T.P

※※※ Logarithmic Function:

Show that:

※※※ (i) $\ln(-1+\sqrt{3}i) = \ln 2 + 2.\left(n+\frac{1}{3}\right).\pi i$

Now,

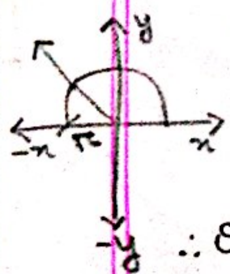
$(-1+\sqrt{3}i)$

Here, $x=-1$, $y=\sqrt{3}$

$\therefore r = \sqrt{(-1)^2+(\sqrt{3})^2}$

$\therefore r = 2$

$\therefore \theta = \tan^{-1}(-\sqrt{3})$



$= \tan^{-1}.\{\tan(-\pi/3)\}$

$= \tan^{-1}.\{\tan(\pi-\pi/3)\}$

$\therefore \theta = \frac{2\pi}{3}$

$\therefore \quad (-1+\sqrt{3}i) = r.e^{i(\theta+2n\pi)}$

$\Rightarrow (-1+\sqrt{3}i) = 2.e^{i\left(\frac{2\pi}{3}+2n\pi\right)}$

$\Rightarrow \ln(-1+\sqrt{3}i) = \ln\left\{2.e^{i\left(\frac{2\pi}{3}+2n\pi\right)}\right\}$

$= \ln 2 + \ln e^{i\left(\frac{2\pi}{3}+2n\pi\right)}$

$= \ln 2 + i\left(\frac{2\pi}{3}+2n\pi\right)$

$\therefore \ln(-1+\sqrt{3}i) = \ln 2 + 2.\left(n+\frac{1}{3}\right).\pi i$ [showed]

*** (ii) $\ln(1-i) = \frac{1}{2}.\ln 2 + (2n+\frac{7}{4}).\pi i$

Now,

$(1-i)$

Here, $x=1$, $y=-1$

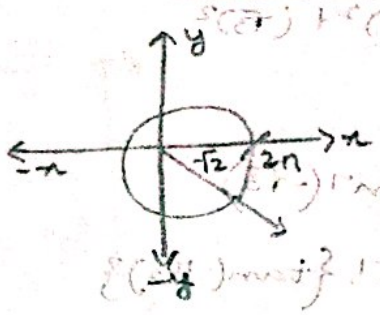
$\therefore r = \sqrt{1^2+(-1)^2}$

$\therefore r = \sqrt{2}$

$\therefore \theta = \tan^{-1}(-1)$

$= \tan^{-1}.\{\tan(-\pi/4)\}$

$= \tan^{-1}.\{\tan(2\pi-\pi/4)\}$



$\therefore \theta = \frac{7\pi}{4}$

$\therefore (1-i) = r.e^{i(\theta+2n\pi)}$

$\Rightarrow (1-i) = \sqrt{2}.e^{i(\frac{7\pi}{4}+2n\pi)}$

$\Rightarrow \ln(1-i) = \ln\{\sqrt{2}.e^{i(\frac{7\pi}{4}+2n\pi)}\}$

$= \ln\sqrt{2} + \ln.e^{i(\frac{7\pi}{4}+2n\pi)}$

$= \ln 2^{1/2} + (2n+\frac{7}{4})\pi i$

$\therefore \ln(1-i) = \frac{1}{2}.\ln 2 + (2n+\frac{7}{4}).\pi i$

[showed]

---x---

*** (iii) $\ln(i^{1/2}) = (n+\frac{1}{4}).\pi i$

Now,

(i)

Here, $x=0$, $y=1$.

$\therefore r = \sqrt{0^2+1^2}$

$\therefore r = 1$

$\therefore \theta = \tan^{-1}(\frac{1}{0})$

$= \tan^{-1}(\alpha)$

$= \tan^{-1}(\tan\pi/2)$

$\therefore \theta = \pi/2$

$\therefore \quad i = r.e^{i(\theta+2n\pi)}$

$\Rightarrow \quad i = e^{i(\pi/2+2n\pi)}$

$\Rightarrow \ln i = \ln.e^{(2n+1/2).\pi i}$

$\Rightarrow \ln i = (2n+\frac{1}{2}).\pi i$

$\Rightarrow \frac{1}{2}.\ln i = \frac{1}{2}.(2n+\frac{1}{2})\pi i$

$\therefore \ln(i^{1/2}) = (n+\frac{1}{4}).\pi i$

[showed]

x

P.S → 2

Functions of complex variables:

$w = f(z)$; $z = (x+iy)$.

$\therefore w = f(x+iy) = u(x,y) + iv(x,y)$

$\therefore w = (u + \underbrace{iv}_{\text{imaginary part}})$

u → real part.

Now, $f(z) = z^2$; $z = (x+iy)$

$\therefore f(x+iy) = (x+iy)^2$

$= x^2 - y^2 + 2xy \cdot i$

$\therefore u(x,y) = (x^2 - y^2)$

$\therefore v(x,y) = 2xy$

$\therefore w = f(z) = (u + iv)$

Theorem on Limits:

Suppose, $f(z) = u(x,y) + iv(x,y)$; $z = (x+iy)$.

Now, if, $z_0 = (x_0 + iy_0)$ and $w_0 = (u_0 + iv_0)$.

then,

$$\lim_{z \to z_0} f(z) = w_0 \text{ ; iff,}$$

$$\boxed{\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0}$$

$$\boxed{\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0}$$

Now, $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$

then,

$$\boxed{\lim_{z \to z_0} \left[f(z) + F(z) \right] = (w_0 + W_0)}$$

$$\boxed{\lim_{z \to z_0} \left[f(z) \times F(z) \right] = (w_0 \cdot W_0)}$$

$$\boxed{\lim_{z \to z_0} \left[\frac{f(z)}{F(z)} \right] = \left(\frac{w_0}{W_0} \right) \text{ ; } W_0 \neq 0}$$

Evaluate:

① $\lim\limits_{z \to (1+i)} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

$= \lim\limits_{z \to (1+i)} \dfrac{z^2 - z - i^2 - i}{z^2 - 2z + 1 + 1}$

$= \lim\limits_{z \to (1+i)} \dfrac{z^2 - i^2 - z - i}{(z-1)^2 + 1}$

$= \lim\limits_{z \to (1+i)} \dfrac{(z+i)\cdot(z-i) - (z+i)}{(z-1)^2 - i^2}$

$= \lim\limits_{z \to (1+i)} \dfrac{(z+i)\cdot(z-i-1)}{(z-1+i)\cdot(z-1-i)}$

$= \lim\limits_{z \to (1+i)} \dfrac{z+i}{z-1+i}$

$= \dfrac{1+i+i}{1+i-1+i}$

$= \dfrac{1+2i}{2i}$

$= \dfrac{i-2}{-2}$

$= \boxed{\dfrac{2-i}{2}}$ (Ans) $= \left(1 - i/2\right)$

***(ii) $\lim_{z\to(1+i)} \left\{\frac{z-1-i}{z^2-2z+2}\right\}^2$

$= \lim_{z\to(1+i)} \left\{\frac{z-1-i}{z^2-2z+1+1}\right\}^2$

$= \lim_{z\to(1+i)} \cdot \left\{\frac{z-1-i}{(z-1)^2+1}\right\}^2$

$= \lim_{z\to(1+i)} \cdot \left\{\frac{z-1-i}{(z-1)^2-i^2}\right\}^2$

$= \lim_{z\to(1+i)} \cdot \left\{\frac{(z-1-i)}{(z-1-i)\cdot(z-1+i)}\right\}^2$

$= \lim_{z\to(1+i)} \left(\frac{1}{z-1+i}\right)^2$

$= \left(\frac{1}{1+i-1+i}\right)^2$

$= \left(\frac{1}{2i}\right)^2$

$= \frac{-1}{4}$ (Ans)

(iii) $\lim_{z \to i} \frac{z^2+1}{z^6+1}$

$= \lim_{z \to i} \frac{z^2 - i^2}{z^6 - i^6}$

$= \lim_{z \to i} \frac{z^2 - i^2}{(z^2)^3 - (i^2)^3}$

$= \lim_{z \to i} \frac{(z^2 - i^2)}{(z^2 - i^2) \cdot (z^4 + z^2 \cdot i^2 + i^4)}$

$= \lim_{z \to i} \frac{1}{z^4 - z^2 + 1}$

$= \lim_{z \to i} \frac{1}{z^4 - z^2 + 1}$

$= \frac{1}{i^4 - i^2 + 1}$

$= \frac{1}{1+1+1}$

$= \frac{1}{3}$ (Ans)

(ii) $\lim_{z\to i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2}$ Evaluate using limits

$$= \frac{(2\cdot i/2 - 3)\cdot(4\cdot i/2 + i)}{(i\cdot i/2 - 1)^2}$$

$$= \frac{(i-3)\cdot(2i+i)}{(-1/2-1)^2}$$

$$= \frac{(i-3)\cdot 3i}{\left(-\frac{3}{2}\right)^2}$$

$$= \frac{3i^2 - 9i}{\frac{9}{4}}$$

$$= \frac{-3-9i}{\frac{9}{4}}$$

$$= -3(1+3i)\cdot\frac{4}{9}$$

$$= -\frac{4}{3}\cdot(1+3i)$$

$$= -(4/3) - 4i \quad \text{(Ans)}$$

Evaluate using L'Hospital's Rule

(i) $$\lim_{z \to 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i}$$

$$= \lim_{z \to 2i} \frac{2z}{4z+3-4i} \quad \text{[L'Hospital's rule]}$$

$$= \frac{2.2i}{4.2i+3-4i}$$

$$= \frac{4i}{4i+3}$$

$$= \frac{4i.(4i-3)}{(4i)^2-3^2}$$

$$= \frac{16i^2-12i}{16i^2-9}$$

$$= \frac{-16-12i}{-16-9}$$

$$= \frac{-(16+12i)}{-25}$$

$$= \left(\frac{16}{25}\right) + \left(\frac{12}{25}\right) \cdot i$$

(Ans)

***(ii) $\lim_{z\to 0} \frac{z-\sin z}{z^3}$

$= \lim_{z\to 0} \frac{1-\cos z}{3z^2}$ [L' Hospital's Law]

$= \lim_{z\to 0} \frac{\sin z}{6z}$ [L' Hospital's law]

$= \lim_{z\to 0} \frac{\cos z}{6}$

$= \lim_{z\to 0} \frac{\cos z}{6}$

$= \frac{\cos 0^\circ}{6}$

$= \frac{1}{6}$ (Ans)

If $f(z) = \left(\frac{2z-1}{3z+2}\right)$; prove that,

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{7}{(3z_0+2)^2} ; \quad z_0 \neq -\frac{2}{3}$$

$\Rightarrow$ Now,

$$f(z) = \frac{2z-1}{3z+2}$$

$$\therefore f(z_0) = \left(\frac{2z_0-1}{3z_0+2}\right)$$

$$\therefore f(z_0+h) = \frac{2(z_0+h)-1}{3(z_0+h)+2}$$

$$\therefore f(z_0+h) - f(z_0) = \frac{2z_0+2h-1}{3z_0+3h+2} - \frac{2z_0-1}{3z_0+2}$$

$$= \frac{(2z_0+2h-1).(3z_0+2)-(2z_0-1).(3z_0+3h+2)}{(3z_0+3h+2).(3z_0+2)}$$

$$= \frac{6z_0^2+4z_0+6hz_0+4h-3z_0-2-(6z_0^2+6hz_0+4z_0-3z_0-3h-2)}{(3z_0+3h+2).(3z_0+2)}$$

$$= \frac{\cancel{6z_0^2}+\cancel{4z_0}+\cancel{6z_0h}+4h\cancel{-3z_0}\cancel{-2}\cancel{-6z_0^2}\cancel{-6hz_0}\cancel{-4z_0}+\cancel{3z_0}+3h+\cancel{2}}{(3z_0+3h+2).(3z_0+2)}$$

$\therefore f(z_0+h)-f(z_0) = \dfrac{7h}{(3z_0+3h+2)\cdot(3z_0+2)}$

$\therefore \lim\limits_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = \lim\limits_{h\to 0} \dfrac{7h}{h\times(3z_0+3h+2)\cdot(3z_0+2)}$

$= \lim\limits_{h\to 0} \dfrac{7}{(3z_0+3h+2)\cdot(3z_0+2)}$

$= \dfrac{7}{(3z_0+2)\cdot(3z_0+2)}$

$\therefore \lim\limits_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = \dfrac{7}{(3z_0+2)^2}$

[Proved]

⊛⊛⊛ **Continuity:**

If the limit point and the function point of a function is equal then the function is continuous at that point.

$\therefore f(z)$ is continuous at z_0 if,

(i) $\lim_{z \to z_0} f(z)$ exist.

(ii) $f(z_0)$ is defined.

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

⊛⊛⊛ Let $f(z) = \left(\frac{z^2+4}{z-2i}\right)$; if $z \neq 2i$, while $f(2i) = (3+4i)$,

is $f(z)$ continuous at $z = 2i$?

$\Rightarrow$ Now,

$$f(z) = \begin{cases} \frac{z^2+4}{z-2i} & ; \ z \neq 2i \\ 3+4i & ; \ z = 2i \end{cases}$$

$$\therefore \lim_{z \to 2i} f(z) = \lim_{z \to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z \to 2i} \frac{z^2 - 4i^2}{z-2i}$$

$$= \lim_{z \to 2i} \frac{(z+2i)\cdot(z-2i)}{(z-2i)}$$

$$= \lim_{z \to 2i} (z+2i)$$

$$= (2i+2i)$$

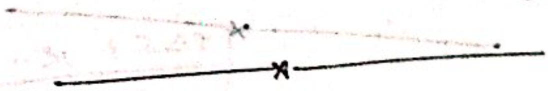
$\therefore \lim_{z \to 2i} f(z) = 4i$

and $f(2i) = (3+4i)$.

Since, $\lim_{z \to 2i} f(z) \neq f(2i)$

$f(z)$ is not continuous at $z = 2i$.

(Ans)



Differentiability:

The derivative of $f(z)$ is denoted by $f'(z)$ or $\frac{d}{dz} f(z)$ or $\frac{dw}{dz}$ and defined as,

$$\boxed{f'(z) = \lim_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\}}$$

If $f(z)$ is differentiable at $z = z_0$.

$$\boxed{f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \right\}}$$

Using the definition, find the derivatives:

(i) $f(z) = \dfrac{2z - i}{z + 2i}$ at $z = -i$

$\Rightarrow$ Here, $z_0 = -i$

$\therefore f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$\Rightarrow f'(-i) = \lim\limits_{\Delta z \to 0} \dfrac{f(-i + \Delta z) - f(-i)}{\Delta z}$ ——— (1)

Now,

$f(-i + \Delta z) = \dfrac{2(-i + \Delta z) - i}{-i + \Delta z + 2i}$

$\therefore f(-i + \Delta z) = \left(\dfrac{-3i + 2\Delta z}{i + \Delta z}\right)$

Again,

$f(-i) = \dfrac{2(-i) - i}{-i + 2i} = \dfrac{-3i}{i} = -3$

$\therefore f(-i) = -3$

$\therefore (1) \Rightarrow f'(-i) = \lim\limits_{\Delta z \to 0} \dfrac{\dfrac{-3i + 2\Delta z}{i + \Delta z} - (-3)}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} \dfrac{-3i + 2\Delta z + 3i + 3\Delta z}{i + \Delta z} \times \dfrac{1}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} \dfrac{5 \cdot \Delta z}{i + \Delta z} \times \dfrac{1}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{5}{i+\Delta z}$$

$$= \frac{5}{i}$$

$$= \frac{5i}{i^2}$$

$$= -5i$$

(Ans)

*** ii $f(z) = 3z^{-2}$ at $z = (1+i)$

$\Rightarrow$ Here, $z_0 = (1+i)$

Now, $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(1+i) = \lim_{\Delta z \to 0} \frac{f(1+i+\Delta z) - f(1+i)}{\Delta z} \quad \text{------ (i)}$$

Now,

$$f(1+i+\Delta z) = 3.(1+i+\Delta z)^{-2}$$

$$= \frac{3}{(1+i+\Delta z)^2}$$

$$= \frac{3}{\{(1+i)+\Delta z\}^2}$$

$$= \frac{3}{1+2i+i^2+2.(1+i).\Delta z+\Delta z^2}$$

$$=\left(\frac{3}{2i+2\Delta z+2\Delta z.i+\Delta z^2}\right)$$

Again,

$$f(1+i)=3.(1+i)^{-2}$$

$$=\frac{3}{(1+i)^2}$$

$$=\frac{3}{1+2i+i^2}$$

$$\therefore f(1+i)=\frac{3}{2i}$$

$$\therefore f(1+i+\Delta z)-f(1+i)=\frac{3}{2i+2\Delta z+2\Delta z.i+\Delta z^2}-\frac{3}{2i}$$

$$=\frac{(6i-6i-6\Delta z-6\Delta z.i-3\Delta z^2)}{(2i+2\Delta z+2\Delta z i+\Delta z^2).2i}$$

$$\therefore f(1+i+\Delta z)-f(1+i)=\frac{-3.(2+2i+\Delta z).\Delta z}{(2i+2\Delta z+2\Delta z.i+\Delta z^2).2i}$$

$\therefore$ from ①,

$$\therefore f'(1+i)=\lim_{\Delta z\to 0}\frac{f(1+i+\Delta z)-f(1+i)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{-3.(2+2i+\Delta z).\Delta z}{(2i+2\Delta z+2\Delta z.i+\Delta z^2).2i}\times\frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3(2+2i+\Delta z)}{(2i+2\Delta z+2\Delta z \cdot i+\Delta z^2)\cdot 2i}$$

$$= \frac{-3(2+2i)}{2i \cdot 2i}$$

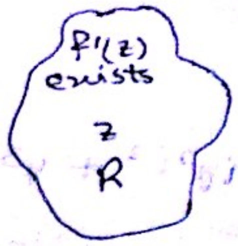
$$= \frac{-2\cdot(3+3i)}{-4}$$

$$= \frac{3}{2}\cdot(1+i)$$

(Ans)

Analytic Functions

If $f'(z)$ exists at all points z of a region R, then $f(z)$ is said to be analytic in R.



[$f'(z)$ is differentiable]

Necessary and sufficient condition:

$f(z) = u(x,y) + i.v(x,y)$ and $f'(z)$ exists at $z_0 = (x_0 + iy_0)$, then 1st order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy,

$$\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}}$$

→ known as $\boxed{\text{Cauchy-Riemann equation}}$

This is known as $\boxed{\text{necessary condition.}}$

Also $f'(z)$ can be written as at z_0,

$$f'(z_0) = u_x(x_0, y_0) + i.v_x(x_0, y_0)$$

$$\therefore \boxed{f'(z_0) = (v_y - iu_y)}$$

which is known as $\boxed{\text{sufficient condition.}}$

Show that if $f(z) = z^3$, $f'(z)$ and $f''(z)$ exist everywhere.

$\Rightarrow$ Now,

$$f(z) = z^3$$

$$= (x+iy)^3$$

$$= x^3 + 3x^2yi + 3xy^2i^2 + y^3 \cdot i^3$$

$$= x^3 - 3xy^2 + 3x^2 \cdot y \cdot i - iy^3$$

$$= (x^3 - 3xy^2) + i \cdot (3x^2y - y^3)$$

$\therefore u(x,y) = (x^3 - 3xy^2)$

$\therefore v(x,y) = (3x^2 \cdot y - y^3)$

$\therefore u_x = (3x^2 - 3y^2)$; $\therefore v_x = 6xy$

$\therefore u_y = -6xy$; $\therefore v_y = (3x^2 - 3y^2)$

$\therefore$ Here, $u_x = v_y$

$u_y = -v_x$

Hence, $f'(z)$ exist everywhere.

Now $f'(z) = u_x + i \cdot v_x$

$\therefore f'(z) = (3x^2 - 3y^2) + i(6xy)$

$\therefore u_{xx} = 6x \quad ; \therefore v_{xx} = 6y$

$\therefore u_{xy} = -6y \quad ; \therefore v_{xy} = 6x$

Here,

$u_{xx} = v_{xy}$

$u_{xy} = -v_{xx}$

Hence $f''(z)$ exists everywhere.

and $f''(z) = u_{xx} + i \cdot v_{xx}$

$\therefore f''(z) = (6x + i\,6y)$ [showed]

⊛⊛⊛ Harmonic Function:

A real-valued function u of two real variables x and y is said to be harmonic in a given domain in the xy-plane if through out that domain it has continuous partial derivatives of 1st and 2nd order and satisfies the partial differential equation:

$$\boxed{u_{xx}(x,y) + u_{yy}(x,y) = 0}$$

$\rightarrow$ Laplace Equation

⊛⊛⊛ Determine u is harmonic. Find conjugate harmonic function v and express $(u+iv)$ as an analytic function of z.

***(i) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

Harmonic:

Now, $u_x = (6xy + 4x)$

$\therefore u_{xx} = (6y + 4)$

$u_y = (3x^2 - 3y^2 - 4y)$

$\therefore u_{yy} = (-6y - 4)$

$\therefore u_{xx} + u_{yy} = (6y + 4 - 6y - 4)$

$\therefore (u_{xx} + u_{yy}) = 0$

$\therefore u$ is harmonic.

<u>conjugate:</u>

Here,

$u_x = (6xy + 4x) = v_y$ —— (i)

$u_y = (3x^2 - 3y^2 - 4y) = -v_x$ —— (ii)

Now,

(i) $\Rightarrow \quad v_y = (6xy + 4x)$

keeping x constant, integrate both sides of (i) with respect to y,

$$\therefore \int_{x.\text{constant}} v_y . dy = \int_{x.\text{constant}} (6xy + 4x) . dy$$

$$\Rightarrow v = 6x \int y . dy + 4x . \int dy$$

$$\Rightarrow v = 6x \frac{y^2}{2} + 4xy + \phi(x)$$

$$\therefore v = \{3xy^2 + 4xy + \phi(x)\} \quad \text{—— (iii)}$$

Differentiate (III) with respect to x :

$\therefore v_x = 3y^2 + 4y + \phi'(x)$

$\Rightarrow -(3x^2 - 3y^2 - 4y) = 3y^2 + 4y + \phi'(x)$ [from (ii)]

$\Rightarrow -3x^2 + \cancel{3y^2} + \cancel{4y} = \cancel{3y^2} + \cancel{4y} + \phi'(x)$

$\Rightarrow \phi'(x) = -3x^2$

$\Rightarrow \int \phi'(x) \cdot dx = -3 \cdot \int x^2 \cdot dx$

$\Rightarrow \phi(x) = -3 \cdot \frac{x^3}{3} + c$

$\therefore \phi(x) = (-x^3 + c)$

$\therefore$ from (iii),

$v = (3xy^2 + 4xy - x^3 + c)$

$\therefore f(z) = (u + iv)$

$= (3x^2y + 2x^2 - y^3 - 2y^2) + i(3xy^2 + 4xy - x^3 + c)$ (Ans)

x

(ii) $u = (x.e^x.\cos y - y e^x.\sin y)$

Harmonic:

Now, $u_x = (e^x.\cos y + x.e^x.\cos y - y.\sin y.e^x)$

$\therefore u_{xx} = e^x.\cos y + e^x.\cos y + x.e^x.\cos y - e^x.y.\sin y$

$u_y = (-x.e^x.\sin y - e^x.\sin y - y.e^x.\cos y)$

$\therefore u_{yy} = -x.e^x.\cos y - e^x.\cos y - e^x.\cos y + y.e^x.\sin y$

$\therefore u_{xx} + u_{yy} = 2e^x.\cancel{\cos y} + \cancel{x.e^x.\cos y} - \cancel{e^x.y.\sin y} - \cancel{x.e^x.\cos y}$
$\quad -2.e^x.\cos y + \cancel{y.e^x.\sin y}.$

$\therefore (u_{xx} + u_{yy}) = 0$

$\therefore u$ is harmonic.

conjugate:

Now, $u_x = (e^x.\cos y + x.e^x.\cos y - y.e^x.\sin y) = v_y$ — (i)

$u_y = -(x.e^x.\sin y + e^x.\sin y + y.e^x.\cos y) = -v_x$ — (ii)

From, (i) $\Rightarrow$

$v_y = (e^x.\cos y + x.e^x.\cos y - y.e^x.\sin y)$ — (i)

Keep x constant, integrate both sides of (i) with respect to y,

$$\therefore \int_{x\,constant} v_y \cdot dy = \int_{x.constant} (e^x \cdot \cos y + x e^x \cdot \cos y - y \cdot e^x \cdot \sin y) \cdot dy$$

$$\Rightarrow v = e^x \int \cos y \cdot dy + x \cdot e^x \int \cos y \cdot dy - e^x \int y \cdot \sin y \cdot dy$$

$$= e^x \cdot \sin y + x \cdot e^x \cdot \sin y - e^x \left[y \cdot \int \sin y \cdot dy - \int \left(\left(\frac{d}{dy} \cdot y\right) \cdot \int \sin y \cdot dy \right) \cdot dy \right]$$

$$= e^x \cdot \sin y + x \cdot e^x \cdot \sin y - e^x \left[-y \cdot \cos y - \int (-\cos y) \cdot dy \right]$$

$$\Rightarrow v = (\cancel{e^x \cdot \sin y} + x \cdot e^x \cdot \sin y + e^x \cdot y \cdot \cos y - \cancel{e^x \cdot \sin y}) + \phi(x)$$

$$\therefore v = (x \cdot e^x \cdot \sin y + y \cdot e^x \cdot \cos y) + \phi(x) \quad \text{--- (III)}$$

Differentiate (III) with respect to x,

$$\therefore v_x = x e^x \cdot \sin y + e^x \cdot \sin y + e^x \cdot y \cdot \cos y + \phi'(x)$$

$$\Rightarrow \cancel{x \cdot e^x \cdot \sin y} + \cancel{e^x \cdot \sin y} + \cancel{y \cdot e^x \cdot \cos y} = \cancel{x \cdot e^x \cdot \sin y} + \cancel{e^x \cdot \sin y} + \cancel{y \cdot e^x \cdot \cos y} + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x) \cdot dx = \int 0 \cdot dx$$

$$\therefore \phi(x) = 0 + c$$

(III) =>

$\therefore v = \left(x.e^{x}.\sin y + y.e^{x}.\cos y + c\right)$

$\therefore f(z) = (u + iv)$

$\therefore f(z) = (x.e^{x}.\cos y - y.e^{x}.\sin y) + i(x.e^{x}.\sin y + y.e^{x}.\cos y + c)$

(Ans)

*** (iii) $u = e^{-x}(x.\sin y - y.\cos y)$

<u>Harmonic:</u>

$\therefore u = \left(e^{-x}.x.\sin y - e^{-x}.y.\cos y\right)$

Now, $u_x = \left(-e^{-x}.x.\sin y + e^{-x}.\sin y + e^{-x}.y.\cos y\right)$

$\therefore u_{xx} = +e^{-x}.x.\sin y - e^{-x}.\sin y - e^{-x}.\sin y - e^{-x}.y.\cos y.$

$\therefore u_{xx} = \left(e^{-x}.x.\sin y - e^{-x}.y.\cos y\right) - 2.e^{-x}.\sin y)$

$u_y = \left(e^{-x}.x.\cos y - e^{-x}.\cos y + e^{-x}.y.\sin y\right)$

$\therefore u_{yy} = \left(-e^{-x}.x.\sin y + e^{-x}.\sin y + e^{-x}.\sin y + e^{-x}.y.\cos y\right)$

$\therefore u_{yy} = \left(-e^{-x}.x.\sin y + 2e^{-x}.\sin y + y.e^{-x}.\cos y\right)$

$\therefore (u_{xx} + u_{yy}) = \cancel{x.e^{-x}.\sin y} - \cancel{y.e^{-x}.\cos y} - \cancel{2e^{-x}.\sin y}$
$\quad - \cancel{e^{-x}.x.\sin y} + \cancel{2e^{-x}.\sin y} + \cancel{y.e^{-x}.\cos y}$

$\therefore (u_{xx} + u_{yy}) = 0 \quad \therefore u$ is harmonic.

conjugate:

Now

$u_x = (-e^{-x}.x.\sin y + e^{-x}.\sin y + e^{-x}.y.\cos y) = v_y$ — ①

$u_y = (x.e^{-x}.\cos y - e^{-x}.\cos y + y.e^{-x}.\sin y) = -v_x$ — ⑪

Now, from ①.

$v_y = (-e^{-x}.x.\sin y + e^{-x}.\sin y + e^{-x}.y.\cos y)$ — ①

Keeping x constant, integrate both sides of ① with respect to y.

$\therefore \int_{x\,constant} v_y.dy = \int_{x.constant} (-e^{-x}.x.\sin y + e^{-x}.\sin y + e^{-x}.y.\cos y).dy$

$\Rightarrow v = -e^{-x}.x.\int \sin y.dy + e^{-x}\int.\sin y.dy + e^{-x}.\int y.\cos y.dy$

$= x.e^{-x}.\cos y - e^{-x}.\cos y + e^{-x}\left[y.\int \cos y.dy - \int\left(\frac{d}{dy}y.\int \cos y.dy\right).dy\right]$

$= x.e^{-x}.\cos y - e^{-x}.\cos y + e^{-x}\left[y.\sin y - \int \sin y.dy\right]$

$= x.e^{-x}.\cos y - \cancel{e^{-x}.\cos y} + e^{-x}.y.\sin y + \cancel{e^{-x}.\cos y} + c$

$\therefore v = (x.e^{-x}.\cos y + e^{-x}.y.\sin y + \phi(x))$ — (III)

Now,

Differentiate (III) with respect to x,

$\therefore v_x = e^{-x}.\cos y - x.e^{-x}.\cos y - e^{-x}.y.\sin y + \phi'(x)$

$\Rightarrow -(x.e^{-x}.\cos y - e^{-x}.\cos y + y.e^{-x}.\sin y) = e^{-x}.\cos y - x.e^{-x}.\cos y - e^{-x}.y.\sin y + \phi'(x)$ [From (II)]

$\Rightarrow -x.e^{-x}.\cancel{\cos y} + \cancel{e^{-x}.\cos y} - \cancel{y.e^{-x}.\sin y} = \cancel{e^{-x}.\cos y} - \cancel{x.e^{-x}.\cos y} - \cancel{e^{-x}.y.\sin y} + \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\Rightarrow \int \phi'(x).dx = \int 0.dx$

$\therefore \phi(x) = c$

$\therefore$ (III) $\Rightarrow v = (x.e^{-x}.\cos y + e^{-x}.y.\sin y + c)$

$\therefore f(z) = (u + iv)$

$\therefore f(z) = (e^{-x}.x.\sin y - e^{-x}.y.\cos y) + i(x.e^{-x}.\cos y + y.e^{-x}.\sin y + c)$

(Ans)

---x---

⊛⊛⊛ Find all points of discontinuity of the function:

$$f(z) = \left(\frac{2z-3}{z^2+2z+2}\right)$$

$\Rightarrow$ Function will be discontinuous at those points where, $(z^2+2z+2)=0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2-4.1.2}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{2(-1 \pm \sqrt{-1})}{2}$$

$$\therefore z = (-1 \pm i)$$

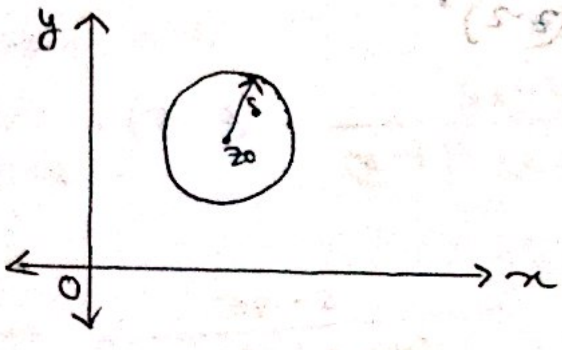
$\therefore$ Function will be discontinuous at $(-1+i)$ and $(-1-i)$ points.

Singular Points:

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$

① Isolated singularities:

The function $f(z)$ has an isolated singularity at z_0, if we can find $\delta > 0$, such that the circle $|z - z_0| = \delta$ enclosed no singular point of $f(z)$ other than z_0.



② Poles:

If z_0 is an isolated singularity, than we can find a positive integer n such that,

$$\boxed{\lim_{z \to z_0} (z - z_0)^n \cdot f(z) = A \neq 0}$$

then z_0 is called a pole of order n. If $n = 1$, z_0 is called a simple pole

*** **Problems: (find the singularities:)**

* ① $f(z) = \frac{1}{(z-2)^3}$

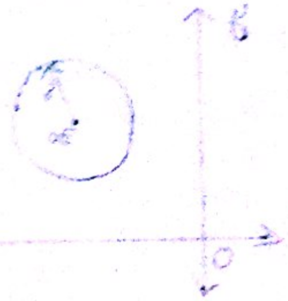
$\Rightarrow$ If, $z=2$ than $f(z) = \frac{1}{(2-2)^3} = \frac{1}{0}$ = undefined

$\therefore f(z)$ has isolated singularity at $z=2$.

Now, for $n=3$,

we can find,

$$\lim_{z\to 2} (z-2)^3 \cdot \frac{1}{(z-2)^3}$$



$$= \lim_{z\to 2} (1)$$

$$= 1$$

$$\neq 0$$

than $z=2$ is a pole of order 3.

(Ans)

***② $f(z) = \dfrac{3z-2}{(z-1)^2 \cdot (z+1) \cdot (z-4)}$

Now,

$$(z-1)^2 \cdot (z+1) \cdot (z-4) = 0$$

$\therefore (z-1)^2 = 0$ | $\therefore (z+1) = 0$ | $\therefore (z-4) = 0$

$\Rightarrow z-1 = 0$ | $\therefore z = -1$ | $\therefore z = 4$

$\therefore z = 1$

$\therefore$ isolated singular points are $1, -1, 4$.

Now, here $n=2$,

$$\lim_{z\to 1} (z-1)^2 \cdot \frac{3z-2}{(z-1)^2 \cdot (z+1) \cdot (z-4)}$$

$$= \lim_{z\to 1} \frac{3z-2}{(z+1)\cdot(z-4)}$$

$$= \frac{3-2}{(1+1)\cdot(1-4)}$$

$$= \frac{1}{2\times(-3)}$$

$$= -\left(\frac{1}{6}\right) \neq 0$$

$\therefore z = 1$ is a pole of order 2.

Again, here, $n=1$,

$$\lim_{z\to(-1)} \frac{3z-2}{(z-1)^2\cdot(z+1)\cdot(z-4)}\cdot(z+1)$$

$$= \lim_{z\to(-1)} \frac{3z-2}{(z-1)^2\cdot(z-4)}$$

$$= \frac{3\cdot(-1)-2}{(-1-1)^2\cdot(-1-4)}$$

$$= \frac{-5}{4\cdot(-5)}$$

$$= \frac{1}{4} \neq 0$$

$\therefore z=-1$ is a simple pole.

Again, here, $n=1$

$$\lim_{z\to4} (z-4)\cdot\frac{3z-2}{(z-1)^2\cdot(z+1)\cdot(z-4)}$$

$$= \lim_{z\to4} \frac{3z-2}{(z-1)^2\cdot(z+1)}$$

$$= \frac{3\cdot4-2}{(4-1)^2\cdot(4+1)}$$

$= \frac{10}{9\times5} = \frac{2}{9} \neq 0$ $\therefore z=4$ is a simple pole (Ans)

——x——

(7) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+4)^4}$

Now,

$$(z^2+2z+4)^4 = 0$$

$$\Rightarrow z^2+2z+4=0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2-4.1.4}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2(-1 \pm \sqrt{1-4})}{2}$$

$$= -1 \pm \sqrt{-3}$$

$$\therefore z = -1 \pm \sqrt{3}i$$

again, $\ln(0) = \text{undefined}$

$$\therefore \ln(z-2) = 0$$

$$\Rightarrow z-2=0$$

$$\therefore z=2$$

$\therefore$ isolated singular points are $(-1+\sqrt{3}i), (-1-\sqrt{3}i), 2.$

$x^2+2x+4=0$

$\Rightarrow (x+2)^2=0$

$\therefore x=-2,-2$

$\rightarrow \{x-(-2)\}.\{x-(-2)\}$

Now, Here, $n=4$

$$\lim_{z\to(-1+\sqrt{3}i)} \{z-(-1+\sqrt{3}i)\}^4 . \frac{\ln(z-2)}{(z^2+2z+4)^4}$$

$$=\lim_{z\to(-1+\sqrt{3}i)} \{z+1-\sqrt{3}i\}^4 . \frac{\ln(z-2)}{\left[\{z-(-1+\sqrt{3}i)\}.\{z-(-1-\sqrt{3}i)\}\right]^4}$$

$$=\lim_{z\to(-1+\sqrt{3}i)} (z+1-\sqrt{3}i)^4 . \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4 . (z+1+\sqrt{3}i)^4}$$

$$=\lim_{z\to(-1+\sqrt{3}i)} . \frac{\ln(z-2)}{(z+1+\sqrt{3}i)^4}$$

$$=\frac{\ln(-1+\sqrt{3}i-2)}{(-1+\sqrt{3}i+1+\sqrt{3}i)^4}$$

$$=\frac{\ln(-3+\sqrt{3}i)}{(2\sqrt{3}i)^4} \neq 0$$

$\therefore z=(-1+\sqrt{3}i)$ is a pole of order 4.

Again, here, $m=4$.

$$\lim_{z\to(-1-\sqrt{3}i)} \{z-(-1-\sqrt{3}i)\}^4 \cdot \frac{\ln(z-2)}{\left[\{z-(-1+\sqrt{3}i)\}.\{z-(-1-\sqrt{3}i)\}\right]^4}$$

$$=\lim_{z\to(-1-\sqrt{3}i)} (z+1+\sqrt{3}i)^4 \cdot \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4 . (z+1+\sqrt{3}i)^4}$$

$$=\lim_{z\to(-1-\sqrt{3}i)} \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4}$$

$$=\frac{\ln(-1-\sqrt{3}i-2)}{(-1-\sqrt{3}i+1-\sqrt{3}i)^4}$$

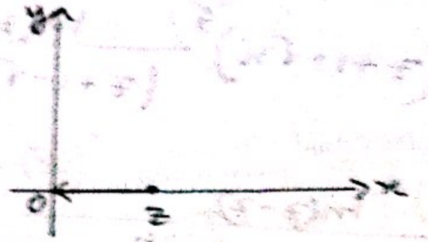
$$=\frac{\ln(-1-\sqrt{3}i-2)}{(-2\sqrt{3}i)^4}$$

$$=\frac{\ln(-3-\sqrt{3}i)}{(-2\sqrt{3}i)^4} \neq 0$$

$\therefore$ $z=(-1-\sqrt{3}i)$ is a pole of order 4. (Ans) ([illegible]) Prove that $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ does not exists

$\Rightarrow$ Let,

$z \to 0$, along x-axis, than obviously $y=0$.

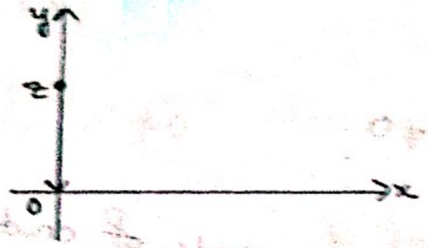


then, $z = (x+iy) = (x+i.0) = x$

$\bar{z} = (x-iy) = (x-i.0) = x$

$\therefore \lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{z\to 0} \dfrac{x}{x} = 1$

Again let, $z\to 0$, along the y-axis, than obviously, $x=0$



$\therefore z = (x+i.y) = (0+iy) = iy$

$\therefore \bar{z} = (x-iy) = (0-iy) = -iy$

$\therefore \lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{z\to 0} \dfrac{iy}{-iy} = -1$

Since, two approaches do not provide same result then the limit does not exist. [Proved]

(4) $f(z) = \dfrac{z}{(z^2+4)^2}$

$\Rightarrow$ Here, $(z^2+4)^2 = 0$

$\Rightarrow (z^2+4) = 0$

$\Rightarrow z^2 - 4i^2 = 0$

$\Rightarrow z^2 - (2i)^2 = 0$

$\Rightarrow (z+2i)(z-2i) = 0$

$\therefore z = 2i, -2i$

$\therefore$ isolated singular points are $2i, -2i$

Now, Here, $n = 2$,

$\therefore \lim\limits_{z \to z_0} (z-z_0)^n . f(z)$

$= \lim\limits_{z \to 2i} (z-2i)^2 . \dfrac{z}{(z^2+4)^2}$

$= \lim\limits_{z \to 2i} . (z-2i)^2 . \dfrac{z}{(z^2-4i^2)^2}$

$= \lim\limits_{z \to 2i} . (z-2i)^2 . \dfrac{z}{\{(z+2i).(z-2i)\}^2}$

$$= \lim_{z \to 2i} (z-2i)^2 \cdot \frac{z}{(z+2i)^2 \cdot (z-2i)^2}$$

$$= \lim_{z \to 2i} \frac{z}{(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{z}{(z+2i)^2}$$

$$= \frac{2i}{(2i+2i)^2}$$

$$= \frac{2i}{(4i)^2}$$

$$= \frac{2i}{16i^2}$$

$$= \frac{1}{8i}$$

$$= \frac{1}{8i} \neq 0$$

$\therefore z = 2i$ is a pole of order 2.

Again, Here, $n = 2$,

$$\lim_{z \to -2i} \{z-(-2i)\}^2 \cdot \frac{z}{(z^2+4)^2}$$

$$= \lim_{z \to -2i} (z+2i)^2 \cdot \frac{z}{(z^2-4i^2)^2}$$

$$= \lim_{z \to -2i} (z+2i) \cdot \frac{z}{\{(z+2i) \cdot (z-2i)\}^2}$$

$$= \lim_{z \to -2i} \cdot (z+2i) \cdot \frac{z}{(z+2i)^2 \cdot (z-2i)^2}$$

$$= \lim_{z \to -2i} \frac{z}{(z-2i)^2}$$

$$= \frac{-2i}{(-2i-2i)^2}$$

$$= \frac{-2i}{(-4i)^2}$$

$$= \frac{-2i}{16i^2}$$

$$= \frac{-1}{8i} \neq 0$$

$\therefore z = -2i$ is a pole of order 2.

(Ans)

(3) $f(z) = \frac{z^8+z^4+2}{(z-1)^3.(3z+2)^2}$

$\Rightarrow$ Now,

$(z-1)^3.(3z+2)^2 = 0$

$\therefore (z-1)^3 = 0$ | $\therefore (3z+2)^2 = 0$

$\Rightarrow z-1 = 0$ | $\Rightarrow 3z+2 = 0$

$\therefore z = 1$ | $\therefore z = -2/3$

$\therefore$ isolated singular points are $1, -2/3$

Here, $n = 3$

$\therefore \lim_{z\to 1} (z-1)^3. \frac{z^8+z^4+2}{(z-1)^3.(3z+2)^2}$

$= \lim_{z\to 1} \frac{z^8+z^4+2}{(3z+2)^2}$

$= \frac{1+1+2}{(3.1+2)^2}$

$= \frac{4}{5^2}$

$= \frac{4}{25} \neq 0$

$\therefore z = 1$ is a pole of order 3.

Here, $n = 2$

$$\therefore \lim_{z \to -2/3} \left\{ z - \left(-2/3\right) \right\}^2 \cdot \frac{z^8 + z^4 + 2}{(z-1)^3 \cdot (3z+2)^2}$$

$$= \lim_{z \to -2/3} \left(z + 2/3\right)^2 \cdot \frac{z^8 + z^4 + 2}{(z-1)^3 \cdot (3z+2)^2}$$

$$= \lim_{z \to -2/3} \left(\frac{3z+2}{3}\right)^2 \cdot \frac{z^8 + z^4 + 2}{(z-1)^3 \cdot (3z+2)^2}$$

$$= \lim_{z \to -2/3} \frac{1}{9} \cdot \frac{z^8 + z^4 + 2}{(z-1)^3}$$

$$= \frac{1}{9} \cdot \frac{(-2/3)^8 + (-2/3)^4 + 2}{(-2/3 - 1)^2}$$

$$= \frac{1}{9} \cdot \frac{\left(\frac{-2}{3}\right)^8 + \left(\frac{-2}{3}\right)^4 + 2}{\left(\frac{-5}{3}\right)^3}$$

$$= -0.054 \neq 0$$

$\therefore z = -2/3$ is a pole of order 2

$$f(z) = \frac{z+3}{(z^2-1)}$$

$\Rightarrow$ Now, $z^2 - 1 = 0$

$\Rightarrow z^2 = 1$

$\therefore z = \pm 1$

$\therefore$ isolated singular points are $+1, -1$.

Here, $n = 1$,

$$\therefore \lim_{z \to 1} (z-1) \cdot \frac{z+3}{z^2-1}$$

$$= \lim_{z \to 1} (z-1) \cdot \frac{z+3}{(z+1) \cdot (z-1)}$$

$$= \lim_{z \to 1} \cdot \frac{z+3}{z+1}$$

$$= \frac{1+3}{1+1}$$

$= 2 \neq 0$

$\therefore z = 1$ is a simple pole.

Here, $n = 1$,

$$\therefore \lim_{z \to -1} \{z - (-1)\} \cdot \frac{z+3}{z^2-1}$$

$$= \lim_{z \to -1} (z+1) \cdot \frac{z+3}{(z+1) \cdot (z-1)}$$

$$= \lim_{\to -1} \frac{z+3}{z-1}$$

$$= \frac{-1+3}{-1-1}$$

$$= \frac{2}{-2} = -1 \neq 0$$

$\therefore z = -1$ is a simlpe pole.

(Ans)

***⑥ $f(z) = \dfrac{z^2-3z}{z^2+2z+2}$

$\Rightarrow$ NOW,

$$z^2+2z+2 = 0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2-4.1.2}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2(-1 \pm \sqrt{1-2})}{2}$$

$$= -1 \pm \sqrt{-1}$$

$$\therefore z = (-1 \pm i)$$

$\therefore$ isolated singular points are $(-1+i)$, $(-1-i)$

Here, $n=1$,

$$\therefore \lim_{z \to (-1+i)} \{z-(-1+i)\} \cdot \frac{z^2-3z}{\{z-(-1+i)\} \cdot \{z-(-1-i)\}}$$

$$= \lim_{z \to (-1+i)} (z+1-i) \cdot \frac{z^2-3z}{(z+1-i) \cdot (z+1+i)}$$

$$= \lim_{z \to (-1+i)} \frac{z^2-3z}{(z+1+i)}$$

$$= \frac{(-1+i)^2-3(-1+i)}{(-1+i+1+i)}$$

$$= \frac{(-1)^2 - 2i + i^2 + 3 - 3i}{2i}$$

$$= \frac{1 - 2i - 1 + 3 - 3i}{2i}$$

$$= \frac{3 - 5i}{2i} = \frac{3i - 5i^2}{2i \cdot i}$$

$$= \frac{3i + 5}{2i^2}$$

$$= \frac{5 + 3i}{-2}$$

$$= -\left(\frac{5}{2}\right) - \left(\frac{3}{2}\right)i \neq 0.$$

$\therefore z = (-1+i)$ is a simple pole.

Now, Here, $n = 1$,

$$\therefore \lim_{z \to (-1-i)} \{z - (-1-i)\} \cdot \frac{z^2 - 3z}{\{z - (-1+i)\} \cdot \{z - (-1-i)\}}$$

$$= \lim_{z \to (-1-i)} (z + 1 + i) \cdot \frac{z^2 - 3z}{(z + 1 - i) \cdot (z + 1 + i)}$$

$$= \lim_{z \to (-1-i)} \cdot \frac{z^2 - 3z}{(z + 1 - i)}$$

$$= \frac{(-1-i)^2-3(-1-i)}{(-1-i+1-i)}$$

$$= \frac{(-1)^2+2i+i^2+3+3i}{-2i}$$

$$= \frac{1+5i-1+3}{-2i}$$

$$= \frac{3+5i}{-2i}$$

$$= \frac{3i+5i^2}{-2.i.i}$$

$$= \frac{-5+3i}{2}$$

$$= -\left(\frac{5}{2}\right)+\left(\frac{3}{2}\right)i \neq 0$$

$\therefore z=(-1-i)$ is a simple pole.

(Ans)

Q. $f(z)=|z|^2$. show that the derivatives of $f(z)=|z|^2$ exist only at $z=0$.

$\Rightarrow$ Now,

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z}$$

$\therefore f(z+\Delta z)=|z+\Delta z|^2$
$\therefore f(z)=|z|^2$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)\cdot(\overline{z+\Delta z})-z\cdot\bar{z}}{\Delta z}$$

$[|z|^2=z\cdot\bar{z}]$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)\cdot(\bar{z}+\overline{\Delta z})-z\cdot\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{z\cdot\bar{z}+z\cdot\overline{\Delta z}+\Delta z\cdot\bar{z}+\Delta z\cdot\overline{\Delta z}-z\cdot\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\left(z\cdot\frac{\overline{\Delta z}}{\Delta z}+\bar{z}+\overline{\Delta z}\right)$$

$$\therefore f'(z)=\lim_{\Delta z\to 0}\left(\bar{z}+z\cdot\frac{\overline{\Delta z}}{\Delta z}+\overline{\Delta z}\right)$$

Now,

approaching along real line, $\Delta z \to 0$ implies, $\Delta y = 0$

$\therefore \Delta z = (\Delta x + i.\Delta y) = \Delta x$

$\therefore \Delta \bar{z} = (\Delta x - i.\Delta y) = \Delta x$

$\therefore \Delta z = \Delta \bar{z}$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \left(\bar{z} + z.\frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z}\right)$$

$$= \lim_{\Delta z \to 0} \left(\bar{z} + z + \Delta z\right)$$

$\therefore f'(z) = (z + \bar{z})$

approaching along imaginary line, $\Delta z \to 0$ implies $\Delta x = 0$.

$\therefore \Delta z = (\Delta x + i.\Delta y) = i.\Delta y$

$\therefore \Delta \bar{z} = (\Delta x - i.\Delta y) = -i.\Delta y$

$\therefore \Delta z = -\Delta \bar{z}$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \left(\bar{z} + z.\frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z}\right)$$

$$= \lim_{\Delta z \to 0} \left(\bar{z} - z - \Delta z\right)$$

$\therefore f'(z) = (\bar{z} - z)$

Since the two limits are unique then,

$\therefore \bar{z}+z=\bar{z}-z$

$\therefore z=0$ and $\bar{z}=0$

$\therefore$ since $z=0$, $\therefore f'(z)$ exists at $z=0$.

[Showed]