Lec: 03 (MAT: 216)
Md. Saddam Hossain
Lecturer (MNS)
BRAC University

Practice Sheet: 1

1. Solve the following matrix equation for a, b, c & d

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

2. (a) AB (e) (3E)D (f) CC^T

(g) $(DA)^T$ (i) $tr(DD^T)$ (j) $tr(4E^T - D)$

Definition: If A is a square matrix, then the trace of A, denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then $trace(A) = a_{11} + a_{22} + a_{33}$

if $B = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ then $tr(B) = 3+5+9$.

Transposes:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Example:

$B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

$B^T = \begin{bmatrix} 6 & 0 & 7 \\ -2 & 1 & 7 \\ 4 & 3 & 5 \end{bmatrix}$

properties of the Transpose:

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$ & $(A-B)^T = A^T - B^T$

(c) $(kA)^T = kA^T$

(d) $(AB)^T = B^T A^T$ (*)

Definitions: If A is a square matrix then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A.

The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}

Example: Find the Minors and Cofactors; $A=\begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

The minor of entry a_{11} is $M_{11}=\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}=16$

" " " " a_{12} is $M_{12}=\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix}=10$

" " " " a_{13} " $M_{13}=\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}=3$

" " " " a_{21} " $M_{21}=\begin{vmatrix} 1 & 4 \\ 4 & 8 \end{vmatrix}=-8$

" " " " a_{22} " $M_{22}=\begin{vmatrix} 3 & 4 \\ 1 & 8 \end{vmatrix}=20$

" " " " a_{23} " $M_{23}=\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}=11$

" " " " a_{31} " $M_{31}=\begin{vmatrix} 1 & 4 \\ 5 & 6 \end{vmatrix}=-14$

" " " " a_{32} " $M_{32}=\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix}=10$

" " " " a_{33} " $M_{33}=\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix}=13$.

The cofactor of a_{11} is $C_{11}=(-1)^{1+1}M_{11}=M_{11}=16$

" " " a_{22} is $C_{22}=(-1)^{2+2}M_{22}=M_{22}=20$.

" " " a_{32} " $C_{32}=(-1)^{3+2}M_{32}=$ $=\cancel{}-M_{32}=-10$

Determinant: Let $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$. Evaluate det(A)

$$\det(A) = 3\begin{pmatrix} -4 & 3 \\ 4 & -2 \end{pmatrix} - 1\begin{pmatrix} -2 & 3 \\ 5 & -2 \end{pmatrix} - 0\begin{pmatrix} -2 & -4 \\ 5 & 4 \end{pmatrix}$$

$$= 3(8-12) - 1(4-15) - 0$$

$$= -12 + 11 = -1$$

Definition: If A is any nxn matrix and C_{ij} is the cofactor of a_{ij}, the the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactor from A. The transpose of this matrix is called the adjoint of A and is denoted by adj(A)

Example: Adjoint of a 3×3 matrix

let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

The cofactors of A are:

$C_{11} = 12 \quad C_{12} = 6 \quad C_{13} = -16$

$C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16$

$C_{31} = 12 \quad C_{32} = -10 \quad C_{33} = 16$

and the matrix of cofactor is

$$C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

Therefore the adjoint of A is

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

* Inverse of a Matrix Using It's adjoint:

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Example: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

Then $\det(A) = 3(0+12) - 2(0-6) - 1(-4-12)$

$= 36 + 12 + 16$

$= 64$

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$ (previous example)

Therefore $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$$= \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$