

4 Vector Spaces

4.2

NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS

Linear Algebra

and its applications

FOURTH EDITION



David C. Lay

NULL SPACE OF A MATRIX

- **Definition:** The null space of an $m \times n$ matrix A , written as $\text{Nul } A$, is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,
$$\text{Nul } A = \{x : x \text{ is in } \mathbf{R}^n \text{ and } Ax = 0\}.$$
- **Theorem 2:** The null space of an $m \times n$ matrix A is a subspace of \mathbf{R}^n . Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of \mathbf{R}^n .
- **Proof:** $\text{Nul } A$ is a subset of \mathbf{R}^n because A has n columns.
- We need to show that $\text{Nul } A$ satisfies the three properties of a subspace.

NULL SPACE OF A MATRIX

- $\mathbf{0}$ is in $\text{Nul } A$.
- Next, let \mathbf{u} and \mathbf{v} represent any two vectors in $\text{Nul } A$.
- Then

$$A\mathbf{u} = \mathbf{0} \text{ and } A\mathbf{v} = \mathbf{0}$$

- To show that $\mathbf{u} + \mathbf{v}$ is in $\text{Nul } A$, we must show that $A(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.
- Using a property of matrix multiplication, compute
$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
- Thus $\mathbf{u} + \mathbf{v}$ is in $\text{Nul } A$, and $\text{Nul } A$ is closed under vector addition.

NULL SPACE OF A MATRIX

- Finally, if c is any scalar, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$$

which shows that $c\mathbf{u}$ is in $\text{Nul } A$.

- Thus $\text{Nul } A$ is a subspace of \mathbf{R}^n .
- **An Explicit Description of $\text{Nul } A$**
- There is no obvious relation between vectors in $\text{Nul } A$ and the entries in A .
- We say that $\text{Nul } A$ is defined *implicitly*, because it is defined by a condition that must be checked.

NULL SPACE OF A MATRIX

- No explicit list or description of the elements in $\text{Nul } A$ is given.
- *Solving the equation $Ax = 0$ amounts to producing an explicit description of $\text{Nul } A$.*
- **Example 1:** Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

NULL SPACE OF A MATRIX

- **Solution:** The first step is to find the general solution of $Ax = 0$ in terms of free variables.
- Row reduce the augmented matrix $[A \ 0]$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{array}$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2 , x_4 , and x_5 free.
- Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

\uparrow \uparrow \uparrow
 u v w

NULL SPACE OF A MATRIX

$$= x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}. \quad \text{----(1)}$$

- Every linear combination of \mathbf{u} , \mathbf{v} , and \mathbf{w} is an element of $\text{Nul } A$.
 - Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $\text{Nul } A$.
1. The spanning set produced by the method in Example (1) is automatically linearly independent because the free variables are the weights on the spanning vectors.
 2. When $\text{Nul } A$ contains nonzero vectors, the number of vectors in the spanning set for $\text{Nul } A$ equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

COLUMN SPACE OF A MATRIX

- **Definition:** The column space of an $m \times n$ matrix A , written as $\text{Col } A$, is the set of all linear combinations of the columns of A . If $A = [a_1 \cdots a_n]$, then

$$\text{Col } A = \text{Span}\{a_1, \dots, a_n\}.$$

- **Theorem 3:** The column space of an $m \times n$ matrix A is a subspace of \mathbf{R}^m .
- A typical vector in $\text{Col } A$ can be written as $A\mathbf{x}$ for some \mathbf{x} because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A . That is,

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}.$$

COLUMN SPACE OF A MATRIX

- The notation $A\mathbf{x}$ for vectors in $\text{Col } A$ also shows that $\text{Col } A$ is the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
- The column space of an $m \times n$ matrix A is all of \mathbf{R}^m if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} in \mathbf{R}^m .

- **Example 2:** Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$
and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

COLUMN SPACE OF A MATRIX

- a. Determine if \mathbf{u} is in $\text{Nul } A$. Could \mathbf{u} be in $\text{Col } A$?
- b. Determine if \mathbf{v} is in $\text{Col } A$. Could \mathbf{v} be in $\text{Nul } A$?

■ Solution:

- a. An explicit description of $\text{Nul } A$ is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

COLUMN SPACE OF A MATRIX

- \mathbf{u} is *not* a solution of $A\mathbf{x} = \mathbf{0}$, so \mathbf{u} is not in $\text{Nul } A$.
- Also, with four entries, \mathbf{u} could not possibly be in $\text{Col } A$, since $\text{Col } A$ is a subspace of \mathbf{R}^3 .

b. Reduce $[A \quad \mathbf{v}]$ to an echelon form.

$$[A \quad \mathbf{v}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

- The equation $A\mathbf{x} = \mathbf{v}$ is consistent, so \mathbf{v} is in $\text{Col } A$.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- With only three entries, \mathbf{v} could not possibly be in $\text{Nul } A$, since $\text{Nul } A$ is a subspace of \mathbf{R}^4 .
- Subspaces of vector spaces other than \mathbf{R}^n are often described in terms of a linear transformation instead of a matrix.
- **Definition:** A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector \mathbf{x} in V a unique vector $T(\mathbf{x})$ in W , such that
 - $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V , and
 - $T(c\mathbf{u}) = cT(\mathbf{u})$ for all \mathbf{u} in V and all scalars c .

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- The **kernel** (or **null space**) of such a T is the set of all \mathbf{u} in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W).
- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some \mathbf{x} in V .
- The kernel of T is a subspace of V .
- The range of T is a subspace of W .

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

Nul A	Col A
1. Nul A is a subspace of \mathbf{R}^n .	1. Col A is a subspace of \mathbf{R}^m .
2. Nul A is implicitly defined; <i>i.e.</i> , you are given only a condition ($Ax = 0$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; <i>i.e.</i> , you are told how to build vectors in Col A.

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

3. It takes time to find vectors in Nul A. Row operations on $[A \ 0]$ are required.

3. It is easy to find vectors in Col A. The columns of a are displayed; others are formed from them.

4. There is no obvious relation between Nul A and the entries in A.

4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

5. A typical vector \mathbf{v} in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.

6. Given a specific vector \mathbf{v} , it is easy to tell if \mathbf{v} is in Nul A. Just compare $A\mathbf{v}$.

5. A typical vector \mathbf{v} in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.

6. Given a specific vector \mathbf{v} , it may take time to tell if \mathbf{v} is in Col A. Row operations on $[A \quad \mathbf{v}]$ are required.

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

7. $\text{Nul } A = \{0\}$ if and only if the equation $Ax = 0$ has only the trivial solution.

8. $\text{Nul } A = \{0\}$ if and only if the linear transformation $x \mapsto Ax$ is one-to-one.

7. $\text{Col } A = \mathbf{R}^m$ if and only if the equation $Ax = b$ has a solution for every b in \mathbf{R}^m .

8. $\text{Col } A = \mathbf{R}^m$ if and only if the linear transformation $x \mapsto Ax$ maps \mathbf{R}^n onto \mathbf{R}^m .