CSE 221: Algorithms

Heapsort

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References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.

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Contents

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- Sorts *in place* – like *insertion sort*.
- Combines the best of both algorithms.
- Uses a data structure called the heap, which is also extensively used in other applications.

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Max vs. Min Heap

Unless explicitly stated as max heap or min heap, heap means max heap in this course.

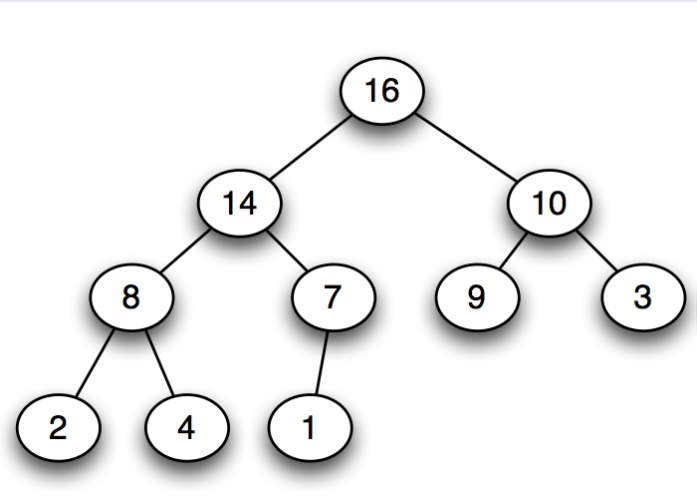
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.

Example of (max) heap



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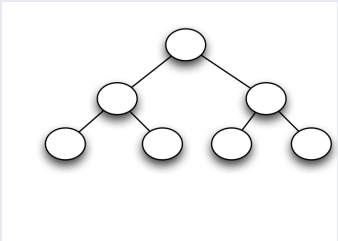
2. the tree is almost-complete.

Example of complete tree (or *not*)



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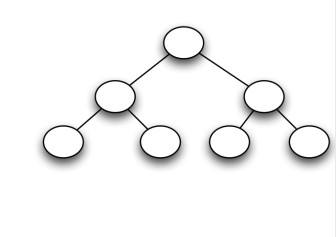

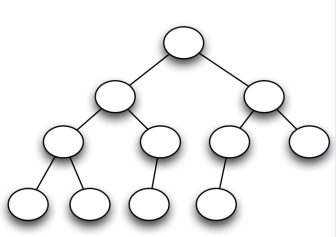

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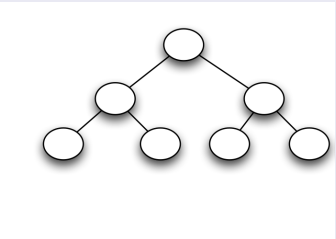

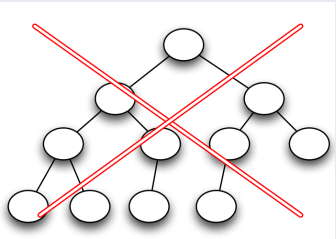

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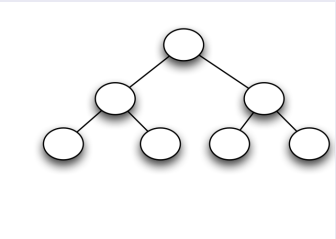

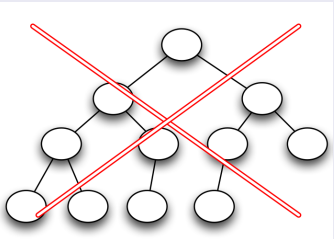

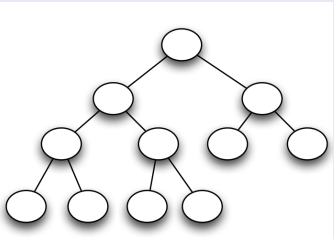

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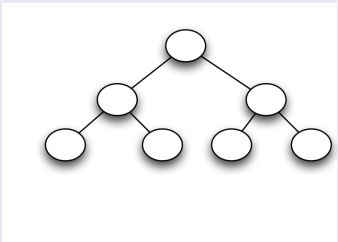
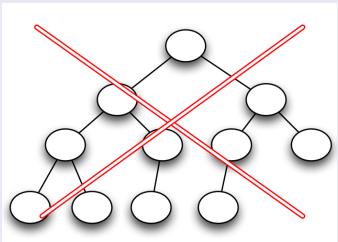
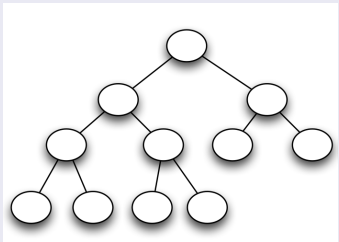
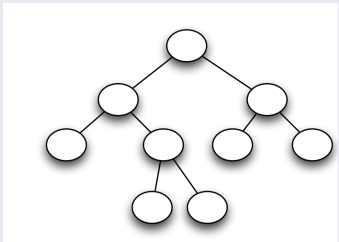

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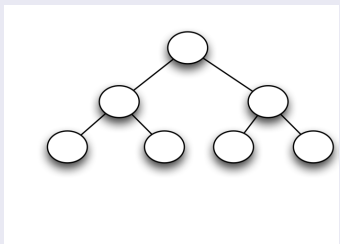
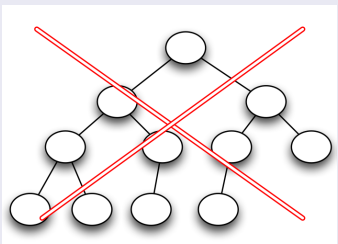
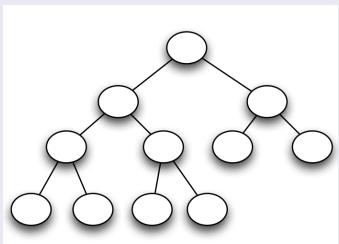
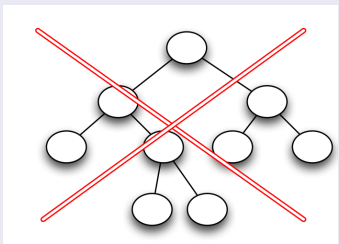

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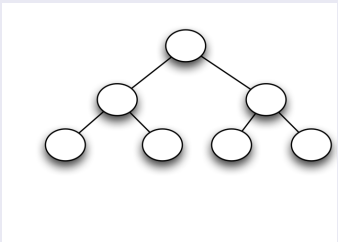
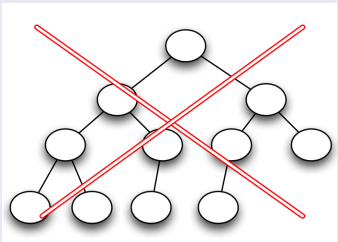
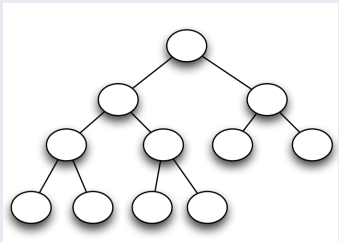
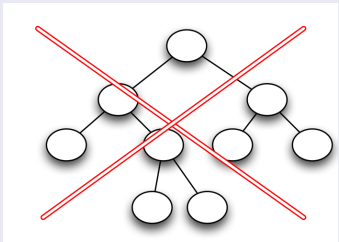

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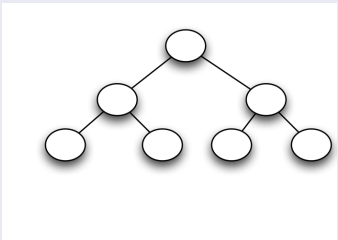
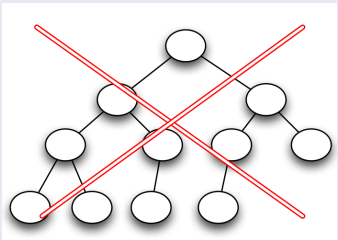
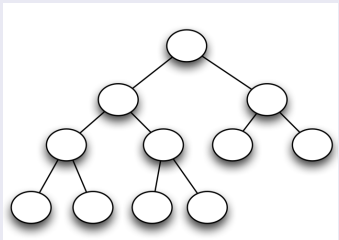
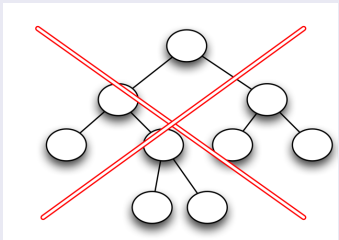

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- A heap of height h has the most number of elements if the tree is complete, so n equals the sum of nodes at each level.

$$\begin{aligned} n &\leq 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^h \\ &= \frac{2^{h+1} - 1}{2 - 1} \\ &= 2^{h+1} - 1. \end{aligned}$$

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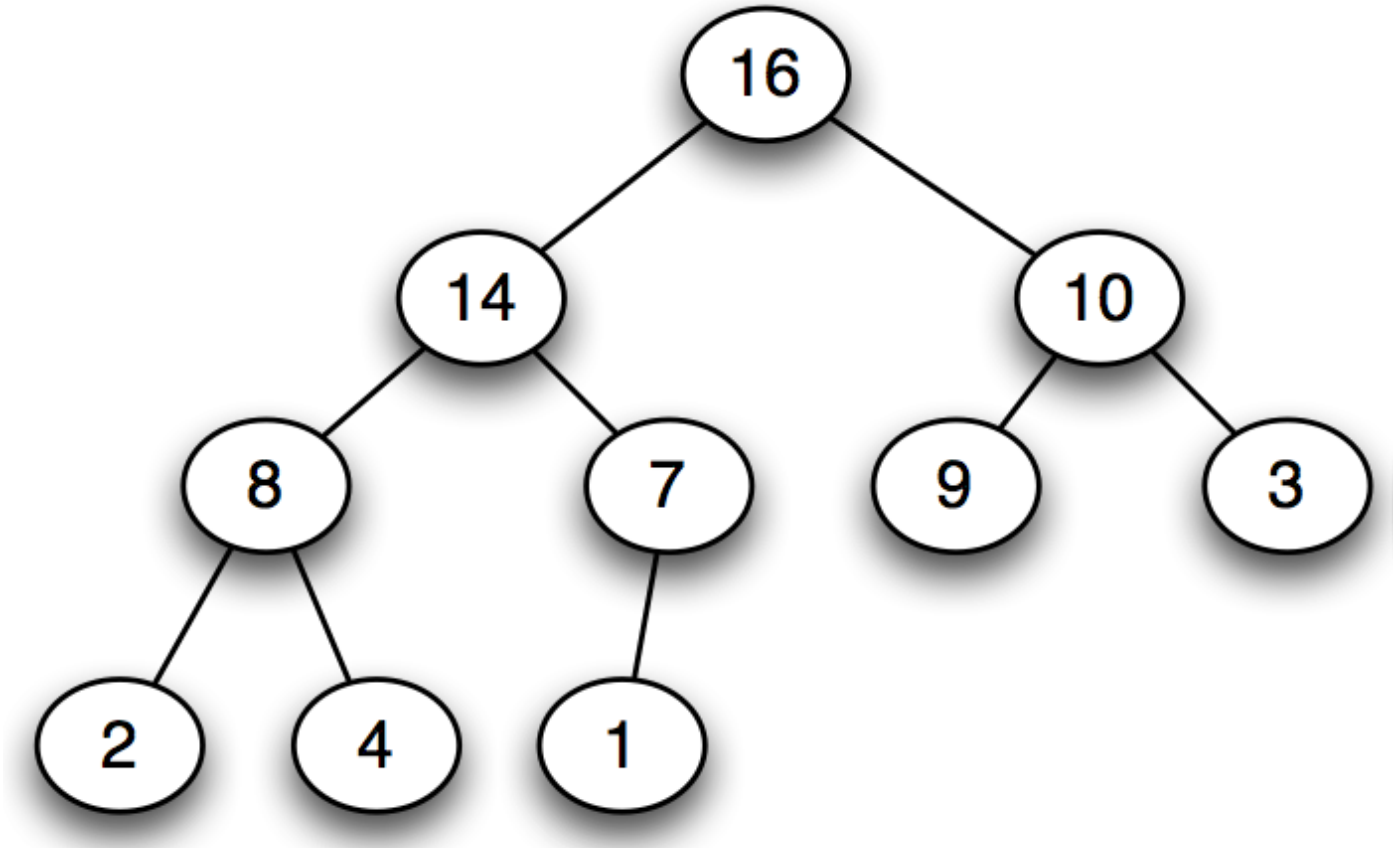
- It has the least number of elements if the lowest level has a single element and all higher levels are complete, so $n \geq 2^h - 1 + 1 = 2^h$.
- $2^h \leq n \leq 2^{h+1} - 1 < 2^{h+1} \quad \Rightarrow \quad h \leq \lg n < h + 1$. Since h is an integer, $h = \lfloor \lg n \rfloor = \Theta(\lg n)$.

Heap – array representation of heap-ordered tree



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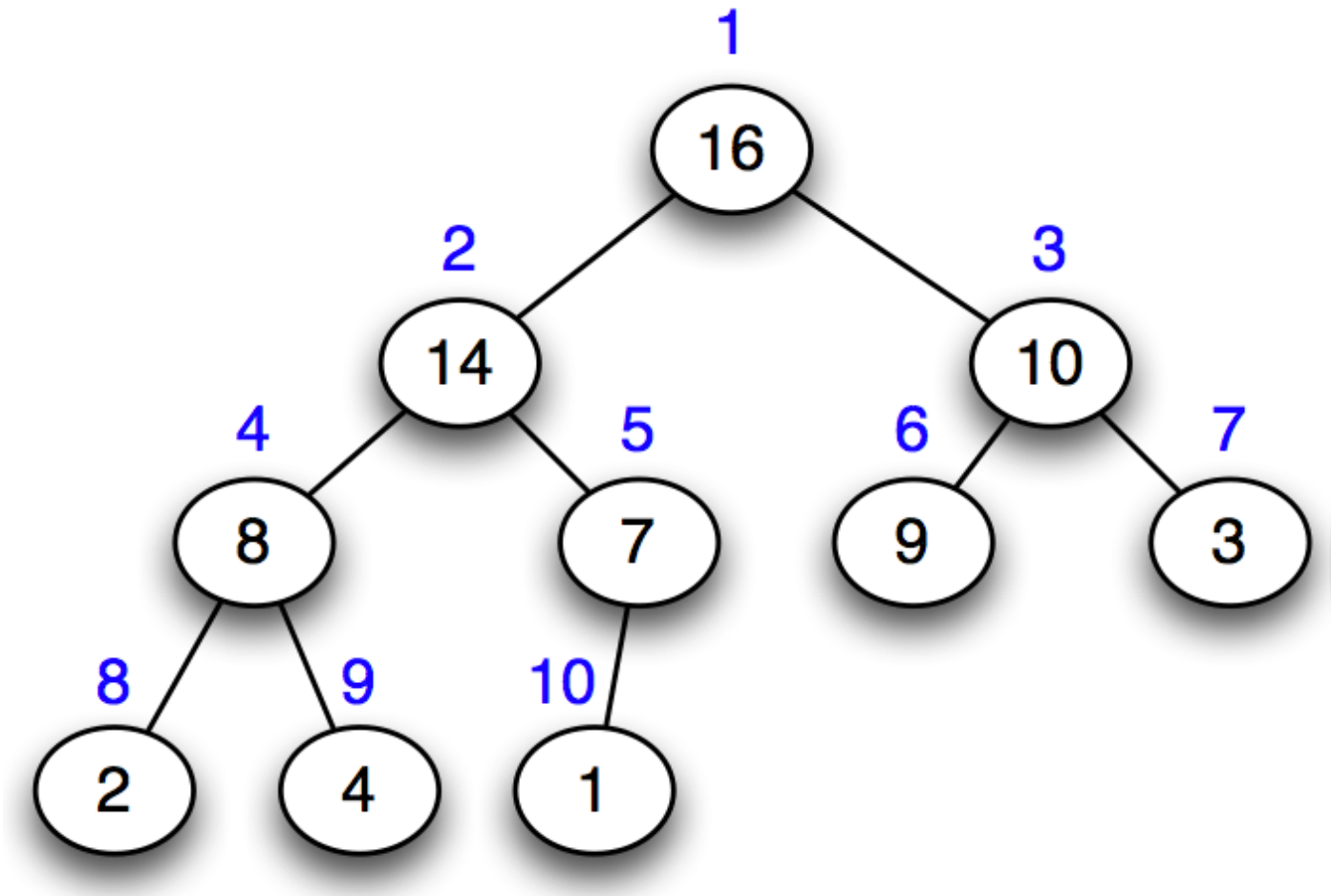

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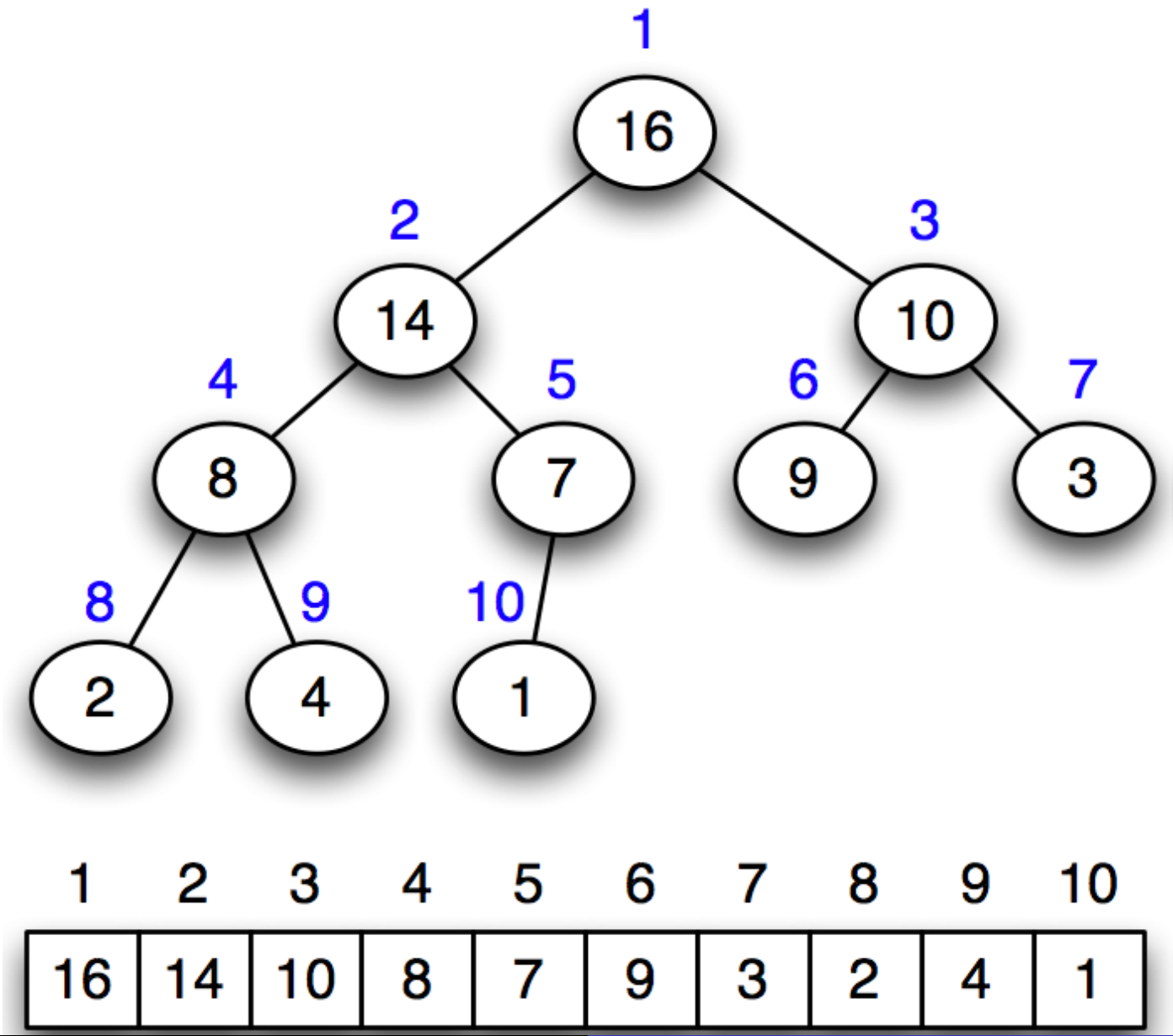

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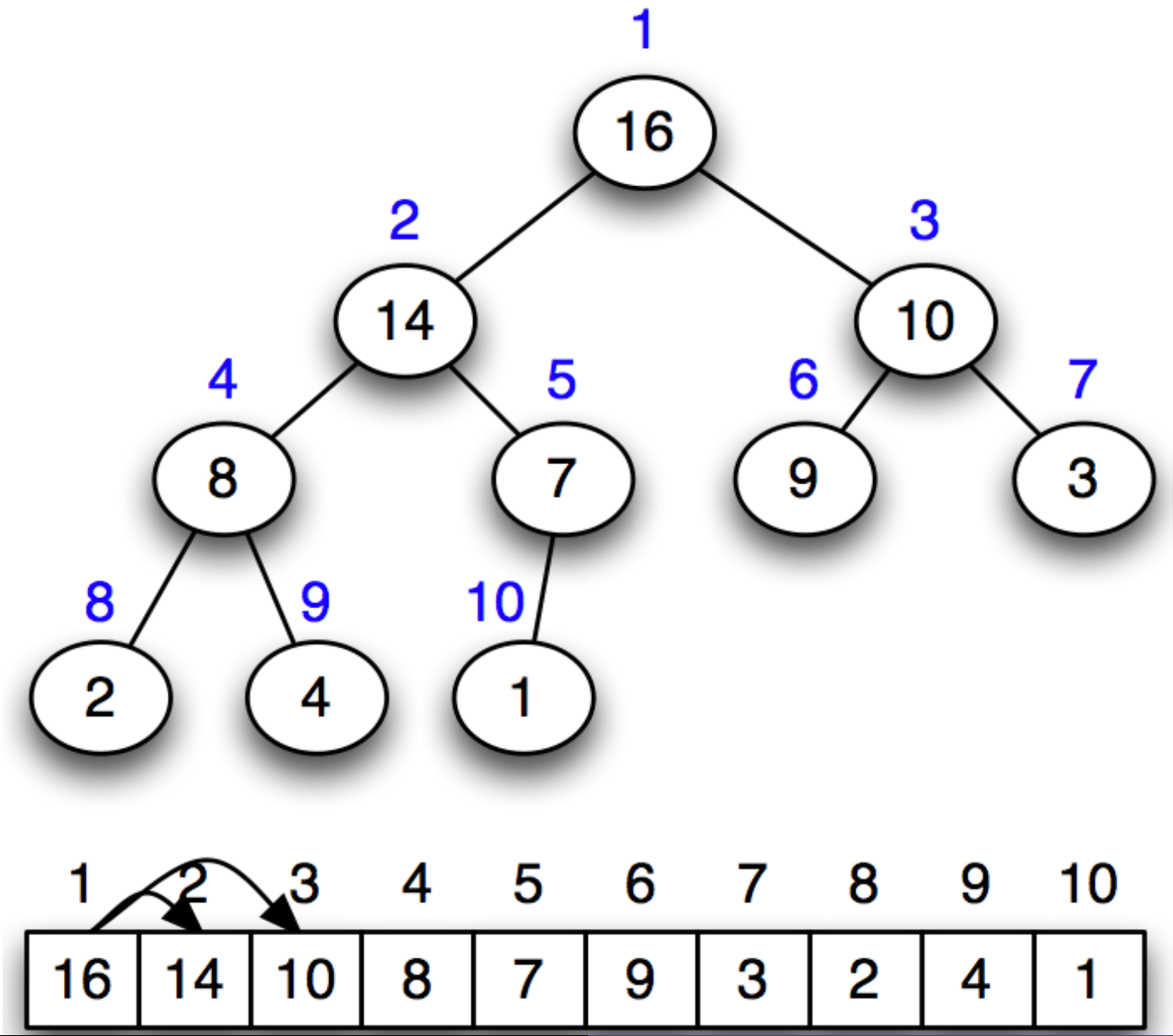

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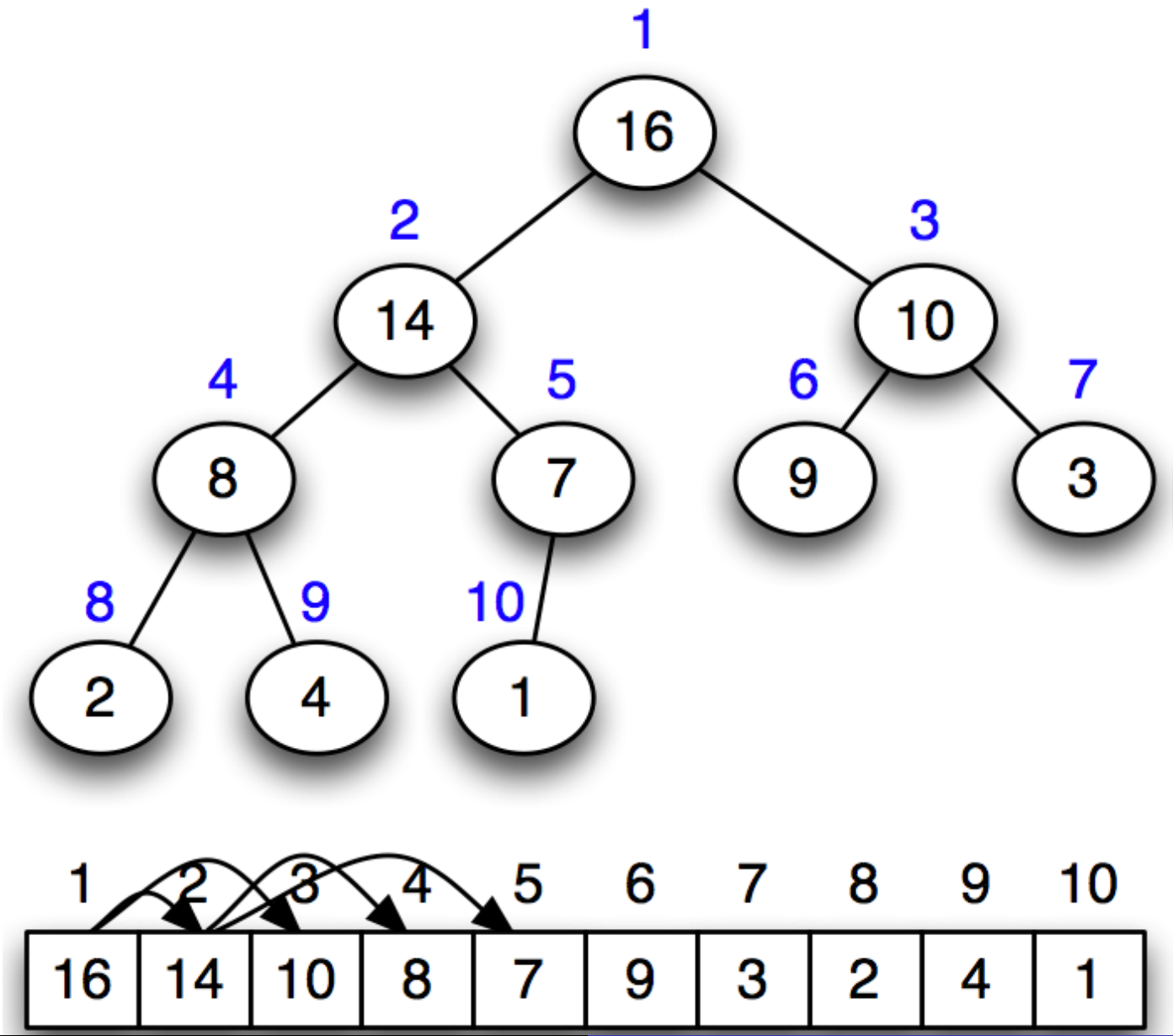

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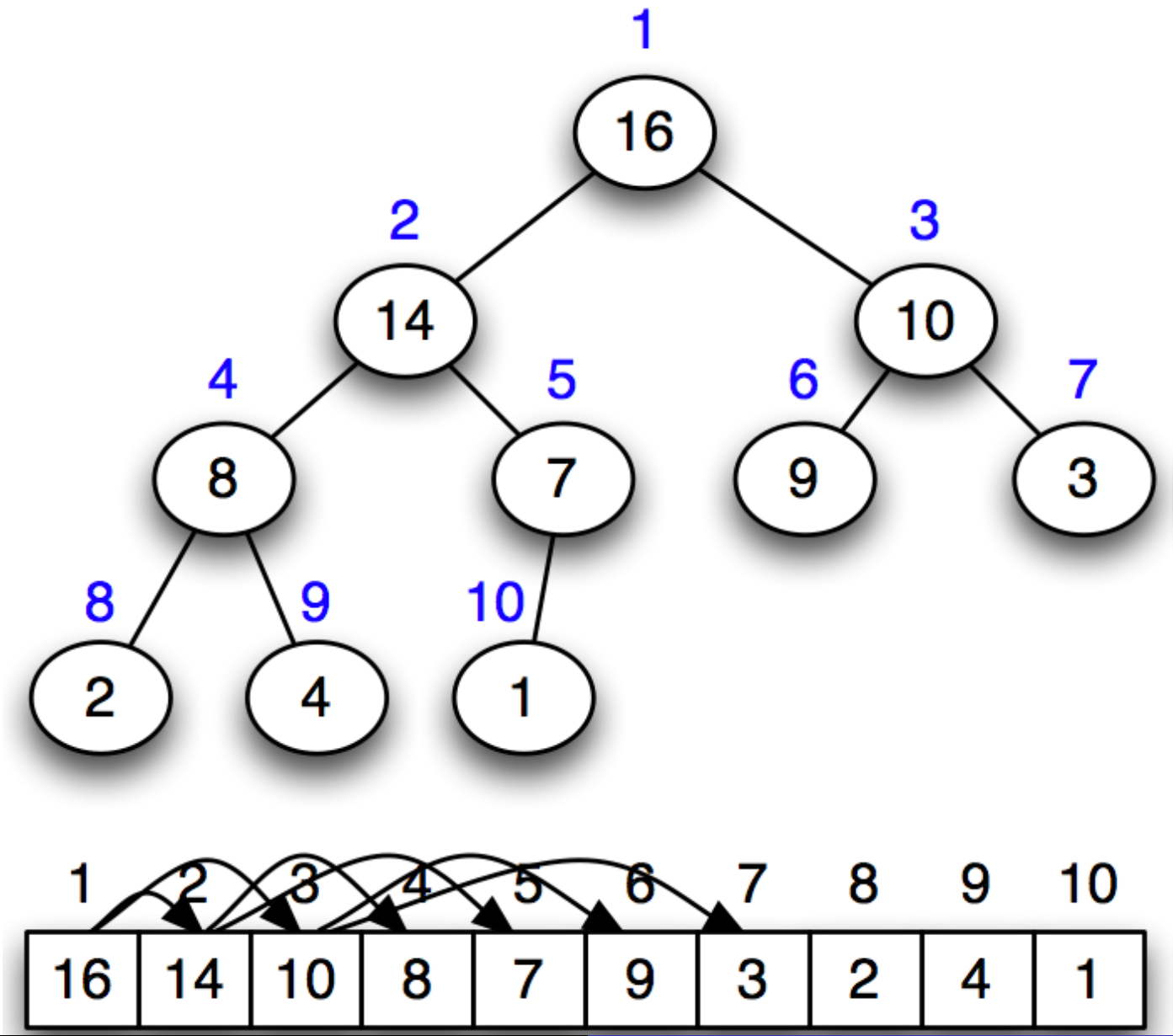

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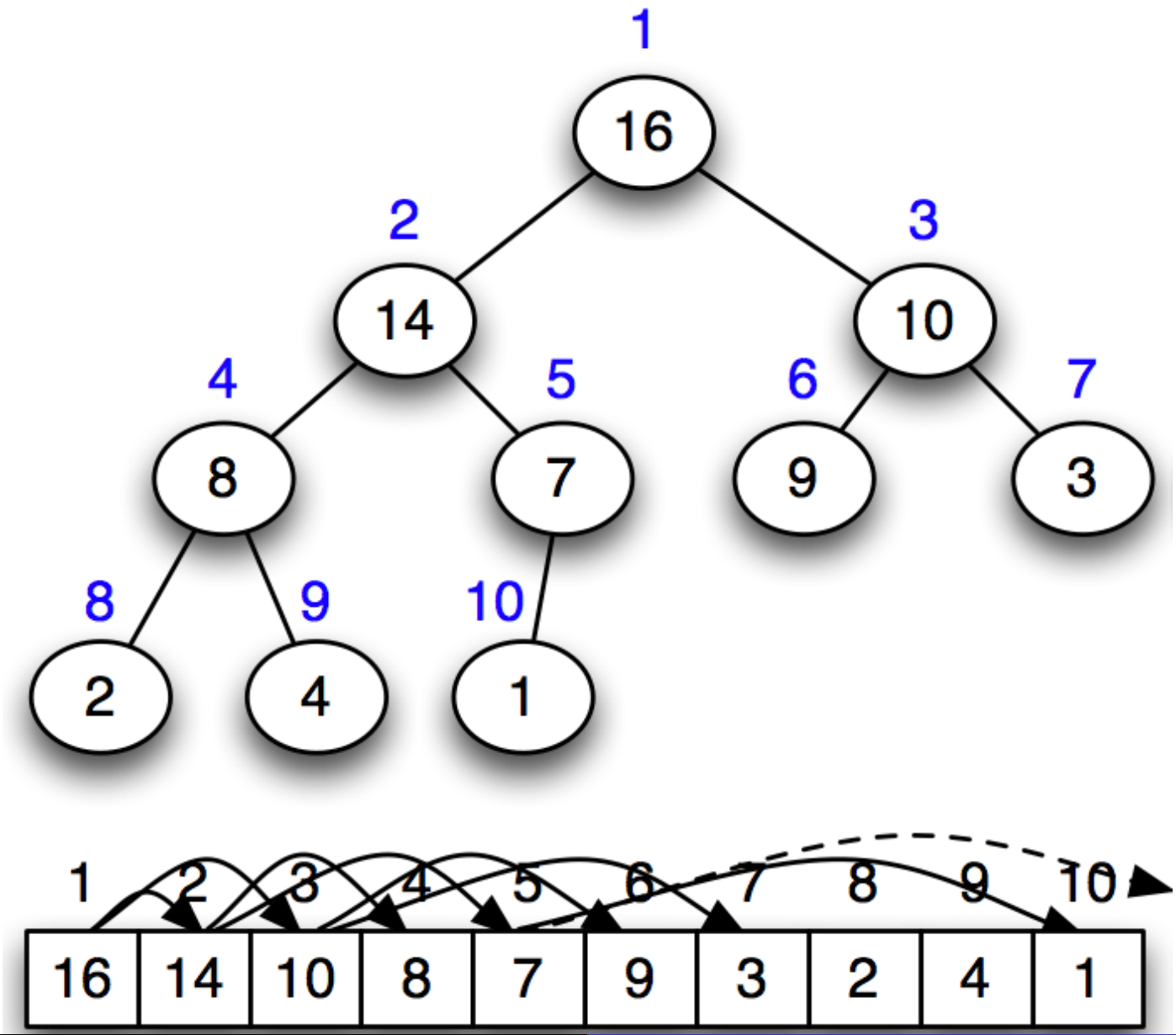
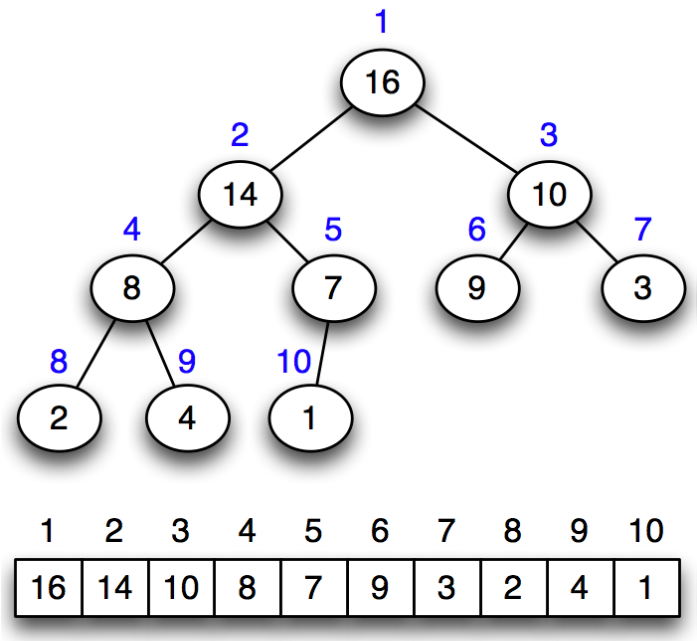


Heap – accessing parent and children



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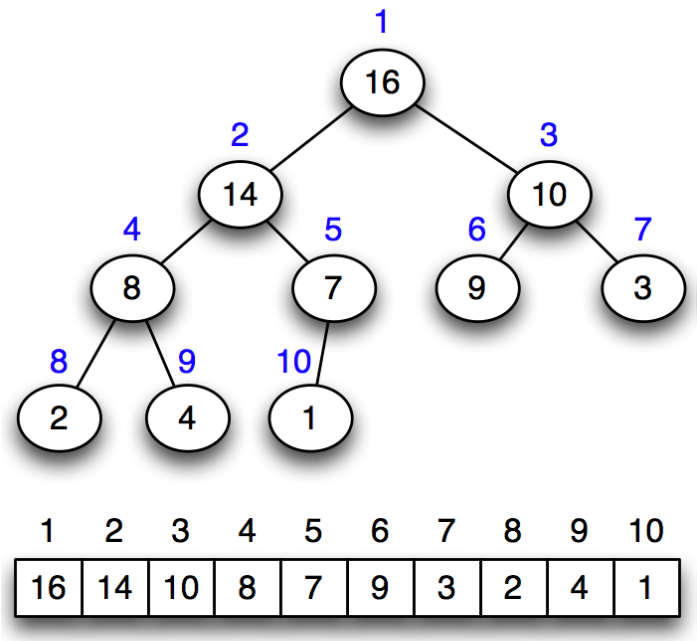

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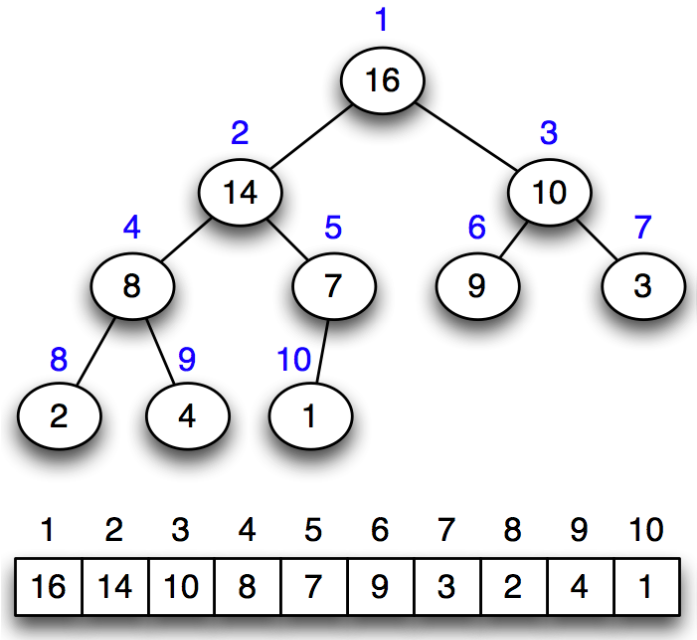

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Question

What if PARENT(i) < 1?

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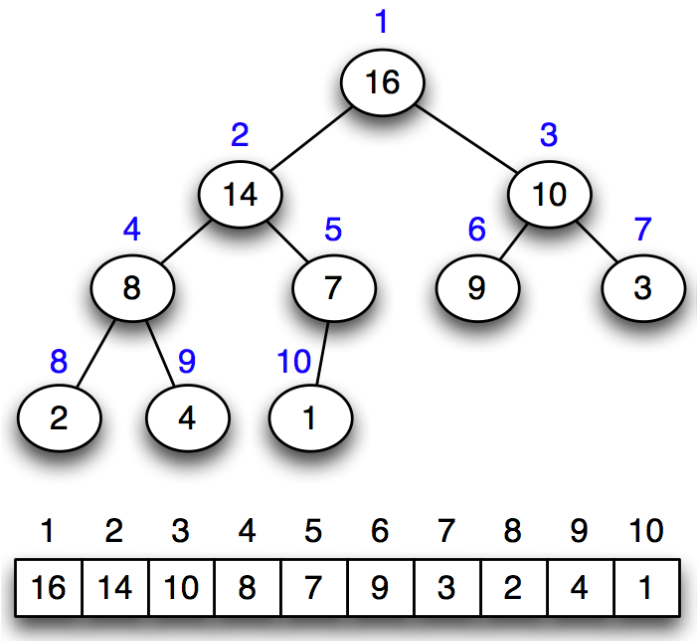

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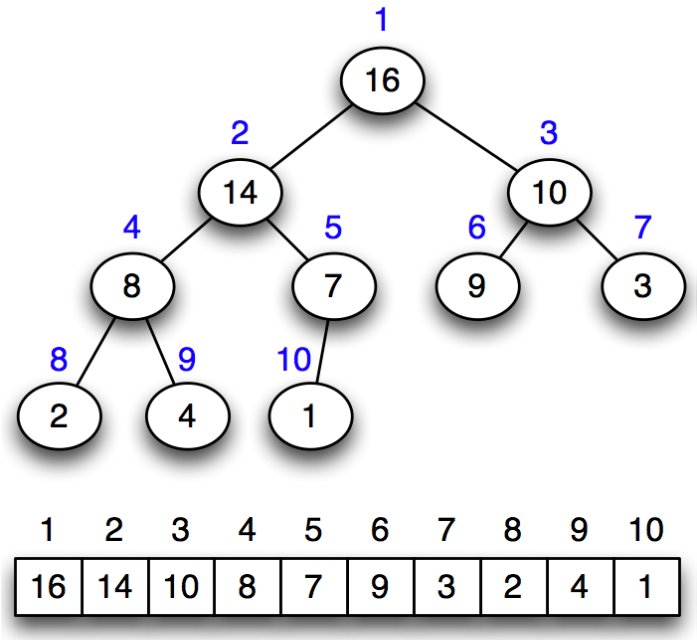

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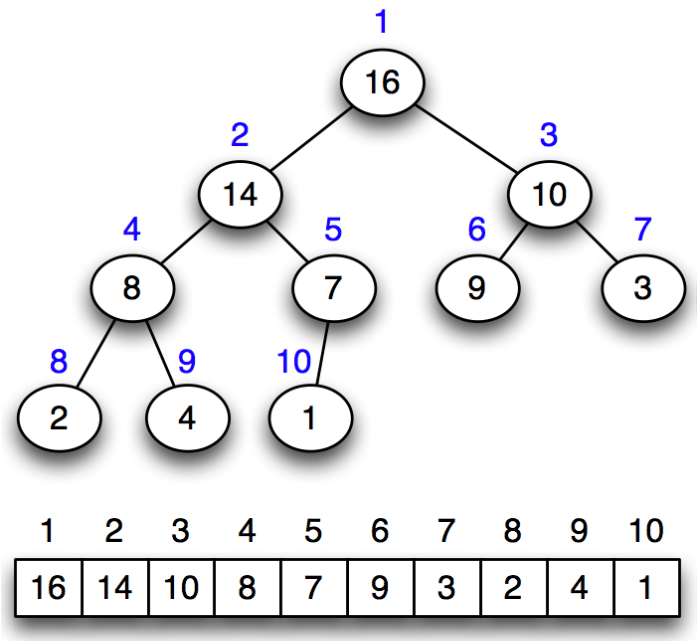
Question

What if LEFT(i) $> n$?

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

PARENT(i)

return $\lfloor i/2 \rfloor$

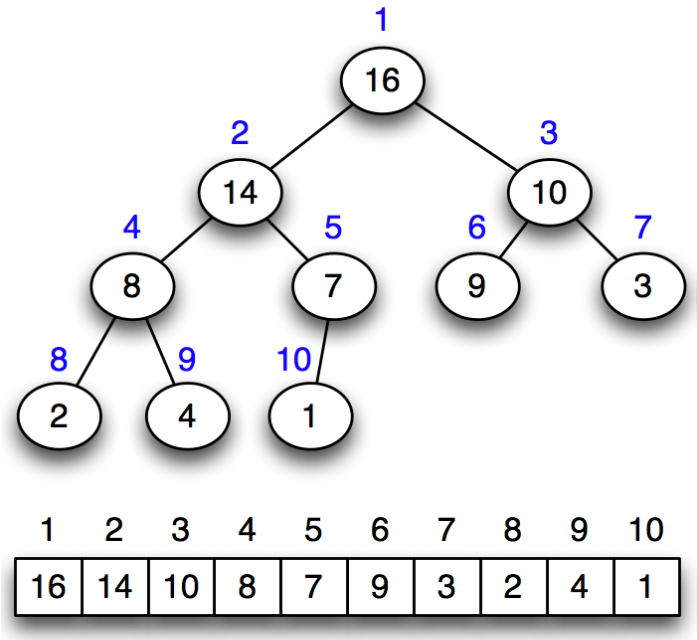
LEFT(i)

return $2i$

RIGHT(i)

return $2i + 1$

Heap – accessing parent and children



MAXIMUM(A)

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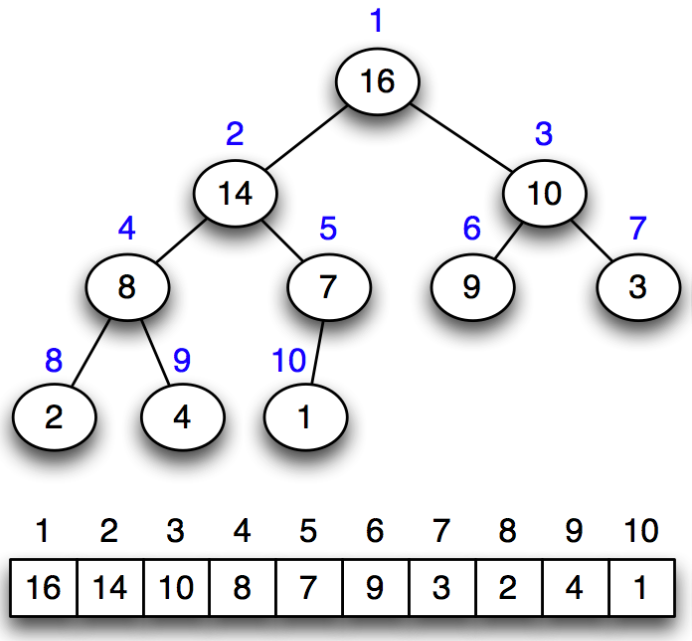
Question

What if RIGHT$(i) > n$?

Heap – accessing parent and children



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return $A[1]$

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LEFT(i)

return $2i$

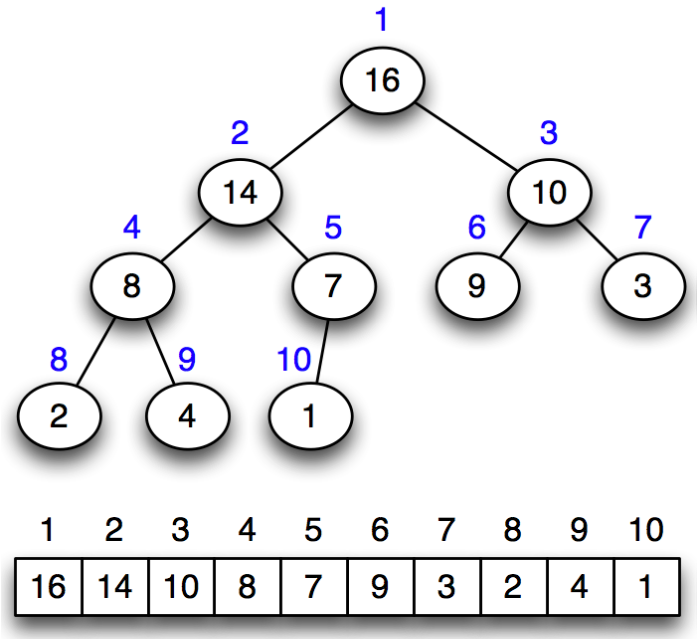
RIGHT(i)

return $2i + 1$

Lemma

All nodes $i > \lfloor length[A]/2 \rfloor$ (or equivalently, $i > \lfloor heap\text{-}size[A]/2 \rfloor$) are leaf nodes.

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

PARENT(i)

return $\lfloor i/2 \rfloor$

LEFT(i)

return $2i$

RIGHT(i)

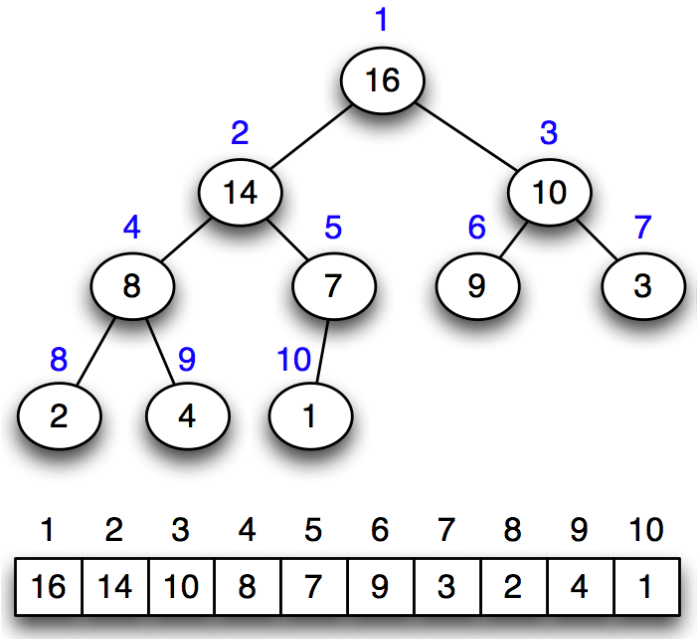
return $2i + 1$

Definition (Heap property)

Heap property: For every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i].$$

Heap – accessing parent and children



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Question

Why do we insist that a heap-ordered tree be a complete binary tree? (Hint: draw the array representation of a tree that is not complete and see the *gaps*).

Contents

Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.

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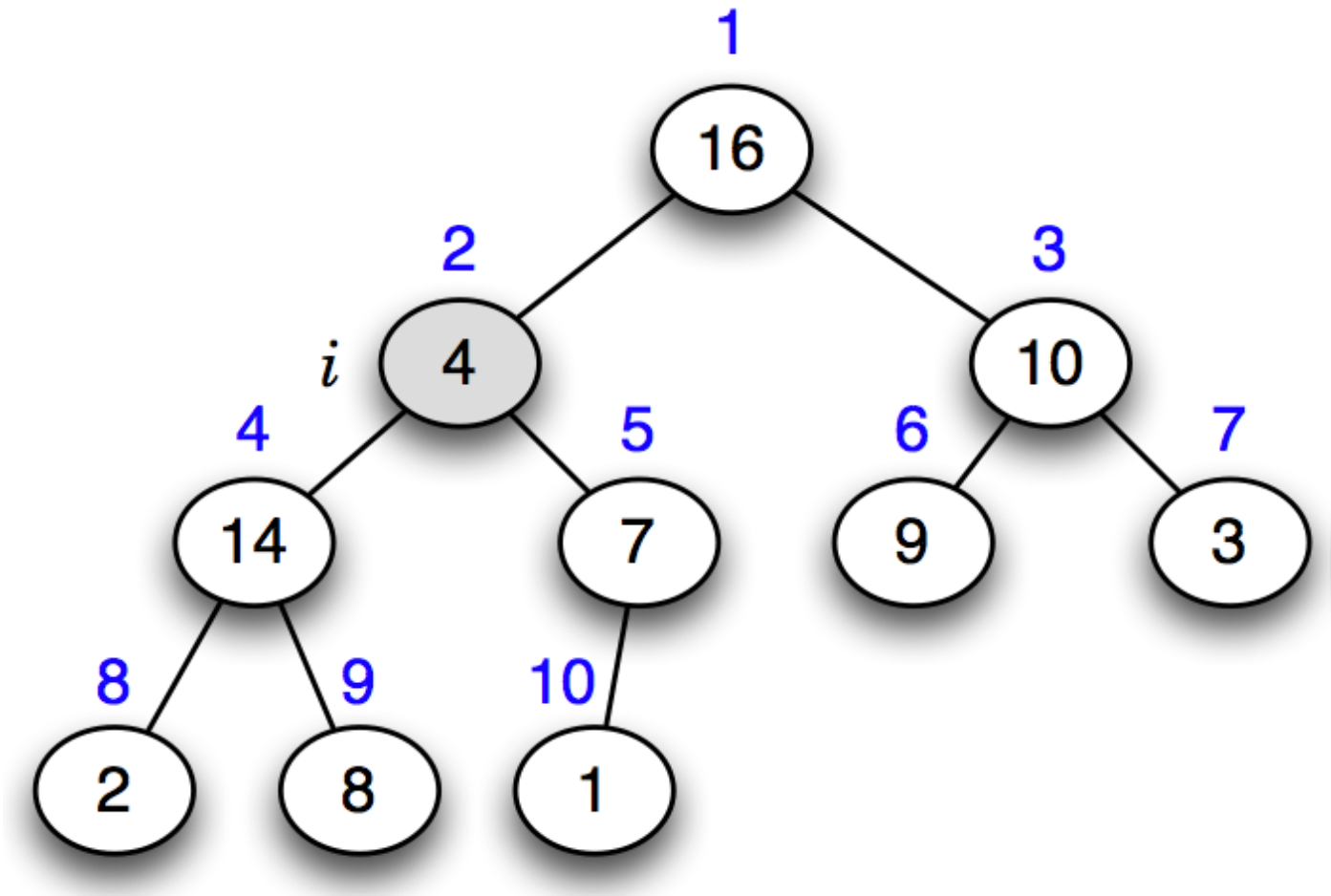
Example of MAX-HEAPIFY ("sink") operation



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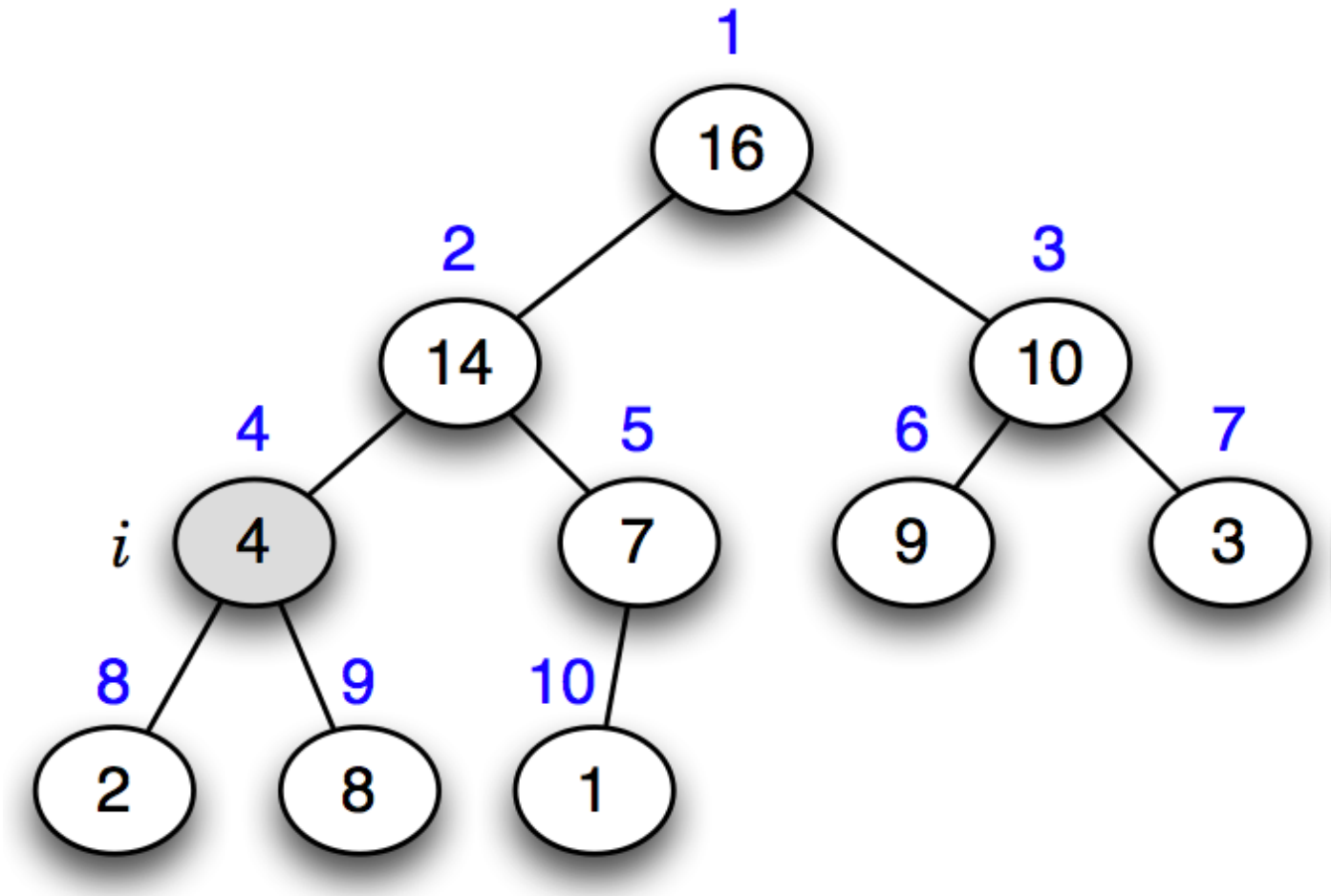

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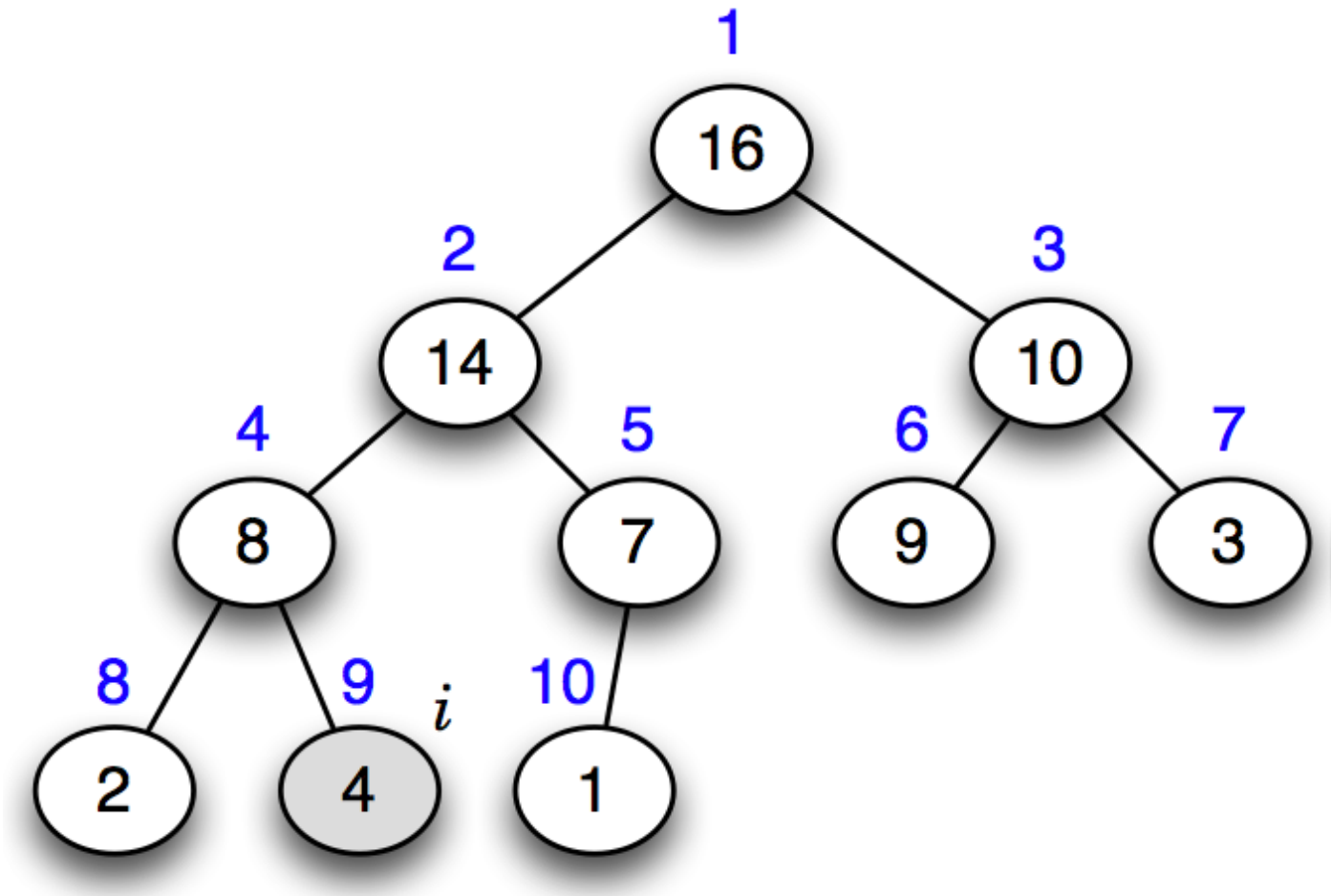

MAX-HEAPIFY algorithm

MAX-HEAPIFY(A, i)

```
 1  l ← left(i)
 2  r ← right(i)
 3  if l ≤ heap-size[A] and A[l] > A[i]
 4      then largest ← l
 5      else largest ← i
 6  if r ≤ heap-size[A] and A[r] > A[largest]
 7      then largest ← r
 8  if largest ≠ i
 9      then exchange A[i] ↔ A[largest]
10           MAX-HEAPIFY(A, largest)
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Analysis – first way

Since the children's subtrees each have at most size of $2n/3$ (when the last row is exactly half full), we have

$$T(n) \leq T(2n/3) + \Theta(1).$$

According to case 2 of the Master theorem, $T(n) = O(\lg n)$.

MAX-HEAPIFY algorithm

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10           MAX-HEAPIFY(A, largest)
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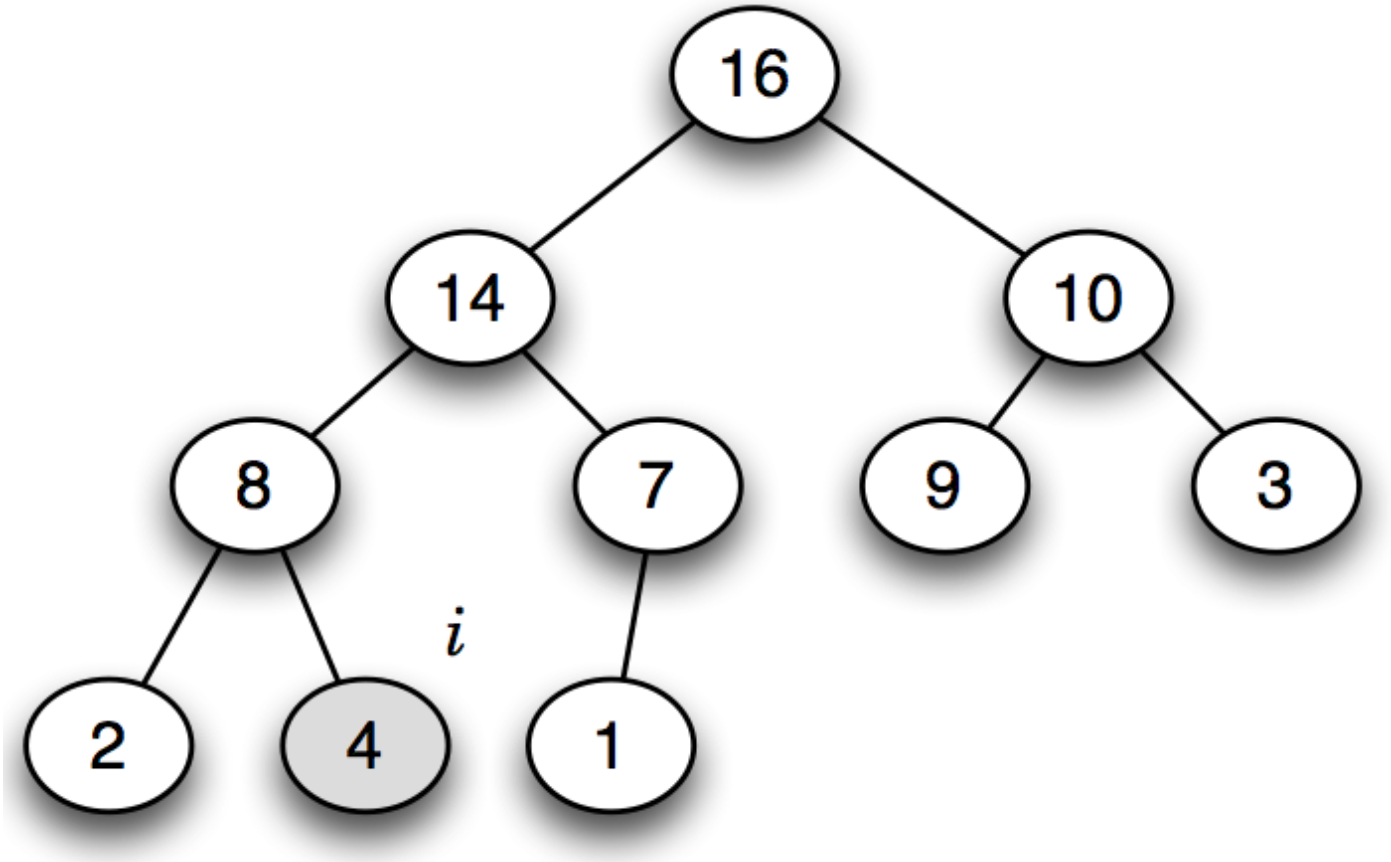
Analysis – second way

The running time of MAX-HEAPIFY on a node of height h is $T(n) = O(h) = O(\lg n)$.

Example of HEAP-INCREASE-KEY ("swim") operation



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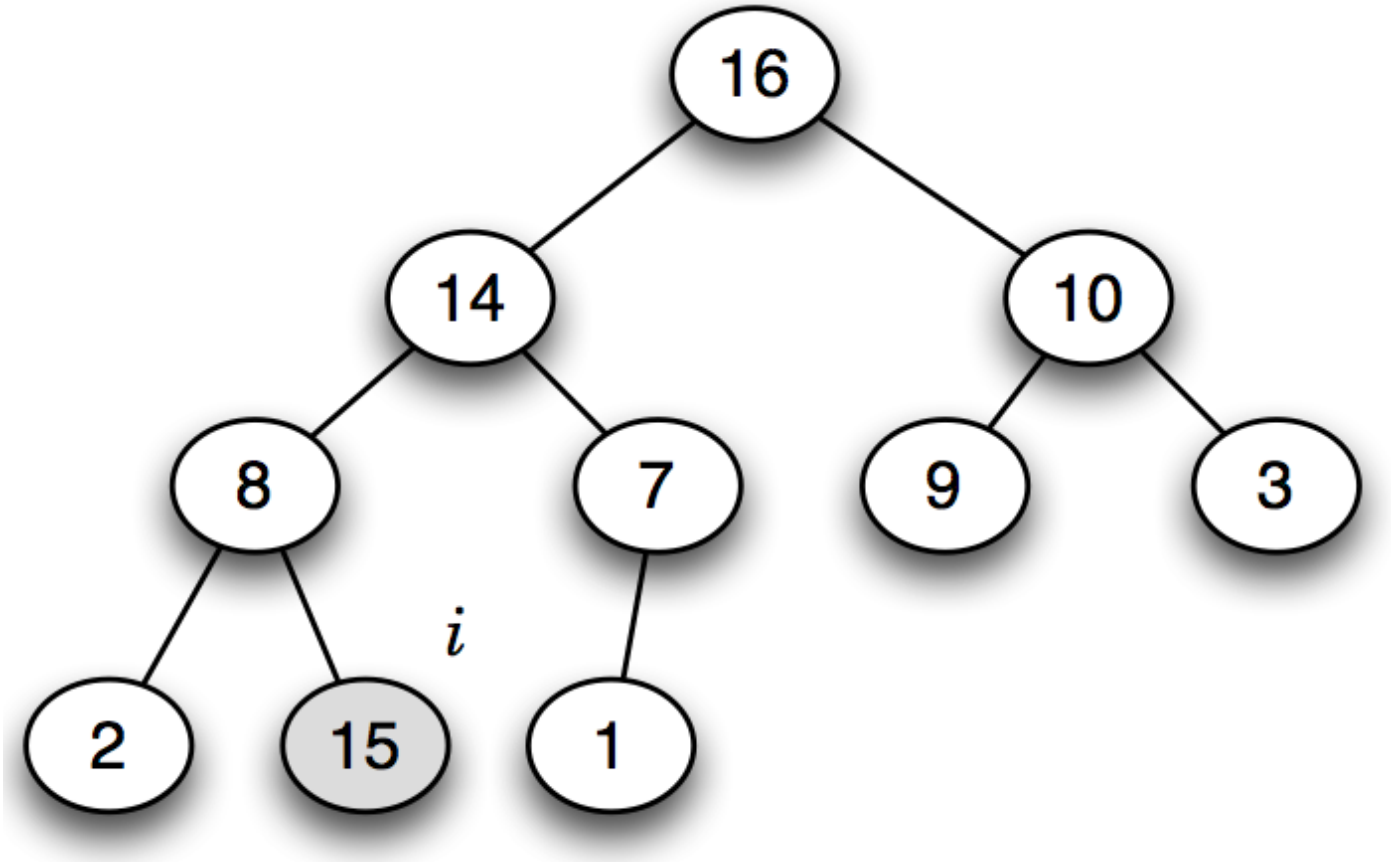

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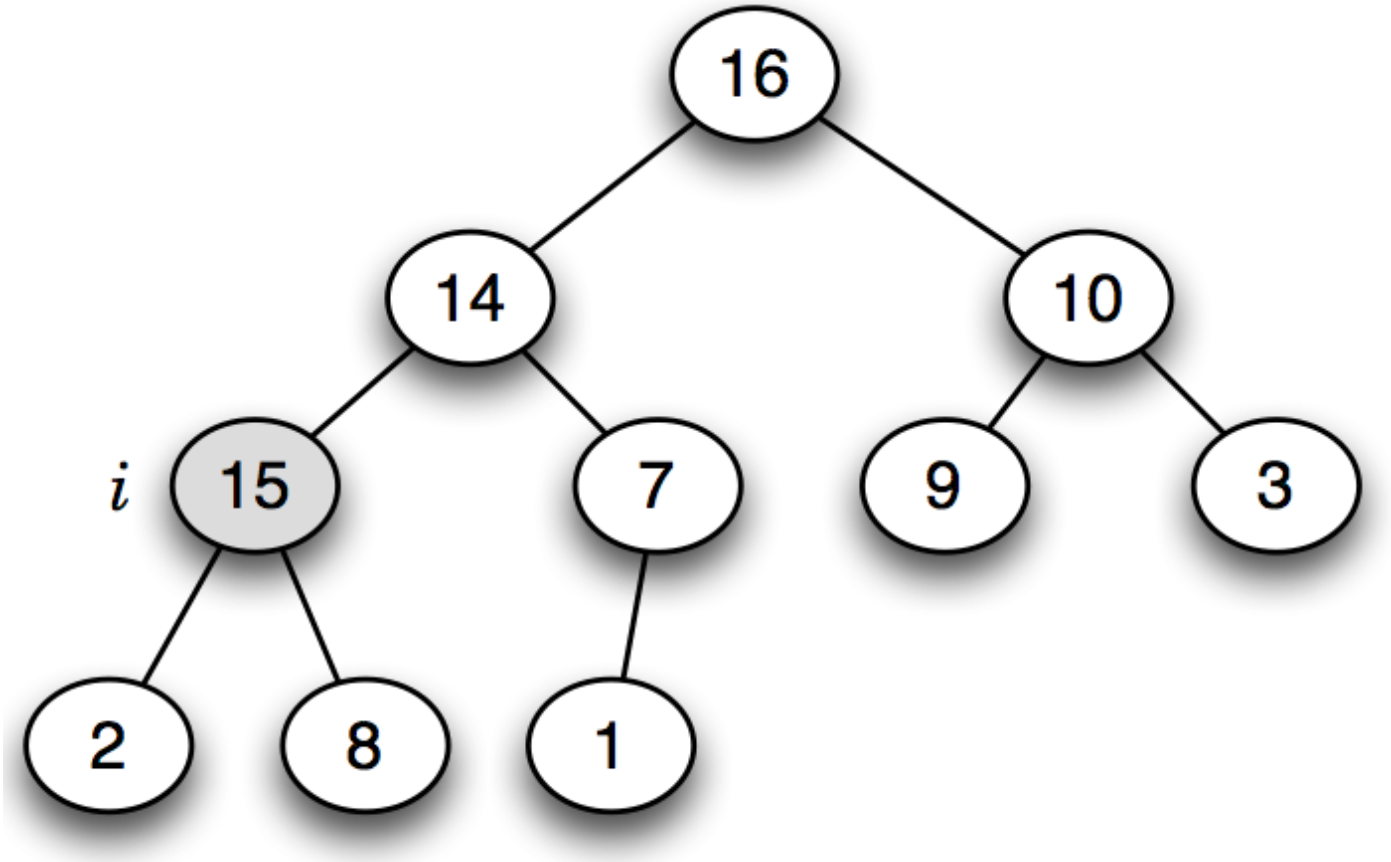

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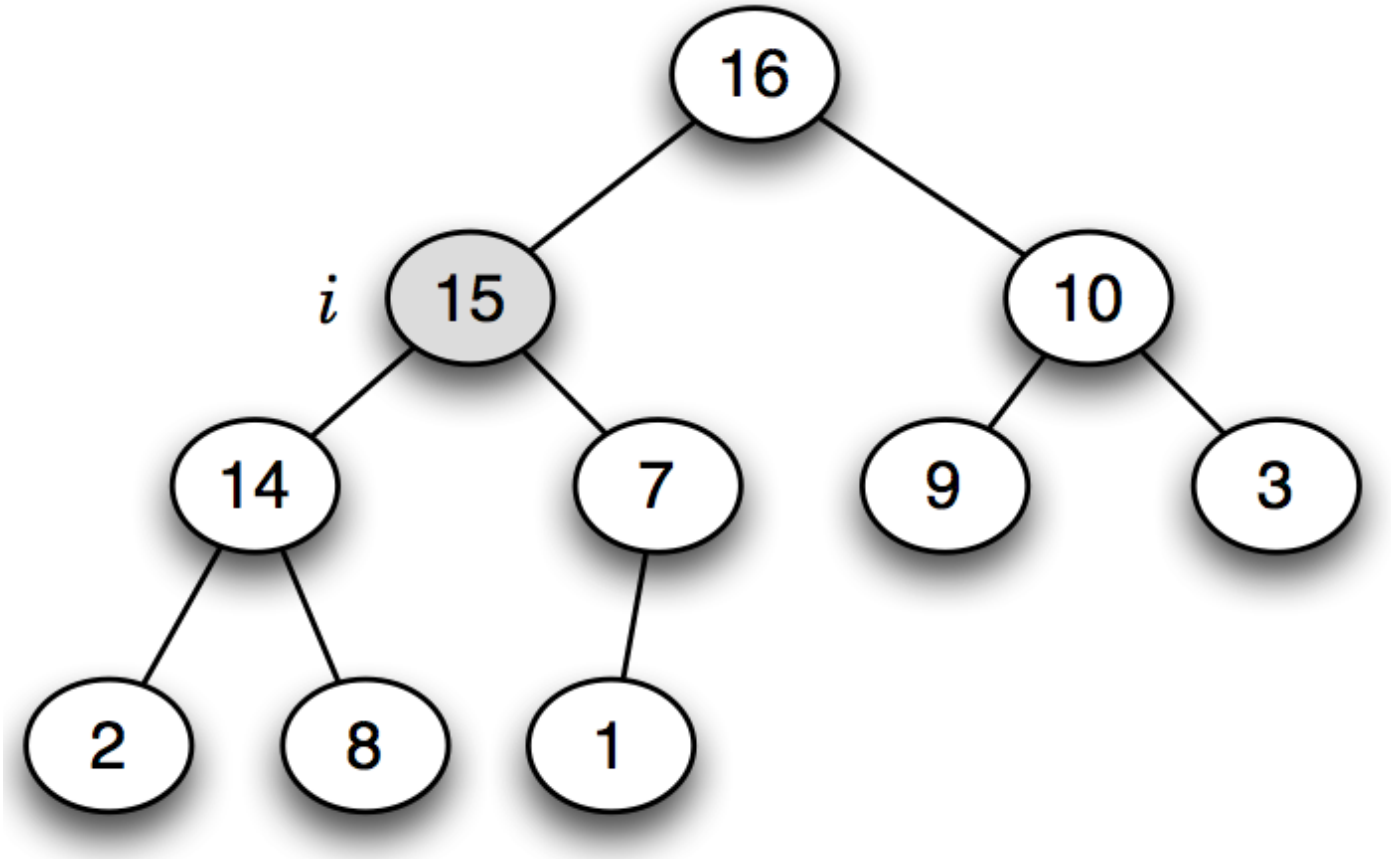

HEAP-INCREASE-KEY algorithm

HEAP-INCREASE-KEY(A, i, key)

```
1  if key < A[i]
2      then error "new key is smaller than current key"
3  A[i] ← key
4  while i > 1 and A[PARENT(i)] < A[i]
5        do exchange A[i] ↔ A[parent(i)]
6           i ← PARENT(i)
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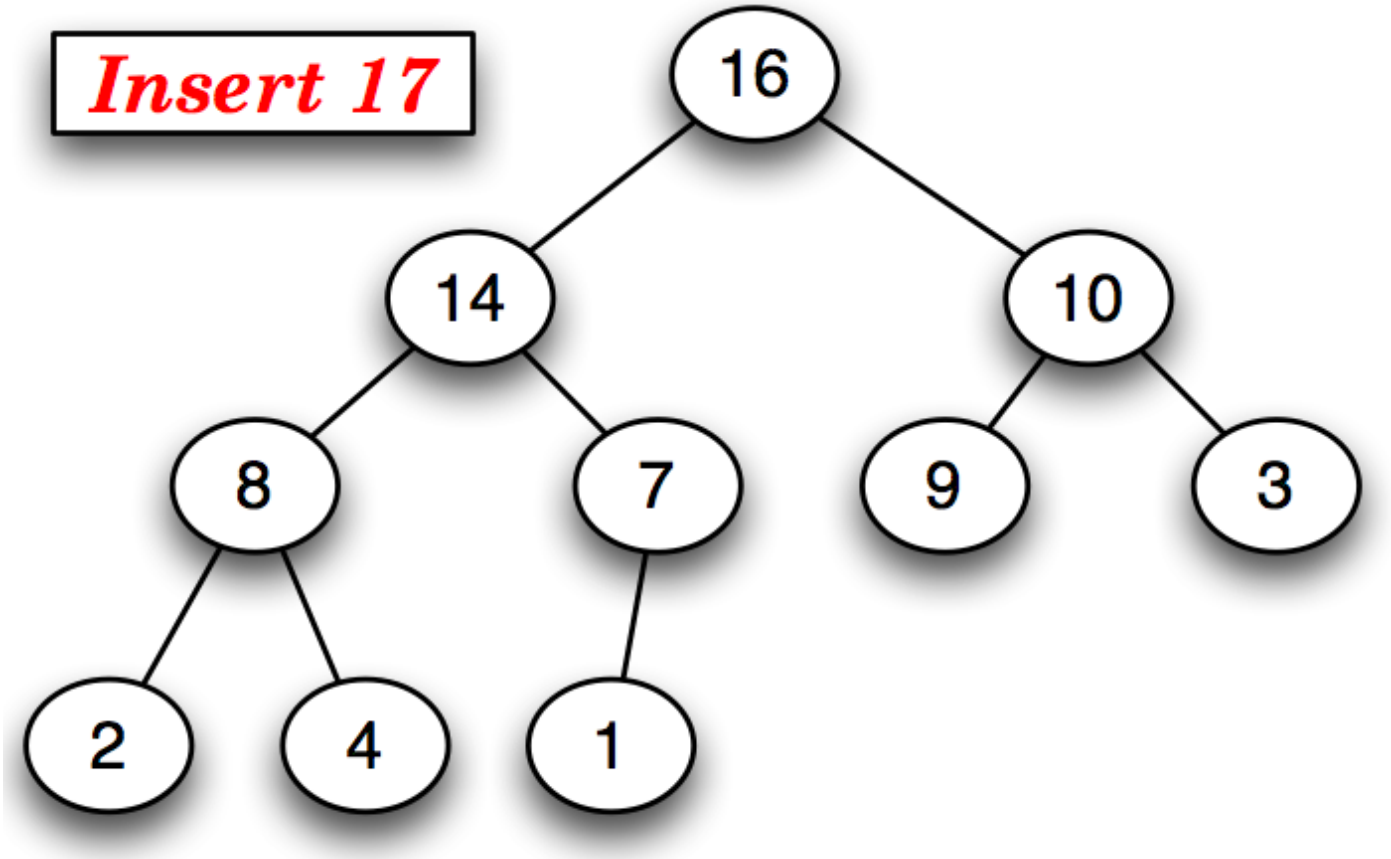
Analysis

A node may move all the way from a leaf node to the root because of increased value, so $T(n) = O(h) = O(\lg n)$.

Example of MAX-HEAP-INSERT operation



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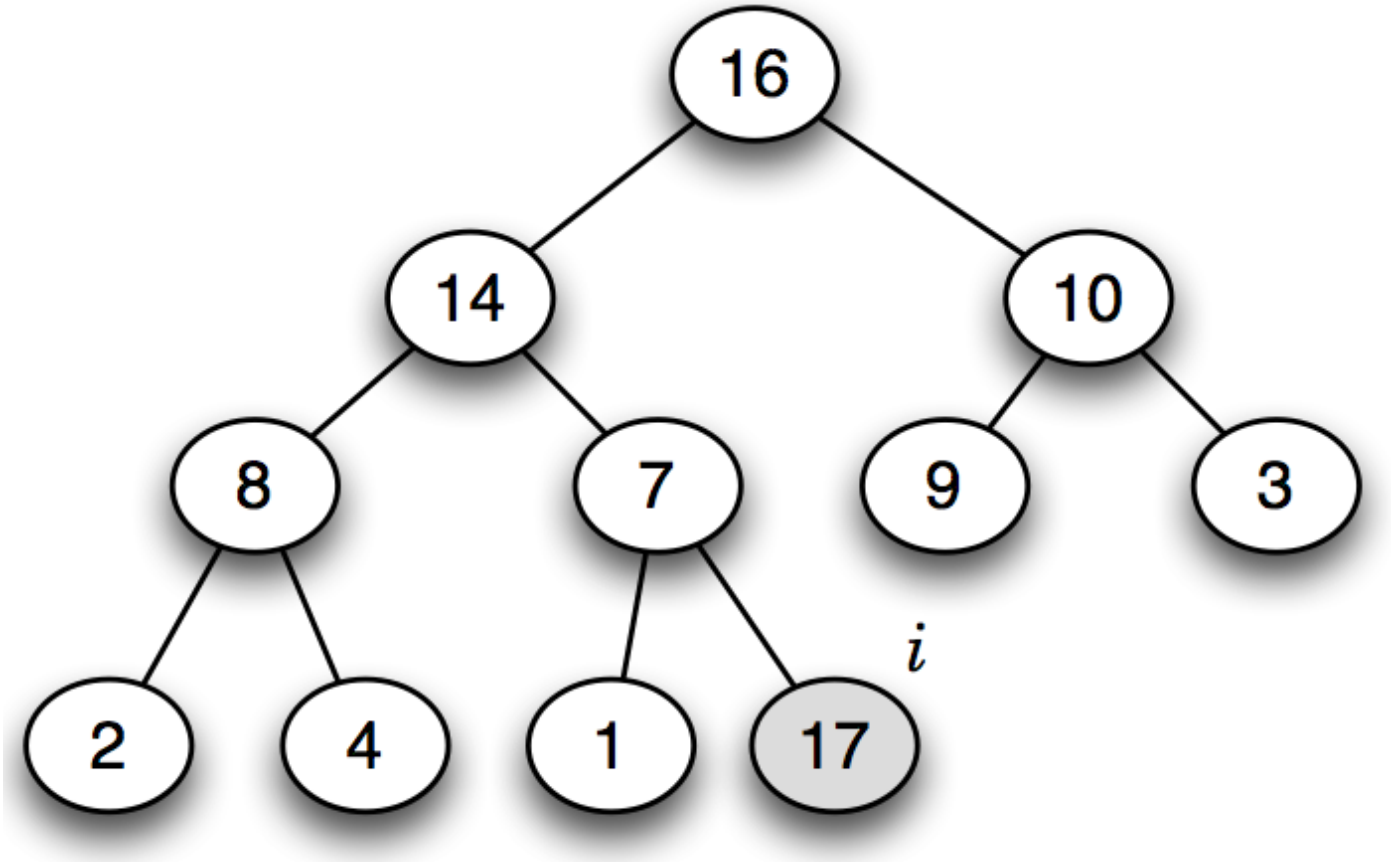

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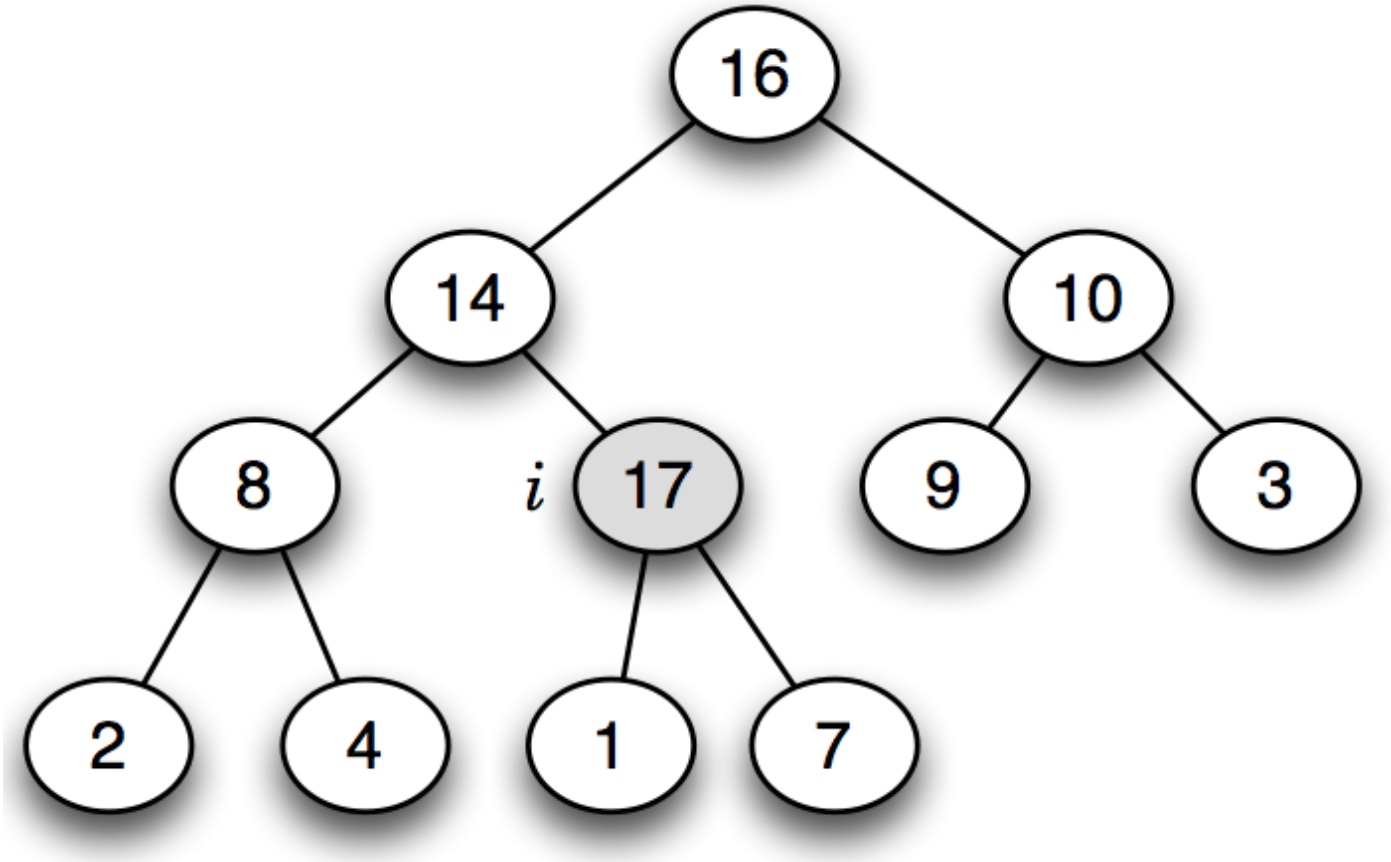

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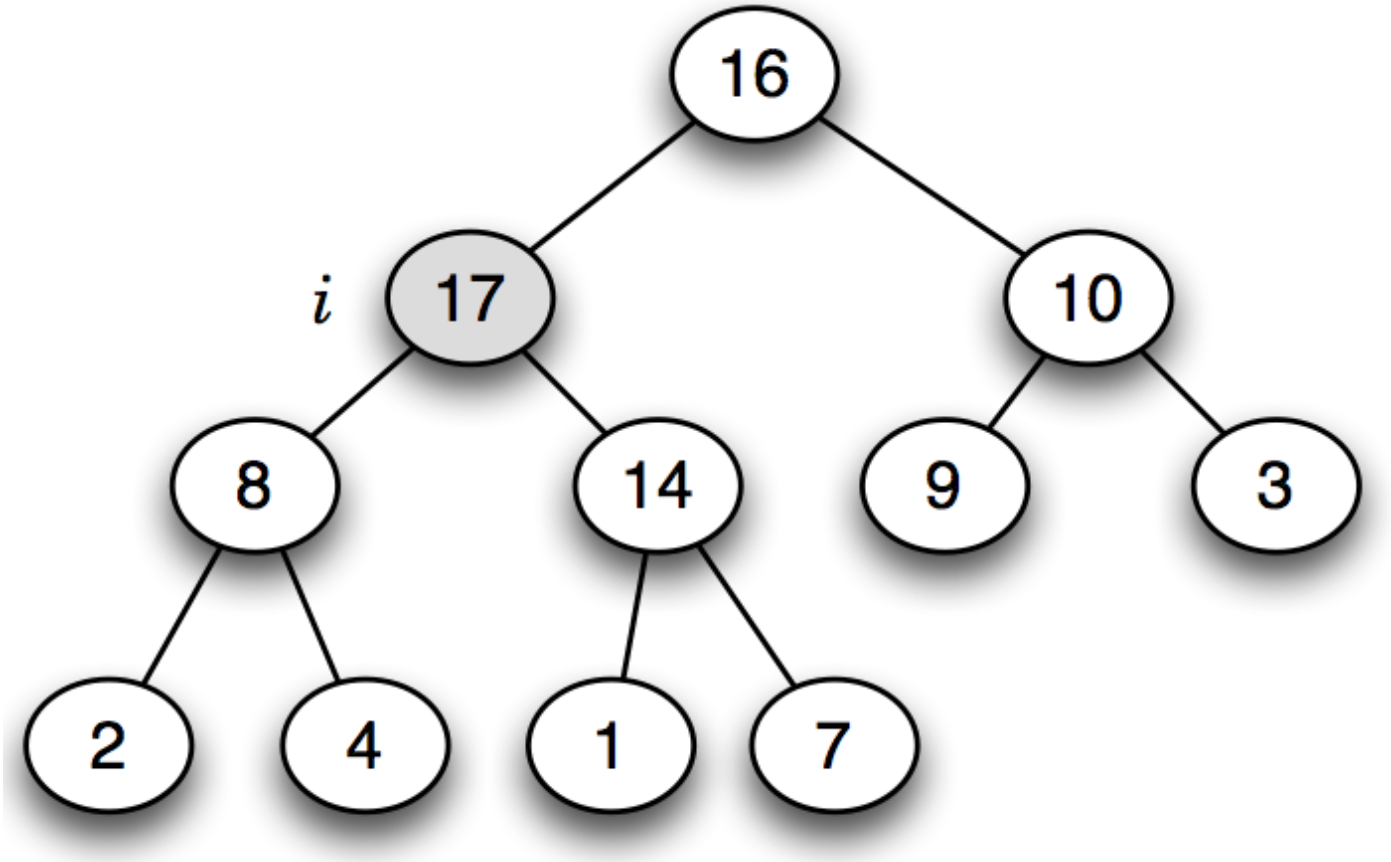

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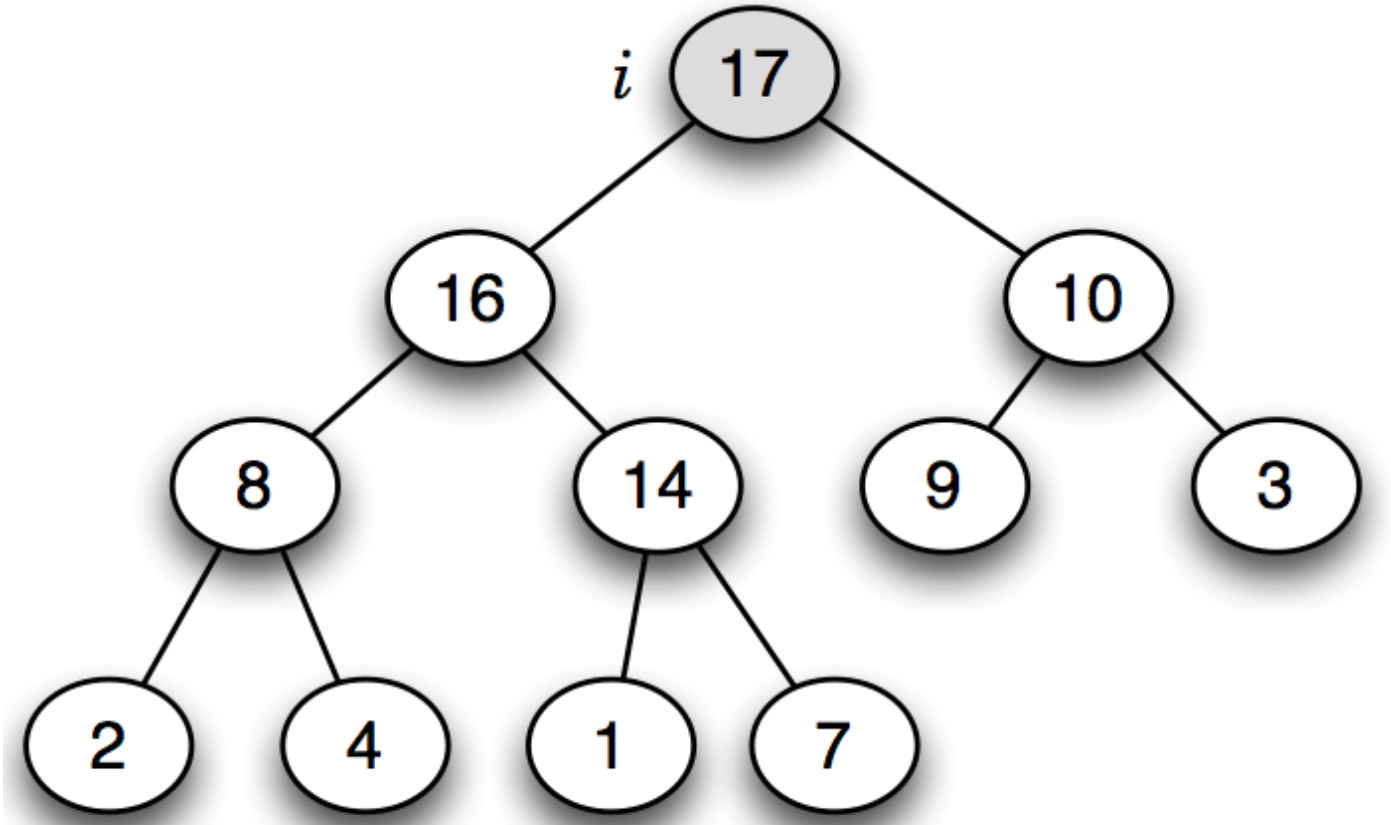

MAX-HEAP-INSERT algorithm

MAX-HEAP-INSERT(A, key)

1 $heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
2 $A[heap\text{-}size[A]] \leftarrow key$
3 $i \leftarrow heap\text{-}size[A]$
4 **while** $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
5 **do** exchange $A[i] \leftrightarrow A[parent(i)]$
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Can also be done using HEAP-INCREASE-KEY.

MAX-HEAP-INSERT(A, key)

1 $heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
2 $A[heap\text{-}size[A]] \leftarrow -\infty$
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Analysis

$T(n) = O(h) = O(\lg n)$.

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

1 $heap\text{-}size[A] \leftarrow 1$
2 **for** $i \leftarrow 2$ **to** $length[A]$
3 **do** MAX-HEAP-INSERT$(A, A[i])$

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There are $n - 1$ calls to MAX-HEAP-INSERT, each taking $O(\lg n)$ time, so $T(n) = O(n \lg n)$.

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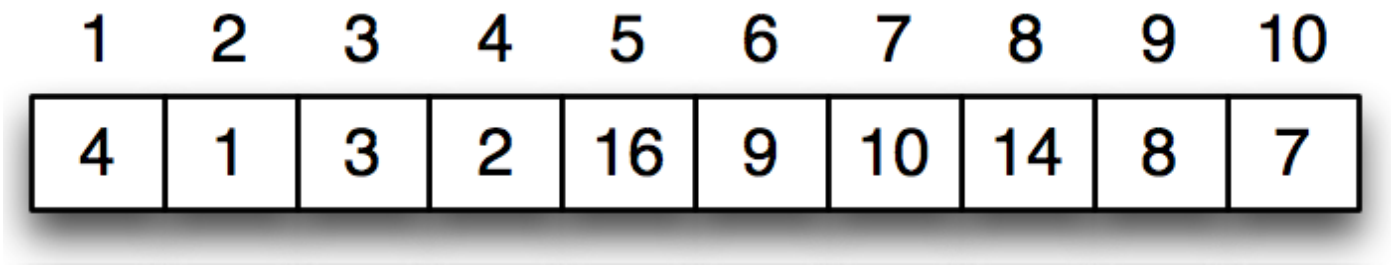
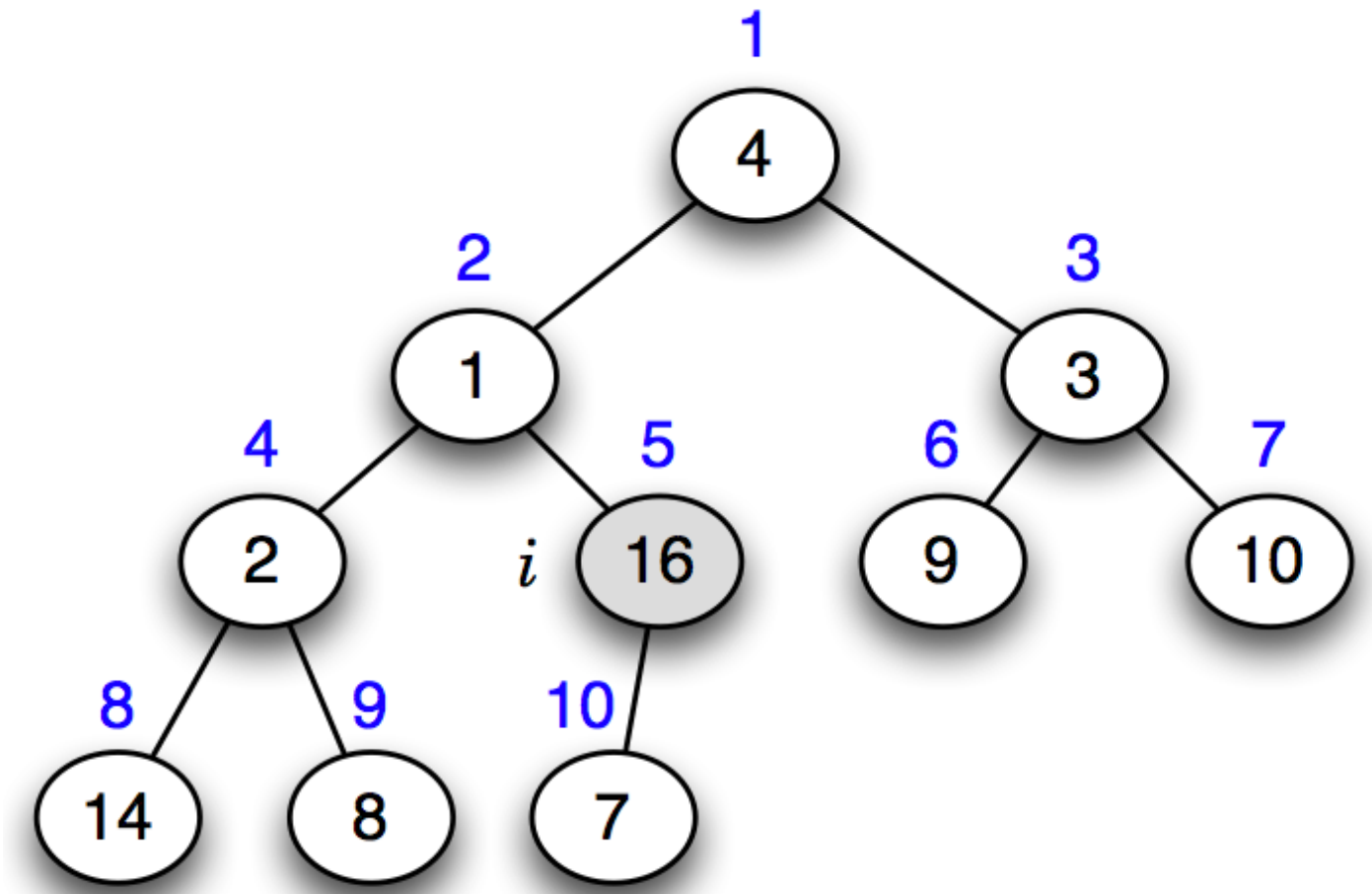
Better way?

A better way is to build up the heap from the smaller trees. See next.

Example of BUILD-MAX-HEAP ("heapify") operation





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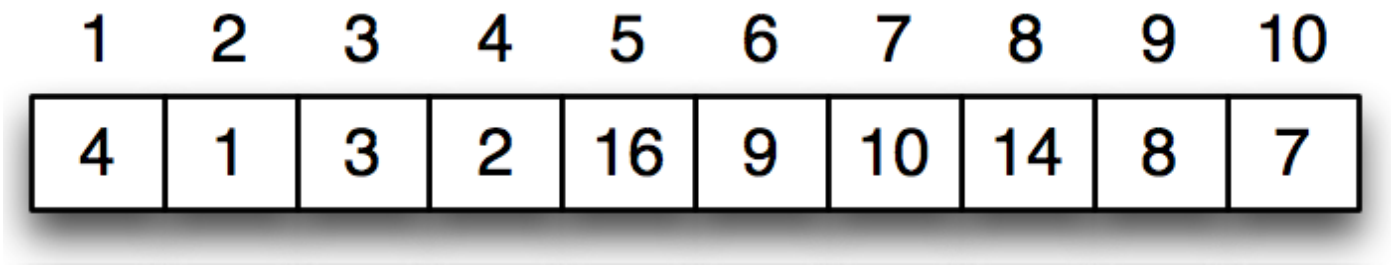

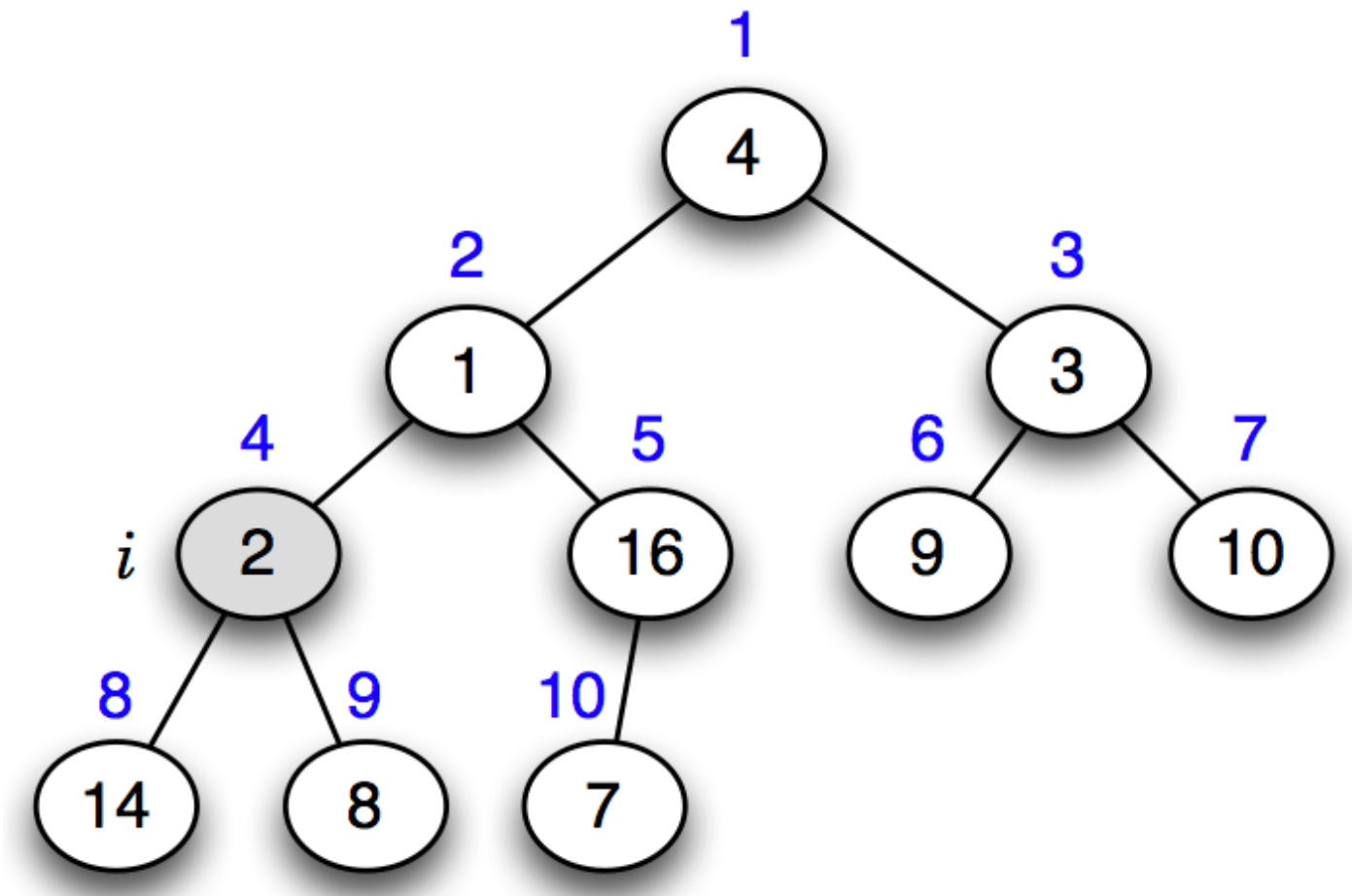

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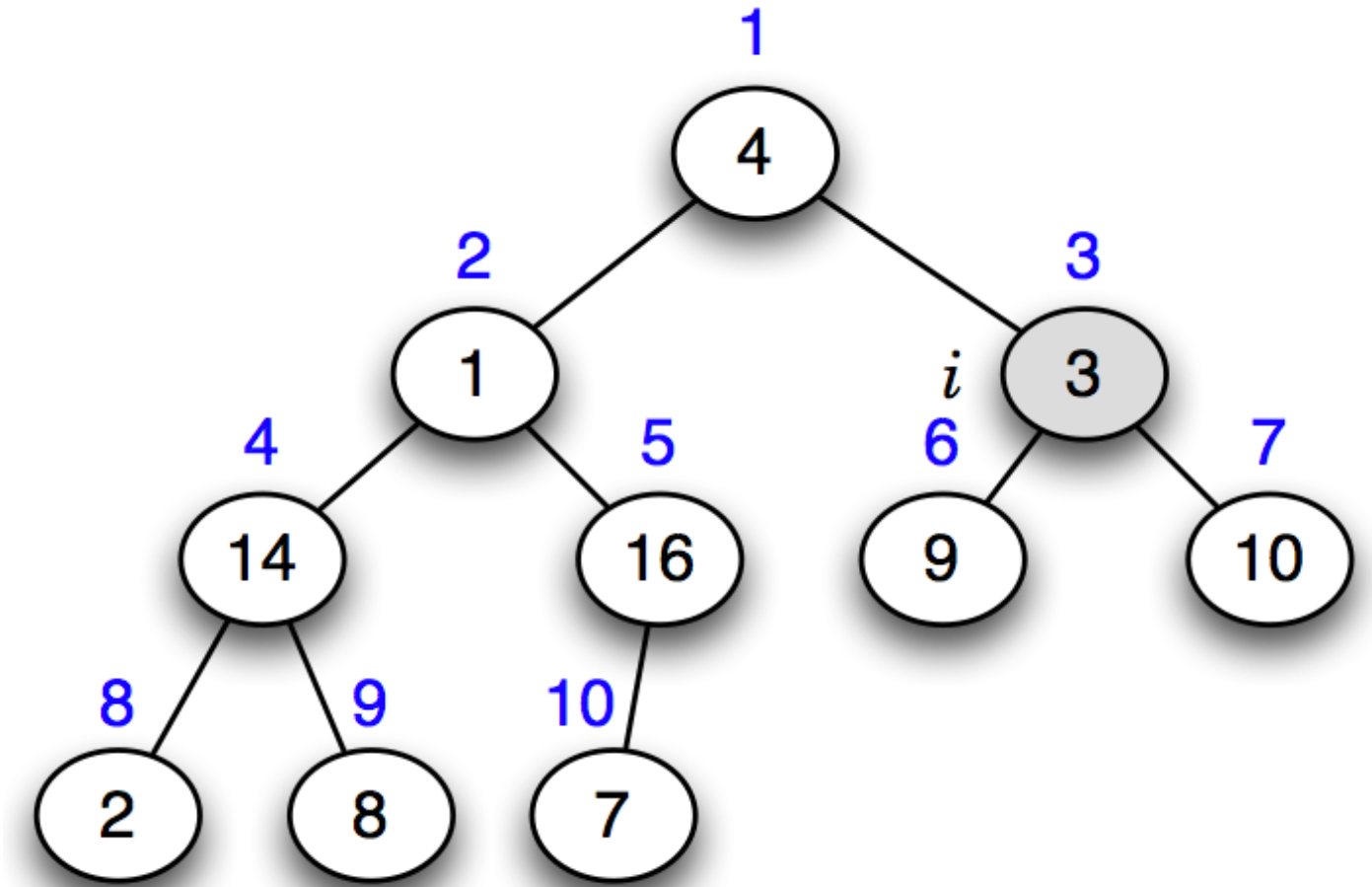
1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



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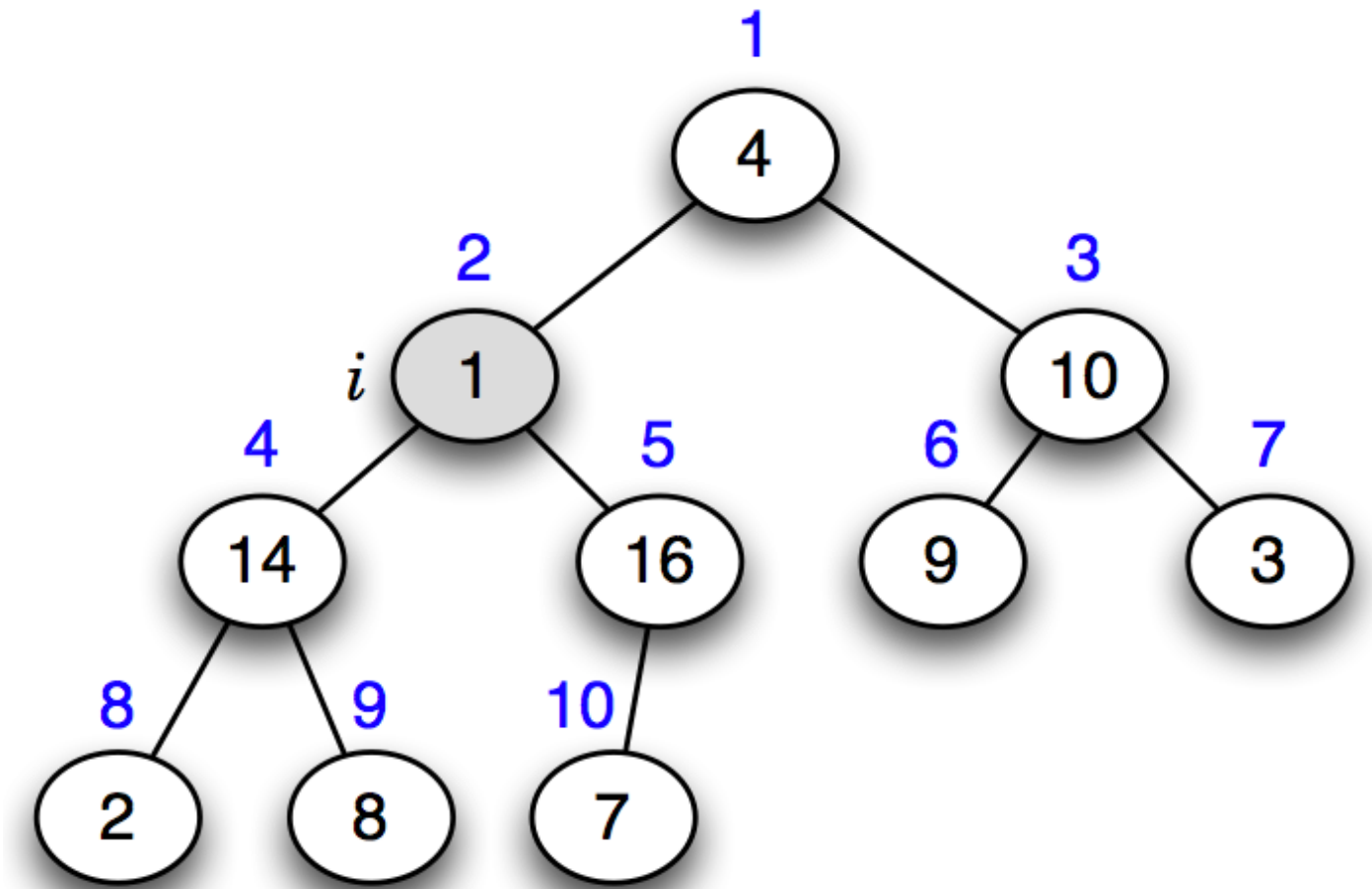

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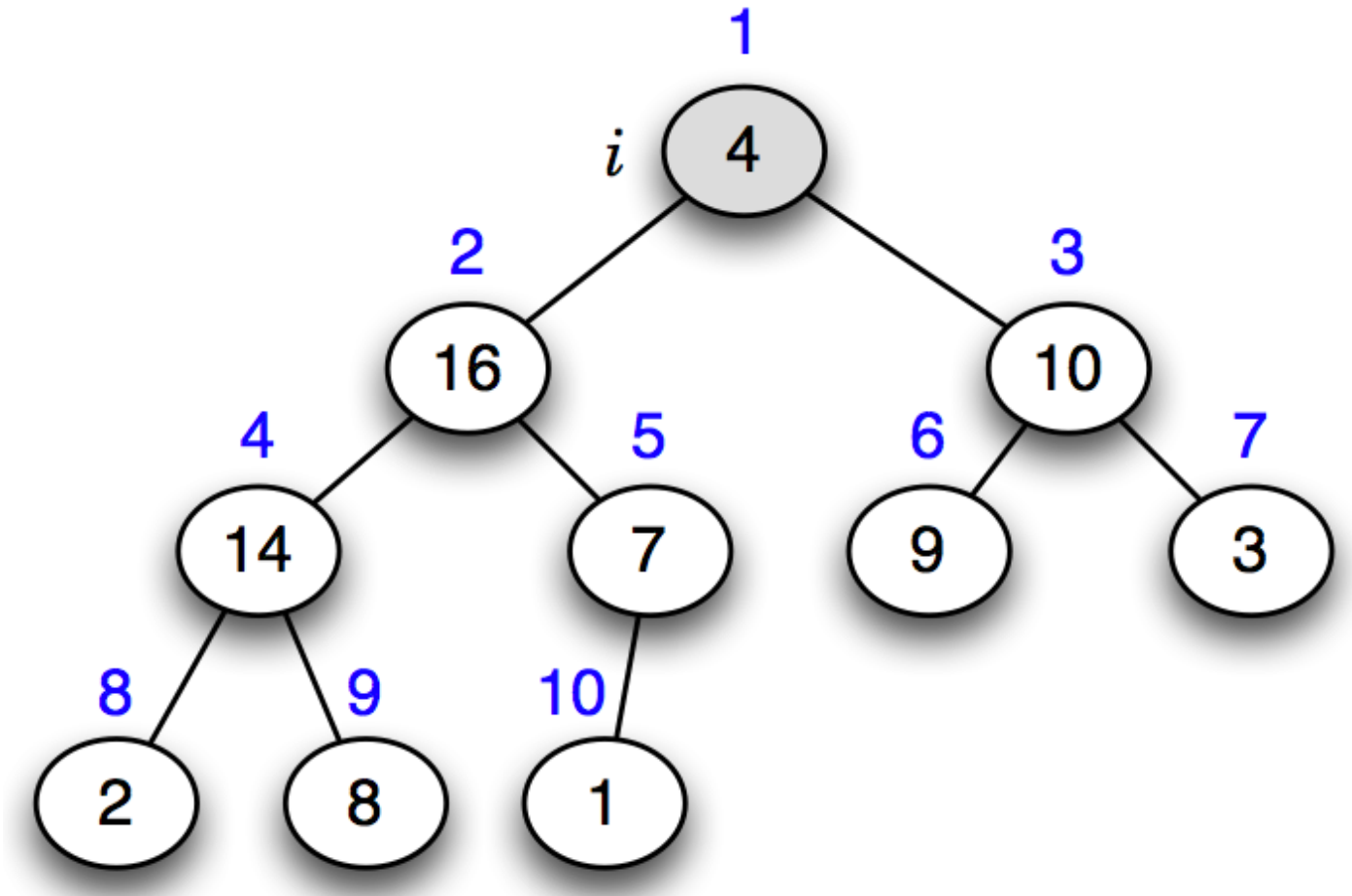

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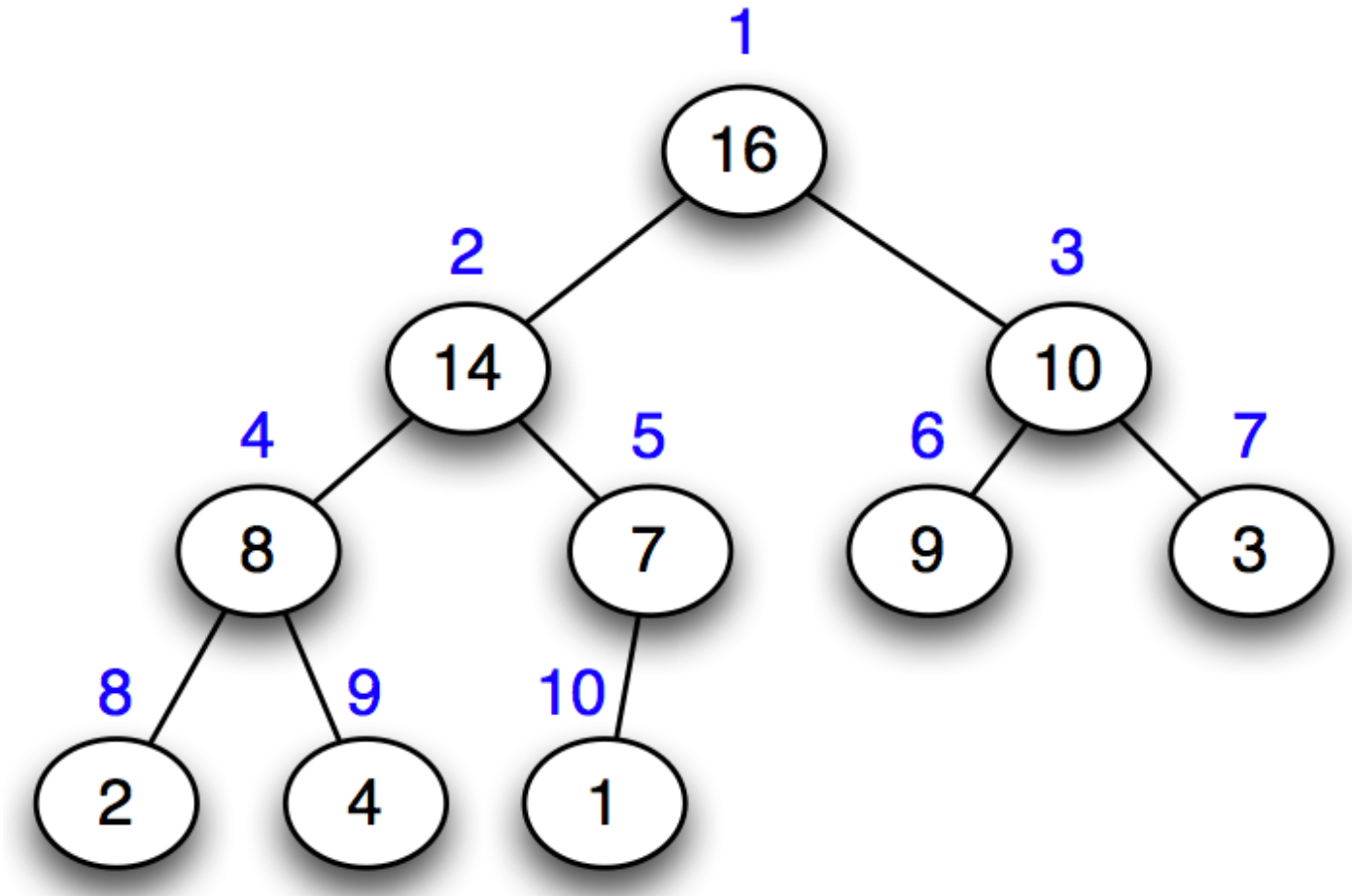

BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

1 $heap\text{-}size[A]] \leftarrow length[A]$
2 **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3 **do** MAX-HEAPIFY(A, i)

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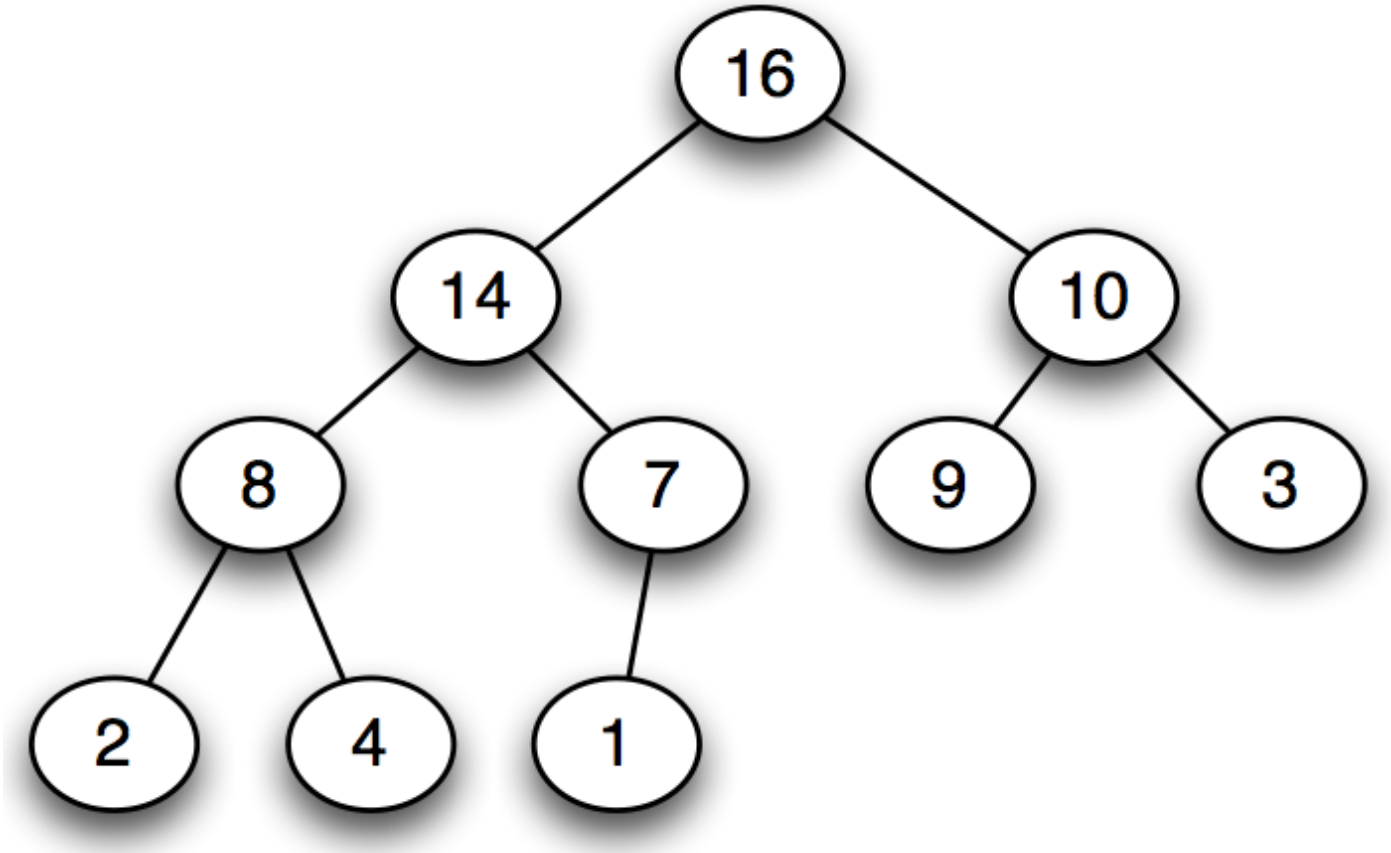
Analysis

$T(n) = O(n)$ (see textbook for details)

Contents

Heap use - heapsort



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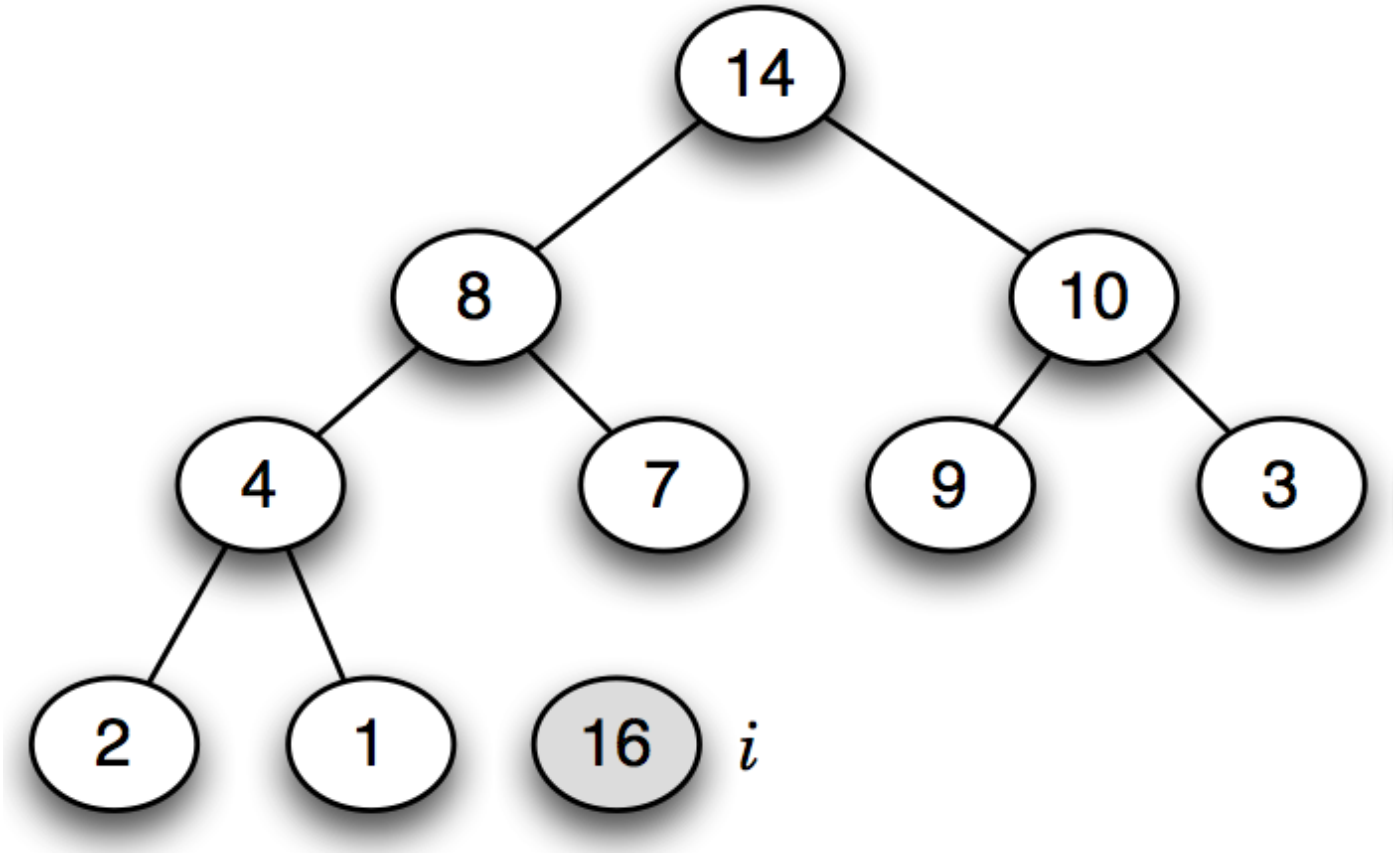
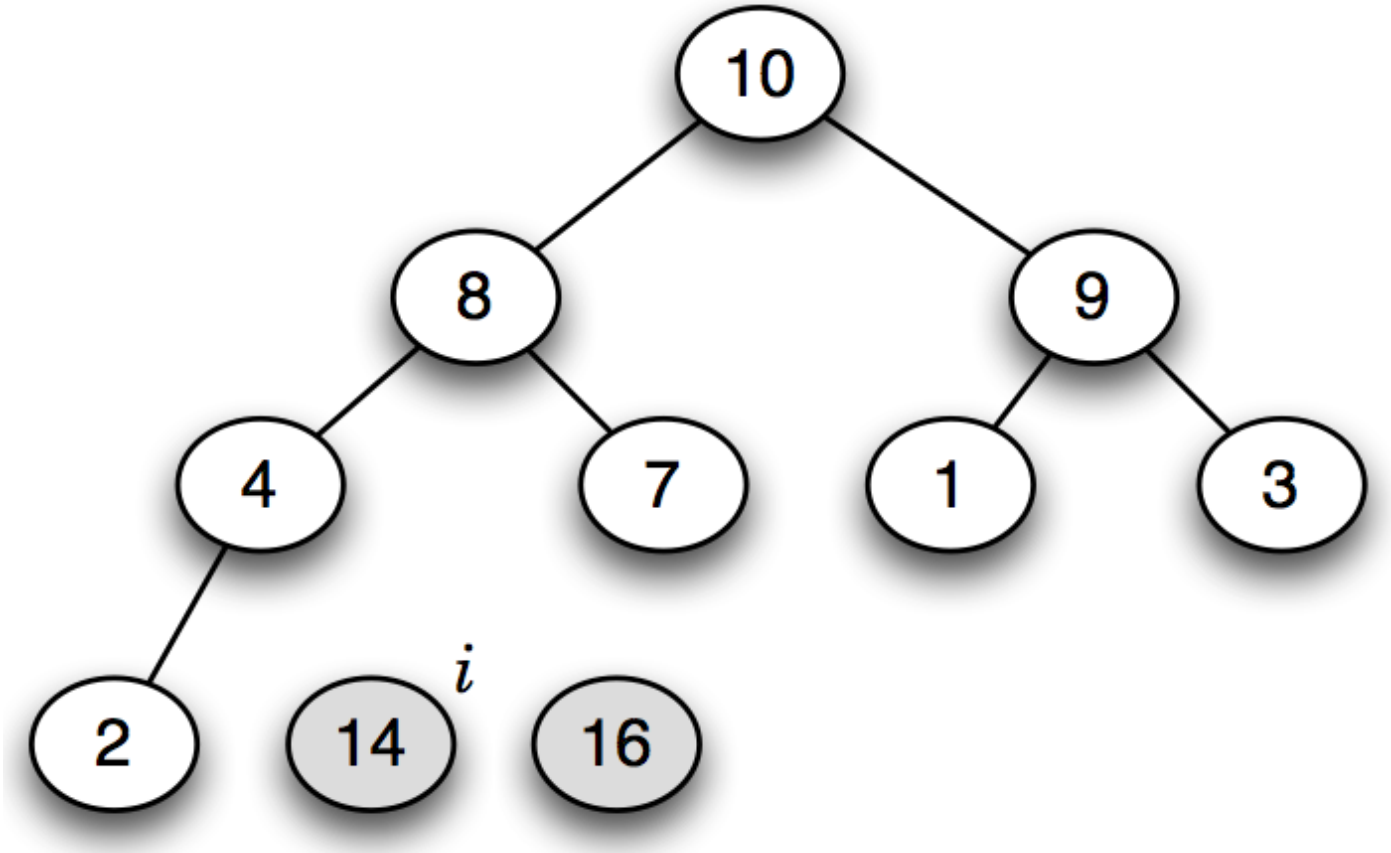


Heap use - heapsort



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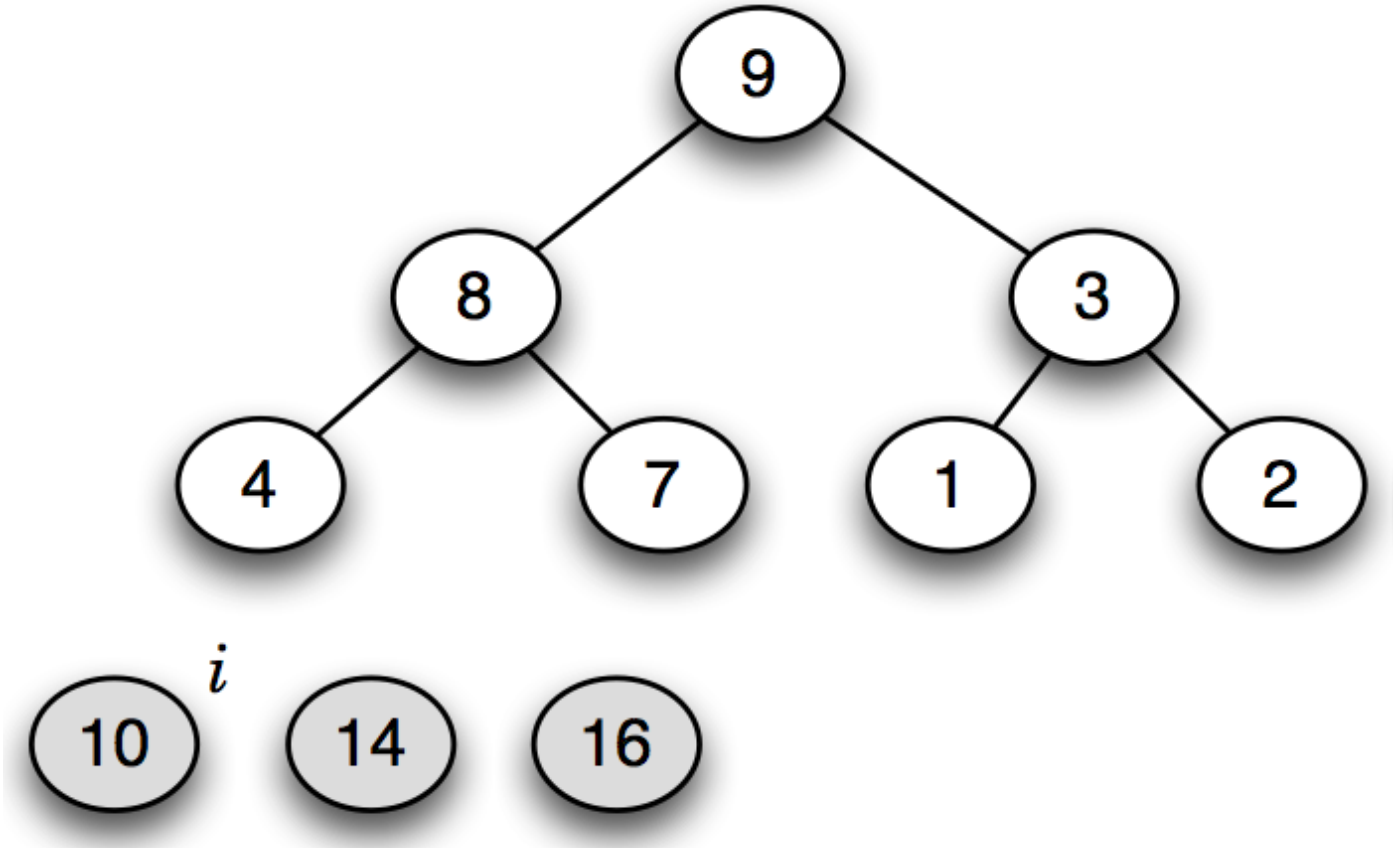

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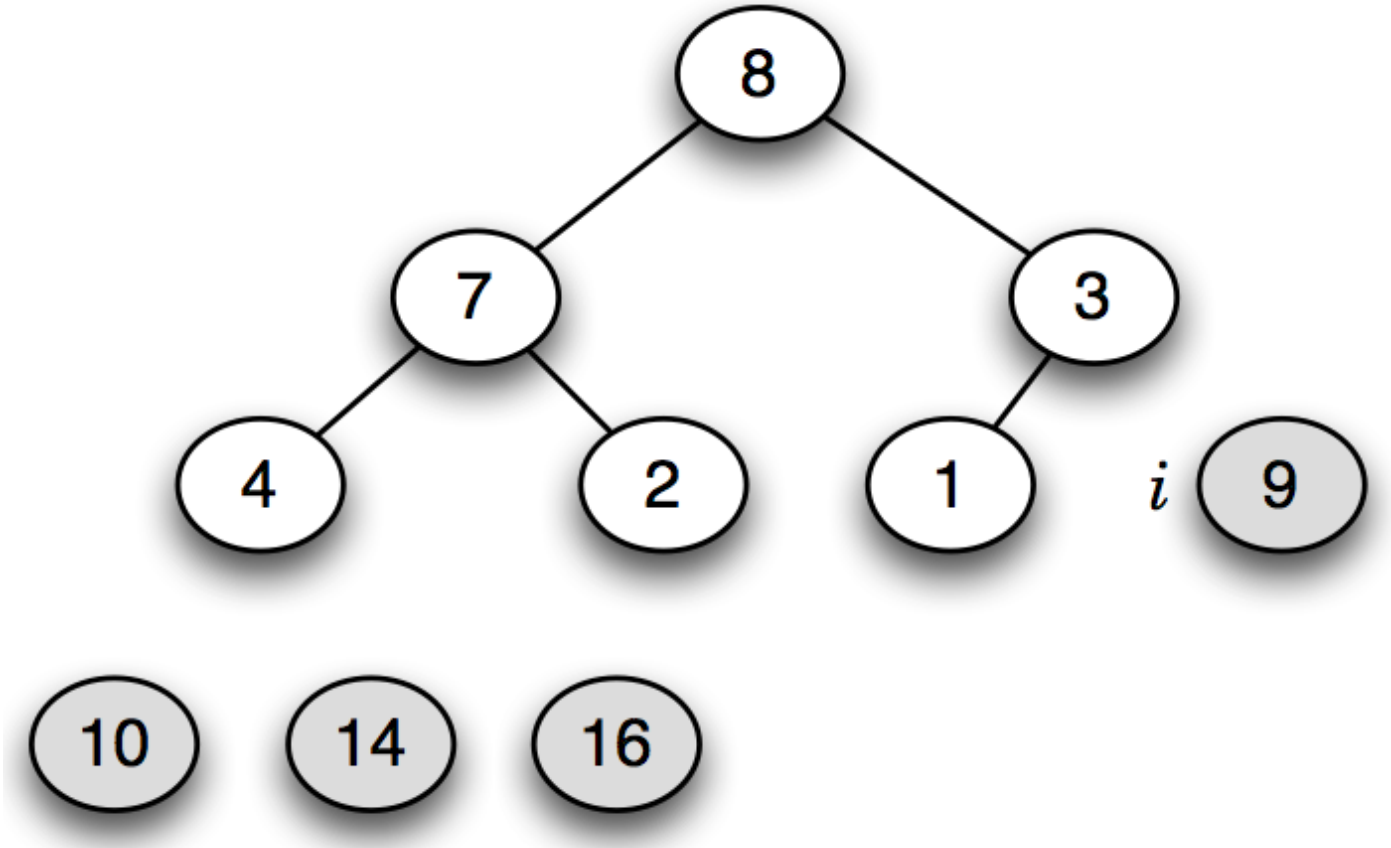

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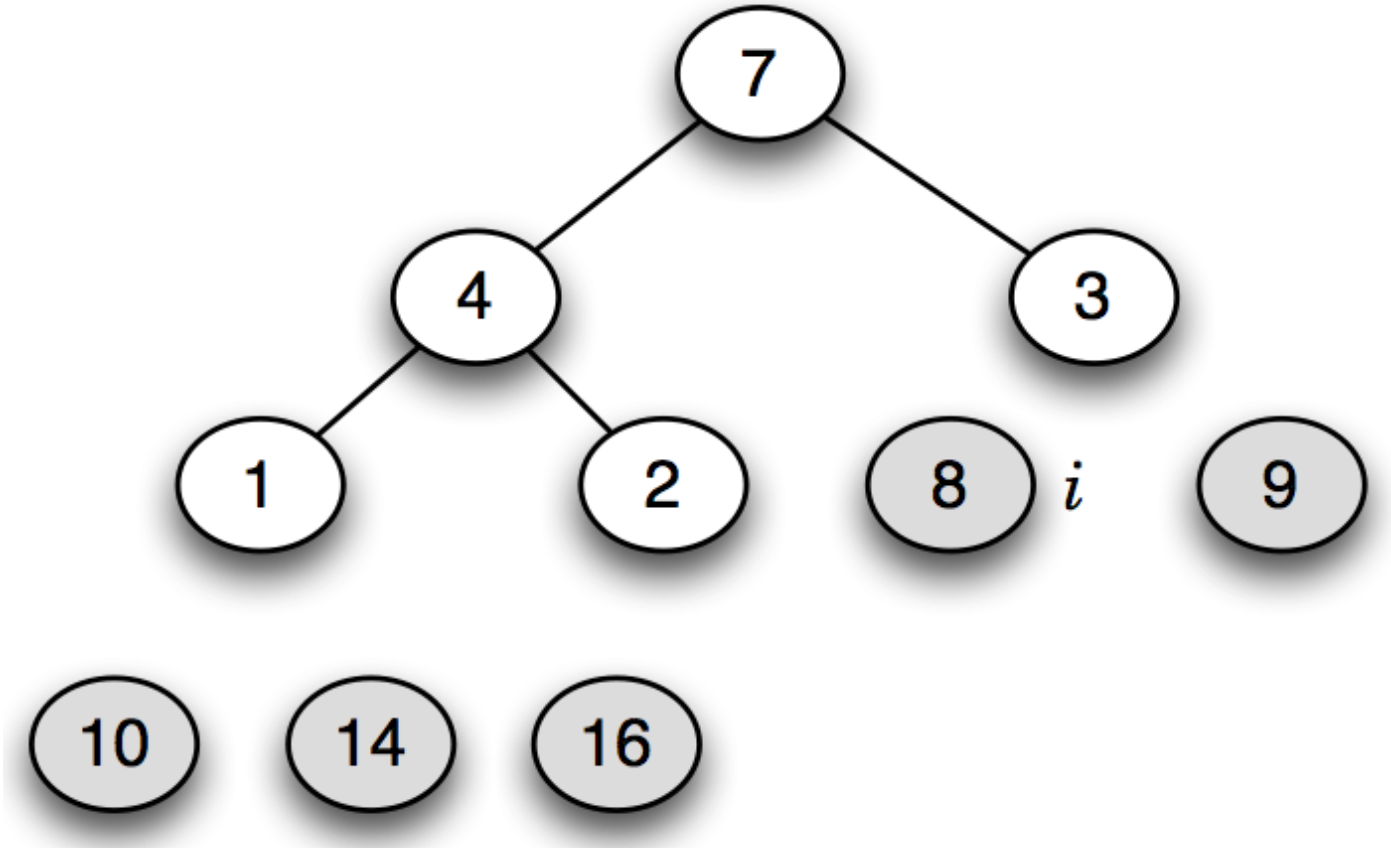

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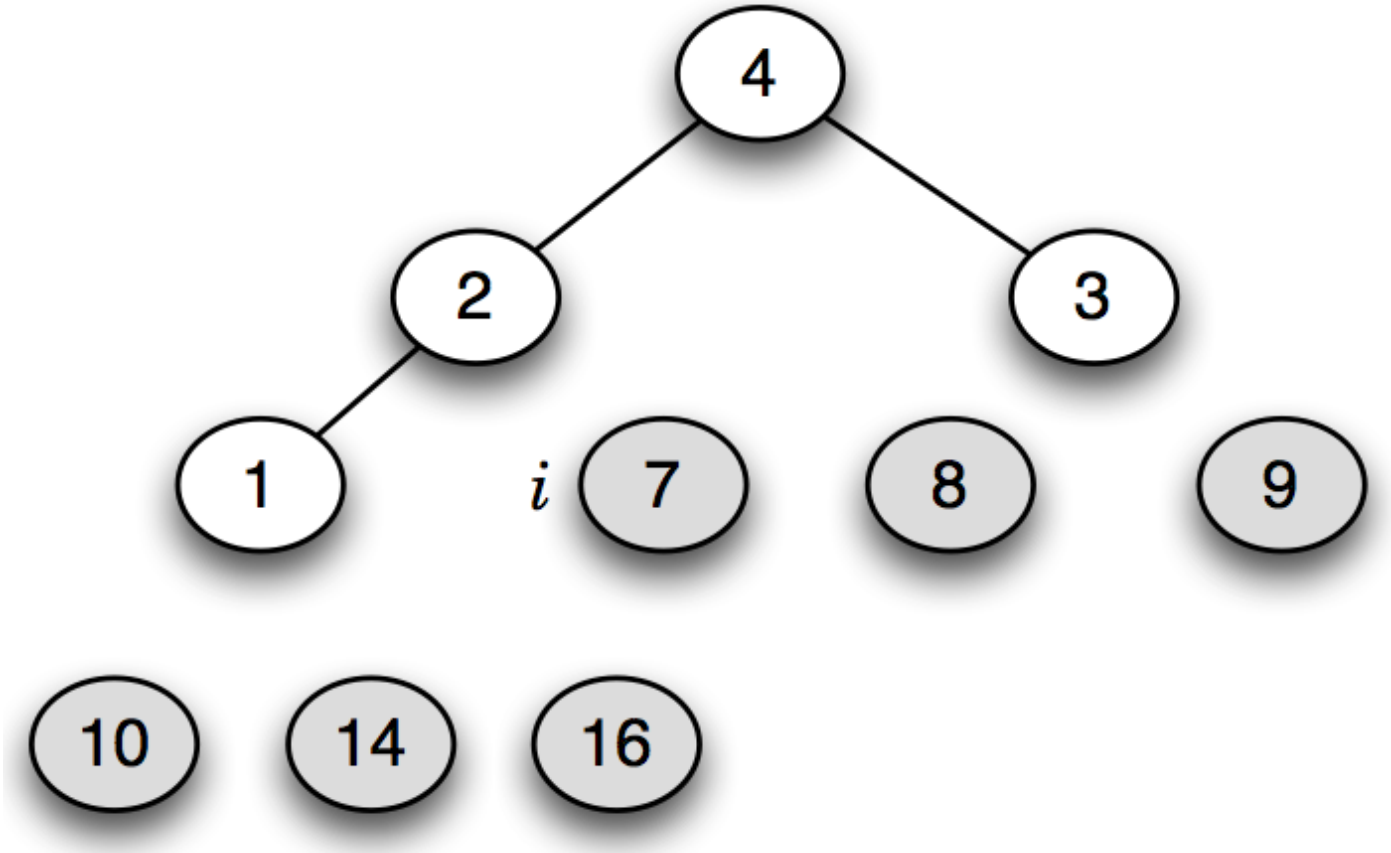

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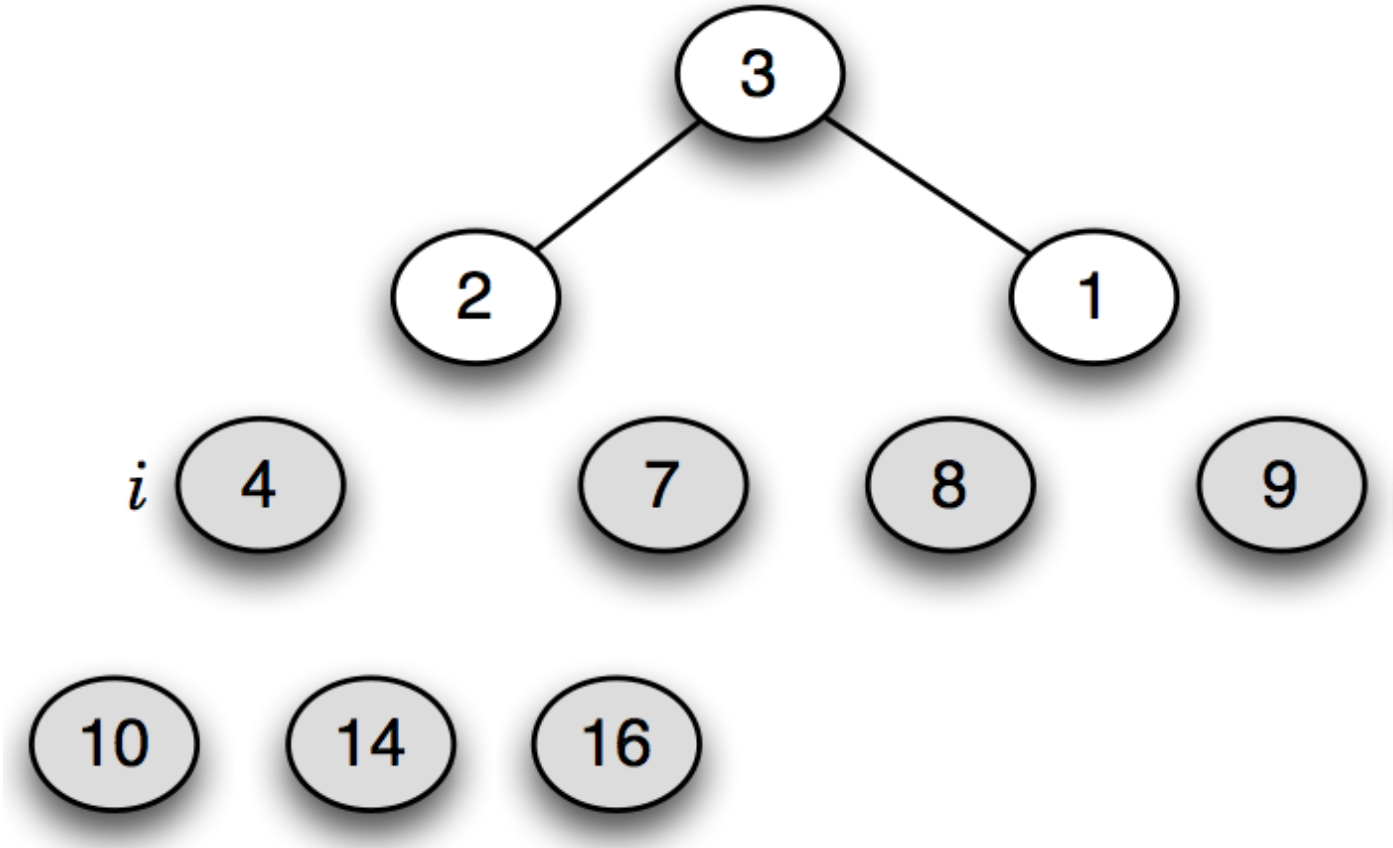

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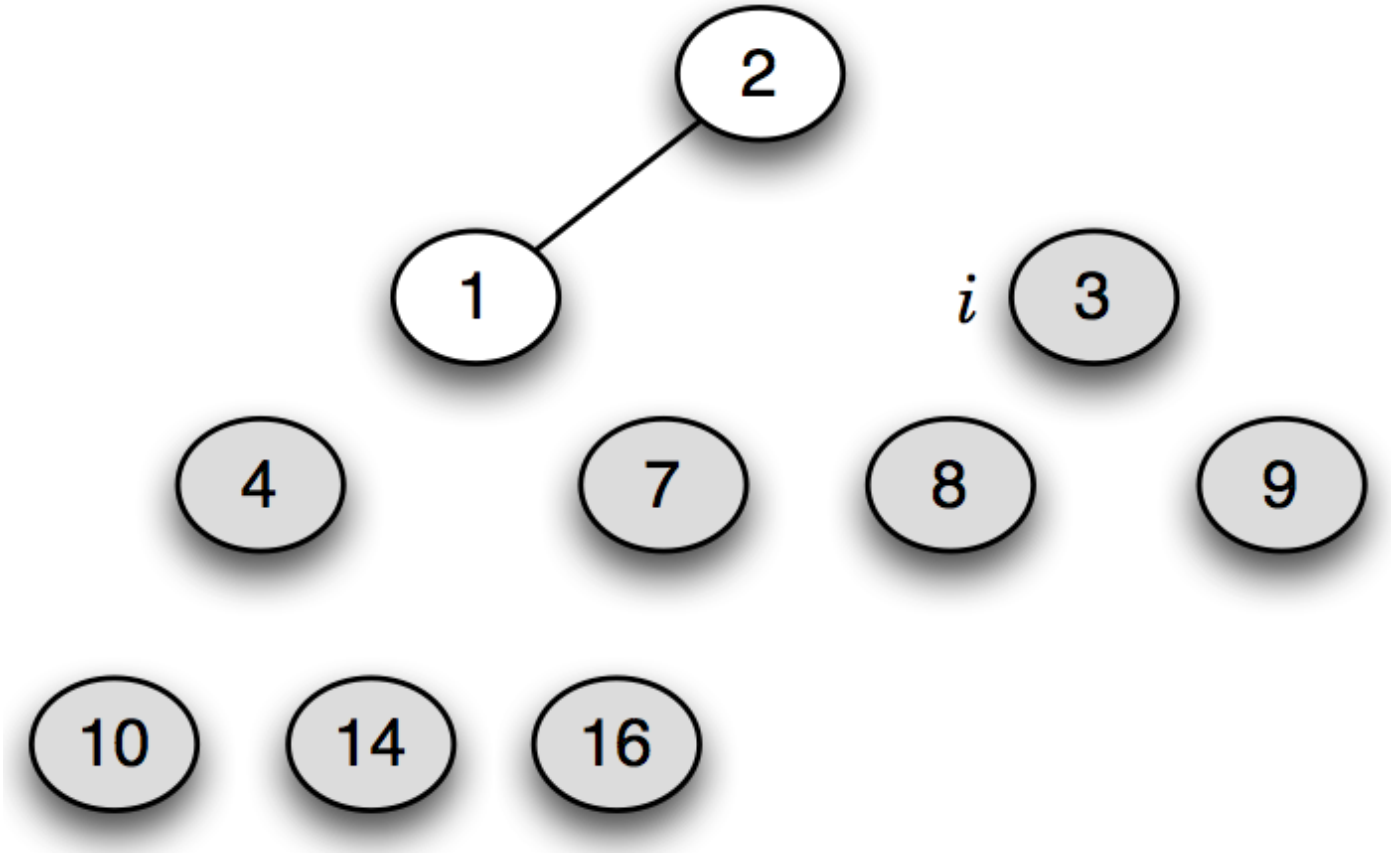

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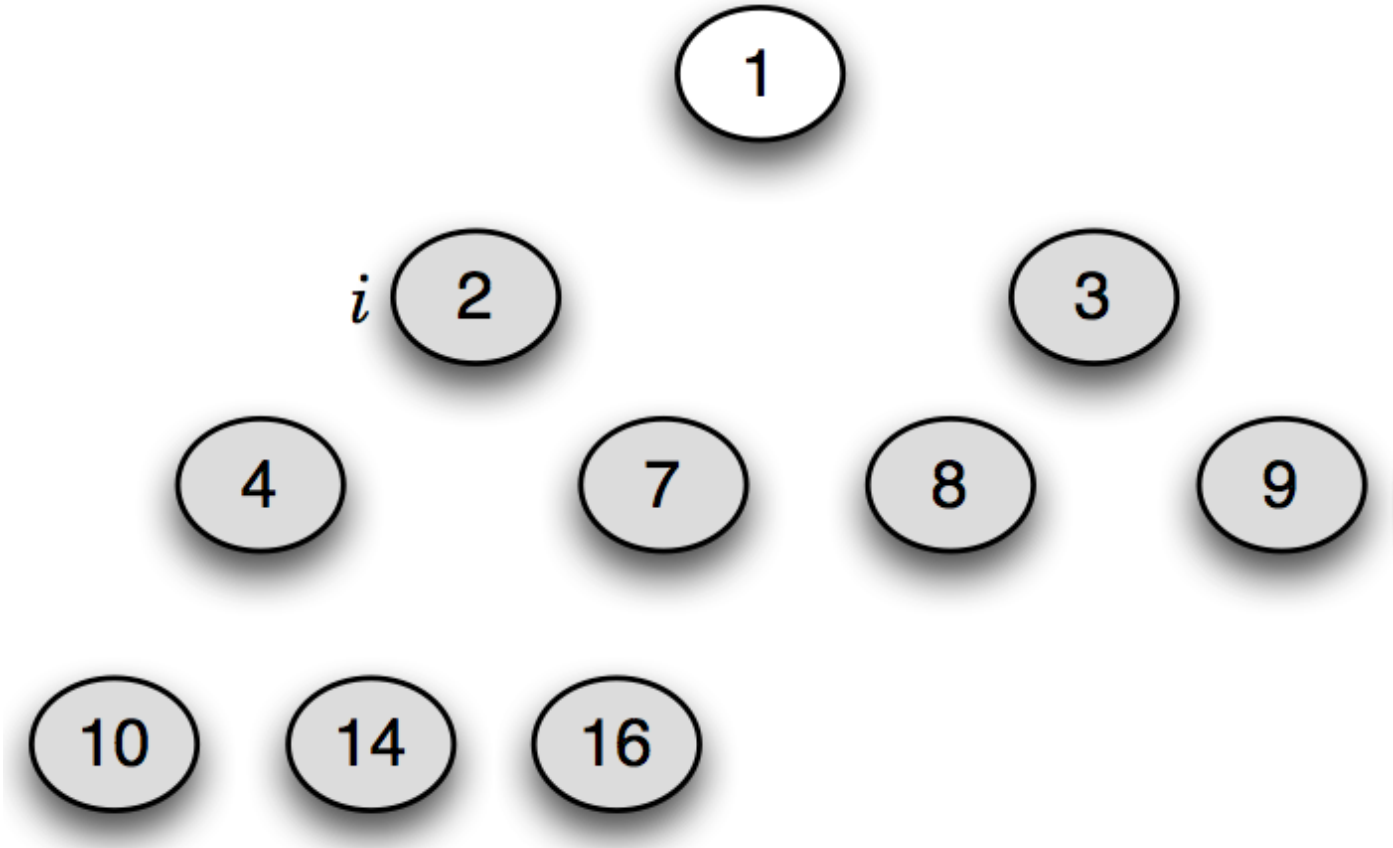

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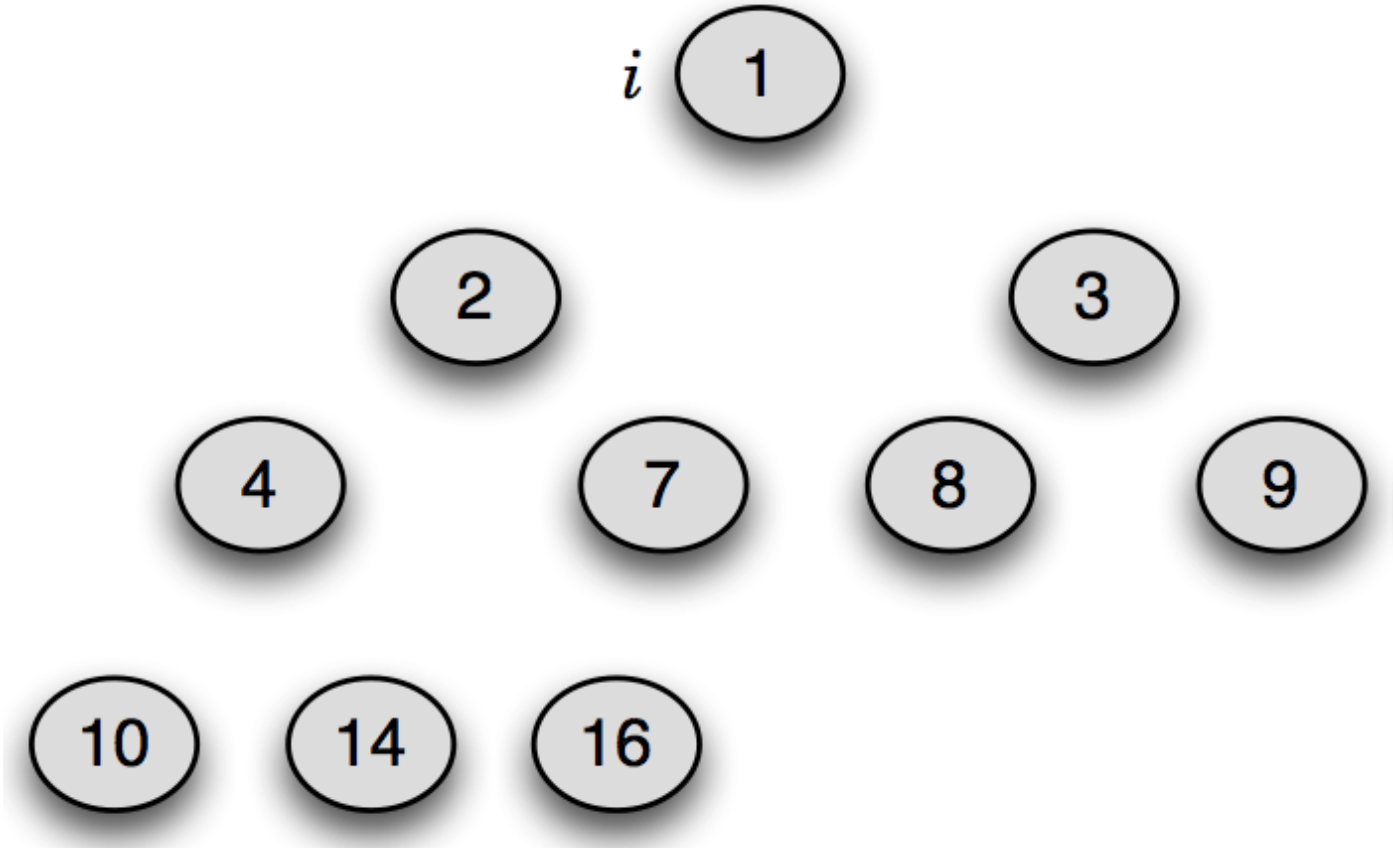

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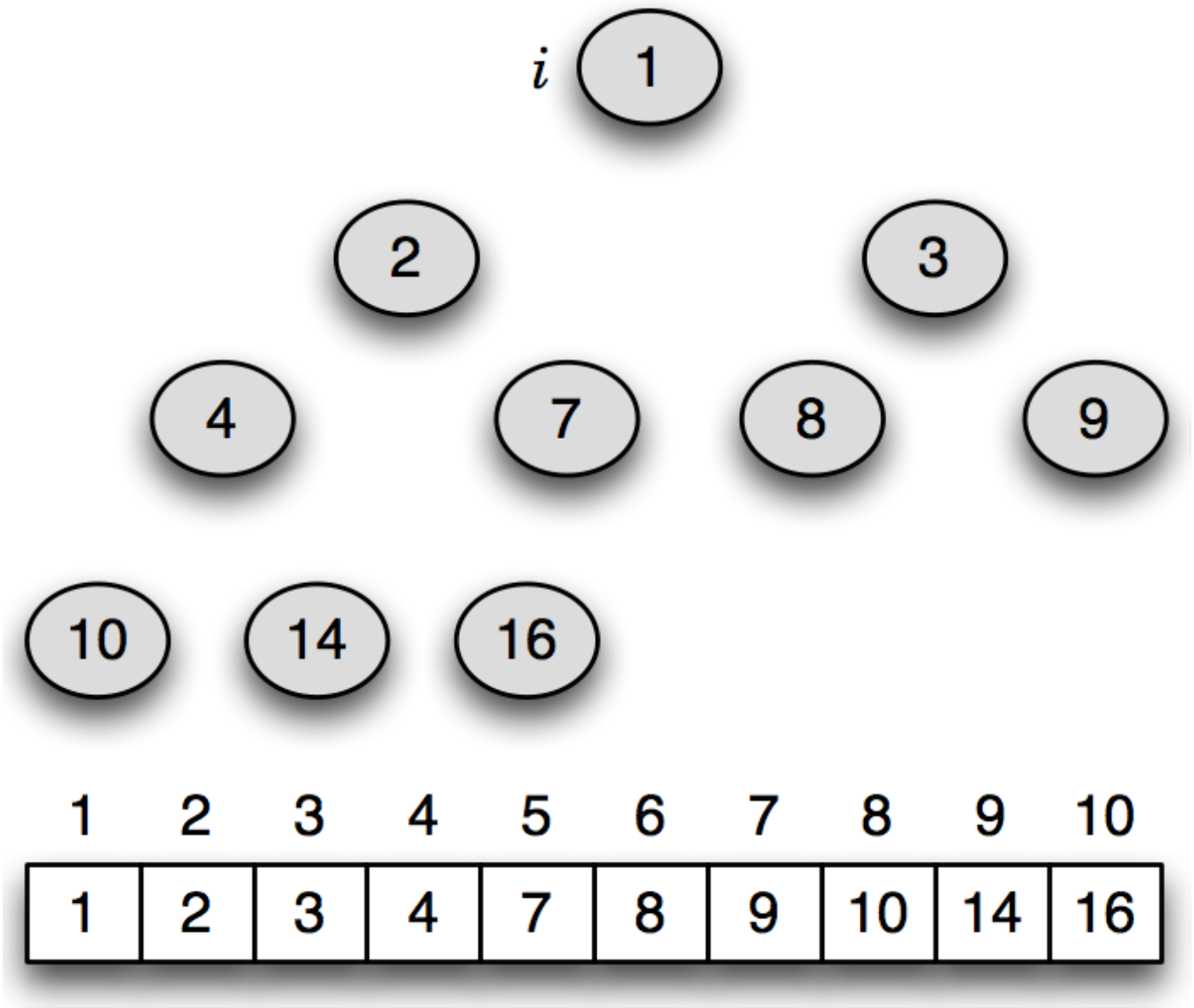

HEAPSORT algorithm

HEAPSORT(A)

		cost	*times*
1	BUILD-MAX-HEAP(A)	$\Theta(n)$	1
2	**for** $i \leftarrow$ *length*$[A]$ **downto** 2	$\Theta(1)$	n
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4	*heap-size*$[A] \leftarrow$ *heap-size*$[A]-1$	$\Theta(1)$	$n-1$
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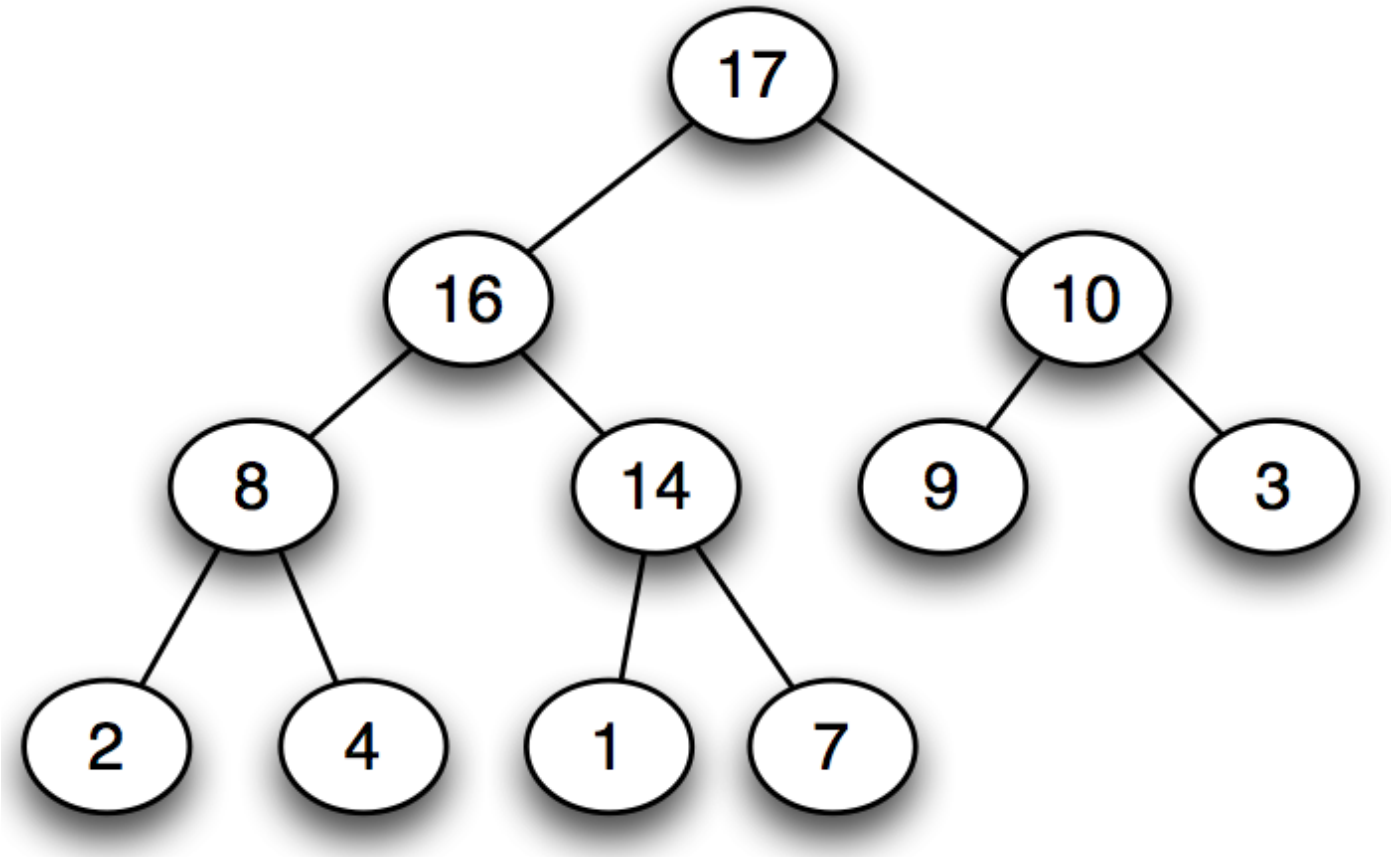

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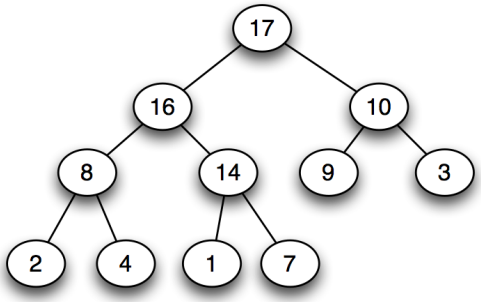

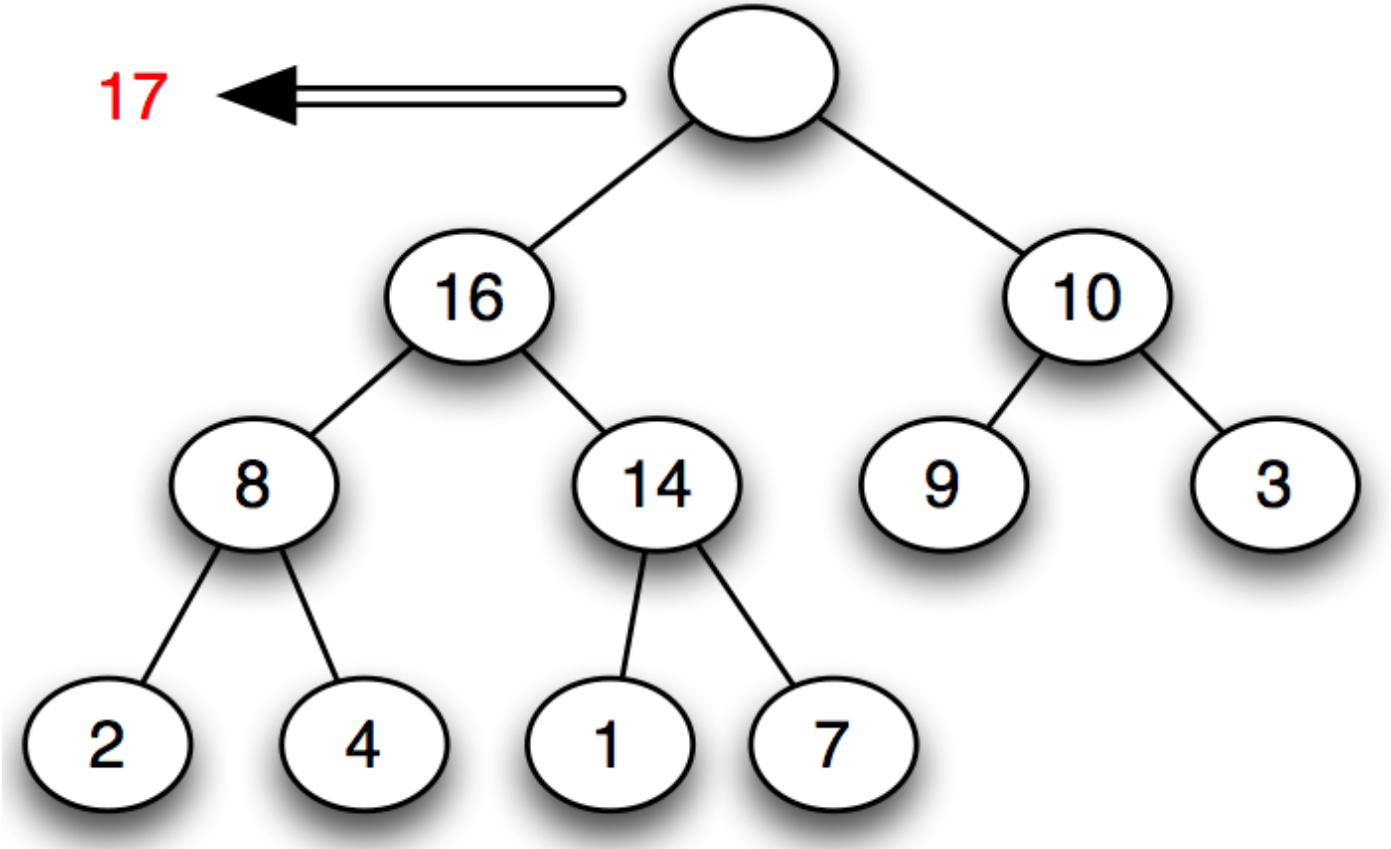

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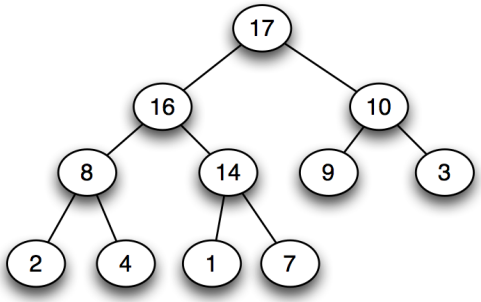

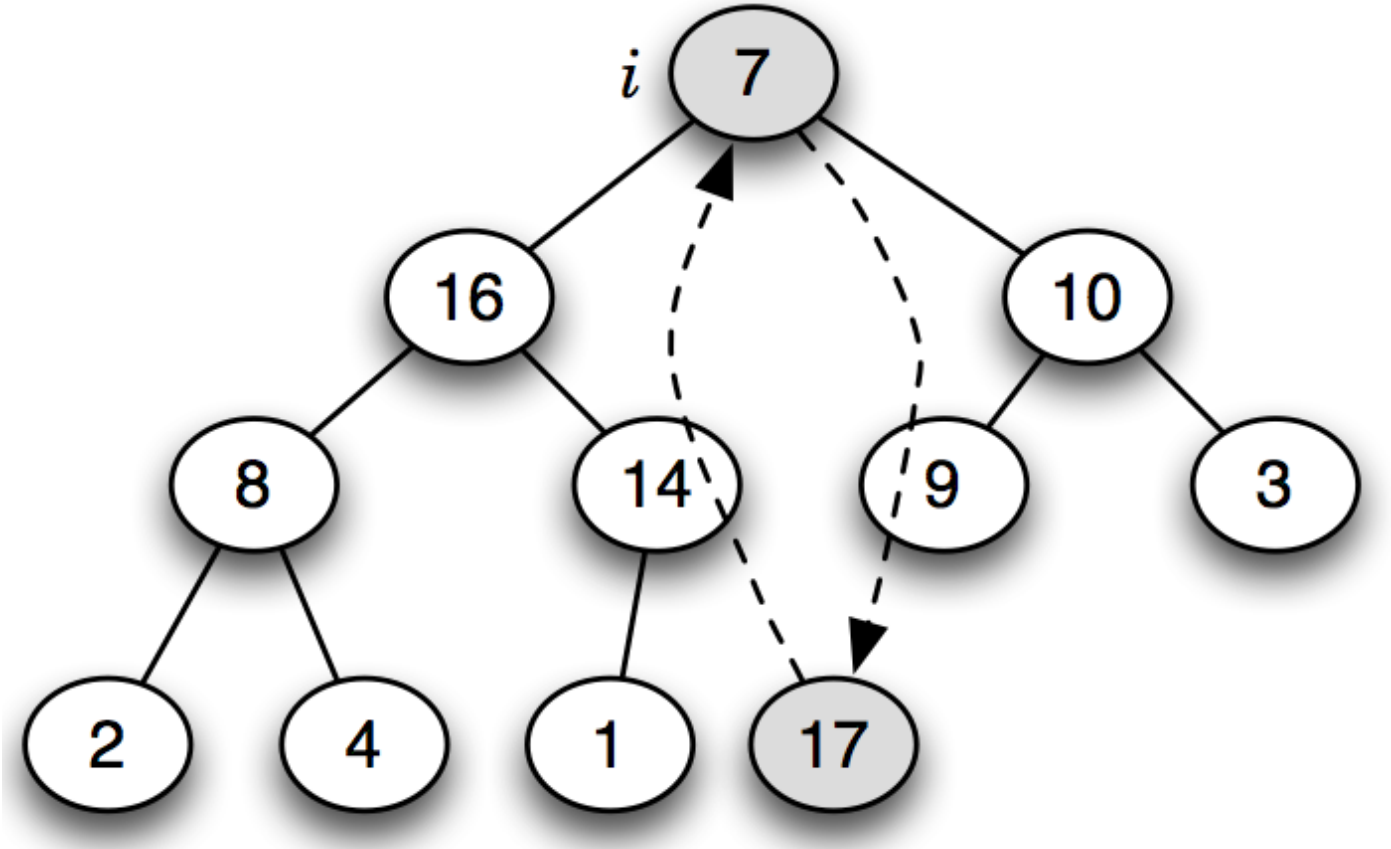

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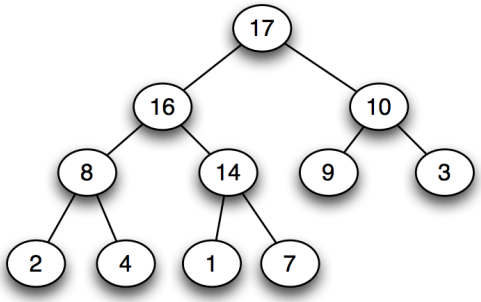

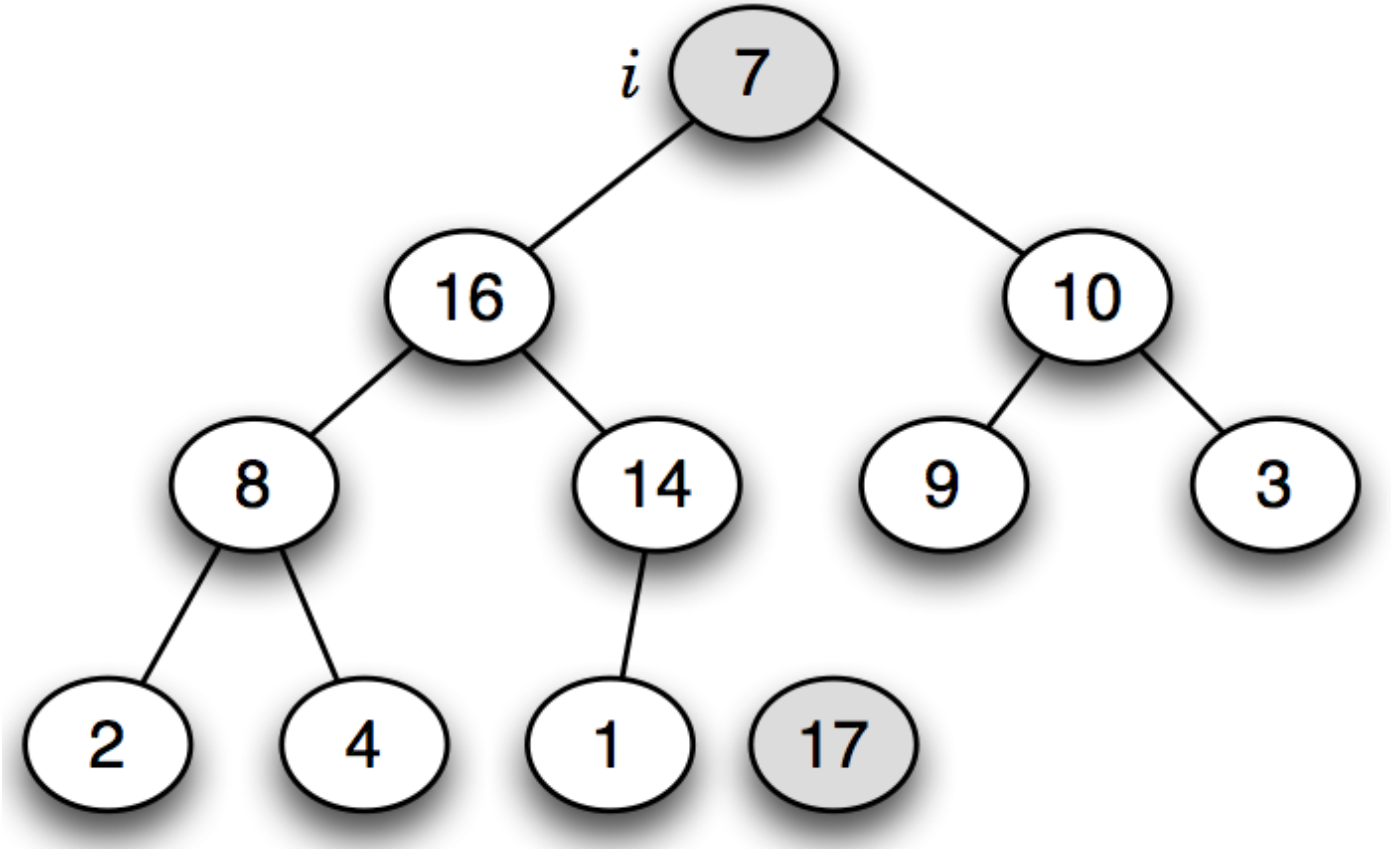

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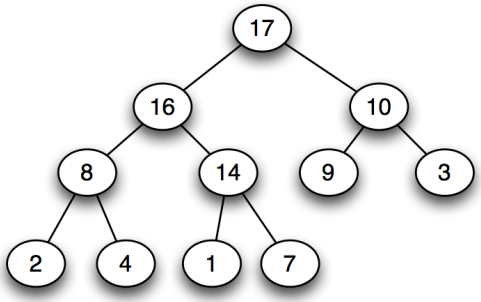

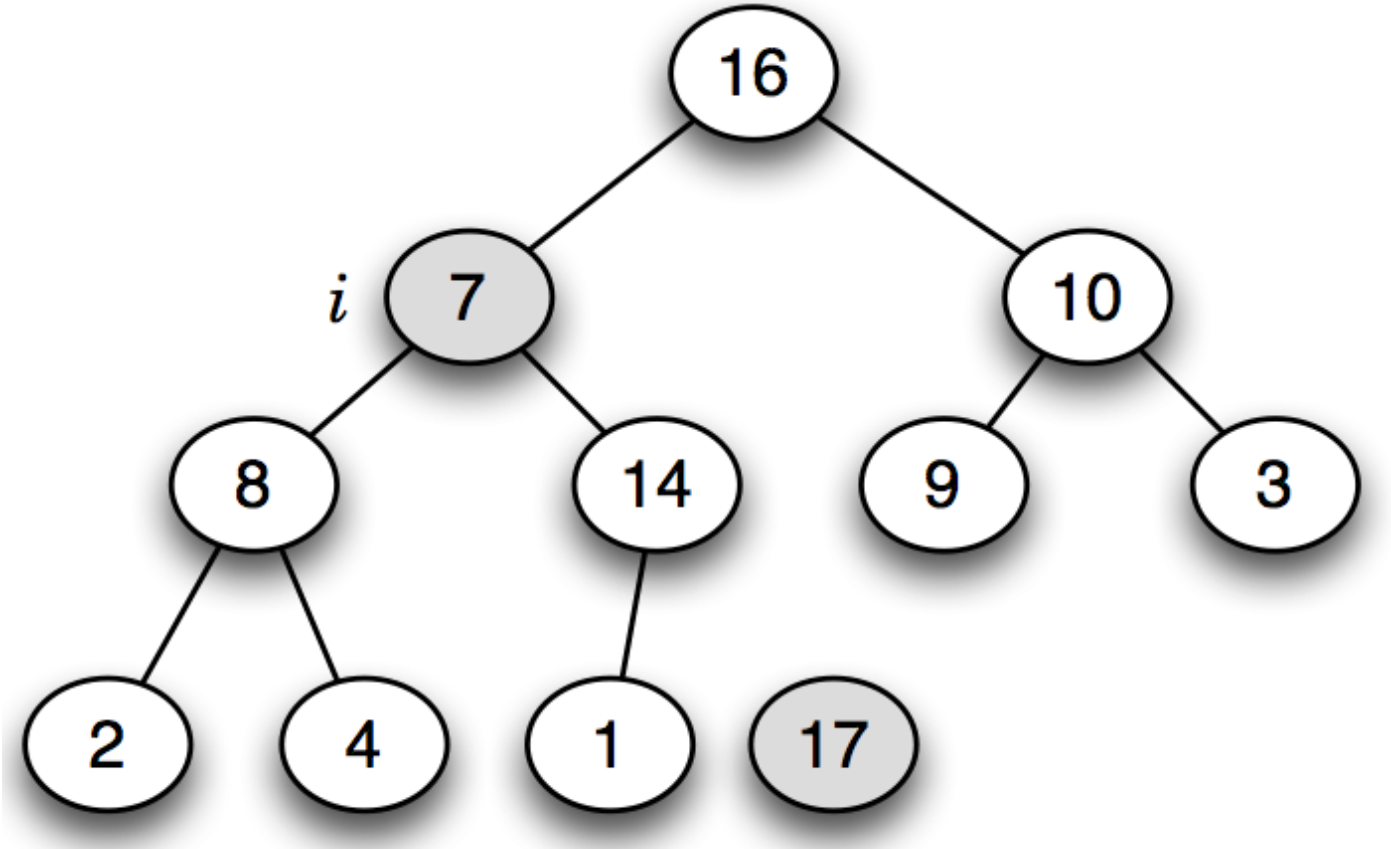

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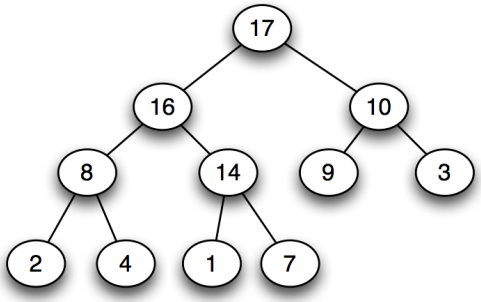

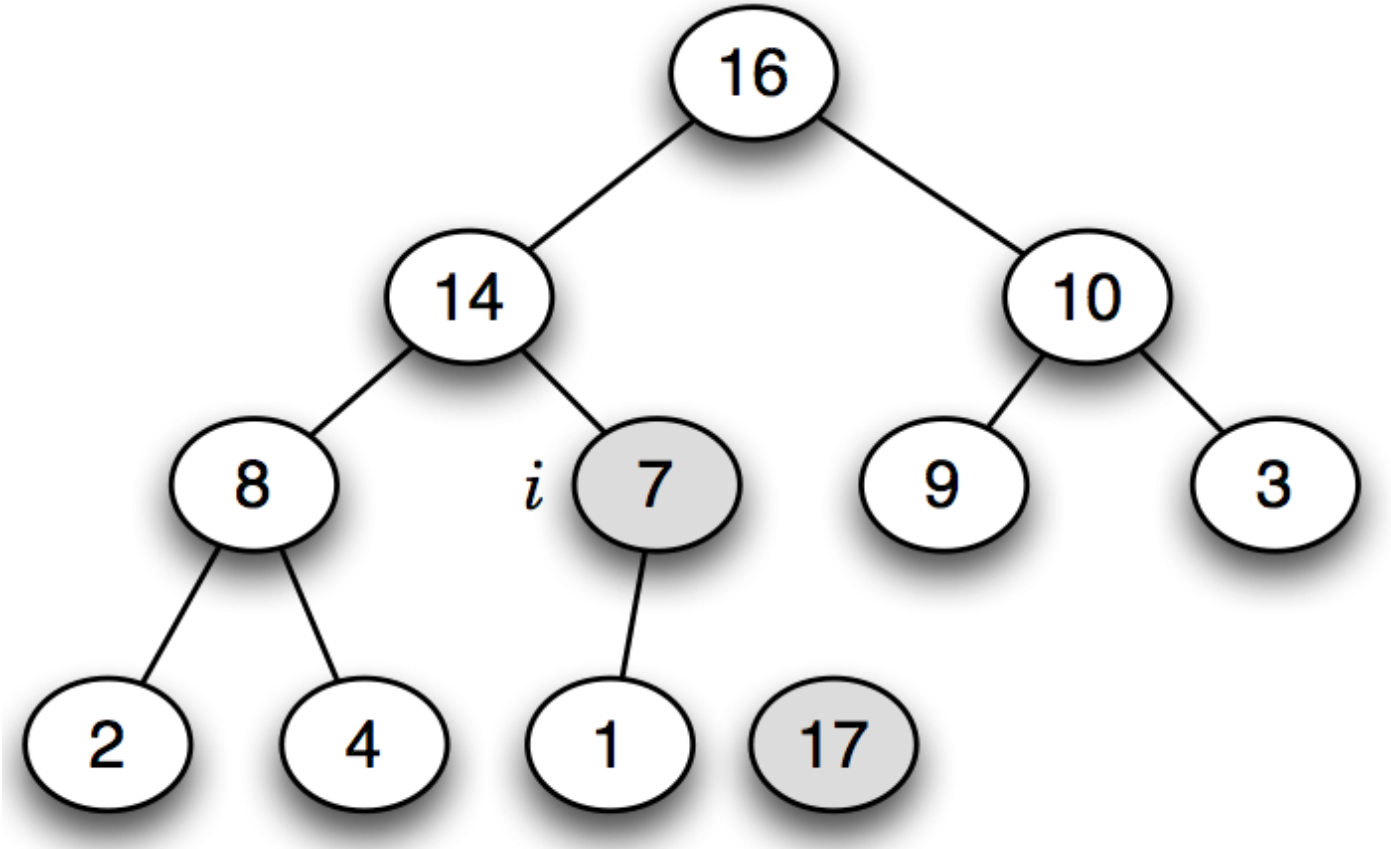

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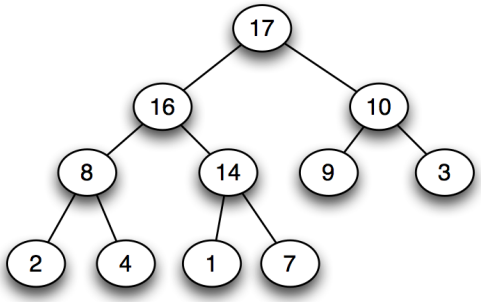

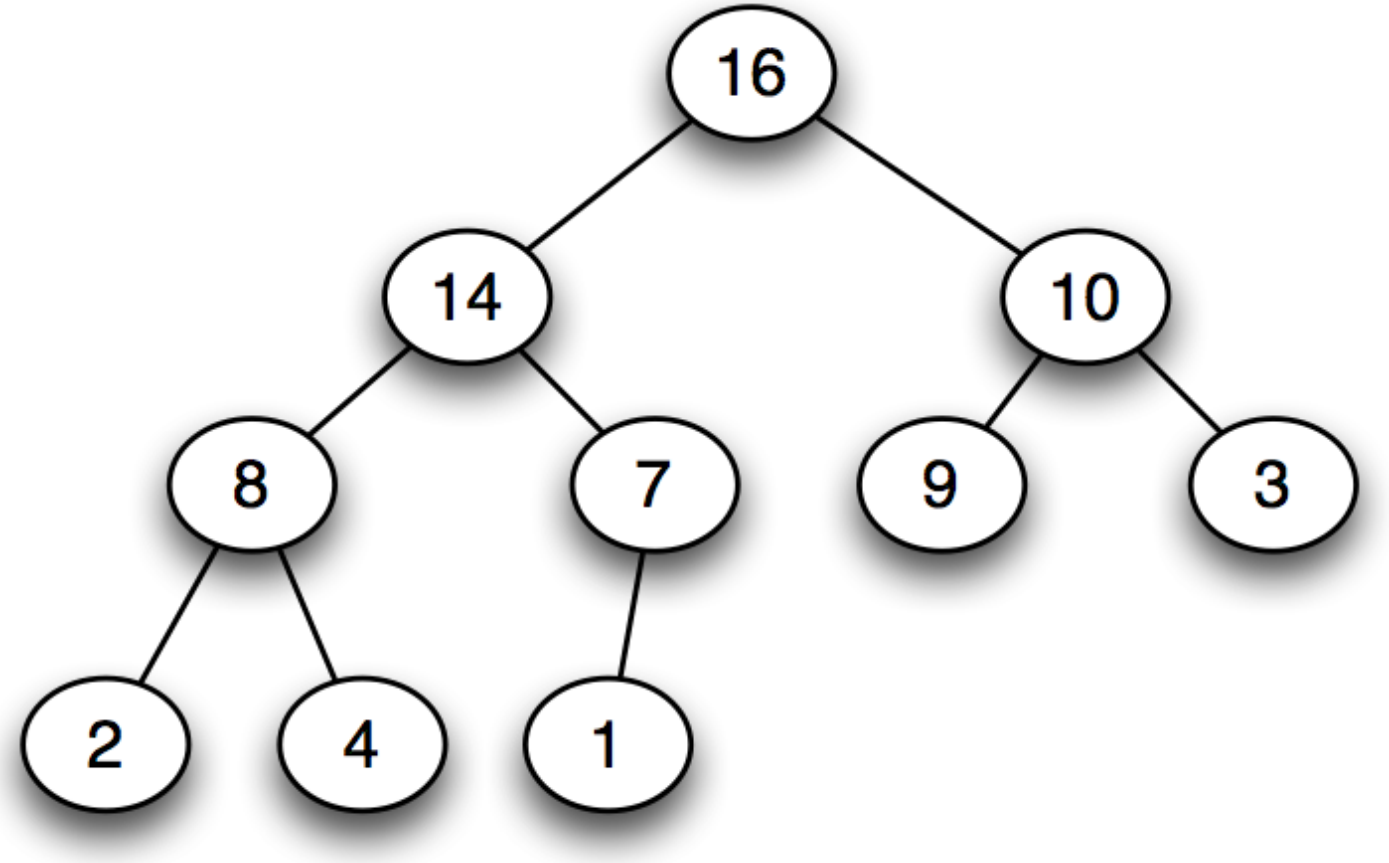

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

		cost	*times*
1	**if** *heap-size*$[A] < 1$	$\Theta(1)$	1
2	**then error** "heap underflow"	$\Theta(1)$	1
3	$max \leftarrow A[1]$	$\Theta(1)$	1
4	$A[1] \leftarrow A[$*heap-size*$[A]]$	$\Theta(1)$	1
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Worst-case analysis

$T(n) = \Theta(\lg n)$

Conclusion

- Heap plays a very important role in many algorithms, either used directly or as part of a priority queue implementation.
- If the size of a queue is known in advance, then an array representation (using a fixed size array) provides compact storage coupled with fast operations.
- Even if the size of the heap is not known in advance, "intelligent" resizing can still provide good benefits.
- Heapsort is a natural application of Heap with two very important properties – $\Theta(n \lg n)$ complexity, and in-place sorting.