Example. *Find the kernel of the linear transformation T from R^5 to R^4 given by the matrix*

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix}$$

Solution We have to solve the linear system $\mathrm{T}(\vec{x}) = \mathrm{A}\vec{0} = \vec{0}$

$$\mathrm{rref(A)} = \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel of T consists of the solutions of the system

$$\left| \begin{array}{ccccc} x_1 & & -6x_3 & & +6x_5 = 0 \\ & x_2 & +2x_3 & & -2x_5 = 0 \\ & & & x_4 & +2x_5 = 0 \end{array} \right|$$

The solution are the vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix}$$

where s and t are arbitrary constants .

$$\text{ker(T)=} \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} : \text{s , t arbitrary scalars}$$

We can write

$$\begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} = \text{s} \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{t} \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Example. *Find the kernel of the linear transformation T from R^3 to R^2 given by*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution

We have to solve the linear system

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

$$rref \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left| \begin{array}{ccccc} x_1 & & - & x_3 = 0 \\ & x_2 & + & 2x_3 = 0 \end{array} \right|$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel is the line spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Rank-Nullity revisited

Suppose T is the matrix transformation with $m \times n$ matrix A.

We know

- $\text{Ker}(T) = \text{nullspace}(A)$,
- $\text{Rng}(T) = \text{colspace}(A)$,
- the domain of T is $\mathbb{R}^n$.

Hence,

- $\dim(\text{Ker}(T)) = \text{nullity}(A)$,
- $\dim(\text{Rng}(T)) = \text{rank}(A)$,
- $\dim(\text{domain of } T) = n$.

We know from the rank-nullity theorem that

$$\text{rank}(A) + \text{nullity}(A) = n.$$

This fact is also true when T is not a matrix transformation:

Theorem

If $T : V \to W$ is a linear transformation and V is finite-dimensional, then

$$\dim(\text{Ker}(T)) + \dim(\text{Rng}(T)) = \dim(V).$$