From the third equation, we have $y = \frac{1}{2}(c - b + 2a)$

Substituting the value of y in the second equation

we get $z = \frac{1}{6}(7b - 5c - 14a)$

Again substituting [illegible] $\frac{1}{6}(7b$ [illegible] the first equation,

we get $x = \frac{1}{6}(7b - 5c - 8a)$

Therefore $(a, b, c) = \frac{1}{6}(7b - 5c - 8a)(1, 2, 0)$

$+ \frac{1}{2}(c - b + 2a)(0, 5, 7) + \frac{1}{6}(7b - 5c - 14a)(-1, 1, 3)$

Thus every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors (1, 2, 0), (0, 5, 7) and (−1, 1, 3). Hence the vectors (1, 2, 0), (0, 5, 7) and (−1, 1, 3) form a basis of $\mathbb{R}^3$.

Example 24. (i) Extend {(2, 0, 1), (1, 1, 1)} to a basis of $\mathbb{R}^3$.

[D. U. P. 1984]

(ii) Extend the set {(3, 2, 1), (0, 1, 1)} to a basis of $\mathbb{R}^3$.

[D. U. S. 1982]

Solution : (i) First we have to show that the given set of two vectors is linearly independent. Set a linear combination of the two given vectors equal to zero by using unknown scalars x and y :

$$x(2, 0, 1) + y(1, 1, 1) = (0, 0, 0)$$

or, $(2x, 0, x) + (y, y, y) = (0, 0, 0)$

or, $(2x + y, y, x + y) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 2x + y &= 0 \\ y &= 0 \\ x + y &= 0 \end{aligned}\right\} \text{ Thus we have } x = 0,\ y = 0.$$

Hence the given two vectors in $\mathbb{R}^3$ are linearly independent. So {(2, 0, 1), (1, 1, 1)} is a part of the basis of $\mathbb{R}^3$ and hence we can extend them to a basis of $\mathbb{R}^3$. Now we seek three independent vectors in IR^3 which include the given two vectors. Thus we can easily verify that (2, 0, 1), (1, 1, 1), (0, 1, 0) are linearly independent. So they form a basis of IR^3 which is an extension of the given set of vectors to a basis of IR^3.

(ii) First we have to show that the given set of the vectors is linearly independent. Set a linear combination of the two given vectors equal to zero by using unknown scalars x and y :

$$x(3, 2, 1) + y(0, 1, 1) = (0, 0, 0)$$

or, $(3x, 2x, x) + (0, y, y) = (0, 0, 0)$

or, $(3x, 2x + y, x + y) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 3x \quad &= 0 \\ 2x + y &= 0 \\ x + y &= 0 \end{aligned}\right\} \quad \text{Thus we have } x = 0,\ y = 0.$$

Hence the given two vectors in IR^3 are linearly independent. So the given set of vectors is a part of the basis of IR^3 and hence we can extend them to a basis of R^3. Now we seek three independent vectors in IR^3 which include the given vectors. Thus we can easily verify that (3, 2, 1) (0, 1, 1), (1, 0, 0) are linearly independent. So they form a basis of IR^3 which is an extension of the given set of vectors to a basis of IR^3.

Example 25. Determine a basis and the dimension for the solution space of the following homogeneous system:

$$\left.\begin{aligned} x - 3y + z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}\right\} \qquad (2)$$

Solution : The given linear system is

$$\left.\begin{aligned} x-3y+z&=0\\ 2x-6y+2z&=0\\ 3x-9y+3z&=0 \end{aligned}\right\} \qquad (1)$$

Reduce the system to echelon form by the elementary transformations. We multiply first equation by 2 and by 3 and then subtract from the second and the third equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} x-3y+z&=0\\ 0&=0\\ 0&=0 \end{aligned}\right\} \Rightarrow x-3y+z=0$$

This system is in echelon form and has only one non-zero equation in three unknowns. So the system has $3-1=2$ free variables which are y and z. Hence the dimension of the solution space is 2 (two).

Set (i) $y=1$, $z=0$ (ii) $y=0$, $z=1$ to obtain the respective solutions $v_1=(3, 1, 0)$, $v_2=(-1, 0, 1)$.

Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

Example 26. Find the solution space W of the following homogeneous system of linear equations :

$$\left.\begin{aligned} x+2y-z+4t&=0\\ 2x-y+3z+3t&=0\\ 4x+y+3z+9t&=0\\ y-z+t&=0\\ 2x+3y-z+7t&=0 \end{aligned}\right\}$$

[D. U. H. T. 1983]
[J. U. H. 1988]

Solution : Reduce the given system to echelon form by the elementary operations. We multiply 1st equation by 2, 4 and 2 and then subtract from 2nd, 3nd and 5th equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} x+2y-z+4t&=0\\ -5y+5z-5t&=0\\ -7y+7z-7t&=0\\ y-z+t&=0\\ -y+z-t&=0 \end{aligned}\right\}$$

We multiply 2nd, 3rd and 5th equations by $-\frac{1}{5}, -\frac{1}{7}$ and (-1) respectively. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y - z + 4t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \end{aligned}\right\}$$

Since 2nd, 3rd, 4th & 5th equations are identical, we can disregard any three of them. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y - z + 4t &= 0 \\ y - z + t &= 0 \end{aligned}\right\}$$

This system is in echelon form having two equations in 4 unknowns. So the system has $4 - 2 = 2$ free variables which are z and t and hence it has non-zero solutions. Let $z = a$ and $t = b$ where a and b are arbitrary real numbers. Putting $z = a$ and $t = b$ in the 2nd equation we get $y = a - b$. Again putting the values of y, z and t in the 1st equation, we get $x = -a - 2b$. Hence the required solution space is

$W = \{(-a - 2b, a - b, a, b) : a, b \in \mathbb{R}\}$.

Example 27. Let S and T be the following subspaces of $\mathbb{R}^4$:

$S = \{(x, y, z, t) \mid y - 2z + t = 0\}$

$T = \{(x, y, z, t) \mid x - t = 0, y - 2z = 0\}$

Find a basis and the dimension of (i) S (ii) T (iii) $S \cap T$.

Solution : (i) We seek a basis of the set of respective solution (x, y, z, t) of the equation $y - 2z + t = 0$.

The free variables are x, z and t. Set

(a) $x = 1, z = 0, t = 0$

(b) $x = 0, z = 1, t = 0$.

(c) $x = 0, z = 0, t = 1$, to obtain the respective solutions

$u_1 = (1, 0, 0, 0), u_2 = (0, 2, 1, 0), u_3 = (0, 1, 0, 1)$

The set $\{u_1, u_2, u_3\}$ is a basis of S and dim S = 3.

(ii) We seek a basis of the set of solutions (x, y, z, t) of the equations

$$\left.\begin{aligned} x - t &= 0 \\ y - 2z &= 0 \end{aligned}\right\}$$

The free variables are z and t. Set (a) z = 1, t = 0, (b) z = 0, t = 1 to obtain the respective solutions $u_1 = (0, 2, 1, 0)$ and $u_2 = (1, 0, 0, 1)$. The set $\{u_1, u_2\}$ is a basis of T and dim T = 2.

(iii) $S \cap T$ consists of those vectors (x, y, z, t) which satisfy all conditions given in S and in T. i. e.

$$\left.\begin{aligned} y - 2z + t &= 0 \\ x - t &= 0 \\ y - 2z \quad &= 0 \end{aligned}\right\}$$

$$\text{or,}\quad \left.\begin{aligned} x \quad - \quad t &= 0 \\ y - 2z \quad &= 0 \\ y - 2z + t &= 0 \end{aligned}\right\} \text{Subtract second equation from the third equation.}$$

$$\text{Then we have}\quad \left.\begin{aligned} x - t &= 0 \\ y - 2z &= 0 \\ t &= 0 \end{aligned}\right\} \text{Which is in echelon form.}$$

The free variable is z. Set z = 1 to obtain the solution $u = (0, 2, 1, 0)$.

Thus $\{u\}$ is a basis of $S \cap T$ and dim $(S \cap T) = 1$.

Example 28. (i) Let U be the subspace of $\mathbb{R}^3$ spanned (generated) by the vectors (1, 2, 1), (0, – 1, 0) and (2, 0, 2). Find a basis and the dimension of U.

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(ii) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors (1, −2, 0, 0, 3), (2, −5, −3, −2, 6), (0, 5, 15, 10, 0) and (2, 6, 18, 8, 6). Find a basis and the dimension of W.

Solution : (i) Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$ we multiply first row by 2 and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$ we multiply second row by 4 and subtract from the third row.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ we multiply second row by 2 and then add with the first row.

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ we multiply second row by − 1

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row–echelon form and the non-zero rows in the matrix are (1, 0, 1) and (0, 1, 0). These non-zero rows form a basis of the row space and consequently a basis of U; that is, Basis of U = {(1, 0, 1), (0, 1, 0)} and dim U = 2.

(ii) Form the matrix whose rows are the given vectors and reduce the matrix to row–echelon form by the elementary row operations :

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Reduce this matrix to row echelon form by the elementary row operations.

we multiply first row by 2 and then subtract from the second and fourth rows respectively.

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

We multiply second row by 5 and then add with the third row.

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

Interchange third and fourth rows

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 10 & 18 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We multiply second row by 10 and then add with the third row.

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We multiply second row by -1 and divide third row by -12.

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form. The non-zero rows (vectors) in the above matrix are (1, 2, 0, 0, 3), (0, 1, 3, 2, 0) and (0, 0, 1, 1, 0). These non-zero rows form a basis for the row space and consequently a basis of W. Thus basis of W = {(1, 2, 0, 0, 3), (0, 1, 3, 2, 0), (0, 0, 1, 1, 0)} and dim W = 3.

Example 29. Let W be the subspace generated by the polynomials $p_1(t) = t^3 + 2t^2 - 2t + 1$,

$p_2(t) = t^3 + 3t^2 - t + 4$ and $p_3(t) = 2t^3 + t^2 - 7t - 7$.

Find a basis and the dimeinsion of W.

Solution : Clearly, W is a subspace of the vector space V (F) of polynomials in t of degree ≤ 3. Thus the set $S_1 = \{1, t, t^2, t^3\}$ is a basis of V (F).

Now the coordinates of the given vectors $p_1(t)$, $p_2(t)$ and $p_3(t)$ relative to the basis S_1 are (1, 2, –2, 1), (1, 3, –1, 4) and (2, 1, –7, –7) respectively.

Forming the matrix whose rows are the above coördinate vectors, we get

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$

Reduce this matrix to row echelon form by the elementary row operations. We multiply 1st row by 1 and 2 and then subtract from 2nd & 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$$

We multiply 2nd row by 3 and then add with the 3rd row.

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form having two non-zero rows (coordinate vectors) (1, 2, –2, 1) and (0, 1, 1, 3) which will form a basis of the vector space generated by the coordinate vectors and so the set of corresponding polynomials is $\{t^3 + 2t^2 - 2t + 1,\ t^2 + t + 3\}$ which will form the basis of W. Thus dim W = 2.

Example 30. Let U and W be the subspaces of $\mathbb{R}^4$ generated by the set of vectors

{(1, 1, 0, –1), (1, 2, 3, 0), (2, 3, 3, –1)} and

{(1, 2, 2, –2), (2, 3, 2, –3), (1, 3, 4, –3)} respectively.

Find (i) dim (U + W) and (ii) dim (U∩W). **[D. U. H. 1998]**

Solution : (i) U + W is subspace spanned (or generated) by all given six vectors. Hence form the matrix whose rows are the given six vectors and then reduce this matrix to row echelon form by the elementary row operations :

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reduce this matrix to row-echelon form by the elementary row operations.

We subtract 1st row from 2nd, 4th and 6th rows. Also we multiply 1st row by 2 and then subtract from 3rd and 5th rows.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

We subtract 2nd row from 3rd, 4th & 5th rows. Also we multiply 2nd row by 2 and then subtract from 6th row.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

We multiply 4th row by 1 and 2 and then subtract from 5th and 6th rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We interchange 3rd and 4th rows.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having three non-zero (1, 1, 0, −1), (0, 1, 3, 1) and (0, 0, −1, −2) which will form a basis of U + W. Thus dim (U + W) = 3.

(ii) Let us first find the dim U and the dim W. Form the matrix whose rows are the generators of U and then reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

We multiply 1st row by 1 and 2 and then subtract from 2nd and 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

We subtract 2nd row from third row.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows (1, 1, 0, −1) and (0, 1, 3, 1) which will form a basis of U.

Thus dim U = 2.

Again form the matrix whose rows are the generators of W and then reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

We multiply 1st row by 2 and 1 and then subtract from 2nd and 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

We add 2nd row with 3rd row.

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows which will form a basis of W. Thus dim W = 2.

Now by theorem we have

$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

or, $\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 2 + 2 - 3 = 1$

$\therefore \dim(U \cap W) = 1$ (one).

Example 31. Let V be the vector space of 2×2 matrices over the real field IR . Find a basis and the dimension of the subspace W of V spanned by

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}.$$

Solution : The coordinate vectors of the given matrices relative to the usual basis of V are as follows :

$[A] = (1, 2, -1, 3)$, $[B] = (2, 5, 1, -1)$, $[C] = (5, 12, 1, 1)$ and $[D] = (3, 4, -2, 5)$.

Form a matrix whose rows are the coordinate vectors and then reduce this matrix to row-echelon form by the elementary row operations and join successive matrices by the equivalence sign ~ :

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & -1 \\ 5 & 12 & 1 & 1 \\ 3 & 4 & -2 & 5 \end{bmatrix}$$

We multiply 1st row by 2, 5 & 3 and then subtract from 2nd, 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 2 & 6 & -14 \\ 0 & -2 & 1 & -4 \end{bmatrix}$$

We multiply 2nd row by 2 and −2 and then subtract from 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -18 \end{bmatrix}$$

Interchange 3rd and 4th rows.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 7 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having three non-zero rows (1, 2, −1, 3), (0, 1, 3, −7) and (0, 0, 7, −18) which are linearly independent.

Hence the corresponding matrices $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & -7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 7 & -18 \end{bmatrix}$ form a basis of W and dim W = 3.

Example 32. Let V be the vector space of 2 ×2 matrices over the real field IR. Find a basis and the dimension of the subspace W of V spanned by the matrices

$$A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}.$$

Solution : The coordinate vectors of the given matrices relative to the usual basis of V are as follows :

[A] = (1, −5, −4, 2), [B] = (1, 1, −1, 5), [C] = (2, −4, −5, 7)
and [D] = (1, −7, −5, 1)

Form a matrix whose rows are the coordinate vectors and then reduce this matrix to row-echelon form by the elementary row operations and join successive matrices by the equivalence sign ~ :

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$$

We multiply 1st row by 1, 2, and 1 and then subtract from 2nd, 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

We multiply 2nd row by 1 and $-\frac{1}{3}$ and then subtract from 3rd & 4th rows respectively.

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows $(1, -5, -4, 2)$ and $(0, 6, 3, 3)$ which are linearly independent. Hence the corresponding matrices

$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}$ form a basis of W and dim W = 2.

Example 33. Prove that the vectors $u_1 = (1, 0, 2)$, $u_2 = (-1, 1, 0)$ and $u_3 = (0, 2, 3)$ form a basis of $\mathbb{R}^3$ and find the co-ordinates of the vectors $v = (1, -1, 1)$ and $w = (-1, 8, 11)$ relative to this basis.

Proof : First Portion

The given vectors will be a basis of $\mathbb{R}^3$ if and only if they are linearly independent and every vector in $\mathbb{R}^3$ can be written as a linear combination of u_1, u_2 and u_3. First we shall prove that the vectors u_1, u_2 and u_3 are linearly independent. For arbitrary scalars x_1, x_2 and x_3

let $x_1 u_1 + x_2 u_2 + x_3 u_3 = 0$

or, $x_1 (1, 0, 2) + x_2 (-1, 1, 0) + x_3 (0, 2, 3) = (0, 0, 0)$

or, $(x_1 - x_2, x_2 + 2x_3, 2x_1 + 3x_3) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_1 + 3x_3 &= 0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by elementary transformations. We multiply first equation by 2 and then subtract from the third equation. Thus the above system reduces to

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_2 + 3x_3 &= 0 \end{aligned}\right\} \quad (2)$$

Again we multiply second equation by 2 and then subtract from the third equation. Then we get the equivalent system.

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ - x_3 &= 0 \end{aligned}\right\} \quad (3)$$

This system is in echelon form and has exactly three equations in three unknowns, hence the system has only the zero solution i. e. $x = 0$, $y = 0$, $z = 0$. Accordingly, the vectors are linearly independent.

To show that u_1, u_2 and u_3 span IR^3, we must show that an arbitrary vector $v = (a, b, c)$ can be expressed as a linear combination $v = x_1 u_1 + x_2 u_2 + x_3 u_3$

or. $v = (a, b, c) = x_1 (1, 0, 2) + x_2 (-1, 1, 0) + x_3 (0, 2, 3)$

Forming the linear system, we get

$$\left.\begin{aligned} x_1 - x_2 &= a \\ x_2 + 2x_3 &= b \\ 2x_1 + 3x_3 &= c \end{aligned}\right\} \quad (4)$$

Reduce this system to echelon form by the elementary transformations.

We multiply first equation by 2 and then subtract from the third equation. Thus we have the equivalent system

$$\left.\begin{array}{rl} x_1 - x_2 \qquad\quad & = a \\ x_2 + 2x_3 & = b \\ 2x_2 + 3x_3 & = c-2a \end{array}\right\} \qquad (5)$$

We multiply second equation by 2 and then subtract from the third equation. Then we get the equivalent system

$$\left.\begin{array}{rl} x_1 - x_2 \qquad\quad & = a \\ x_2 + 2x_3 & = b \\ - x_3 & = c - 2a - 2b \end{array}\right\} \qquad (6)$$

From the third equation, we have $x_3 = 2a + 2b - c$.

Substituting the value of x_3 in the second equation, we get $x_2 = -4a - 3b + 2c$.

Again, substituting the value of x_2 in the first equation, we get $x_1 = -3a - 3b + 2c$. Therefore,

$v = (-3a - 3b + 2c)\, u_1 + (-4a - 3b + 2c)\, u_2 + (2a + 2b - c)\, u_3$

or, $(a, b, c) = (-3a - 3b + 2c)\,(1, 0, 2) + (-4a - 3b + 2c)\,(-1, 1, 0)$
$+ (2a + 2b - c)\,(0, 2, 3)$

Thus every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors $(1, 0, 2)$, $(-1, 1, 0)$ and $(0, 2, 3)$.

Hence the vectors u_1, u_2 and u_3 form a basis of $\mathbb{R}^3$.

Second Portion : Let $v = (1, -1, 1) = x_1 u_1 + x_2 u_2 + x_3 u_3$

or, $(1, -1, 1) = x_1\,(1, 0, 2) + x_2\,(-1, 1, 0) + x_3\,(0, 2, 3)$.

Forming linear system, we have

$$\left.\begin{array}{rl} x_1 - x_2 \qquad\quad & = 1 \\ x_2 + 2x_3 & = -1 \\ 2x_1 \qquad + 3x_3 & = 1 \end{array}\right\} \qquad (7)$$

Solving the system, we get $x_1 = 2, x_2 = 1, x_3 = -1$.

Thus $v = (1, -1, 1) = 2u_1 + 1u_2 + (-1)\, u_3$.

So the vector v has co-ordinates (2, 1, -1).

Similarly, let $w = (-1, 8, 11) = y_1u_1 + y_2u_2 + y_3u_3$

or, $(-1, 8, 11) = y_1\,(1, 0, 2) + y_2\,(-1, 1, 0) + y_3\,(0, 2, 3)$.

Forming linear system, we have

$$\left.\begin{aligned} y_1 - y_2 \qquad &= -1 \\ y_2 + 2y_3 &= 8 \\ 2y_1 \qquad + 3y_3 &= 11 \end{aligned}\right\} \qquad (8)$$

Solving the system, we get $y_1 = 1, y_2 = 2, y_3 = 3$

Thus $w = (-1, 8, 11) = 1u_1 + 2u_2 + 3u_3$

So the vector w has co-ordinates (1, 2, 3).

Example 34. Given the vectors (2, 1, 1), (1, 3, 2), (1, 3, -1) and (1, -2, 3). Test whether they are linearly independent by **Sweep out method.**

Solution : Let $\alpha_1\ (2, 1, 1) + \alpha_2\ (1, 3, 2) + \alpha_3\ (1, 3, -1)$

$$+ \alpha_4\ (1, -2, 3) = O = (0, 0, 0).$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are scalars. then

$(2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + 3\alpha_2 + 3\alpha_3 - 2\alpha_4,\ \alpha_1 + 2\alpha_2 - \alpha_3 + 3x_4)$ $= (0, 0, 0)$. Equating corresponding components from both sides and forming linear system, we get

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\ \alpha_1 + 3\alpha_2 + 3\alpha_3 - 2\alpha_4 &= 0 \\ \alpha_1 + 2\alpha_2 - \alpha_3 + 3\alpha_4 &= 0 \end{aligned}\right\}$$

α_1	α_2	α_3	α_4	
2	1	1	1	$O \to R_1$
1	3	3	−2	$O \to R_2$
1	2	−1	3	$O \to R_3$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$O \to R_4 = \frac{R_1}{2}$ 1st pivotal row
0	$\frac{5}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$	$O \to R_5 = R_2 - R_4$
0	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$	$O \to R_6 = R_3 - R_4$
	1	1	−1	$O \to R_7 = \frac{R_5}{\frac{5}{2}}$ 2nd pivotal row
	0	−3	4	$O \to R_8 = R_6 - \frac{3}{2} R_7$
		1	$-\frac{4}{3}$	$O \to R_9 = \frac{R_8}{-3}$ 3rd pivotal row

From the pivotal rows, we have

$\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_4 = 0$ (i)

$\alpha_2 + \alpha_3 - \alpha_4 = 0$ (ii)

$\alpha_3 - \frac{4}{3}\alpha_4 = 0$ (iii)

This system is in echelon form and has three equations in 4 unknowns and hence 4 − 3 = 1 free variable which is α_4.

Thus the system has an infinite number of non-zero solutions.

Let $\alpha_4 = t$, where t is a scalar.

Then $\alpha_3 = \frac{4}{3}t$, $\alpha_2 = -\frac{1}{3}t$, and $\alpha_1 = -t$.

Since all α's are not zero, so the given vectors are linearly dependent.

Example 35. Are these vectors (2, 1, 1), (2, 4, 7), (4, −9, 11) dependent ?. Test by **Sweep out method.**

Solution : Let α_1 (2, 1, 1) + α_2 (2, 4, 7) + α_3 (4, −9, 11) = (0, 0, 0)

Or, $(2\alpha_1 + 2\alpha_2 + 4\alpha_3, \alpha_1 + 4\alpha_2 - 9\alpha_3, \alpha_1 + 7\alpha_2 + 11\alpha_3) = (0, 0, 0)$

Equating corresponding components from both sides and forming the linear system, we get

$$\left.\begin{aligned} 2\alpha_1 + 2\alpha_2 + 4\alpha_3 &= 0 \\ \alpha_1 + 4\alpha_2 - 9\alpha_3 &= 0 \\ \alpha_1 + 7\alpha_2 + 11\alpha_3 &= 0 \end{aligned}\right\}$$

α_1	α_2	α_3	
2	2	4	$O \to R_1$
1	4	-9	$O \to R_2$
1	7	11	$O \to R_3$
1	1	2	$O \to R_4 = \frac{R_1}{2}$ 1st pivotal row
O	3	-11	$O \to R_5 = R_2 - R_4$
O	6	9	$O \to R_6 = R_3 - R_4$
	1	$-\frac{11}{3}$	$O \to R_7 = \frac{R_5}{3}$ 2nd pivotal row
	O	31	$O \to R_8 = R_6 - 6R_7$
		1	$O \to R_9 = \frac{R_8}{31}$ 3rd pivotal row

From the pivotal rows, we have

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \qquad \text{(i)}$$

$$\alpha_2 - \frac{11}{3}\alpha_3 = 0 \qquad \text{(ii)}$$

$$\alpha_3 = 0 \qquad \text{(iii)}$$

Therefore, the system has zero solution i. e $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence the given vectors are linearly independent.

Example 36. Find the rank and the basis of a given set of vectors {(2, –1, 5, 4), (0, 1, 2, 3), (4, 0, 6, 1), (0, – 2, 4, 7)} by using **Sweep out method.**

Solution : Rank of a given set of vectors is the number of linearly independent vectors of that set and these linear independent vectors form a basis of that set.

2	–1	5	4	$\to R_1$
0	1	2	3	$\to R_2$
4	0	6	1	$\to R_3$
0	–2	4	7	$\to R_4$
1	$-\frac{1}{2}$	$\frac{5}{2}$	2	$\to R_5 = \frac{R_1}{2}$ 1st pivotal row
0	1	2	3	$\to R_6 = R_2 - 0R_5$
0	2	–4	–7	$\to R_7 = R_3 - 4R_5$
0	–2	4	7	$\to R_8 = R_4 - 0R_5$
	1	2	3	$\to R_9 \frac{R_6}{1}$ 2nd pivotal row
	0	–8	–13	$\to R_{10} = R_7 - 2R_9$
	0	8	13	$\to R_{11} = R_8 + 2R_9$,
		1	$\frac{13}{8}$	$\to R_{12} = \frac{R_{10}}{-8}$ 3rd pivotal row
		0	0	$\to R_{13} = R_{11} - 8R_{12}$

Now the rank of the given set of vectors is equal to the number of pivotal rows = 3. A basis of the given set of vectors is $\{(1, -\frac{1}{2}, \frac{5}{2}, 2), (0, 1, 2, 3), (0, 0, 1, \frac{13}{8})\}$

Since $\alpha_1\ (1, -\frac{1}{2}, \frac{5}{2}, 2) + \alpha_2(0, 1, 2, 3) + \alpha_3\ (0, 0, 1, \frac{13}{8}) = (0, 0, 0, 0)$

implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$. (Zero solution).

EXERCISES – 6 (C)

1. Prove that the vectors (1, 1) and (1, 0) form a basis of $\mathbb{R}^2$.

2. (i) Prove that $\{(2, -\frac{1}{2}, 1), (3, 2, 1), (0, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

(ii) Prove that $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1)\ (0, 0, 0, 1)\}$ is a basis of $\mathbb{R}^4$.

3. (i) Extend $\{(2, 0, 0, -1), (1, 3, -1, 0)\}$ to a basis for $\mathbb{R}^4$.

[D. U. S. 1984]