Lec: 09

Linear Transformations:

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$T : \mathbb{R}^m \to \mathbb{R}^n$

denote a function T that takes vectors in $\mathbb{R}^m$ as input and produces vectors in $\mathbb{R}^n$ as output.

put another way, the domain of T is $\mathbb{R}^m$ and the codomain is $\mathbb{R}^n$.

For every vector u in $\mathbb{R}^m$, the vector $T(u)$ is called the image of u under T.

The set of all images of vectors u in $\mathbb{R}^m$ under T is called the range of T, denoted range (T). Thus the range of T is a subset of the codomain of T.

Definition: A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if for all vectors u and v in $\mathbb{R}^m$ and all scalars r we have

(i) $T(u+v) = T(u) + T(v)$

(ii) $T(ru) = rT(u)$

condition (i) and (ii) can be combined into a single condition. (*) $T(ru+sv) = rT(u) + sT(v)$

Theorem: Let A be an $n \times m$ matrix, and define $T(x) = Ax$. Then $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

Theorem: Let $T : \mathbb{R}^m \to \mathbb{R}^n$. Then $T(x) = Ax$, where A is an $n \times m$ matrix, if and only if T is a linear transformation.

(*) One-to-One and Onto Linear Transformations.

Example: Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$; $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; $v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} * \\ 4 \end{bmatrix}$

Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^3$ with $T(x) = Ax$. Compute $T(u)$ & $T(v)$, and determine if w is in the range of T.

$$T(u) = Au = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-4+4 \\ 3+0-5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$T(v) =$

$Ax = w$

Example: Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $\qquad T(x) = Ax$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ -x_1 + 2x_2 \\ x_1 - 3x_2 + 5x_3 \\ 4x_2 \end{bmatrix} ; \quad A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 5 \\ 0 & 4 & 0 \end{bmatrix}$$

Show that T is a linear transformation.

(11) Suppose that a linear transformation T satisfies
$T(u_1) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, $T(u_2) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $T(u_3) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$
Find $T((-u_1) + (4u_2) + (-3)u_3)$

 # Theorem: Let T be a linear transformation. Then T is one-to-one if and only if $T(x) = 0$ has only the trivial solution $x = 0$.

Kernel and Range of Linear Transformation:

Theorem: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then the kernel of T is a subspace of the domain $\mathbb{R}^m$ and the range of T is a subspace of the codomain $\mathbb{R}^n$.

Definition of Kernel: The kernel of T is the set of vectors x such that $T(x) = 0$. The kernel of T is denoted by $\ker(T)$.

$$\ker(T) = \text{null}(A)$$

Example: Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -3x_1 + 6x_2 \\ 2x_1 - 4x_2 \end{bmatrix}$$

Find $\ker T$ and range (T).

Solution: We have $T(x) = Ax$ for $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \\ 2 & -4 \end{bmatrix}$

To find the null space of A, we solve the homogeneous linear system $Ax = 0$. We have

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 6 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is equivalent to the single equation $x_1 - 2x_2 = 0$. Since $\ker(T) = \text{null}(A)$; it follows that if we let $x_2 = s$, then $x_1 = 2s$

and thus $\ker(T) = s\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or $\ker(T) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

Because the range of T is equal to the span of the column of A, we have range $T = \text{span}\{a_1\} = \text{span}\left\{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}\right\}$

Find an Example that meets the given specifications:

(i) A 2×3 matrix A with nullity $(A) = 1$

(ii) A 9×4 matrix A with rank $(A) = 3$

(iii) A matrix A with rank $(A) = 2$ and nullity $(A) = 2$.

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Theorem: Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [a_1, \dots, a_n]$, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(x) = Ax$.

Then the following are equivalent:

(a) $\mathcal{A}$ spans $\mathbb{R}^n$.

(b) $\mathcal{A}$ is linearly independent.

(c) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$.

(d) T is one-to-one

(e) T is onto

(f) A is invertible.

(g) $\ker(T) = \{0\}$.

(h) $\mathcal{A}$ is a basis for $\mathbb{R}^n$.

(i) $Col(A) = \mathbb{R}^n$.

(j) $Row(A) = \mathbb{R}^n$.

(k) $Rank(A) = n$.

Suppose that A is a 6×11 matrix and that $T(x) = Ax$. If nullity $(A) = 7$; what is the dimension of the range of T?

Suppose that A is a 13×5 matrix and that $T(x) = Ax$. If T is one-to-one, then what is the dimension of the null-space of A?

Suppose that A is a 9×5 matrix and that B is an equivalent matrix in echelon form

(i) If B has two pivot columns, what is rank (A)?

(ii) If B has ~~two pivot~~ three non-zero rows, what is rank (A)?

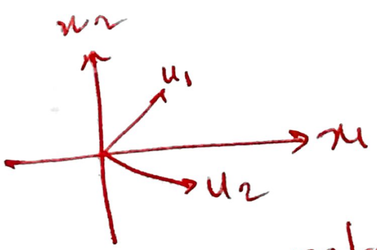
(iii) If B has three non-zero rows, what is nullity (A)?

(iv) If rank $(A) = 3$, how many non-zero rows does B have?

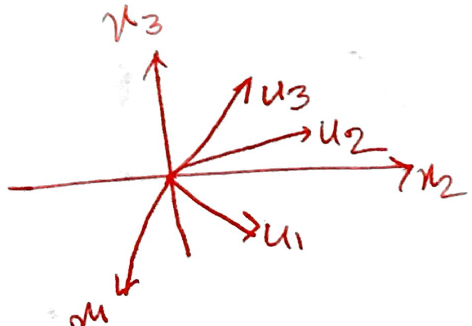
(v) If " $(A) = 1$, how many pivot columns does B have.

A 7×11 matrix A has rank 7. What is the dimension of the null space of A?

Suppose that A is a 5×13 matrix. What is the maximum possible value for the rank of A, and what is the minimum possible value for the nullity of A.



Any two non-zero vectors that do not lie on the same line forms a basis for $\mathbb{R}^2$



Any three non-zero vector that do not lie in the same plane forms a basis for $\mathbb{R}^3$.

Find all values of x so that rank $(A) = 2$

(i) $\begin{bmatrix} 1 & -4 \\ -2 & x \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 11 \\ 4 & 3 & x \end{bmatrix}$