CSE 221: Algorithms

Balanced trees

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Computer Science and Engineering
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References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.
2. Erik Demaine and Charles Leiserson, *6.046J Introduction to Algorithms*. MIT OpenCourseWare, Fall 2005. Available from: `ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/CourseHome/index.htm`
3. Robert Sedgewick, *Left-Leaning Red-Black Trees*. 2008.

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Contents

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Need for balanced trees

- The lookup and insertion time in a binary search tree is $O(h)$:

  Best case when the tree is balanced, $h = \lfloor \lg n \rfloor = O(\lg n)$
  Worst case when the tree is *linear*, then $h = O(n)$

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2. Bounded depth n-ary trees – 2-3-4, B, etc. trees.

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- **3-node** two keys, three children
- **4-node** three keys, four children

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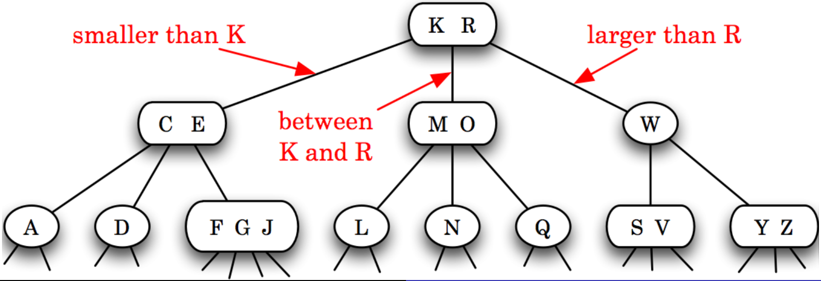
Courtesy of Robert Sedgewick `http://www.cs.princeton.edu/~rs/talks/LLRB/RedBlack.pdf`



Searching in a 2-3-4 tree

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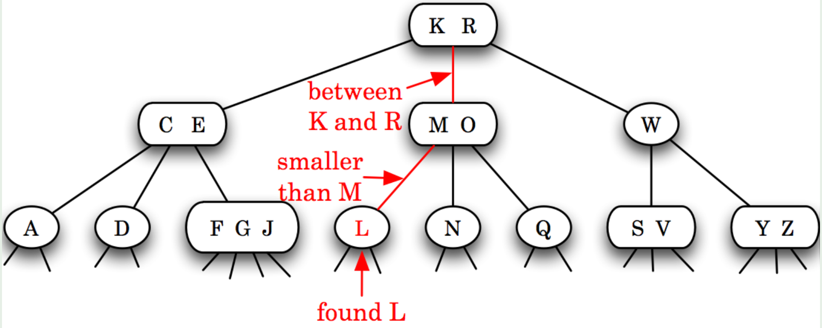
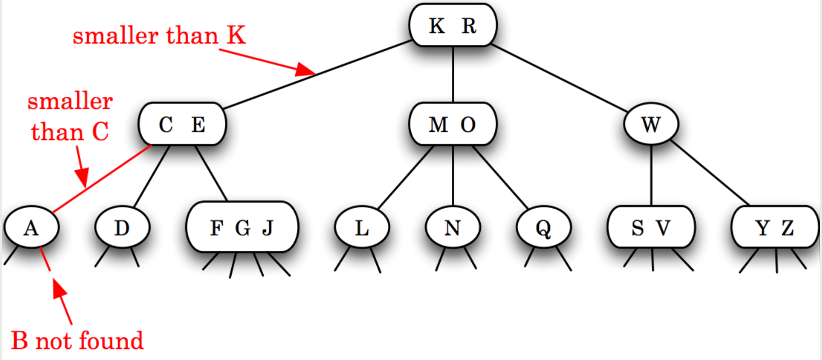
Example (Searching for L)



Inserting into a 2-3-4 tree

- Search to bottom for insertion position of key B.
- 2-node at bottom: convert to 3-node

Example (Insert B)



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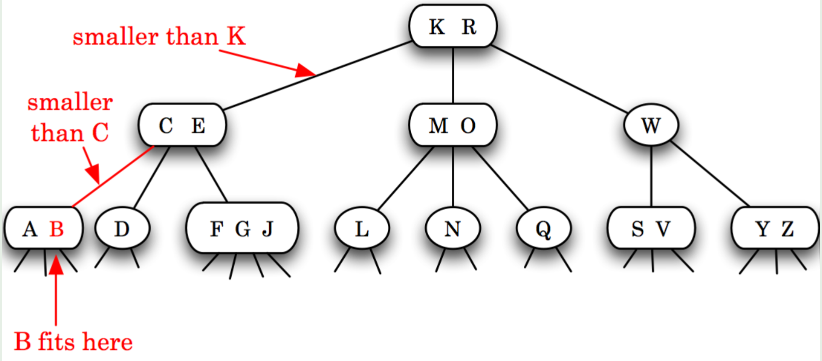

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- Search to bottom for insertion position of key X.
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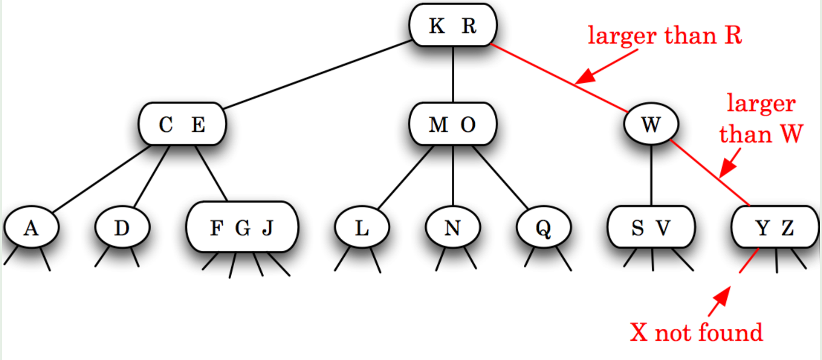

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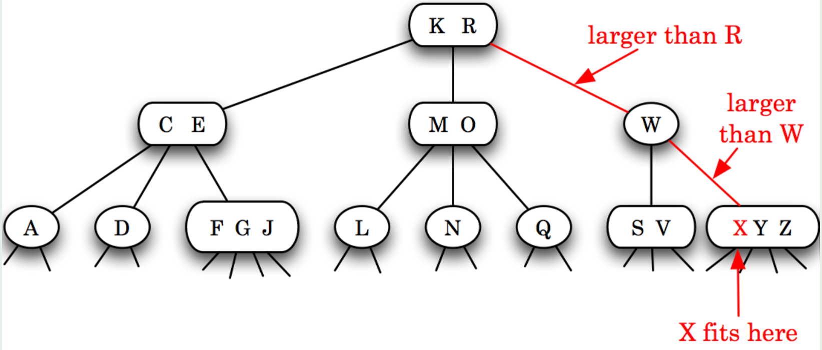

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- Search to bottom for insertion position of key H.
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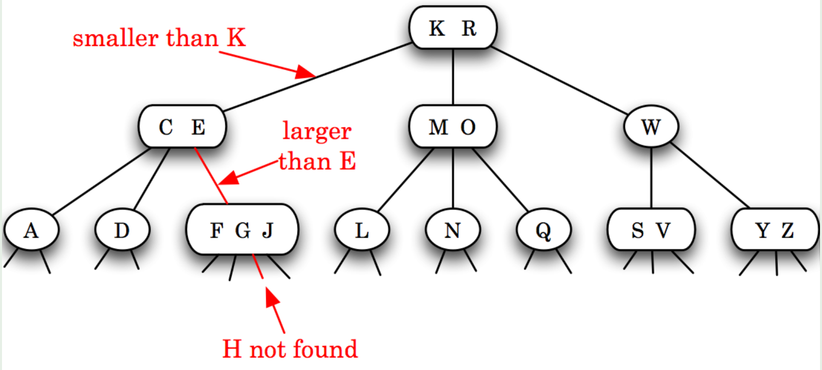
- Must split node to make room for new key.

Example (Insert H)



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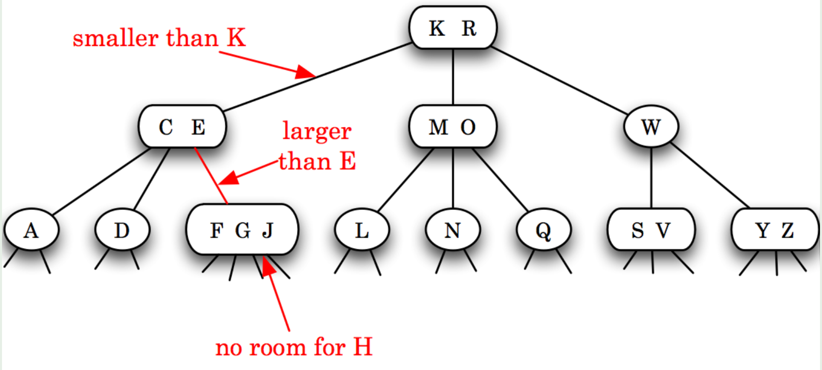

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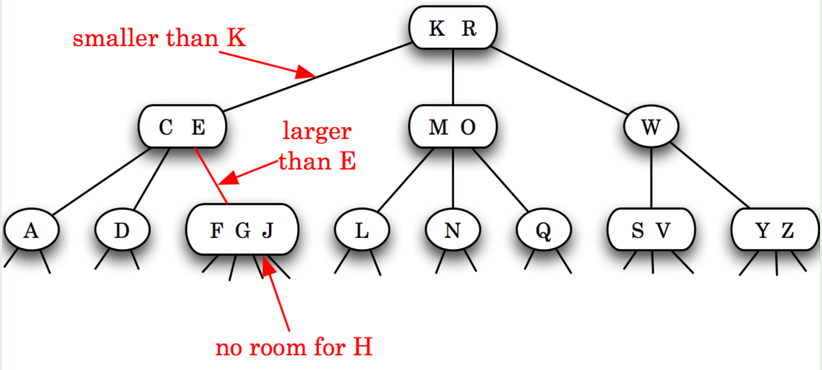
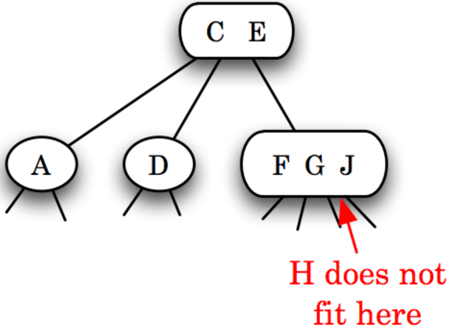


Splitting a 4-node in a 2-3-4 tree

Idea is to move the middle element to the parent, making room for one more key.



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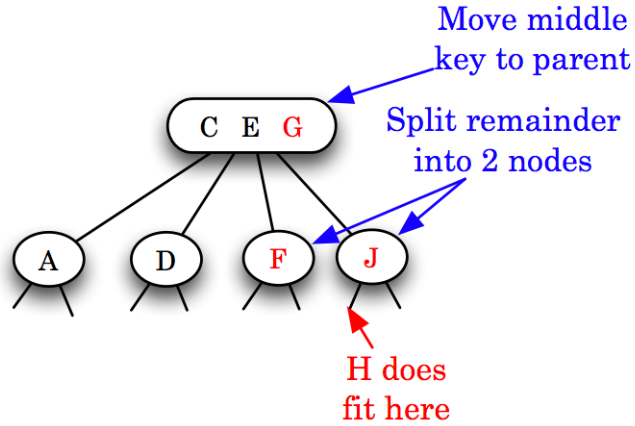

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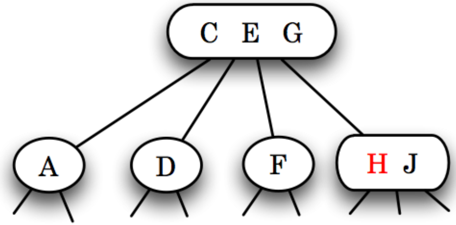

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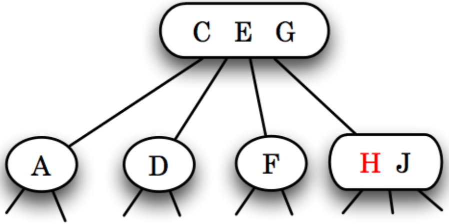

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What if the parent is a 4-node too!

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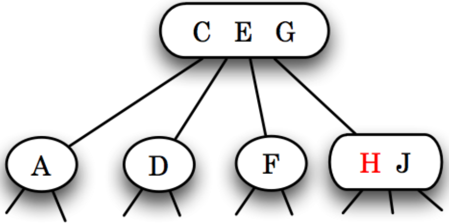


Question

What if the parent is a 4-node too!

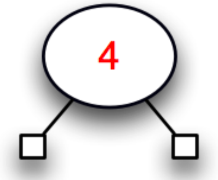
Solution: Split the parent too, potentially creating a new root.

Insertion in action

Insert 4 into an empty 2-3-4 tree

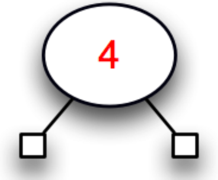
Insertion in action

Insert 4 into an empty 2-3-4 tree – done



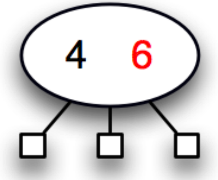
Insertion in action

Insert 6



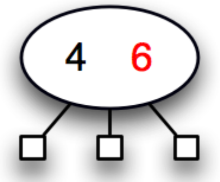
Insertion in action

Insert 6 – done



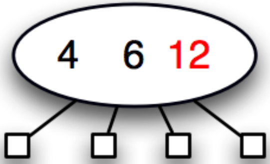
Insertion in action

Insert 12



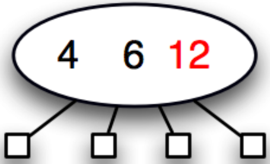
Insertion in action

Insert 12 – done



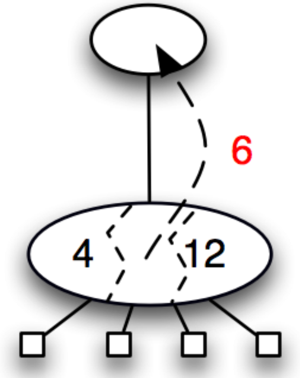
Insertion in action

Insert 15



Insertion in action

Insert 15: No room, so split node



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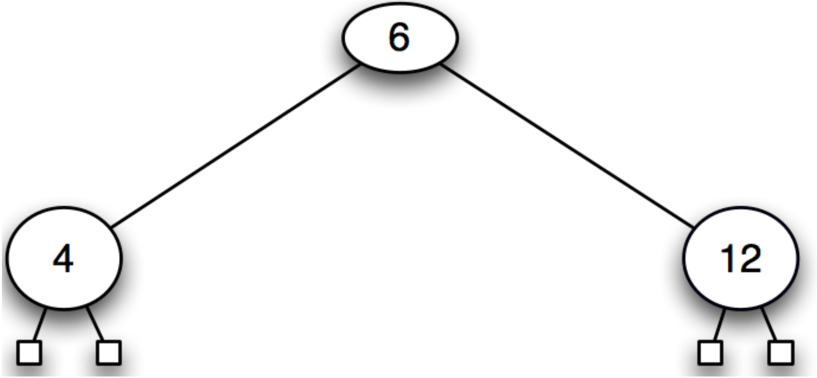
Insert 15: Room available after splitting



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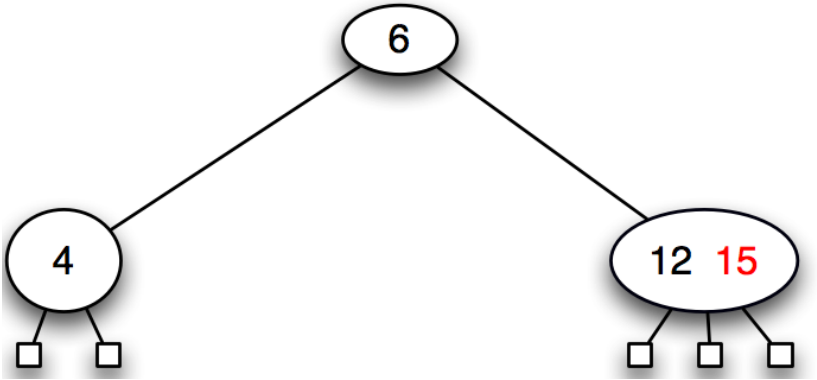

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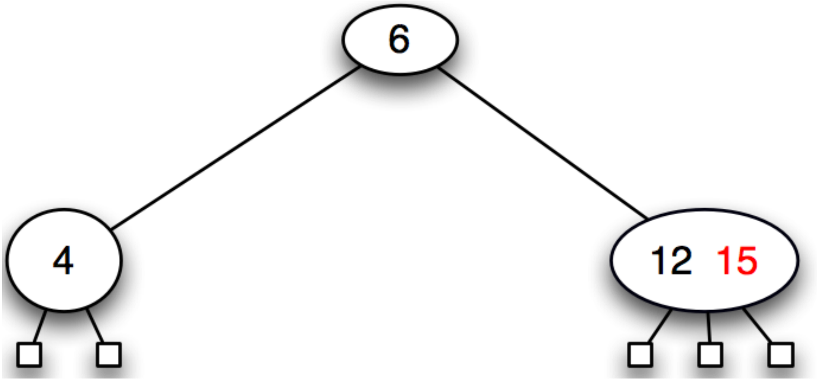
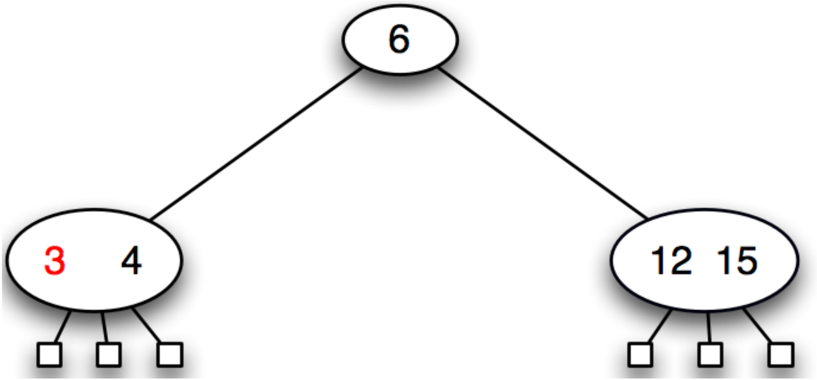


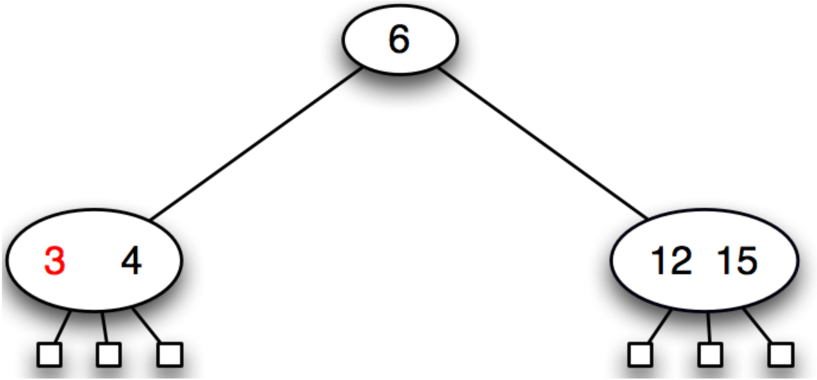
Insertion in action

Insert 3 – done



Insertion in action

Insert 5



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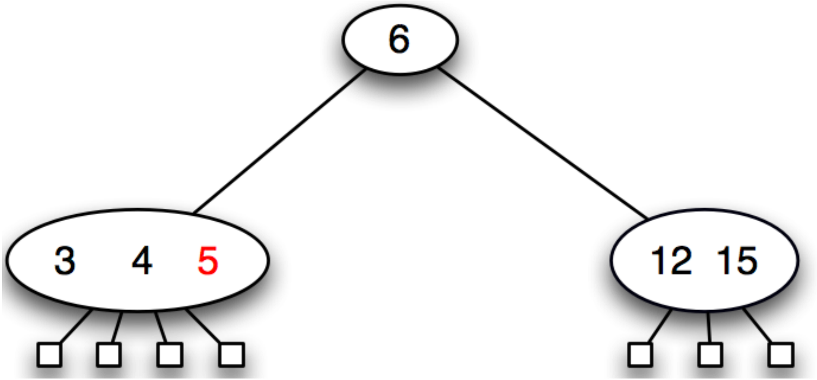

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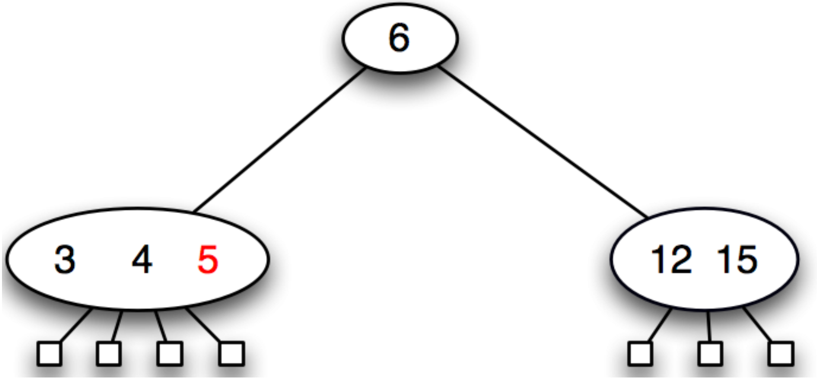
Insert 10



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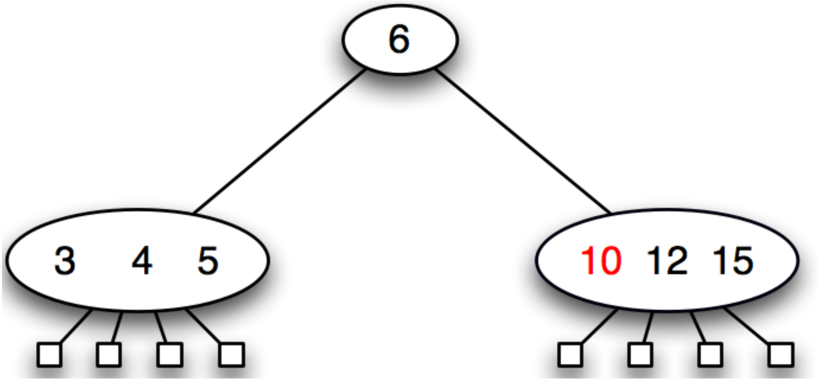

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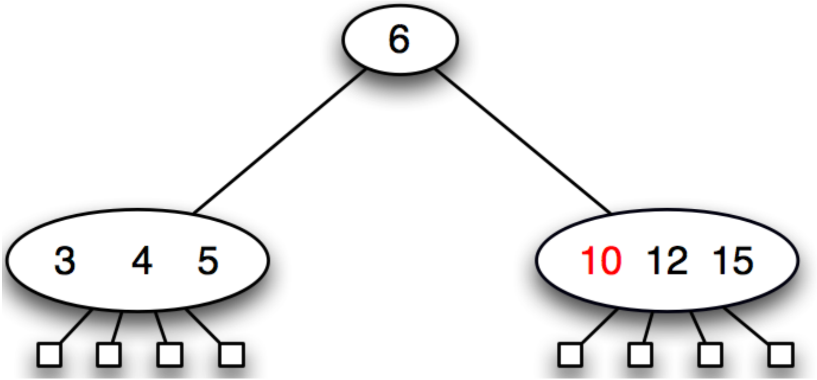
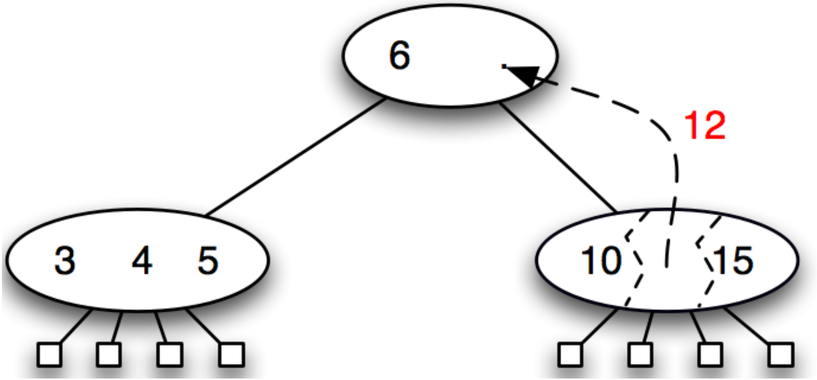
Insert 8



Insertion in action

Insert 8: No room, split node



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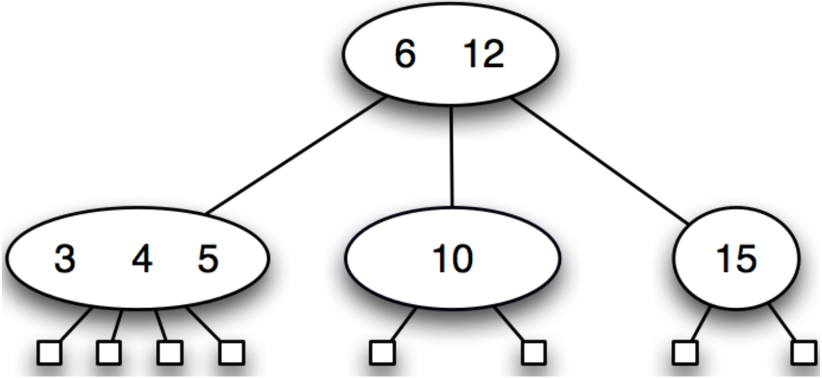

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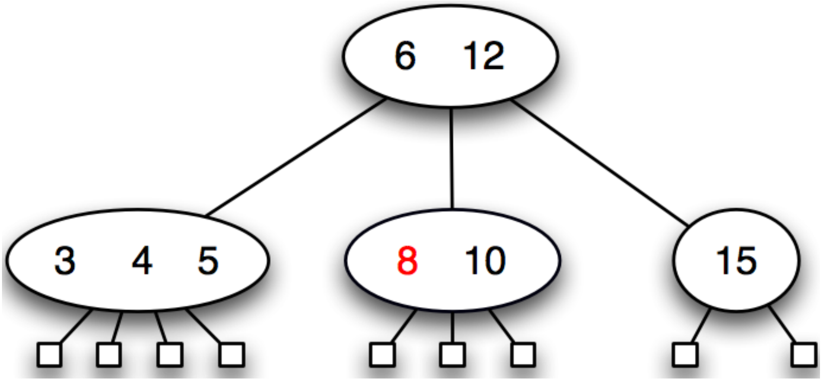

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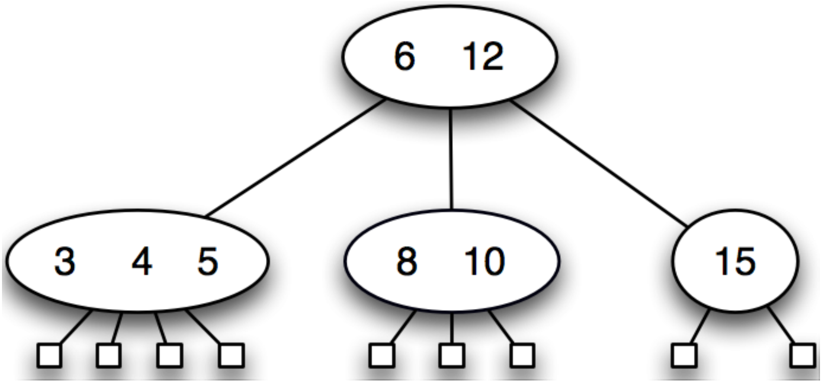

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This proves that $h = \Theta(\lg n)$.

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  This proves that $h = \Theta(\lg n)$.
- The bounded depth property guarantees that all operations are $O(h) = O(\lg n)$ in a 2-3-4 tree.

Summary of 2-3-4 trees

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Key question

Is there something that provides $O(\lg n)$ performance with the same advantages of binary tree format? YES – Red-Black trees!

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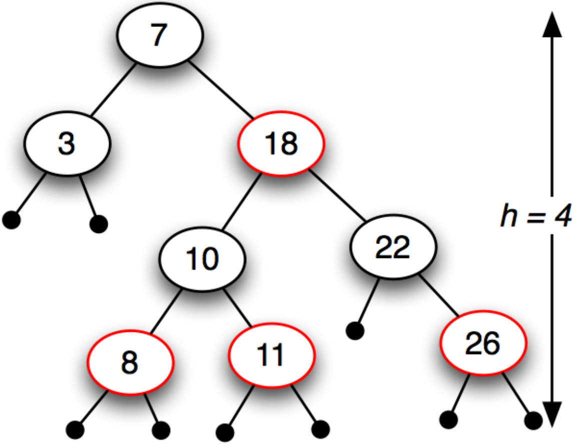
The data structure needed for a Red-Black tree is a binary search tree with an extra color bit for each node.

Example of red-black tree



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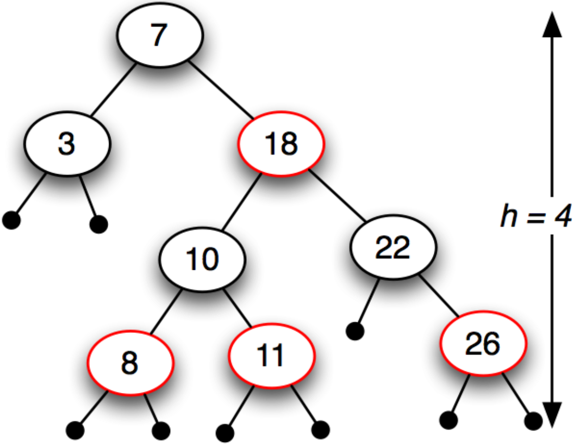

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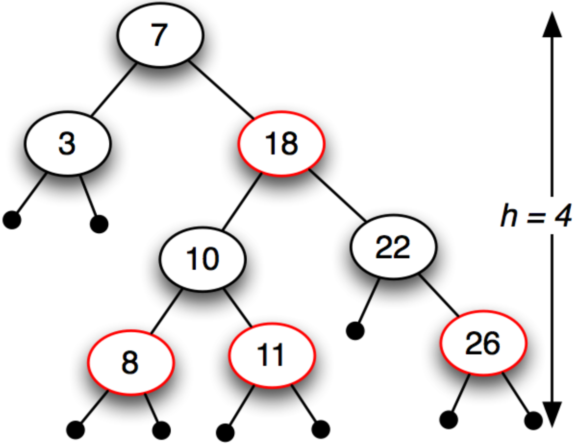

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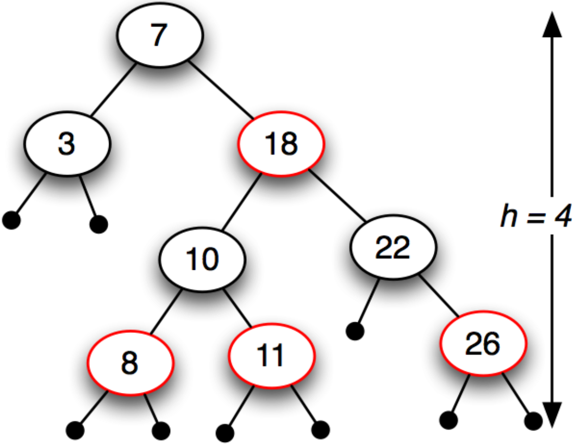
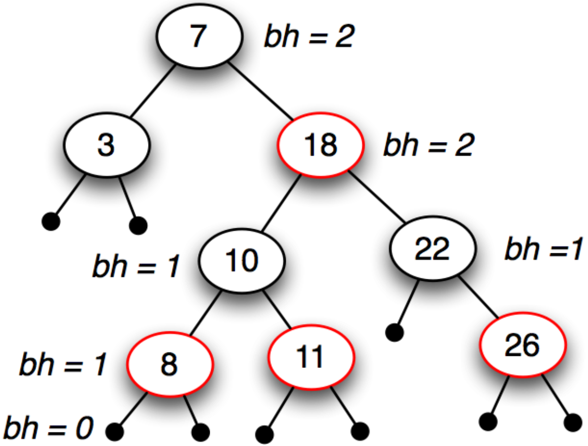


3. If a node is red, then its parent is black.

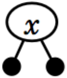
Example of red-black tree



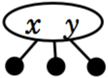
4. All simple paths from any node x to a descendant external node or leaf have the same number of black nodes = black-height(x).

Equivalence of red-black tree and a 2-3-4 tree

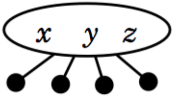
2-node



3-node



4-node



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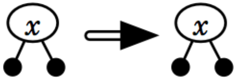

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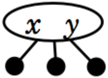

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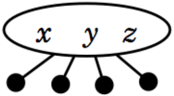

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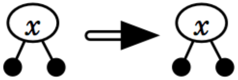
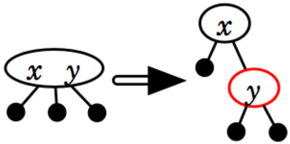


3-node



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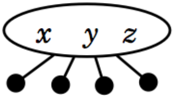
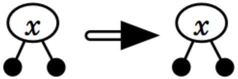
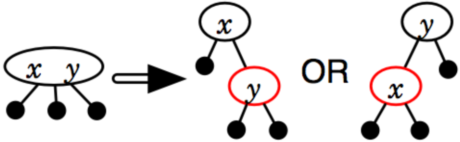


Equivalence of red-black tree and a 2-3-4 tree

2-node



3-node



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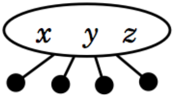

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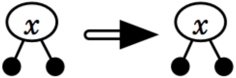

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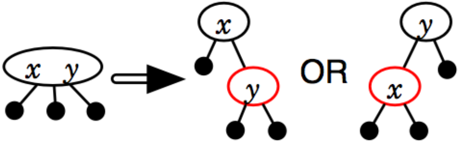
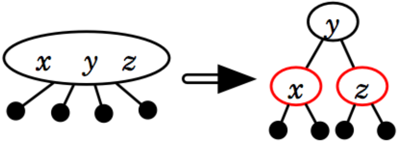


4-node

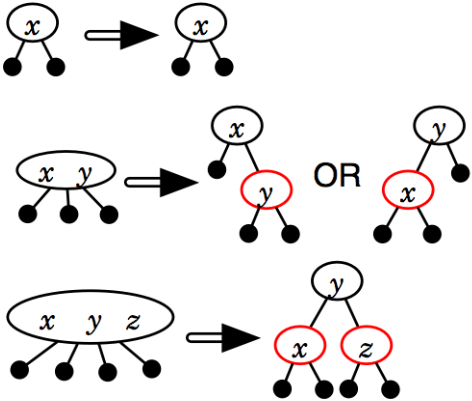


Equivalence of red-black tree and a 2-3-4 tree

2-node

3-node

4-node

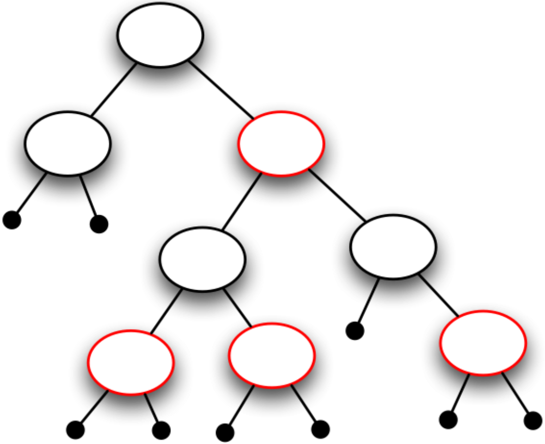


Key observation

Red-black tree is simply another way of representing a 2-3-4 tree!

Height of a red-black tree

- Merge the red nodes into their black parents.



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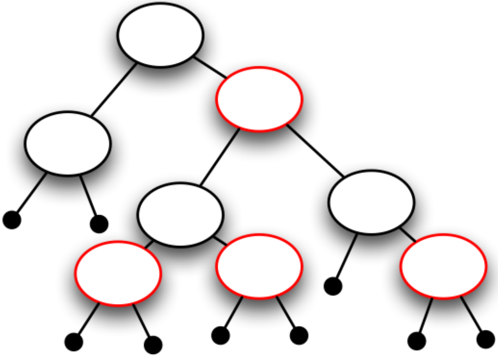

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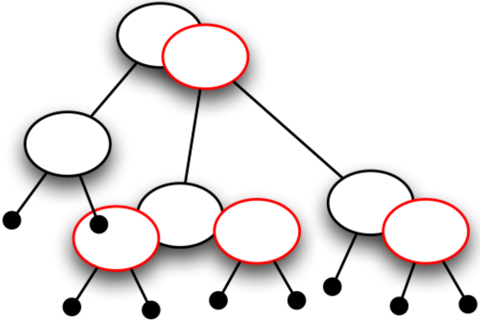

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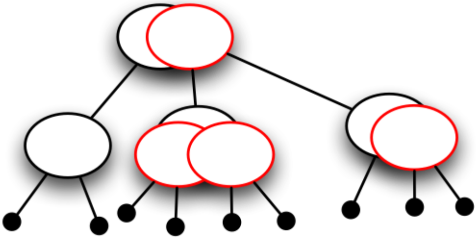

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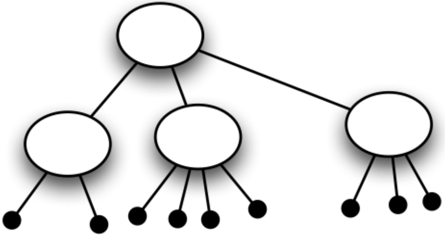

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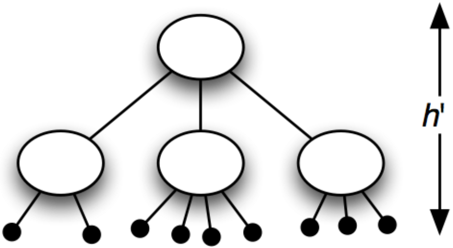
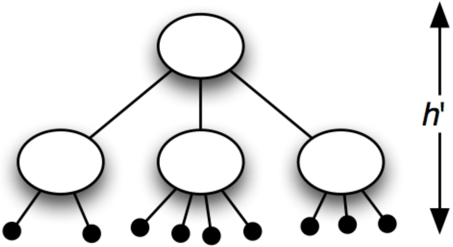
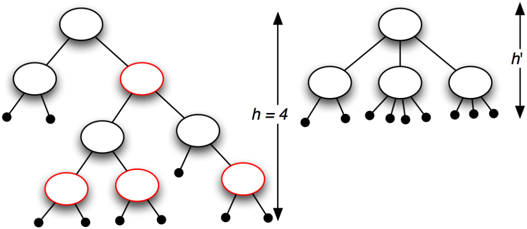


Height of a red-black tree

- Merge the red nodes into their black parents.
- Produces a 2-3-4 tree with height h'.



Height of a red-black tree (continued)



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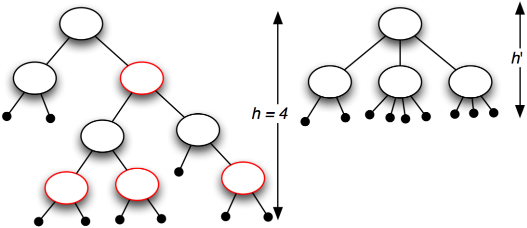

- We have $h' \geq h/2$, since at most half the nodes on any path are red.

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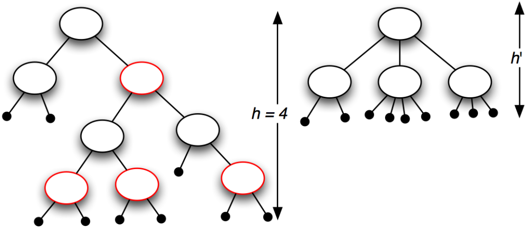

- We have $h' \geq h/2$, since at most half the nodes on any path are red.
- Number of external nodes or leaves is $n + 1$, so we have:

$$n + 1 \geq 2^{h'} \Rightarrow \lg(n+1) \geq h' \geq h/2 \Rightarrow h \leq 2\lg(n+1).$$

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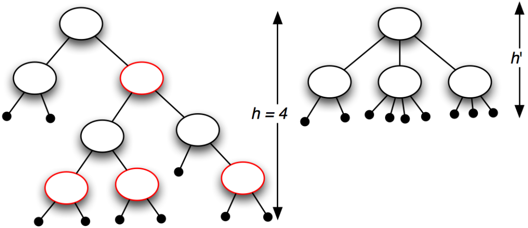

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Theorem

A red-black tree with n keys has height $h \leq 2\lg(n+1) = O(\lg n)$.

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Key question

How do Red-black trees compare with 2-3-4 trees in terms of performance and data structure complexity?

Conclusion

- Binary search tree guarantees a performance of $O(h)$, but h can vary from $O(\lg n)$ in the best-case to $O(n)$ in the worst-case.

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Questions to ask (and remember)

- What's the equivalence of a 2-3-4 tree and Red-Black tree?
- Why is the data structure in implementing a 2-3-4 tree considered complex?
- What are some of the disadvantages of a Red-Black tree?