# CSE 221: Algorithms

## Quicksort

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**References**

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.
2. Erik Demaine and Charles Leiserson, *6.046J Introduction to Algorithms*. MIT OpenCourseWare, Fall 2005. Available from: `ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/CourseHome/index.htm`

Last modified: June 21, 2009

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**Why do we want to study Quicksort?**

One of the most widely used, and extensively studied, sorting algorithms.

# Divide and conquer

Quicksort an $n$-element array:

1. *Divide* Partition the array into subarrays around a *pivot* $x$ such that the elements in lower subarray $\leq x \leq$ elements in the upper subarray.


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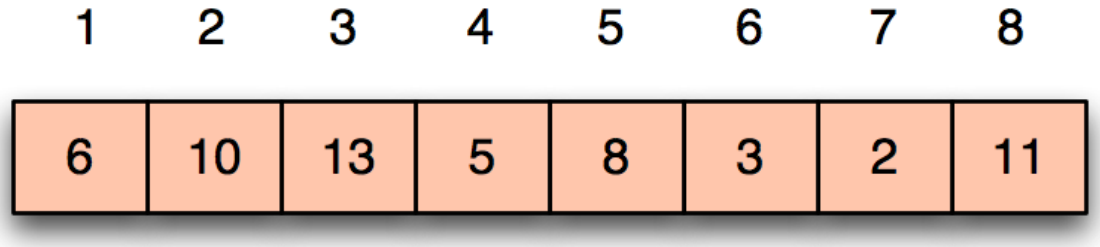
**Key**

*Linear-time partitioning algorithm.*

# Quicksort in action



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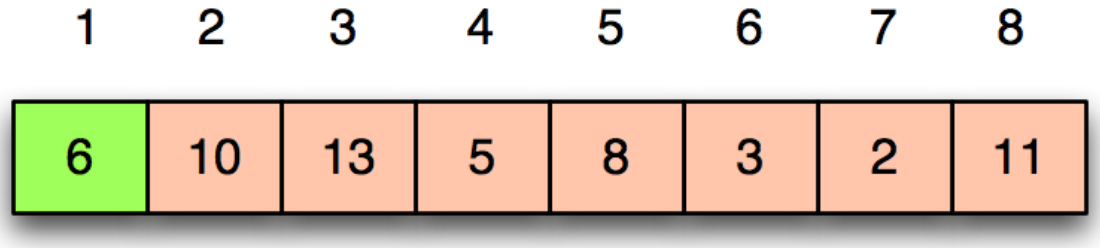

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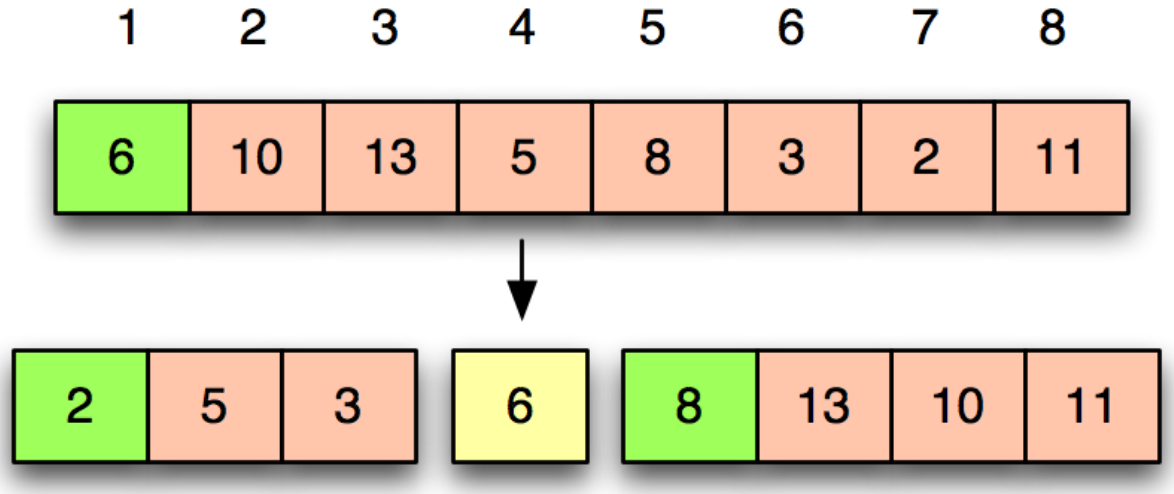

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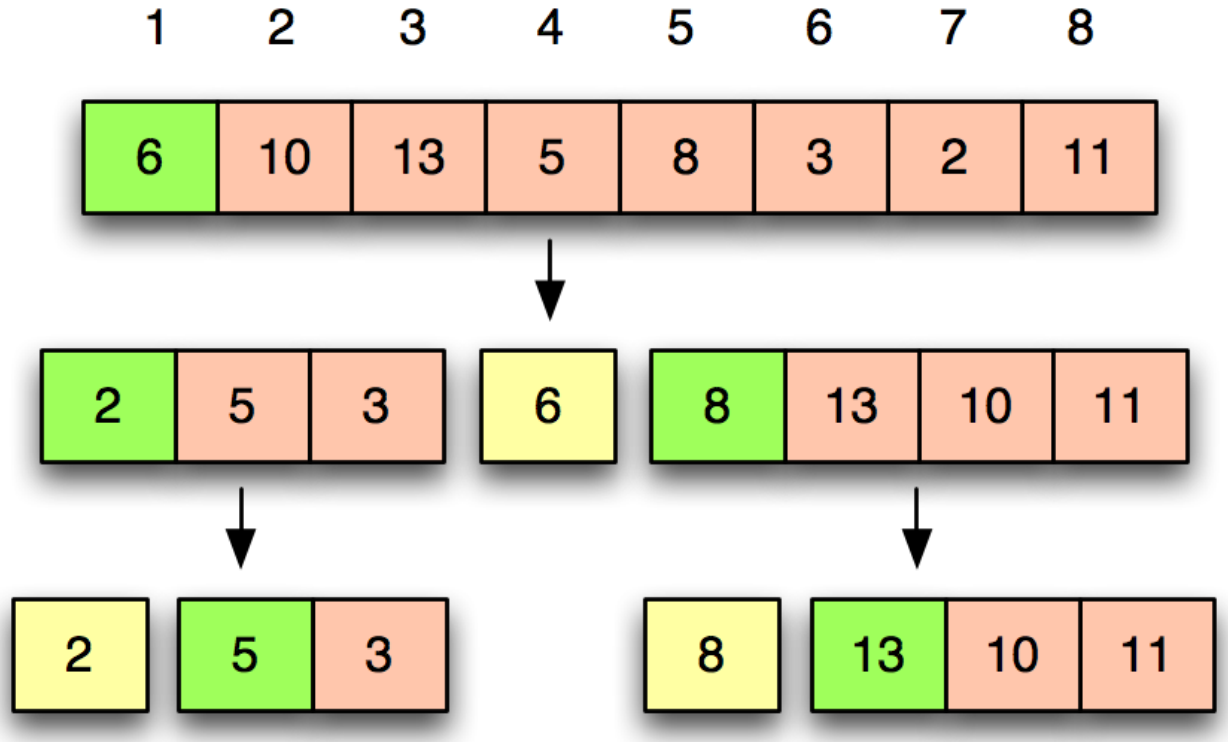

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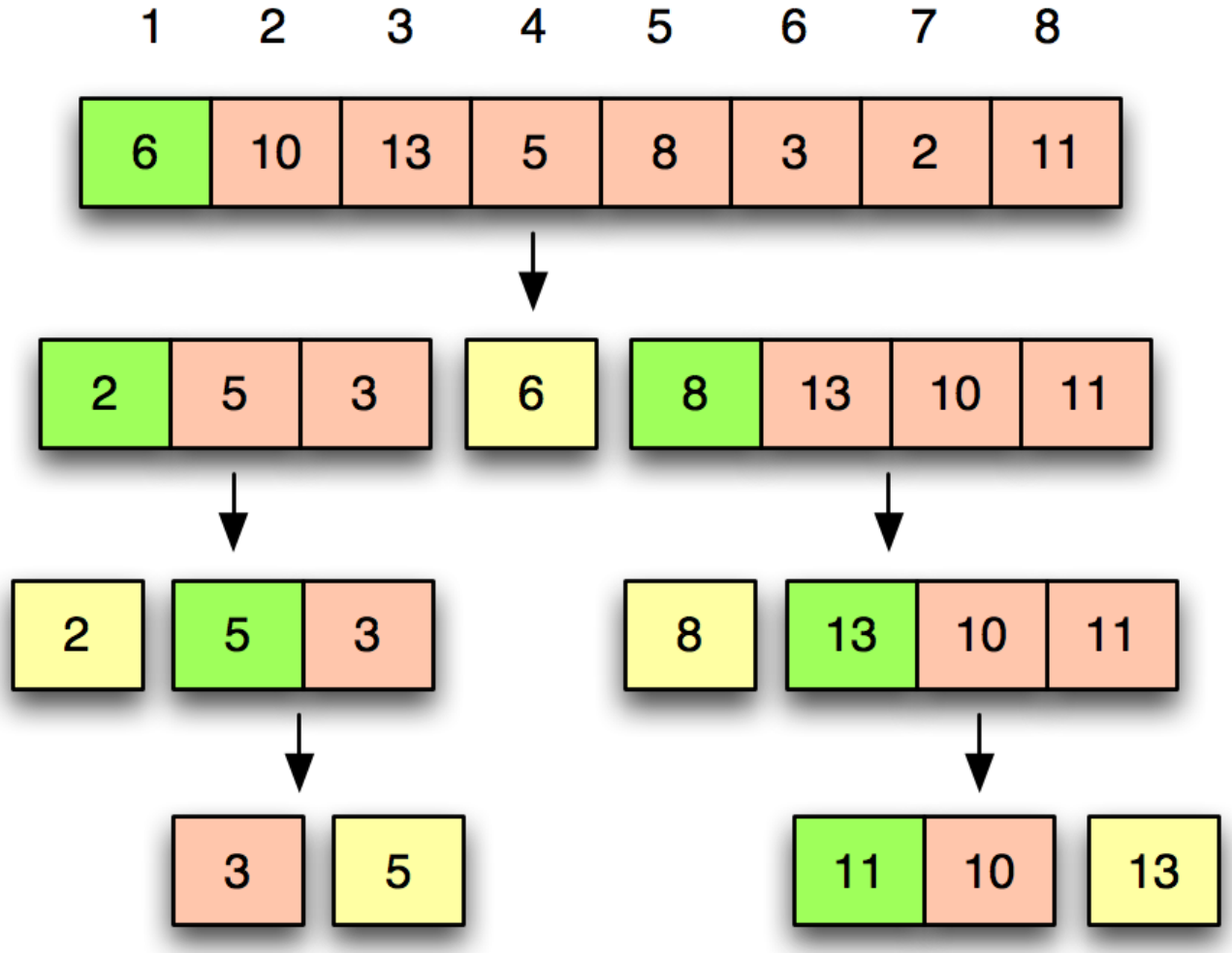

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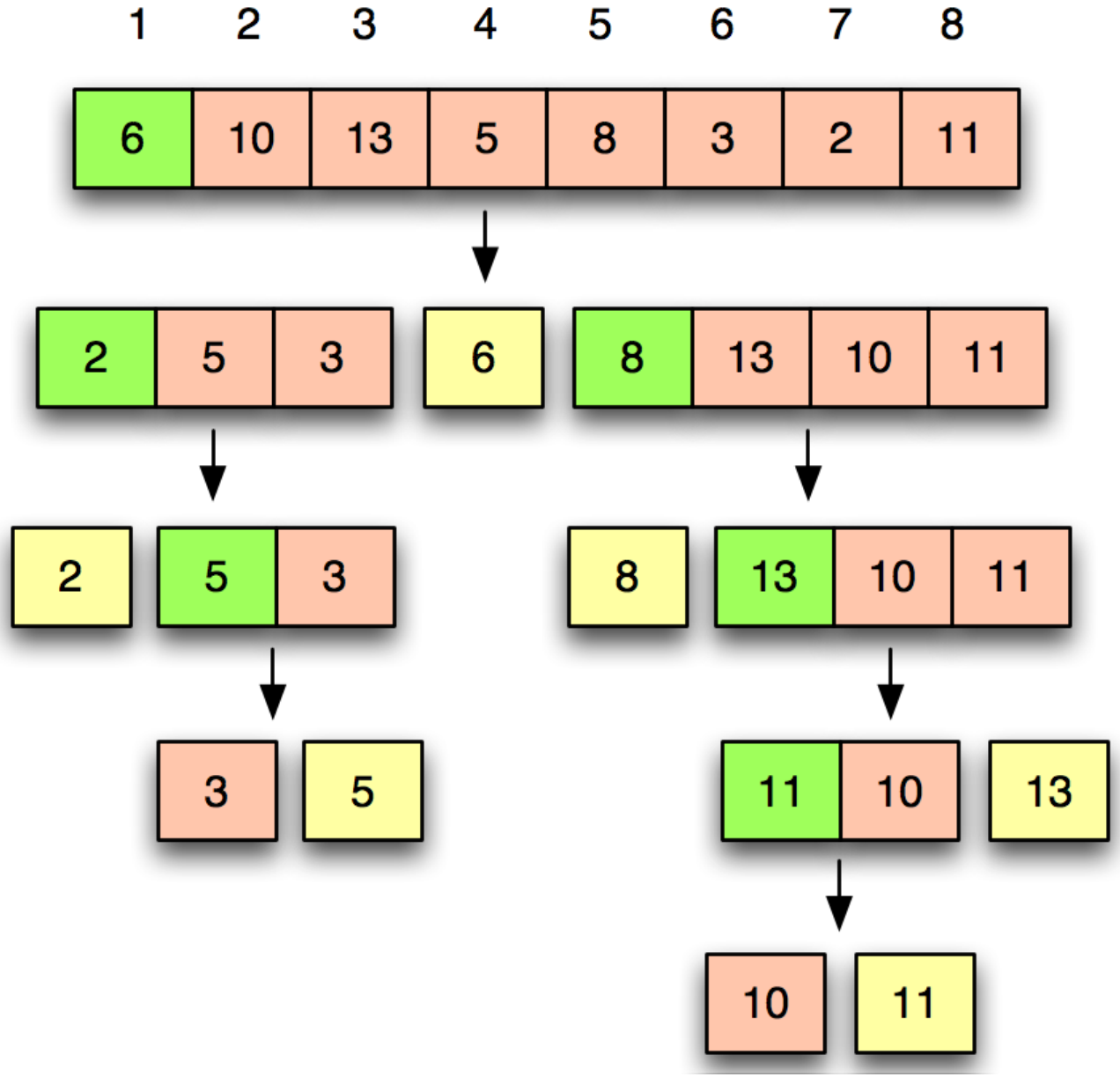

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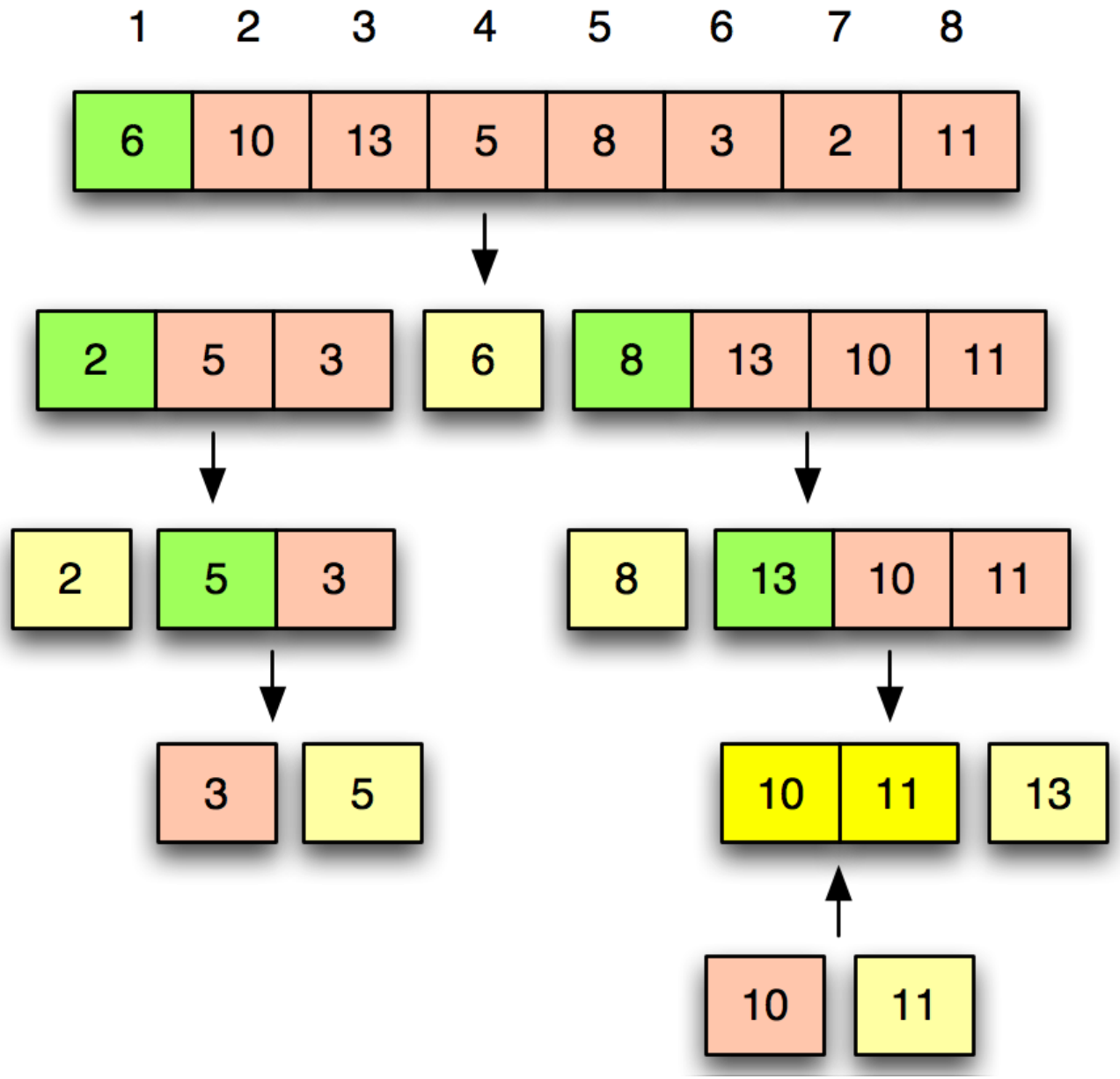

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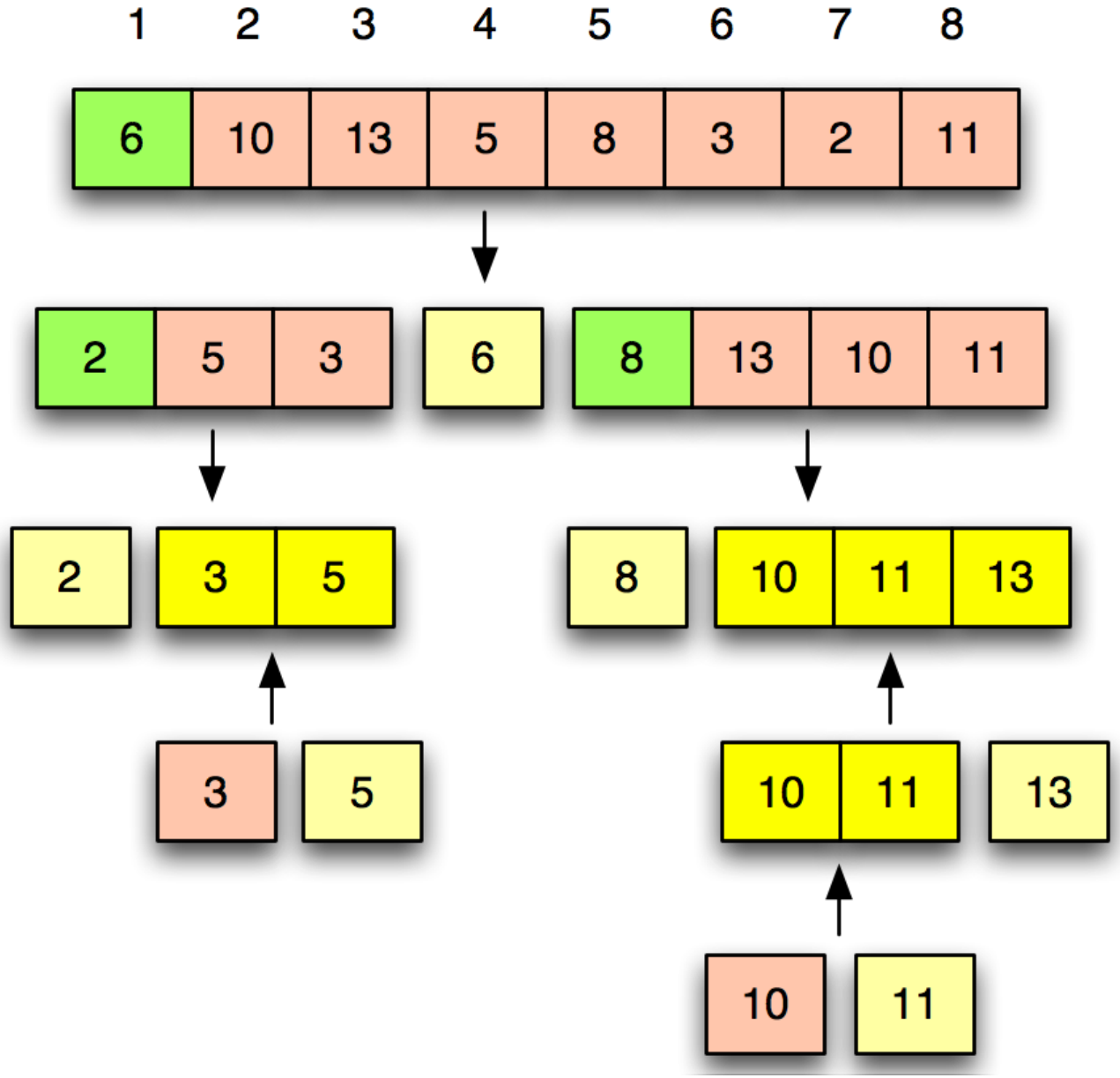

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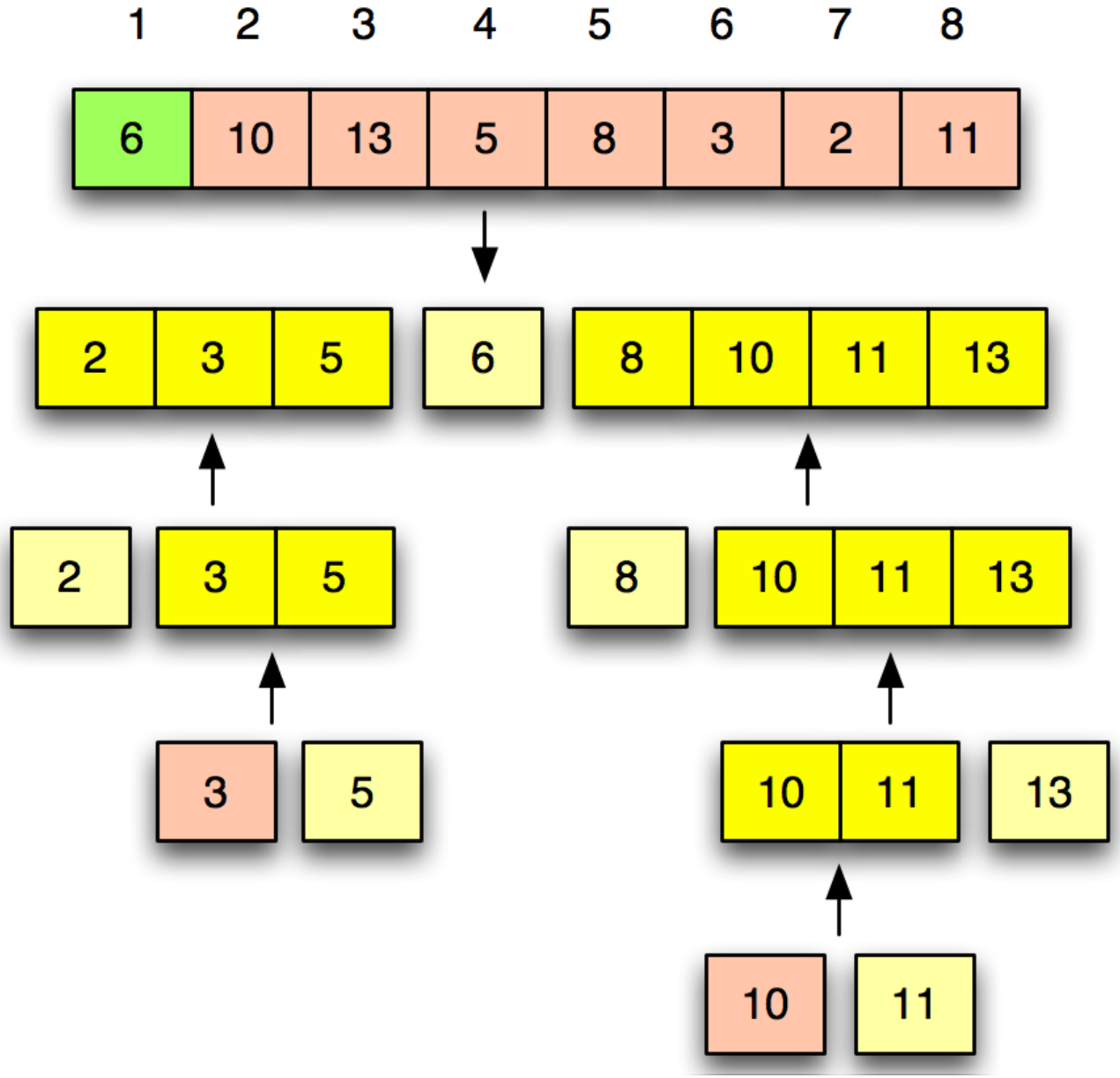

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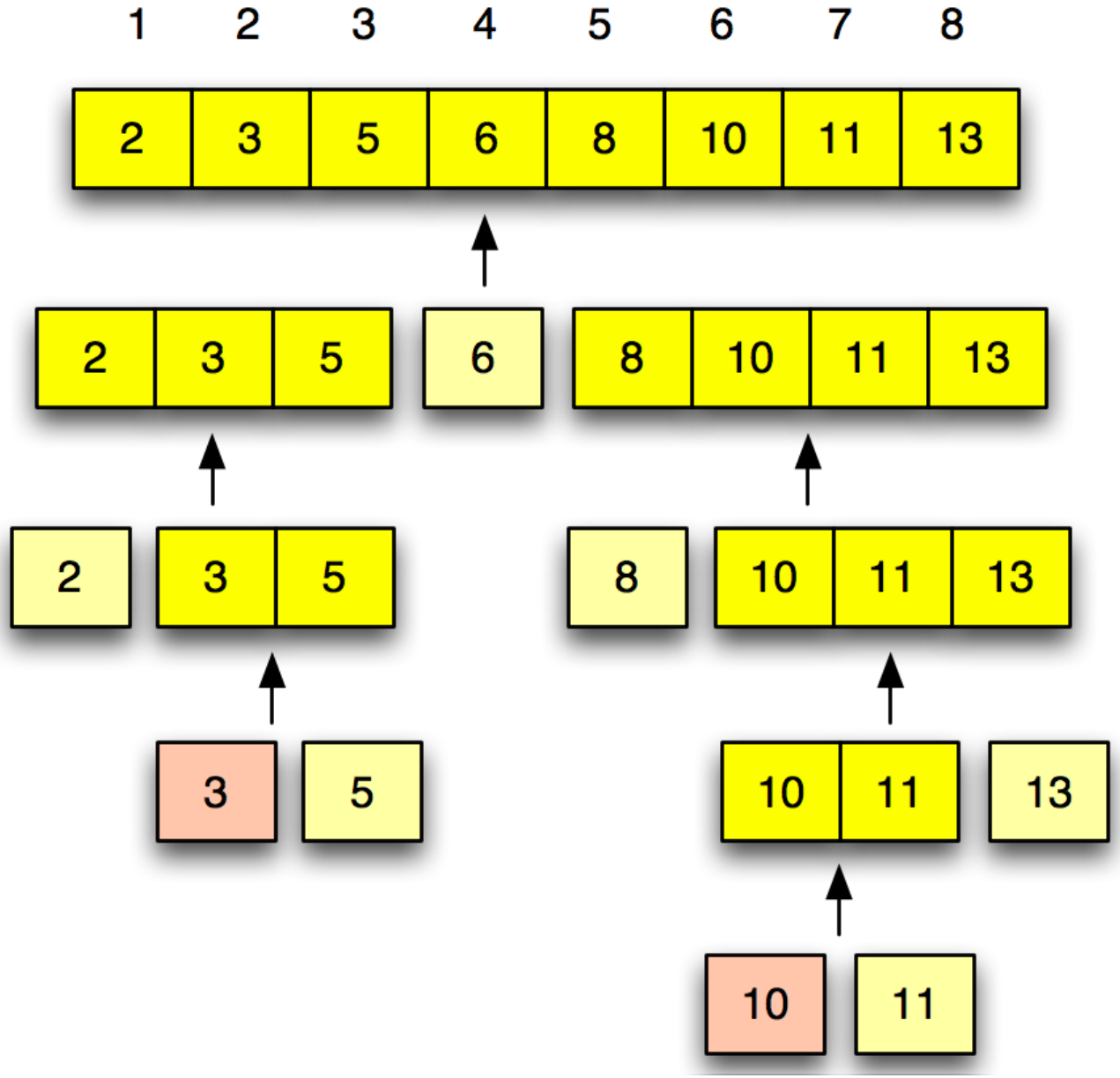

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# Partitioning algorithm

## Algorithm

PARTITION$(A, p, q) \triangleright A[p\,..\,q]$

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1  x ← A[p]              ▷ pivot = A[p]
2  i ← p
3  for j ← p + 1 to q
4        do if A[j] ≤ x
5              then i ← i + 1
6                   exchange A[i] ↔ A[j]
7  exchange A[p] ↔ A[i]
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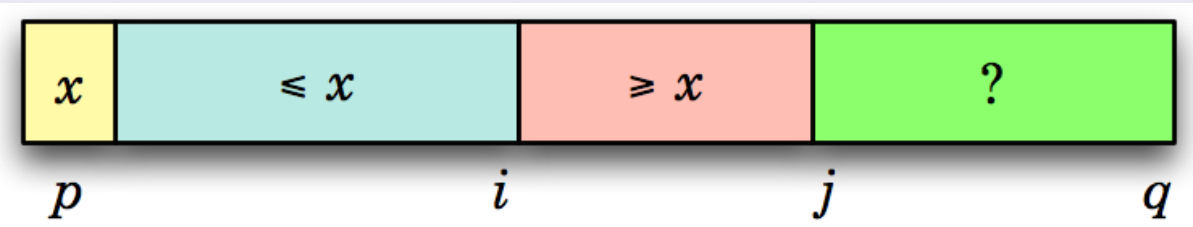
## Invariant



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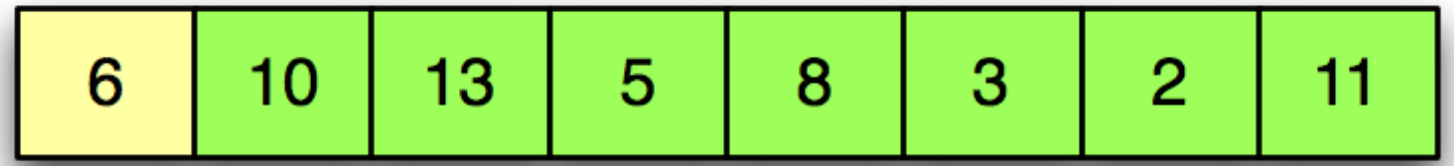


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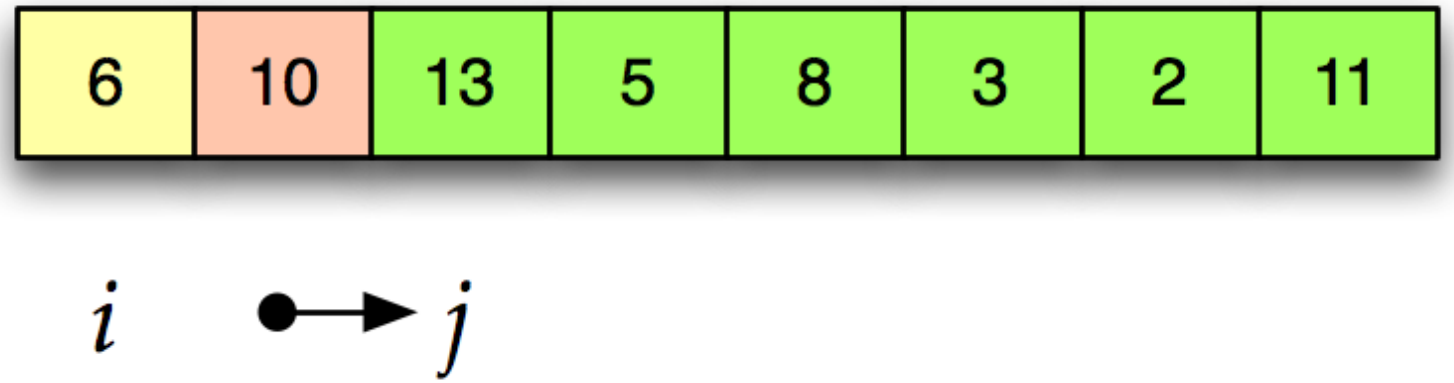

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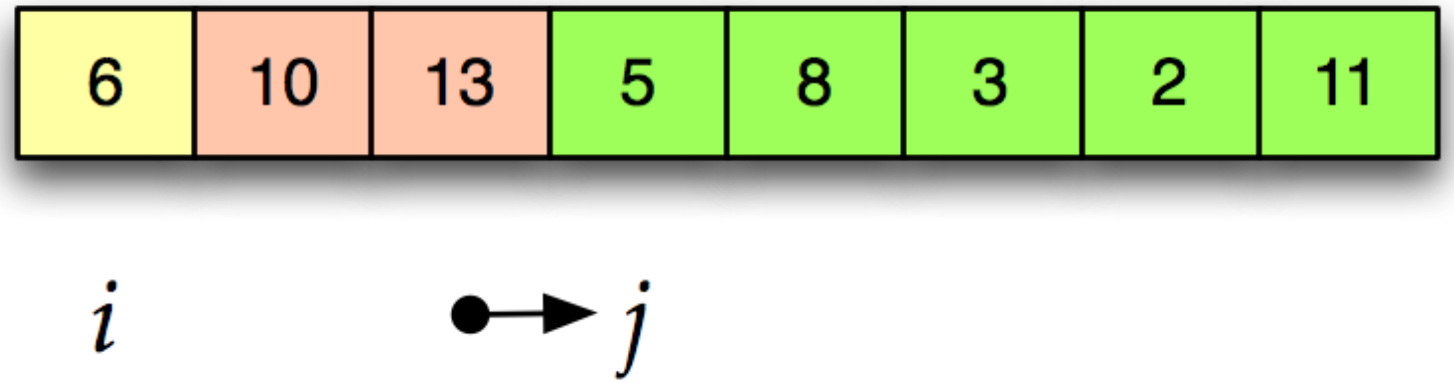

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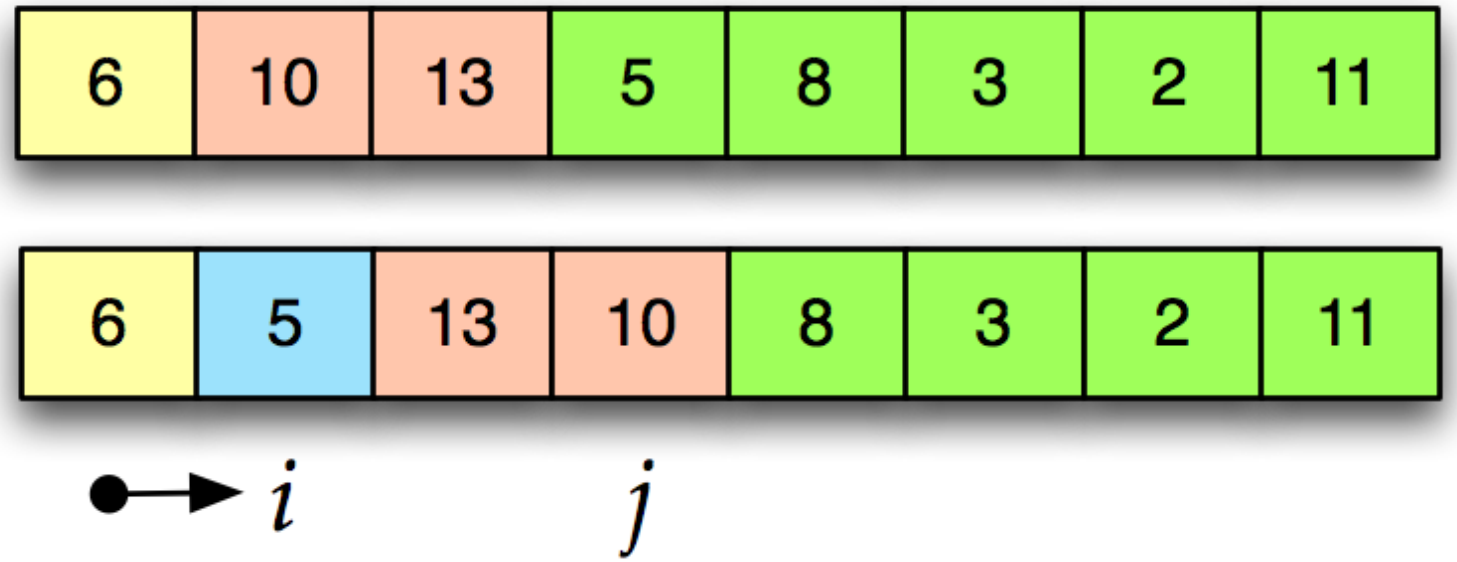

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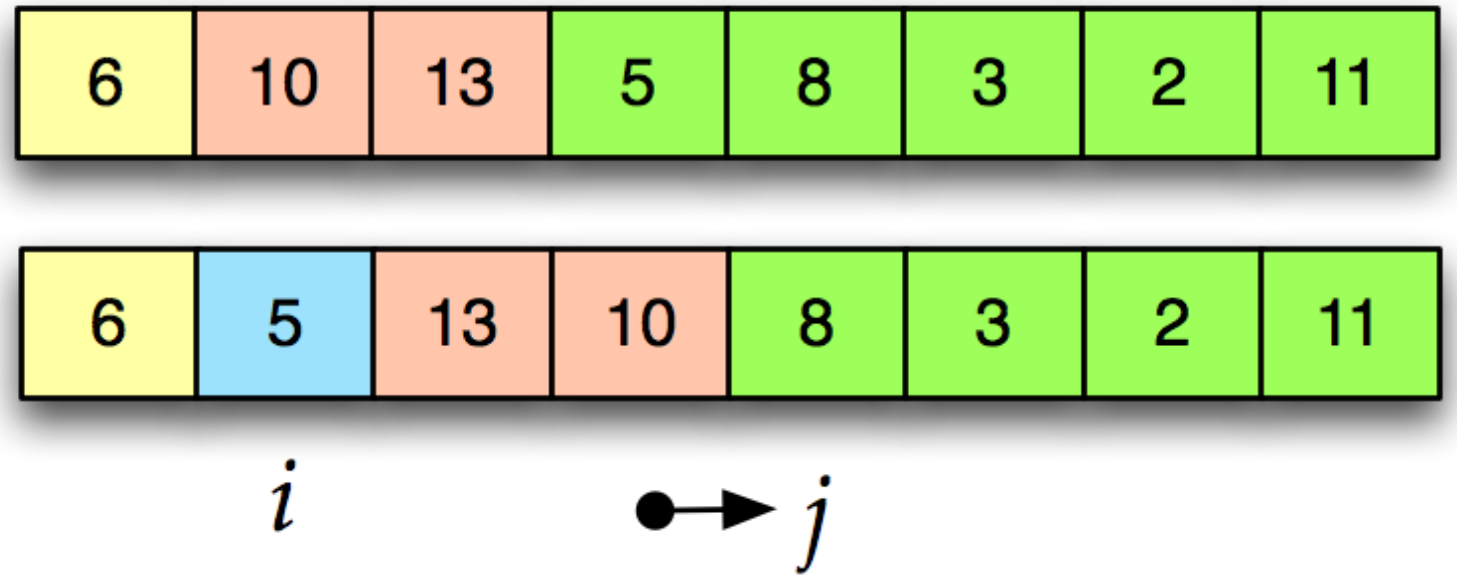

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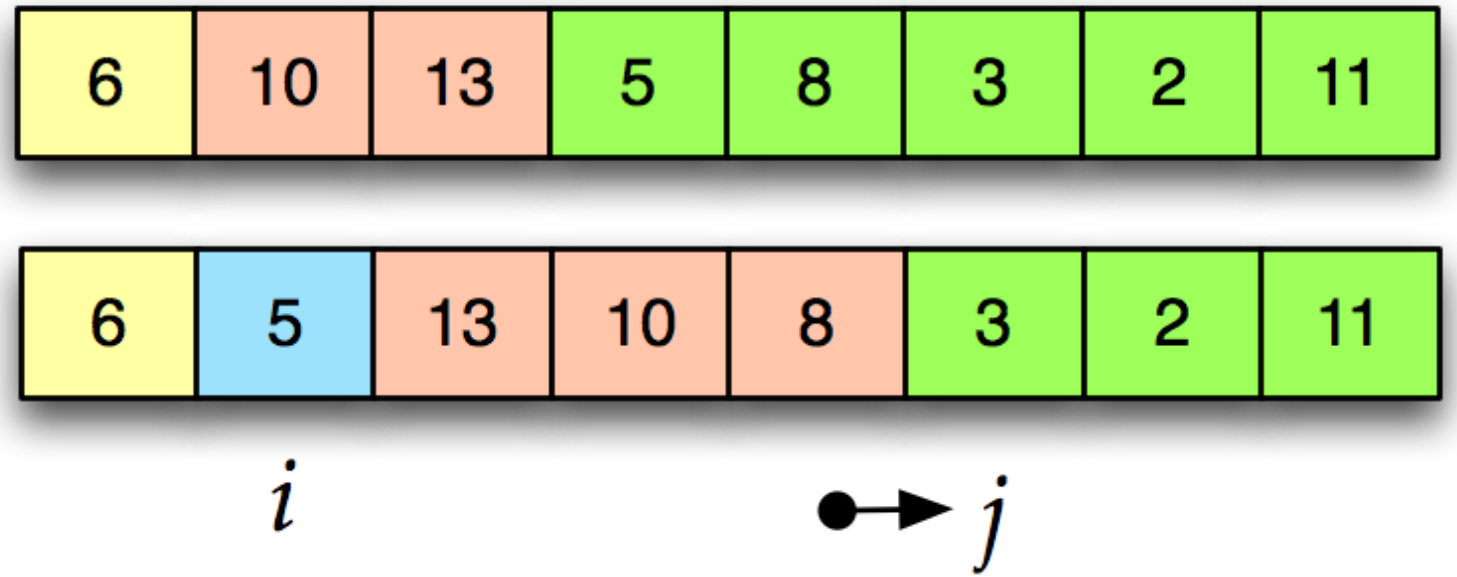

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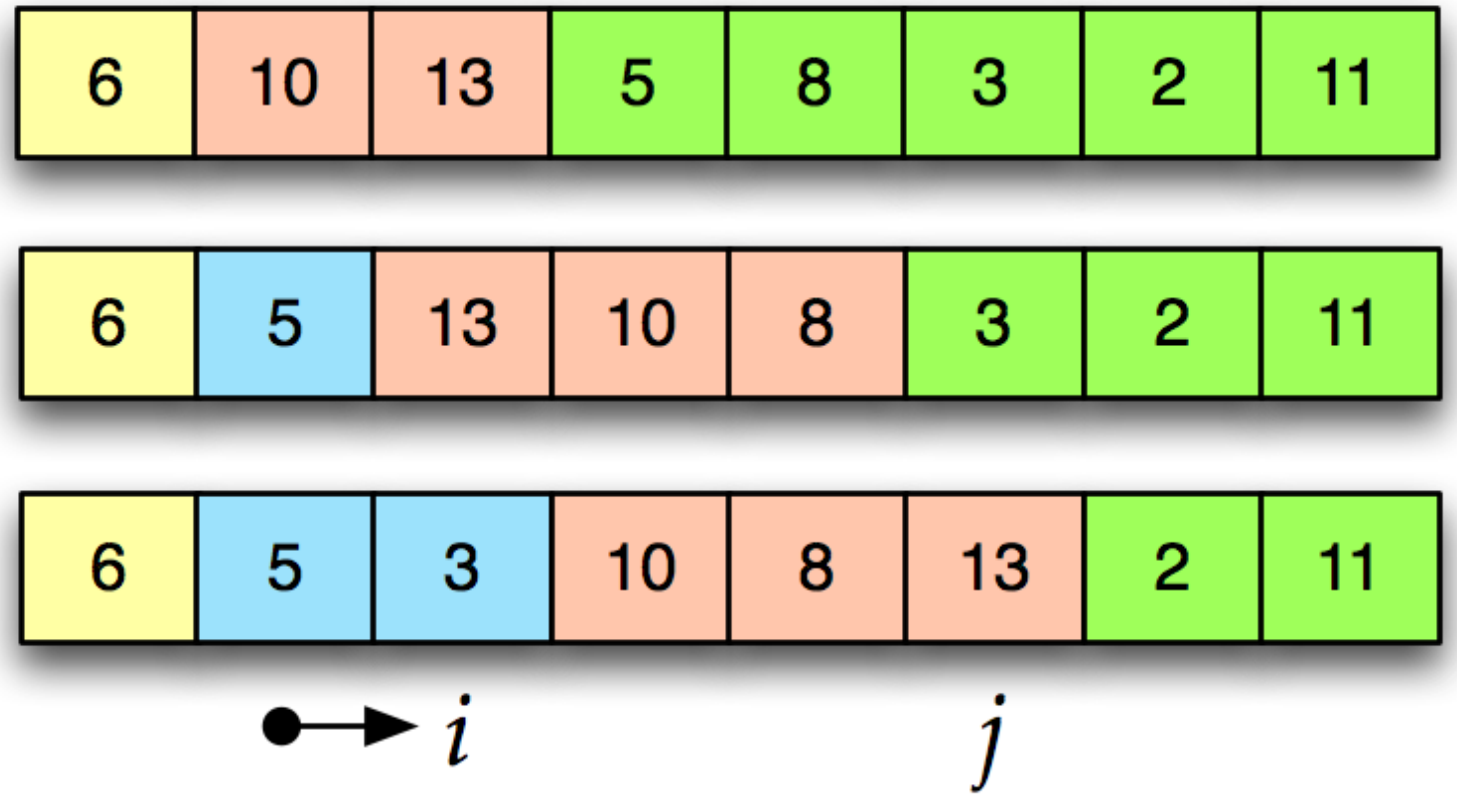

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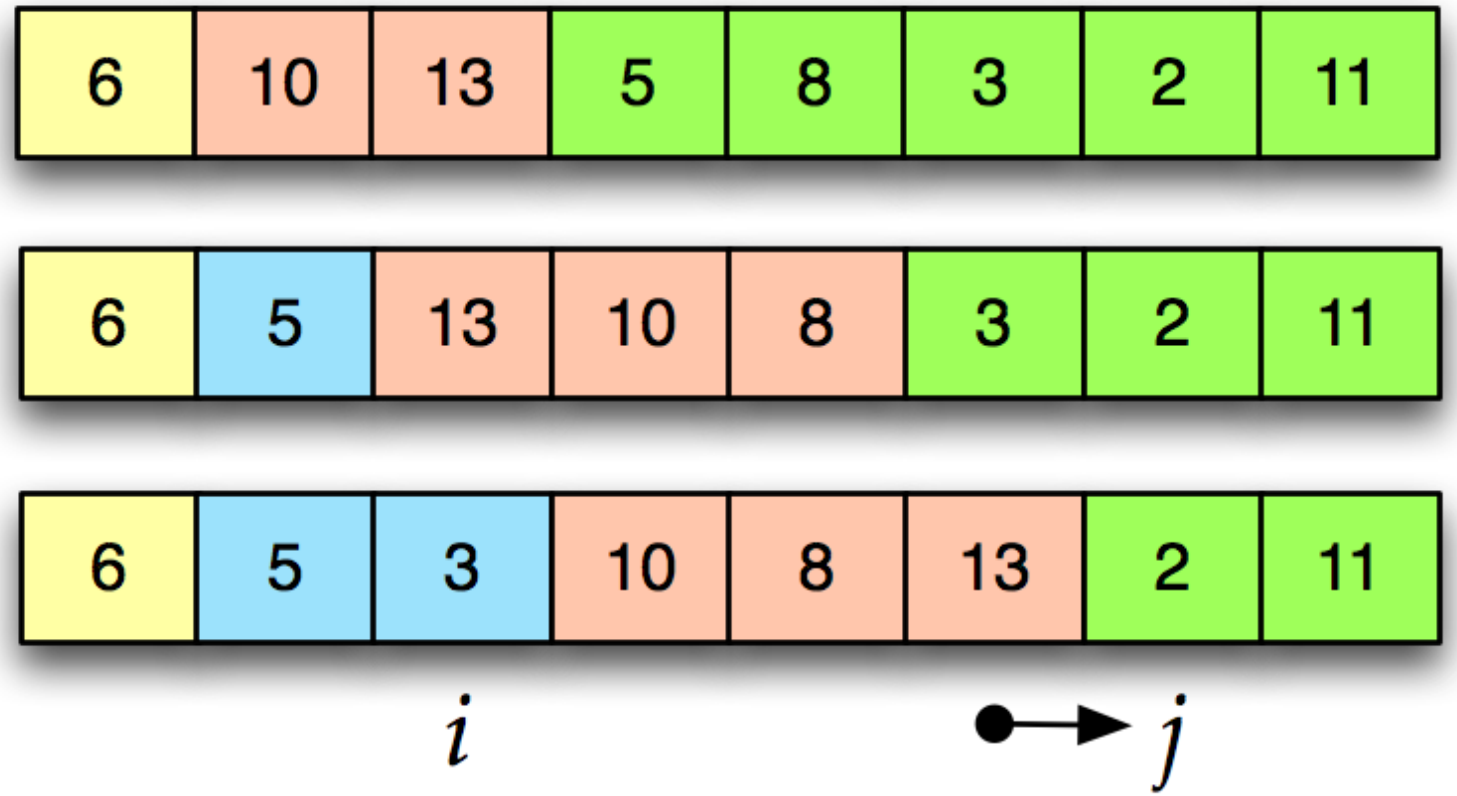

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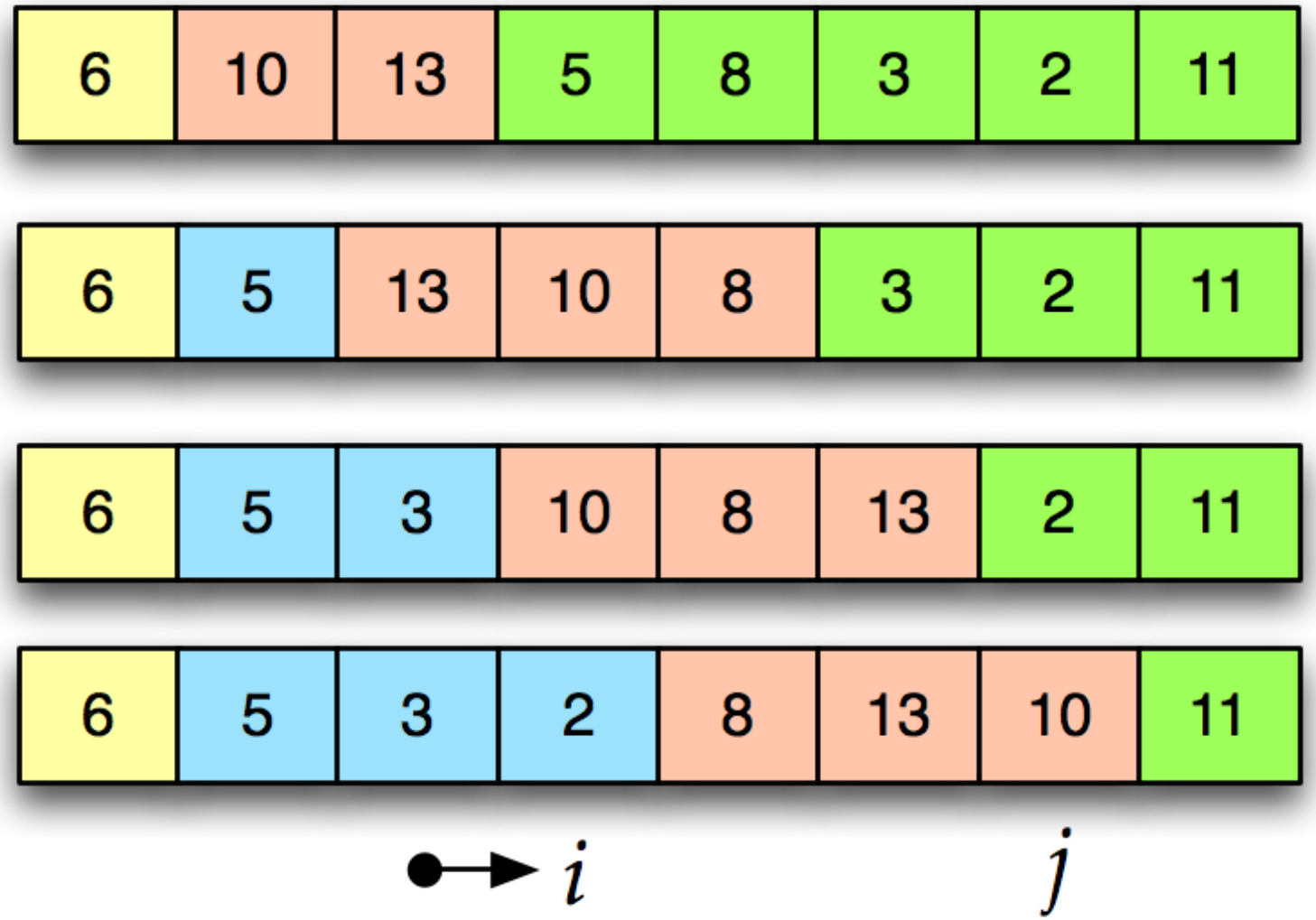

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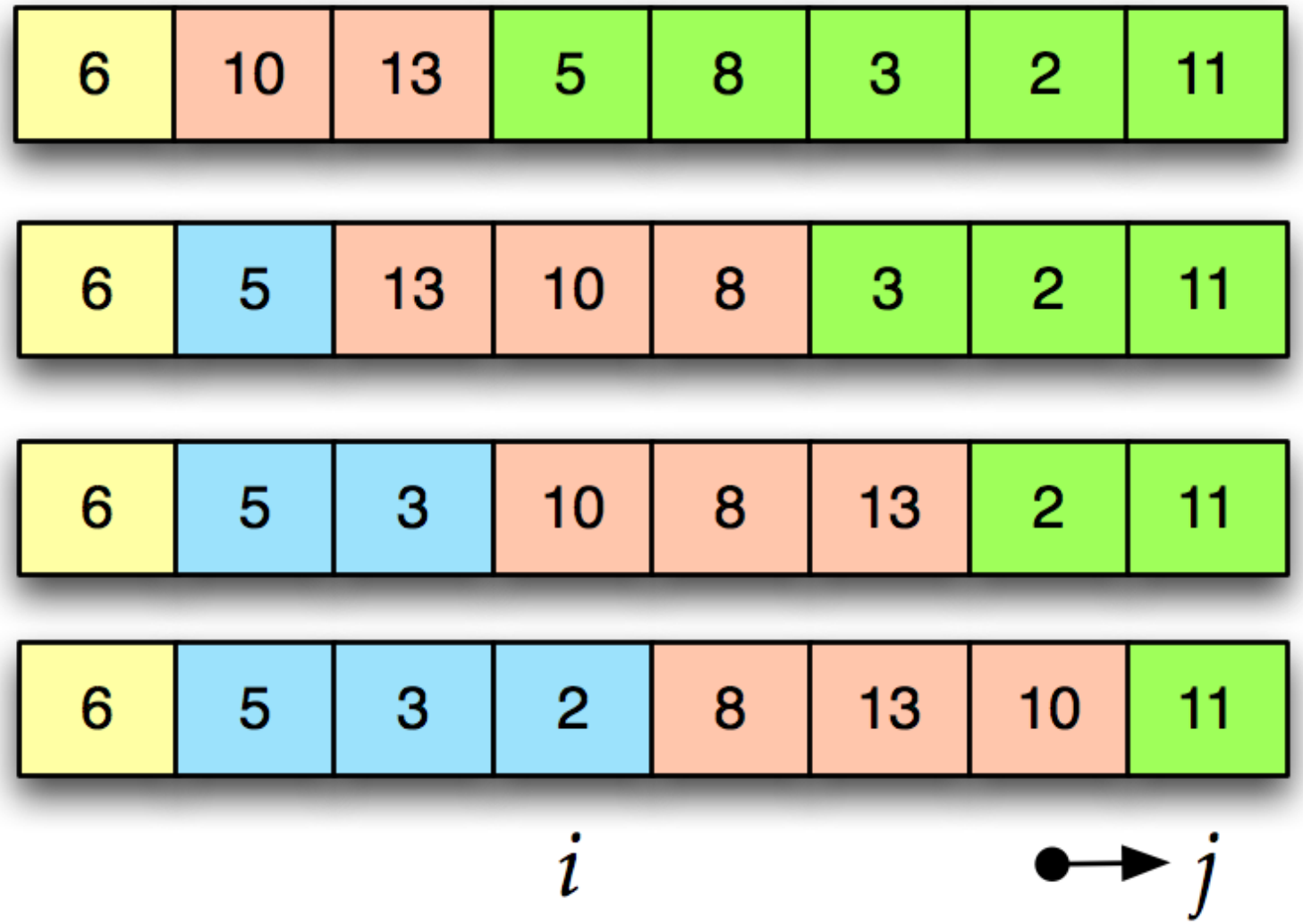

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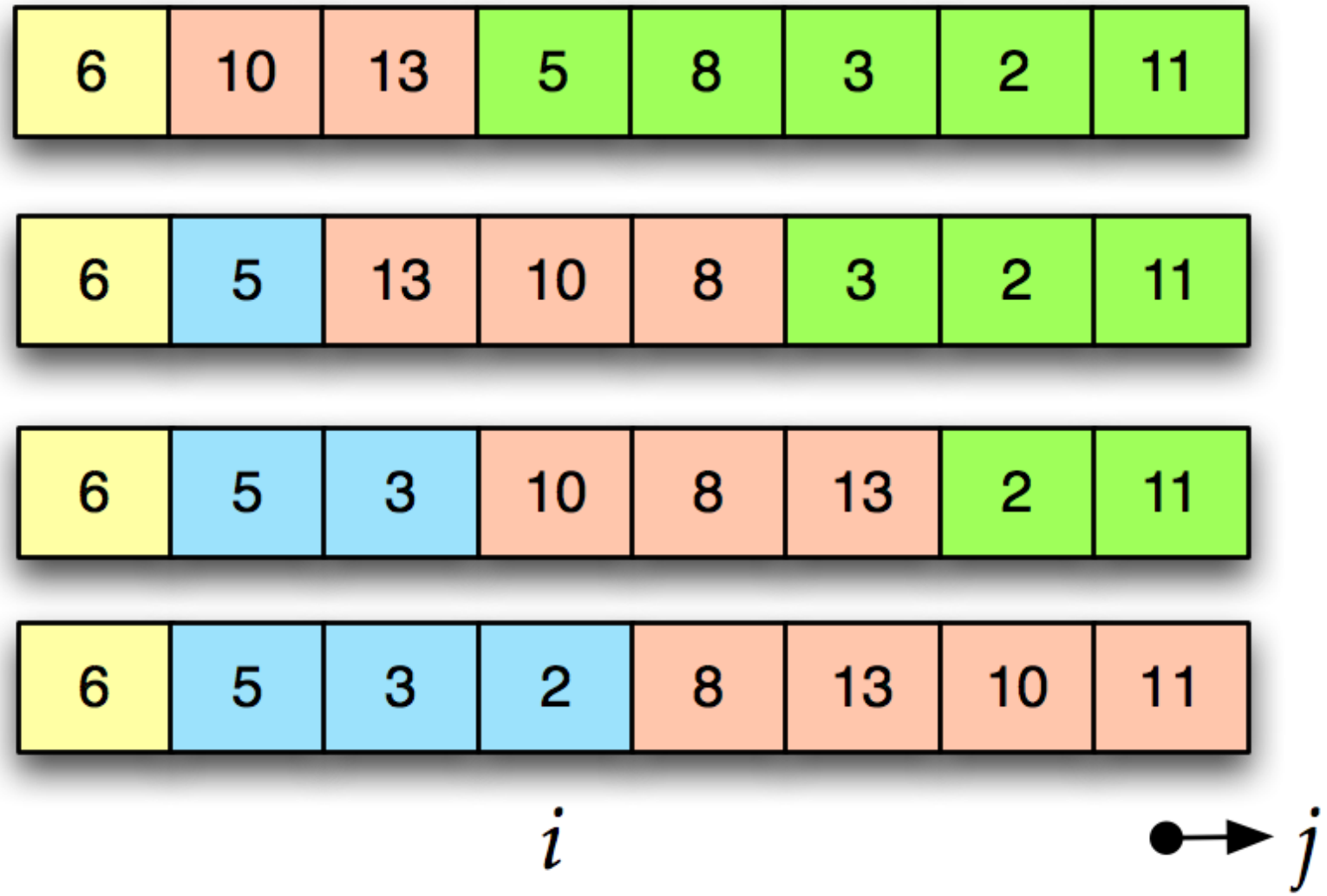

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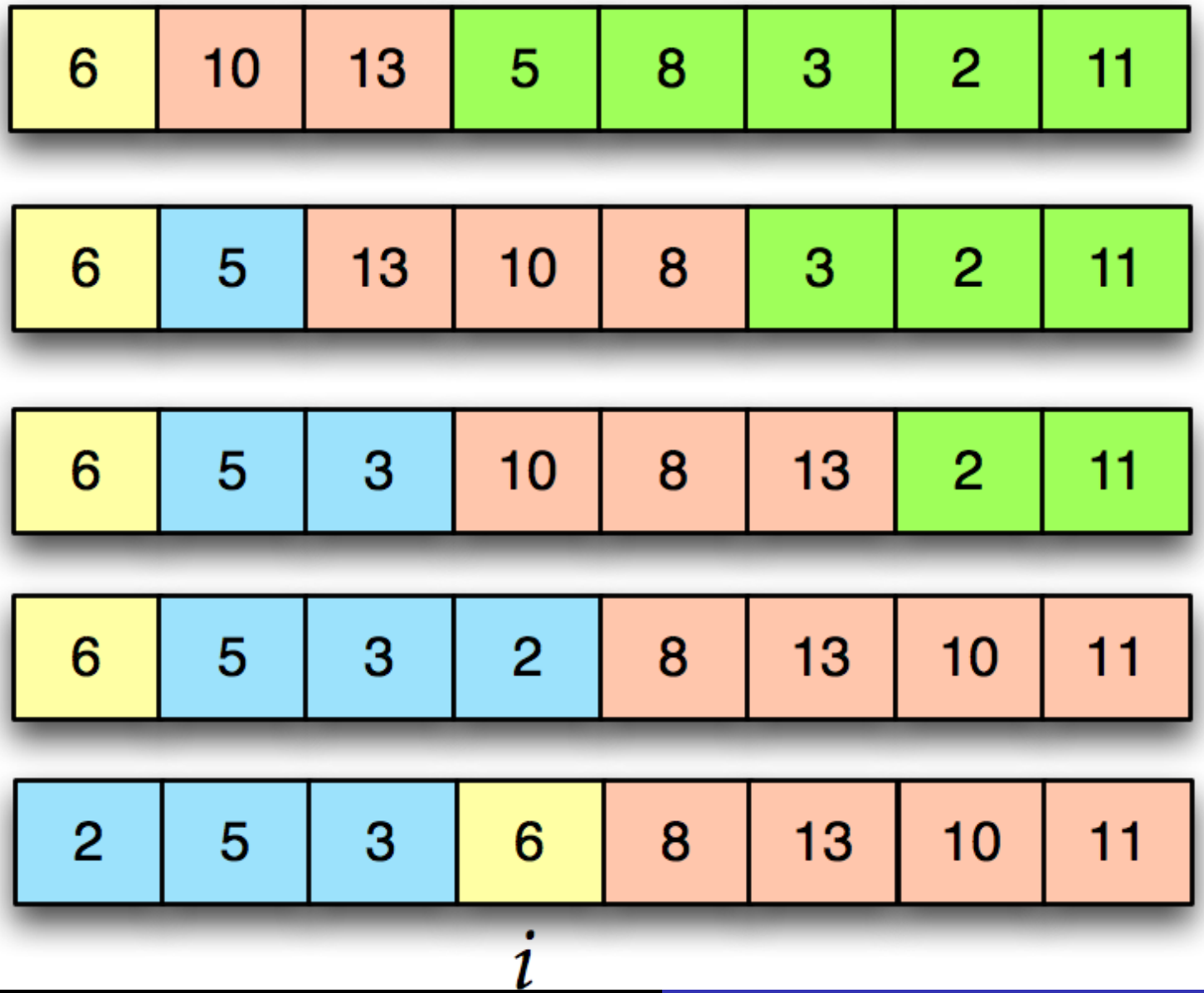

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## Initial call

$\textsc{quicksort}(A, 1, n)$

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## Worst-case analysis

(Note: the worst-case running time for partitioning is $\Theta(n)$.)

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \boxed{\Theta(n^2)} \end{aligned}$$

# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

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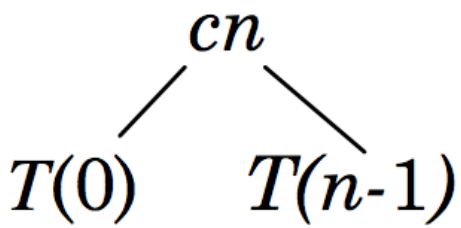

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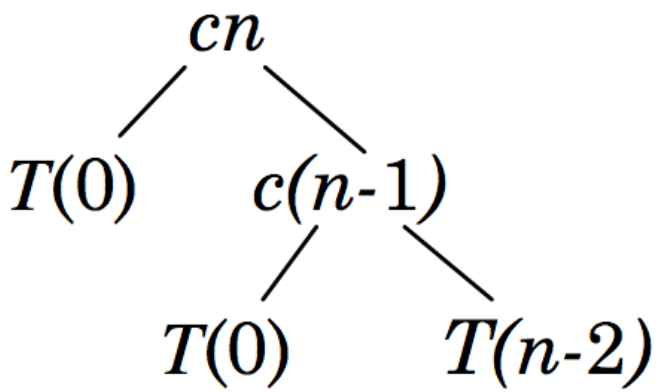

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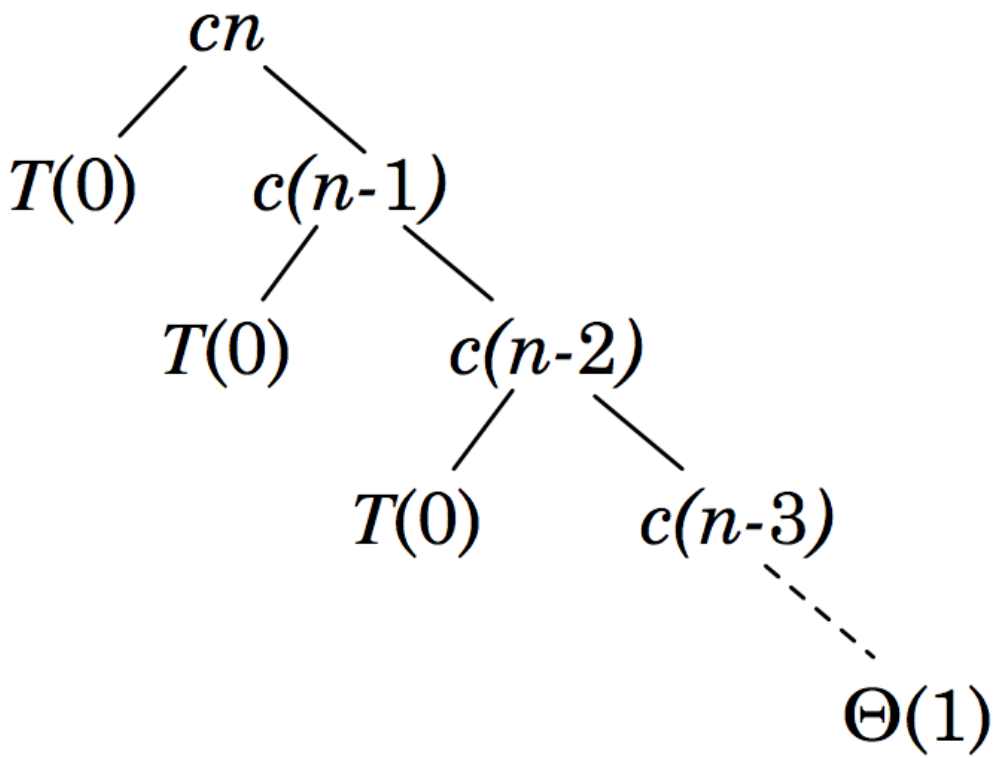

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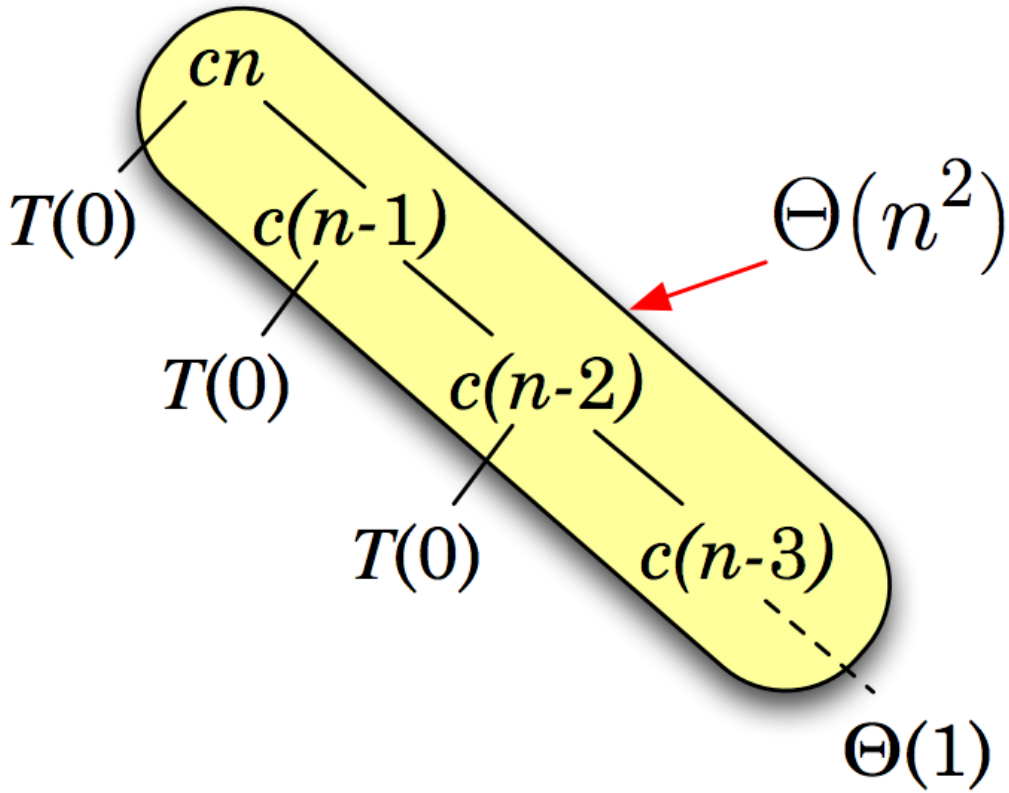

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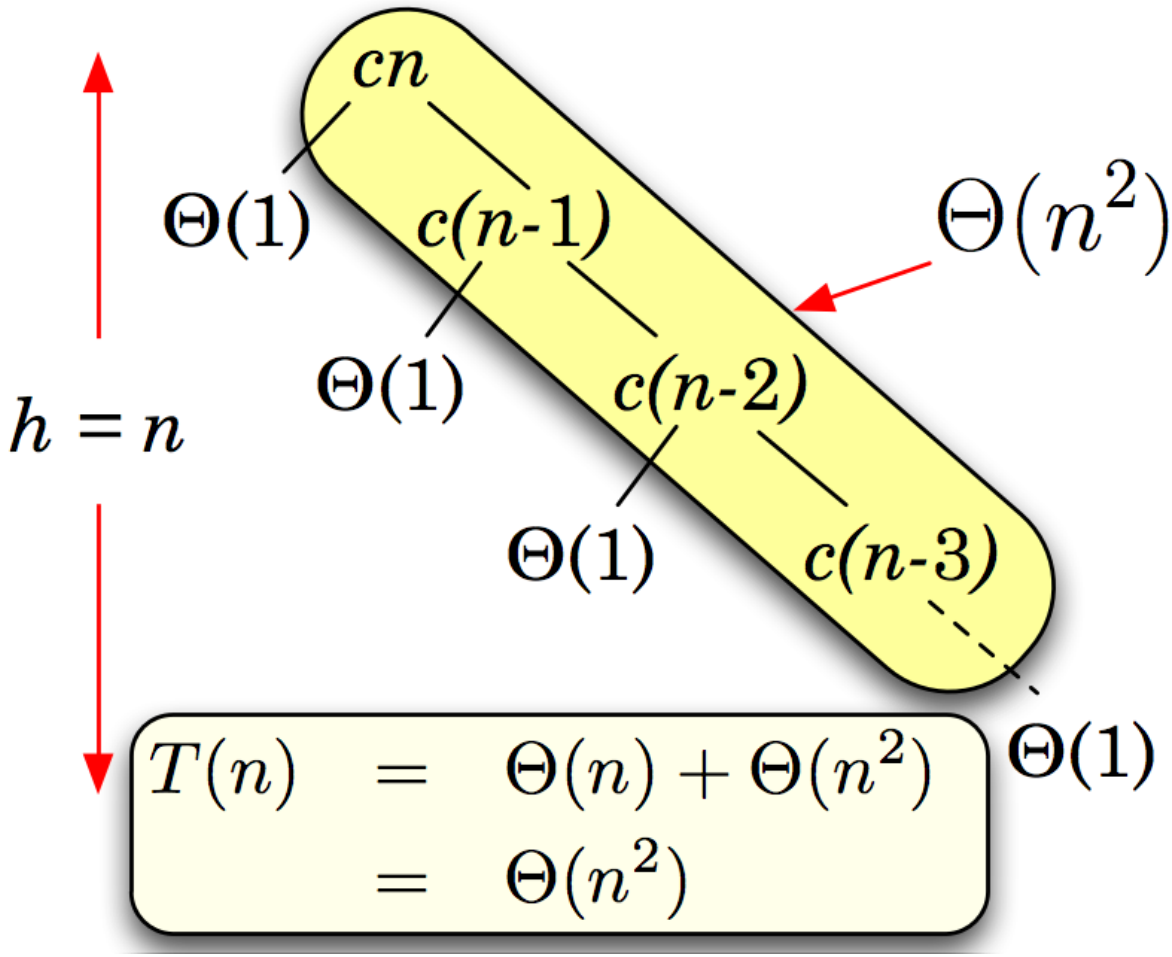

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How to ensure that we don't *usually* hit the worst-case?

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1 $i \leftarrow$ RANDOM$(p, r)$ $\triangleright$ $i = [p\,..\,r]$
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3 RANDOMIZED-QUICKSORT$(A, p, q - 1)$
4 RANDOMIZED-QUICKSORT$(A, q + 1, r)$

# Conclusion

- One of the most widely used sorting algorithm.
- While it runs in $O(n^2)$ time in the worst-case, it runs in $O(n \lg n)$ time on the average.
- Runs almost twice as fast as merge-sort.
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