$\Rightarrow \lambda$ is real.

Thus the eigenvalues of the Hermitian matrix are real.

Example 2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{bmatrix}$$

Now the determinant of $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2$$

Therefore, the characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$

or, $\lambda^2 - 2\lambda - \lambda + 2 = 0$ or, $(\lambda - 2)(\lambda - 1) = 0$

$\therefore$ $\lambda = 1, \lambda = 2$ which are the eigenvalues of A.

Example 3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -1 & 0 \\ -3 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix} = (\lambda-4)\{(\lambda-2)^2 - 3\}$$

Therefore, the characteristic equation of A is

$$(\lambda-4)\{(\lambda-2)^2 - 3\} = 0$$

or, $(\lambda-4)(\lambda^2 - 4\lambda + 4 - 3) = 0$

or, $(\lambda-4)(\lambda^2 - 4\lambda + 1) = 0$

$\therefore$ $\lambda = 4$ and $\lambda^2 - 4\lambda + 1 = 0$

Here $\lambda^2 - 4\lambda + 1 = 0$ is a quadratic equation which can be solved by the quadratic formula

$$\lambda = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

or, $\lambda = 2 \pm \sqrt{3}$

Hence the eigenvalues of A are

$\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2 - \sqrt{3}$.

Example 4. Find the eigenvalues and the corresponding eigevectors of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ (the characteristic polynomial of A) is $|\lambda I - A| = \begin{vmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4) - 3.$

Therefore, the characteristic equation of A is

$$(\lambda-2)(\lambda-4) - 3 = 0$$

or, $\lambda^2 - 6\lambda + 8 - 3 = 0$

or, $\lambda^2 - 6\lambda + 5 = 0$

or, $(\lambda-5)(\lambda-1) = 0$ $\therefore$ $\lambda = 5$, $\lambda = 1$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$, that is, of

$$\begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

If $\lambda = 5$, equation no (1) becomes

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} 3x_1 - 3x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0$$

This system is in echelon form and consistent. Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = a$ (a is an arbitrary real number). Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 5$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ a \end{bmatrix}$

In particular, let a = 1, then $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

If $\lambda = 1$, equation no (1) becomes

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} -x_1 - 3x_2 &= 0 \\ -x_1 - 3x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 + 3x_2 = 0$$

This system is in echelon form and consistent. Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = b$ (b is an arbitrary real number). $x_1 = -3b$.
Therefore, the eigen vectors of A corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors of the form

$$X = \begin{bmatrix} -3b \\ b \end{bmatrix}$$

In particular, let $b = 1$, then $X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

Example 5. For the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 3y, x + 5y)$, find all eigenvalues and a basis of eigenspace.

Solution : First find a matrix representation of T, say relative to the usual basis of $\mathbb{R}^2$.

$$A = [T] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

The characteristic polynomial $\Delta(\lambda)$ of T is then

$$\Delta = |\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda - 3 & -3 \\ -1 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 5) - 3$$

Hence the characteristic equation is $(\lambda - 3)(\lambda - 5) - 3 = 0$

or, $\lambda^2 - 8\lambda + 12 = 0$

or, $(\lambda - 2)(\lambda - 6) = 0 \quad \therefore \lambda = 2, \lambda = 6$

Thus 2 and 6 are the eigenvalues of T.

Now we find a basis of the eigenspace of the eigenvalue 2.

Putting $\lambda = 2$ into $\lambda I - A$ to obtain

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{matrix} -x_1 - 3x_2 = 0 \\ -x_1 - 3x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 + 3x_2 = 0$

The system has only one independent solution, e. g. $x_1 = 3$, $x_2 = -1$.

Thus $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector which generates the eigenspace of 2. i.e $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ forms a basis of eigenspace of 2.

Again, we find a basis of the eigenspace of the eigenvalue 6. Putting $\lambda = 6$ into $\lambda I - A$ to obtain

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} 3x_1 - 3x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0$$

The system has only one independent solution, e. g. $x_1 = 1$, $x_2 = 1$. Thus $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector which generates the eigenspace of 6. i. e $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms basis of the eigenspace of 6.

Example 6. Find all eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix} \qquad \text{[D. U. P. 1991]}$$

Solution : The characteristic matrix of A is

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix}$$

The characteristic polynomial of A is

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda + 2)(\lambda - 2)$$

Therefore, the characteristic equation of A is

$$(\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$$

$$\therefore \lambda = 1, \lambda = -2, \lambda = 2$$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A

corresponding to the eigenvalue λ if and only if X is a non-trivial solution of $(\lambda I - A) X = 0$.

i.e $= \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda = 1$, $\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we have

$$\left.\begin{aligned} -2x_2 + x_3 &= 0 \\ 3x_2 \quad &= 0 \\ 5x_2 - x_3 &= 0 \end{aligned}\right\} \text{ Solving we get } x_2 = x_3 = 0$$

Hence x_1 is a free variable. Let $x_1 = a$ where a is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 1$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$.

In particular, let a = 1, then $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

When $\lambda = -2$, $\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we get

$$\left.\begin{aligned} -3x_1 - 2x_2 + x_3 &= 0 \\ 5x_2 - 4x_3 &= 0 \end{aligned}\right\}$$

This system is in echelon form and has one free variable which is x_3. Let $x_3 = b$ where b is any real number. Then from second equation we get $x_2 = \frac{4b}{5}$. Putting $x_2 = \frac{4b}{5}$ and $x_3 = b$ in the first equation, we get $x_1 = -\frac{b}{5}$.

Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = -2$ are the non-zero vectors of the form

$$X = \begin{bmatrix} \frac{-b}{5} \\ \frac{4b}{5} \\ b \end{bmatrix}$$

In particular, let b = 5, then $X = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$.

When $\lambda = 2$, $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming linear system, we get

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 4x_2 &= 0 \\ 5x_2 &= 0 \end{aligned}\right\} \text{i.e.} \quad \left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\}$$

Here x_3 is a free variable. Let $x_3 = c$ where c is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 2$ are non-zero vectors of the form.

$X = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$ In particular, let c = 1, then

$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

9.4 Diagonalization

A square matri: called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diangonalize A.