

1

Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS

Linear Algebra

and its applications FOURTH EDITION



VECTOR EQUATIONS

Vectors in \mathbf{R}^2

- A matrix with only one column is called a **column vector**, or simply a **vector**.
- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by \mathbf{R}^2 (read “r-two”).

VECTOR EQUATIONS

- The \mathbf{R} stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains 2 entries.
- Two vectors in \mathbf{R}^2 are **equal** if and only if their corresponding entries are equal.
- Given two vectors \mathbf{u} and \mathbf{v} in \mathbf{R}^2 , their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of \mathbf{u} and \mathbf{v} .
- Given a vector \mathbf{u} and a real number c , the **scalar multiple** of \mathbf{u} by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in \mathbf{u} by c .

VECTOR EQUATIONS

- **Example 1:** Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

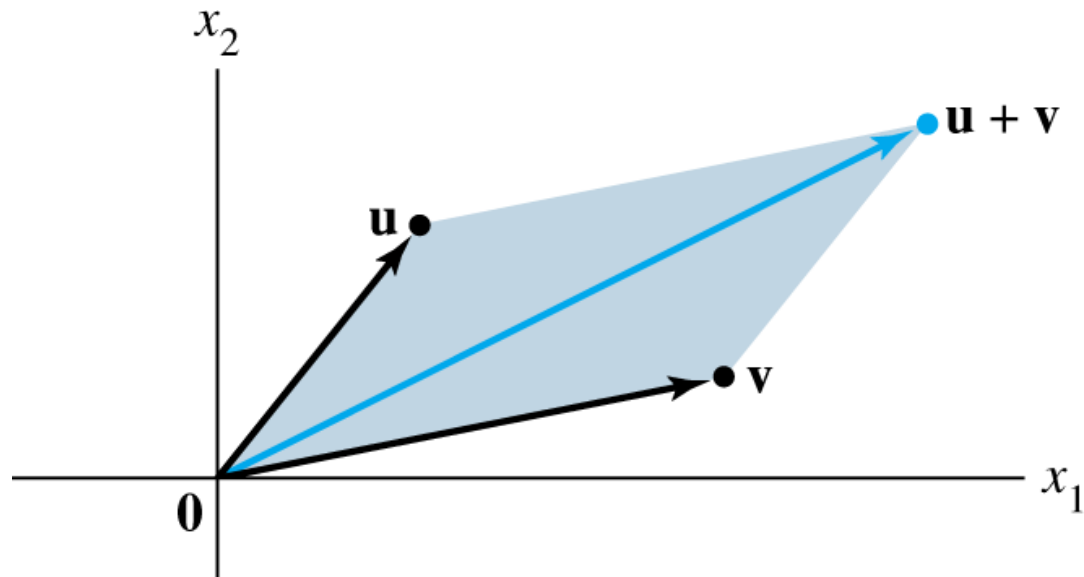
$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

GEOMETRIC DESCRIPTIONS OF \mathbf{R}^2

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.
- So we may regard \mathbf{R}^2 as the set of all points in the plane.

PARALLELOGRAM RULE FOR ADDITION

- If \mathbf{u} and \mathbf{v} in \mathbf{R}^2 are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and \mathbf{v} . See the figure below.



VECTORS IN \mathbf{R}^3 and \mathbf{R}^n

- Vectors in \mathbf{R}^3 are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, \mathbf{R}^n (read “r-n”) denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

ALGEBRAIC PROPERTIES OF \mathbf{R}^n

- The vector whose entries are all zero is called the **zero vector** and is denoted by **$\mathbf{0}$** .
- For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbf{R}^n and all scalars c and d :
 - (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
 - (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
 - (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
 - (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
 - (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
 - (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

LINEAR COMBINATIONS

$$(vii) \quad c(du) = (cd)(u)$$

$$(viii) \quad 1u = u$$

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbf{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** c_1, \dots, c_p .

- The weights in a linear combination can be any real numbers, including zero.

LINEAR COMBINATIONS

- **Example 2:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine whether \mathbf{b} can be generated (or written) as a linear combination of \mathbf{a}_1 and \mathbf{a}_2 . That is, determine whether weights x_1 and x_2 exist such that

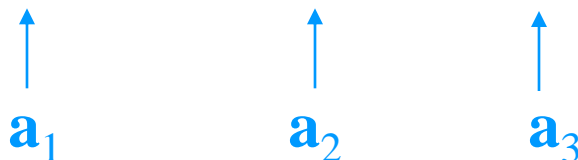
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \quad \text{----(1)}$$

If vector equation (1) has a solution, find it.

LINEAR COMBINATIONS

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix},$$



which is same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

LINEAR COMBINATIONS

$$\text{and } \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \quad \text{----(2)}$$

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \quad \text{----(3)} \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence \mathbf{b} is a linear combination of \mathbf{a}_1 and \mathbf{a}_2 , with weights $x_1 = 3$ and

$x_2 = 2$. That is,

$$3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{b} are the columns of the augmented matrix that we row reduced:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

\mathbf{a}_1 \mathbf{a}_2 \mathbf{b}

- Write this matrix in a way that identifies its columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}] \quad \text{----(4)}$$

LINEAR COMBINATIONS

- A vector equation

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \quad \text{----(5)}$$

- In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

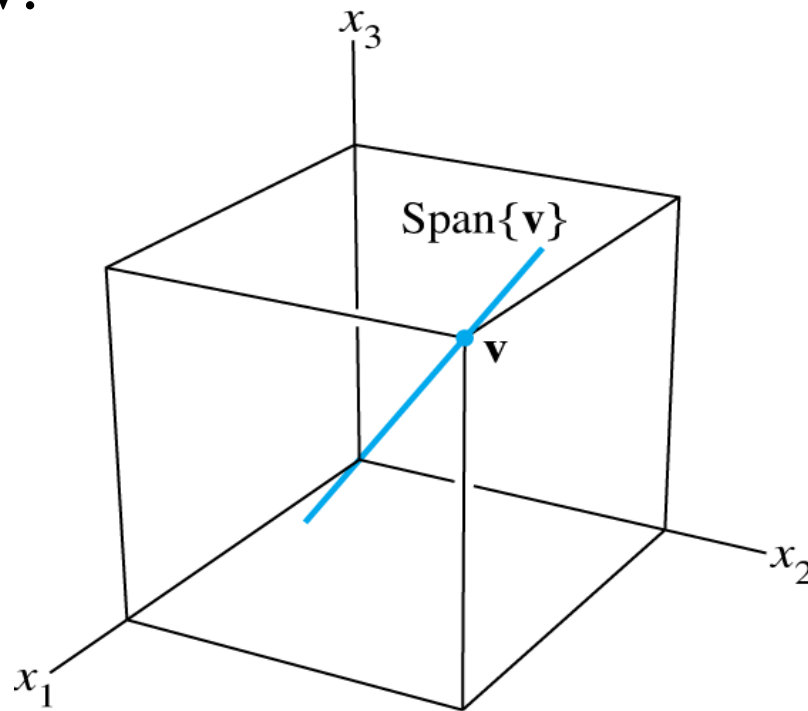
- **Definition:** If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbf{R}^n , then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of \mathbf{R}^n spanned (or generated) by $\mathbf{v}_1, \dots, \mathbf{v}_p$** . That is, $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p$$

with c_1, \dots, c_p scalars.

A GEOMETRIC DESCRIPTION OF $\text{Span}\{\mathbf{v}\}$

- Let \mathbf{v} be a nonzero vector in \mathbf{R}^3 . Then $\text{Span}\{\mathbf{v}\}$ is the set of all scalar multiples of \mathbf{v} , which is the set of points on the line in \mathbf{R}^3 through \mathbf{v} and $\mathbf{0}$. See the figure below.



A GEOMETRIC DESCRIPTION OF SPAN $\{\mathbf{u}, \mathbf{v}\}$

- If \mathbf{u} and \mathbf{v} are nonzero vectors in \mathbf{R}^3 , with \mathbf{v} not a multiple of \mathbf{u} , then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in \mathbf{R}^3 that contains \mathbf{u} , \mathbf{v} , and $\mathbf{0}$.
- In particular, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line in \mathbf{R}^3 through \mathbf{u} and $\mathbf{0}$ and the line through \mathbf{v} and $\mathbf{0}$. See the figure below.

