

# 4 Vector Spaces

## 4.5

### THE DIMENSION OF A VECTOR SPACE

## Linear Algebra

*and its applications* FOURTH EDITION



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# DIMENSION OF A VECTOR SPACE

- **Theorem 9:** If a vector space  $V$  has a basis  $B = \{b_1, \dots, b_n\}$ , then any set in  $V$  containing more than  $n$  vectors must be linearly dependent.
- **Proof:** Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  be a set in  $V$  with more than  $n$  vectors.
- The coordinate vectors  $[\mathbf{u}_1]_B, \dots, [\mathbf{u}_p]_B$  form a linearly dependent set in  $\mathbf{R}^n$ , because there are more vectors ( $p$ ) than entries ( $n$ ) in each vector.

# DIMENSION OF A VECTOR SPACE

- So there exist scalars  $c_1, \dots, c_p$ , not all zero, such that

$$c_1 [\mathbf{u}_1]_{\mathbf{B}} + \dots + c_p [\mathbf{u}_p]_{\mathbf{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{The zero vector in } \mathbb{R}^n$$

- Since the coordinate mapping is a linear transformation,

$$[c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p]_{\mathbf{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

# DIMENSION OF A VECTOR SPACE

- The zero vector on the right displays the  $n$  weights needed to build the vector  $c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$  from the basis vectors in  $B$ .
- That is,  $c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p = 0 \cdot \mathbf{b}_1 + \dots + 0 \cdot \mathbf{b}_n = \mathbf{0}$ .
- Since the  $c_i$  are not all zero,  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is linearly dependent.
- Theorem 9 implies that if a vector space  $V$  has a basis  $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ , then each linearly independent set in  $V$  has no more than  $n$  vectors.

# DIMENSION OF A VECTOR SPACE

- **Theorem 10:** If a vector space  $V$  has a basis of  $n$  vectors, then every basis of  $V$  must consist of exactly  $n$  vectors.
- **Proof:** Let  $B_1$  be a basis of  $n$  vectors and  $B_2$  be any other basis (of  $V$ ).
- Since  $B_1$  is a basis and  $B_2$  is linearly independent,  $B_2$  has no more than  $n$  vectors, by Theorem 9.
- Also, since  $B_2$  is a basis and  $B_1$  is linearly independent,  $B_2$  has at least  $n$  vectors.
- Thus  $B_2$  consists of exactly  $n$  vectors.

# DIMENSION OF A VECTOR SPACE

- **Definition:** If  $V$  is spanned by a finite set, then  $V$  is said to be **finite-dimensional**, and **the dimension of  $V$ , written as  $\dim V$ , is the number of vectors in a basis for  $V$ .** The dimension of the zero vector space  $\{0\}$  is defined to be zero. If  $V$  is not spanned by a finite set, then  $V$  is said to be **infinite-dimensional**.
- **Example 1:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

# DIMENSION OF A VECTOR SPACE

- $H$  is the set of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly,  $\mathbf{v}_1 \neq \mathbf{0}$ ,  $\mathbf{v}_2$  is not a multiple of  $\mathbf{v}_1$ , but  $\mathbf{v}_3$  is a multiple of  $\mathbf{v}_2$ .
- By the Spanning Set Theorem, we may discard  $\mathbf{v}_3$  and still have a set that spans  $H$ .

# SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- Finally,  $\mathbf{v}_4$  is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .
  - So  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$  is linearly independent and hence is a basis for  $H$ .
  - Thus  $\dim H = 3$ .
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- **Theorem 11:** Let  $H$  be a subspace of a finite-dimensional vector space  $V$ . Any linearly independent set in  $H$  can be expanded, if necessary, to a basis for  $H$ . Also,  $H$  is finite-dimensional and
$$\dim H \leq \dim V$$



# SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- **Proof:** If  $H = \{0\}$ , then certainly  $\dim H = 0 \leq \dim V$ .
- Otherwise, let  $S = \{u_1, \dots, u_k\}$  be any linearly independent set in  $H$ .
- If  $S$  spans  $H$ , then  $S$  is a basis for  $H$ .
- Otherwise, there is some  $u_{k+1}$  in  $H$  that is not in  $\text{Span } S$ .

# SUBSPACES OF A FINITE-DIMENSIONAL SPACE

- But then  $\{u_1, \dots, u_k, u_{k+1}\}$  will be linearly independent, because no vector in the set can be a linear combination of vectors that precede it (by Theorem 4).
- So long as the new set does not span  $H$ , we can continue this process of expanding  $S$  to a larger linearly independent set in  $H$ .
- But the number of vectors in a linearly independent expansion of  $S$  can never exceed the dimension of  $V$ , by Theorem 9.

# THE BASIS THEOREM

- So eventually the expansion of  $S$  will span  $H$  and hence will be a basis for  $H$ , and  $\dim H \leq \dim V$ .
- **Theorem 12:** Let  $V$  be a  $p$ -dimensional vector space,  $p \geq 1$ . Any linearly independent set of exactly  $p$  elements in  $V$  is automatically a basis for  $V$ . Any set of exactly  $p$  elements that spans  $V$  is automatically a basis for  $V$ .
- **Proof:** By Theorem 11, a linearly independent set  $S$  of  $p$  elements can be extended to a basis for  $V$ .

# THE BASIS THEOREM

- But that basis must contain exactly  $p$  elements, since  $\dim V = p$ .
- So  $S$  must already be a basis for  $V$ .
- Now suppose that  $S$  has  $p$  elements and spans  $V$ .
- Since  $V$  is nonzero, the Spanning Set Theorem implies that a subset  $S'$  of  $S$  is a basis of  $V$ .
- Since  $\dim V = p$ ,  $S'$  must contain  $p$  vectors.
- Hence  $S = S'$ .

# THE DIMENSIONS OF NUL $A$ AND COL $A$

- Let  $A$  be an  $m \times n$  matrix, and suppose the equation  $A\mathbf{x} = \mathbf{0}$  has  $k$  free variables.
- A spanning set for Nul  $A$  will produce exactly  $k$  linearly independent vectors—say,  $\mathbf{u}_1, \dots, \mathbf{u}_k$ —one for each free variable.
- So  $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  is a basis for Nul  $A$ , and the number of free variables determines the size of the basis.

# DIMENSIONS OF NUL A AND COL A

- Thus, the dimension of  $\text{Nul } A$  is the number of free variables in the equation  $Ax = 0$ , and the dimension of  $\text{Col } A$  is the number of pivot columns in  $A$ .
- **Example 2:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

# DIMENSIONS OF NUL A AND COL A

- **Solution:** Row reduce the augmented matrix  $[A \ 0]$  to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable— $x_2$ ,  $x_4$  and  $x_5$ .
- Hence the dimension of  $\text{Nul } A$  is 3.
- Also  $\dim \text{Col } A = 2$  because  $A$  has two pivot columns.