

4 Vector Spaces

4.1

VECTOR SPACES AND SUBSPACES

Linear Algebra *and its applications* FOURTH EDITION



VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and for all scalars c and d .
 1. The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$, is in V .
 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
 3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
 4. There is a zero vector $\mathbf{0}$ in V such that
$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

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5. For each \mathbf{u} in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
 6. The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$, is in V .
 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
 10. $1\mathbf{u} = \mathbf{u}$.
- Using these axioms, we can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the **negative** of \mathbf{u} , in Axiom 5 is unique for each \mathbf{u} in V .

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- For each \mathbf{u} in V and scalar c ,

$$0\mathbf{u} = \mathbf{0} \quad \text{----(1)}$$

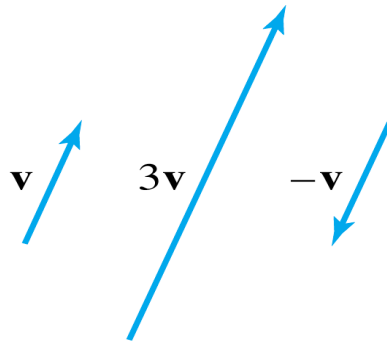
$$c\mathbf{0} = \mathbf{0} \quad \text{----(2)}$$

$$-\mathbf{u} = (-1)\mathbf{u} \quad \text{----(3)}$$

- **Example 1:** Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule, and for each \mathbf{v} in V , define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of \mathbf{v} , pointing in the same direction as \mathbf{v} if $c \geq 0$ and otherwise pointing in the opposite direction.

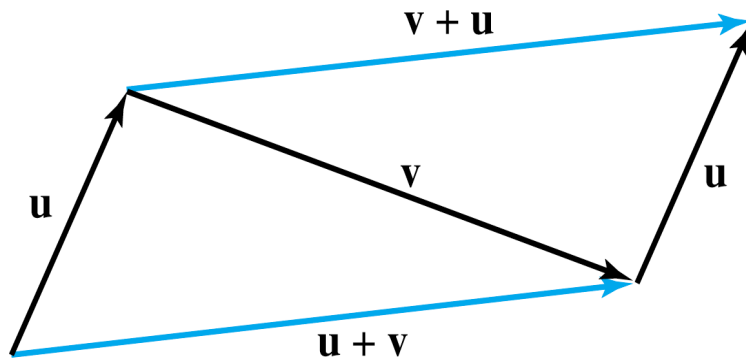
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- See the following figure below. Show that V is a vector space.

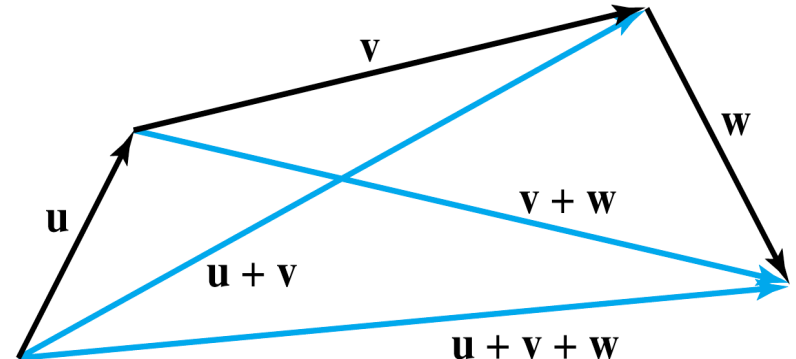


- **Solution:** The definition of V is geometric, using concepts of length and direction.
- No xyz -coordinate system is involved.
- An arrow of zero length is a single point and represents the zero vector.
- The negative of \mathbf{v} is $(-1)\mathbf{v}$.
- So Axioms 1, 4, 5, 6, and 10 are evident. See the figures on the next slide.

SUBSPACES



$$u + v = v + u.$$



$$(u + v) + w = u + (v + w).$$

- **Definition:** A subspace of a vector space V is a subset H of V that has three properties:
 - a. The zero vector of V is in H .
 - b. H is closed under vector addition. That is, for each u and v in H , the sum $u + v$ is in H .

SUBSPACES

c. H is closed under multiplication by scalars.

That is, for each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

- Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V .
- Every subspace is a vector space.
- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V , called the **zero subspace** and written as $\{\mathbf{0}\}$.
- As the term **linear combination** refers to any sum of scalar multiples of vectors, and **Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$** denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$.

A SUBSPACE SPANNED BY A SET

- **Example 2:** Given \mathbf{v}_1 and \mathbf{v}_2 in a vector space V , let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V .
- **Solution:** The zero vector is in H , since $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
- To show that H is closed under vector addition, take two arbitrary vectors in H , say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \text{ and } \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

- By Axioms 2, 3, and 8 for the vector space V ,

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2\end{aligned}$$

A SUBSPACE SPANNED BY A SET

- So $\mathbf{u} + \mathbf{w}$ is in H .
- Furthermore, if c is any scalar, then by Axioms 7 and 9,
$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$
which shows that $c\mathbf{u}$ is in H and H is closed under scalar multiplication.
- Thus H is a subspace of V .

A SUBSPACE SPANNED BY A SET

- **Theorem 1:** If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V , then $\text{Span} \{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$ is a subspace of V .
- We call $\text{Span} \{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$ **the subspace spanned (or generated)** by $\{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$.
- Give any subspace H of V , a **spanning (or generating)** set for H is a set $\{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$ in H such that

$$H = \text{Span} \{ \mathbf{v}_1, \dots, \mathbf{v}_p \}.$$