MAT: 216; Lecture: 05

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Definition

A *vector space* is a nonempty set V of objects, called *vectors* on which are defined two operations, called *vector addition* and *scalar multiplication*, respectively, are defined such that, for $x, y \in V$ and $\alpha \in F$, $\mathbf{x} + \mathbf{y}$ and $\alpha\mathbf{x}$ are well defined elements of V with the following properties:

- *commutativity of addition*: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- *associativity of addition*: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
- *additive identity*: there is a vector $\mathbf{0}$ such that $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$
- *additive inverse*: for each vector $\mathbf{x}$, there exists another vector $\mathbf{y}$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$
- *scalar associativity*: $\alpha(\beta\, \mathbf{x}) = (\alpha\, \beta)\mathbf{x}$
- *scalar distributivity*: $(\alpha + \beta)\mathbf{x} = \alpha\, \mathbf{x} + \beta\, \mathbf{x}$
- *vector distributivity*: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\, \mathbf{x} + \alpha\, \mathbf{y}$
- *scalar identity*: $1\mathbf{x} = \mathbf{x}$

Subspaces

A *subspace* is a vector space inside a vector space. When we look at various vector spaces, it is often useful to examine their subspaces.

The subspace S of a vector space V is that S is a sub*set* of V and that it has the following key characteristics

- S is closed under scalar multiplication: if $\lambda \in \mathbf{R}$, $\mathbf{v} \in S$, $\lambda\, \mathbf{v} \in S$
- S is closed under addition: if $\mathbf{u}, \mathbf{v} \in S$, $\mathbf{u} + \mathbf{v} \in S$.
- S contains $\mathbf{0}$, the zero vector.

Any subset with these characteristics is a subspace

OR;

Definition:

Let V be a vector space and let S be a subset of V such that S is a vector space with the same + and * from V. Then S is called a *subspace* of V.

Important Result:
W is a subspace of a real vector space V if and only if
1. If ***u*** and ***v*** are any vectors in W, then $u + v \in W$.
2. If ***c*** is any real number and ***u*** is any vector in W, then $cu \in W$.

Example:

Let V be the vector space R^3 and let S be the set of points that lie on the plane

$$z = x - y$$

Then S is a subspace of V.

This is true since S is closed under + and *. A point belonging to S has the form

$$(x, y, x - y)$$

If

$$(x_1, y_1, x_1 - y_1) \quad \text{and} \quad (x_2, y_2, x_2 - y_2)$$

are in S then

$$(x_1, y_1, x_1 - y_1) + (x_2, y_2, x_2 - y_2) = (x_1 + x_2, y_1 + y_2, x_1 - y_1 + x_2 - y_2)$$

$$= (x_1 + x_2, y_1 + y_2, (x_1 + x_2) - (y_1 + y_2))$$

is in S. We also have

$$c(x_1, y_1, x_1 - y_1) = (cx_1, cy_1, cx_1 - cy_1)$$

is in S. The rest of the properties follow immediately since they are true in V. In fact, the two closure properties are all we need to show when we want to check that any subspace S is a subspace of any vector space V.

Example:

Let S be the subset of M^{2x2} of trace 0, that is the sum of the diagonal entries is zero. Then S is a subspace of M^{2x2}. Elements of S have the form

$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$$

so if

$$A = \begin{pmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{pmatrix} \qquad B = \begin{pmatrix} x_2 & y_2 \\ z_2 & -x_2 \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & -x_1 - x_2 \end{pmatrix}$$

has zero trace. And

$$cA - c\begin{pmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{pmatrix} - \begin{pmatrix} cx_1 & cy_1 \\ cz_1 & -cx_1 \end{pmatrix}$$

also has trace zero. Hence S is closed under + and *. We can conclude that S is a subspace of V.

Example:

$W_1 \equiv$ the subset of R^3 consisting of all vectors of the form,

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad a \in R \quad ,$$

together with standard addition and scalar multiplication. Is W_1 a subspace of R^3?

We need to check if the conditions (1) and (2) are satisfied. Let

$$u = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix},\ v = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix},\ c \in R.$$ Then,

(1):

$$u + v = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 0 \end{bmatrix} \in W_1.$$

(2):

$$cu = \begin{bmatrix} ca_1 \\ 0 \\ 0 \end{bmatrix} \in W_1.$$

$\Rightarrow$ W_1 is a subspace of R^3.

Example:

Let the real vector space V be the set consisting of all $n \times n$ matrices together with the standard addition and scalar multiplication. Let

$W_2 \equiv$ the subset of V consisting of all $n \times n$ diagonal matrices.

Is W_2 a subspace of V?

Let $u = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \in W_2$, $v = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \in W_2$, and $c \in R$.

(1):

$$u + v = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix} \in W_2$$

since $u + v$ is still a diagonal matrix.

(2):

$$cu = \begin{bmatrix} ca_{11} & 0 & \cdots & 0 \\ 0 & ca_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ca_{nn} \end{bmatrix} \in W_2$$

since cu is still a diagonal matrix.

$\Rightarrow W_2$ is a subspace of V.

Example:.

$P_n \equiv$ the set consisting of ***all*** polynomials of degree n or less with the form together with standard polynomial addition and scalar multiplication**.** P_n **is a vector space.**
$P \equiv$ the set consisting of ***all*** polynomials with the form together with standard

polynomial addition and scalar multiplication. Then, P **is also a vector space. Then,** P_n **is a subspace of** P**.**

Example:

$V_4 \equiv$ the set consisting of ***all*** real-valued ***continuous functions*** defined on the entire real line together with standard addition and scalar multiplication. **Let** $V_4^* \equiv$ **t**he set of all differentiable functions defined on the entire real line together with standard

addition and scalar multiplication. Then, V_4 is a real vector space. Also. V_4^* is a subspace of V_4.

Example:

$W_3 \equiv$the subset of R^3 consisting of all vectors of the form,

$$\begin{bmatrix} a \\ a^2 \\ b \end{bmatrix}, \; a, \; b \in R ,$$

together with standard addition and scalar multiplication. Is W_3 a subspace of R^3?

Let $$u = \begin{bmatrix} a_1 \\ a_1^2 \\ b_1 \end{bmatrix}, \; v = \begin{bmatrix} a_2 \\ a_2^2 \\ b_2 \end{bmatrix}, \; c \in R .$$

Then,

(1):

$$u+v=\begin{bmatrix} a_1 \\ a_1^2 \\ b_1 \end{bmatrix}+\begin{bmatrix} a_2 \\ a_2^2 \\ b_2 \end{bmatrix}=\begin{bmatrix} a_1+a_2 \\ a_1^2+a_2^2 \\ b_1+b_2 \end{bmatrix}\neq\begin{bmatrix} a_1+a_2 \\ (a_1+a_2)^2 \\ b_1+b_2 \end{bmatrix}\in W_3.$$

Therefore, $u+v\notin W_3$.

$\Rightarrow$ W_3 is **not** a subspace of R^3.

Example:

$V_3 \equiv$ the set consisting of ***all*** polynomials of degree 2 or less with the form together with standard polynomial addition and scalar multiplication. V_3 is a vector space. Let

$W_4 \equiv$ the subset of V_3 consisting of all polynomials of the form

$$ax^2+bx+c,\ a+b+c=2.$$

Is W_4 a subspace of V_3?

Let

$$u=a_2x^2+a_1x+a_0\in W_4$$

and

$$v=b_2x^2+b_1x+b_0\in W_4.$$

Then, $a_2 + a_1 + a_0 = 2$ and $b_2 + b_1 + b_0 = 2$. Thus,

$$\begin{aligned} u + v &= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\ &= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \notin W_4 \end{aligned}$$

since

$$(a_2 + b_2) + (a_1 + b_1) + (a_0 + b_0) = (a_2 + a_1 + a_0) + (b_2 + b_1 + b_0) = 2 + 2 = 4.$$

$\Rightarrow$ W_4 is **not** a subspace of V_3.

Good Luck