CSE 221: Algorithms

Divide and Conquer

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References

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2. Erik Demaine and Charles Leiserson, *6.046J Introduction to Algorithms*. MIT OpenCourseWare, Fall 2005. Available from: `ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/CourseHome/index.htm`

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Divide-and-Conquer design strategy

1. *Divide* the problem (instance) into subproblems.
2. *Conquer* the subproblems by solving these recursively.
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Example (of D&C strategy)

1. Binary search – divide the problem into half, and recursively search the appropriate 1 subproblem.
2. Mergesort – divide the problem into half, and recursively sort 2 subproblems, and then merge the results into a complete sorted sequence.
3. Computing x^n, computing fibonacci numbers, multiplying matrices (using *Strassen's algorithm*), etc.

Binary search

The problem

Find an element in a sorted array:

1. *Divide:* Check the middle element.
2. *Conquer:* Recursively search 1 subarray.
3. *Combine:* Trivial.

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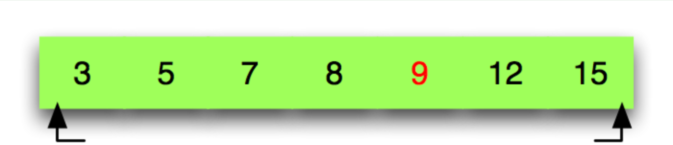
Example (Searching for 9)



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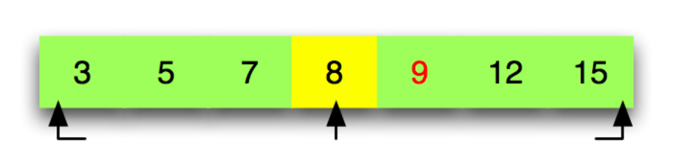

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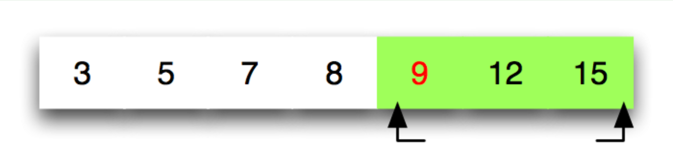

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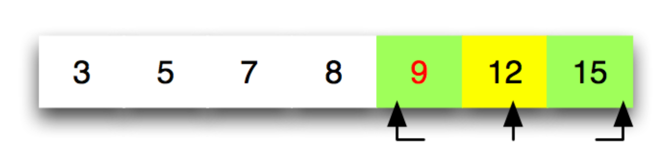

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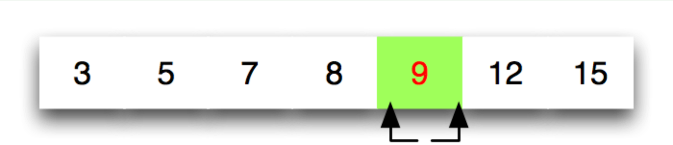

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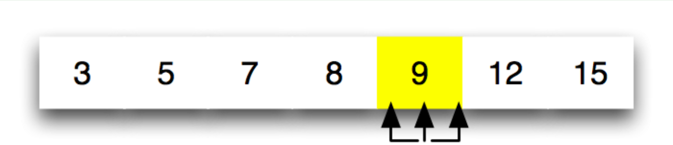

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Analysis

$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow$ CASE 2 $(k = 0)$

$$\Rightarrow \boxed{T(n) = \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(\lg n)}$$

Merge sort

The problem

Find an element in a sorted array:

1. *Divide:* Trivial.
2. *Conquer:* Recursively sort 2 subarrays.
3. *Combine:* Merge the sorted subarrays in $\Theta(n)$ time.

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Key subroutine

MERGE – to merge two sorted arrays in linear-time.

Merge sort in action

1	2	3	4	5	6	7	8
9	4	5	2	1	7	4	6

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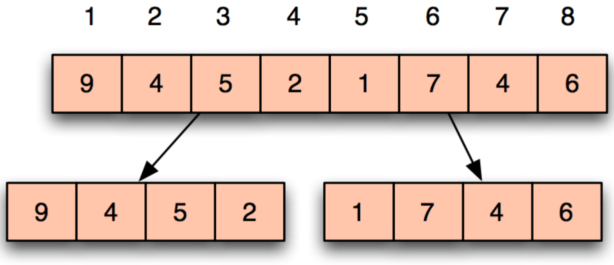

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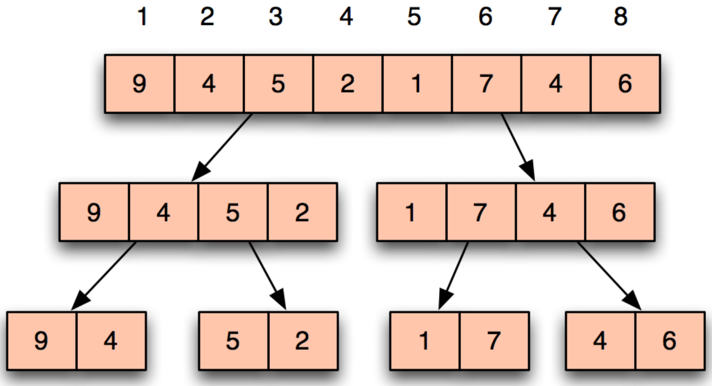

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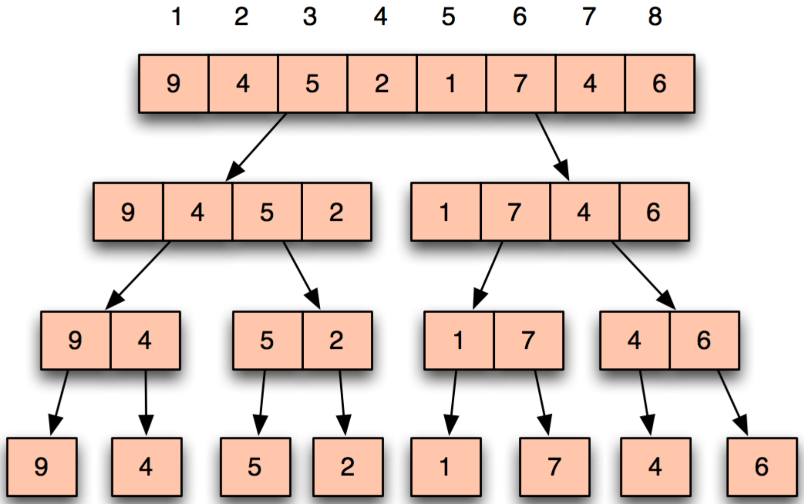

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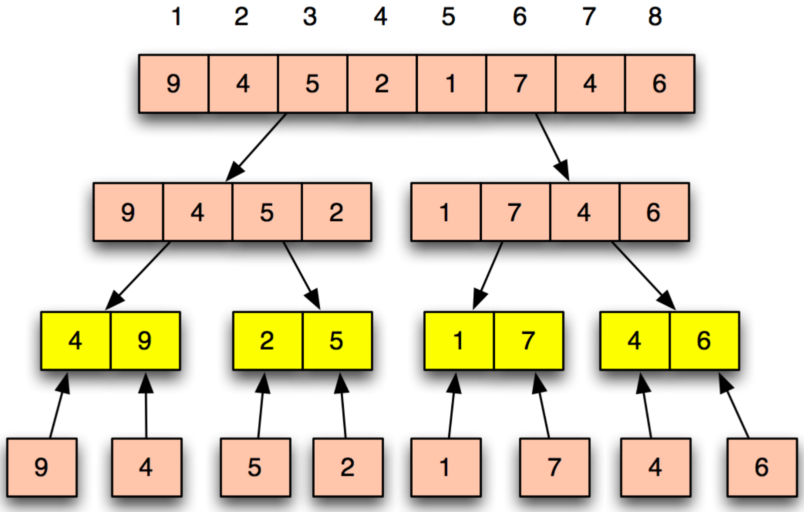

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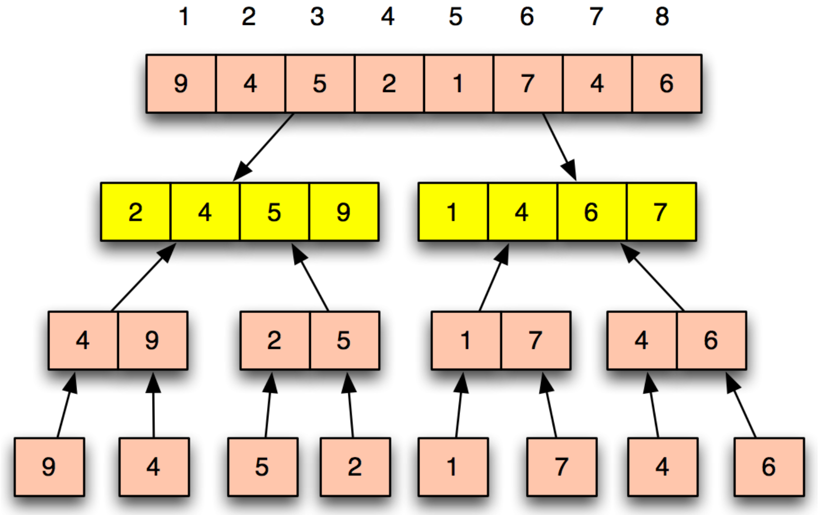

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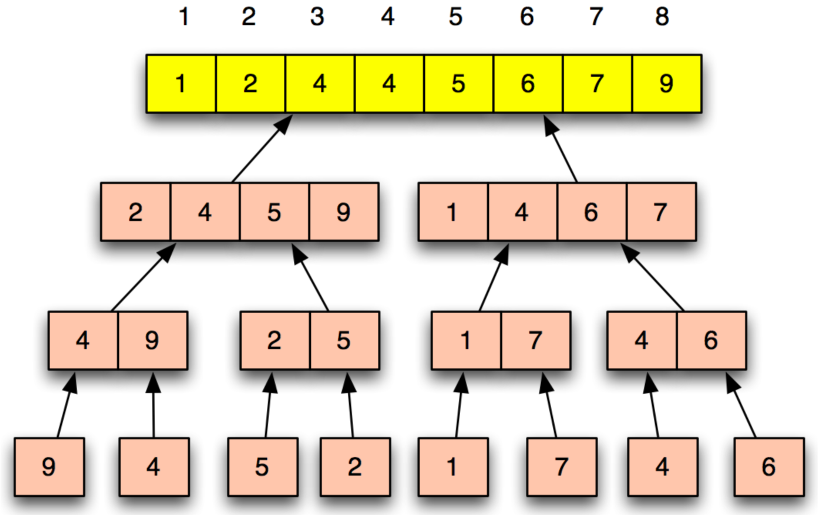

Merging in $\Theta(n)$ time

2 7 13 20 | 1 9 11 12

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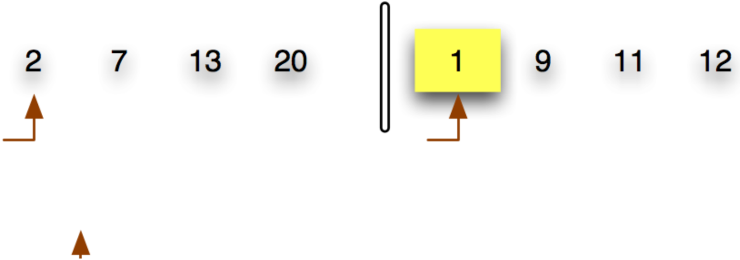

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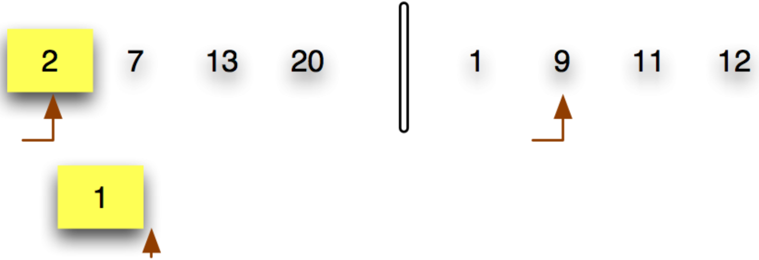

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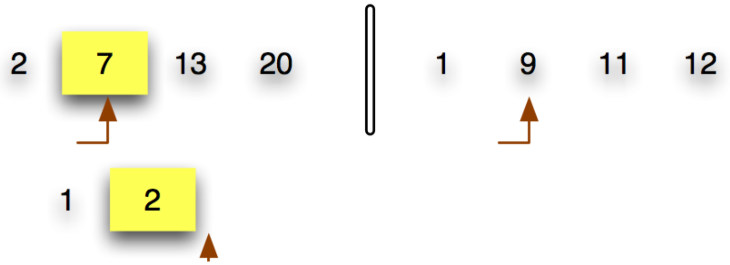

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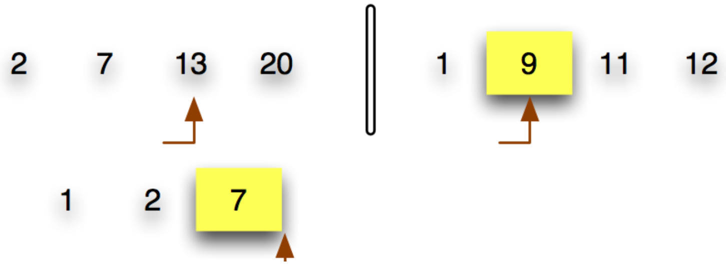

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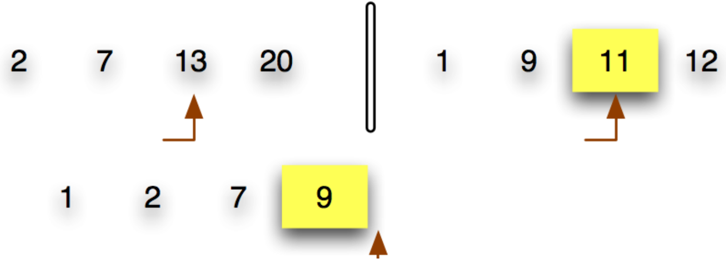

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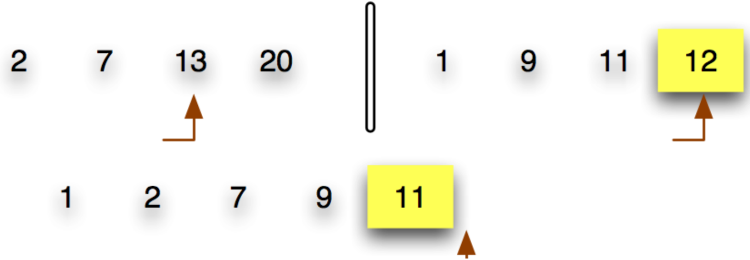

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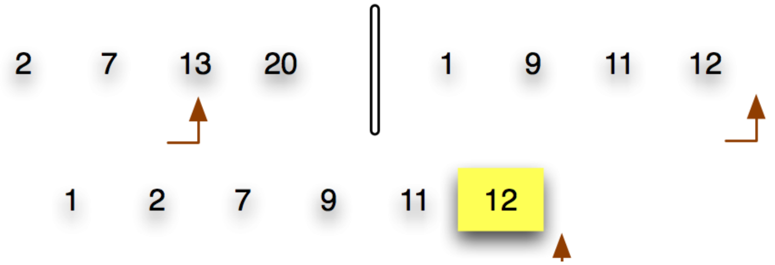

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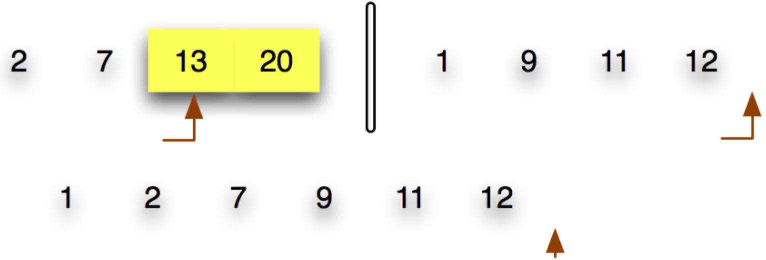

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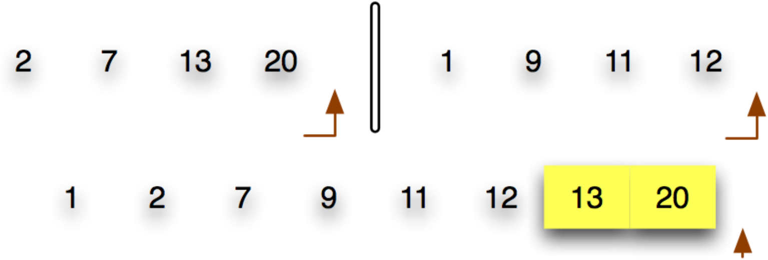

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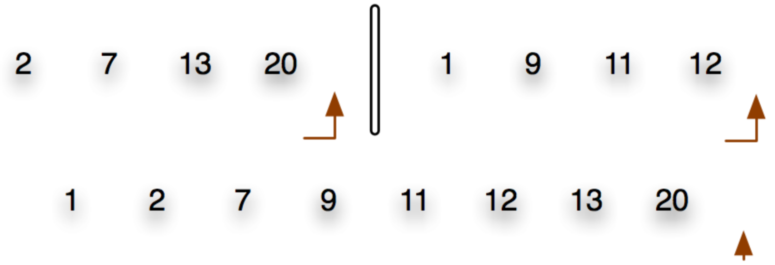

A $\Theta(n)$ time merge algorithm

```
MERGE(A, B)
    INPUT: Two sorted arrays A and B
    OUTPUT: Returns C as the merged array
    ▷ n1 = length[A], n2 = length[B], n = n1 + n2
    Create C[1 . . n]
    Initialize two indices to point to A and B
    while A and B are not empty
          do Select the smaller of two and add to end of C
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    if A or B is not empty
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Issue: Out-of-place algorithm. Can it be made in-place?

Merge sort algorithm

$\text{MERGE-SORT}(A) \triangleright A[1..n]$

```
if n = 1
    then return
    else                                    ▷ recursively sort the two subarrays
        A1 = MERGE-SORT(A[1 .. ⌈n/2⌉])
        A2 = MERGE-SORT(A[⌈n/2⌉ + 1 .. n])
        A = MERGE(A1, A2)                   ▷ merge the sorted arrays
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Merge sort algorithm

$\textsc{Merge-Sort}(A)$ ▷ $A[1\,..\,n]$

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    $A = \textsc{Merge}(A_1, A_2)$ ▷ merge the sorted arrays

Few notes on the algorithm

1. Dividing is trivial, but it's the merging that requires most time.
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$$T(n) = T(n/2) + \Theta(1) \Rightarrow \boxed{T(n) = \Theta(\lg n)}.$$

Conclusion

- Divide and Conquer is just one of several algorithm design strategies.
- Used by many of the commonly used algorithms
  - Binary search
  - Merge sort
  - Fast Fourier Transform (FFT)
  - Finding closest pair of points
  - Matrix multiplication (Strassen's algorithm)
  - Matrix inversion
  - Quicksort and (k^{th}) selection
  - ...
- Can be easily analyzed using recurrences
- Often leads to efficient algorithms