⊛ Lec:02 (MAT:216)

Md. Saddam Hossain
Lecturer, Dept. of MNS
BRAC University

Gaussian Elimination

Solve
$$x+y+2z=9$$
$$2x+4y-3z=1$$
$$3x+6y-5z=0$$
by Gaussian elimination and back-substitution.

Solution:

The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Add -2 times the first row to the second and add -3 times the first row to the third to obtain

$$\begin{array}{l} r_2' \to r_2+(-2)r_1 \\ r_3 \to r_3+(-3)r_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\xrightarrow{r_2' \to r_2/2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

Add -3 times the second row to the third to obtain

$$\xrightarrow{r_3' \to r_3 + (-3) r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right]$$

Multiply the third row by -2 to obtain

$$\xrightarrow{r_3' \to (-2) r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

which is in row-echelon form. Therefore the system corresponding to this matrix is

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -17/2 \\ z &= 3 \end{aligned}$$

Solving for the leading variable yields

$$\begin{aligned} x &= 9-y-2z \\ y &= -17/2+7/2\,z \\ z &= 3 \end{aligned}$$

Substituting the bottom equation into those above yields

$$\begin{aligned} x &= 3-y \\ y &= 2 \\ z &= 3. \end{aligned}$$

and substituting the second equation into the top yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3. \end{aligned}$$

Gauss-Jordan Elimination

Solve

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 \quad + 8x_4 + 4x_5 + 18x_6 &= 6 .
\end{aligned}
$$

Solution: The augmented matrix for the given system is

$$
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]
$$

Adding -2 times the first row to the second and fourth rows gives

$$
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]
$$

Multiplying the second row by -1 gives

$$
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]
$$

Adding -5 times the second row to the third row &

times the second row to the fourth row gives

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row-echelon form

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row-echelon form

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equation is

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = 1/3$$

If we assign the free variables x_2, x_4 and x_5 arbitrary values r, s, and t respectively, the general solutions is given by the formulas

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = \frac{1}{3}$$

problem: Determine the value of parameters such that the following system has (i) no solution (ii) a unique solution (iii) more than one solution.

(1)
$$\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\lambda z &= \mu \end{aligned}$$

Solution:

The augmented matrix for the given system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Add -1 times the first row to the second & third row to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

Add -1 times the second row to the third row to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

which is in row-echelon form

Case I: If $\lambda \neq 3$ [redacted], then a unique solution exist.

Case II: If $\lambda = 3$ & $\mu = 0$, then there exist ~~two~~ equation and three variable

i.e one free variable

Therefore more than one solution exist.

Case III: if $\lambda = 3$ & $\mu \neq 10$, then the system is inconsistent and there is no solution.

Similar example:

① $x + y - z = 1$
$2x + 3y + \lambda z = 3$
$x + \lambda y + 3z = 2$.

② $x - 3z = -3$
$2x + \lambda y - z = -2$
$x + 2y + \lambda z = 1$

③ $x + y + kz = 2$
$2x + 3y - z = 1$
$2x + 3y - z = 1$