Diagonalization:

Lec: 09 (MAT: 216)

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Definition: A square matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, the matrix P is said to diagonalize A.

Example 1: Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution: The characteristic equation of matrix A is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$\lambda_1 = 1$, $\lambda_{2,3} = 2$, so there are two eigenspace of A.

By definition $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if x is a nontrivial solution of $(\lambda I - A)x = 0$ i.e.

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(i)}$$

If $\lambda=2$, then (1) becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1=-s,\ x_2=t,\ x_3=s$$

Thus, the eigenvectors of A corresponding to $\lambda=2$ are the nonzero vectors of the form

$$x=\begin{bmatrix} -s \\ t \\ s \end{bmatrix}=\begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}+\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}=s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}+t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

are linearly independent, these vectors form a basis for the eigenspace corresponding to $\lambda=2$.

If $\lambda=1$, then (1) becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this yields

$$x_1=-2s,\ x_2=s,\ x_3=s$$

Thus the eigenvectors corresponding to $\lambda=1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ so that } \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda=1$.

Therefore $P_1=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $P_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $P_3=\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ are three basis vectors in total. So the matrix A is diagonalized and

$$P=\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ diagonalizes } A$$

Now $P^{-1}AP=\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: 2 (page 372); Exercise: 13, 14