Matrix:
- Rectangular arry of real number
- m Rows by n columns
- Named using capital letters
- First subscript is row, second subscript is column.

$$A_{mn} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Terminology:
- A matrix with m rows and n columns is called a matrix of order $m \times n$
- A square matrix is a matrix with an equal number of rows and columns. Since the number of rows & columns are the same, it is said to have order n.
- The main diagonal of a square matrix are the elements from the upper left to the lower right of the matrix.
- A row matrix is a matrix that has only one row
- A column matrix is a matrix that has only one column.
- A matrix with only one row or column is called a vector.

* Converting System of Linear Equations to Matrices

Definition 1: A linear equation in the variable $x_1, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b;$$

where $a_1, \cdots, a_n$ and b are constants, $x_1, \cdots, x_n$ are variables.

Example: The equation

$$2x_1 + x_2 - 7x_3 = \sqrt{5} \quad \text{is linear}$$

The equation

$$3x_1x_2 + 2x_3^2 = 1 \quad \text{is NOT linear.}$$

Definition 2: A system of linear equations (or a linear system) is a collection of one or more linear equations

Example:

$$\begin{aligned} 3x_1 + 2x_2 + 7x_3 - x_4 &= 6 \\ x_1 + x_2 - x_3 + x_4 &= 1 \\ 4x_1 + 3x_2 + 6x_3 \quad\quad &= 8 \end{aligned}$$

$\boxed{\text{Convertion:}}$ Each equation in the system becomes a row. Each variable in the system becomes a column. The variable are dropped and the coefficient are placed into a matrix.

If the right hand side is included it's called an augmented matrix. If the right hand side isn't included it is called a coefficient matrix.

Example:

Given a system

$$\left.\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}\right\}$$

The matrix $\begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}$ is called the coefficient matrix of the system,

and $\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$ is called the augmented matrix of the system.

Elementary Row Operations: When ~~working with~~ a systems of linear equations is converted to an augmented matrix, each equation become a row. So, there are three elementary row operations which will produce a row-equivalent matrix.

1. Interchange two rows
2. Multiply a row by a non-zero constant
3. Multiply a row by a non-zero constant & add it to another row, replacing that row.

Types of Solutions:

There are three types of solutions which are possible when solving a system of linear equations.

Independent:

- Consistent
- Unique Solution.
- A row-reduced matrix has the same number of non-zero rows as variable
- The left hand side is usually the identity matrix, but not necessarily
- There must be at least as many equations as variable to get an independent solution.

Example:

$$\begin{array}{cccc} x & y & z & rhs \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

When you convert the augmented matrix back into equation forms you get

$x=3$, $y=1$, and $z=2$.

Dependent:

- Consistent.
- Many solutions.
- Write answer in parametric form
- A row-reduced matrix has more variable than non-zero rows.
- There doesn't have to be a row of zeros but there usually is
- This could also happen when there are less equations than variables.

Example:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & rhs \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The first equation will be $x+3z=4$
$\Rightarrow x = 4-3z$

the second equation will be $y-2z = 3$
$\Rightarrow y = 3+2z$

Here the variables x & y depends on z.
There-fore; z will be the parameter t and the solution is $x = 4-3t,\ y = 3+2t,\ z=t$

Inconsistent: • No solution
• A row-reduced matrix has a row on the left side, but the right hand side isn't zero.

Example:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & rhs \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right] \end{array}$$

There is no solution here. You can write that as the null set $\emptyset$, the empty set $\{\ \}$, or no solutions.

Row-Echelon Form:

A matrix is in rowechelon form when the following conditions are met.

1. If there is a row of all zeros, then it is at the bottom of the matrix.

2. The first non-zero element of any row is a one. That element is called the leading one.

3. The leading one of any row is to the right of the leading one of the previous row.

Note: • The row-echelon form of a matrix is not necessarily unique.

*** • Gaussian Elimination places a matrix into row-echelon form, and then back substitution is required to finish finding the solutions to the system.

Example:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

which is in row-echelon form.

Reduce Row - Echelon Form:

A matrix is in reduce row-echelon form when all of the conditions of row-echelon form are met and all elements above, as well as below, the leading ones are zero.

1. If there is a row of all zeros, then it is at the bottom of the matrix
2. The first non-zero element of any row is a one. That element is called the leading one.
3. The leading one of any row is to the right of the leading one of the previous row
4. All elements above and below a leading one are zero.

Note: • The reduce row-echelon form of a matrix is unique.

• Gauss-Jordan Elimination places a matrix into reduced row-echelon form.

Example:
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -6 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
which is in reduce-row echelon form.