

5

Eigenvalues and Eigenvectors

5.3

DIAGONALIZATION

Linear Algebra

and its applications

FOURTH EDITION



DIAGONALIZATION

- **Example 1:** Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for

A^k , given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- **Solution:** The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

DIAGONALIZATION

- Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD \underbrace{(P^{-1}P)}_I DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

- Again,

$$A^3 = (PDP^{-1})A^2 = (PD \underbrace{P^{-1}P}_I) D^2 P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

DIAGONALIZATION

- In general, for $k \geq 1$,

$$\begin{aligned} A^k &= PD^k P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix} \end{aligned}$$

- A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal, matrix D .

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A . In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P .

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of \mathbf{R}^n .

We call such a basis an **eigenvector basis** of \mathbf{R}^n .

THE DIAGONALIZATION THEOREM

- **Proof:** First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then

$$AP = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} \quad \text{----(1)}$$

while

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix} \quad \text{----(2)}$$

THE DIAGONALIZATION THEOREM

- Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P , we have
 $AP = PD$.
- In this case, equations (1) and (2) imply that
$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix}$$
----- (3)
- Equating columns, we find that
$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \dots, A\mathbf{v}_n = \lambda_n \mathbf{v}_n$$
----- (4)
- Since P is invertible, its columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ must be linearly independent.

THE DIAGONALIZATION THEOREM

- Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \dots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are corresponding eigenvectors.
- This argument proves the “only if” parts of the first and second statements, along with the third statement, of the theorem.
- Finally, given any n eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ to construct D .

THE DIAGONALIZATION THEOREM

- By equation (1)–(3), $AP = PD$.
- This is true without any condition on the eigenvectors.
- If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$.

DIAGONALIZING MATRICES

- **Example 2:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
- *Step 1. Find the eigenvalues of A .*
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

DIAGONALIZING MATRICES

$$\begin{aligned} 0 &= \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

- The eigenvalues are $\lambda = 1$ and $\lambda = -2$.
- *Step 2. Find three linearly independent eigenvectors of A .*
- *Three* vectors are needed because A is a 3×3 matrix.
- This is a critical step.
- If it fails, then Theorem 5 says that A cannot be diagonalized.

DIAGONALIZING MATRICES

- Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- Basis for $\lambda = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

DIAGONALIZING MATRICES

- *Step 3. Construct P from the vectors in step 2.*
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- *Step 4. Construct D from the corresponding eigenvalues.*
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P .

DIAGONALIZING MATRICES

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- To avoid computing P^{-1} , simply verify that $AD = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
- **Proof:** Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A .
- Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1.
- Hence A is diagonalizable, by Theorem 5.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
- Theorem 6 provides a *sufficient* condition for a matrix to be diagonalizable.
- If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and if $P = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.
- **Theorem 7:** Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.
 - a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k .

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n , and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k .
- c. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k , then the total collection of vectors in the sets B_1, \dots, B_p forms an eigenvector basis for \mathbf{R}^n .