Mechanism of the Production of Small Eddies from Large Ones

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Introduction

The connexion between the statistical representation of turbulence and dissipation of energy has been discussed* in relation to the decay of the isotropic turbulence which is produced in a wind tunnel by means of regular grids. It was shown that a length λ can be defined which may be taken as a measure of the scale of the small eddies which are responsible for dissipation. This λ can be found by measuring the correlation R_y between the indications of two hot wire anemometers set at a distance y apart on a line perpendicular to the axis of the tunnel. Then

$$\frac{1}{\lambda^2} = \underset{y \to 0}{\mathrm{Lt}} \frac{1 - R_y}{y^2},$$

and the mean rate of dissipation of energy per unit volume is

$$\overline{W} = 15\mu\overline{u^2}/\lambda^2, \tag{1}$$

where $\overline{u^2}$ is the mean of the square of one component of velocity.

When turbulence is generated in a wind stream by a grid of regularly spaced bars it may be expected to possess a definite scale proportional to the linear dimensions of the grid. In any complete statistical description of turbulence this scale must be implicitly or explicitly involved. One way in which the scale can be defined is to measure the distance y apart by which the two hot wires must be separated before the correlation between the indications disappears. Another way is to define the scale as

$$l_2 = \int_0^y R_y \, dy. \tag{2}$$

It is to be expected that if the turbulence is entirely produced by a regular grid (*i.e.*, there is no turbulence in the stream before it reaches

* Taylor, 'Proc. Roy. Soc.,' A, vol. 151, p. 421 (1935).

the grid) l_2 and y will be proportional to M, the mesh-length of the grid. In one set of experiments with a square-mesh grid it was found that

$$l_2 = 0 \cdot 195\text{M}. \tag{3}$$

In these experiments, R_y was not determined accurately when its value was small, but the R_y curve seems to cut the axis $R_y = 0$ at*

$$y = \frac{1}{2}\text{M}. \tag{4}$$

It has not yet been shown experimentally whether these figures apply generally.

If the speeds all over a turbulent field are increased in the ratio $m:1$ the Reynolds stresses increase in the ratio $m^2:1$, so that the rate of dissipation of energy increases in the ratio $m^3:1$. Now

$$\overline{\text{W}} = 15\mu\overline{u^2}/\lambda^2, \tag{5}$$

so that $1/\lambda^2$ must increase in the ratio $m:1$. This result can be expressed by the formula

$$\frac{\lambda}{\text{L}} = \text{const} \sqrt{\frac{\nu}{u'\text{L}}}, \tag{6}$$

where L is a length representing the scale of the turbulence and $u'^2 = \overline{u^2}$. It has, in fact, been found in a large number of experiments on the rate of dissipation of energy in a stream behind regular square-mesh grids that

$$\frac{\lambda}{\text{M}} = 2 \cdot 0 \sqrt{\frac{\nu}{u'\text{M}}}, \tag{7}$$

and if $y = \frac{1}{2}\text{M}$ this may be written

$$\frac{\lambda}{y} = 1 \cdot 41 \sqrt{\frac{\nu}{u'y}}, \tag{8}$$

or if $l_2 = 0 \cdot 195\text{M}$

$$\frac{\lambda}{l_2} = 0 \cdot 88 \sqrt{\frac{\nu}{u'l_2}}. \tag{9}$$

Of these equivalent formulae (7) seems to be firmly established, but (8) and (9) depend on only one set of measurements.

* *See* Taylor, 'Proc. Roy. Soc.,' A, vol. 151, p. 445 (1935), Pt. II, fig. 1, where it will be seen that R_y tends to zero at $y = 0 \cdot 45$ inches when $\text{M} = 0 \cdot 9$ inches.

Object of the Present Work

The theory outlined above represents in a satisfactory manner the dissipation of energy in turbulent flow. The alternative formulae (7), (8), or (9) which relate the scale of the small scale turbulence to the main scale of turbulence are, however, quite empirical. They are merely the formulae which are necessary in order that the observed square law of resistance in turbulent fields may hold. They represent the effect of the fundamental process in turbulent flow, namely the grinding down of eddies produced by solid obstructions (and on a scale comparable with these obstructions) into smaller and smaller eddies until these eddies are of so small a scale that they die away owing to viscosity more rapidly than they are produced by the grinding down process.

To explain this process is, perhaps, the fundamental problem in turbulent motion. It seems clear that it is intimately associated with diffusion. Suppose that eddying motion of some definite scale is generated in a non-viscous fluid. Consider two particles A, B situated on the same vortex line in a turbulent field of flow and separated initially by a small distance d_0.

If the turbulence is diffusive, in the sense that a concentrated collection of particles spreads into a diffuse cloud (and turbulence is always found to be diffusive), the average distance d between pairs of particles like A and B increases continually.

If the fluid were non-viscous the continual increase in the average value of d^2 would necessarily involve a continual increase in ω^2, ω being the resultant vorticity at any point. In fact, the equation for conservation of circulation in a non-viscous fluid is

$$\frac{\omega}{d} = \frac{\omega_0}{d_0} \quad \text{or} \quad \frac{\omega^2}{d^2} = \frac{{\omega_0}^2}{{d_0}^2}, \tag{10}$$

where ω_0 is the initial resultant vorticity when $d = d_0$. Hence ω^2 increases continually as d^2 increases.

The mean rate of dissipation of energy in a viscous fluid is

$$\overline{\mathrm{W}} = \mu\,(\overline{\xi^2 + \eta^2 + \zeta^2}) = \mu\overline{\omega^2}, \tag{11}$$

so that if turbulence is set up in a slightly viscous fluid by the formation of large scale eddies (*e.g.*, as in a wind tunnel when the wind meets a large scale obstruction) we may expect first an increase in $\overline{\omega^2}$ in accordance with (10).

When $\overline{\omega^2}$ has increased to some value which depends on the viscosity, it is no longer possible to neglect the effect of viscosity in the equation for the conservation of circulation, so that (10) ceases to be true. Experiment shows, in fact, that in a wind tunnel $\overline{W}$ reaches the definite value indicated by (5) and (6).

It is difficult to express these ideas in a mathematical form without assuming some definite form for the disturbance, but it is almost impossible to suggest an initial form which has the characteristics of the statistical isotropic turbulent motion to which (5) and (6) apply. Accordingly, we have searched for types of initial motion which have a definite scale and also have some of the properties of statistically uniform isotropic turbulence with a view to tracing the subsequent motion and finding out whether anything analogous to the process of the grinding down into smaller and smaller eddies occurs.

At the outset the extreme limitations of mathematical methods are very evident, for it is only in special cases where the initial motion is such that one of the essential features of turbulent motion (*i.e.*, extension along vortex lines) is absent that the subsequent motion has so far been calculated. By far the largest class of fields of flow which has been analysed mathematically is two-dimensional. Since the vortex lines are then perpendicular to the plane of motion, they are not extending, and this essential characteristic of turbulent flow is therefore absent.

The largest class of three-dimensional motions which has been solved is the irrotational motions of a non-viscous fluid. Here again there are no vortex lines, so that no motions of this type are significant in connexion with turbulence. Another class of motions which can be treated by existing methods is deviations from states of rest or steady motion. Such motions are only significant in discussing the first beginning of turbulence arising in a steady flow.

Decay of a Special Class of Vortices

It appears that nothing but a complete solution of the equations of motion in some special case will suffice to illustrate the process of grinding down of large eddies into smaller ones. In the following pages an attempt is made to trace the subsequent motion of a viscous incompressible fluid when the initial motion is represented by

$$\left.\begin{aligned} u &= \mathrm{A} \cos ax \sin by \sin cz \\ v &= \mathrm{B} \sin ax \cos by \sin cz \\ w &= \mathrm{C} \sin ax \sin by \cos cz \end{aligned}\right\}. \qquad (12)$$

These equations are consistent if

$$\mathrm{A}a + \mathrm{B}b + \mathrm{C}c = 0. \tag{13}$$

One equation of motion is

$$-\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x} - \nu\nabla^2 u, \tag{14}$$

if, therefore, $\partial \mathrm{P}/\partial x$ is known when the initial velocity components are known, the initial value of $\partial u/\partial t$ is known, and it is possible to take the first step in a step-by-step solution.

Eliminating $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$ between the three equations of motion and the equation of continuity,

$$-\frac{1}{\rho}\nabla^2 \mathrm{P} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right). \tag{15}$$

Hence, substituting for $\partial u/\partial x$, ...

$$-\frac{1}{\rho}\nabla^2 \mathrm{P} = \frac{1}{2}\sum_{xyz} \mathrm{A}^2 a^2 (\cos 2by \cos 2cz - \cos 2ax). \tag{16}$$

The periodic solution of (16) is

$$\frac{\mathrm{P}}{\rho} = \frac{1}{8}\sum_{xyz} \mathrm{A}^2 \left(\frac{a^2}{b^2 + c^2}\cos 2by \cos 2cz - \cos 2ax\right). \tag{17}$$

Substituting for u, v, w and P from (12) and (17) in (14) gives the initial value of $\partial u/\partial t$ as

$$\frac{\partial u}{\partial t} = -\theta\nu \mathrm{A}\cos ax \sin by \sin cz + \frac{\mathrm{A}_3}{a}\sin 2ax \cos 2by$$

$$-\frac{\mathrm{A}_2}{a}\sin 2ax \cos 2cz, \tag{18}$$

where

$$\theta = a^2 + b^2 + c^2 \tag{19}$$

and the constants A_2, A_3 are obtained by cyclic permutation of the letters a, b, c from a constant A_1 where

$$\mathrm{A}_1 = \frac{b}{4}\left(\mathrm{A}^2 b\frac{a^2}{b^2 + c^2} + \mathrm{AB}a\right) = -\frac{c}{4}\left(\mathrm{A}^2 c\frac{a^2}{b^2 + c^2} + \mathrm{AC}a\right). \tag{20}$$

The initial values of $\partial v/\partial t$, $\partial w/\partial t$ are obtained from (18) by simultaneous cyclic permutation of the letters a, b, c and x, y, z.

First Approximation

The first approximation to the value of u after time t is obtained from (18) by a simple integration. It is

$$u = \mathrm{A}\,(1 - \theta\nu t)\cos ax \sin by \sin cz + \frac{\mathrm{A}_3}{a}\,t \sin 2ax \cos 2by$$
$$- \frac{\mathrm{A}_2}{a}\,t \sin 2ax \cos 2cz \qquad (21)$$

with corresponding values for v, w.

Successive Approximations

Since all the terms in u given by (21) are the same type as those in the original u, namely

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} lax \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} mby \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} ncz,^*$$

the same process may be applied to find u to a second approximation when the value of u is taken to be that given by (21) instead of the original u. By successive repetitions the complete solution could be obtained for the motion which ensues when the flow represented by (12) is established at time $t = 0$. The resulting expressions for u, v, w will consist entirely of terms of the type

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} lax \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} mby \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} ncz,$$

multiplied by a power series of the type $\mathrm{A}_0 + \mathrm{A}_1 t + \mathrm{A}_2 t^2 + \ldots$. In fact, if a solution of the type

$$\left.\begin{matrix}u\\ v\\ w\end{matrix}\right\} = \left\{\begin{matrix}{}_0\mathrm{A}^{csc}_{lmn} + {}_1\mathrm{A}^{csc}_{lmn}\,t + {}_2\mathrm{A}^{csc}_{lmn}\,t^2 + \ldots\\ {}_0\mathrm{B}^{csc}_{lmn} + {}_1\mathrm{B}^{csc}_{lmn}\,t + {}_2\mathrm{B}^{csc}_{lmn}\,t^2 + \ldots\\ {}_0\mathrm{C}^{csc}_{lmn} + {}_1\mathrm{C}^{csc}_{lmn}\,t + {}_2\mathrm{C}^{csc}_{lmn}\,t^2 + \ldots\end{matrix}\right\} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} lax \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} mby \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} ncz \qquad (22)$$

is assumed and $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$ are calculated as explained above, and equated to the values obtained by differentiating (22) with respect to t, the coefficients of

$$t^r \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} lax \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} mby \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} ncz$$

* l, m, n, integers.

on the two sides can then be equated and the coefficients ${}_rA^{csc}_{lmn}$, ... determined. The upper suffices c, s, c are used to show which of the two alternatives sine or cosine occurs in the Fourier term, and the meaning of the other suffices is obvious. Thus the coefficient of

$$t^r \cos lax \sin mby \cos ncz$$

would be ${}_rA^{csc}_{lmn}$.

Second Approximation

The above process will now be examined in more detail. The first approximation to u is already given in (21). The second application of the process gives the following expression for u as a second approximation, namely

$$\begin{aligned} u = {} & \delta_1 \cos ax \sin by \sin cz + \frac{x_3}{a} \sin 2ax \cos 2by - \frac{x_2}{a} \sin 2ax \cos 2cz \\ & + \frac{\alpha_1}{3} \cos 3ax \sin by \sin cz + \gamma_2 \cos ax \sin 3by \sin cz \\ & + \beta_3 \cos ax \sin by \sin 3cz + \frac{y_3}{a} \cos 3ax \sin 3by \sin cz \\ & - \frac{y_2}{a} \cos 3ax \sin by \sin 3cz + \frac{L_1}{2} \sin 4ax \cos 2by \cos 2cz \\ & + N_2 \sin 2ax \cos 4by \cos 2cz + M_3 \sin 2ax \cos 2by \cos 4cz. \end{aligned} \tag{23}$$

As before, v, w are obtained from u by simultaneous permutation of a, b, c; x, y, z, and the suffices 1, 2, 3. The coefficients are

$$\left.\begin{aligned} \delta_1 &= A\left(1 - \theta\nu t + \frac{1}{2}\theta^2\nu^2 t^2\right) - \frac{1}{2a}(CcA_3 - BbA_2)\left(\frac{1}{2}t^2 - \frac{1}{3}\theta\nu t^3\right) \\ x_1 &= A_1\left(t - \theta\nu t^2 + \frac{1}{3}\theta^2\nu^2 t^2\right) - 2A_1(b^2 + c^2)\nu t^2 \\ &\qquad\qquad - \frac{A_2A_3}{3}\left(\frac{b^2 - c^2}{b^2 + c^2}\right)t^3 \\ y_1 &= \frac{1}{2}\left(\frac{1}{2}t^2 - \frac{1}{3}\theta\nu t^3\right)A_1Aa \\ \frac{\alpha_1}{3} &= -\left(\frac{1}{2}t^2 - \frac{1}{3}\theta\nu t^3\right)\left(\frac{6a\{A_3(Aa - Bb) + A_2(Cc - Aa)\}}{9a^2 + b^2 + c^2}\right. \\ &\qquad\qquad \left. - AA_3 + AA_2 - \frac{1}{2}A_2\frac{Cc}{a} + \frac{1}{2}A_3\frac{Bb}{a}\right) \\ \beta_1 &= -\left(\frac{1}{2}t^2 - \frac{1}{3}\theta\nu t^3\right)\left(\frac{2b\{A_3(Aa - Bb) + A_2(Cc - Aa)\}}{9a^2 + b^2 + c^2}\right. \\ &\qquad\qquad \left. + \frac{1}{2}\frac{a}{b}AA_3 + \frac{1}{2}BA_2\right) \end{aligned}\right\}, \tag{24}$$

$$\left.\begin{aligned}
\gamma_1 &= -\left(\frac{1}{2}t^2 - \frac{1}{3}\theta\nu t^3\right)\left(\frac{2c\{A_3(Aa-Bb)+A_2(Cc-Aa)\}}{9a^2+b^2+c^2}\right. \\
&\qquad\qquad\qquad\qquad\qquad\left. - \frac{1}{2}\frac{a}{c}AA_2 - \frac{1}{2}CA_3\right) \\
\frac{L_1}{2} &= -\frac{1}{3}t^3\left(\frac{4a}{4a^2+b^2+c^2} - \frac{2}{a}\right)A_2A_3 \\
M_1 &= -\frac{1}{3}t^3\left(\frac{2b}{4a^2+b^2+c^2} + \frac{1}{b}\right)A_2A_3 \\
N_1 &= -\frac{1}{3}t^3\left(\frac{2c}{4a^2+b^2+c^2} + \frac{1}{c}\right)A_2A_3
\end{aligned}\right\}, \qquad \text{(24) (cont.)}$$

the coefficients with suffices 2, 3 being obtained by cyclic permutation of the letters a, b, c. These coefficients are not all independent but are connected by the following relations

$$a\delta_1 + b\delta_2 + c\delta_3 = 0 \tag{25}$$

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 0, \tag{26}$$

with two similar relations obtained by cyclic permutation, and

$$aL_1 + bM_1 + cN_1 = 0, \tag{27}$$

with two similar relations as before.

We have obtained the third approximation for u, v, w in the general case, but it is too long to set out here. In order to give some idea of the length of the work, it is sufficient to say that the expression for u consists of 518 terms of the type

$$P(t)\left(\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} l_1ax \pm \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} l_2ax\right)\left(\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m_1by \pm \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m_2by\right) \times \left(\begin{matrix}\cos\\ \sin\end{matrix}\, n_1cz \pm \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} n_2cz\right),$$

where l_1, l_2, m_1, m_2, n_1, n_2 are integers or zero and $P(t)$ is a power series in t. At this stage it seems impossible to obtain significant results in the general case when the initial motion is represented by (12), so attention is now confined to a special case.

Special Case

The special case which is considered is when $a = b = c$; $A = -B$, and $C = 0$. Equation (13) is then satisfied, while (19) and (20) give

$$\theta = 3a^2; \quad A_1 = -A_2 = -\frac{1}{8}A^2a^2; \quad A_3 = 0. \tag{28}$$

The expressions in (24) reduce to

$$\left.\begin{aligned}
&\delta_3 = x_3 = y_3 = \alpha_3 = \mathrm{L}_1 = \mathrm{M}_1 = \mathrm{N}_1 = \mathrm{L}_2 = \mathrm{M}_2 = \mathrm{N}_2 = 0\\
&\delta_1 = -\delta_2 = \mathrm{A}\left(1 - 3a^2\nu t + \frac{9}{2}a^4\nu^2t^2\right) + \frac{1}{2}\mathrm{AA}_1\left(\frac{1}{2}t^2 - a^2\nu t^3\right)\\
&x_1 = -x_2 = \mathrm{A}_1\,(t - 7a^2\nu t^2 + 3a^4\nu^2t^3)\\
&\frac{1}{2}\alpha_1 = -\frac{1}{2}\alpha_2 = -\beta_1 = -\gamma_1 = \beta_2 = \gamma_2 = \frac{15}{22}\mathrm{AA}_1\left(\frac{1}{2}t^2 - a^2\nu t^3\right)\\
&-\beta_3 = \gamma_3 = \frac{y_1}{a} = \frac{y_2}{a} = \frac{1}{2}\mathrm{AA}_1\left(\frac{1}{2}t^2 - a^2\nu t^3\right)\\
&-\frac{1}{2}\mathrm{L}_3 = \mathrm{M}_3 = \mathrm{N}_3 = \frac{4}{9}\frac{\mathrm{A}_1^{\,2}}{a}t^3
\end{aligned}\right\}. \tag{29}$$

Before proceeding any further, it is necessary to recall that we are seeking an expression for the mean rate of dissipation of energy $\overline{\mathrm{W}}$ and

$$\frac{\overline{\mathrm{W}}}{\mu} = \overline{\xi^2} + \overline{\eta^2} + \overline{\zeta^2}, \tag{30}$$

where ξ, η, ζ are the components of vorticity and the mean value is taken throughout each cubical volume in which the initial motion lies. The only terms which contribute to this mean value are those of the type $\left.\begin{smallmatrix}\cos^2\\ \sin^2\end{smallmatrix}\right\} lax \left.\begin{smallmatrix}\cos^2\\ \sin^2\end{smallmatrix}\right\} mby \left.\begin{smallmatrix}\cos^2\\ \sin^2\end{smallmatrix}\right\} ncz$, which have a mean value of $1/8$. The mean value of each of the "cross terms" in ξ^2, η^2, ζ^2 is zero. Now, it can be seen that the first application of the above process for finding $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$ or, in other words, the first approximation to u, v, w at time t, gives u, v, w correct to the first power of t. The second approximation gives u, v, w correct to t^2 and so on. In general the rth repetition of the process or the rth approximation gives u, v, w correct to t^r. So that after the third approximation we can at once obtain the value of $\overline{\mathrm{W}}$ correct to t^3. If, however, the expression (23) for u is examined, it will be seen that the terms whose coefficients are $\mathrm{L}_1, \ldots, \mathrm{M}_1, \ldots, \mathrm{N}_1, \ldots$ have t^3 as the lowest power of t, and, further, that any terms, besides those given in (23), which are introduced during the third and succeeding approximations will not contain powers of t less than t^3. The first eight terms in the expressions for u, v, w are therefore the only terms which contribute to $\overline{\mathrm{W}}$ when developed as far as terms in t^5. However, in order to obtain $\overline{\mathrm{W}}$ to this approximation, it is not necessary to obtain all of these eight terms correct to t^5. It is sufficient to obtain the first term of u, v, w correct to t^5, the next two terms correct to t^4, and the next five terms correct to t^3.

After the third approximation, the first eight terms are known correct to t^3, and it would appear that two further repetitions of the process are necessary in order to obtain the first three terms to the required order. It will be shown in the following pages that it is not necessary to do this owing to the fact that only the first eight terms of u, v, w as set forth in (23) make any contribution to the first term, to the order t^5, and to the next two terms, to the order t^4, however many repetitions of the process are made.*

After the third approximation, let the coefficients which are given in (29) be represented by the same letters with dashes, *e.g.*, δ_1 now becomes δ'_1. These new coefficients are then given by the following expressions:

$$\left.\begin{aligned}
\frac{d\delta'_1}{dt} &= -\frac{d\delta'_2}{dt} = \frac{1}{2}\delta_1 x_1 - x_1\left(\gamma_2 + \frac{1}{2}\beta_3\right) - \theta\nu\delta_1 \\
\frac{dx'_1}{dt} &= -\frac{dx'_2}{dt} = -\frac{1}{8}a^2\delta_1^{\,2} - a^2\delta_1\left(\frac{y_2}{4a} + \frac{\alpha_1}{12} + \frac{\gamma_3}{4}\right) + 8a^2\nu x_2
\end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned}
\frac{1}{2}\alpha'_1 = -\frac{1}{2}\alpha'_2 = -\beta'_1 = -\gamma'_1 = \beta'_2 &= \gamma'_2 \\
&= \frac{15}{22}\mathrm{AA}_1\left(\frac{1}{2}t^2 - \frac{31}{6}a^2\nu t^3\right) \\
-\beta'_3 = \gamma'_3 &= \frac{1}{2}\mathrm{AA}_1\left(\frac{1}{2}t^2 - \frac{31}{6}a^2\nu t^3\right) \\
y'_1 = y'_2 &= \frac{1}{2}\mathrm{AA}_1 a\left(\frac{1}{2}t^2 - \frac{13}{2}a^2\nu t^3\right)
\end{aligned}\right\}. \tag{32}$$

The coefficients δ_3, x_3, y_3, α_3 remain zero throughout. Substituting the expressions (29) in (31) and integrating it is found that

$$\begin{aligned}
\delta'_1 = -\delta'_2 = \mathrm{A}\Big\{1 - 3a^2\nu t &+ \frac{1}{2}t^2\left(9a^4\nu^2 - \frac{1}{16}\mathrm{A}^2a^2\right) \\
&- \frac{1}{3}t^3\left(\frac{27}{2}a^6\nu^3 + \frac{23}{32}a^4\nu\mathrm{A}^2\right) \\
+ \frac{1}{4}\mathrm{A}_1 t^4\left(\frac{\mathrm{A}^2a^2}{88} + \frac{63}{4}a^4\nu^2\right) &\quad \mathrm{A}_1 t^5\left(\frac{9}{88}a^4\nu\mathrm{A}^2 + \frac{81}{4}a^6\nu^3\right)\Big\}
\end{aligned} \tag{33}$$

* Contributions to the first three terms arise from the product terms in the equations of motion. The last three terms of (23), together with any further terms that arise during the third and succeeding approximations, either give no term of the required type when forming products with the first eight terms of (23) or if they give terms of the required type their coefficients are of a higher order of t than is required. This is true for the general case as well as in the particular case we are considering.

and

$$x'_1 = -x'_2 = A_1\left\{t - 7a^2\nu t^2 + \frac{74}{3}a^4\nu^2 t^3 - \frac{43}{44.12}A^2a^2t^3 + \frac{5.43}{44.16}A^2a^4\nu t^4 - \frac{51}{4}a^6\nu^3t^4\right\}. \quad (34)$$

In (33) powers of t above t^5 are neglected and in (34) powers of t above t^4 are neglected, but in both cases the expressions are only correct as far as t^3.

Now let the coefficients δ'_1, δ'_2, x'_1, x'_2 be denoted by δ''_1, δ''_2, x''_1, x''_2 after the fourth approximation. Expressions for these latter coefficients can then be obtained by adding a dash to each coefficient in the relations (31). If now the expressions (32), (33), and (34) be substituted in the right-hand side of the resulting equations and one integration is performed, we obtain

$$\begin{aligned}\delta''_1 = -\delta''_2 = A\Big\{1 - 3a^2\nu t &+ \frac{1}{2}t^2\left(9a^4\nu^2 - \frac{1}{16}A^2a^2\right) \\ &- \frac{1}{3}t^3\left(\frac{27}{2}a^6\nu^3 - \frac{23}{32}a^4\nu A^2\right) \\ &+ \frac{1}{4}t^4\left(\frac{31}{132}\frac{A^4a^4}{64} - \frac{185}{48}A^2a^6\nu^2 + \frac{27}{2}a^8\nu^4\right) \\ &+ \frac{1}{5}A_1t^5\,\frac{39}{44.32}A^2a^4\nu - \frac{1171}{16}a^6\nu^3\Big)\Big\},\end{aligned} \quad (35)$$

and

$$x''_1 = -x''_2 = A_1\left\{t - 7a^2\nu t^2 + \frac{74}{3}a^4\nu^2t^3 - \frac{43}{44.12}A^2a^2t^3 + \frac{43}{44}A^2a^4\nu t^4 - \frac{175}{3}a^6\nu^3t^4\right\}. \quad (36)$$

The coefficients x_1'', x_2'' are now obtained to the required order, but we must go one step further with δ_1'', δ''_2. Let the fifth approximation to these latter coefficients be noted by δ'''_1, δ'''_2. These are obtained from δ''_1, δ''_2 in exactly the same way as δ''_1, δ''_2 were obtained from δ'_1, δ'_2. The values of δ'''_1, δ'''_2, which are correct to the order t^5, are

$$\begin{aligned}\delta'''_1 = -\delta'''_2 = A\Big\{1 - 3a^2\nu t &+ \frac{1}{2}t^2\left(9a^4\nu^2 - \frac{1}{16}A^2a^2\right) \\ &- \frac{1}{3}t^3\left(\frac{27}{2}a^6\nu^3 - \frac{23}{32}a^4\nu A^2\right) \\ &+ \frac{1}{4}t^4\left(\frac{31}{132}\frac{A^2a^4}{64} - \frac{185}{48}A^2a^6\nu^2 + \frac{27}{2}a^8\nu^4\right) \\ &- \frac{1}{5}t^5\left(\frac{555}{44.32.8}A^4a^6\nu - \frac{2575}{192}A^2a^8\nu^3 + \frac{81}{8}a^{10}\nu^5\right)\Big\}.\end{aligned} \quad (37)$$

The final expressions for u, v, w which will give the value of the mean rate of dissipation of energy $\overline{W}$ correct to the order t^5 are

$$
\begin{aligned}
u = {} & \delta'''_1 \cos ax \sin ay \sin az - \frac{x''_2}{a} \sin 2ax \cos 2az \\
& + \frac{\alpha'_1}{3} \cos 3ax \sin ay \sin az + \gamma'_2 \cos ax \sin 3ay \sin az \\
& + \beta'_3 \cos ax \sin ay \sin 3az - \frac{y'_2}{a} \cos 3ax \sin ay \sin 3az
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
v = {} & \delta'''_2 \sin ax \cos ay \sin az + \frac{x_1''}{a} \sin 2ay \cos 2az \\
& + \frac{\alpha'_2}{3} \sin ax \cos 3ay \sin az + \gamma'_3 \sin ax \cos ay \sin 3az \\
& + \beta'_1 \sin 3ax \cos ay \sin az + \frac{y'_1}{a} \sin ax \cos 3ay \sin 3az,
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
w = {} & \frac{x''_2}{a} \cos 2ax \sin 2az - \frac{x''_1}{a} \cos 2ay \sin 2az \\
& + \gamma'_1 \sin 3ax \sin ay \cos az + \beta'_2 \sin ax \sin 3ay \cos az \\
& + \frac{y'_2}{a} \sin 3ax \sin ay \cos 3az - \frac{y'_1}{a} \sin ax \sin 3ay \cos 3az,
\end{aligned} \tag{40}
$$

the coefficients being given by (32), (36), and (37).

Results in Non-Dimensional Form

It is now convenient to express the velocity components in a non-dimensional form. Put

$$T = Aat \qquad R = A/a\nu, \tag{41}$$

then R is a Reynolds number and

$$
\left.
\begin{aligned}
\frac{1}{2}\alpha'_1 = -\frac{1}{2}\alpha'_2 = -\beta'_1 = -\gamma'_1 = \beta'_2 = \gamma'_2 & \\
= -\frac{15}{176} A \left(\frac{1}{2} T^2 - \frac{31}{6}\frac{T^3}{R}\right) & \\
-\beta'_3 = \gamma'_3 = -\frac{1}{16} A \left(\frac{1}{2} T^2 - \frac{31}{6}\frac{T^3}{R}\right) & \\
y'_1 = y'_2 = -\frac{1}{16} Aa \left(\frac{1}{2} T^2 - \frac{13}{2}\frac{T^3}{R}\right) &
\end{aligned}
\right\} \tag{42}
$$

$$
\begin{aligned}
x''_1 = -x''_2 = -\frac{1}{8} Aa \Big\{ & T - \frac{7T^2}{R} + \left(\frac{74}{3R^2} - \frac{43}{44 \,.\, 12}\right) T^3 \\
& - \left(\frac{175}{3R^2} - \frac{43}{44}\right)\frac{T^4}{R} \Big\},
\end{aligned} \tag{43}
$$

and

$$\delta'''_1 = -\delta'''_2 = A\left\{1 - \frac{3T}{R} + \left(\frac{9}{R^2} - \frac{1}{16}\right)\frac{T^2}{2} - \left(\frac{27}{2R^2} - \frac{23}{32}\right)\frac{T^3}{3R}\right.$$
$$+ \left(\frac{31}{132 \, . \, 64} - \frac{185}{48R^2} + \frac{27}{2R^4}\right)\frac{T^4}{4}$$
$$\left. - \left(\frac{555}{44 \, . \, 32 \, . \, 8} - \frac{2575}{192R^2} + \frac{81}{8R^4}\right)\frac{T^5}{5R}\right\}. \qquad (44)$$

The formula (1) is true for isotropic turbulence. In our present problem, where the turbulence is not isotropic, it seems reasonable to replace $\overline{u^2}$, which is the mean value of the square of one component of velocity in the case where all three are equal, by

$$\overline{u_1^2} = \tfrac{1}{3}(\overline{u^2} + \overline{v^2} + \overline{w^2}). \qquad (45)$$

Using expressions (38), (39), and (40) for u, v, w where the coefficients of the terms are given by (42), (43), and (44), we obtain, after some calculation,

$$\frac{\overline{W}}{\mu} = \frac{3A^2a^2}{4} W', \qquad (46)$$

where

$$W' = 1 - \frac{6T}{R} + \left(\frac{5}{48} + \frac{18}{R^2}\right)T^2 - \left(\frac{5}{3} + \frac{36}{R^2}\right)\frac{T^3}{R}$$
$$+ \left(\frac{50}{99 \, . \, 64} + \frac{1835}{9 \, . \, 16R^2} + \frac{54}{R^4}\right)T^4$$
$$- \left(\frac{361}{44 \, . \, 32} + \frac{761}{12R^2} + \frac{324}{5R^4}\right)\frac{T^5}{R}. \qquad (47)$$

The expression for W' is correct to T^5. In the same way the value of $\overline{u_1^2}$ can be obtained correct to the order T^5. It is, however, possible to obtain $\overline{u_1^2}$ correct to T^6 by making use of the energy equation

$$\overline{W} = -\frac{1}{2}\rho \frac{d\overline{u_1^2}}{dt}$$

which can be expressed as

$$\frac{\overline{W}}{\mu} = -\frac{3}{2}a^2 R \frac{d\overline{u_1^2}}{dT}. \qquad (48)$$

From (46), (47), and (48) the value of $\overline{u_1^2}$ correct to T^6 is given by

$$\overline{u_1^2} = \frac{A^2}{12} u'^2, \qquad (49)$$

where

$$u'^2 = 1 - \frac{6T}{R} + \frac{18T^2}{R^2} - \left(\frac{5}{24} + \frac{36}{R^2}\right)\frac{T^3}{R} + \left(\frac{5}{2R^2} + \frac{54}{R^4}\right)T^4$$
$$- \left(\frac{5}{44\,.\,12} + \frac{367}{24R^2} + \frac{4\,.\,81}{5R^4}\right)\frac{T^5}{R}$$
$$+ \left(\frac{361}{44\,.\,32} + \frac{761}{12R^2} + \frac{324}{5R^4}\right)\frac{T^6}{R^2}. \tag{50}$$

A check on the work is provided by the fact that the first five terms in (50) agree with those obtained by direct calculation of $\overline{u_1^2}$. Also for a non-viscous fluid where $1/R = 0$ it is at once verified that $\overline{u_1^2}$ remains constant for all time.

Numerical Discussion

The equations (47) and (50) contain the information which is required. They express, in fact, the way in which $\overline{W}$ and $\overline{u_1^2}$ change with time when the eddies represented by (12) are started at time $T = 0$. Since, however, (47) and (50) contain only terms up to T^5 and T^6 respectively, they cease to be valid representations of the flow when T is too great. In the following tables the terms in (47) and (50) have been calculated for increasing values of T for the following values of R:—20, 50, 100, 200, 300.

The values of W′ and u'^2, so far as they are represented by the approximations (47) and (50), are given in Tables I–V. These tables are not carried beyond the values of T for which the approximation is likely to be reasonably good, but calculations were made using the approximate expressions (47) and (50) beyond the limit of T, at which they can be regarded as reasonably accurate, because it is thought that the results may have at least a qualitative interest.

In fig. 1 the values of W′ for increasing values of T are shown by means of five curves, one for each of the chosen values of R. Each curve is marked with a full line as far as the terms are given in Tables I–V and a broken line for the higher values of T, for which the approximation of (47) and (50) may be taken as of qualitative value only.

Initially W′ begins to decrease, but when $R \geqslant 100$ this decrease is very small and W′ subsequently increases. The broken line indicates that at a time which depends on R, W′ reaches a maximum and afterwards begins to decrease. This decrease is, no doubt, due to the fact that u'^2, *i.e.*, the total energy of the eddies, is decreasing continually from the start, so that even though λ, the scale of the smallest eddies, may be

TABLE I*

$R = 300$

$$W' = 1 - 0{\cdot}02T + 0{\cdot}10436T^2 - 0{\cdot}0055568T^3 + 0{\cdot}0080329T^4 - 0{\cdot}00085702T^5$$

$$u'^2 = 1 - 0{\cdot}02T + 0{\cdot}0002T^2 - 0{\cdot}0006957T^3 + 0{\cdot}000027784T^4 - 0{\cdot}000032132T^5 + 0{\cdot}0000028567T^6$$

T	Values of individual terms					Total
	0·02T	T^2	T^3	T^4	T^5	W′
0·5	0·01	0·02609	0·0007	0·0005	—	1·0159
1·0	0·02	0·1043	0·0055	0·0080	0·0008	1·0859
1·5	0·03	0·2348	0·0187	0·0407	0·0065	1·2202
2·0	0·04	0·4175	0·0444	0·1285	0·0274	1·4341
2·5	0·05	0·6523	0·0868	0·3138	0·0837	1·7455
3·0	0·06	0·9393	0·1500	0·6507	0·2082	2·1717
3·5	0·07	1·2785	0·2382	1·2055	0·4501	2·7256

T	0·02T	T^2	T^3	T^4	T^5	T^6*	u'^2	$\lambda a/\pi$
0·5	0·01	—	—	—	—	—	0·9899	0·4055
1·0	0·02	0·0002	0·0007	—	—	—	0·9795	0·3905
1·5	0·03	0·0004	0·0023	0·0001	0·0002	—	0·9680	0·3660
2·0	0·04	0·0008	0·0055	0·0004	0·0010	0·0002	0·9548	0·3352
2·5	0·05	0·0012	0·0109	0·0011	0·0031	0·0007	0·9390	0·3015
3·0	0·06	0·0018	0·0189	0·0022	0·0078	0·0021	0·9195	0·2675
4·0	0·08	0·0032	0·0445	0·0071	0·0329	0·0117	0·8645	0·2068
5·0	0·10	0·0050	0·0869	0·0173	0·1005	0·0446	0·7795	0·1597
6·0	0·12	0·0072	0·1502	0·0360	0·2500	0·1333	0·6563	0·1242

* The coefficients of T^2 to T^6 at the head of the columns of figures are omitted. The upper set of figures are the values of the terms of W′ for various values of T; the lower set of figures are the values of the terms of u'^2.

decreasing or constant $W = 15\mu\overline{u^2}/\lambda^2$ decreases. It will be seen in fig. 1 that if $R < 50$, W′ is never greater than its initial value 1·0.

COMPARISON BETWEEN CALCULATED λ AND THAT OBSERVED IN TURBULENT FLOW

The value of λ may be calculated from (1), but in order to express it in a non-dimensional form so that it may be comparable with that which

TABLE II

R = 200

$$W' = 1 - 0{\cdot}03T + 0{\cdot}104616T^2 - 0{\cdot}00833783T^3 + 0{\cdot}0082099T^4 - 0{\cdot}0012899T^5$$

$$u'^2 = 1 - 0{\cdot}03T + 0{\cdot}00045T^2 - 0{\cdot}00104616T^3 + 0{\cdot}0000625337T^4 - 0{\cdot}00004926T^5 + 0{\cdot}0000064496T^6$$

T	0·03T	T^2	T^3	T^4	T^5	W′
0·2	0·006	0·0042	0·0001	—	—	0·9975
0·4	0·012	0·0167	0·0005	0·0002	—	1·0044
1·0	0·030	0·1046	0·0083	0·0082	0·0013	1·0732
1·5	0·045	0·2354	0·0281	0·0415	0·0098	1·1940
2·0	0·060	0·4185	0·0667	0·1313	0·0413	1·3819
2·5	0·075	0·6539	0·1303	0·3207	0·1260	1·6433
3·0	0·090	0·9416	0·2251	0·6650	0·3134	1·9780

T	0·03T	T^2	T^3	T^4	T^5	T^6	u'^2	$\lambda a/\pi$
1·0	0·030	0·0004	0·0010	0·0001	—	—	0·9694	0·3910
1·5	0·045	0·0010	0·0035	0·0003	—	—	0·9527	0·3671
2·0	0·060	0·0018	0·0084	0·0010	0·0015	0·0004	0·9333	0·3379
2·5	0·075	0·0028	0·0163	0·0024	0·0048	0·0016	0·9107	0·3060
3·0	0·090	0·0041	0·0282	0·0051	0·0119	0·0047	0·8836	0·2748
4·0	0·120	0·0072	0·0669	0·0160	0·0504	0·0264	0·8122	0·2214
5·0	0·150	0·0112	0·1308	0·0391	0·1540	0·1008	0·7163	0·1854

occurs in the turbulence produced by a grid of regular bars of mesh-length M, for which λ has been measured, it is necessary to consider how the wave-length of the periodic disturbance (12) may be expected to compare with the mesh-length M of the grid which produces the turbulence. One obvious standard of comparison is the distance apart of two points where the correlation between the velocities is zero. In the case of the periodic eddies (12), this is obviously one-quarter of the wave-length, *i.e.*, $\pi/2a$, while in the case of the turbulence produced by a regular grid of mesh-length M it is $\frac{1}{2}$M* Thus we may take

$$\frac{\pi}{2a} = \frac{M}{2} \text{ or } M = \pi/a, \tag{51}$$

* *See* (4), p. 500.

TABLE III

$R = 100$

$$W' = 1 - 0{\cdot}06T + 0{\cdot}10596T^2 - 0{\cdot}0167T^3 + 0{\cdot}009166T^4 - 0{\cdot}002627T^5$$

$$u'^2 = 1 - 0{\cdot}06T + 0{\cdot}0018T^2 - 0{\cdot}002119T^3 + 0{\cdot}0002505T^4 - 0{\cdot}00011T^5 + 0{\cdot}00002627T^6$$

T	0·06T	T^2	T^3	T^4	T^5	W'
0·6	0·036	0·0381	0·0036	0·0012	0·0002	0·9995
1·0	0·060	0·1059	0·0167	0·0091	0·0026	1·0358
1·5	0·090	0·2384	0·0564	0·0464	0·0199	1·1185
2·0	0·120	0·4238	0·1336	0·1467	0·0841	1·2328
2·5	0·150	0·6622	0·2610	0·3581	0·2566	1·3527
3·1	0·186	1·0181	0·4976	0·8465	0·7522	1·4288

T	0·06T	T^2	T^3	T^4	T^5	T^6	u'^2	$\lambda a/\pi$
1·0	0·060	0·0018	0·0022	0·0003	0·0001	—	0·9399	0·3915
1·5	0·090	0·0041	0·0071	0·0013	0·0008	0·0003	0·9077	0·3705
2·0	0·120	0·0072	0·0168	0·0040	0·0035	0·0017	0·8726	0·3460
2·5	0·150	0·0113	0·0328	0·0098	0·0107	0·0064	0·8339	0·3228
3·1	0·186	0·0173	0·0626	0·0231	0·0315	0·0233	0·7836	0·3043
3·6	0·216	0·0233	0·0980	0·0421	0·0665	0·0572	0·7421	0·3072
4·0	0·240	0·0288	0·1344	0·0641	0·1126	0·1076	0·7134	0·3405

TABLE IV

$$W' = 1 - 0{\cdot}12T + 0{\cdot}11136T^2 - 0{\cdot}033621T^3 + 0{\cdot}012997T^4 - 0{\cdot}0056355T^5$$

$$u'^2 = 1 - 0{\cdot}12T + 0{\cdot}0072T^2 - 0{\cdot}0044546T^3 + 0{\cdot}0010086T^4 - 0{\cdot}00031193T^5 + 0{\cdot}00011271T^6$$

T	0·12T	T^2	T^3	T^4	T^5	W'
0·6	0·072	0·0401	0·0073	0·0017	0·0004	0·9621
1·0	0·120	0·1113	0·0336	0·0130	0·0056	0·9651
1·5	0·180	0·2505	0·1135	0·0658	0·0428	0·9801
1·6	0·192	0·2851	0·1377	0·0852	0·0591	0·9815
2·0	0·240	0·4454	0·2690	0·2080	0·1803	0·9641

T	0·12T	T^2	T^3	T^4	T^5	T^6	u'^2	$\lambda a/\pi$
0·6	0·072	0·0026	0·0009	0·0001	—	—	0·9298	0·4040
1·0	0·120	0·0072	0·0044	0·0010	0·0003	0·0001	0·8837	0·3930
1·5	0·180	0·0162	0·0150	0·0051	0·0024	0·0013	0·8252	0·3770
1·6	0·192	0·0184	0·0182	0·0066	0·0032	0·0019	0·8135	0·3743
2·0	0·240	0·0288	0·0356	0·0161	0·0100	0·0072	0·7665	0·3665
2·4	0·284	0·0415	0·0616	0·0335	0·0248	0·0215	0·7261	0·3745

Table V

$R = 20$

$$W' = 1 - 0\cdot 3T + 0\cdot 14916T^2 - 0\cdot 08783T^3 + 0\cdot 040086T^4 - 0\cdot 020767T^5$$

T	0·3T	T^2	T^3	T^4	T^5	W′
0·2	0·06	0·0059	0·0007	—	—	0·9453
0·6	0·18	0·0537	0·0189	0·0052	0·0016	0·8583
1·0	0·30	0·1492	0·0878	0·0401	0·0207	0·7806
1·5	0·45	0·3356	0·2963	0·2028	0·1577	0·6344

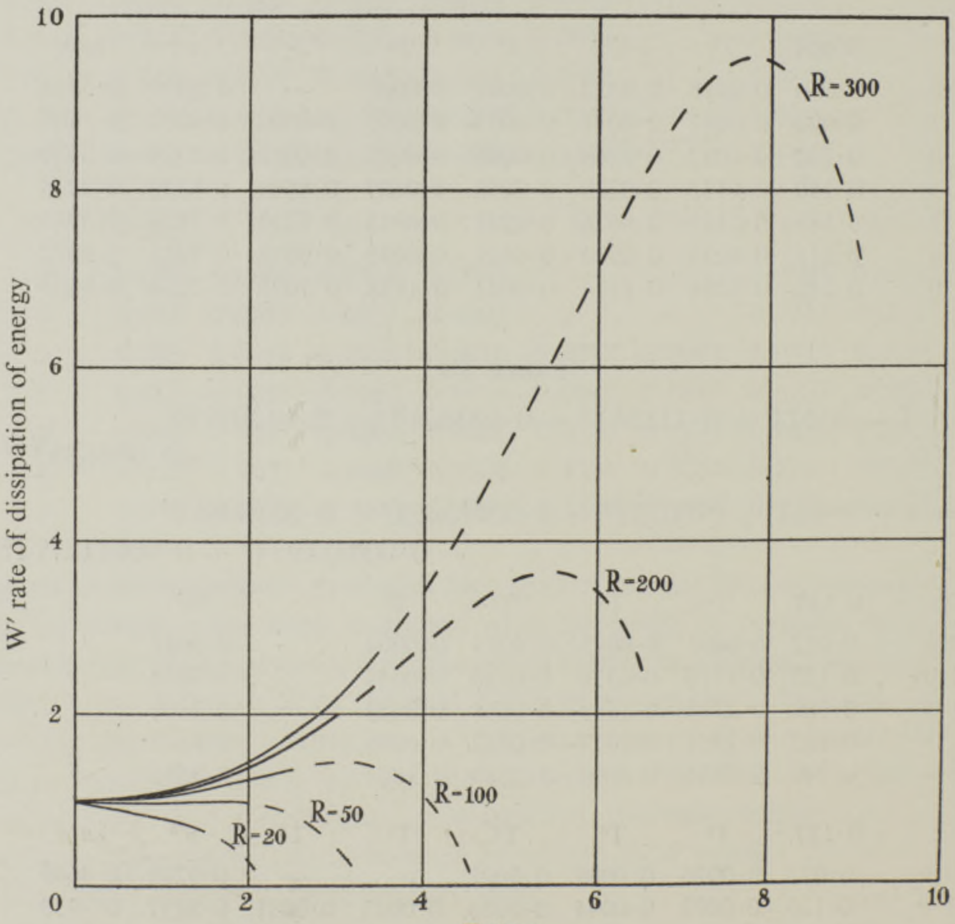

Fig. 1.

and we may therefore compare λ/M measured in a wind tunnel with the calculated values of $\lambda a/\pi$.

The calculated values of $\lambda a/\pi$ are given in Tables I–IV, and they are shown graphically in fig. 2. The initial value of $\lambda a/\pi$ is $0\cdot41$ whatever the value of R may be. For very low values of R the decay of energy would be exponential,* corresponding to the constant value of

$$\lambda a/\pi = 0\cdot41.$$

For larger values of R, $\lambda a/\pi$ decreases with the time, but the indications of the broken lines, particularly in the cases of R = 50 and R = 100, seem to suggest that $\lambda a/\pi$ would not decrease indefinitely.

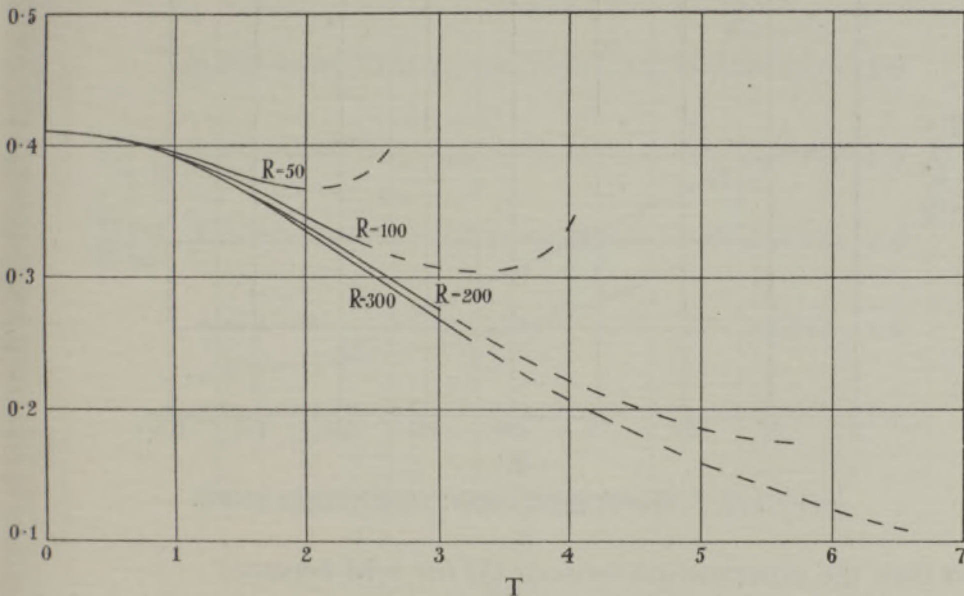

FIG. 2.

For comparison with observation, it must be remembered that λ/M has been found experimentally to be $2\cdot0\sqrt{\dfrac{\nu}{M\sqrt{\overline{u^2}}}}$. To compare the observed rate of dissipation with that calculated, it is necessary to compare λ/M with $\lambda a/\pi$ when the total energy of the turbulence is the same in the two cases, *i.e.*, $\overline{u^2}$ in the wind tunnel must be taken as $\overline{u^2}_1 = \dfrac{A^2}{12}u'^2$.† Then, since M is comparable with π/a and $R = A/a\nu$, we must compare

* This is easily shown by using (5) with the energy equation.
† *See* (45) and (49).

$\dfrac{\nu}{M\sqrt{\overline{u^2}}}$ with $\dfrac{\sqrt{12}}{\pi R u'}$. The ordinates of the curve (fig. 3) in which the comparison between observation and the present calculations is made may be taken as

$$\left.\begin{aligned} R' &= \frac{\pi R}{\sqrt{12}}\, u' \text{ for the calculated points} \\ \text{or} \qquad & \\ R' &= \frac{M\sqrt{\overline{u^2}}}{\nu} \text{ for the observed points} \end{aligned}\right\} \tag{52}$$

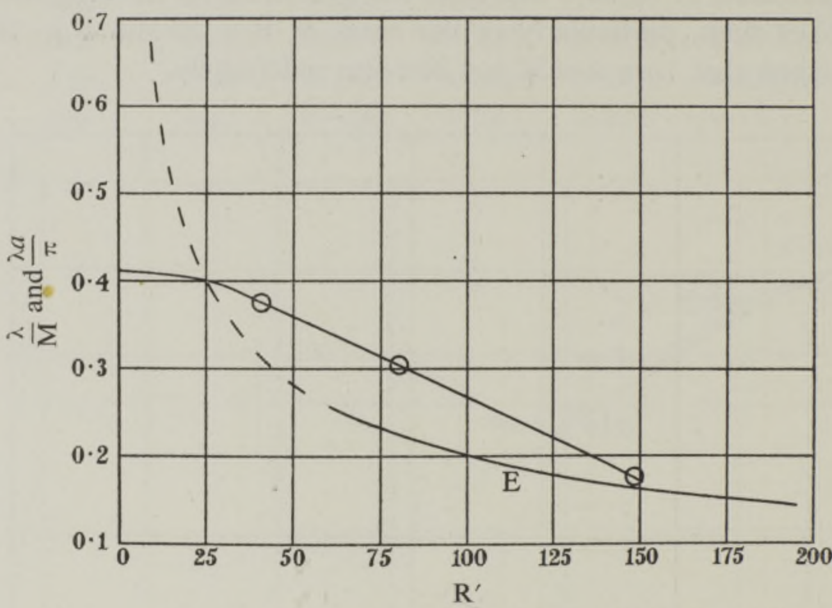

FIG. 3—E, experimental curve; O, calculated points.

and then the experimental formula (7) for λ/M becomes

$$\frac{\lambda}{M} = 2\cdot 0 \sqrt{\frac{\nu}{M\sqrt{\overline{u_1^2}}}} = 2\cdot 0\,(R')^{-\frac{1}{2}}.$$

The calculated values of $\lambda a/\pi$ corresponding to the maximum $\overline{W}$ are shown in fig. 3. It will be seen that provided $R' > 25$ the calculated values of $\lambda a/\pi$ are in good agreement, so far as order of magnitude is concerned, with the formula deduced from observation.

Lower Limit of Application of (7)

In the work on dissipation it was suggested that formula (7) would only hold provided $\dfrac{M\sqrt{\overline{u^2}}}{\nu}$ is less than 60. The corresponding value of R' is

60, and the part of the observed curve which falls below this is marked by a broken line.

Upper Limit of R for Exponential Decay of Turbulence

The lower limit below which formula (7) may be expected to hold is necessarily above the highest value of R for which λ does not appreciably decrease after the motion is started, *i.e.*, it is above the range for which the decay is exponential.

To find this limit the rate of decay may be calculated on the assumption that $\lambda a/\pi$ has its initial value 0·41. If $W'_{\text{max.}}$ is the maximum value of

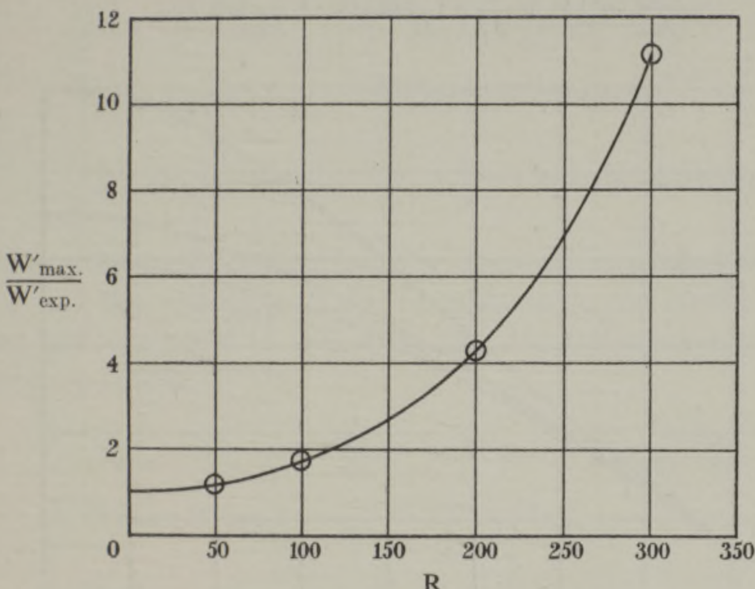

Fig. 4.

W', as shown in fig. 1, we may calculate $W'_{\text{max.}}/W'_{\text{exp.}}$ where $W'_{\text{exp.}}$ is the value which W' would have, assuming exponential decrement at the time corresponding with $W'_{\text{max.}}$. The increase of $W'_{\text{max.}}/W'_{\text{exp.}}$ above its value for $R \to 0$, namely 1·0, then represents, qualitatively at any rate, the effect we are discussing, namely the decrease in λ due to increase in the mean square of the vorticity.

The values of $W'_{\text{max.}}/W'_{\text{exp.}}$ are shown in fig. 4. It will be seen that up to $R = 50$ the effect is very small, the increase being only 19% at $R = 50$. Above $R = 50$ the effect rapidly increases.

Local Increase in Vorticity

The progressive increase in the mean square of the vorticity which the present investigation reveals is due to comparatively large local increases in vorticity in certain parts of the field. One region where the vorticity increases is along the axis $y = 0$, $z = 0$. Taking the case when $1/R = 0$, *i.e.*, the fluid is non-viscous, we may take the component ξ where $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$.

If we use the formula correct to T^3, then

$$\frac{\xi}{Aa} = \left(1 + \frac{3}{32}T^2\right)\sin ax - \frac{T^2}{32}\sin 3ax.$$

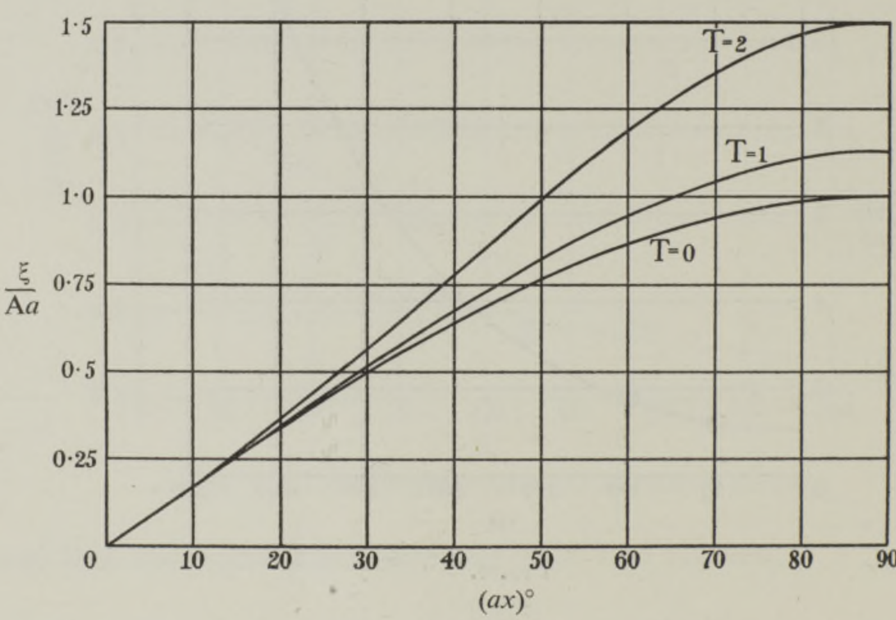

Fig. 5.

The values of ξ/Aa are shown in fig. 5 as functions of ax for various values of T. It will be seen that at $ax = \frac{1}{2}\pi$ the value of ξ/Aa increases 50% between $T = 0$ and $T = 2$.

APPENDIX

In the text only those terms necessary for calculation of $\overline{W}$ and $\overline{u_1^2}$ are given.

The complete expressions for u, v, w for the particular case considered were, however, calculated correct to T^3. They are:

$$
\begin{aligned}
u = {} & \delta'''_1 \cos ax \sin ay \sin az + \frac{x''_3}{a} \sin 2ax \cos 2ay - \frac{x''_2}{a} \sin 2ax \cos 2az \\
& + \frac{\alpha'_1}{3} \cos 3ax \sin ay \sin az + \gamma'_2 \cos ax \sin 3ay \sin az \\
& + \beta'_3 \cos ax \sin ay \sin 3az + \frac{y'_3}{a} \cos 3ax \sin 3ay \sin az \\
& - \frac{y'_2}{a} \cos 3ax \sin ay \sin 3az + \frac{L'_1}{2} \sin 4ax \cos 2ay \cos 2az \\
& + N'_2 \sin 2ax \cos 4ay \cos 2az + M'_3 \sin 2ax \cos 2ay \cos 4az \\
& + z_1 \sin 2ax \cos 2ay \cos 2az + \frac{u_3}{a} \sin 4ax \cos 4ay \\
& - \frac{u_2}{a} \sin 4ax \cos 4az + \frac{2G_3}{a} \sin 2ax \cos 4ay - \frac{G_2}{a} \sin 4ax \cos 2az \\
& + \frac{H_3}{a} \sin 4ax \cos 2ay - \frac{2H_2}{a} \sin 2ax \cos 4az \\
& + \frac{J_3}{a} \sin 4ax \cos 4ay \cos 2az - \frac{J_2}{a} \sin 4ax \cos 2ay \cos 4az,
\end{aligned}
$$

with two similar expressions for v, w obtained by cyclic permutation of x, y, z and the suffices 1, 2, 3. In addition to the coefficients already defined and evaluated in (24), (29), etc., the following values complete the solution correct to T^3:

$$
\begin{gathered}
L'_1 = L'_2 = -\frac{AT^3}{99}; \qquad M'_1 = N'_2 = \frac{49AT^3}{66\,.\,24\,.\,8}; \\
M'_2 = N'_1 = \frac{79AT^3}{66\,.\,24\,.\,8}; \\
\frac{1}{2}L'_3 = -M'_3 - N'_3 = -\frac{AT^3}{96}; \\
z_1 = z_2 = -\frac{1}{2}z_3 = \frac{19AT^3}{396\,.\,8} \\
u_1 = -u_2 = -\frac{AaT^3}{192}; \qquad u_3 = 0 \\
H_1 = -G_2 = \frac{27AaT^3}{440\,.\,8}; \\
H_2 = H_3 = -G_1 = -G_3 = \frac{AaT^3}{384}; \\
J_1 = -J_2 = -\frac{AaT^3}{384}; \qquad J_3 = 0.
\end{gathered}
$$