Combining the fundamental eqns of fluid mechanics and statistical relations gives equations which describe the development of statistical relations in a flow which obeys fundamental equations.

6 The basics is the NS Eqns where a translational flow over a flat plate is presumed i.e. $\bar{u} = \text{const}$, $\bar{v} = \bar{w} = 0$, $\nabla \bar{u_i} = 0$.

↓

NS − RANS = Eqn. 6.1. Pg 117

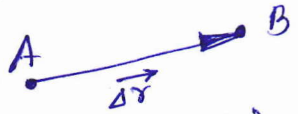
↓

One writes Eqn 6.1 for a point A in flow field and multiplies the eqn throughout by fluctuations velocities at another point B i.e. Fluctuations at B are correlated to the behaviour of fluid at A. ⟹ Eqn 6.2 Pg 117.

When differentiating eqn 6.2, the terms representing fluctuating velocities at B are ~~taken~~ observed as constants.

Same is done with respect to point B: Eqn 6.1 written for B and multiplied throughout by fluctuating velocities at A.

⟹ Eqn 6.3 Pg 118.



Quantities at B depend on $\vec{\Delta r}$. At A, they are independent of $\vec{\Delta r}$.

↓

(6.2) Pg 117 + (6.3) Pg 118 and then averaging gives

Eqn 6.4 Pg 118.

where $Q_{ij} = \overline{u_i'(\vec{r})\, u_j'(\vec{r}+\vec{\Delta r})}$ = Euler 2 point correlation tensor.

$$\left.\begin{aligned} S_{ik,j} &= \overline{u_i'(\vec{r})\, u_j'(\vec{r}+\vec{\Delta r})\, u_k'(\vec{r})} \\ S_{i,kj} &= \overline{u_i'(\vec{r}-\vec{\Delta r})\, u_j'(\vec{r})\, u_k'(\vec{r})} \end{aligned}\right\} \text{Euler triple correlation tensor.}$$

Essential summary: In order to find $\frac{\partial Q_{ij}}{\partial t}$, (i.e. to solve eqn 6.4) which is the ~~timely~~ temporal development of double-correlation, a description of the triple product is required. In order to know triple correlation one needs ~~to~~ ~~know~~ a description for quadruple-correlation and so on.

So always we will have more unknowns than equations → Closure problem.

Hence to find a simplification for (6.4), assumptions /simplifications for following terms must be found:

- 1st order correlations tensor $\overline{p'u'}$ (pressure-velocity correlation)
- 2nd order " " $(Q_{ij}')_{AB}$
- 3rd " triple corr. " $S_{ik,j}$

<u>Properties of isotropic Turbulence</u>

- $\overline{p'(A)\, u_j'(B)} = 0$. So isotropic turbulence has no diffusion due to pressure forces.
- $(Q_{ij})_{AB}$ is symmetric & has 6 elements. Because isotropic turbulence must be rotationally invariant, mixed correlat cannot occur. Correlation coefficients $f(\Delta x_i)$ and $g(\Delta x_i)$ formulated.
- $S_{ik,j}$ Correlation coeffs $k(\Delta x_i)$, $h(\Delta x_i)$ and $q(\Delta x_i)$ formulated.

⋮

$$\left[Eq\ 6.4\right] \xrightarrow[\text{of isotropic turbulence}]{\text{Assumptions}} \text{Give eqns } (6.23)\ ;\ (6.24)$$

Introduction of coefficients ~~f(Δ~~ f, g for double correlat. ~~k~~ h, k, q for triple correlation

↓ gives

$\left[Eqn\ 6.25\right]$

von Kármán - Howarth - Gleichung:

- vKHG ~~Eqn~~ [$\underbrace{\text{von Kármán - Howarth - Gleichung}}_{\text{Eqn (6.25)}}$] represents a new simplified relationship for the study of "Development of statistical relations in isotropic turbulence". However, inspite of simplifying assumptions of isotropic turbulence, an analytical solution is not possible, because the relationship contains 2 unknowns and hence it is not closed (closure problem). However if vKHG is transformed into spectral space, then we can gain some fundamental insight into dynamics of isotropic turbulence.

So. vKHG $\xrightarrow{\text{Fourier transformation}}$ Spektralgleichung (spectral eqn)
→ describes the development of Energy spectrum w.r.t time.

$$\Rightarrow \begin{pmatrix} \text{Eqn } 6.25 \\ \text{Pg-124} \end{pmatrix} \xrightarrow[\text{Transform}]{\text{Fourier}} \frac{\partial}{\partial t} E(k,t) = F(k,t) - 2\nu k^2 E(k,t)$$

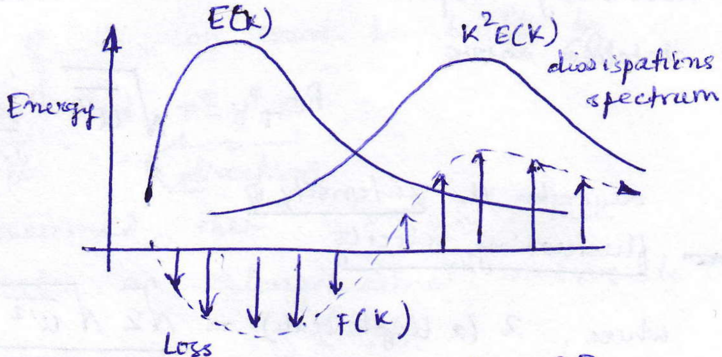
Now,

$$\frac{\partial}{\partial t} E(k,t) = \underbrace{F(k,t)} - \underbrace{2\nu k^2 E(k,t)}$$

⊙ Describes the specific dissipations rate "ε"
where ε can be defined as $\varepsilon = 2\nu \int_0^\infty k^2 E(k) dk$

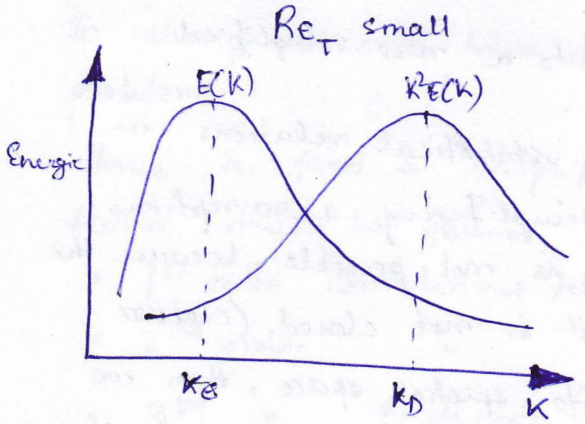
⊙ The term $k^2 . E(k)$ is described as "Dissipations spectrum".

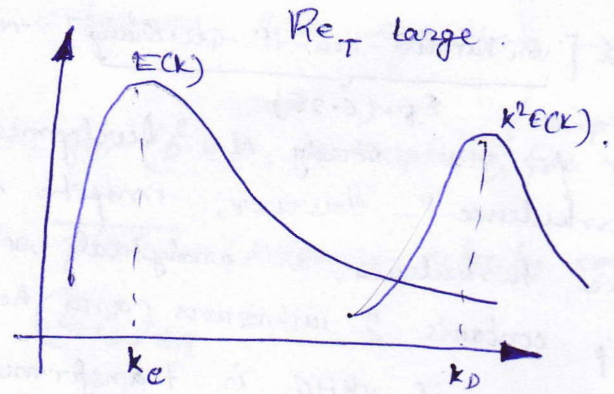
The model function $F(k,t)$ describes the transport of energy along the k-axis i.e. the transport of energy from small wave numbers (large structures) to the big wave numbers (small structures) and this is the "Energy Cascade".
This describes the steady decay of large eddies to smaller eddies



Model function F(k) when $\frac{\partial E}{\partial t} < 0$

This diagram shows that spectral Energy is being lost with the progress of time.





In the above diagrams, two wavenumbers K_e and K_D are ~~def~~ shown.

K_E : is the wavenumber at which the spectral density $E(K)$ has its maximum. Hence K_e is defined as the wavenumber of the "energy carrying eddy". The corresponding "structure length", i.e., the size of the eddy which carries the largest share of turbulent energy is given by $\boxed{L = \frac{1}{K_e}}$

K_D : is the wavenumber at which the dissipations spectrum $K^2E(K)$ has its maximum. Hence K_D is defined as the wavenumber of the "energy dissipating eddy".

The relationship between Energy spectrum & Dissipations spectrum is essentialy dependant on the turbulent Reynolds number Re_T, as sketched above.

$$Re_T = \sqrt{\overline{u'^2}}\,\frac{\lambda_g}{\nu}$$

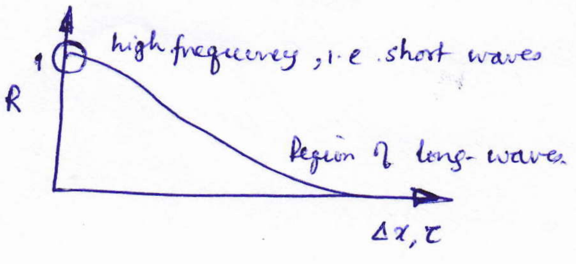
$\sqrt{\overline{u'^2}}$ describes the intensity of fluctuating velocity

λ_g describes dissipation length scale

ν kinematic viscosity.

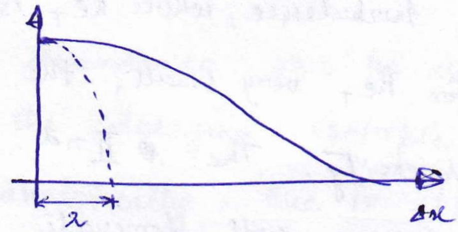
where, λ (a length scale) $= \dfrac{\sqrt{2}\,\sqrt{\overline{u'^2}}}{\sqrt{\overline{\left(\dfrac{\partial u(x)}{\partial x}\right)^2}}\Bigg|_{\Delta x=0}}$.

λ describes the "micro scale" i.e. length scale describing the size of the smallest eddies. This information is contained in the following Correlations curve.



It is known that the smallest eddy has the shortest life span. Hence in order to find information about the smallest eddies (i.e. the fluctuations with the highest frequencies), one must look at the region of the curve corresponding to smallest values of Δx or Time τ. This region ~~is~~ happens to be the <u>Peak area</u> ("Scheitelbereich") of the correlation curve.

<u>It can be shown that the correlations curve $R(\Delta x)$ is a parabolic function of Δx for very small values ~~for~~ of Δx.</u> This parabola has a very large curvature at $\Delta x = 0$ and if one continues to draw this parabola with ~~high~~ large curvature, then it intersects the Abscissa of the correlation curve. This intersection gives the "λ". As the curvature is large, so "λ" is small.



Now, recalling that λ is given by
$$\lambda = \frac{\sqrt{2}\sqrt{\overline{u'^2}}}{\sqrt{\overline{\left(\frac{\partial u'(x)}{\partial x}\right)^2}}}\Bigg|_{\Delta x = 0}$$

we realize that 9 different types of λ can ~~exist~~ be built by using the combination of $\underbrace{u', v', w'}_{\text{velocity fl.}}$ and $\underbrace{x, y, z}_{\text{directions}}$

Now, if isotropic turbulence is assumed, then $u' = v' = w'$. Then we can distinguish the microscales as "longitudinal" microscale (λ_f) and "latitudinal" microscale (λ_g).

$$\lambda_f = \frac{\sqrt{2}\sqrt{\overline{u'^2}}}{\sqrt{\overline{\left(\frac{\partial u'\cancel{(xy)}}{\partial x}\right)^2}}}$$

$$\lambda_g = \frac{\sqrt{2}\sqrt{\overline{u'^2}}}{\sqrt{\overline{\left(\frac{\partial u'}{\partial y}\right)^2}}}$$

The microscale used to define the turbulent Reynolds number Re_T.

Generally, λ can be interpreted as the average dimension of the smallest (dissipating) eddies in the flow. Hence such length scales are mentioned as "micro-structure length" or "dissipating length".

When Re_T is large, then there are typically many orders of magnitude between K_e and K_D. Hence the wavenumber of the energy dissipating eddy (K_D) becomes ~~smaller~~ bigger and ~~smaller~~ bigger as Re_T becomes bigger and bigger. As K_D (wavenumber) $\uparrow$ so frequency $\uparrow$ i.e. size of smallest eddies become smaller and smaller. This means that for methods like LES and DNS, one has to keep refining the mesh to capture turbulence energy. Hence as $Re_T \uparrow$ so computational effort $\uparrow$.

In contrast, as $Re_T \downarrow$ so K_D also $\downarrow$ and hence the separation between K_e and K_D decreases.

6.4 The spectrum due to small Re_T, Final stage of decay

We observe an extreme case of turbulence, where Re_T is very very small. As $Re = \frac{\text{Inertial}}{\text{Viscous}}$ forces, so for Re_T very small, the flow will be ~~described~~ decided through viscosity. The II^{nd} term in the eqn for temporal development of energy will dominate

$$\frac{\partial E(k,t)}{\partial t} = \underbrace{F(k,t)}_{\text{Term I}} - \underbrace{2\nu k^2 E(k,t)}_{\text{Term II}}$$

Hence

$$\frac{\partial}{\partial t} E(k,t) = -2\nu k^2 E(k,t) \qquad (6.28)$$

It follows immediately that the borderline case of Re_T being very small is decided through the fact that no transport of energy along k-axis takes place. i.e. $F(k,t)=0$. In other words, the decaying turbulence shall reach a state, in which the energy cascade comes to a "Standstill" and no further decay of larger to smaller eddies take place.

The solution of eqn (6.28) is then:

$$E(k,t) = E(k,t_0)\exp\left[-2\nu k^2 (t-t_0)\right] \qquad (6.29)$$

The physical interpretation of (6.29) is that, smaller eddies dissipate faster than bigger eddies.

Eq (6.29) applies not only for isotropic, but more generally for homogeneous turbulence, where the energy at a certain wavenumber is dissipated exponentially with time. So larger the wavenumber k, faster will the E(K) subside with time t.

After a sufficiently long time in the so called "end-stage" or the "final stage of decay", the shape of the energy spectrum is determined exclusively by the energy at small wave numbers. (i.e. large eddies)

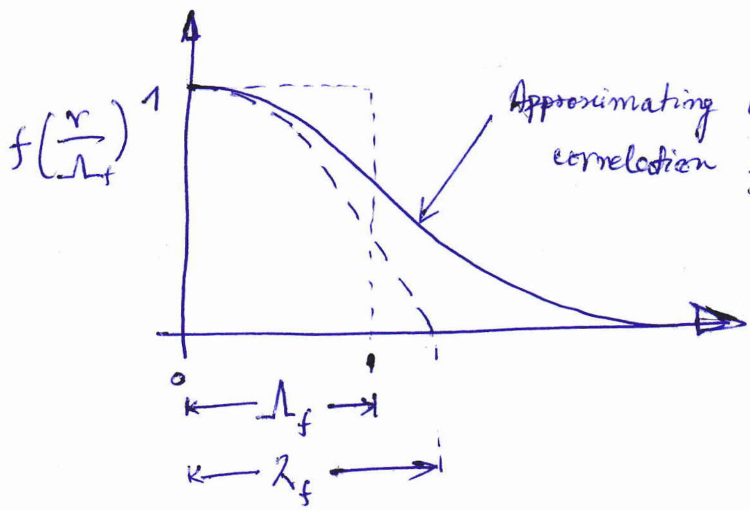
Following 2 relations are significant:

Macroscale $\Lambda = \sqrt{2\pi\nu t}$ $\longrightarrow$ (6.37) Relation 1

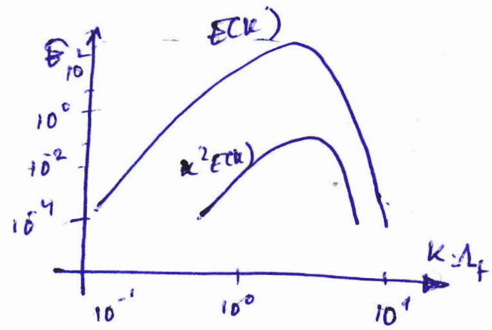
This simplified dependance can be universally found in the end stages of the decaying isotropic turbulenz. This result can be found analytically. Due to it's analytical finding and simple universal validity, it is a popular test case to validate intense numerical computation procedures (e.g. DNS).

$$\lambda_f = \sqrt{\frac{4}{\pi}}\,\Lambda_f \Rightarrow \lambda_f > \Lambda_f \quad (6.40) \text{ Relation 2}$$

The inequality $\lambda_f > \Lambda_f$ states that as the Dissipation is dominative, the Taylor' micro scale λ_f will be larger than the macroscale Λ_f.



Approximating curve for the longitudinal correlation $f\left(\frac{r}{\Lambda_f}\right)$ for very small Re.



Spectral distribution in final stage of decay.