The understanding and prediction of turbulent flow

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1. INTRODUCTION

It was easy for me to choose a subject for this lecture. First, turbulence is an important part of engineering fluid dynamics which has not so far been treated in the Reynolds-Prandtl lectures; secondly, it is the most obvious link between the careers of our two heroes; and thirdly it is the only scientific subject on which I am even remotely qualified to lecture before such a distinguished audience. The first and second propositions are sufficiently demonstrated by the annual consumption of something of the order of 10^{10}kg of kerosene to overcome the effects of Reynolds' stresses in Prandtl's boundary layers. Have any two human beings ever had such a spectacular memorial? (Yes, Lanchester and Prandtl, to whose induced drag we sacrifice another 10^{10}kg of fuel each year.)

My title mentioned understanding and prediction, in that order. Perhaps the most difficult part of any science is to assess how much understanding one must acquire to make adequate predictions for engineering purposes. In turbulence studies we are fortunate in having a complete set of equations, the Navier Stokes equations, whose ability to describe the motion of air at temperatures and pressures near atmospheric is not seriously in doubt (it is easy to show that the smallest significant eddies are many times larger than a molecular mean free path). We are unfortunate because numerical solution of the full time-dependent equations for turbulent flow is not practicable with present computers. Useful research results can be obtained from computer solution of intelligently simplified versions of the time-dependent equations, but even to calculate the simple case of parallel duct flow, with a rather coarse mesh, takes several hours on the fastest computers[1]. At present, engineering calculations are necessarily based on the time-averaged equations. At first sight this is no shortcoming because engineers want time-averaged values: however, averaging eliminates some of the information contained in the Navier Stokes equations and, by substituting apparent mean (Reynolds) stresses for the actual transfer of momentum by the velocity fluctuations, increases the number of unknowns above the number of equations.

The Sixth Reynolds-Prandtl Lecture, given in Munich on 14th April 1972.

The problem faced by an engineer, then, is to supply the information missing from the time-averaged equations (the "Reynolds equations") by formulating a model to describe some or all of the six independent Reynolds stresses $-\rho\overline{u_i u_j}$.* By weighting the Navier Stokes equations before averaging we can obtain differential equations which express conservation of each Reynolds stress, in the same way that the Navier Stokes equations express conservation of each component of momentum. We call these the Reynolds-stress transport equations. They are the simplest exact equations for turbulence quantities, their terms represent the basic processes by which Reynolds stresses are created, transported and destroyed; and they provide a convenient, though not unique, framework for a discussion of turbulent flow. Unfortunately, just as the Reynolds equations for the mean velocity components contain the unknown Reynolds stresses, the Reynolds-stress transport equations themselves contain further unknowns. These unknowns include pressure-fluctuation terms and terms involving eddy length scales, as well as higher-order products of fluctuating velocities. An infinite number of weighted equations would be needed to restore all the information lost by time averaging; the alternative is to use a finite number of equations and supply the missing information from experiments.

Having established the framework of discussion, let us examine the tasks facing the developer of a calculation method, the experimenter, and the user of calculation methods.

1.1. The developer

The developer of a calculation method must decide how many of the transport equations and other weighted equations he can profitably use in his model of the Reynolds stresses. The extra unknowns must be represented by dimensionally correct and physically plausible combinations of the mean velocity, Reynolds stresses and any other quantities for which transport equations are available. These combinations will involve empirical functions or

*When discussing generalities it is helpful to use the suffix notation: i or j can be 1, 2 or 3, denoting components in the x, y, z directions respectively. If a suffix is repeated in a single term (e.g. $\overline{u_j\,\partial u_i/\partial x_j}$ or $\overline{\partial u_i/\partial x_i}$) the term is summed over all values of the suffix: $\overline{u_i^2} \equiv \overline{u_i u_i} = \overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}$.

constants, whose values must be derived in some way from experiment. This problem of reducing the number of unknowns to equal the number of equations is known as the closure problem. It is closely related to the closure problem in homogeneous turbulence, and those who know the sad story of attempts to solve the latter problem may wince at the thought of a similar quest for the Philosopher's Stone in the much more difficult field of shear-flow turbulence. However, our ambition is limited to the prediction of Reynolds stresses rather than the complete spectrum tensor, and we are willing to make full use of experimental data. On the other hand we require turbulence models which are potentially accurate to within a few per cent; this disqualifies many interesting pieces of work which give only order-of-magnitude results.

When calculations had to be done on a desk calculator or slide rule the most realistic models that could be used—and then only for painfully few cases—were the "eddy viscosity" and "mixing length" models, which relate Reynolds stresses directly to the local mean velocity gradient without necessarily considering the transport equations at all. Computers have changed all that. However, we must remember that our descriptions of the physical world, including all our equations of motion, rest entirely on observation. All a computer can do is to rearrange the information we give it, and the information does not acquire extra merit or accuracy by being subjected to a few million arithmetic operations. The value of the computer is that it can rearrange more complicated descriptions of the physical world than can be handled by analytic methods or desk calculators. In particular it allows us to couple almost any turbulence model we can conceive with a suitable numerical procedure to make a complete calculation method. Fig. 1 shows the whole development process. Careful attention to the input (at the top) reduces the number of iterations required. I shall not say much about numerical procedures in this paper because, for every one person who knows enough about turbulence to produce a plausible set of differential equations to describe it, there are tens or hundreds who know (or can learn) enough about numerical analysis to solve those equations. This does not imply that numerical analysis is a lower-grade subject but merely that we know more about it. In particular there are now several well-developed numerical procedures for the sort of equations and the sort of boundary conditions found in turbulence problems (excepting, of course, the more complex flows for which even laminar solutions are difficult to obtain).

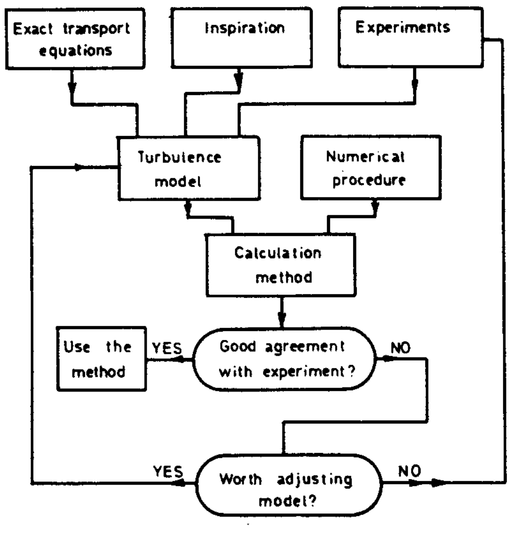

Figure 1. Flow chart for development of calculation methods.

Although the computer has reduced the computational constraints on our choice of a turbulence model, we still face the physical constraints of our lack of measurements of turbulence quantities—the statistical properties of the fluctuation field. The question concerning how many equations to use now becomes "How many equations can we handle before the benefits of increasing refinement are outweighed by decreasing understanding of the turbulence quantities in the equations?". Donaldson[3] has asked—and answered—much the same question. Clearly we should seek, not a perfect and all-embracing model to use for all time, but the best model to use for the next few years until experimental work has improved our understanding: let us call this the optimum model. The views of different research groups on the optimum model are nearer unanimity today than in the past: most of the models being developed or coming into engineering use at the present time are based on the transport equations for the Reynolds stresses themselves, with an associated equation for eddy length scale which may be a transport equation or, in restricted cases, an algebraic formula. Closure of the Reynolds-stress transport equations requires third-order mean products of velocity fluctuations, $\overline{u^2v}$ for instance, to be expressed in terms of second-order mean products (the Reynolds stresses themselves) and is known as second-order closure or Reynolds stress closure. First-order closure or mean-velocity-field closure expresses the Reynolds stresses as functions of the mean velocity field as in the mixing length or eddy viscosity formulae: I shall call these algebraic formulae to distinguish them from higher-order closures involving differential equations for turbulence quantities. All but the very simplest second-order closures require empirical information about turbulence quantities which have not been measured to the accuracy needed in calculation methods—if they have ever been measured at all. This is not an impossible situation: one simply has to rely more on physical intuition and trial-and-error adjustment of constants than on turbulence measurements (Fig. 1). However unless more effort is devoted to experimental work the trial-and-error part of the process will predominate, leading to the unhealthy situation of too many computers chasing too few facts (Fig. 2).

1.2. The experimenter

For some time, experimenters have used the Reynolds-stress transport equations, especially the turbulent energy equation, as a framework for basic research on turbulence processes. Now that developers of calculation methods are starting to do the same, we may hope for significant interaction between theory and experiment, which is a prerequisite of true science. The experimenter's task can now be defined, helpfully instead of just hopefully as in the past, as the provision of qualitative inspiration and quantitative data for the developers of calculation methods: they in turn can provide the experimenter with a point of view. There seems little danger of an experimenter trusting a particular calculation method so completely as to perform experiments that are useful for that calculation method and no other, so that this choice of a point of view can do nothing but good. The only danger is that an interest in complicated turbulence quantities may deter experimenters from making the accurate measurements of mean flows and Reynolds stresses that are needed to test calculation methods.

Techniques for turbulence measurement have improved steadily with advances in electronics (Fig. 2) but the basic sensor is still the hot wire anemometer. The laser doppler anemometer will probably supplement rather than

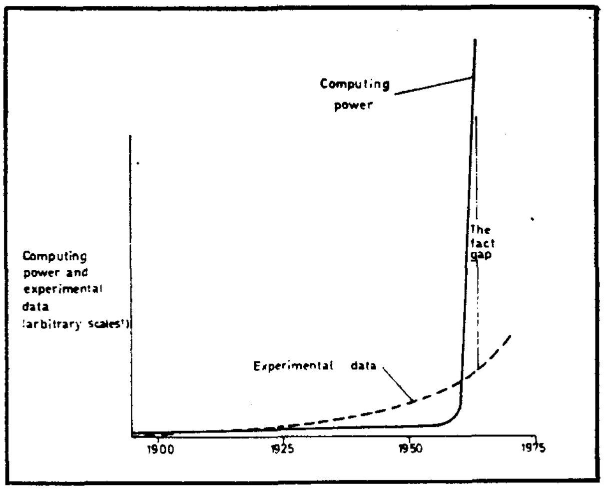

Figure 2. Too many computers chasing too few facts.

supplant it. Perhaps the biggest advance in recent years is in the ability to extract complicated statistical properties and conditional probabilities ("conditional sampling") by either analogue or digital processing. This ability is doing for the experimenter what the ability to handle transport equations has done for the developer of calculation methods—given him more degrees of freedom to use for good or ill.

Since one of the biggest difficulties in closing the Reynolds-stress transport equations is in dealing with the pressure-fluctuation terms, the recent appearance of plausible measurements of pressure fluctuations within the atmospheric boundary layer(3), using a disc-shaped probe, is particularly timely. Probe shape appears critical and the construction of probes small enough for laboratory use may be extremely difficult. Basic data on pressure fluctuations might be obtainable most economically by "computer experiments"—numerical solutions of the time-dependent Navier Stokes equations.

1.3. The user

The newer calculation methods, using second-order closure, are more realistic than the old, but, even when successfully developed, require more knowledge of turbulence to assess. A calculation method should be assessed by the user, who will have to bear the consequences of inadequate prediction: especially in aeronautics these consequences can be very serious, and originators of calculation methods for turbulent flow sometimes make over-enthusiastic claims on the basis of insufficient comparisons with experiment (see Thompson's review(4) for some examples). Assessment by the user implies that individual users, or the community of users in industry and research establishments, must keep in touch with the advances in physical understanding on which prediction methods are based, and I hope this review may help. The prospective user of the newer calculation methods should approach them with his eyes open: this may seem an irritating demand to make of a busy engineer, but it is at least a more realistic demand than asking him to keep his eyes shut, which is what he has sometimes been expected to do in the past! Prospective users of the newer methods may have misgivings about the amount of computer programming or computer running time needed: certainly the days when an industrial user could program the method himself are over, and he must accept a program package from the originator, but the running times involved are short by modern standards. About 20 seconds on a machine like the IBM 360-65 or a minute or so on an IBM 7094, suffice for a typical boundary layer calculation by the most advanced "field" methods using two or three transport equations for turbulence quantities and various short cuts can be taken. It is not wise to distinguish—or choose—calculation methods on the basis of the numerical procedure employed, even though much of the work in developing a calculation method may be numerical analysis and computer programming: a numerical procedure without a turbulence model stands in the same relation to a complete calculation method as an ox does to a bull.

1.4. Complex turbulent flows(5)

Experiments, calculation methods and reviews often concentrate on the velocity field of the two-dimensional incompressible boundary layer, ignoring the problems of heat or pollutant transfer, three-dimensionality, compressibility and flows other than boundary layers. This can be partly justified because

1. the accuracy required in heat or pollutant transfer problems is either fairly low (so that Reynolds' analogy is acceptable in many cases) or impossibly high (in determining creep limits);
2. turbulence is always three-dimensional, so that moderate three-dimensionality of the mean flow should not affect the turbulence structure and rational extensions of turbulence models for two-dimensional flow should give unimpaired accuracy (but see next paragraph);
3. compressibility should not directly affect the turbulence if the Mach number fluctuation is small (Morkovin's hypothesis(6)), although mean density gradients, or extra strain rates resulting from longitudinal pressure gradients, may affect the turbulence;
4. viscosity and its dependence on temperature should not directly affect the turbulence except in the universal sublayer of a turbulent wall flow,‡ or in certain low-Reynolds-number situations.

However the extension of calculation methods to flows other than boundary layers is a more serious matter. Even other types of thin shear layer, such as jets or wakes, make greater demands on accuracy than boundary layers. More importantly, the presence of rates of strain other than a simple shear $\partial U/\partial y$ (e.g. strong deflections or accelerations) may significantly affect turbulence structure, in a way that is not fully represented by including terms involving the extra rates of strain in the exact transport equation. To quote a case from my own experience, the presence of a small value of $\partial V/\partial x$ (i.e. longitudinal curvature) changes the Reynolds shear stress by a fraction which is an order of magnitude larger than $(\partial V/\partial x)/(\partial U/\partial y)$. I think it needs emphasising quite strongly that thin shear layers alone are not likely to be a sufficient test of a model intended for complex flows. This is not just an academic point—many users are concerned with complex flows whether they like it or not.

The present second-order-closure candidates for the title of "optimum model" include methods that aim at complete generality—applicability to any turbulent flow. There is no reason to suppose that any general model independent of, and simpler than, the Navier Stokes equations can be found. Therefore engineering calculation methods must be based on, or compatible with, empirical simplifications of the Navier Stokes equations. These simplifications are virtually certain to have a smaller range of validity than the Navier Stokes equations as well as less accuracy in a given case, so we must not expect too much of our optimum model.

1.5. Scope of this paper

A sad feature of most previous combined reviews of turbulence physics and calculation methods, a feature which is no fault of the authors, is the almost complete lack of connection between the two parts of the review. This is true even of the masterly survey by Rotta[7]—indeed his exposure of the lack of physical relevance of the calculation methods current in 1962 has inspired much later work. I hope that recent advances in calculation methods will help me to present a more unified view in this paper. Section 2 is a discussion of the physics of turbulence, related to the behaviour of the Reynolds-stress transport equations: Ref. 8 may be a helpful introduction for non-specialists, though not too successful as a unified view! Section 3 is a discussion of turbulent shear layers, especially wall layers, ending with a list of the questions that must be answered in the development of any turbulence model. At the cost of some damage to my unified image I have given, in Section 4, an introduction to the older turbulence models in current use, arranged as a historical review starting with the "eddy viscosity" model of Boussinesq and ending with the turbulence models presented at the Stanford meeting in 1968. In my view none of the models described in Section 4 is even potentially suitable for flows other than thin shear layers. Section 5 returns to the main theme with a discussion of the most recent models based on the Reynolds-stress transport equations: this section can be regarded as an addendum to the broader-ranging specialists' review by W. C. Reynolds[9], and as an introduction to methods potentially suitable for complex flows.

2. THE MECHANISM OF TURBULENCE

2.1. Eddies

The only short but satisfactory answer to the question "what is turbulence?" is that it is the general solution of the Navier Stokes equations. The motion is rotational and the dependent variables U, V, W and p are functions of all the independent variables x, y, z and t. The non-linearity of the equations causes interaction between fluctuations of different wavelengths and directions and, as a result, the wavelengths of the motion usually extend all the way from a maximum set by the width of the flow to a minimum set by viscous dissipation of energy. From the physical viewpoint the mechanism that spreads the motion over a large range of wavelengths is vortex stretching[10]. Energy enters the turbulence if the vortex elements are mainly orientated in the right sense to be stretched by the mean velocity gradients. Naturally the part of the motion that can best interact with the mean flow is that whose wavelengths are not too small compared to the mean-flow width, and this larger-scale motion carries most of the energy and Reynolds stresses in the turbulence.

One of the most striking things about a turbulent shear flow is the way that large bodies of fluid migrate across the flow, carrying smaller-scale disturbances with them. The arrival of these large eddies near the interface between the turbulent region and non-turbulent fluid distorts this interface into a highly re-entrant shape (Fig. 3) and appears to control the rate of spread of the turbulence (another way of saying that the large eddies carry most of the Reynolds stress). Not only do the large eddies migrate across the flow, but their lifetime is so long that during it they may travel downstream for a distance many times the width of the flow (a rough calculation suggests 30δ in the case of a boundary layer of thickness δ). Therefore the Reynolds stress at a given position depends significantly on upstream history and is not uniquely specified by the local mean velocity gradient(s) as in a laminar flow. This is why we need transport equations to describe turbulence. Fig. 3 is a sufficient answer to those who hope for a simple description of this complicated phenomenon.

Vortex stretching and large-eddy migration between them dominate the Reynolds-stress transport equations: vortex stretching also dominates equations for turbulent length scales, since it is the mechanism of energy exchange between eddies of different sizes. Unfortunately it is not possible to write the equations solely in terms of vorticity and an effort is needed to remember the connection between

1. The term in a transport equation.
2. The process represented (e.g. energy dissipation).
3. The mechanism (e.g. transfer of energy by vortex stretching to viscosity-dependent eddies).

2.2. Behaviour of the Reynolds stresses

To avoid proliferation of tensors we will consider only the transport equation for the Reynolds shear stress in the (x, y) plane, $-\rho\overline{uv}$. Even if the flow does not strictly obey Prandtl's boundary-layer approximation, shear-stress gradients are usually more important than normal-stress gradients. In a three-dimensional flow the shear stress has an additional component in the (y, z) plane, $-\rho\overline{wv}$. For $\overline{uv}$ in steady incompressible flow we have, with the useful "D/Dt" notation,

Transport by mean flow

$$\left(\underline{\underline{U\frac{\partial}{\partial x}}}+\underline{\underline{V\frac{\partial}{\partial y}}}+W\frac{\partial}{\partial z}\right)(-\overline{uv})\equiv\frac{D}{Dt}(-\overline{uv})$$

(1) Generation by interaction with mean flow

$$\underline{\underline{\overline{v^2}\frac{\partial U}{\partial y}}}+\overline{u^2}\frac{\partial V}{\partial x}+\overline{uv}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)$$

$$+\overline{wv}\frac{\partial U}{\partial z}+\overline{uw}\frac{\partial V}{\partial z}$$

(2) Redistribution by pressure fluctuations

$$\underline{\underline{-\overline{\frac{p'}{\rho}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)}}}$$

(3a) Transport by velocity fluctuations

$$\left(\frac{\partial\overline{u^2v}}{\partial x}+\underline{\underline{\frac{\partial\overline{uv^2}}{\partial y}}}+\frac{\partial\overline{uvw}}{\partial z}\right)$$

(3b) and pressure fluctuations

$$+\frac{1}{\rho}\left(\underline{\underline{\frac{\partial\overline{p'u}}{\partial y}}}+\frac{\partial\overline{p'v}}{\partial x}\right)$$

(4) Transport and destruction by viscous forces

$$-\nu\,\underline{\underline{\overline{(u\nabla^2 v+v\nabla^2 u)}}} \qquad (2.1)$$

In any two-dimensional flow all the z-wise gradients of time-average quantities would be zero and in a two-dimensional thin shear layer only the terms marked with a double underline would remain. These simplifications do not eliminate any group completely since each represents an essential physical process: usually the thin-shear-layer terms are larger than the others even if the shear layer is not strictly "thin". The transport equation for any other individual Reynolds stress looks similar to (2.1), with appropriate changes in symbols. "Transport" terms (e.g. [3]) can be distinguished from "source/sink" terms

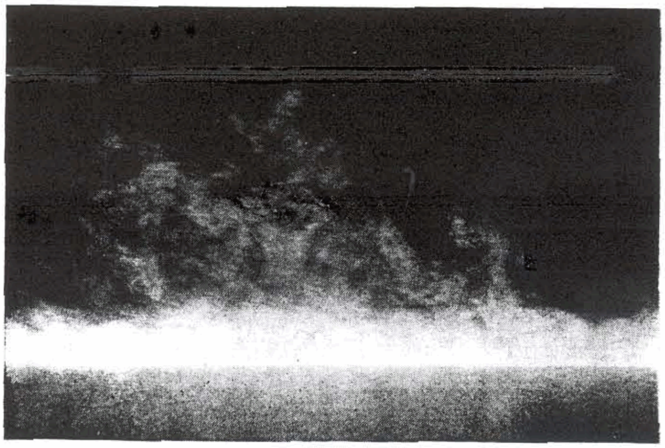

Figure 3. Illuminated cross-section of smoke-filled boundary layer. Flow right to left.

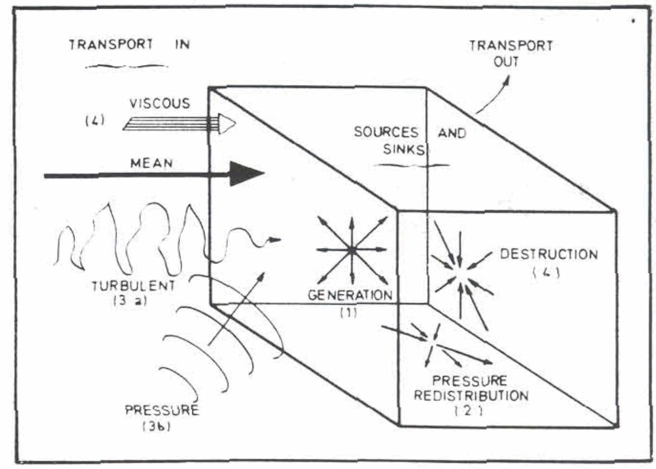

Figure 4. General form of Reynolds-stress transport equations.

(e.g. [1]) because they can be written as spatial gradients of mean or turbulence quantities ("divergences" in the context of control-volume analysis—*see* Fig. 4). The only transport equation whose terms have been measured extensively[2] is the turbulent energy equation for half the sum of the Reynolds normal stresses, $\frac{1}{2}(\overline{u^2}+\overline{v^2}+\overline{w^2})$. The redistribution term analogous to [2] sums to zero in the turbulent energy equation, and the other terms are given special names: the left hand side is called "advection" and the terms analogous to [1], [3] and [4] are called "production", "diffusion" and "dissipation" respectively.

Since the prediction of Reynolds stresses is explicitly or implicitly the solution of the Reynolds-stress transport equations, let us discuss the current understanding of (2.1). The left hand side and group [1] on the right hand side are straightforward in mathematical content and physical meaning. Our chief enemy both in calculation and in experiment is group [2], the rate at which $-\overline{uv}$ is destroyed by the interaction of pressure fluctuations with the fluctuating rate of strain. Elliott[3] has recently measured the analogous term in the $\overline{u^2}$ equation but at this stage our preconceived ideas about the term lend credence to the measurements rather than the other way round. Group [3] is the net rate of spatial transport of $-\overline{uv}$ by the turbulence. It seems that [3*b*] is considerably smaller than [3*a*] and nobody worries about it too much: again few measurements are available. Group [4] is the sum of net spatial transport, and destruction, of Reynolds stress by viscous effects. Since second derivatives appear, these effects are confined to the smallest eddies which contribute little to the Reynolds stress, so that viscous transport of Reynolds stress is negligible. The small eddies are supplied with energy from the larger eddies by the "cascade" process of random vortex stretching: the rate of energy supply, and therefore the rate of energy dissipation, is determined by the larger eddies, which do not depend on viscosity. Again, the random nature of the vortex-stretching cascade process makes the smallest eddies virtually unconscious of the preferred directions of the mean flow and of the larger eddies, so that they are statistically isotropic. This means that cross-product terms like $\overline{u\nabla^2 v}$ appearing in [4] are negligible (terms like $\overline{u\nabla^2 u}$ appearing in the normal-stress equations are not negligible but constitute the dissipation). We conclude that these two properties of the vortex-stretching cascade make the Reynolds stresses, normal or shear, independent of viscosity; this great simplification is sometimes ignored by developers of calculation methods. The only exception to these conclusions about term [4] is turbulence at low local Reynolds number—in the viscous sublayer (Section 3.1) for instance—where the ratio of the largest eddy size to the smallest may be small enough for the two to interact and for significant viscous transport to occur.

The problem of closure at second order, outlined in Section 1, is now more clearly seen as a representation of terms like [2] to [4] as functions of the Reynolds stresses and the mean velocity field. The left hand side of (2.1) and term [1] on the right hand side are already of this form although [1] may have to be approximated unless transport equations are used for all the Reynolds stresses. In a thin shear layer where only shear stress gradients are important we may try to avoid solving for the normal stresses. In all cases we need eddy length scales, as well as the velocity scales provided by the Reynolds stresses themselves, because the terms in the Reynolds-stress transport equations all have dimensions (velocity)^3/length. Transport equations for length scales, or quantities like dissipation which imply a length scale, are rather more complicated than those for Reynolds stress, even when one has made one of the many possible choices (Section 5).

Perhaps (2.1) and its companions are not the only way of posing the problem—Gupta, Laufer and Kaplan[12] have emphasised that even our distinction between mean and fluctuating velocities is rather arbitrary—but since it is undoubtedly the Reynolds stress that the user of a calculation method needs to know, contenders for the title of "optimum model" can usually be interpreted as approximations to these equations. "Approximation" of course, can include complete neglect of some of the terms or the processes they represent.

3. CURRENT UNDERSTANDING OF THE PHYSICS OF TURBULENT SHEAR LAYERS

3.1. The inner layer of a turbulent wall flow (Figure 5)

Near a solid surface, the largest stress-containing eddies have a wavelength of the order of the distance from the surface, y say. Eddies with a larger longitudinal wavelength are too flat to carry much shear stress. Since $y \ll \delta$, where δ is the total width of the shear layer, the eddies are little affected by the turbulent motion in the outer part of the flow, and their life time is short compared to the time scale of significant change in the x direction. Thus both mean and turbulent transport of Reynolds stress to or from regions at a distance much larger than y are negligible, and the typical velocity scale of the turbulence depends only on conditions in this "inner layer", specifi-

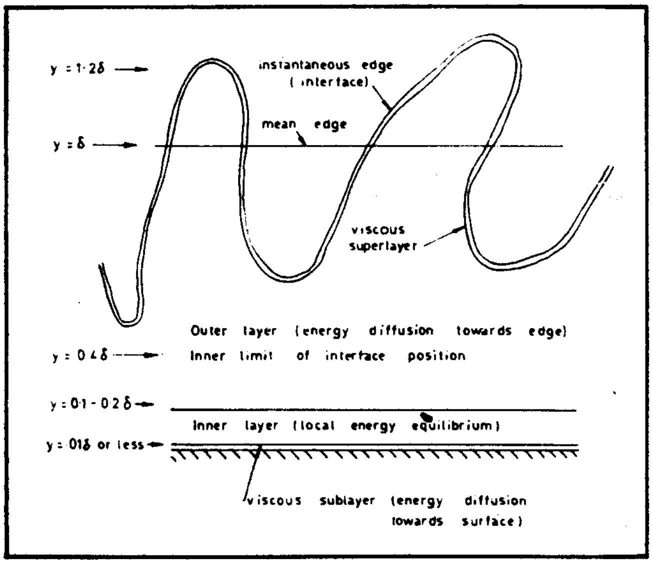

Figure 5. Boundary layer nomenclature.

cally on the shear stress transmitted through the layer to the surface, τ_w. If the velocity scale of the turbulence is independent of y and the length scale dependent only on y we expect turbulent transport of Reynolds stress within the layer to be small, so that the exact Reynolds-stress equations like (2.1) reduce to "generation" = "destruction", all transport terms being zero. This is called a state of local equilibrium[13]. Given that the turbulence is specified by a length scale y and a velocity scale $\sqrt{(\tau_w/\rho)} \equiv u_\tau$, either substitution of the appropriate dimensional forms into the Reynolds-stress equations or direct dimensional analysis based on the argument that the local equilibrium must also include the mean velocity gradient (which comes to the same thing) leads to

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{Ky} \quad . \quad . \quad . \quad (3.1)$$

where K is an absolute constant found experimentally to be about 0·41. The point of repeating this familiar analysis is to show how it depends on quite subtle and special arguments about the turbulence which are emphatically not true in general. The arguments break down very near a solid surface where viscous effects are important and the turbulence scales are no longer simply u_τ and y. Further dimensional analysis leads to the "law of the wall"

$$\frac{U}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right) \quad . \quad . \quad . \quad (3.2)$$

or

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{Ky} f_1\left(\frac{u_\tau y}{\nu}\right)$$

of which (3.1) is a special case:

$f_1 \longrightarrow 1$ at $u_\tau y/\nu \approx 30$, or $y/\delta = 0{\cdot}01$ in a boundary layer with $U_\infty \theta/\nu = 10^4$). This does not mean that the flow in the viscous sub-layer is in local equilibrium, and in fact there is a significant transport of Reynolds stress towards the surface. Integration of (3.1) and use of (3.2) gives the well-known logarithmic profile for $u_\tau y/\nu > 30$

$$\frac{U}{u_\tau} = \frac{1}{K} \log \frac{u_\tau y}{\nu} + C \quad . \quad . \quad (3.3)$$

where C is about 5.

If the total shear stress is a function of y, dimensional reasoning suggests that (3.1) be replaced by

$$\frac{\partial U}{\partial y} = \frac{\sqrt{\tau/\rho}}{Ky} f_2\left(\frac{y}{\tau}\frac{\partial \tau}{\partial y}, \quad \frac{y^2}{\tau}\frac{\partial^2 \tau}{\partial y^2}, \quad \ldots\right) \quad (3.4)$$

but measurements show that $f_2 = 1$ is a good approximation. Closures of (2.1) usually reduce to (3.4), with $f_2 = 1$, if mean and turbulent transport terms are neglected and the appropriate inner-layer scales inserted. This is of course a warning that (3.4) is not to be trusted if transport terms are likely to be important, e.g. if $\partial\tau/\partial x$ or $\partial\tau/\partial y$ are large. Equation (3.4) can be integrated in many cases of interest[14]. It is usual to rearrange the integral so that its right hand side is the same as that of (3.3). The constant of integration C, representing the effect of the viscous sub-layer, depends significantly on the shear stress distribution in the sub-layer: for instance it becomes greater than five if $(\nu/\rho u_\tau^3)\,\partial\tau/\partial y$ is more negative than about $-0{\cdot}003$, as it is in pipe flow at bulk Reynolds numbers, based on diameter, of less than about 10 000.

The chances of understanding the behaviour of the viscous sub-layer well enough to predict C in all cases seem small but, since C is usually a function of one or two parameters only, correlations of experimental data are not too difficult to establish. An entirely adequate reason for doing experiments on eddy structure in the sub-layer[12, 15, 16, 17] is that the sub-layer eddies are likely to have a particularly simple shape, close to the most efficient shape for extracting energy from the mean shear and so maintaining themselves against strong viscous damping. Several experimenters have clearly identified "bursts" or "eruptions", in which organised bodies of slowly-moving fluid move away from the surface in a manner reminiscent of Grant's[18] "mixing jets" and Kovasznay's[19] model of the behaviour of eddies near the outer edge of a boundary layer. The experiments suggest a strong interaction between the motion at different distances from the surface: an embarrassing difficulty is that flow models based on the experiments are liable to predict that the interaction continues beyond $u_\tau y/\nu = 30$, which conflicts with the ideas of local equilibrium and viscosity-independence on which (3.1) and (3.4) depend. At present, direct experimental checks of (3.1) have made it one of our firmer foundations for a treatment of turbulent wall flows and, though one would be foolish to regard it as exact, one feels that models for the sub-layer and inner layer should be compatible to a first approximation with (3.1).

3.2. The outer layer (Figure 5)

This is the region of a boundary layer $y > 0{\cdot}2\delta$ approximately, where the stress-carrying eddies are not strongly affected by the boundary conditions at the surface. The free-stream edges of other flows are qualitatively like the outer layer. The probability distribution of the "interface" between turbulent and irrotational (non-turbulent) fluid in a boundary layer is roughly Gaussian with a mean of 0·85δ and a standard deviation of 0·15δ (so that the interface occasionally extends as far in as 0·4δ or as far out as 1·3δ). Most of a boundary layer, then, is strongly affected by the surface or the interface: the outer layer is dominated by large eddies which transport low momentum and high turbulent energy from the outer part of the inner layer and deposit both near the outer edge of the boundary layer. This behaviour can be deduced from measurements of turbulent transport of turbulent energy (Fig. 6) but it is shown most spectacularly by still photographs or motion pictures in a smoke-filled flow (Fig. 3). Townsends large eddy equilibrium hypothesis[11] focused attention on the large eddies; although some of the assumptions of Townsend's theory have been disproved, the very disproof has enhanced the importance of the large eddies, which are now seen to carry much of the Reynolds stress in any turbulent shear layer. The behaviour of the large eddies depends on their interaction with the smaller-scale motion via the vortex-stretching mechanism, and it seems probable that the smaller-scale motion is far from the state of spatial homogeneity assumed in theories such as that of Kolmogorov (Ref. 20 is the latest of several experimental papers on this subject).

Some progress has been made in deducing the typical shape of the large eddies from measurements of the coefficient of correlation between velocity fluctuations at two points[18,19]. A recently-conceived technique which seems ideal for studying the large eddies and their interactions with the small-scale motion is "conditional sampling"[12,16,19]. Since this technique is likely to become quite common in basic research, a brief outline may be helpful to all three sections of the community. The standard form of a hypothesis, about turbulence or any other phenomenon, is that something (S, say) depends on something else (S', say). If S and S' are simple, like Reynolds stress and mean velocity gradient, an experiment to test the hypothesis is simple at least in principle. If the hypothesis is, say, that the large eddy structure in the outer layer depends on the aforementioned "bursts" in the viscous sub-layer then difficulties arise because bursts at a given point occur for only part of the time. To begin with we must choose measurable quantities S and S' to represent large eddy structure and sub-layer bursts respectively. Suppose we take $S=v^3$ at $y/\delta=0{\cdot}8$, say; $\overline{v^3}$ appears in the turbulent transport term in the turbulent energy equation, which is supposed to depend greatly on the large eddies. We can choose S' to be a short-term average of $-uv$ in the sub-layer. Then, we require the average value of v^3 at $y/\delta=0{\cdot}8$ conditional on the short-term average of $-uv$ in the sub-layer being greater than, say, twice the conventional average $-\overline{uv}$: that is, we accumulate contributions to the average of v^3 only during periods of high $(-uv)$ in the sub-layer. If the average value of v^3, in the outer layer during bursts in the sub-layer is significantly different from the conventional average $\overline{v^3}$, then there is a significant connection between sub-layer bursts and transport of turbulent energy by the large eddies. Which causes which, or whether a third phenomenon causes both, is another question! The hypothesis mentioned above has been seriously suggested—though for reasons mentioned at the end of the last section I do not believe it (the experiment has not yet been done). Conditional sampling is not necessarily restricted to hypotheses about terms in the exact transport equations, but this seems to be its main relevance to the development of calculation methods.

When making hypotheses about eddy behaviour one is always tempted to seek pattern and order where none exist. Mollo-Christensen[21] has compared theories of turbulence based on measurements of gross statistical properties with the sort of theory about road traffic that might be based solely upon the statistic that a given group of cars and motor cycles has an average of 3·6 wheels per vehicle. I feel that a greater error is to classify turbulent eddies into two wheel and four wheel types when in reality they are as complicated as a vehicle which does have 3·6 wheels. It is a great pity that of the three concepts of turbulence:

1. all eddies small compared to mean flow dimensions (Boussinesq "eddy viscosity" hypothesis, 1877);
2. some eddies large but weak (Townsend's hypothesis[11], 1956);
3. some eddies large and strong, interacting with smaller and weaker eddies (current view);

the most recent is both the most realistic and the most complicated. But what else can one expect of a phenomenon which is defined as the fluid motion of maximum possible complexity?

3.3. Transition to turbulence, and turbulent flow at low Reynolds numbers

The problem of tracing the growth of given disturbances and the eventual breakdown to turbulent flow is, of course, more difficult than the calculation of fully-turbulent flow, but similar methods can be used in principle, and some progress has already been made (Hall[22], Donaldson[23]). Unfortunately it is rarely possible to specify the details of initial disturbances that exist in a real flow, because they arise from free-stream turbulence or structural vibration (note that surface roughness cannot itself produce time-dependent disturbances). Only in the case of turbomachines and certain other types of internal flow can we hope to know even the statistical properties of the free stream turbulence. Therefore the detailed prediction of transition, as opposed to the growth of given disturbances, is often an ill-posed problem. The main need at present is for better empirical correlations between the global parameters of the transition region and the statistical properties of the free-stream turbulence. Theoretical work may help us to choose relevant statistical properties: certainly the length scale as well as the intensity is relevant since large-scale unsteadiness has a different effect from small-scale turbulence. Most of the above remarks also apply to the effect of free stream turbulence on a turbulent boundary layer[24,25]: in this case, if the turbulence intensity is only a few per cent of the mean velocity the direct effect is confined to the outer layer and the law of the wall is unaltered.

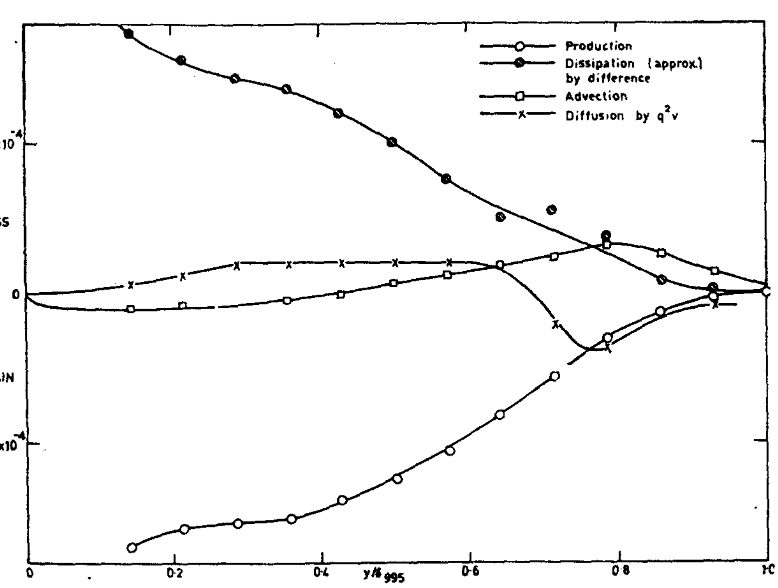

Figure 6. Energy balance in typical retarded boundary layer.

Another phenomenon whose direct effect seems to be confined to the outer layer and the interface is the influence of viscosity on turbulent flow at low bulk Reynolds number. Coles[26] showed that, on the assumption that the law of the wall is unaltered (an assumption disputed by some) the velocity-defect law alters if the Reynolds number based on momentum thickness is less than about 5000: in compressible flow the effects appear to be stronger. Once more, we need experiments rather than hopeful adjustments of calculation methods. The ultimate in low-Reynolds number effects is reverse transition or "re-laminarisation": several local criteria for reversion to laminar flow have been devised but Jones and Launder[27] have suggested that the inner layer ceases to become a local-equilibrium region before reverse transition occurs so that the phenomenon must be treated by using a transport equation for Reynolds stress (whose terms will of course depend on Reynolds number, in a way that may be sensitive to the details of the flow).

In summary, the effect of viscosity on turbulent flow makes itself felt in several different phenomena of engineering importance, but we must be prepared to acknowledge that turbulence influenced by viscosity presents a more difficult problem than ordinary turbulence.

3.4. Outstanding questions

The questions about turbulence posed by our inspection of the Reynolds-stress transport equations (Section 2), and the discussion in the earlier parts of the present section, are mainly questions about the large eddies and their interaction with the smaller eddies and with the irrotational-turbulent interface. Let us list these questions as they apply to a thin shear layer in two-dimensional flow, remembering that the same questions also apply to more general flows.

1. How is the turbulent transport, in the y direction, of turbulent energy or Reynolds stress (term [3] of (2.1)) related to the profile of turbulent energy or Reynolds stress? So far, calculation methods have used only simple gradient-diffusion or bulk-convection hypotheses. The former is inappropriate to turbulence dominated by large eddies; the latter hypothesis, introduced by Townsend, is more plausible though strictly applicable only if the large eddies are weak. Either hypothesis is adequate for correlating existing experimental data and Lawn[28] has recently found that his admirably accurate data are better fitted by the worse hypothesis, so that the situation is fluid.
2. How are the length scales of the turbulence related to the thickness of the shear layer? Since the shear-stress-producing motion is strongly linked to the large eddies, whose size is strongly linked to the shear layer thickness, the length scales should be strongly linked to the shear layer thickness (though the relation will be different in different flows). If the shear layer thickness is changing rapidly this argument will be less reliable and we must use a transport equation for length scale, which poses new questions.
3. What determines the difference between the different components of the Reynolds stress tensor—that is, the anisotropy of the turbulence? We know that the exchange of intensity between the different components is effected by pressure fluctuations, but, as mentioned above, a plausible measurement of one of these "pressure-strain" terms has only recently been made, and further work is needed to obtain measurements accurate enough to be used directly in calculation methods.
4. How long is the "memory" of the turbulence for its upstream history? If the three preceding questions could be answered, the answer to the present one would follow because it is given by the balance of the generation, destruction and turbulent transport terms in the exact Reynolds stress equations. Nevertheless it is worth asking the question directly, and trying to answer it by tracing the downstream development of the large eddies. The work of Favre[29], Kovasznay[19] and their collaborators springs to mind: modern conditional-sampling techniques can be used to follow the motion of selected parts of the turbulence.
5. What are the effects on the larger eddies of rates of strain, additional to the simple shear $\partial U/\partial y$ found in two-dimensional thin shear layers? This is a question that embraces all the difficulties of flows other than two-dimensional thin shear layers and at present it must be answered separately for each flow if at all. Even the case of small extra strain rates poses problems. "Complex flows"[5] have not yet received adequate attention theoretically or experimentally: the new generation of calculation methods is the first that is even potentially capable of dealing with flows other than thin shear layers. When considering three-dimensional flows we must distinguish three very different rate-of-strain fields. The first case is the true three-dimensional boundary layer, in the x, z plane say, in which $\partial U/\partial y$ greatly exceeds $\partial U/\partial x$ or $\partial U/\partial z$. Since, in any part of the layer, the mean flow direction changes by only a few degrees over a distance in the y direction equal to a typical eddy size, we do not expect great changes in eddy behaviour. Three-dimensional boundary layers can be satisfactorily predicted by logical extensions of turbulence models for two-dimensional flow: transport-equation models can reproduce the difference in direction between shear stress and velocity gradient that occurs in most flows of this type. The second case, typified by the flow in the corner of a duct or a wing-body junction, satisfies what may be called the slender shear layer approximation based on the assumption that $\partial U/\partial x$ is much smaller than $\partial U/\partial y$ or $\partial U/\partial z$. Normal-stress and shear-stress gradients in both the y and z directions are important and, because $\partial U/\partial z$ is of the same order as $\partial U/\partial y$, we cannot expect simple extensions of two-dimensional models to work. The third case of three-dimensional flow, in which $\partial U/\partial x$, $\partial U/\partial y$ and $\partial U/\partial z$ are all of the same order, is likely to deter all but the boldest developers of calculation methods for some time to come.

4. TURBULENCE MODELS AND CALCULATION METHODS

We now turn to the representation in calculation methods of the eddy processes discussed above. This constitutes something of a discontinuity because only the most recent methods deserve the title of "representation of eddy processes", and I shall postpone consideration of these to Section 5. Section 4 describes models that are still used in engineering practice: some at least may be used for a long time to come because they are simple and give adequate results in simple cases, but in my opinion none is even potentially suitable for flows other than thin shear layers. Most of this section refers to boundary layers because most calculation methods have been developed only for boundary layers, but many of the points made also apply to other types of thin shear layer or to complex flows.

The main distinction between turbulence models is whether turbulence properties are related directly to the mean flow or obtained from transport equations. Particularly in the case of thin shear layers, one must also distinguish between methods based on relations for the turbulence properties at each point in the field and those in which the relations are based on integral parameters such as the velocity profile parameter H. The latter distinction is sometimes difficult to make when reading the description of a calculation method, because partial differential equations can be converted into ordinary differential equations for integral parameters by the generalised Galerkin method, alias GKD method, alias "method of integral relations" alias "method of weighted residuals" (*see* Ref. 30, p. 16): the test is whether the original assumptions, rather than the final equations to be solved, are formulated at each point in the field. Thus the four classes are Mean-flow / Field, Mean-flow / Integral, Transport-equation / Integral and Transport-equation / Field methods (Fig. 7).

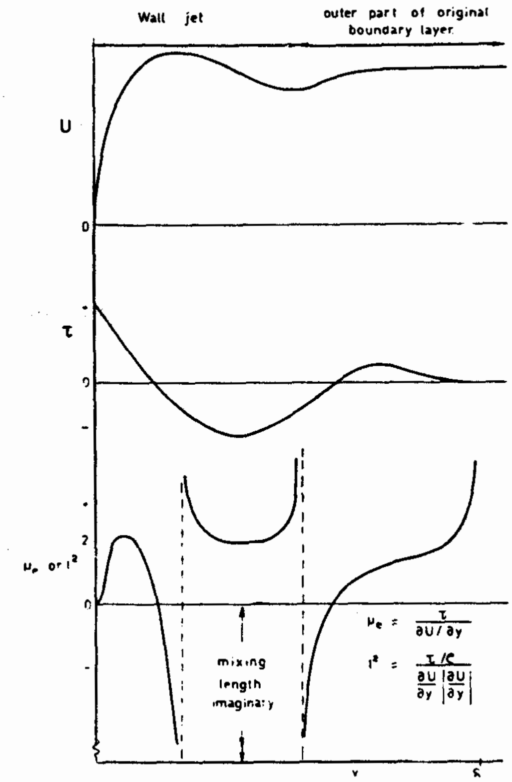

Figure 7. Distribution of eddy viscosity and mixing length in a wall jet beneath a boundary layer. Qualitative trends—for quantitative values *see* Ref. 33.

4.1. Mean-flow/Field methods

Only two of the early ideas about the behaviour of Reynolds stress are still used in a recognisable form: they are the eddy-viscosity and mixing-length hypotheses, which are closely-related hypotheses of a connection between the Reynolds stress and the local mean velocity gradient. Boussinesq's 1877 paper introducing eddy viscosity was actually written in 1872, so we may celebrate its centenary this year. The mixing-length hypothesis is perhaps Prandtl's best-known contribution to turbulence studies (Ref. 31, p. 714). Therefore this time and this lecture are appropriate for a reasoned assessment of these two hypotheses.

We can define a quantity μ_e, having the dimensions of viscosity, by an equation like

$$-\rho\overline{uv} = \mu_e \partial U/\partial y \quad . \quad . \quad . \quad (4.1)$$

and a length l by an equation like

$$-\rho\overline{uv} = \rho l^2 (\partial U/\partial y)^2. \quad . \quad . \quad (4.2)$$

Further, we can establish empirical correlations for the behaviour of μ_e or l in a given flow just as we can do for $-\rho\overline{uv}$ itself. The question is whether working with μ_e or l gives more accurate or more reliable answers than working with $-\rho\overline{uv}$ directly.

If μ_e and l are regarded as given, equations like (4.1) and (4.2) represent "gradient diffusion": (4.1), for instance, states that the rate of transport of U-component momentum in the y direction ($\rho\overline{uv}$) is directly proportional to the mean gradient of U-component momentum in the y direction. Gradient diffusion is an accurate model of molecular transport of mass, heat or momentum in a gas, where the mean free path between molecular collisions is small compared to the flow dimensions but large compared to the molecule diameter. Boussinesq and Prandtl hoped that the transport of momentum by turbulent eddies might be similar to that of gas molecules but on a larger spatial scale. However, eddies or vortex motions in turbulence are not all small compared to the flow dimensions, and the interaction between different eddies is a continuous process and not a series of discrete collisions. Therefore μ_e and l are likely to depend strongly on flow conditions (perhaps even more strongly than $-\rho\overline{uv}$ itself) instead of approximating to constant properties of the fluid like molecular viscosity or mean free path. G. I. Taylor[32], describing his own early reasoning on the subject, gives a characteristically clear and simple demonstration of the inadequacy of results based on a near-constant diffusivity. "Any theory of diffusion which is based on a virtual coefficient of diffusion must predict a mean shape for a smoke plume that is paraboloidal and it was clear to me that near the emitting source the mean outline of a smoke plume is pointed". In some cases, like the wall jet[33] shown in Fig. 7, the apparent eddy viscosity or mixing length does some very strange things.

How are we to reconcile this unphysical behaviour with the success of eddy-viscosity and mixing-length correlations in a large number of practical cases? I think the general view could be put as follows: "mixing length theory in particular, with its concept of momentum transfer by fairly well-organised lumps of fluid, is not too bad a first approximation to the behaviour of turbulence dominated by large eddies as shown in Fig. 3, and the result of using it should therefore be not too bad". However the main reason for the success of mixing length and eddy viscosity is that most of the basic fluid flows used as test cases for Reynolds stress models are nearly in a state of self-preservation or of local equilibrium, and in these cases eddy viscosity and mixing length are simply-behaved purely for dimensional reasons.

A self-preserving flow (sometimes rather misleadingly called an "equilibrium" flow) is one in which profiles of mean velocity (measured from a suitable origin) and of other quantities like Reynolds stress are geometrically similar at all streamwise positions. Turbulent flows usually take up self-preserving forms if the boundary conditions allow. For instance, the outer layer of a boundary layer on a flat plate at high Reynolds number is very close to self-preservation with

$$(U_\infty - U)/u_\tau = f_1(y/\delta) \quad . \quad . \quad (4.3)$$

and

$$\tau/\tau_w \equiv \tau/\rho u_\tau^2 = f_2(y/\delta) \quad . \quad . \quad (4.4)$$

where δ and u_τ are functions of x. These expressions and the definitions of eddy viscosity and mixing length give, at once,

$$\mu_e/\rho u_\tau \delta = f_3(y/\delta) \quad . \quad . \quad . \quad (4.5)$$

$$l/\delta = f_4(y/\delta) \quad . \quad . \quad . \quad (4.6)$$

(note that, using (4.3), $u_\tau\delta$ is proportional to $U_\infty\delta^\star$, the usual choice for a mean-flow scaling factor for eddy viscosity). As mentioned in Section 3, the inner layer of a boundary layer in any pressure gradient and at any Reynolds number is close to local equilibrium (a special case of self-preservation) where the only relevant length scale is y. In this case

$$\mu_e = \rho K (\tau/\rho)^{1/2} y \quad . \quad . \quad . \quad (4.7)$$

$$(\text{or } \mu_e = \rho K u_\tau y \quad \text{if} \quad \tau = \rho u_\tau^2 \text{ everywhere})$$

and $$l = Ky \quad . \quad . \quad . \quad (4.8)$$

(note that these formulae correspond to $f_3 = f_4 = Ky/\delta$, with $f_1 \sim \log y/\delta + \text{constant}$ and $f_2 = \text{constant}$). Now the only physical input to this analysis is the statement that the flow is self-preserving, and there is nothing in what we know of these flows to suggest that the turbulence arranges itself so as to approximate to the Prandtl or Boussinesq models: these simple formulae for μ_e and l are the results solely of dimensional analysis. Any dimensionally-correct "theory" with a disposable constant would give the same results irrespective of its physical correctness.

Closures of the exact Reynolds-stress transport equations reduce to formulae of eddy viscosity type if the transport terms are negligible (as in local-equilibrium flow: see Section 3.1) or if their ratio to the generation or destruction terms is a function of y/δ only (as in self-preserving flow). This strongly suggests that eddy viscosity formulae can be no better than first approximations in non-self-preserving flows where the behaviour of the transport terms is more complicated. All the early tests of the mixing length and eddy viscosity models were made on self-preserving flows, for the simple reason that no computers were available to solve the partial differential equations that appear in more general flows. Only in recent years have these models been tested in non-self-preserving flows such as boundary layers on aerofoils. Even here, providing that the boundary layer or other flow is not too far from self-preservation, fairly accurate predictions can be made by using values of μ_e or l from self-preserving flow. The fact that $f_3 = \text{constant}$ and $f_4 = \text{constant}$ have both been used successfully in the outer part of a boundary layer and in wakes and jets is simply a demonstration that our present standards of accuracy are not high: it can easily be shown that at the free-stream edge of a flow $\mu_e \longrightarrow 0$ and $l \longrightarrow \infty$.

One of the major obstacles to the use of more refined prediction methods is the lingering belief that the success of the eddy viscosity and mixing length models in fairly simple cases is a general *a posteriori* justification of the physical concepts used to derive those models, or at least a licence to use them in more complex flows. However, the mixing-length concept of identifiable "lumps" of turbulence is approached most closely near a free stream edge while the local-equilibrium "justification" of the mixing-length formula (4.8) is valid only in the inner layer of a wall flow . . . similar arguments to those above apply to the use of eddy-viscosity formulae for Reynolds stresses other than $-\rho\overline{uv}$, and to eddy-conductivity concepts in heat transfer.

We now come to a crucial question for the future: granted that (for whatever reasons) the eddy-viscosity and mixing-length formulae are useful first approximation in simple cases, can we use them as the basis of a second approximation for use in more complex cases or where higher accuracy is needed? The only such second approximations suggested to date are closely related to "second-order closure" as described in Section 1, being transport equations for eddy viscosity rather than Reynolds stress. Prandtl in 1945 (Ref. 31, p. 874) suggested a model in which the eddy viscosity was equated to $\rho\sqrt{\overline{q^2}}\,l$ where $q^2 = u^2 + v^2 + w^2$. The model was by implication restricted to thin shear layers, l being related to the shear layer width while $\overline{q^2}$ was obtained from a closure of the turbulent energy equation. Several later workers (Refs. 34–36 and Mellor and Herring in Ref. 30) have used Prandtl's 1945 model: others[37–40] have formulated transport equations for l in addition to one for $\overline{q^2}$ so that the model is nominally able to deal with flows other than thin shear layers. Nee and Kovasznay[30] have devised a wholly-empirical transport equation for μ_e. A good deal of effort has been devoted to these methods and numerical solutions have been obtained in quite complicated flows. In view of this I am sorry to say that I do not think eddy-viscosity transport equations are worth pursuing, for the following three reasons.

1. The simple behaviour of eddy viscosity and mixing length in simple thin shear layers is not maintained in more complicated cases like three-dimensional flow, multiple shear layers (*see* Fig. 7) or flows with significant extra rates of strain. In cases where the rate of strain changes rapidly (in the x or y direction) the Reynolds stress will respond slowly and not at once as implied by eddy-viscosity formulae.
2. There is no independent exact equation for eddy viscosity or mixing length analogous to the transport equations for Reynolds stress: therefore any independent eddy-viscosity transport equation must be completely empirical.
3. Any transport equation for eddy viscosity or mixing length can be converted into a transport equation for Reynolds stress by substituting for the velocity gradients, obtained by differentiating the time-average Navier Stokes equations.

The eddy-viscosity transport equations suggested to date look rather like the exact Reynolds-stress transport equations: however the empirical Reynolds-stress transport equations deduced from them by the process just described in (3) contain second derivatives of mean pressure and Reynolds stresses, and velocity-gradient factors, which are absent from the exact Reynolds-stress equations. Assuming that the eddy viscosity remains finite, the shear stress $-\rho\overline{uv}$ necessarily goes to zero when the mean rate of shear strain, $\partial U/\partial y$ in a thin shear layer, goes to zero: there are corresponding results for the other Reynolds stresses. This characteristic feature of eddy-viscosity models implies that there is no smooth transition from them to explicit Reynolds-stress closure. Formulation of explicit Reynolds-stress closures in terms of eddy viscosity, so that laminar-flow numerical procedures can be used, is unsuitable for the more complicated flows because the eddy viscosity is in general a fourth-order tensor which may (Fig. 7) take infinite values.

Various modifications of eddy-viscosity formulae have been suggested to remove their qualitative deficiencies (e.g. the addition of a term in $\partial^2 U/\partial y^2$ to maintain non-zero shear stress with zero $\partial U/\partial y$, and the use of an anisotropy factor to maintain a difference between the directions of shear stress and of velocity gradient in three-dimensional flow). However these modifications seem to take us further away from the exact Reynolds-stress equations, and I am not convinced that there is any good dynamical reason for formulating assumptions in terms of eddy viscosity rather than Reynolds stress, except in the simple cases where first approximations like (4.5) or (4.6) suffice.

In this discussion of eddy viscosity and mixing length I have had to criticise the first steps taken in our subject by great men no longer living. We may be sure that the pioneers, were they still alive, would themselves have superseded their earlier theories, so I hope my comments will not be taken amiss.

4.2. Mean-flow/Integral methods

Because of the difficulty of solving partial differential equations without a computer, the methods developed in the period 1930 to 1960 for aerofoil boundary layer calculations were based on the momentum integral equation. Information about the Reynolds-stress was inserted via an empirical skin friction formula and an auxiliary equation for the rate of change with x of the velocity-profile shape parameter $H \equiv \delta^\star/\theta$ or its equivalent. The usual form of the auxiliary equation is

$$\theta \frac{Hp}{dx} = \frac{\theta}{U_\infty}\frac{dU_\infty}{dx} f_1\left(H, \frac{U_\infty\theta}{\nu}\right) + f_2\left(H, \frac{U_\infty\theta}{\nu}\right). \qquad (4.9)$$

The exact equation for dH/dx, derivable from the boundary-layer form of the momentum equation, contains a weighted integral of the shear stress profile, so we see that (4.9) implies that the shear stress profile depends on H and $U_\infty\theta/\nu$ and possibly on dU_∞/dx. If an algebraic eddy-viscosity formula is combined with a velocity-profile "family" of the type $U/U_\infty = f_3(y/\theta,\ U_\infty\theta/\nu,\ H)$—and families exist which fit experimental profiles quite accurately—then $\tau/\rho U_\infty^2 = f_4(y/\theta,\ U_\infty\theta/\nu,\ H)$ which is of the form implied by (4.9). This is an example of the use of the generalised Galerkin technique.

The older equations of the type (4.9) were not derived from eddy-viscosity assumptions: instead, f_1 and f_2 were adjusted by trial and error so as to optimise predictions of H. As shown by Rotta[7] and Thompson[4] results were generally poor, especially in pressure distributions differing significantly from those used to find f_1 and f_2. One of the best wholly-empirical versions of (4·9) is that of Head[30,41] who did not adjust f_1 and f_2 directly but instead assumed that the rate of entrainment of fluid into a boundary layer is equal to U_∞ times a function of H

$$\frac{d}{dx}\int_0^\delta U dy \equiv \frac{d}{dx}\{U_\infty(\delta - \delta^\star)\} = U_\infty f_5(H). \qquad (4.10)$$

This still leads to an equation of the form (4.9) but is a simple and explicit assumption involving one empirical function of one variable. Good results were obtained for aerofoil-type flows not too different from those used to determine the function f_5. More recently Head and Patel[42] have improved the entrainment method, in effect by allowing f_5 to depend on $(\theta/U_\infty)\,dU_\infty/dx$ as well as H. This helps the method to distinguish between flows in which H is high because of high shear stress (leading to high entrainment) and those in which H is high because of strong adverse pressure gradient. A similar method in which an allowance for flow history is made by inserting the pressure gradient in what is nominally a turbulence function is that of Walz[30] and his collaborators, which makes use of a great deal of carefully-organised empirical information and is probably the ultimate method of its kind. By making this kind of allowance for flow history, wholly-empirical versions of (4.9) can in principle achieve better accuracy than methods based on algebraic eddy-viscosity formulae. Strictly speaking, however, the history is specified by the whole pressure distribution between the start of the boundary layer and the point considered, and the local mean pressure gradient does not appear in any exact Reynolds-stress transport equation in incompressible flow.

4.3. Transport-equation/Integral methods

A more formal way of representing the effect of flow history on the turbulence in a thin shear layer while still dealing only with ordinary differential equations is to leave the shear-stress integral in the shape-parameter equation in the exact form, so that (4.9) is replaced by

$$\theta \frac{dH}{dx} = f\left(\frac{\theta}{U_\infty}\frac{dU_\infty}{dx},\ H,\ \frac{U_\infty\theta}{\nu},\ c_\tau\right) \qquad (4.11)$$

where c_τ is a dimensionless form of the shear-stress integral, and the only empirical content is that provided by the velocity-profile "family" or its equivalent. We then write an empirical ordinary differential equation for c_τ as

$$\theta \frac{dc_\tau}{dx} = f\left(H,\ \frac{U_\infty\theta}{\nu},\ c_\tau\right). \qquad (4.12)$$

Several calculation methods with partly or wholly empirical versions of (4.12) have been produced and some give very good results.

Methods of this sort would result from the application of the Galerkin technique to the partial differential equations for shear stress obtained as closures of the exact Reynolds-stress transport equations: the latter will be discussed in Section 5. The disadvantage of the Galerkin technique—and indeed of integral methods as a whole—is the computational difficulty of extending the calculations to more complicated flows even if the turbulence assumptions are still sound. For instance, quite large changes, particularly in the profile "families", may be needed to incorporate suction, injection or surface roughness. In a field method extra effects can be incorporated without changing the basic calculation: in the example above one would simply change the surface boundary condition, while to extend the method to compressible flow one would add a density calculation to the existing program. For this reason I do not feel very enthusiastic about using the Galerkin technique on closures of the Reynolds-stress transport equations.

A general but apparently little-employed technique[43] is to use calculations by a field method, instead of experiments, to derive the empirical functions used in an integral method. The technique is most simply applied to functions of one variable, such as occur in Head's method, but it may be attractive for more elaborate integral methods, such as those based on (4.12). It has some relation to the more formal Galerkin technique.

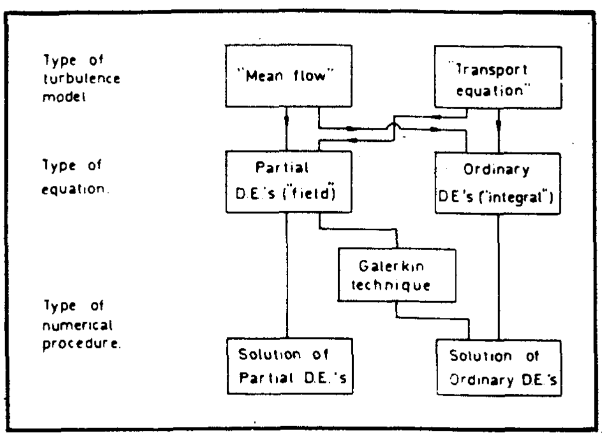

Figure 8. Classification of turbulence models and numerical procedures.

4.4. The 1968 Stanford meeting[30]

This meeting was a competition between all the available calculation methods for two-dimensional incompressible boundary layers, based on predictions of all the available test cases. Nearly all the methods in current use were presented. There were eight mean-flow/integral methods, 10 mean-flow/field methods, three transport-equation/integral methods and four transport-equation/field methods. In many cases the authors' presentations did not make the shear-stress assumption clear: W. C. Reynolds' introductory paper "A morphology of the prediction methods" is helpful although he counts Galerkin methods as Integral methods.

The sample of calculation methods was too small for the best type of method to stand out clearly. There were good and bad examples of each type, and good performance was as much a function of the state of development as of the realism of the assumptions. The evaluation committee which assessed the results emphasized that, for these reasons, its assessments were not durable. The main comments made about future developments were that only field methods were likely to be able to deal with complex (three-dimensional) flows and that methods allowing for turbulence history (transport equation methods) were likely to give better results, when properly developed, than methods which did not allow for turbulence history. It is mainly field methods that have been extended to take account of three-dimensionality, compressibility, heat transfer and other real-life effects, and to flows other than attached boundary layers, although Head's Integral entrainment method has also been extended to these cases. Also, most of the new calculation methods developed since the Stanford meeting have made allowances for turbulence history.

The proceedings of the Stanford meeting are still an excellent user's guide to well-established methods for predicting turbulent flow but a few shortcomings have to be borne in mind. Quite apart from the restriction of the meeting to two-dimensional incompressible isothermal flow, boundary layers are a less severe test of turbulence assumptions than, say, jets, because most of the work is done by the law of the wall and the assumptions made to calculate the small velocity defect in the outer layer are not critical except for the cases of strong adverse pressure gradient or strong injection. Near separation, wind-tunnel boundary layers tend to converge laterally because of boundary layer growth on the side walls, and the omission of any allowance for this makes it difficult to judge the reliability of the Stanford methods near separation. I believe that a boundary layer calculation method should be able to predict surface shear stress (in a known surface pressure distribution) up to a point very close indeed to separation, because failure of the boundary layer approximation in the outer layer does not affect the flow near the surface. The Stanford meeting did not attempt to assess numerical accuracy or computing times: some methods showed obvious shortcomings in one or other area. I made some comments about comparative running times for the different field methods, and for several quasi-Galerkin versions of my own calculation method, in Ref. 44. It appears that the better numerical procedures used in field methods all take very roughly the same computing time, that the Galerkin versions of field methods can be up to half an order of magnitude faster and that a simple method like Head's, programmed without especial care, is roughly one order of magnitude faster than a field method. Only the user can decide whether an integral method or a slower-running but (perhaps) more accurate field method is the better choice for a particular case. We may expect field methods to be used in future for most cases unless speed is overwhelmingly important.

5. TRANSPORT-EQUATION/FIELD METHODS

At the time of the Stanford meeting, calculation methods using partial differential (transport) equations for turbulence quantities were mostly in an early stage of development and their performance as a group was not impressive. However most of the new calculation methods (as distinct from extensions of existing methods to more complicated cases) that have appeared since the Stanford meeting have been of this type: evidently workers have heeded the remarks of the evaluation committee mentioned above, especially those relating to complex flows. For a specialist review of individual methods up to 1970 see the paper by W. C. Reynolds[9]. New methods continue to appear but it is possible to distinguish the general type of method that will receive most attention from developers of calculation methods in the next few years and most attention from users of calculation methods in the few years after that. Perhaps I am crowing too long before sunrise but I think it is clear that transport equations for eddy viscosity, after flourishing just after the Stanford meeting, are being abandoned by some of their former adherents and that the newest methods are based solely on equations derived from the Navier Stokes equations: exact "transport" equations for the Reynolds stresses, and in general for turbulence length scales, are approximated term by term, in exactly the way suggested by Rotta[45] in 1951. Moreover, people are now giving Dr. Rotta credit for it, which is almost as big a breakthrough. My guesses at the range of validity of the various assumptions used in this "closure" process are given in Fig. 9.

The simplest type of transport-equation closure[44] uses one Reynolds-stress transport equation, necessarily for $\overline{uv}$, and an algebraic relation between length scale and shear-layer thickness. Both of these features restrict it to thin or quasi-parallel shear layers. The basic assumption is that the shear-stress profile sufficiently defines the turbulence field. The next step in sophistication[2] is to use transport equations for all the non-zero Reynolds stresses (whether their gradients affect the mean motion or not) still using an algebraic length scale. Here the whole set of Reynolds stresses is used to define the turbulence field. This will be an advance on the first type of closure only if the latter is limited by the $\overline{uv}$ assumption rather than the length scale assumption and I am not sure myself that this is so (compare rows two and three of Fig. 9). To deal with complex flows, in which the shear layer thickness may be changed rapidly by mean-flow acceleration, a transport equation for length scale is needed in addition to transport equations for all the Reynolds stresses whose gradients are significant. Several of the new methods use length-scale equations but I doubt whether any of them can yet be relied on to reproduce all the curious effects of extra rates of strain which are the essence of complex flows. However, second-order closure has already shown great promise and, when based more firmly on measurements, the new methods should produce results of engineering accuracy for a much wider range of flows than could be treated by any of the earlier generations of calculation methods.

5.1. Length-scale equations

Transport equations for length scale are less well understood than the Reynolds-stress transport equations discussed in Section 2. To begin with, there is no agreement on what length scale one should choose to represent the energy-containing eddies. More than one may be needed[46]

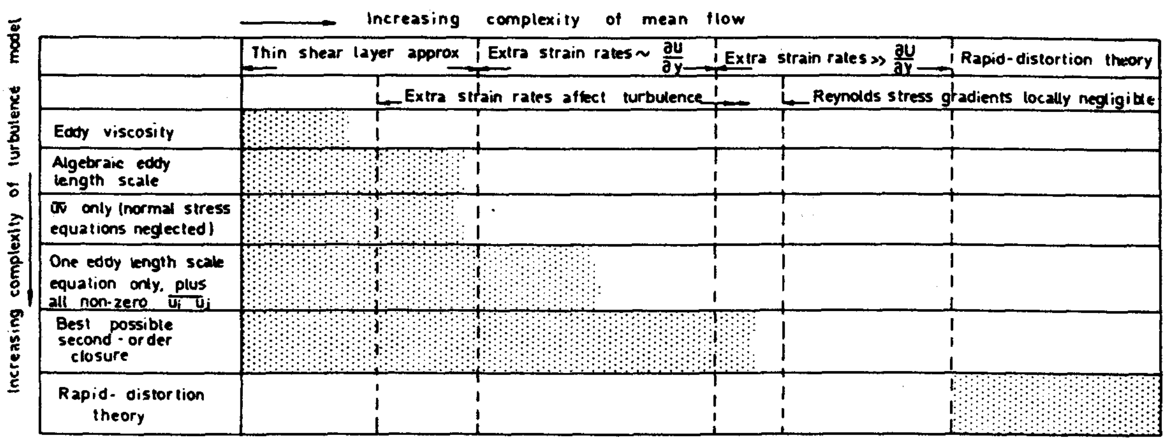

Figure 9. Estimated range of validity of various turbulence models (left hand column) in flows of increasing complexity (top two rows).

because length scales in different co-ordinate directions are no more likely to be proportional than are the different Reynolds stresses, particularly in flows with complicated rates of strain. This is an example of an ever-present difficulty: we are always in danger of ruining a high-grade model by making just one low-grade assumption. The types of "length scale" so far suggested are the correlation integral scales[45], the dissipation rate (implying a dissipation length parameter $(\overline{u_i^2})^{3/2}/\epsilon$)[46, 48] and the mean-square vorticity or frequency of the energy-containing eddies, w (implying a length scale $(\overline{u_i^2}/w)^{1/2}$)[40]. The first two definitions are exact, and exact transport equations can be derived; the third is empirical. All the transport equations have terms whose meanings are the same as those of the terms in (2.1) except that the pressure "redistribution" term [2] is absent if the length scale or its equivalent has no preferred direction. A new type of "destruction" term appears, representing the tendency of the length scale to decrease because of transfer of energy to smaller scales by vortex stretching.

The attraction of using the dissipation rate ϵ to define a length scale is that ϵ itself appears (equipartitioned at high Reynolds number) in the transport equations for the Reynolds normal stresses. However the ϵ equation has a number of disadvantages: for instance, each term contains the viscosity although we know that ϵ is independent of viscosity at high Reynolds numbers. Rodi[47] has recently pointed out that the vortex-stretching term, just mentioned, and the viscous-destruction term, analogous to [4] in (2.1), are vastly larger than all the other terms at high Reynolds number. Clearly the difference between the two terms must be of the order of the other terms and Rodi suggests that the two should be grouped together for modelling purposes. The dissipation ϵ is equal, at high Reynolds numbers, to the rate of transfer of energy from the energy-containing eddies to the smaller eddies, for which a transport equation can be written. This transport equation would avoid some of the above difficulties but, since the energy transfer depends on certain triple correlations, it is not a feasible choice for a length-scale equation at present. However the equality of ϵ and energy transfer implies that use of a transport equation for ϵ is a first step towards third-order closure.

It is of course arguable that in our present state of knowledge none of the "length-scale" equations amounts to more than an empirical equation like

$$\frac{Dl}{Dt} = a_{ij} l \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j}(lV_j) - (b_{ij}\overline{u_i u_j})^{1/2} \quad . \quad (5.1)$$

where the dimensionless tensors a and b and the turbulent transport velocity V are to be determined by trial and error. Transport equations in which the length scale is implicit (e.g. the dissipation equation, whence $l=(\overline{u_i^2})^{3/2}/\epsilon$, can be reduced to the form (5.1) by using the Reynolds-stress transport equations. Physical interpretations of the terms provide only qualitative constraints. A difficulty which has not so far been explicitly faced, but which I regard as serious in the length scale equation if not the Reynolds stress equations, is that the local Reynolds stresses may not adequately represent the turbulence terms. Firstly, if l is a meaningful eddy length scale its behaviour at a given point will depend on conditions in a volume of order l^3 surrounding that point. Also, especially in the case of flows with solid boundaries, the mean flow width may be signalled to the eddies mainly by pressure fluctuations or irrotational ("inactive") velocity fluctuations, dependent on integrals over the whole flow volume. Ng and Spalding[39] allowed for the wall effect by introducing the distance from the wall into b_{ij} and Rotta[45] suggested that (5.1) should contain third derivatives of mean velocity, thus allowing, on a Taylor-series basis, for spatial variation of the mean rate of strain: however both sets of authors still used local Reynolds stresses to provide the velocity scales.

5.2. The state of the art

One subject on which all recent workers are agreed is the representation of turbulent transport processes in the transport equations: they use the gradient-diffusion hypothesis

$$\overline{\phi' u_j} = c\,\frac{\partial \bar{\phi}}{\partial x_j} \quad . \quad . \quad . \quad . \quad (5.2)$$

where ϕ stands for Reynolds stress or length scale as appropriate and primes and overbars denote fluctuations and means respectively. It is of course a pity that gradient diffusion should be abandoned for mean-flow quantities but retained for turbulence quantities, but if one regards gradient diffusion as a first approximation it can be used with consistency for small terms in a second approximation (i.e. a transport-equation model). The alternative "bulk convection" hypothesis[11, 44],

$$\overline{\phi' u_j} = c\bar{\phi} V_j \quad . \quad . \quad . \quad . \quad (5.3)$$

where V_j is a velocity scale of the large eddies, has more physical justification and leads to differential equations of lower order: one would have expected the latter consideration, if not the former, to have some appeal. Fortunately the workers who used (5.2) with ϕ equal to the pressure have now abandoned this *reductio ad absurdum* and included the pressure-transport terms with the triple-velocity-product terms. Hanjalic and Launder[48] have considered the exact transport equation for the triple products and, using experimental data and heuristic arguments, reduced it to the form (5.2) with $\phi = u_i u_j$. However their reasons for neglecting the explicit or implicit appearance of mean velocity gradients in the exact equations are not completely convincing.

Agreement is in sight on another of the controversial subjects, whether the empirical representation of the pressure-strain terms [2] in (2.1) should contain mean velocity gradients or only turbulence terms. Undoubtedly the Poisson equation for p' (another exact equation derivable from the Navier Stokes equations[11]) contains mean velocity gradients and Rotta (1951)[45] suggested that the pressure-strain term should do so as well: Rotta later (1962)[7] reverted to assuming that only the turbulence terms were important and suggested that the pressure-strain term was directly proportional to the anisotropy

$$\overline{p'\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}\ \alpha\ \left(\overline{u_i u_j}-\frac{1}{3}\,\delta_{ij}\overline{u_i^2}\right).$$

Donaldson[49] and Herring and Mellor[50] used this form. Champagne *et al*[51] stated that their Reynolds-stress measurements in a homogeneous shear flow supported the "anisotropy" model, but in fact the discrepancies between their measurements and the "anisotropy" model were larger than one could accept in a calculation method. Bradshaw[52] showed that his shear-stress equation, derived from closure of the turbulent energy equation, implied the presence of the mean velocity gradient in the pressure-strain term. More recently Hanjalic and Launder[48] have closely followed Rotta (1951): Daly and Harlow[46] have produced a form containing mean velocity gradients but say that there is evidence (unspecified: Champagne?) that they are negligible.

The "length scale" equations proposed in the literature differ widely even when one has reduced the equations for dissipation, (energy/length) and vorticity to the explicit length-scale form (5.1)—for instance the equation of Rodi and Spalding[38] proves to have $a_{ij}=0$, and a_{ij} is small in refs. 39 and 48. Some of these equations[38-40] are intended for use with eddy-viscosity formulae of the type $\mu_e = c\rho\sqrt{\overline{u_i^2}}\,l$ and therefore form part of an eddy-viscosity transport equation, but they could also be used, in principle, for true Reynolds-stress closure.

As representatives of the post-Stanford developments I will select the methods of Daly and Harlow[46], Donaldson[2, 49] and Hanjalic and Launder[48], which are all true Reynolds-stress closures (Table I). My own 1967 method[44], using only one Reynolds-stress equation and an algebraic length scale, is in the same spirit as these but rather less ambitious. Other recent work (e.g. that of Rotta[53], Shamroth and Elrod[54], Lundgren[55], and Mellor and Herring[50]) has not progressed as far. Earlier work by the groups to which Harlow and Launder belong used eddy-viscosity transport equations, so their abandonment of eddy-viscosity can be regarded as significant, though Launder (private communication) considers that at present his own method is less computationally flexible than that of Ref. 39 which uses an isotropic eddy viscosity.

So far, only Daly and Harlow's method has been applied to a case other than a thin shear layer (they give results for a comparatively simple case of turbulence distortion). Hanjalic and Launder have treated the flow in

TABLE I

Comparison of essential features of the latest transport-equation methods

Authors	*Donaldson*	*Harlow*	*Launder*
Reynolds-stress transport equations	$\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$	$\overline{u_i u_j}$	$\overline{uv}$, $\overline{u_i^2}$ *
"Length scale"	algebraic	Transport equation for dissipation rate	Transport equation for dissipation rate
Mean strain rate in pressure-strain?	no	yes but assumed zero	yes
Reynolds number dependent?	maybe	yes	no
Turbulent transport?	gradient diffusion	gradient diffusion	gradient diffusion†

*In thin shear layer version: ratio of normal stresses to $\overline{u_i^2}$ assumed constant.

†Obtained by "firm pruning" of transport equation for triple products.

an asymmetric duct (in which velocity gradient and shear stress change sign at different points) and Donaldson has calculated an axisymmetric vortex flow: these are quite severe tests of the basic concepts, and the asymmetric duct would defeat an eddy-viscosity method. Hanjalic and Launder have done most of the analytical work needed to extend their method to general flows, but in its present form Donaldson's method is restricted to flows where a length scale is imposed by the boundary conditions—in effect, to thin shear layers.

Apart from the differences listed in Table I the three methods agree on most general points, though there are numerous details over which they disagree with each other (and with my own views). Hanjalic and Launder's treatment of the dissipation equation seems more realistic than that of Daly and Harlow (*see* Ref. 47). All three methods are still in progress of development (Harlow's in particular has changed greatly since the publication of Ref. 37) and the empirical content of each may be further refined, by physical reasoning or the use of turbulence data. At present most of the numerical input consists of constants even where the methods would admit functions of dimensionless quantities. For the sake of ending this paper with a positive statement after so many negative and cautionary ones, I would choose Hanjalic and Launder's method as the "optimum model", the most promising of the post-Stanford generation: it seems to be the best compromise between flexibility (requiring a refined turbulence model) and tractability (requiring a simple model so as not to outstrip our empirical knowledge of turbulence).

6. CONCLUSIONS

In the last decade, the digital computer has enabled the developers of prediction methods to make fuller use of our experimental understanding of turbulence in devising realistic models of Reynolds-stress behaviour[56]. In particular, it is now possible to use the exact Reynolds-stress transport equations as a framework for calculation methods, just as they have long been used as a framework for experiments. For the first time, significant interaction is occurring between predictor and experimenter and we may hope for rapid progress in future. The experimenter has the new techniques of conditional sampling, laser anemometry, and measurement of pressure fluctuations within the stream. Solutions of the full time-dependent Navier Stokes equations, though too expensive for engineering use even with approximations for the small-scale motion, can be used for "numerical experiments" on quantities which are difficult to measure directly. Calculation methods and basic experiments are starting to treat "complex" turbulent flows (flows other than simple shear layers) which should increase the possibility of interaction between research workers and engineers. Fig. 10 shows the ideal state of turbulence studies, incorporating interactions between developers of calculation methods, experimenters and users. I hope this lecture has given some of the engineers who use calculation methods an introduction to the latest developments, which should soon start to replace the methods of the Stanford era[30] in cases where the highest accuracy or the greatest flexibility are needed. I hope the lecture may also be a useful introduction to Dr. Rotta's book[57], which I did not see until after my lecture was written. I have not had space to deal with geophysical problems although I acknowledge their importance and also the significant contribution made to our knowledge by meteorologists and oceanographers. Nor have I dealt with more than a small fraction of the exciting experimental work on turbulence that is being done in this country and elsewhere, but I hope I have made clear my belief that experiments are the foundation of our understanding and should be the foundation of our prediction methods.

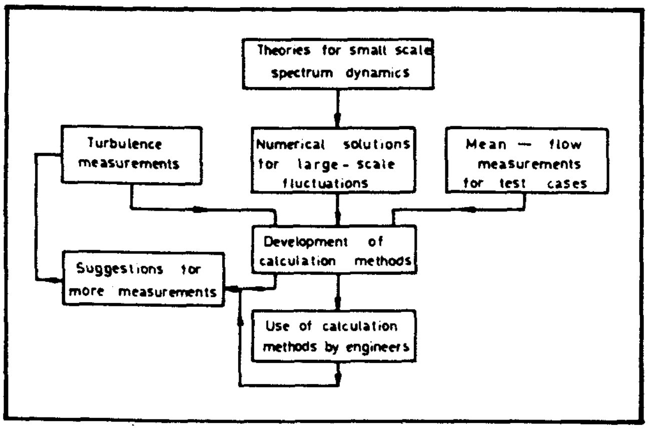

Figure 10. The ideal state of turbulence studies.

What would our heroes say to all this, Reynolds who never saw hot-wire measurements of his turbulent stresses, Prandtl who never saw computer solutions of his turbulence models? Would they be amazed at the spectacular progress we have made? Perhaps they would be amused to find that with all our hot wires and computers we have still not achieved an engineering understanding of turbulence, and that it is still as important and fascinating and difficult a phenomenon as when the first steps in studying it were taken by Reynolds and Prandtl.

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