Turbulent Flows: General Properties

Turbulence is ubiquitous in natural fluids: atmosphere, ocean, lakes, rivers, Earth's interior, planetary atmospheres and their convective interiors, stars, and space gases (neutral and ionized). From a mathematical perspective, its essential behaviors arise because the equations of fluid dynamics are a nonlinear partial differential equation (PDE) system. From an accompanying physical perspective, the advection (*i.e.,* movement following fluid parcels moving with the velocity field) causes the generic behavior of the entanglement of neighboring material parcels; this causes chaotic evolution, transport, and mixing.

The subject of this course is turbulence in Earth's tropospheric atmosphere and ocean on spatial scales less than the general circulation and climate.

Turbulent flows manifest a complexity that has thus far exceeded scientists' abilities to measure, theorize, or simulate comprehensively. Nevertheless, much has been and is being learned about the behaviors of turbulence, and these lectures provide a survey of the fruits of this research. The major themes are

- statistical symmetries and observed statistical regularities
- power of dimensional reasoning (a.k.a. similarity or scaling theory)
- patterns, processes, and effects of coherent structures in different physical regimes
- interaction of turbulence with mean flow and density stratification in statistical equilibrium regimes
- utility of simple conceptual and parameterization models of turbulence for the use in simulating larger-scale flows.

1 Incompressible Fluid Dynamics

The basic equations of fluid dynamics are the *Navier-Stokes Equations* for parcel acceleration in the presence of forces[1]. In this course we will further make the approximation of *incompressibility*, not because it is a very accurate approximation in the ocean and atmosphere but because it allows a simpler dynamical characterization of the essential turbulent phenomena[2] The result is the *Boussinesq Equations*:

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= -\nabla\phi + \nu\nabla^2\mathbf{u} + \Big[\,\hat{\mathbf{z}}b - f\hat{\mathbf{z}}\times\mathbf{u}\,\Big] \\ \nabla\cdot\mathbf{u} &= 0 \\ \Big[\,\frac{Db}{Dt} &= \kappa\nabla^2 b\,\Big]\,, \end{aligned} \tag{1}$$

[1]If viscosity is neglected, they are called the *Euler Equations*.

[2]There is a useful generalization of the incompressible approximation for flows that extend over a significant vertical range in the presence of a background density stratification, $\overline{\rho}(z)$. It is called the *analastic approximation*, with a mass balance equation, $\nabla\cdot(\overline{\rho}(z)\mathbf{u}) = 0$. It includes some effects of compressibility while still excluding extraneous acoustic motions.

where $\mathbf{u}$ is the velocity; p is the pressure (*n.b.,* Appendix A); $\phi = p/\rho_o$; f is the Coriolis frequency associated with (planetary) rotation; ν is the molecular viscosity; and κ is the molecular diffusivity. b is the buoyancy field,

$$b \;=\; g\left(1 - \frac{\rho}{\rho_0}\right) , \tag{2}$$

where g is the gravitational acceleration constant; ρ is the density; and ρ_0 is a constant reference value. The vertical direction $\hat{\mathbf{z}}$ is assumed to be parallel to both gravity and the axis of rotation for simplicity. A simple thermodynamic equation of state is implicit in (1) whereby density is adiabatically conserved following a parcel when $\kappa = 0$. (This is too simple a model for moist convection in the later chapter on *Planetary Boundary Layer Turbulence*.) The square-bracketed terms in (1) include the effects of rotation and gravitation, which are not part of the "classical" turbulence problem (a.k.a. homogeneous or shear turbulence; Batchelor (1967) and Tritton (1988)). These equations must be completed by forcing and boundary and initial conditions for a well-posed problem.

We can also analyze the evolution of a passive tracer (*i.e.,* a material concentration field). It satisfies the same equation as b above, but it has no influence on the evolution of $\mathbf{u}$.

The essential nonlinearity is contained in the advective time derivative,

$$\frac{D}{Dt} \;=\; \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \, . \tag{3}$$

Thus D/Dt is the time derivative following the flow, *i.e.,* in the Lagrangian reference frame. The second term in (3) is called *advection*. The essential effect of ν and κ is to spatially smooth the fields of $\mathbf{u}$ and b, respectively, through *diffusion*.

These equations have conservative (*i.e.,* with $\nu = \kappa = 0$) spatial *integral invariants* for the energy and all powers or functionals of the buoyancy and passive tracer fields. For non-conservative dynamics, the *energy* and *scalar variance* principles are

$$\frac{dE}{dt} \;=\; -\,\mathcal{E} \quad \& \quad \frac{dV}{dt} \;=\; -\,\mathcal{E}_b \, , \tag{4}$$

where

$$[E,\, V,\, \mathcal{E},\, \mathcal{E}_b] \;=\; \int\!\!\int\!\!\int d\mathbf{x}\, [e,\, b^2,\, \varepsilon,\, \chi] \, , \tag{5}$$

and

$$e \;=\; \frac{1}{2}\mathbf{u}^2 + zb, \quad \varepsilon \;=\; \nu\, \nabla\mathbf{u} : \nabla\mathbf{u}, \quad \chi \;=\; 2\kappa(\nabla b\cdot\nabla b) \, . \tag{6}$$

e is the *energy density*, comprised of kinetic and potential energy components. The right-hand-side terms in (4), ε and χ, are the *dissipation* terms that act to decrease the kinetic energy and scalar variance. They are written in tensor notation in (6), or they can alternatively be written in index notation as

$$\varepsilon \;=\; \nu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}, \quad \chi \;=\; 2\kappa\frac{\partial b}{\partial x_j}\frac{\partial b}{\partial x_j} \, ,$$

where a repeated index implies index summation. In deriving (4), it is assumed that there are no boundary fluxes of energy or scalar variance (which certainly is not always true). In the absence of forcing, sources, or boundary fluxes, these integral measures of the flow can only decrease with time through the action of molecular viscosity and diffusivity. The right-hand-side terms in (4), ε and χ, are the *dissipation* terms that act to decrease the kinetic energy and scalar variance.

Another parcel invariant for conservative flows governed by (1) is the *potential vorticity*, defined by

$$Q \;=\; (f\hat{\mathbf{z}} + \zeta) \;\cdot\; \nabla b \,, \tag{7}$$

where $\zeta = \nabla \times \mathbf{u}$ is the 3D vorticity. Note that it becomes trivial (*i.e.,* dynamically meaningless) in the absence of gravitational effects[3]. Its integral variance is called *potential enstrophy*. A related quantity is the integral variance of ζ, called *enstrophy*.

2 Transition to Turbulence

If we nondimensionalize (1) by characteristic scales,

$$\mathbf{x} \sim L, \quad \mathbf{u} \sim V, \quad t \sim \frac{L}{V}$$

(the latter assumes that advection dominates the evolution), then the momentum and scalar diffusion terms will have nondimensional prefactors that are the inverse of

$$Re \;=\frac{VL}{\nu} \quad \& \quad Pe = \frac{VL}{\kappa} \,, \tag{8}$$

respectively. Re is the *Reynolds number*, and Pe is the *Peclet number*. Thus, the physical statement that advection dominates diffusion is equivalent to $Re,\ Pe \;\gg\; 1$. For historical reasons turbulent flows dominated by gravitational instability, called *convection*, have a different parameter that expresses the degree of advective dominance, the *Rayleigh number*,

$$Ra \;=\frac{BL^3}{\nu\kappa} \,, \tag{9}$$

where B (*e.g.,* $B \,=\, \alpha g \Delta T$ in thermal convection) is a characteristic scale for the buoyancy difference forcing the flow. Again, $Ra \;\gg\; 1$ is the advectively dominated, turbulent regime. Note that for $\nu \sim \kappa$ and $V \sim \sqrt{BL}$ (often called the *free-fall velocity*, associated with the velocity of a parcel gravitationally accelerated over a distance L),

$$Ra \;\sim\; Re^2 \,.$$

As a traditional shorthand, Re is used as the representative parameter among Re, Pe, and Ra; all three parameters are essentially related for fluids like air and water, in which ν and κ have similar values.

A basic perspective for fluid dynamics is to view Re as a *control parameter*: as Re varies the nature of the flow varies. The sequence of flow regimes that occur as Re increases is called the *transition to turbulence*.

[3] In a 2D or barotropic fluid with $\partial_z = 0$, $f + \zeta^z$ plays the role of potential vorticity without gravity influence.

If $Re \ll 1$, the advective terms can be neglected, and the governing dynamics of the fluid become linear. In the case of steady flow, solutions are simpler yet, with a simple balance between pressure gradient and viscous momentum diffusion (here using index notation):

$$\frac{\partial \phi}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_j^2} \,, \tag{10}$$

which is called *creeping flow*. Equation (10) has been solved for a variety of situations and geometries. It is generally of no geophysical interest, with two possible exceptions: *laminar Poiseuille Flow* and *Stokes Flow*. The former describes the flow profile in pipes or channels and can be important in observational instruments. The latter describes the flow around an object and is important in understanding the hydrodynamics of small *hydrometeors* in suspension (cloud droplets) and other *aerosols* and *aquasols*. For larger hydrometeors such as rain-drops and ice particles, *inertial effects* (*i.e.,* involving the finite mass of the particle) become increasingly important in modifying (10).

For almost every other flow of interest in the atmosphere and ocean, Re is very much greater than one. For instance, in the atmospheric boundary layer (ABL), $L \approx 1000$ m, $U \approx 1$ ms^{-1}, and $\nu \approx 1.5 \times 10^{-5}$ m^2 s^{-1}, leads us to estimate $Re \approx 10^8$. This means that, if one were to measure the contribution of various terms in (1) to the momentum balance of u_i,, one would expect to find that the viscous term is entirely negligible — fully eight orders of magnitude smaller than the other terms. And while this is true for components of the flow of scale L this is not to say that viscosity plays no essential role. Paradoxically, even when Re is large, the dissipation terms in (1) are usually not small. The paradox disappears when one realizes that simple scale estimation or dimensional analysis is not reliable in this matter, since turbulence comprises an exceedingly broad range of scales: for larger scales of motion, which may contain most of the energy or scalar variance, Re is large and the evolution is essentially conservative over a characteristic advective (or *eddy turnover*) timescale of L/V; whereas for small enough scales of motion, which may contain most of the variance of vorticity and scalar gradients, the relevant Re value is small and the dissipation rate is dynamically significant. Thus, there can be profound differences in solution behavior between the asymptotic tendency as $Re \to \infty$ and the Euler limit, $Re = \infty$ or $\nu = 0$, and one must be very careful about this situation. A hallmark of high-Re flows is their ability to generate small scale fluctuations on which molecular effects can operate efficiently.

Viewing Re as a control parameter, we can ask what will happen to a flow as Re is steadily increased, from a small number less than one, to a very large value, such as is characteristic of the ABL. In doing this we are asking how does the flow change as advective effects become increasingly dominant and viscosity becomes increasingly relegated to smaller and smaller scales. We find that as Re becomes large relative to unity the flow behaves increasingly erratically. This is illustrated with the help of Fig. 1 for the wake flow behind a cylinder. Similar figures, differing only in the details, could be constructed for most other flow regimes. In each case, once Re becomes sufficiently large, we will find a completely erratic flow field, with no evidence of global order; we call *fully developed turbulence*.

The sequence of steps flows go through on their way to becoming turbulent is called the transition to turbulence. Exactly how flows transition to turbulence is a wide area of study. In Fig. 1 the route is through something called *frequency doubling*. This is evident because the flow goes from being completely steady for $Re < 1$ (not shown) to oscillatory at $Re = 85$. When Re increases to 185 the period of the oscillations doubles, and there is some evidence of low-frequency

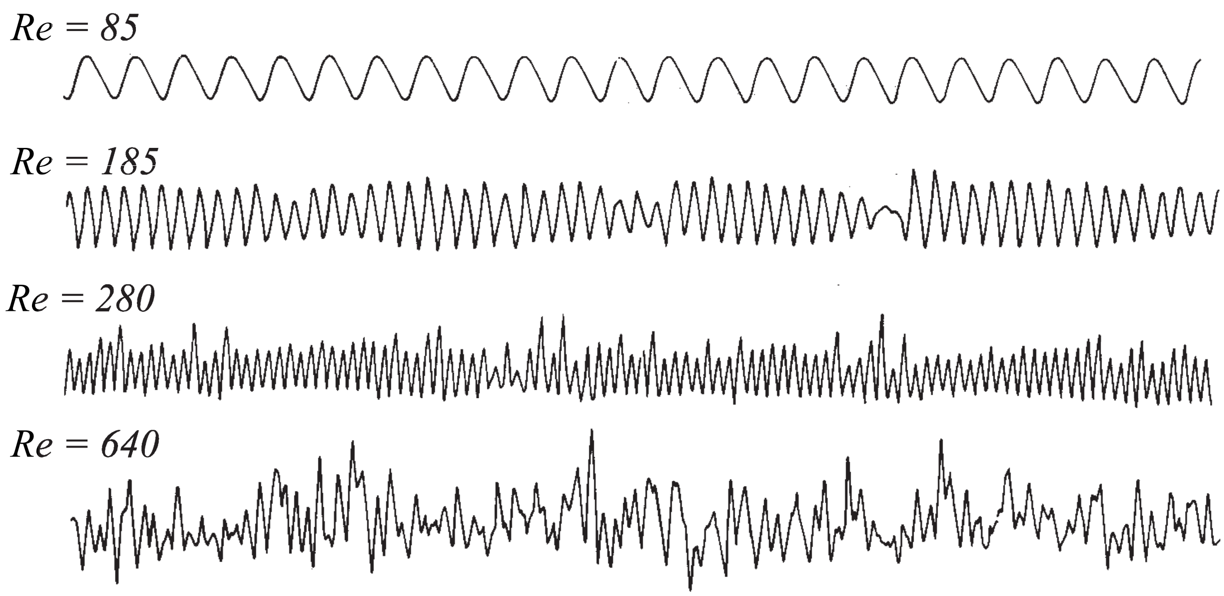

Figure 1: Illustration of the transition to turbulence as a function of Re for a cylindrical wake. Note that the timescale of the last panel has been increased by a factor of about 3. Adapted from Tritton (1988, Fig. 3.10).

modulation. Increasing Re further leads to yet another doubling and to even less uniformity in the amplitude of the signal. By the time Re increases to 640, there is no longer evidence of a single dominant period, but rather the time-series of the flow velocity is rather erratic and seemingly unpredictable.

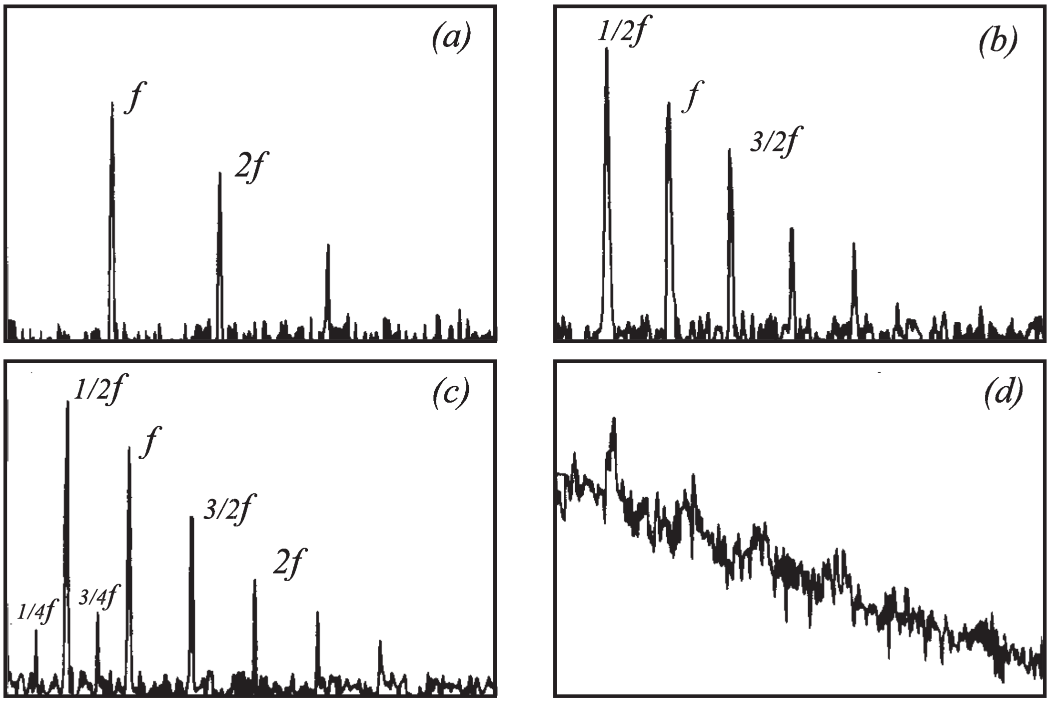

Figure 2: Illustration of the transition to turbulence as a function of Ra for Rayleigh-Benard convection. Shown is the power spectra of velocity for: (a) $Ra/Ra_c = 21$, (b) $Ra/Ra_c = 26$, (c) $Ra/Ra_c = 27$, (d) $Ra/Ra_c = 37$. ($Ra_c \sim 10^3$ is the "critical" value at which the conducting state is unstable to the onset of steady convective cellular circulations.) In each panel the abscissa is linear in frequency and logarithmic in power. Adapted from Gollub *et al.* (1980); see also Tritton (1988, Fig. 24.8).

This process can also be illustrated by considering the frequency spectrum of the signal. In Fig. 2 we do just that, except it is the case of *Rayleigh-Benard convection*, which is the flow between two horizontal plates with an imposed destabilizing buoyancy gradient between them (*e.g.,* a hot plate below a cold one). Instead of looking at the transition in terms of Re, we look at things in terms of Ra, as discussed above. For $Ra < Ra_c = 1700$ we have a motionless state with diffusive heat flux between the plates, and the first instability of this state develops at Ra_c with further instabilities (*i.e.,* bifurcations) for $Ra > Ra_c$. At $Ra = 21Ra_c$ we find the dominant unstable mode and its harmonics. As Ra increases further to $26Ra_c$ sub-harmonics with frequencies of $f/2$ appear, indicating period doubling. A slightly greater increase leads to further harmonics, $2f$, $3f/2$, $3f/4$, $f/4$, etc. before the onset of turbulent flow (*i.e.,* the absence of any dominant periodicity, but rather fluctuations at all frequencies) by $Ra = 37Ra_c$.

A recurring idea in the above is that flows become turbulent through a well ordered sequence of steps, but once they are turbulent (or if you like, once Re is large enough) further increases in Re lead to no qualitative changes in the flow, in effect implying that the flow becomes independent of the exact value of viscosity in such a regime. This state of affairs is associated with fully developed turbulence and is often referred to as *Reynolds number similarity*.

The following represent typical Re values for successive regimes:

$Re \to 0$ is laminar flow, which is smoothly varying in space and time, hence predictable in principle.

$Re \geq Re_c \sim 10 - 10^3$ is the transitional flow regime, in which there is instability in any hypothesized laminar (or mean) flow and often some degree of approximate periodicity of fluctuations in space and/or time. Therefore, the wavenumber and/or frequency spectra usually show sharp peaks. This dynamics can often be analyzed linearly, either in terms of normal mode instabilities or non-normal (algebraic) growth, or in terms of low-order, nonlinear ordinary differential equations in time (sometimes called amplitude equations). The spatial patterns in this regime are usually rather simple and often have a global (*i.e.,* domain filling) extent; however, the temporal behavior can be more complex. Often there are several different Re_c values associated with different bifurcations (*e.g.,* in non-rotating Rayleigh-Benard convection), and temporal complexity occurs only after multiple bifurcations.

$Re \to \infty$ (or at least $Re \gg Re_c$) is fully developed turbulence, which is rapidly varying in space and time, hence unpredictable in practice over many fluctuation cycles. Its spatial and temporal patterns are complex and lack global order, although they typically have clearly evident locally ordered patterns, called coherent structures. Therefore, the wavenumber and frequency spectra are broadband and without sharp peaks. This dynamics is fundamentally nonlinear, and this regime will be the focus of this course.

It is very much an open question whether there are usually any important bifurcations beyond $Re \sim 10^4$ or $Ra \sim 10^8$, and as yet there is not convincing evidence that this is so. This question has been extensively studied in the Rayleigh-Benard problem, where experimental apparati are capable of producing very large Ra values using liquid helium. The way the problem has been posed for these flows is to ask how the scaling properties, rather than the flow patterns, change as a function of Ra. Specifically the question has been asked as to how the nondimensional heat flux (called the *Nusselt number,* Nu, and defined to be the ratio of the total heat flux to the diffusive heat flux), scales with Ra. The hope has been that once Ra has been increased far enough the $Nu - Ra$ scaling ceases to change, and then we can obtain a universal, essentially Ra-independent regime. Experiments have shown that for Ra values that are large but less than 10^8, $Nu \propto Ra^{1/3}$; however, for larger values of Ra the scaling regime changes and $Nu \propto Ra^{2/7}$. It was originally thought that the 1/3 regime was the asymptotic regime, but the discovery of the new regime led to revisions in that thinking. Because both regimes are turbulent yet distinct, they are often referred to as *soft and hard turbulence*, respectively. Recently this thinking as had to be revised yet again: at increasingly large values of Ra there is evidence that the 2/7 regime also ceases to persist. Thus the question as to whether truly hard turbulence, and by implication Reynolds number similarity, actually exists remains unanswered.

The lesson above illustrates that attempts at a strict definition of turbulence can be problematic. We have taken the opposite approach, loosely associating turbulence with erratic behavior in the flow. One can be a bit more precise without being restrictive. Perhaps the most general and useful definition of a flow as turbulent is if it exhibits spatial **and** temporal complexity through advective dynamics. This is a concise way of saying that turbulent flows are fundamentally nonlinear, and that their nonlinearities lead to broad-band wave-number and frequency spectra. This is particularly evident in Fig. 2 where we note with increasing Ra an increasing range of scales becomes evident in the velocity power spectrum. By this broader definition turbulence is not limited to smaller scales in the atmosphere and ocean, and we can usefully interpret winter storms and oceanic mesoscale eddies as manifestations of turbulence on the planetary scale.

3 Turbulent Cascades

The naive scaling of the momentum equation above suggests that the viscous terms would be negligible. But how can that be? For instance we know that even high Re fluids obey no-slip boundary conditions at a surface. For the case of bounded flows, Prandtl developed the concept of a boundary layer; a thin layer attached to the surface wherein viscosity is important. For viscosity to be important implies that this layer has a scale,

$$\lambda \propto \sqrt{\nu L/U}. \tag{11}$$

This relation is an example of what might be called *dimensional analysis* or *dimensional reasoning*, *i.e.,* representing dynamical balances in the governing PDE by scale estimates. It follows from the statement that a viscous diffusion time on the scale λ, *viz.,* λ^2/ν, is balanced against an eddy turnover time on the dominant flow scale L/U. Such layers are indeed observed in real flows. For the ABL numbers above, $\lambda \approx 10^{-1}$ m, which is quite small. Thus, we must understand the role of viscosity in flows with large Re values in a *singular perturbation* sense: viscosity is mostly unimportant to the dominant flow patterns, but there are ($\sim$ singular) locations where the spatial scale is small enough and the higher-order derivatives in the diffusion operator are large enough that viscous effects are significant in comparison with advection.

The development of thin layers, over which viscosity acts, is not confined to boundaries of the flows. They are also readily apparent in unbounded regions of the flow. If this were not the case, fuel would not mix efficiently in combustion chambers, and chemical reactions and dissipation would only occur at the boundaries. Thus, despite the startlingly small magnitude of the molecular terms in our 'scaled' governing equations, they can never be neglected: the flow always contrives to produce scales on which molecular processes operate efficiently. For this reason, rather than viewing the Reynolds number as a measure of the relative importance of viscous *vs.* advective effects, it is perhaps better to view $Re = (L/\lambda)^2$ as a measure of the ratio of the scales at which each is commensurate. The quantity λ appears often enough to be given a separate name; it is called the *Taylor microscale*. Often it is defined based on the magnitude of the fluctuating velocity u' and its shear:

$$\lambda = \sqrt{\frac{\langle\, u'^2 \,\rangle}{\langle\, (\partial_x u')^2 \,\rangle}}\,, \tag{12}$$

where the angle brackets indicate an average over many fluctuation cycles. The dimensional reasoning behind this definition of λ is that u' characterizes the most energetic scale while $\partial_x u'$ is as large as it can be, given u', limited only by the action of viscosity to set a smallest scale with substantial associated variance. The subsidiary quantity $Re_\lambda = L/\lambda$ is called the *microscale Reynolds number*. (The relation between Re and Re_λ is more fully explained in *3D Homogeneous Turbulence*.)

Above we have claimed that some hallmarks of turbulent flows are the wide range of scales they allow and the associated presence of fine-scale structure. In a flow where only one scale is forced, which for instance is the case in the Rayleigh-Benard problem where energy is put in at the scale H, the additional scales are primarily associated with non-linear interactions and secondary instabilities. The filling up of the spectrum, which we see in Fig.2, reflects the development of something we call the turbulent cascade. This is the process whereby all scales of the fluid are excited, and through a sequence of non-linear interactions, inviscid invariants of the flow (*e.g.,* energy), are carried to scales where molecular processes operate efficiently. This cascade is illustrated schematically in Fig. 3 for the case where energy is inviscidly converted from larger scales to smaller scales.

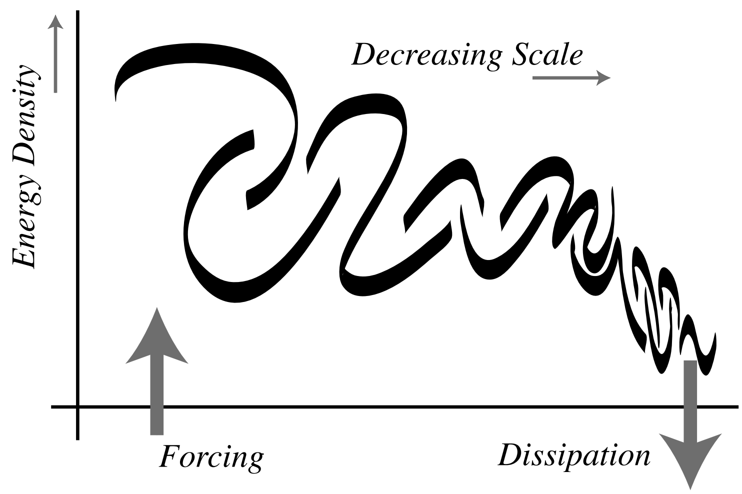

Figure 3: Cartoon of a *forward cascade* whereby energy or some other scalar variance (the ordinate) is systematically transferred by conservative advection from larger scales where it is forced to smaller scales where it is dissipated by molecular viscosity or diffusivity (the abscissa).

Vortex Stretching: One classical conception of why a cascade occurs is closely related to the process of 3D *vortex stretching*, although as illustrated below in a purely horizontal flow this is not the only cascade mechanism. The vortex-stretching mechanism for the energy cascade can be illustrated mathematically by considering small amplitude vorticity superimposed on a large-scale straining field. Recall that the strain rate S_{ij} is simply the second-order tensor describing the symmetric part of the velocity gradients:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{13}$$

and the vorticity, $\zeta_i \equiv -\epsilon_{ijk}\partial u_j/\partial x_k$, is the curl of the velocity field (with $\epsilon_{ijk} = 1$ for an even permutation of the indices (123), $= -1$ for an odd permutation, and $= 0$ for any other combination, *i.e.,* with any index repeated). An equation for ζ_i can readily be derived by taking the curl of the momentum equations. In the absence of viscosity, rotation, and stratification, this equation takes the form:

$$\frac{d\zeta_i}{dt} = \zeta_j \frac{\partial u_i}{\partial x_j} \ . \tag{14}$$

A simple straining flow can be represented in two dimensions as follows:

$$(u, v, w) = (sx, -sy, 0) \quad \Longrightarrow \quad \frac{\partial u_i}{\partial x_j} \ = \begin{cases} s & \text{if } i = j = 1 \\ -s & \text{if } i = j = 2 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

for constant s, as illustrated in Fig. 4. Thus we take u_i as given above as our (large-scale) background flow. Note that the vorticity of the background flow is zero, so that the vorticity equation simply describes the effect of the background straining flow on the vorticity of some other (small-scale) embedded flow component. Non-trivial relations exist only for the first two components of the vorticity field:

$$\frac{d\zeta_1}{dt} = \zeta_1 s \quad \text{and} \quad \frac{d\zeta_2}{dt} = -\zeta_2 s. \tag{16}$$

These equations imply exponential growth for ζ_1 and damping for ζ_2. But because the magnitude of a damped solution is lower-bounded by zero, overall the vorticity in the system will grow. Physically we interpret this result as the manifestation of vortex tubes aligned with the flow being stretched while those opposed to the flow are compacted. This vortex stretching leads to energy being associated with smaller scales (Fig. 4), and it is one way to think about the energy cascade in three-dimensional (3D) flows. Although we have illustrated stretching with a particular flow and vorticity orientation, we can imagine that a full range of orientations occur in a statistically isotropic 3D flow. However, note that this mechanism does not exist for strictly two-dimensional (2D) flows because of the geometrical restriction that the only non-trivial vorticity component is ζ_3, and ζ_3 has a trivial evolution equation; vortex stretching does not occur; and vorticity is an inviscid parcel invariant for the flow.

Line Stretching: More generally, turbulent flows exhibit cascades for many flow properties, not just kinetic energy, even in the absence of vortex stretching. Nevertheless, the distinctive roles of vorticity and strain in cascades is fundamental. To explore further some of the roles of these velocity-gradient fields, we will consider a special type of flow field that is purely horizontal in direction but fully 3D in spatial variation,

$$\mathbf{u} \ = \ \hat{\mathbf{z}} \times \nabla\psi(x, y, z, t) \ , \tag{17}$$

where ψ is a scalar potential called the *streamfunction*. This is often a good approximation for large-scale geophysical flows because of the strong influences of the Earth's rotation and stable stratification (*i.e.,* geostrophic flows). For this flow the vertical vorticity is

$$\zeta = \zeta_3 \ = \ \partial_x v - \partial_y u \ = \ \nabla_h^2 \psi \ , \tag{18}$$

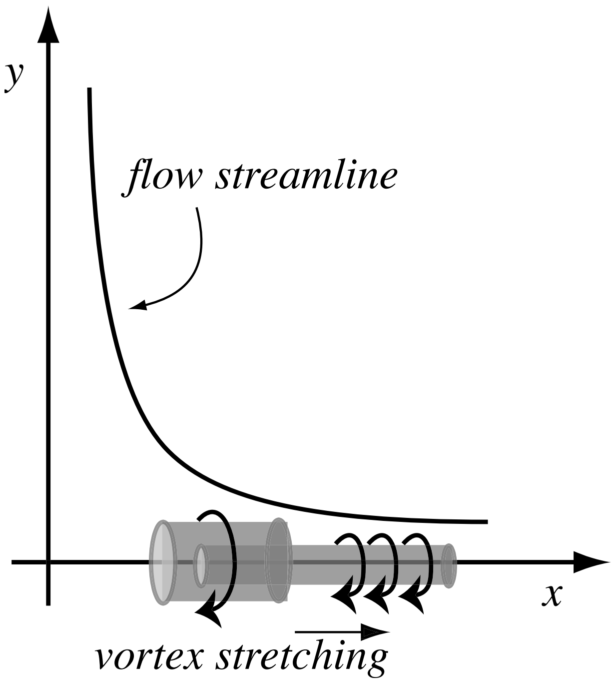

Figure 4: A vortex tube aligned with the axis of the strain-rate tensor stretches its length, shrinks its diameter (*i.e.,* reduces its spatial scale), preserves its circulation, and increases its vorticity magnitude.

where the subscript h denotes purely horizontal components. The magnitude s of the strain rate associated with horizontal shear is given by

$$s^2 \;=\; (\partial_x v + \partial_y u)^2 + (\partial_x u - \partial_y v)^2 \;=\; (\partial_{xx}\psi)^2 + (\partial_{yy}\psi)^2 - 2(\partial_{xx}\psi)(\partial_{yy}\psi) + 4(\partial_{xy}\psi)^2 \,. \quad (19)$$

The advective dynamics in this special case is essentially 2D since

$$\frac{D}{Dt} \;=\; \frac{D}{Dt_h} \;=\; \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla \;=\; \frac{\partial}{\partial t} + J_h(\psi,\ ) \,, \quad (20)$$

where J is the horizontal Jacobian operator:

$$J_h(A, B) \;=\; (\partial_x A)(\partial_y B) - (\partial_y A)(\partial_x B) \,.$$

For $\nu = 0$, the curl of the horizontal momentum equations gives

$$\frac{Dq}{Dt_h} \;=\; 0 \,, \quad (21)$$

where $q = f + \zeta_3$ is the potential vorticity that is conserved following the flow. Equation (21) also holds for a more generally defined potential vorticity q in other circumstances, *e.g.,* quasi-geostrophic flow. Equation (21) also has the same form as a material tracer equation, assuming (17) and $\kappa = 0$.

Taking the horizontal gradient of (21),

$$\nabla_h \frac{Dq}{Dt_h} = 0$$
$$\frac{D}{Dt_h}\nabla_h q = -J_h(\nabla_h \psi, q)$$
$$\frac{d}{dt_L}\nabla_h q = M \cdot \nabla_h q \;, \tag{22}$$

where $d \cdot / dt_L$ indicates the Lagrangian derivative following a fluid parcel and $\mathcal{M}$ is a 2×2 matrix of second spatial derivatives of ψ,

$$\mathcal{M} \;=\; \begin{pmatrix} \partial_{xy}\psi & -\partial_{xx}\psi \\ \partial_{yy}\psi & -\partial_{xy}\psi \end{pmatrix} \;. \tag{23}$$

If we assume that $\mathcal{M}$ is slowly varying in the Lagrangian frame compared to $\nabla_h q$ (*e.g.,* because the spectrum has more power on small scales in the latter than in the former, as certainly is true for $\nabla_h q \sim \nabla_h \nabla_h \nabla_h \psi$ and $\mathcal{M} \sim \nabla_h \nabla_h \psi$), then we can solve (22) as if it were a linear ODE system with constant coefficients. In particular, there are eigensolutions of the form,

$$\nabla_h q \;=\; (\nabla_h q)_o e^{\sigma t} \;, \tag{24}$$

where the local growth rate σ in the Lagrangian frame is an eigenvalue of the matrix $\mathcal{M}$ and thus satisfies

$$\begin{aligned}(\partial_{xy}\psi - \sigma)(-\partial_{xy}\psi - \sigma) \;&+\; (\partial_{xx}\psi)(\partial_{yy})\psi \;=\; 0 \\ \sigma^2 \;&=\; -J_h(\partial_x \psi, \partial_y \psi) \\ \sigma^2 \;&=\; \frac{1}{4}[s^2 - \zeta^2] \;, \end{aligned} \tag{25}$$

using (18)-(19) and the steps shown in (50). Thus, the temporal behavior of the gradient of a conservative scalar is also governed by the competition between vorticity and strain rate magnitudes. Furthermore, one can show by horizontally integrating by parts and discarding boundary integral terms (assuming periodicity or zero velocity) that

$$\int\!\!\int dx\, dy\, \sigma^2 \;=\; 0 \;\text{ and }\; \int\!\!\int dx\, dy\, \zeta^2 \;=\; \int\!\!\int dx\, dy\, s^2 \;. \tag{26}$$

Thus this competition is an equal one between dominance by vorticity or strain-rate, averaged over the domain.

Now consider the evolutionary consequences of this competition:

$s^2 > \zeta^2$ **(strain-rate dominance)** This $\Rightarrow \sigma^2 > 0 \Rightarrow \sigma$ is $\pm$ a positive real number. Thus there is exponential growth and decay of initial scalar gradients. General forcing or initial conditions will project equally onto the two eigenmodes, and after some time the solution will be dominated by the growing modes. This implies an impermanence of initial patterns in q. Growth of gradients is also a representation of the turbulent cascade to smaller scales, hence eventually to dissipation with small but finite (ν, κ). It occurs when a local Taylor series expansion of ψ has a hyperbolic pattern at second order (*e.g.,* as in $\psi \sim xy$), and since parcel trajectories follow isolines of ψ, the flow in this regime has locally diverging parcels. The local topology of ψ is a saddle node.

$s^2 < \zeta^2$ **(vorticity dominance)** This $\Rightarrow \sigma^2 < 0 \Rightarrow \sigma$ is $\pm$ a positive imaginary number. Thus the time behavior of scalar gradients is oscillatory. Spatial patterns of q will be termporally recurrent. Lack of growth of gradients $\Rightarrow$ a lack of turbulent cascade, hence a lack of dissipation whenever $Re \gg 1$. It occurs when a local Taylor series expansion of ψ has an elliptic pattern at second order (*e.g.,* as in $\psi \sim x^2 + y^2$), and the flow in this regime has locally confined parcels. The local topology of ψ is a central extremum.

If we assume that a trajectory is random over a long time interval, it will wander through different ψ environments, experiencing intervals of growth, decay, and oscillation in its scalar gradient. However, the net effect is growth (*e.g.,* the expected value of e^{at}, for a a Gaussian random variable with zero mean and unit variance, shows growth with t like $e^{t^2/4}$). Thus on average, scalar variance is cascaded toward smaller scales and dissipation. In contrast with the 3D vortex stretching cascade mechanism discussed above, the mechanism embodied in this example can be considered as *line stretching*, *i.e.,* elongation of a material contour (labeled in this case by particular values of the tracer field q). Both mechanisms contribute to cascades in turbulent flows.

However, if there is a long-time correlation between the trajectory and its ψ environment—in particular, if a trajectory remains trapped within a region of vorticity dominance—then there need not be a cascade. As we will see later in the discussion of 2D homogeneous turbulence, a 2D coherent vortex has an extremum in ζ and a minimum in s at its center, hence a region of vorticity dominance in its core, and it sustains a pattern of recirculating trajectories and trapped passive scalar concentrations for a very long period of time (much longer than L/V). On the other hand, it is more common for strain rate to be dominant outside of the cores of 2D coherent vortices; this is where scalar gradients grow most strongly, and the cascade of scalar variance is more clearly evident.

For general flows (*i.e.,* without the restricted form of $\mathbf{u}$ in (17)), one cannot make such a simple analysis of the roles of vorticity and strain rate[4]. Nevertheless, experience shows that there are often associations between regions of strong vorticity and persistent flow patterns and between regions of strong strain rate and pattern deformation and cascade to smaller scales. Think of this a rule of thumb based on experience, but not a theorem.

Cascade Inertial Range: With these ideas we can now return to Fig. 3 and imagine some stationary state, in which case energy must be put into the flow at the same rate at which it is dissipated. If we denote this rate of viscous dissipation (forcing) by ε, we note that by dimensional argument that a new length scale, η, where

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}, \tag{27}$$

is implied. This length scale is called the *Kolmogorov scale*. It is related to the Taylor microscale by the square root, $\eta = \sqrt{\lambda\nu/U}$. It is more directly characteristic of the dissipative scales, as opposed to the scales where dissipation starts becoming important. In terms of η, the Reynolds number scales as $Re \propto (L/\eta)^{4/3}$. This dissipation scaling also affords us the possibility of introducing dissipation time and velocity scales:

$$\tau_\eta = \sqrt{\nu/\varepsilon}\,, \qquad u_\eta = (\nu\varepsilon)^{1/4}. \tag{28}$$

[4] As another characterization of their distinctive roles, Appendix A shows their relation to the pressure field.

The Kolmogorov scale is also of fundamental importance because it bounds the range of scales necessary to accurately represent all the scales in a high Reynolds number flow, i.e., $L/\eta \propto Re^{3/4}$. Thus to simulate the atmospheric boundary layer we, in principle, should represent six decades of scales, from millimeters to kilometers.

If $Re \gg 1$ then $L \gg \eta$ in which case we can posit an *inertial range* of scales l such that $L \gg l \gg \eta$ (*n.b., inertial* is sometimes used as a synonym of advective). Because these scales are not directly influenced either by the forcing or by the dissipation, we can posit that their dynamics are predominantly advective. Furthermore, in the well-known phenomenological cascade theory of Kolmogorov for 3D homogeneous turbulence, it is often assumed that the behavior of the flow in this regime is local in wavenumber space (Sec. 4), depending on only the non-linear interactions among commensurate scales. In this view, the dissipation rate ε occurring on small scales on average must be equal to the large-scale energy generation rate by forcing, as well as to the cascade rate at all intermediate scales in the inertial range.

We can now complete the schematic of Fig. 3. Doing so for the case of 3D turbulence we find an inertial range whose scaling is that proposed by Kolmogorov, which is used to connect the dissipation range and energy-containing ranges thereby giving a more complete and somewhat generic description of the spectral energy density of 3D turbulent flows.

A way to summarize these characteristics is that turbulence generally has the properties of *cascade and stirring* on the outer and intermediate scales that connect to *dissipation and mixing* on the finest scales where molecular viscosity and diffusivity are important. The outer scale is defined as the largest scale in the turbulent regime where the advective tendency dominates the pointwise evolution; this is opposed to the even larger scales of the turbulent environment where other evolutionary influences (*e.g.,* solar heating) are at least as important. Cascade is defined as the transfer of the variance of a fluid property from one spatial scale to another one (usually smaller). Dissipation is defined as the removal of variance at small scales. Stirring is defined as increasing the variance of gradients through stretching and folding of isolines of a fluid property, through its cascade. Mixing is defined as the removal of inhomogeneities when the scale of property gradients reach the dissipation scale.

4 Statistical Descriptions and Dynamics

Because of the complexity of turbulence, it is usual to seek some kind of statistical description as a condensation of the excessive amount of information involved. A statistical measure is an average of some kind. It can be over the symmetry coordinates if any are available (*e.g.,* a time average in a stationary regime; a spatial average in a homogeneous regime; or a directional average in an isotropic regime). Or it can be over multiple realizations (*i.e.,* an *ensemble*), taking advantage of the property of *sensitive dependence* to assure that the outcomes will differ among ensemble members. Or it can be over the phase space of the solutions if the dynamics is *ergodic* so that all possible outcomes will be realized. In practice, exact averages are not obtainable in measurements or computations because of finite samples sizes and because exact symmetries do not occur in nature, so compromises and approximations are required.

The governing equations are *deterministic*, not random. So fundamentally turbulent fields are not random because their generating dynamics is not random. Nevertheless, it is useful to use statistical methods for random variables in order to describe turbulence, specifically averages, Fourier transforms, spectra, and covariance functions. The view of turbulence as akin to random motion is not incommensurate with its characterization as erratic behavior, and it is perhaps most clearly stated in the Introduction of Batchelor's (1953) book on homogeneous turbulence:

> It is a well-known fact that under suitable conditions, which normally amount to a requirement that the kinematic viscosity ν be sufficiently small, some of these motions are such that the velocity at any given time and position in the fluid is not found to be the same when it is measured several times under seemingly identical conditions. In these motions the velocity takes random values which are not determined by the ostensible, or controllable, or 'macroscopic' data of the flow, although we believe that the *average* properties of the motion are determined uniquely by the data. Fluctuating motions of this kind are said to be turbulent ... The problem is to understand the mechanics, and to determine analytically the average properties, of this kind of motion.

The language of random fields is probability theory, whose basic unit is the probability distribution function (PDF). Here is a brief summary (but also see Lumley and Panofsky, 1964, for a more extensive discussion). If u denotes a fluid property that fluctuates with zero mean, then we denote the PDF of u by $p(u)$. To be a PDF, $p(u)$ must satisfy two properties: first its range is the unit interval ($0 \leq p(u) \leq 1$); and second

$$\int p(u)\mathrm{d}\,u = \int \mathrm{d}\,p_u = 1\ . \tag{29}$$

Given $p(u)$ the nth moment of u is

$$\langle\, u^n \,\rangle \equiv \int u^n p(u)\mathrm{d}\,u\ . \tag{30}$$

Hence $p(u)$ determines all of the moments of u. By working with a centered distribution, *i.e.,* one with zero mean, these moments are called the central moments. They tell us something about the shape of the distribution. Roughly speaking, even ordered moments describe symmetric properties of the distribution (*i.e.,* its width) and odd order moments describe asymmetric aspects of the distribution. In addition to the mean, the second, third and fourth moments are encountered frequently. $\langle\, u^2 \,\rangle$ is called the variance and sometimes denoted as σ_u^2. The normalized third and fourth order moments,

$$Sk(u) \equiv \frac{\langle\, u^3 \,\rangle}{\sigma_u^3} \quad \text{and} \quad Ku(u) \equiv \frac{\langle\, u^4 \,\rangle}{\sigma_u^4}\ , \tag{31}$$

are called the *skewness* and *kurtosis*, respectively, The skewness is indicative of asymmetry between the positive and negative fluctuations, and the kurtosis is indicative of the likelihood of occurrence of extreme amplitudes. Just as the PDF determines the moments of a distribution, the infinity of moments determine the PDF. Particularly useful are those families of distributions that can be determined in terms of only a small number of moments, the best known such example being the *Gaussian or normal distribution*,

$$p(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-\mu)^2/(2\sigma^2)}\ , \tag{32}$$

that is completely determined by its mean μ and variance σ_u^2. Because a Gaussian distribution is symmetric its skewness is identically zero, $Sk(u) = 0$, while its kurtosis is $Ku(u) = 3$. This is the common comparison standard for the measured or computed PDFs of turbulent quantities.

Another useful distribution for inherently positive quantities, $u \geq o$, is the *log-normal distribution*,

$$p(u) = \frac{1}{Su\sqrt{2\pi}} e^{-(\ln u - \ln u_g)^2/(2(ln\sigma_g)^2)} \ , \tag{33}$$

that is also described by two parameters u_g and σ_g related to the mean and standard deviation, respectively. It has the property that extreme values of u are very much more likely than they are for Gaussian PDF. This PDF is often used, *e.g.,* to describe the statistics of the dissipation ε.

One interesting aspect of turbulent flows is that strong, or extreme, events are more common than one would expect based on the variance of the distribution, *i.e.,* relative to a Gaussian PDF. This is referred to as *intermittency*, and is evident by large kurtosis, or a flattening of the PDF away from the origin $u = 0$. An example in terms of the velocity tendency (*i.e.,* , the difference in velocity at two times close together, and thus related to the acceleration) exemplifies this behavior (Fig 5). (A survey of PDFs from different geophysical turbulent flows is in McWilliams, 2007.) Broadly speaking, the higher the value of Re, the broader are the PDFs, and the more intermittent are the fluctutations.

For coupled nonlinear systems this type of statistical description of the flow requires not only a determination of the PDFs of the individual variables, but also the joint PDFs among variables. For instance, in attempting to write an equation for the expected value of some flow field, the advective terms in the governing equations naturally give rise to expected values of products $\langle\, uv \,\rangle$, which by decomposing the flow into its expected values and fluctuations according to

$$u = \langle\, u \,\rangle + u', \tag{34}$$

takes the form

$$\langle\, uv \,\rangle = \langle\, u \,\rangle \langle\, v \,\rangle + \langle\, u'v' \,\rangle. \tag{35}$$

Hence, determining $\langle\, uv \,\rangle$ requires not only knowledge of $p(u)$ and $p(v)$ but also $p(u, v)$, the probability of some u given some v. When we say that u and v are independent, we mean $p(u, v) = p(u)p(v)$, which is another way of saying that $uv = \langle\, u \,\rangle \langle\, v \,\rangle$. However, in almost all problems of interest these higher order covariances (equivalently, joint PDFs describing the correlation among flow properties) are important to the overall evolution of the flow.

Because most geophysical flows are neither homogeneous, nor stationary, the PDFs of flow properties depend on space and time, *i.e.,* we must adopt the language of conditional probability. That is we speak of $p(u; x, t)$, the probability of some u occurring at some point x and time t. In practical situations, where the PDFs of flow properties are not known we often assume homogeneity or stationarity on some scale x_0, respectively τ. By which we mean that $p(u; x, t) = p(u; x + r, t)$ for $r \gg x_0$, or $p(u; x, t) = p(u; x, t + s)$ for $s \gg \tau$. In such situations the fluid properties on large-scale cease to depend on time t or position x, and the expected values of fluid properties may, as a practical matter, be replaced by spatial or temporal averages.

The common means of representing the scale distribution of a field is through its *Fourier transform*, *spectrum*, and *covariance function*. For example, the Fourier transform for a spatially varying field $\psi(\mathbf{x})$ in an infinite, homogeneous domain is

$$\psi(\mathbf{x}) \ = \ \int d\mathbf{k} \, \hat{\psi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \ , \tag{36}$$

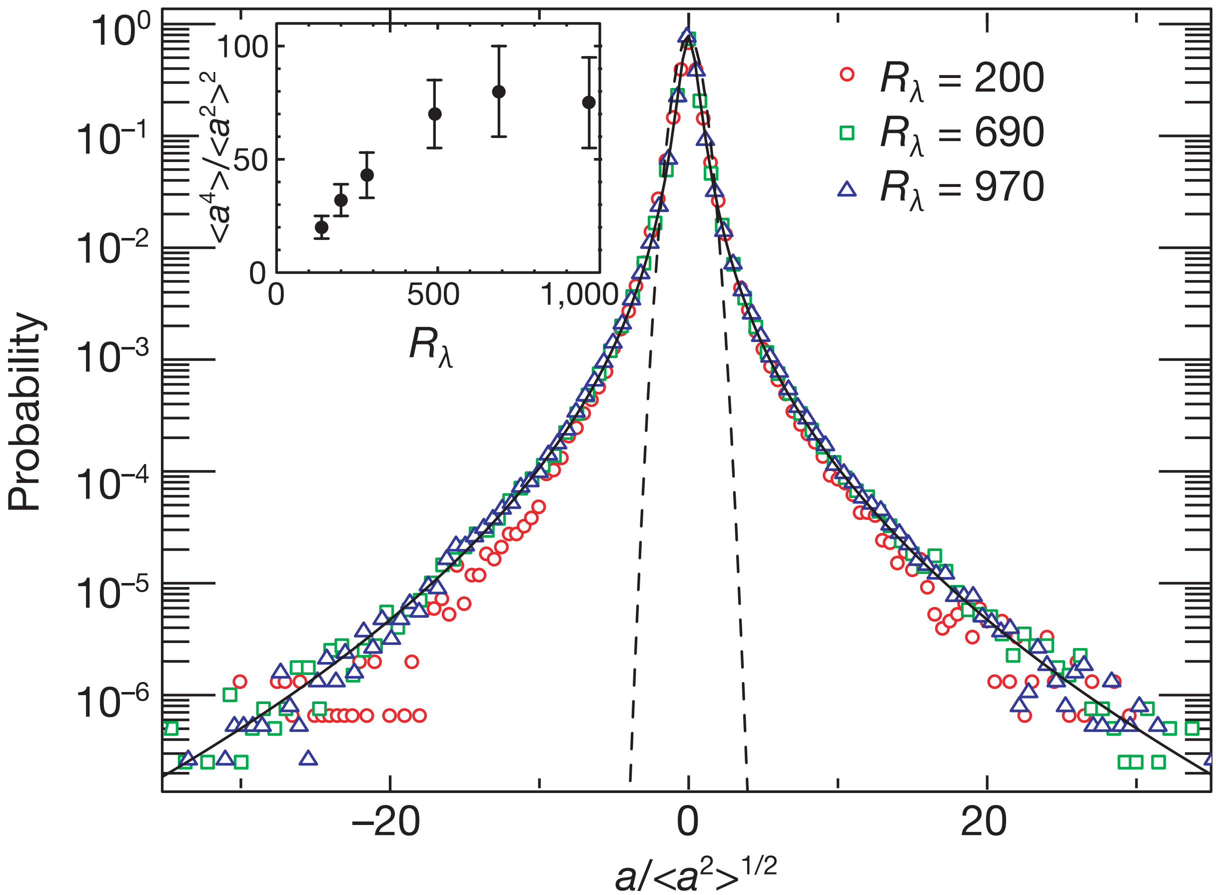

Figure 5: Measurements from a laboratory flow of the normalized transverse velocity derivative in time, or acceleration, PDF at different values of the microscale Reynolds number, $Re_\lambda = R_\lambda$. From La Porta *et al.* (2001, Fig. 3).

together with the inverse transform relation,

$$\hat{\psi}(\mathbf{k}) \;=\; \frac{1}{(2\pi)^3} \int d\mathbf{x}\, \psi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \tag{37}$$

$\mathbf{k}$ is the vector wavenumber, and $\hat{\psi}(\mathbf{k})$ is the complex Fourier transform coefficient. With this definition the spectrum of ψ is

$$S(\mathbf{k}) \;=\; \langle\, |\hat{\psi}(\mathbf{k})|^2 \,\rangle \,. \tag{38}$$

$S(k)$ can be interpreted as the variance of ψ associated with a spatial scale, $L = 1/k$, with $k = |\mathbf{k}|$, such that the total variance, $\langle\, \psi^2 \,\rangle$, is equal to $\int\, d\mathbf{k}\, S$ (sometimes called *Parceval's Theorem*). The covariance function is defined by

$$C(\mathbf{x}) \;=\; \langle\, \psi(\mathbf{x}')\psi(\mathbf{x}' + \mathbf{x}) \,\rangle \,, \tag{39}$$

where the assumption of homogeneity implies that C is independent of $\mathbf{x}'$. In this formula $\mathbf{x}$ is the difference in position between the two quantities being multiplied and is called the *spatial lag* (so C can also be called the spatial lag covariance function). Note that $C(0)$ is the variance $\langle\, \psi^2 \,\rangle$. It can also be shown that $S(\mathbf{k})$ and $C(\mathbf{x})$ are related by a Fourier transform and its inverse, *e.g.,*

$$C(\mathbf{x}) \;=\; \int d\mathbf{k}\, S(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \,.$$

Finally, sometimes turbulent fields are represented by their *structure function*,

$$St(\mathbf{x}) \;=\; \langle\, (\psi(\mathbf{x}') + \psi(\mathbf{x}' + \mathbf{x}))^2 \,\rangle \;=\; 2(\, C(0) - C(\mathbf{x})\,) \,. \tag{40}$$

Such transforms may be performed for variables which depend on time, in which case we speak of the frequency spectra $S(f)$ or $S(\omega)$ and $C(t)$ the temporal lag covariance function. Figure 2 shows examples of frequency spectra associated with a Fourier transform for a function of time. For the $S(f)$ with sharp peaks at smaller Ra values, the associated $C(t)$ are oscillatory functions of the time lag, while for broad-band $S(f)$ at larger Ra, $C(t)$ decays to zero as t increases.

If we now bring a statistical description to the dynamics of turbulence, it is natural to construct various averages from the governing equations. For example, governing equations for the mean fields (*i.e.,* $\langle\, \mathbf{u} \,\rangle$ and $\langle\, b \,\rangle$) can be by integrating (1) over the PDFs of $\mathbf{u}$ and b respectively:

$$\begin{aligned}
\frac{\partial\langle\, \mathbf{u} \,\rangle}{\partial t} + \langle\, \mathbf{u} \,\rangle \cdot \nabla\langle\, \mathbf{u} \,\rangle &= \; -\nabla\langle\, \phi \,\rangle + \nu\nabla^2\langle\, \mathbf{u} \,\rangle + \hat{\mathbf{z}}\langle\, b \,\rangle - f\hat{\mathbf{z}} \times \langle\, \mathbf{u} \,\rangle - \nabla\cdot\langle\, \mathbf{u}'\mathbf{u}' \,\rangle \\
\nabla\cdot\langle\, \mathbf{u} \,\rangle &= \; 0 \\
\frac{\partial\langle\, b \,\rangle}{\partial t} + \langle\, \mathbf{u} \,\rangle \cdot \nabla\langle\, b \,\rangle &= \kappa\nabla^2\langle\, b \,\rangle - \nabla\cdot\langle\, \mathbf{u}'b' \,\rangle \,.
\end{aligned} \tag{41}$$

These equations are isomorphic to the unaveraged equations except for the additional, final terms in the momentum and tracer equations. These terms are divergences of the eddy momentum flux (or *Reynolds stress*) and eddy tracer flux, respectively, where the word eddy refers to the fact that they are associated with the fluctuations. Equations like (41) are a central framework for the theory of *eddy – mean interaction*, but they obviously need to be combined with some prescription for the eddy dynamics in order to be a complete system.

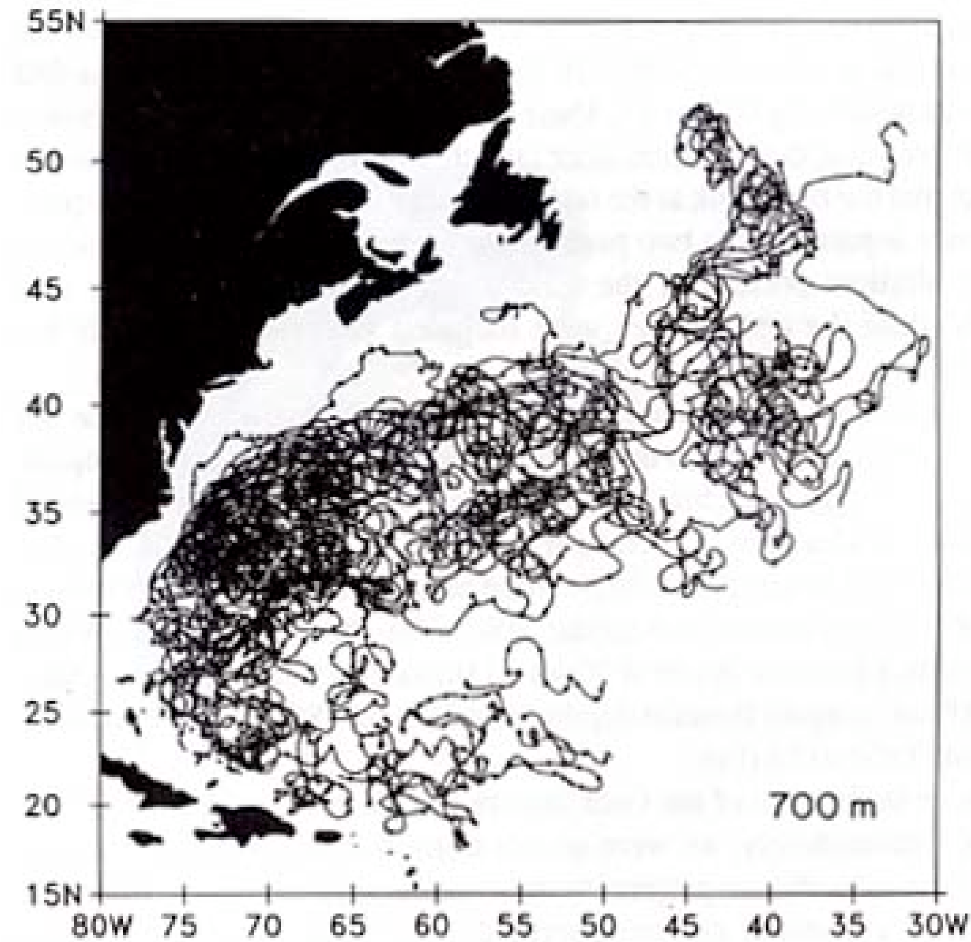

Figure 6: Trajectories for acoustically tracked neutrally buoyant floats at a nominal depth of 700 m in the Northwest Subtropical Gyre in the North Atlantic Ocean. Arrows along trajectories occur aton 30 day intervals. Note the general mixing and dispersion behaviors due to stirring by mesoscale eddies. Adapted from Owens (1991, Fig. 2).

This procedure can suffice if the goal is only to determine the mean fields. Implicit here is an assumption that the solutions of the averaged equations do not exhibit fluctuation behavior, and at the least *a posteriori* consistency must be demonstrated. Note, however, that to determine the mean fields, the eddy flux fields must be known, so the mean-field equations are incomplete. If one seeks to complete them by calculating the eddy fluxes, then an average must be taken of the products of a fluctuating field and a fluctuating equation, and the result will depend on third moments; to calculate the third moments requires knowledge of the fourth moments; *etc.*Escaping this infinite regression is called the *closure problem*, and, because it is simply a reformulation of the original problem, it is not really any easier to solve. The usual computational approach is the more direct one of calculating the unaveraged PDE system and taking averages of the solution. This involves a truncation in spatial scale (and a redefinition of the effective Re, often through modeling, or parameterization, of the effects of the deleted scales of motion) rather than a truncation or modeling in the statistical moment hierarchy. The former is referred to as *Large-Eddy Simulation* (LES) and the latter as direct closure modeling, but even LES must invoke some form of closure modeling for the effects of its subgrid-scale turbulence. Almost all computations of oceanic and atmospheric turbulence are LES, because the target phenomena are on scales too large to connect to the $\sim$ 1 mm scale where molecular diffusion begins to dominate advection. Even climate and general circulation models are LES, albeit ones quite distantly separated from the scales on which micro- and mesoscale turbulence motions occur.

5 Characteristics of Turbulence

Let us now list, in a highly qualitative way, some of the essential characteristics of turbulence that apply in almost all physical situations. The first three entries have been discussed above. The next two entries, dispersion and loss of correlation, are certainly central aspects of the historical view of turbulence. The remaining six entries are perhaps the more modern elements in the evolving understanding of turbulence.

- Cascades
- Scale breadth in space and time
- Dissipation of fluctuation variance for various quantities
- Dispersion of material concentrations and entangling of parcel trajectories (Fig. 6))
- Aperiodicity and loss or correlation with increasing spatial and temporal lags
- Deterministic chaos (*vs.* randomness and ergodicity)
- Sensitive dependence and limited predictability from imperfect initial data
- Intermittency and occurrence of extreme events
- Irreversibility of outcome — due both to mixing and dissipation and due to evolutionary complexification – even though the Euler equations have a time-reversal symmetry (Prigogine, 1980)

- Differentiable but non-smooth fields on all but the microscale (*vs.* fractal: continuous but non-differentiable)
- Locally ordered (coherent structures; Fig. 7) but globally disordered (chaotic) (Prigogine & Stengers, 1984)

Figure 7: Oceanic (left, in the marginal ice zone) and atmospheric (right, in a stratus cloud deck) coherent vortices in Davis Strait (north of the Labrador Sea, west of Greenland) during June 2002. Both vortex types are mesoscale vortices with horizontal diameters of 10s-100s km. Courtesy of Jacques Descloirest (NASA Goddard Space Flight Center).

6 Theoretical Foundations and Models

The mathematical characterization of the Navier-Stokes Equation, and *a fortiori* the Boussinesq Equations, is incomplete, essentially because of the advection operator in (1). There are no mathematical proofs of the long-term existence and uniqueness of PDE solutions for large Re starting from smooth initial conditions and forcing. And certainly there is no general analytic solution method for these equations. It is still very much an open question whether singularities can arise in the PDE solutions. Nevertheless, most fluid dynamicists are persuaded by the cumulative number of particular successes comparing measurements with analytical and computational solutions that these equations are physically reliable.

Because of the lack of a general theory, turbulence is primarily an experimental problem in the broad sense that we know much more about it from experience than from fundamental theory. There is a great deal of data from turbulence experiments (in controlled laboratory conditions), measurements (in uncontrolled natural conditions), and computations. The role of theory is to provide concepts which organize the facts. Many, perhaps most, of the concepts are mathematically embodied in much simpler models than the full fluid equations. For example, a random walk is a paradigm for the spreading of material tracers (*i.e.,* dispersion), and a random walk is a solution of a linear PDE with stochastic excitation (rather than deterministic fluid dynamics). Even though we lack a full theory, we are able to make rather accurate calculations in many turbulent situations; an example is in the design of aircraft, which obviously works rather well.

As remarked above there are no deep or complete theories of turbulence. Nevertheless there are many models of turbulence which have simpler mathematical structure than the Navier-Stokes or Euler Equations, some have solutions that skillfully mimic turbulent behaviors under certain circumstances. Merely as an introduction of terminology, the following are some common model types:

Eddy Diffusion: Replacing turbulent advection by a diffusion operator under the hypothesis that turbulence mixes the quantities that comprise its environment, as if the fluid evolution were laminar with an increased diffusivity.

Instantaneous Adjustment: Assuming that turbulence mixes with such great efficiency that the environmental distributions instantaneously adjust to a condition of marginal stability (*e.g.,* convective adjustment, a well-mixed boundary layer). (Also called a stability-bounds model.)

Moment Closure: Making closure assumptions for the averaged dynamical equations for low-order statistical moments, *e.g.,* (41), to break the infinite regress of moment coupling. (Mean-field closures, second-moment single-point closures, turbulent kinetic energy equations, *etc.*)

Stochastic Dynamics: Devising dynamical equations in which difficult nonlinear advection is replaced by a stochastic forcing term (*e.g.,* inducing mixing).

Vortex Population Dynamics: Devising dynamical equations for the evolution of a population of coherent structures, such as point- or line-vortices.

Rapid-Distortion Theory: Making a quasi-linearization of the Navier-Stokes Equation based on the presumption that small-scale fluctuations $\mathbf{u}$ in the presence of large-scale background flow $\mathbf{U}$ evolve in a mean-Lagrangian frame primarily through the straining effect of the latter on the former, without significant nonlinear self-interaction, *viz.,*

$$\left[\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right]\mathbf{u} + \nabla\phi - \nu\nabla^2\mathbf{u} \;=\; -(\nabla\mathbf{U})\cdot\mathbf{u}, \quad \nabla\cdot\mathbf{u} \;=\; 0\,. \tag{42}$$

The essential eddy – mean interaction implied here is a combination of bulk advection by the mean velocity and differential advection by the mean shear; both have the effect of putting the fluctuation dynamics into a mean Lagrangian reference frame. Implicit in (42) is the possibility of unstable normal modes, as well as transient (non-modal) fluctuation amplification (Farrell, 1988).

Finally there are mathematical relations, sometimes confusingly called models, that are adynamical representations of the empirically determined statistical structure of turbulent fields. In this category are (multi-)fractals (Frisch, 1995), log-normal distributions (*e.g.,* of the single-point PDF of dissipation rate), and inertial-range scaling laws (She & Leveque, 1994; Lundgren, 2008).

We will revisit all these types of turbulence models in particular situations. The reader is referred to, *e.g.,* Lesieur (1997), Frisch (1995), Monin & Yaglom (1971, 1975), and Pope (2000) for extensive elaboration on models.

7 Atmospheric and Oceanic Regimes of Turbulence

There are many different combinations of physical conditions for the turbulence in the ocean and atmospheric troposphere (Fig. 8). Here we survey the distinctive principal regimes, though all combinations of conditions can and do occur. In all regimes the assumption of $Re \gg 1$ is implicit. (A particularly hot topic is the mechanisms for transition out of geostrophic turbulence into smaller-scale, less rotatingly constrained flows; McWilliams, 2008.)

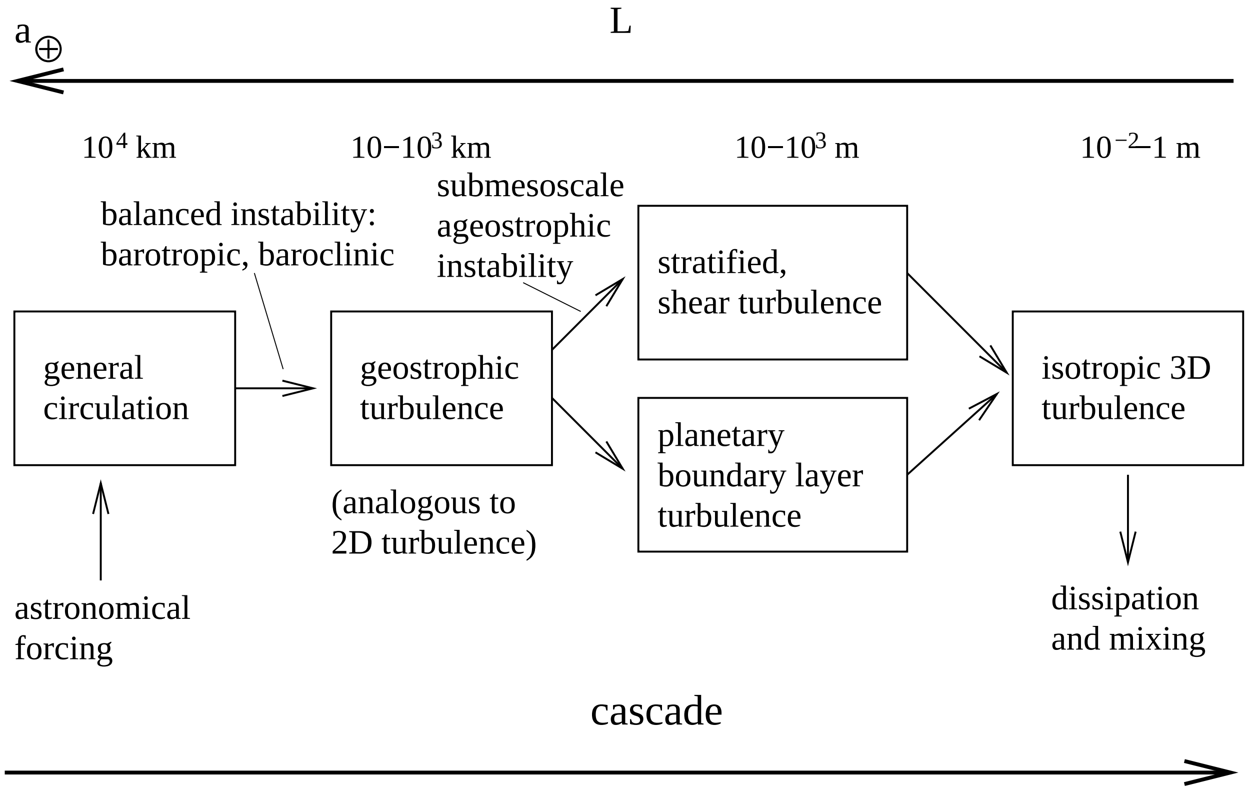

Figure 8: Schematic diagram of the regimes of turbulence in the atmosphere and ocean in a broad sweep of energy from the astronomically-forced planetary scale down to the microscale where mixing and dissipation occur.

Homogeneous Turbulence: Spatially isotropic and homogeneous (hence without boundaries) with $f = g = 0$ (hence without buoyancy forces). (Note the dual meanings of homogeneous: uniform in space and uniform in density.)

Shear Turbulence: In the presence of a mean shear flow, $\nabla \overline{\mathbf{u}}$, with

$$Sh = \frac{V}{SL} \leq 1 , \tag{43}$$

where $S = | \nabla \overline{\mathbf{u}} |$, V is a characteristic velocity of the turbulent flow, and L is a characteristic scale across the mean shear.

Stratified Turbulence: In the presence of a stable buoyancy stratification, with

$$Fr = \frac{V}{NL} \leq 1 , \tag{44}$$

where L is a vertical length scale and N is a characteristic scale for the buoyancy frequency profile defined by

$$N^2(z) = \frac{d\overline{b}}{dz} . \tag{45}$$

Convection: In the presence of an unstable density stratification, with $\partial_z \overline{b} < 0$ (*e.g.,* Rayleigh-Benard convection). If the convecting region is bounded by a stably stratified region, then it is called *penetrative convection* because it will tend to spread into the stable region by *entrainment*.

Boundary Layer Turbulence: Near a material boundary through which fluid parcels mostly do not pass but fluxes of momentum and/or buoyancy do pass. The primary types of boundary layer turbulence are (a) *shear*, with friction velocity $u_* = \sqrt{|\tau|/\rho_o}$ and boundary tangential stress τ; (b) *convection*, with convective velocity $w_* = [\mathcal{B}H]^{1/3}$, surface buoyancy flux $\mathcal{B}$, and boundary layer depth H; and (c) *Langmuir* in the ocean, with Stokes drift velocity from the surface gravity waves, $u^{St} = \sqrt{gk^3}a^2 \sim u_*$, where a and k are the dominant wave amplitude and wavenumber plus additional mixing from breaking waves.

Weak Wave Turbulence: In the presence of weakly nonlinear waves (*e.g.,* surface or internal gravity, inertial, coastal, or Rossby), with $ak \ll 1$ and $V/\sigma L \ll 1$, where a is wave amplitude, k is wavenumber, and σ is wave frequency. (This topic is not otherwise discussed in this course.)

Geostrophic Turbulence: In the presence of rotation and stable stratification, with $Fr \sim Ro$ and

$$Ro = \frac{V}{fL} \leq 1 . \tag{46}$$

This regime also includes influences of the spatial variation of the planetary vorticity, $\beta = df(y)/dy$, and topographic variations of the fluid depth δH, with

$$\frac{\beta L}{f}, \frac{\delta H}{H} \sim Ro . \tag{47}$$

It also includes interactions with the general circulation $\overline{\mathbf{u}}$, with $V' \sim \overline{V}$.

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Appendix A: Role of Pressure

We can simplify the turbulence problem by formally eliminating pressure as a dependent variable. This is possible because of the incompressibility condition in (1). We take the divergence of the momentum equations in (1) and note that $\nabla \cdot \partial_t \mathbf{u} \ = \ 0$. This yields a relation called the *Pressure Poisson Equation*:

$$\nabla^2 \phi \ = \ \nabla \cdot [\, -(\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} + \hat{\mathbf{z}} b - f \hat{\mathbf{z}} \times \mathbf{u}\,]. \tag{48}$$

Since there are no time derivatives in (48) , $p\ (=\ \rho_o \phi)$ is a purely diagnostic field which is wholly slaved to $\mathbf{u}$. After solving (48) for ϕ, it can be substituted for the pressure-gradient force in the momentum equations. Its role is to maintain incompressibility under the action of all other forces. Therefore, it is merely a question of mathematical or computational convenience whether ϕ is explicitly retained as a dependent variable.

We can illustrate this for a the special flow in (17). Since $w = \hat{\mathbf{z}} \cdot \mathbf{u} = 0$ here, the vertical momentum equation is simply hydrostatic balance,

$$\phi_z \ = \ b,$$

and the incompressibility condition is only horizontal,

$$\nabla_h \cdot \mathbf{u}_h \ = \ 0 \ ,$$

where the subscript "h" denotes a horizontal vector. (Note that horizontal incompressibility is automatically satisfied by the form (17).) Thus, the important remaining constraint on ϕ in this case comes from taking the horizontal divergence of the horizontal momentum equations; the result is

$$\begin{aligned} \nabla_h^2 \phi \ &= \ \hat{\mathbf{z}} \cdot [\nabla_h \times f \mathbf{u}_h + 2 \nabla_h u \times \nabla_h v] \\ &= \ \nabla_h \cdot [f \nabla_h \psi] + 2 J_h(\psi_x, \psi_y) \ . \end{aligned} \tag{49}$$

The relation (49) is a simplified form of the general pressure relation (48), and it is called *gradient-wind balance*. It says that the pressure Poisson equation is forced by the divergence of the Coriolis force and an advective force that can be interpreted as a generalized centrifugal force in a curvilinear coordinate frame defined by the streamlines (*i.e.,* isolines of ψ).

We can alternatively express the generalized force divergence in (49) as

$$\begin{aligned} 2J(\psi_x, \psi_y) \ &= \ 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) \\ &= \ \frac{1}{2}[\ (\psi_{xx} + \psi_{yy})^2 - (\psi_{xx} - \psi_{yy})^2 - 4\psi_{xy}^2\ ] \\ &= \ \frac{1}{2}[\ (v_x - u_y)^2 - \{(v_x + u_y)^2 + (u_x - v_y)^2\}\ ] \\ &= \ \frac{1}{2}[\ \zeta^2 - \{s^2\}\ ] \ , \end{aligned} \tag{50}$$

where ζ is the vertical component of vorticity and s is the magnitude of the horizontal strain rate, here defined by reference to the terms in the preceeding line. Thus, the sense of curvature (*i.e.,* the topology of the level surfaces) of ϕ is established by a competition between the magnitudes of the vorticity and strain rate in this special case, apart from the influence of the Coriolis force. This is another indication of the important and competing roles of vorticity and strain rate in advective fluid dynamics.

If we further simplify (49) to axisymmetric flow and $f = f_o$,

$$\psi(x,y,z,t) \ = \ \psi(r,z,t)$$

for $r^2 = x^2 + y^2$, then (12) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial \phi}{\partial r} \right] \ = \ \frac{1}{r}\frac{\partial}{\partial r}\left[f_o r V + V^2 \right] \ , \tag{51}$$

or

$$\frac{\partial \phi}{\partial r} \;=\; f_o V + \frac{V^2}{r}\;, \tag{52}$$

for $V = \partial_r \psi$ the azimuthal velocity, which makes explicit the preceding interpretation of the force terms in (49).