Chapter 2

Reynolds experiment: Below a particular Re, the flow in the pipe was laminar, but above it instable behavior was exhibited. This "Instability" generally leads to turbulent decay/decomposition (Zerfall).

The laminar-turbulent boundary layer envelope observed (in the pipe flow experiment of Reynolds) belongs to the category "Natural Transition". Furthermore, the following different types of Boundary Layer Envelopes exist:

- Natural transition
- Bypass transition
- Detachment induced (Ablösungsinduzierte) transition.
- Cross flow induced (Querströmungsinduzierte) transition.
- Compressibility conditioned (Kompressibilitätsbedingte) transition (in hypersonic flows).

In the case of transition from laminar to turbulent flow for flow over a flat plate (where the incoming flow contains small fluctuations/disturbances), the following chain of events can be postulated:

1. Stability: In this stage all fluctuations are damped and the boundary layer remains laminar.
2. Instability: After exceeding the "Indifference Reynolds number, Re_{ind}" , 2 dimensional fluctuations will be fanned (developed), which form the so called "Tollmien-Schlichting" waves. In further course, additionally 3 dimensional fluctuations will be fanned (developed) and "longitudinal vortex" (Lambda-structures) are formed. Initially, neither the Tollmien-Schlichting waves nor the Lambda structures have any influence on the integral boundary layer parameters, e.g. the boundary layer thickness (delta) or the wall shear stress.
3. Onset of turbulence: When $Re = Re_{st}$ (st = start of the development of turbulence) is reached, the 3 dimensional fluctuations start to decay and the fluctuations are fanned non-linearly (non-linear development of fluctuations). "Spots" start to form and disappear arbitrarily in space and time. The boundary layer is at an "intermittent" or "sporadic" state. This means that for a certain length of time it is laminar while for the other it is turbulent. From Re_{st} onwards, a change in the boundary layer parameters can be observed: the wall shear stress rises while the form factor (H_{12}) falls.[Ref: Wikipedia: Form factor or Shape Factor (H) is used in boundary layer flow to determine the nature of the flow. It is the ratio of the displacement thickness to the momentum thickness. The higher the value of H, stronger is the adverse pressure gradient. A high adverse pressure gradient can greatly reduce the Reynolds number at which transition into turbulence may occur.

Conventionally, H = 1.3 – 1.4 is typical of turbulent flows. And H = 2.59 is typical for laminar flows (Blasius boundary layers)]

4. Turbulent boundary layer: Various "Spots" grow together to form a fully developed turbulent boundary layer, when Re = Re_{end} is reached.

REFER Abbildung 2.1 on Page 11: Schematic representation of the phases of laminar-turbulent transition.

First Stage (Laminar instable)

In the first stage ($Re_{ind} < Re < Re_{st}$) of laminar-turbulent boundary layer envelope, at first, 2 dimensional fluctuations will be fanned. For the investigation of this process, "Primary Stability Theory" will be used. The "Secondary Stability theory" serves the purpose of describing the development of 3 dimensional fluctuations.

- Primary Stability Theory

Primary stability theory for parallel flows is a method for small perturbations (fluctuations/disturbances). As the fluctuations are very small in contrast to the mean flow, a linearization of the equations can be made:

Fundamental equations:

The fluid motion is decomposed into a laminar stationary base flow and a superimposed fluctuating flow. Then under the prerequisite that u' << u, the following are valid:

(Refer equation 2.2 page 12)

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$
$$w = \overline{w} + w'$$
$$p = \overline{p} + p'$$

Say, the base flow (u, v, w, p) is a solution of the Navier-Stokes or the Boundary layer equations. Following prerequisites will be assumed:

Incompressible base low with parallel streamlines. This corresponds to a pipe or channel flow and is denoted as parallel flows. For flow past a body the following assumptions are approximations for the boundary layer:

(Eq 2.3)

2 dimensional fluctuations are of the form:

(Eq 2.4)

The resulting motion will be characterized through:

(Eq 2.5)

In addition to the premise that EQ. (2.3) a solution of Navier-Stokes equation, it is required that also EQ. (2.5) is a solution of the same. In addition, it is assumed that the fluctuations links from GL. (2.4) are so small compared to the basic flow EQ. (2.3), that the quadratic elements in the equation can be neglected.

The Navier Stokes equations for 2D, instationary incompressible (the fluctuation being the instationary element) is equation 2.6 page 12. Further, the continuity equation is eq 2.7 page 13.

By substituting eqn 2.5 in eqn 2.6 and eqn 2.7 and neglecting the quadratic elements of the fluctuation terms, one gets the **system of equations for the resulting motion**. Now, if one subtracts the mean flow field (i.e. the system of equations built only with the parameters of mean flow) from the above-mentioned **system of equations for the resulting motion,** then one gets eqns 2.8, 2.9 and 2.10 with the boundary conditions u' = v' = 0 at the corresponding boundaries.

These 3 equations 2.8, 2.9 and 2.10 have 3 unknowns u',v' and p' (u is already given).

By subtracting the eqns (2.8) and (2.9) can eliminate the pressure, thus remain two equations with the two unknown u', v'.

- **Ansatz for the 2D fluctuations**

  Refer equation 2.11

$$u' = \frac{\partial \psi}{\partial y} \qquad v' = \frac{-\partial \psi}{\partial x}$$

With the help of the "Stream function (Psi)" (which fulfills the continuity criteria) defined in eqn 2.11 page 13, the system of equations 2.8 – 2.10 can be reduced to one equation.

Now, Equation 2.12 defines the stream function Psi(x,y,t), which is a complex Fourieransatz with damping parameter inserted in it.

Equation 2.13 gives the real part of this complex stream function, where:

- Phi( ) is a complex amplitude function.
- Beta ( ) is also a complex function consisting of angular frequency of partial oscillation ( ) (the real part) and the damping parameter ( ) (the complex part)

If damping parameter < 0, then damping → stability
If damping parameter > 0, then amplification/fanning → instability

Equation 2.14 shows how the u' and v' would look like when the complex fourieransatz for the stream function is substituted in equation 2.11

Here **alpha ( ) and c** are defined, where:

- Alpha is the wavenumber of the fluctuation with the wavelength Lambda

- C = Beta / Alpha = $C_r + iC_i$

Where C_r, the real part represents the phase velocity and iC_i, the imaginary part represents the damping parameter.

By using the complex fluctuation function (given by eqn 2.12) and equation 2.11 (where this complex fluctuation function is used to define the u' and v') in the system of differential equations 2.8 – 2.10, one obtains a **"differential equation for describing the fluctuations"** . This is the **Orr-Sommerfeld** Equation (Equation 2.15).

Investigation of stability

For a given velocity profile, the solution of the Orr-Sommerfeld equation depends on :

- Reynolds number (Re)
- Wavenumber (alpha) α

(That's why we have a plot of Wavenumber vs Re, where the indifference curves are shown. Any point falling inside a curve is instable while any point outside that same curve is stable.)

Fluctuation development can happen due to following 2 mechanism:

- Presence of point of inflexion in the velocity profile (Wendepunktkriterium) which is due to adverse pressure gradients.
- Viscous instabilities.

Refer Abb 2.3 : Curve "a" encloses a larger area than curve "b": Adverse pressure gradients cause greater instabilities than the viscous effects and flow transition from laminar to turbulent happens much easily as compared to the flow transition due to viscous instabilities only.

Rayleigh equation........

2.5 Prediction of Transition (Transitionsvorhersage)

In the modeling of the laminar-turbulent transition of boundary layer, the forecast of the following three pieces of information of particular interest :

- Re_{ind} **: Indifference Re**: From this point onwards, begins the amplification of fluctuations
- R_{est} : **Beginning of turbulent production**: Turbulent structures start to build up
- Re_{end} **: Regime of turbulent production**: The length of the line between the Re_{st} and Re_{end} tells that how quickly the turbulent structures form and grow together.

Some methods for prediction of transition:

- DNS
- Stability analysis: (Refer Primary Stability theory)
- LES
- e^N method
- Experimental correlation: These methods are based on empirically determined correlations. Here, a connection between the turbulence formation is made rest and external factors (degree of turbulence, pressure gradients, surface roughness, etc.). In the same manner, the Intermittenzverlauf and the site of fully qualified turbulence Re_{end} are correlated.

eN Method for 2D boundary layers

In this method, the fanning rate of instable fluctuations is examined. The point at which turbulence production begins will be recognized as the point at which a certain "Amplitude ratio" exceeds a particular value. This "Amplitude ratio" is defined by equation 2.18

$N = \ln\left(\frac{A}{A_0}\right) = \int_{x_0}^{x} -\alpha_i \, dx.$

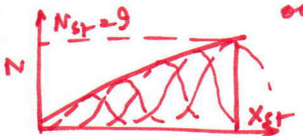
In eqn 2.18, the α_i is defined as follows:

If we calculate 'N' as a function of x for a range of frequencies (fluctuations), we will get a set of N-curves; The envelope of these curves gives the maximum amplification factor N which occurs at any x.

$\alpha = \alpha_r + i\alpha_i$

where $-\alpha_i$ = spatial growth rate of disturbances

α_r = wave number



Where the amplitude of the interference movement and the index 0 which occurs the fault motion in the unstable area place, cf. Fig. 2.18 is A. A fault movement with the frequency f is upstream of the point x 0 steamed. For downstream this positions (x 0 < x < x 1) this fault movement is first kindled, and then steamed (was after the unstable area left, x > x 1). So, a maximum logarithm of the amplitude ratio, as shown in the lower part of Fig. 2.18 stems for each frequency. This is the so-called N-factor.

$N = \ln(A/A_0) =$ If we calculate N as a function of x for a range of frequencies (of fluctuations/disturbances/Störungen) then we will get a set of 'N' curves, each representing the integration formula which is equal to N. The envelope of these curves gives the maximum amplification factor N which occurs at any x.

The comparison to the experiments shows that a value of NSt = 9 delivers very good correspondence to the appropriate Transitionsstellen. This N-factor is an amplitude ratio A/A0 = e9 = 8103. The corresponding x-spot, as well as frequency can be read from the envelope. It can be concluded, that the largest fluctuation is the first one to decay into turbulent structures. The choice of coefficient of NSt = 9 leads the external flow on reliable Transitionsvorhersagen for low degrees of turbulence. However, for higher levels of turbulence, a much earlier Transitionsstelle is observed in experiments.

Eqn 2.20

$N_{st} = -8.43 - 2.4 \ln Tu$

Summary of eN

According to the eN method, the prediction of transition can be summarized as follows:

- Calculation of laminar boundary layer
- Stability analysis of laminar boundary layer – Stability diagram
- Integration of amplifications rate (Anfachungsrate) and determining the envelope for different fluctuation frequencies.
- Choosing a N_{st} factor.
- Determining the corresponding point X_{tr}; this is the flow transition point.

e^N for 3D boundary layers

At the leading edge of a swept wing a three-dimensional boundary layer is formed, where in addition to the velocity profile in the flow direction u(y) also a speed profile across it w(y) exists (cf. Fig. 2.20). The speed profile w(y) has a point pf inflexion and is therefore unstable. Therefore transverse flow instabilities exist in addition to the instability of Tollmien-Schlichting. As a result of the cross flow instability cross flow waves and transverse flow Eddy, whose Achse is approximately aligned in the direction of flow, develop. These waves decay in turbulent structures and lead to a laminar turbulent boundary layer envelope close to front edges. Therefore, the laminar attitude of swept wings with serious problems is connected. With the increase in the angle of attack, the w(y) increases and hence the effect of cross flow instabilities are strengthened.

To extend the eN method for the QA instabilities, a factor of NCF ("crossflow instability") is introduced in addition to the factor of NSt. When crossing a NCF value the cross-flow-induced transition takes place. The corresponding NCF value can be quite different for different use cases, as shown in Figure 2.21.

Ch3

Averaged values :-

- **Time average** : The time average of a stochastic function $f(\underline{r},t)$ eliminates it's time dependancy. Δt must be so big that only the turbulent part is averaged and not the mean flow.

$$\overline{f(\underline{r})} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{T}^{T+\Delta t} f(\underline{r},t)\, dt \longrightarrow (3.1)$$

Usually, $\Delta t \geq \frac{100}{\nu_{min}}$, where ν_{min} is the lowest frequency (energy carrying) of spectrum.

- **Space average** : ... eliminates it's space dependancy → problem becomes trivial because distribution of flow parameters w.r.t space is the most sought after soln. It is however possible to perform a space averaging over a very small distance which is < characteristic dimensions of the flow.

- **Ensemble average** : of $\langle f \rangle$ is defined as the arithmetic average of a large number (N) of values of a stochastic function $f(\underline{r},t)$ i.e. N identical experiments are performed to gather the information.

$$\langle f \rangle = \langle f(\underline{r},t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} f_m(\underline{r},t)$$

So, the ensemble average does not observe only one flow, but an ensemble of all possible flows which can develop under the given boundary conditions.

e.g. Determination of a mean velocity profile as a function of time in a average instationary turbulent flow (blow out from a kettle)

- **Transient / Instationary flows** :

→ Turbulent flows are fundamentally Instationary.

→ A flow is Transient when it cannot be reasonably represented by stationary boundary conditions.

e.g. Moving profiles (blown air) and pulsed free jets] MUST be Transient

Turbulent pipe flow] Need not be classified as Transient as they can be well represented by constant boundary conditions.

W.r.t eqn (3.1), Processes with time constant $T < \Delta t$: Instationary

" " " " $T > \Delta t$: Transient

Ergodicity.

Ergodicity theorem gives a dependance between the Ensemble average and the time & space average. This states that for a stationary stochastic process, the temporal average $\overline{f}$ converges to ensemble average $\langle f \rangle$ for $\Delta t \to \infty$.

This is always then guaranteed, when ALL STATES in an ensemble of systems occur in EACH system of the ensemble.

NS Eqn & RANS

$$\underline{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \quad ; \quad \underline{u} = \overline{\underline{u}} + \underline{u}'(t)$$

$\nabla \cdot \underline{u} = 0$ Continuity. As $\nabla \cdot \overline{\underline{u}} = 0$ so $\nabla \cdot \underline{u}' = 0$.

$$\boxed{\frac{\partial \overline{u_i}}{\partial x_i} = 0} \qquad \boxed{\frac{\partial u_i'}{\partial x_i} = 0}$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \underline{u} \quad \text{NS. Incomp.}$$

Index nota: $$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

RANS: $$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} - \underbrace{\frac{\partial (\overline{u_j' u_i'})}{\partial x_j}}_{\text{Reynolds stress.}}$$

$$(\tau_{ij})_{turb} = -\rho \overline{u_i' u_j'} = (-\rho) \cdot \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{pmatrix}$$

Turbulent kinetic energy: $q = u'^2 + v'^2 + w'^2$

Ex. ① 2D shear flows (homogeneous along Z dirn)

$\overline{u'^2} = f_1(x,y)$, $\overline{v'^2} = f_2(x,y)$, $\overline{w'^2} = f_3(x,y)$, $\overline{u'v'} = f_4(x,y)$

$\overline{u'w'} = \overline{v'w'} = 0$.

② Homogeneous turbulence: $\overline{u'^2} = c_1$ $\overline{v'^2} = c_2$, $\overline{w'^2} = c_3$, $\overline{u'v'} = c_4 = \text{const}$

$\overline{u'w'} = \overline{v'w'} = 0$.

③ Isotropic $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = C = \text{const}$.

$\overline{u'v'} = \overline{u'w'} = \overline{v'w'} = 0$.

⊙ Physical interpretation of Reynolds stresses.

→ Turbulent Normal stresses : $-\rho\overline{u'^2}$, $-\rho\overline{v'^2}$, $-\rho\overline{w'^2}$

By dimensional analysis it may be said that turbulent normal stresses are comparable to local pressure on any sectional area "A" on which impulse forces of turbulent fluctuations are acting.

→ Turbulent Shear Stresses : $-\rho\overline{u'v'}$, $-\rho\overline{u'w'}$, $-\rho\overline{v'w'}$

Stress due to shear force (force acting parallel to surface) on surface A due to turbulent fluctuations.

⊙ The Boundary Layer equation

In flows with boundary layer character, the mean velocity gradient in the dirn of flow are relatively small as compared to the transverse velocity gradient. $\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$

Therefore the only effective term that remains, is the turbulent shear stress $-\rho\overline{u'v'}$.

Although $|u_i'| << \overline{u}_i$, the Boundary layer eqns for turbulent flows cannot be derived from the Boundary layer eqn for laminar flow because $\left|\frac{\partial u_i'}{\partial x_j}\right| >> \left|\frac{\partial \overline{u}_i}{\partial x_j}\right|$

For a stationary, incompressible 2D flow where laminar (molecular) stress (terms $\vartheta\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right)$) and volume forces (e.g. weight) are neglected, the RANS eqn will look like:

$$\left.\begin{aligned}\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y} &= -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}-\frac{\partial}{\partial x}\overline{u'^2}-\frac{\partial}{\partial y}\overline{u'v'} \rightarrow (1)\\ \overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y} &= -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial y}-\frac{\partial}{\partial x}\overline{u'v'}-\frac{\partial}{\partial y}\overline{v'^2} \rightarrow (2)\end{aligned}\right] \quad (3.34)$$

To get the boundary layer eqn for turbulent flows, following estimations are introduced:

1. All dimensions in the flow direction are of order $O(1)$. The longitudinal gradient is of order $O(1/1)$.
2. All dimensions in the transverse dirn are of order $O(\delta)$. with $\delta << 1$; The transverse gradient is of the order $O(1/\delta)$.
3. All mean velocity component in flow dirn are of order $O(1)$.
4. All mean velocity component in transverse dir are of order $O(\delta)$.

⑤ $O(\overline{u'^2}) = O(\overline{v'^2}) = O(\overline{u'v'}) = O(\delta)$.

⑥ $O(p) = O(\delta)$.

Neglecting the terms of order $O(\delta)$, the turbulent boundary layer eqn for mean flow is:

$$\boxed{\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &\approx -\frac{\partial}{\partial y}\overline{u'v'} \qquad (3.39) \\ \frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} &\approx -\frac{\partial}{\partial y}\overline{v'^2} \qquad (3.40) \\ \text{Conti:} \quad \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \qquad (3.41) \end{aligned}}$$

As $\frac{\partial \overline{p}}{\partial x} \approx 0$, so $\frac{\partial \overline{p}}{\partial y} = \frac{d\overline{p}}{dy}$ and hence eqn (3.40) is integrable.

$$\overline{p} - p_0 \approx -\rho\,\overline{v'^2}$$

Hence. in turbulent flows, there is – albeit low – static pressure against the environment.

Energy Equation for Turbulent flows

$$\underbrace{\frac{1}{2}\frac{\partial \overline{q'^2}}{\partial t}}_{\text{lokale instat. Änderung}} = \underbrace{-\frac{1}{2}\frac{\partial}{\partial x_i}(\overline{u_i}\,\overline{q'^2})}_{\text{Konvektion}} \underbrace{-\frac{\partial}{\partial x_i}\left(\overline{u_i'\left(\frac{p'}{\rho} + \frac{q'^2}{2}\right)}\right)}_{\text{Diffusion}} \underbrace{-\overline{u_i'u_j'}\frac{\partial \overline{u_j}}{\partial x_i}}_{\text{Produktion}} +$$

$$\underbrace{+\nu\frac{\partial}{\partial x_i}\left(\overline{u_j'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right)}_{\text{zähe Diffusion}} \underbrace{-\nu\left(\overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial u_j'}{\partial x_i}}\right)}_{\text{Dissipation}}$$

local instationary change of turbulent energy

= Convection + $\underbrace{\text{Diffusion}}_{\text{(contains pressure term)}}$ + Production + Viscous Diffusion − Dissipation

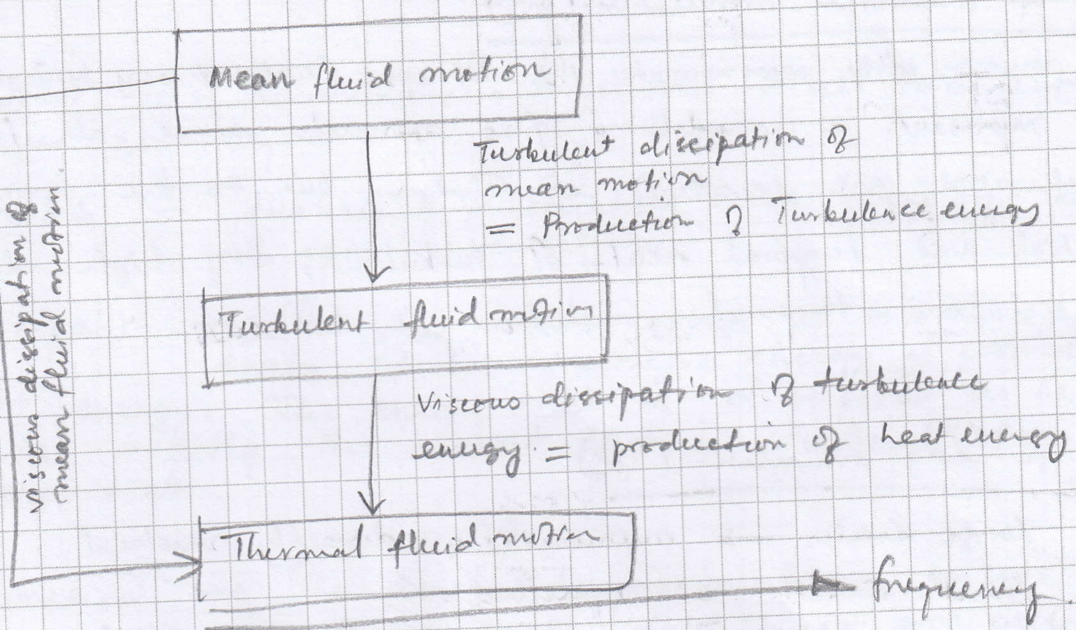
Convection: Transport of turbulent energy due to mean motion.

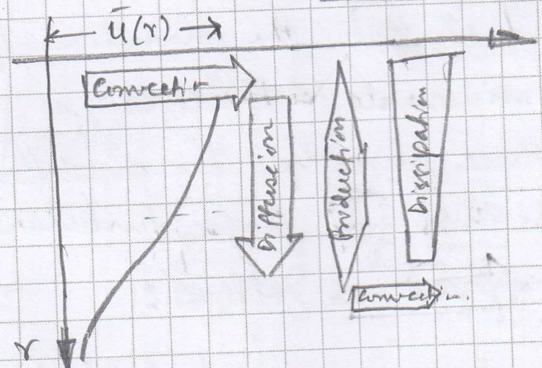
Diffusion: Transport of turbulent energy through turbulent fluctuations.

Production: Production (acquisition) of turbulent energy of the mean motion due ? to turbulent stresses.

Dissipation: Loss of turbulent energy by transition into heat energy, caused by laminar (viscous) friction.

Viscous Diffusion: Molecular (viscous) diffusion of turbulent energy (negligibly small).





Energy transfer in turbulent free jet (turbulenten Freistahl).

Total temporal change of Turbulence energie:

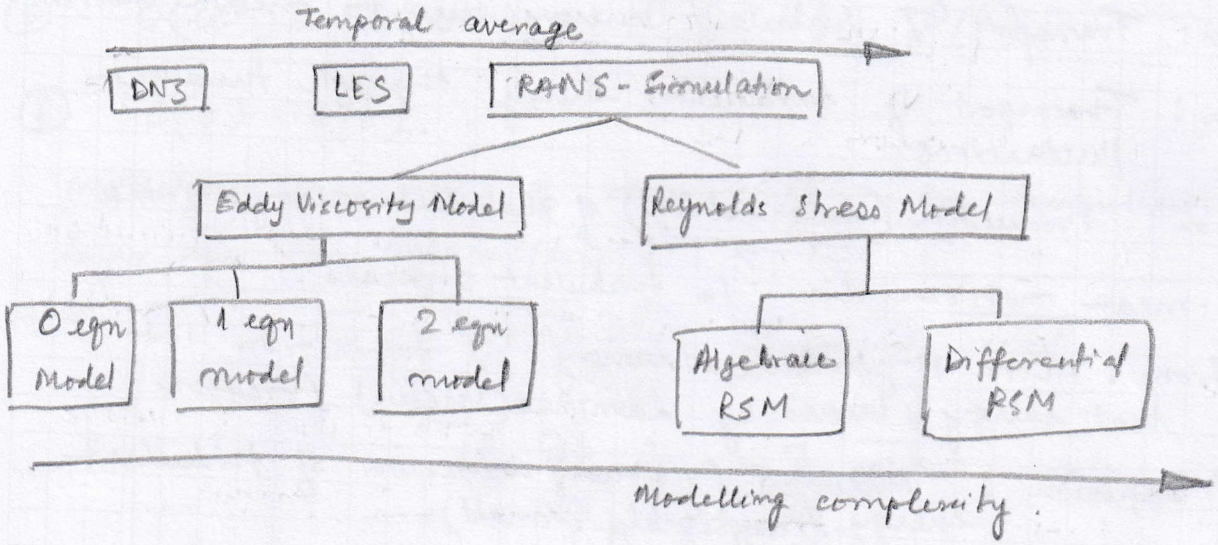
Total temporal change of turbulent energy

= averaged local temporal change of turbulent energy $\quad \dfrac{\partial \overline{q'^2}}{\partial t}$

+ change due to mean flow (convection) $\quad \dfrac{\partial}{\partial x_j}\left(\overline{u_j\, q'^2}\right)$

+ change due to fluctuating motions (turbulent diffusion). $\quad \dfrac{\partial}{\partial x_j}\left(\overline{u_j'\, q'^2}\right)$

Chapter 4



Direct Numerical Simulation (DNS)

It numerically approximates the NS eqns (without any kind of averaging). It represents a modelling free approach since all turbulence mechanisms are considered in NS eqn. Due to the variety of spatial and temporal scales of turbulence, very high demands on spatial and temporal resolution can be seen.

Large Eddy Simulation (LES)

Large scales → numerically directly resolved.
Small scales → modelled.

This approach is motivated by the fact that the modelling of large turbulent structures have considerable difficulties, so it is more advantageous to simulate these directly. The modelling difficulties arise from the properties of large turbulent structures that can be characterized as follows:

→ Vortex formation (maintenance): In shear layers, at first large turbulent structures are formed. Energy is extracted from the base (mean) flow.

→ Anisotropy: In shear layers with a dominating velocity gradient, the turbulent eddies are formed around a preferred axis. The rotational motion of the turbulent structure around the other 2 axis is then relatively weak.

→ Redistribution of turbulent rotational motion: As per Helmholtz, due to the rotational motion of eddy about an axis, rotational motion about the other 2 axis are induced.

→ Decomposition of turbulent Structures: The large structures decompose to always small structures, such that one finds turbulent structures over several orders of magnitude.

→ Turbulent Energy: Large scales contain 80% of turbulent energy. Therefore the coarse structures should include the production area and inertial sub range

In contrast, it is advantageous to model small turbulent structures as they are:

→ Isotropic — turbulent fluctuations have same size in 3 spatial directions

→ Dissipation — Through viscous effects in the small scales, turbulent kinetic energy is converted to heat energy. The energy that is dissipated as heat energy is exactly the amount that is "delivered" from the large scales.

"Filterfunction" → The derived approximation ansatz for small scale turbulence are called "Subscale Models".

RANS

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij} - \rho \overline{u_i' u_j'}\right)$$

Non-transient processes are modelled based on time average. However, transient processes can be entirely covered with RANS approach, then here the flow is stationary but viewed with variable boundary conditions.

Boussinesq Approximation: In a Newtonian fluid, the molecular shear stress $(\tau_{ij})_{mol}$ is proportional to molecular viscosity μ and the mean velocity gradient $\overline{u}_i$

$$(\tau_{ij})_{mol} = \mu \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right)$$

Analogously, now an assumption will be made that the turbulent stresses are proportional to a eddy viscosity μ_T and the mean velocity gradient

$$(\tau_{ij})_{tur} = -\rho \overline{u_i' u_j'} = 2\mu_T S_{ij} - \frac{2}{3}\rho k \delta_{ij} \;?$$

where S_{ij}, the shear rate tensor, represents the mean flow.

<u>Some characteristics of Reynolds stresses:</u>

- A linear relationship between the Reynold's stresses and gradient of mean velocities are assumed, which is not always applicable. At this point, it must be noted that non linear extensions of eddy viscosity models also exist.

- Since only one eddy viscosity model is used for all Reynolds stresses, prediction of <u>anisotropy</u> of Reynold's stress tensor will only be limitedly possible.

- By analogy with molecular stresses, which represents the Diffusion term in NS eqns, a diffusive character is attributed to the turbulent stresses, although turbulent stresses arise from the non-linearity of the convective terms of the average NS eqns.

Turbulence modes based on Boussinesq hypothesis are called Eddy Viscosity Models. A common classification technique is the number of differential equations used in a EVM.

- 0 equation model (algebraic model)
- 1 eqn "
- 2 " "

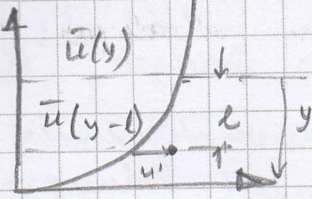
0 equation model

→ Algebraic dependancy between mean flow parameters and eddy viscosity.

This is based on Prandtl's mixing length hypothesis, where the displacement of a fluid particle through turbulent fluctuations is viewed through a length "l":

It is assumed that the velocity fluctuation u' corresponds to the velocity difference $\Delta\bar{u}$ between the original position "y" and the displaced position "$y-l$".

$$u' = \Delta\bar{u} = \bar{u}(y) - \bar{u}(y-l) = l\frac{\partial\bar{u}}{\partial y}$$



For v' it is valid:

$$v' \approx u' = l\frac{\partial\bar{u}}{\partial y}.$$

Then for the Reynolds stress $-\rho\overline{u'v'}$ we have:

$$-\rho\overline{u'v'} = (\Delta\bar{u})^2 = \rho l^2 \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial y}.$$

Generally written: $-\rho\overline{u'v'} = \rho l^2\left|\frac{\partial\bar{u}}{\partial y}\right|\frac{\partial\bar{u}}{\partial y}$.

a $\quad -\rho\overline{u'v'} = \mu_T\frac{\partial\bar{u}}{\partial y}$.

So $\mu_T = \rho l^2\left|\frac{\partial\bar{u}}{\partial y}\right|$, where l is the only unknown.
= Prandtl's mixing length.

Simplest assumption is a linear dependancy of distance from wall 'y': $l = ky$ with $k = 0.41$

1 eqn Model

e.g. Spalart Allmaras (1992) model.

$\mu_T = \rho\nu_T$ with $\nu_T = \tilde{\nu} f_{v1}$

where a differential equation for the "modified" viscosity $\tilde{\nu}$ will be solved.

So, only one quantity is needed to characterize the entire turbulent flow.

$\boxed{\text{2 eqns model}}$ In 1 eqn model, 1 characteristic parameter is u[sed] for describing the essential characteristics of turbulence and transported in the form of differential eqn. Usually this parameter includes information about the turbulent velocities u', v' and w' (e.g. turbulent kinetic energy q). However one can show by dimensional analysis that (at least) one more parameter is required to describe the turbulent structures accurately.

First idea for a 2nd parameter: Kolmogorov 1942 - "Specific turbulent dissipation rate (ω)" - gave an algebraic relation between μ_T and ω

$$\mu_T = \frac{\rho K}{\omega}$$

turbulent kinetic energy.

(Prandtl - Kolmogorov Hypothesis)

Another 2 eqn model is from Wilcox: K - ω model.

K $\longrightarrow$ turbulent kinetic energy (corresponds to the total turbulent kinetic energy q)

ω $\longrightarrow$ specific turbulent dissipations rate, $\omega = \frac{\varepsilon}{K}$.

Kolmogorov assumed that turbulence is in equilibrium.

These 2 assumptions: (i) Boussinesq App. (ii) Turbulence in Equilibrium can sometimes be unrealistic. This is typically the case with:

1. Strong stream line curvature
2. Highly anisotropic turbulence. e.g. region near the wall in boundary layer.

Extensions of 2-eqn models for such flow types are available, but a generalized model is formally ruled out.

Reynolds Stress Model (RSM)

Through multiplication of NS eqns with the fluctuating velocities and then averaging gives the exact equation for the Reynolds-stresses; (Eqn 4.41)

The main advantages of RSM:

(i) Can capture ("Erfassen") the Redistribution mechanism

(ii) Can reproduce the turbulence history.

(iii) Gives non-linear relationships between Reynolds stresses and mean velocities.

(iv) Accurate modelling of turbulent productions.

Also, there are disadvantages:

(i) Difficulty in modeling double-triple correlation, as well as the dissipation rate.

(ii) Individual terms of RSM exhibit large non-linearities which vary strongly in the flow field. Therefore it is difficult for these models to provide robust solutions.

(iii) As there are so many (6) equations, RSM clearly requires higher computational effort than EVM.

Detached Eddy Simulation (DES)

<u>Basic idea</u>: Turbulent boundary layers can be fairly well represented with RANS approach; so one may not use LES. Because the (correct) treatment of turbulent boundary layers by LES demands extremely fine spatial and temporal discretization, as turbulent scales that are to be resolved in thin boundary layers, are very very small. On the other hand, detached areas, trailing wakes and free shear layers can be rather well simulated by LES at reasonal computational costs; the RANS models fail here frequently.

Flow is monitored by a suitable detector, where based on detected flow conditions, either RANS or LES is applied.

The original form of DES was based on Spalart-Allmaras RANS model. With a slight change of formulation, this RANS model can also be used as a Subscale model (LES). After detection of a suitable LES areas, the integral length of turbulence in Spalart-Allmaras model, given by distance from wall "d", is replaced by a DES-length scale.

$$l_{DES} = \min(d, C_{DES}\Delta)$$

with $\Delta = \max[\Delta x, \Delta y, \Delta z]$, $C_{DES} = 0.65$.

With this, a dependancy of the Dissipation on the local cell width Δ and the model constant C_{DES} introduced.

In flow regions where $C_{DES}\Delta < d \Rightarrow l_{DES} = C_{DES}\Delta$.

when $C_{DES}\Delta > d \Rightarrow l_{DES} = d$.