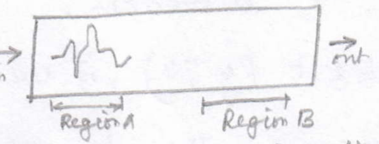
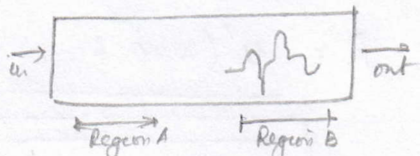
Cross correlation:

The concept of "Frozen Pattern": The shape and size of fluctuations do not change with time. Suppose you have taken a photo of a certain flow field where fluid flows from left to right. In that photo, in a certain region you saw a fluctuation pattern e.g (say "Region A")



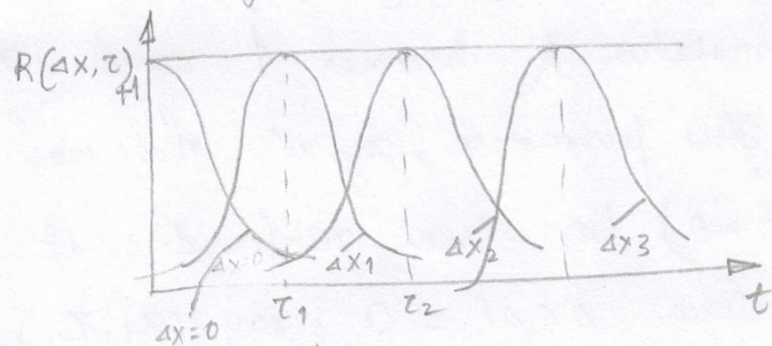
Now if you take another photo after τ seconds, you will find that same fluctuating pattern in a "Region B"



Essentially, we have "frozen" that "fluctuating pattern" which we saw at an arbitarily chosen position (Region A) when we took the first photo, i.e., started observing the flow

So, if we divide the distance covered by this "frozen pattern" by τ, then we get the "Transportgeschwindigkeit" u_c.

Now let us go to the explanation of Abb 5.5 Pg 74.



$\Delta x = 0$
(=autocorrelation

When $t=0$, it is seen that for the curve $\Delta x = 0$, R is 1. This is obvious because at $t=0$ (observation started, our first "photo"), we are correlating: $R =$

$$\frac{u'(x=0,t=0) \cdot u'(x=0,t=0)}{\sqrt{\overline{u'^2}(x=0,t=0)}\,\sqrt{\overline{u'^2}(x=0,t=0)}}$$

Now, we want to stay fixed w.r.t position (i.e. we keep looking at $x=0$) but let time pass on. In other words, we can say that we continue looking at Region A. As time passes by, some different "frozen pattern" will come in from the left and occupy Region A. Longer the elapse of time, more dissimilar will be the frozen patterns.

So we are correlating $u'(x=0,t=0)$ with $u'(x=0, t=\tau>0)$. So as τ increases, R reduces. This is why for $\Delta t>0$ (Abb 5.5 Pg 74), if we hold our observation position constant i.e. $\Delta x=0$, the R goes on reducing.

$$R = \frac{\overline{u'(x=0,t=0) \cdot u'(x=0, t=\tau>0)}}{\sqrt{\overline{u'^2(x=0,t=0)}}\sqrt{\overline{u'^2(x=0,t=\tau>0)}}} < 1.$$

Now, if along with the passing time $(t>0)$ if we also move our eyes away from $x=0$ towards right (i.e. in the flow direction) then we see that after travelling a certain distance to the right $(\Delta x>0)$ in a certain time $(t>0)$, the correlation R becomes 1 again. This means that the flow parameter, say u', that was observed at $(x=0, t=0)$ has been "re-found" at $(x=\Delta x, t=\Delta t)$ where $\Delta x, \Delta t = 0$, say (x_1, τ_1).

With respect to the "frozen pattern", we can hence say that the frozen pattern that was observed at $(x=0, t=0)$ has now "arrived" at $(x=x_1, t=\tau)$ or has been connected to position x_1 in τ_1 time units.

Hence the "Transportgeschwindigkeit"

$$= \frac{x_1 - 0}{\tau_1 - 0} = \frac{\Delta x}{\tau}$$ which is eqn (5.29).

Carrying forward the same concept of transport of frozen pattern, we can say

$$u_c = \frac{\Delta x_1}{\tau_1} = \frac{\Delta x_2}{\tau_2} = \frac{\Delta x_3}{\tau_3} = \dots = \tan\alpha \quad \text{(Abb 5.6)}.$$

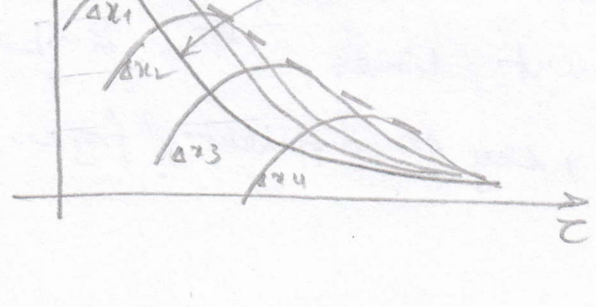
Now, it is worthy to be repeated that "that which is being observed (u') at position $\Delta x > 0$ and $t = \tau$, had also been observed at position $x = 0$ and $t = \tau - \frac{\Delta x}{u_c}$ time units ago"

$$\Rightarrow \quad u'(\Delta x, \tau) = u'\left(0, \tau - \frac{\Delta x}{u_c}\right)$$

$$\text{So} \quad R(\Delta x, \tau) = R\left(0, \tau - \frac{\Delta x}{u_c}\right) \quad \text{(Eqn 5.30)}$$

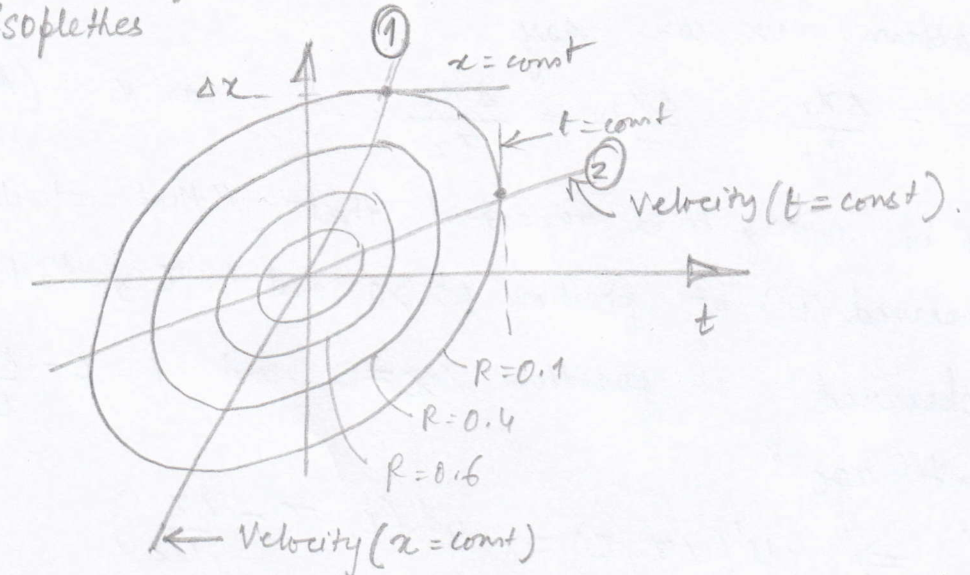
Abb 5.5 assumes "ideal" situation where the fluctuation patterns are not distorted over space and time due to real physical effects like diffusion. So the envelope of R curves is // to t axis. This envelope is the locus of R_{max} for a given Δx = constant and > 0.

Abb 5.7 takes into consideration such physical effects. So the u' found at $x=0, t=0$ can never be exactly "re-found" at $x = \Delta x > 0$ and $t = \tau > 0$. The u'_s at $x = \Delta x > 0, t = \tau > 0$ will be LESSER THAN u'_s at $x=0, t=0$. So the R curve (cross-correlation) never reaches 1, and the peak of R curves keep getting lower as Δx and τ keeps on increasing.



$\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4 > 0$.

These cross-correlational observations can also be plotted as "isoplethes"



All our previous discussions on cross-correlograms refer to velocity 1 as we dealt with constant Δx curves. However, the real velocity is given by straight line 2, i.e. velocity observed when time is kept constant.

This can be explained as follows: Suppose we take a photo of the entire flow field. From that photo we can see the different "frozen patterns" of fluctuations at different position. If we find 2 similar "frozen pattern" (NB. Similar but not exact, due to diffusion) separated from each other by a certain distance, then we can estimate the velocity by dividing that distance by the time which the "diffusing" frozen pattern would have taken to cover that distance. This is the top figure in Abb 5.9, which gives an estimate of velocity as per straight line 2.

Now, if we are observing at one fixed position and allowing time to run, then we will gain no understanding of the flow if we keep on taking photos of the same spot at different times. This is because all we will have is, say 10 different "frozen pattern"

equi-spaced time intervals,
actually need to physically
the flow field to measure u's
t different times. Such a
he probe will definitely DISTURB
l and hence the velocity u'
ny flow field parameter) will
the case for all the cross
with $\Delta x = $ const curves (and hence
l by straight line 1 in the
velocity 1 is not the true

lor - Hypothesis

$$- u \frac{\partial}{\partial x} \Big|_{\text{at fixed position.}}$$

ime dependant quantities can be
dependant quantities and vice-versa.
lid for (approximately) for homogeneous
time-averaged constant velocity $\bar{u}$
boundary condition that $\boxed{u' << \bar{u}}$

The occurence of fluctuating motions
ixed) position in the flow field
the "frozen patterns" swim
position.

We assume that in a given flow the turbulent structures are "frozen". Then applying the Taylor hypothesis ~~for~~ to a quantity q we have:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0$$ (As by definition, ~~the~~ a frozen pattern does not change in time }; so the total differential has to be 0.).

Reynold's arg: $q = \overline{q} + q'$ and $u = \overline{u} + u'$.

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial q'}{\partial t} = -(\overline{u}+u')\frac{\partial \overline{q}}{\partial x} - (\overline{u}+u')\frac{\partial q'}{\partial x}$$

For an averaged stationary and homogeneous turbulent flow, it is valid that:

$$\frac{\partial \overline{q}}{\partial t} = 0 \quad \text{und} \quad \frac{\partial \overline{q}}{\partial x} = 0$$ why?

$$\Rightarrow \quad \frac{\partial q'}{\partial t} = -(\overline{u}+u')\frac{\partial q'}{\partial x}.$$

With $u' \ll \overline{u}$ gives:

$$\frac{\partial q'}{\partial t} = -\overline{u}\frac{\partial q'}{\partial x}$$

Generally one ~~may~~ can write

$$\frac{\partial}{\partial t} = -u_{Tr}\frac{\partial}{\partial x}$$

where u_{Tr} = averaged transformation velocity of "frozen" turbulent structures.

As "Transformation" velocity, the previously described Convective velocity u_c of the frozen pattern, u_c, ~~can~~ is ~~be~~ used.

$$u_{Tr} \equiv u_c.$$

However, $u_{Tr} \equiv u_c$ is not true for other forms of turbulence.

Again, if the fluctuations are small as compared to the mean flow (which is already a condition for Taylor Hypothesis) then ~~also~~ the approximation $u_c \approx \overline{u}$ is also valid.

It is to be noted that for shear flows, homogeneous or isotropic turbulence cannot be assumed. Hence for shear flows, $\overline{u} \neq u_c \neq u_{Tr}$.

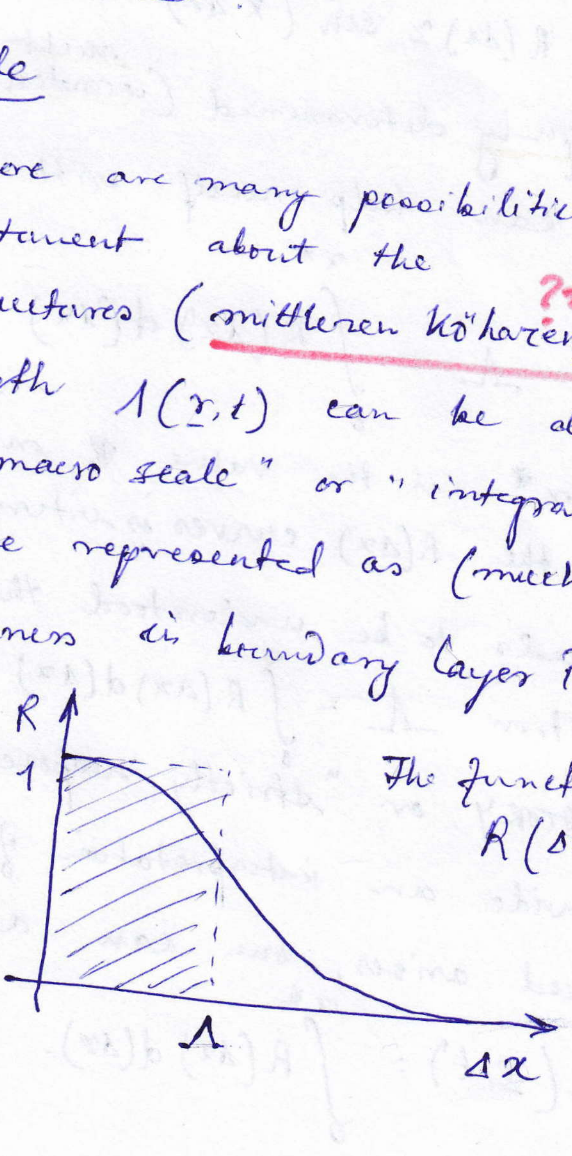
5.4 The Characteristic lengths of Turbulence Structures

From the double-correlations (w.r.t space or time), the measures (length scales) for characterizing turbulent structures can be obtained:

The Integral Scale

For characterization, there are many possibilities. If one wants a statement about the average Coherence structures (mittleren Köharenzbereich) ?* then an integral length $\Lambda(\underline{r}, t)$ can be defined, which is known as "macro scale" or "integral scale". Conventionally, it can be represented as (much like the displacement thickness in boundary layer theory)

$$\Lambda(\underline{r}, t) = \int_0^\infty R(\Delta x)\, d(\Delta x)$$



The function $R(\Delta x)$

* Der Begriff „kohärente Struktur" wird in dieser Arbeit synonym mit den Begriffen „Wirbel" oder „Wirbelstruktur" verwendet und meint damit eine raum-zeitlich organisierte Ansammlung von Vortizität ...

(ref: diploma thesis_wilczek_07.pdf).

≡ The term "coherent structure" is used synonymously in this work by the terms "vortex" or "vortex structures", referring a spatio-temporally organized collection of vorticity

Difficulty with the defn $\Lambda = \int_0^\infty R(\Delta x)\, d(\Delta x)$ arises when

(i) $R(\Delta x)$ is periodic

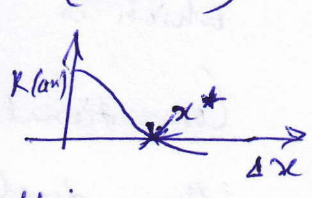
(ii) $R(\Delta x)$ has an overswing i.e. it can have negative values.

Say, if $R(\Delta x) \simeq \sin(K\Delta x)$, then $\Lambda(\underline{r}, t)$ cannot be uniquely determined (nicht eindeutig bestimmbar).

So one can help oneself with the following special defn:

$$\Lambda = \int_0^{x^*} R(\Delta x)\, d(\Delta x)$$

where x^* is the value on the abscissa (Δx axis) where the $R(\Delta x)$ curves is intersecting it.



It needs to be understood that the definition $\Lambda = \int_0^\infty R(\Delta x)\, d(\Delta x)$ is not something MANDATORY or "strictly imposed". It is only intended to provide an interpretation of length scales. So when the need arises, one can adopt the definition

$$\Lambda(\underline{r}, t) = \int_0^{x^*} R(\Delta x)\, d(\Delta x).$$

A better statement for "integral scales" can be obtained if the following would be possible:

(i) The signal (say u') can be split ~~up~~ into different types of signal under the assumption that such "different types" are not correlated to each other;

say $u' = \underbrace{u'_p}_{\text{periodic}} + \underbrace{u'_s}_{\text{stochastic}}$.

and $\overline{u_p' u_s'} = 0$.

(ii) Then, $R_{u'}(\Delta x)$ could be expressed as an additive composite (Korrelationen additiv zusammensetzen) of $R_{u_p'}(\Delta x)$ and $R_{u_s'}(\Delta x)$.

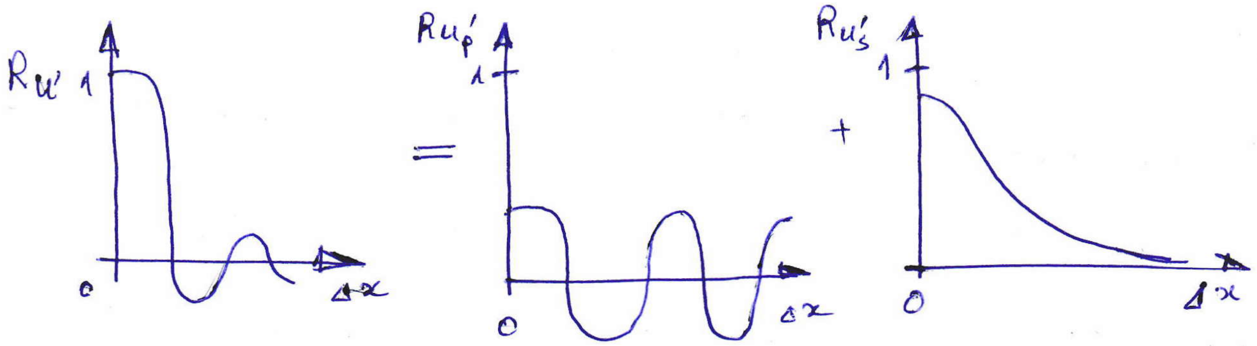


Abbildung 5.13: Composite ~~relat~~ Correlations

From such composite correlations, the turbulent flow can be characterized by 2 characteristic lengths; one deterministic, one probabilistic. These are differently defined and hence not compatible with each other.

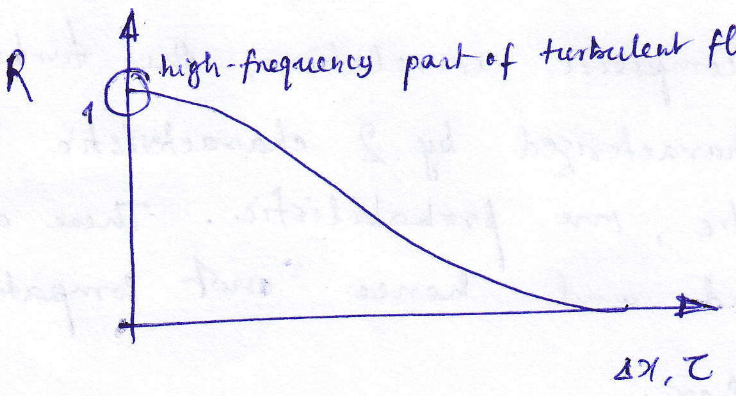
A quantitatively tangible definition of these "turbulence balls" or ~~macro scale~~ corresponds approximately to the middled size (diameter) of the energy carrying vortices. They are characterized by the large-scale, low frequency, part of turbulent structures.

Analogously we can obtain an integral scale measurement for $\underbrace{R(\tau) \text{ and } \tau}_{\text{Auto correlation}}$ instead of $\underbrace{R(\Delta x) \text{ and } \Delta x}_{\text{space correlation}}$.

where $\Lambda(\underline{x}_1, t_1) \approx u_c \Lambda_T(\underline{x}_1, t_1)$.

The Micro-scale

The macro scale provides no information about the high-frequency part of ~~the~~ a turbulent flow i.e. about the processes or structures that dissipate energy (Dissipationsbereich). However, information about the structures in the "Dissipationsbereich" is also included in the correlation curves: In the apex zone of the correlation curve where Δx and τ is small, since smallest eddies have minimal life.



It can be shown that for very small values of Δx, $R(\Delta x)$ is a parabolic function of Δx.

$$R(\Delta x) = 1 - \frac{(\Delta x)^2}{2} \frac{1}{u'^2} \left(\frac{\partial u'}{\partial x}\right)^2 \Bigg|_{\Delta x = 0}.$$

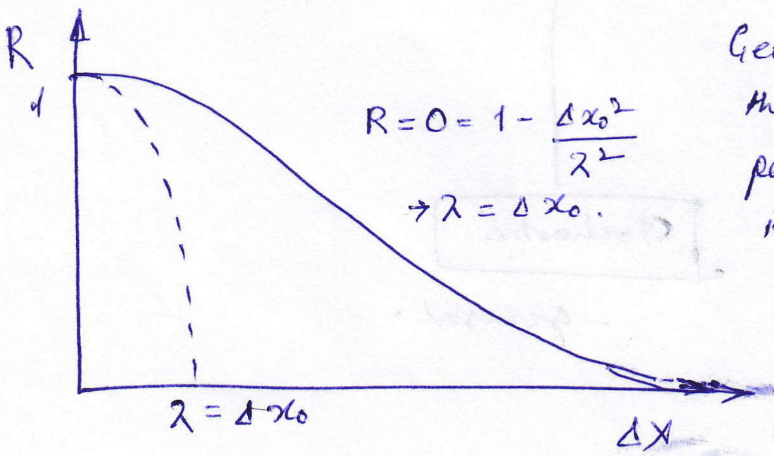
Because parabolas are uniquely defined solely by their curvature at origin, a characterizing length λ can be introduced that characterizes the curvature at the apex of the Correlogramms.

$$R(\Delta x) = 1 - \frac{(\Delta x)^2}{\lambda^2}$$

where $\cancel{\lambda}$ $$\frac{1}{\lambda^2} = -\frac{1}{2} \frac{\partial^2 R}{\partial (\Delta x)^2}\Bigg|_{\Delta x = 0}$$

$$= \cancel{\frac{1}{2}}$$

The parabola has a very big curvature and ~~at~~ due to that it delivers a very small value of λ. Generally one can interpret λ as the average measurement of the smallest (dissipating) eddies. This λ is called the "Micro-Scale" or „Dissipation length".



Geometrically, λ is the value of the abscissa where the Oskulation parabola of the correlation curve intersects the abscissa.

Turbulente Reynolds number

A further characterizing parameter for turbulent flows ~~is~~ ~~built~~ can be build using:

(i) Intensity of velocity fluctuation $\sqrt{\overline{u'^2}}$

(ii) Dissipation length (micro scale) λ_g (g = lateral micro scale)

$$\lambda_g = \frac{\sqrt{2}\sqrt{\overline{u'^2}}}{\sqrt{\overline{\left(\frac{\partial u'}{\partial y}\right)^2}}}$$

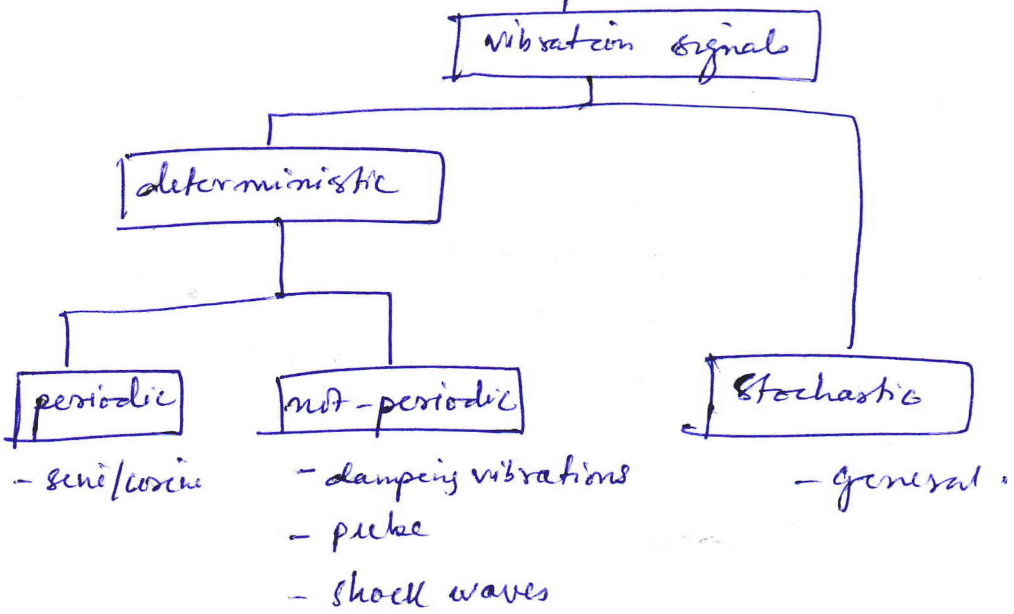
(iii) Molecular, kinematic viscosity

making it particularly important for the characterization of the structure in terms of dissipation.

$$Re_T = \sqrt{\overline{u'^2}}\,\frac{\lambda_g}{\nu} = Re_{\lambda_g}$$

= Reynolds number of Turbulence (as per Taylor)

Spectral analysis of flows



Turbulent motion is the consequence of superimposing Eddies of different frequencies and Sizes/Strength/Intensities. In reality a certain frequency does not occur constantly in turbulent flow. However it is possible to allocate a certain proportion of the total fluctuation energy to a certain frequency. i.e. a harmonic analysis of the fluctuating motion can be performed.

Although turbulent fluctuations are stochastic, by Fourier analysis the periodic vibrations can be outlined.

A real function $u(t)$ with the period T_0 i.e. $f = \frac{1}{T_0}$ can be uniquely represented ~~through~~ as a sum of sin & cosine functions.

$$u(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \sin(n \cdot 2\pi f_0 t) + b_n \cdot \cos(n \cdot 2\pi f_0 \cdot t) \right)$$

a_0 = arithmetic average of the signal. As $\overline{u'} = 0$, so $a_0 = 0$.

f_0 = ~~ground~~ fundamental frequency.

a_n, b_n = Fourier coefficients.

$$a_n = \frac{2}{T} \int_0^T u(t) \sin(n \cdot 2\pi f_0 \cdot t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T u(t) \cos(n \cdot 2\pi f_0 \cdot t)\, dt.$$

Using Euler's identity $e^{ix} = \cos x + i \sin x$.

$$u(t) = \sum_{n=-\infty}^{+\infty} \cancel{\cos}\ C_n \cdot e^{i \cdot n \cdot 2\pi f_0 \cdot t}.$$

?

where $C_0 = a_0 = 0$ $\quad C_n = \frac{1}{2}(a_n + i b_n)$

$$C_{-n} = \frac{1}{2}(a_n - i b_n).$$

The problem with the analysis of non-periodic & stochastic signals is that ~~they~~ a particular fundamental frequency (f_0) can't be specified, ~~as~~ They contain all (not only just discrete) frequencies. To capture all possible frequencies one must go over a infinitely big ~~period~~ Time period $T \longrightarrow \infty$

$$u(t) = \int_{-\infty}^{\infty} \hat{c}(f) \cdot e^{i \cdot 2\pi f \cdot t} \, df .$$

$\hat{c}(f)$: continuous complex valued function, the so called $\boxed{\text{Spectral function}}$.

The Fourier tranformation is unique and thus also reversible i.e. $\hat{c}(f)$ can be uniquely reconstructed from $u(t)$.

Parceval's equation says: $\displaystyle\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{c}(f)|^2 df$. ?

As $u(t)$ is velocity so the square of the corresponding spectral function is a measure of the turbulent kinetic energy contained in certain frequency ranges "df".

$$|\hat{c}(f)|^2 = \hat{c}(f) \cdot \underbrace{\hat{c}^*(f)}_{\text{complex conjugate of } \hat{c}(f)}$$

Based on this consideration, the $\boxed{\text{Energy spectrum}}$ is defined as : $\displaystyle E(f) = \frac{|\hat{c}(f)|^2}{df} = \frac{\hat{c}(f) \cdot \hat{c}^*(f)}{df}$.

The total turbulent Energy can be calculated from the E(f):

$$\overline{u'^2} = \int_0^\infty E(f)\,df. \qquad E(f) = \text{Energy per frequency.}$$

E(f) is also known as Spectral density, Energy spectrum or simply Spectrum

As $[E(f)] = \dfrac{[u'^2]}{\text{frequency}} = [u']^2 \cdot \text{Time}$

Similarly one has $[E(k)] = \dfrac{[u'^2]}{\text{wave number}} = [u'^2] \cdot \text{Length}$

$E(k)$ = Wavenumber spectrum.

Necessity of FFT: To calculate Fourier transform, the knowledge of analytic function $u(t)$ or $u(x)$ is necessary. Usually this is not available and in reality one works with discrete measurement whose functional ~~int~~ dependance's are not explicitly known. Formally, one must now transform the differentials to differences and integrals to sums.. The largest frequency is taken as fundamental frequency and ~~$n \to \infty$~~ in place of $n \to \infty$, discrete measurement points $n = 1, 2, \ldots, N$ are taken. So FFT can transf