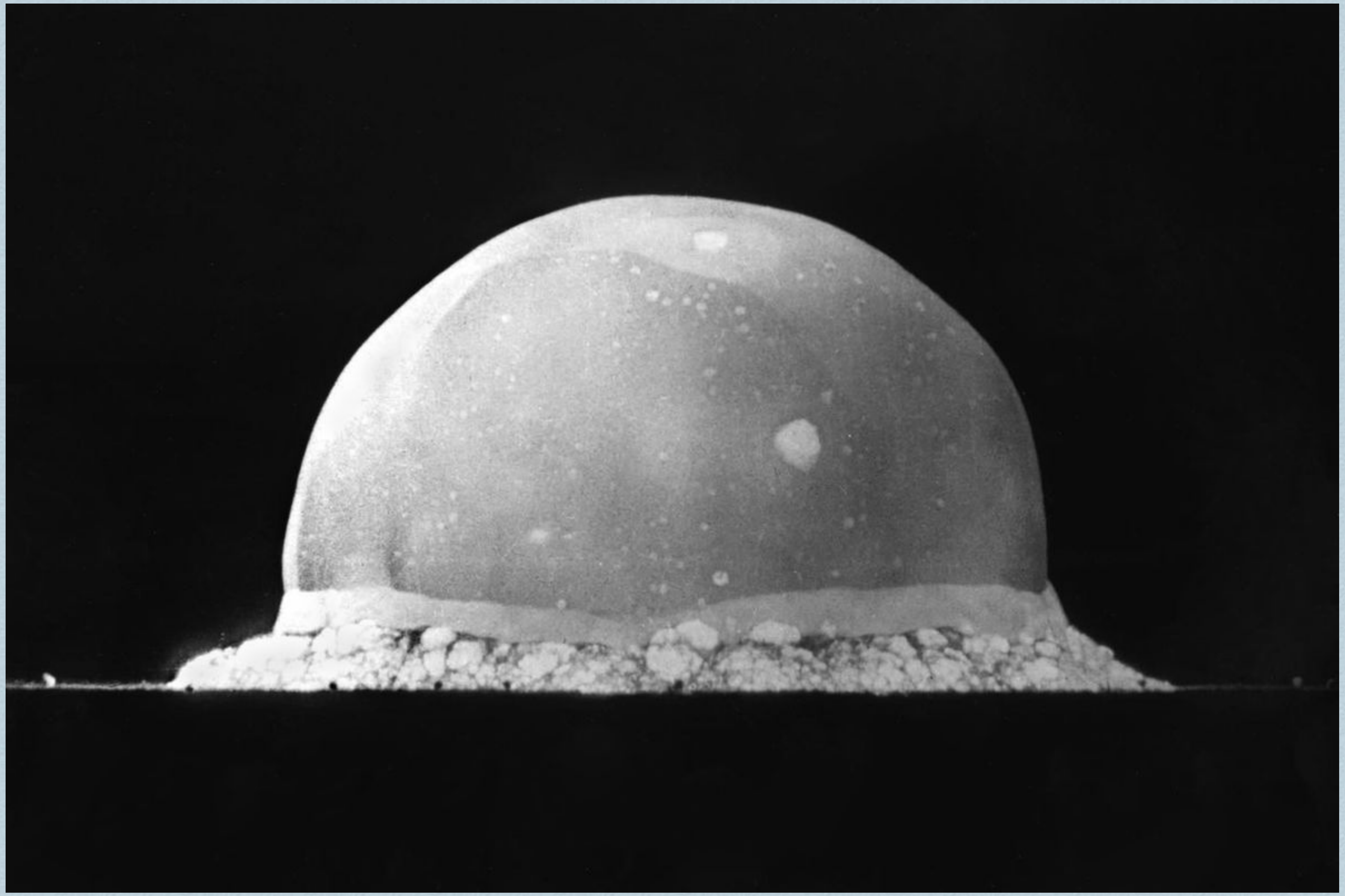
From Viscous to Inviscid: the Math behind shocks



Constant density Navier Stokes/Euler Equations

Momentum $$\boxed{\frac{D\vec{u}}{Dt}} = \boxed{-\frac{1}{\rho_0}\nabla p}\boxed{\nu\,\nabla^2\vec{u}}$$

Mass $$\boxed{\vec{\nabla}\cdot\vec{u} = 0}$$

Dropping viscosity (red box) lowers the order of spatial derivatives by one. Mathematically this is called a “singular limit”.

In these slide we show a way to reconcile viscous and inviscid phenomena using a model equation.

Subscripts denote partial derivatives

Primes denote ordinary derivatives

$$A_t + AA_x = \nu A_{xx} \text{ Burgers Equation}$$

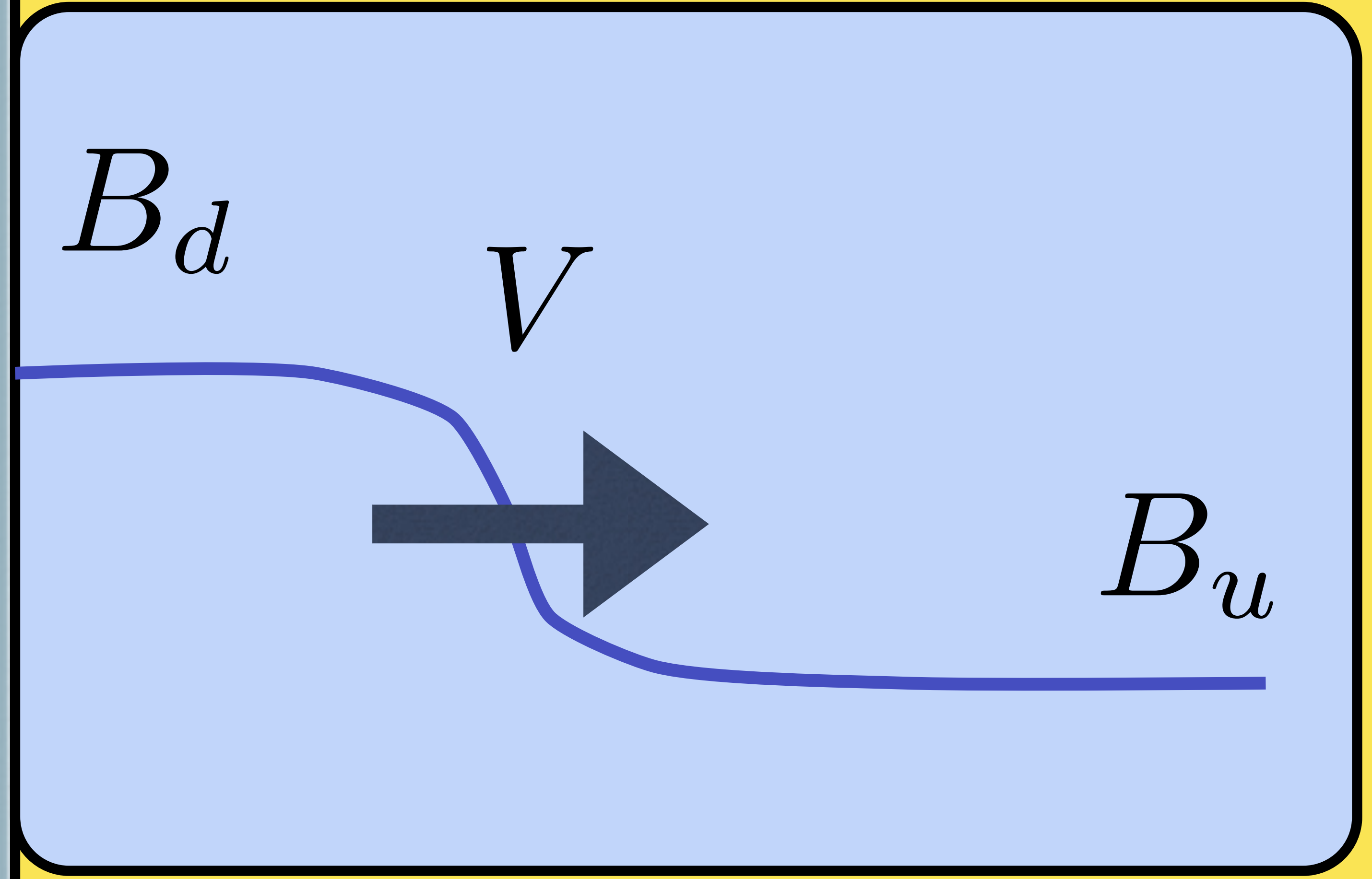
$$\xi = x - Vt \text{ travelling variable}$$

$$A(x,t) = B(\xi) \text{ travelling wave solution}$$

$$-VB' + \frac{1}{2}(B^2)' - \nu B'' = 0$$

$$-VB + \frac{1}{2}B^2 - \nu B' = C_1 \text{ integrating once}$$

- The Burgers equation allows us to understand the mathematics of the gas dynamics setting in a simpler context (one equation).
- First we will keep the second derivative term, then consider what happens when we drop it.
- A basic strategy for a nonlinear equation is to look for travelling solutions (notice these can't be plane waves since Burgers equation is nonlinear).
- We want to find travelling waves that are sharp transitions between two essentially constant regions.



$$-VB + \frac{1}{2}B^2 - \nu B' = C_1$$

Need to find V and C_1

$$B \to B_u, B_d \text{ as } \xi \to \pm\infty$$

$$B' \to 0 \text{ as } \xi \to \pm\infty$$

$$V = \frac{1}{2}(B_u + B_d)$$

$$C_1 = -\frac{1}{2}B_u B_d$$

- We can use the upstream (or in front of shock) and downstream (or behind shock) state to integrate once and to determine the speed of the travelling disturbance and the constant of integration.

- To integrate a second time we factor and use partial fractions.

$$(B - B_u)(B_d - B) = -2\nu B' \quad \text{factoring}$$

$$\frac{\xi}{\nu} = \frac{2}{B_d - B_u} \log \frac{B_d - B}{B - B_u} \quad \text{integrating}$$

$$\Delta B = B_d - B_u \quad \text{define the difference}$$

$$B(\xi) = B_u + \frac{\Delta B}{1 + \exp[\frac{\Delta B}{\nu}\xi]} \quad \text{final solution}$$

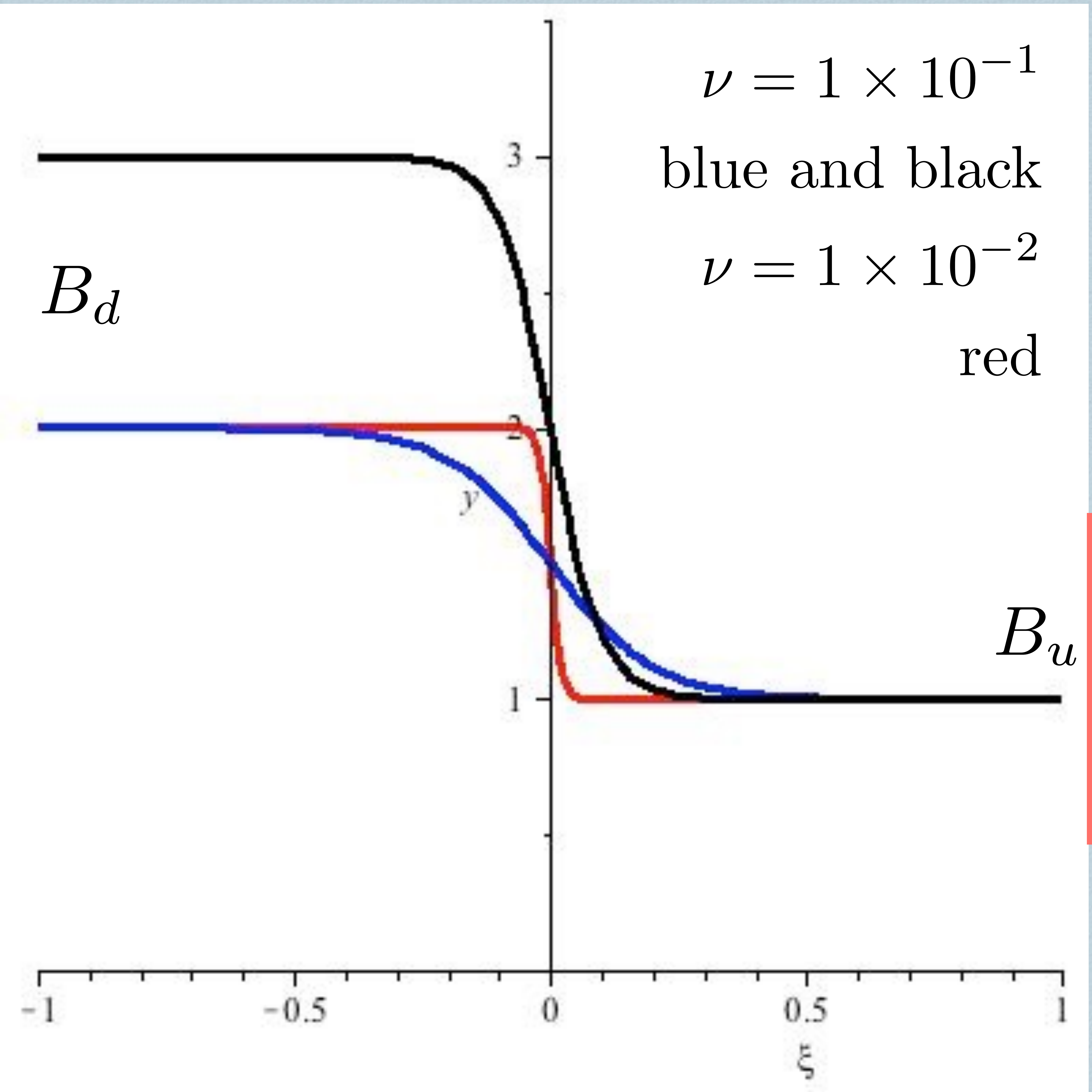
$$V = \frac{1}{2}(B_u + B_d)$$

- The first key point to notice is that the velocity of the travelling wave does NOT depend on the viscosity.

- The second key point to note is that the width of the wave depends inversely on viscosity, so lower viscosity means thinner waves.

- The third key point to note is that that thickness of the wave depends on the wave amplitude with larger waves being 'thinner'.

down
stream
or
behind
shock



upstream
or
ahead
of
shock

- We next want to figure out what happens when the viscosity tends to zero.
- This is the same sort of limit that gives the ideal gas equations.
- From the Burgers equation solution we see that the wave width goes to zero and thus the travelling wave solution tends to a discontinuity called a **shock.**
- We want to use some PDE theory to understand these solutions.
- The simplest problem is called the Riemann problem.

$$A_t + AA_x = 0 \text{ inviscid Burgers equation}$$
$$A(x,0) = A_d > A_u \text{ when } x < 0$$
$$A(x,0) = A_u > 0 \text{ when } x > 0$$

- We could define a characteristic curve in analogy with the xi variable in the travelling wave solution for the viscous Burgers equation:

$$\frac{dx}{dt} = A(x,t)$$

$$\frac{dx}{dt} = A(x,t)$$

- But for the Riemann problem on the last slide we have that

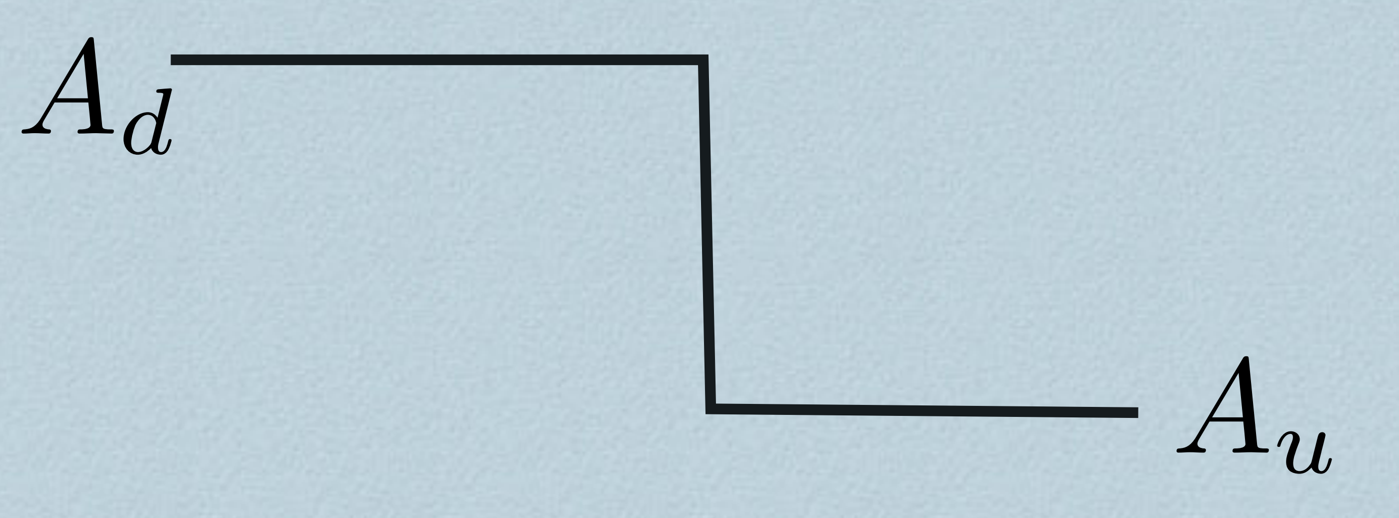
$$\frac{dx}{dt} = A_d \text{ when } x < 0$$

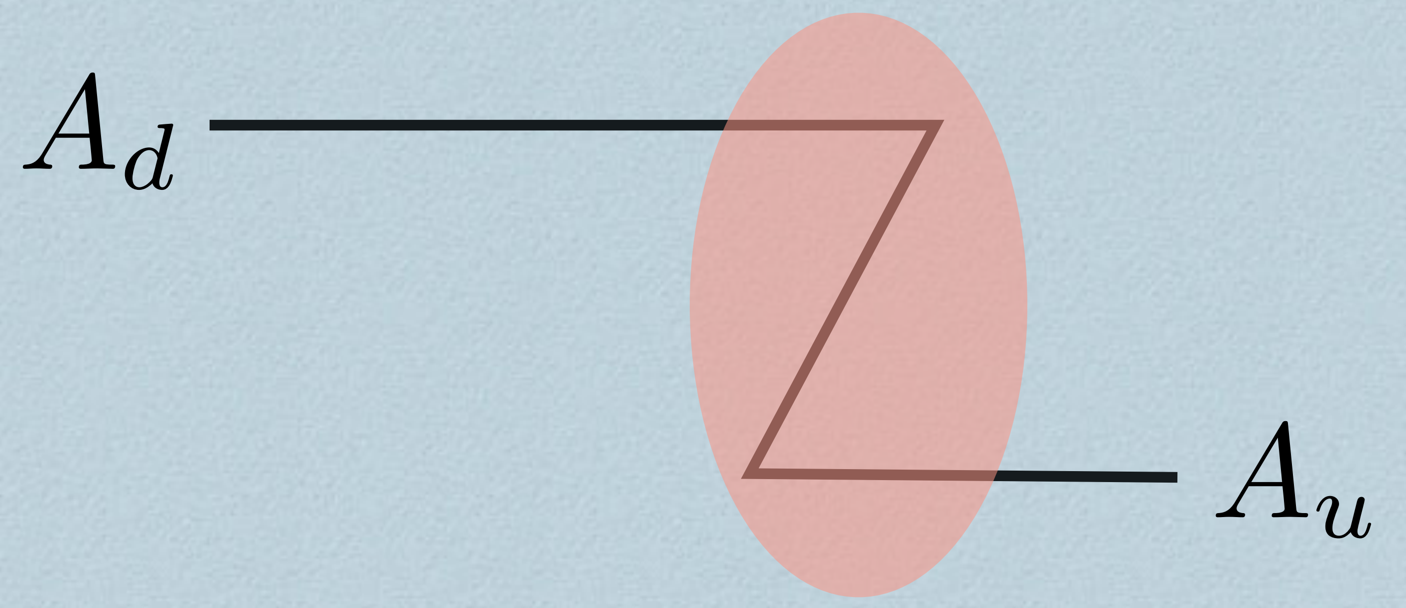
$$\frac{dx}{dt} = A_u \text{ when } x > 0$$

$$\text{But } A_d > A_u \text{ so the curves cross}$$

- Another way to see this is to write down an implicit solution using D'Alembert's solution for the advection equation (F(.) is given by the initial condition):

$$A(x,t) = F[x - A(x,t)t]$$





- So the implicit solution loses uniqueness immediately!

- How can we fix things? Let's return to the Burgers equation with viscosity.

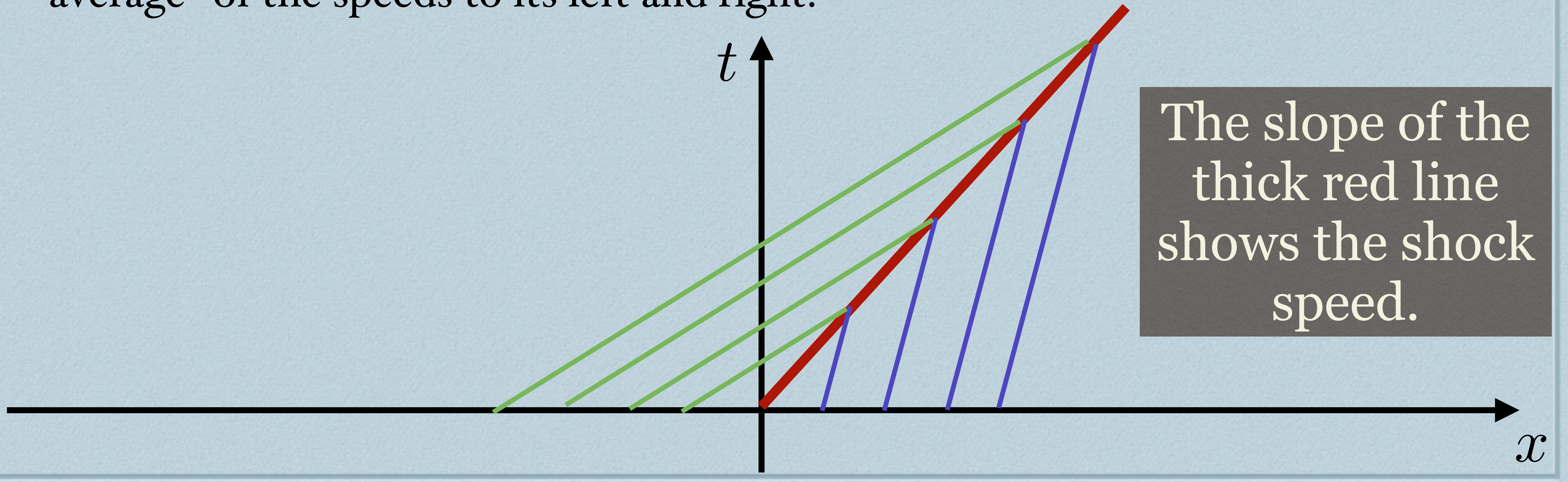
- We found for this case that the wave speed did not depend on viscosity so take the wave speed to match the viscous case and solve the initial value problem as:

$$A(x,t) = A_d + (A_u - A_d)H\left(x - \frac{A_u + A_d}{2}t\right)$$

- Where H(.) is the Heaviside step function.

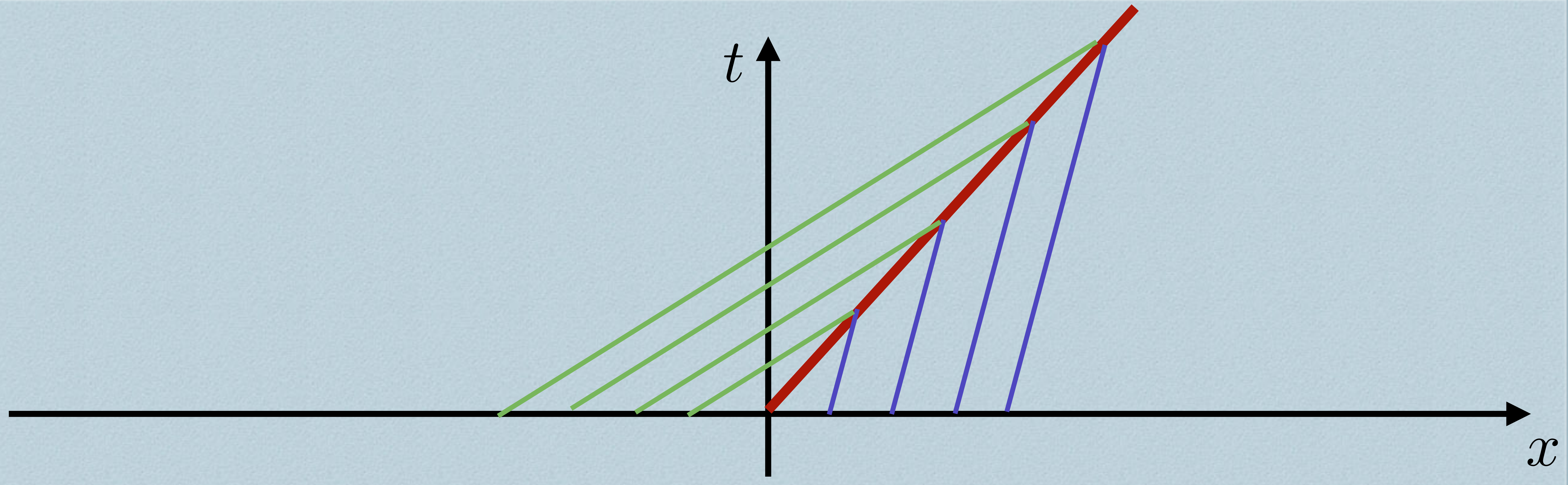
- All that remains to do is to fix the characteristic curves.

- To do this start with the discontinuity. At t=0 it is at x=0 and moves with the "average" of the speeds to its left and right.



The slope of the thick red line shows the shock speed.

The algebra shows the shock moves at speed set by the average of the states



- The steeper characteristics that start from the positive x axis are shown in blue.
- The less steep characteristics that start from the negative x axis are shown in green.
- The path the shock follows is shown in red.
- Returning to the original PDE you might notice that we have gone slightly out of bounds from a mathematical point of view.
- This is because the shock solution is not differentiable (it isn't even continuous!).

- To make things mathematically correct we need to slightly weaken our notion of what a solution is.
- The traditional solution is a formula which, when we substitute it into the equation gives the relation 0=0.
- For this to be true the solution must be differentiable as many times as the order of the PDE.
- For our case the solution is smooth away from the shock, so we seem to be left with the task of somehow "smudging" the single point at which the discontinuity happens.
- Of course this is exactly what viscosity does in the viscous Burgers equation which we were able to solve exactly for "shock-like" waves.
- Mathematicians call this type of solution a "weak" solution and weak solutions form a theoretical topic of study in PDE theory as well as a foundation for many modern numerical techniques like finite element methods.

$$\int_{-\infty}^{\infty} \phi(x)\,(A_t + AA_x)\,dx = 0$$

Where ϕ is an $\boxed{\text{arbitrary,}}$ differentiable function

- You will recall that the integral "overlooks" values that occur at a single point.
- The above "weak formulation" of the PDE allows us to generalize the classical solution to situations like the shock.
- If A(x,t) is differentiable than the weak formulation is equivalent to the classical solution (for analysis fans, you can prove this).
- Numerical methods that handle shocks begin with the weak formulation and build approximation schemes based on it.
- The function ϕ is called a test function.

- There is one more piece of standard applied math notation that comes from gas dynamics and that is the notion of **Conservation Form**

$$A_t + AA_x = 0 \text{ usual inviscid Burgers eqn}$$

$$A_t + \left(\frac{1}{2}A^2\right)_x = 0 \text{ Conservation Form}$$

$$Q = \frac{1}{2}A^2 \text{ Definition of Q}$$

$$\frac{\partial}{\partial t}\int_{x_L}^{x_R} A(x,t), dx = -\left[Q(x_R) - Q(x_L)\right]$$

Matrix

- In 3D **Conservation Form** is an equation of the form:

$$\frac{\partial \vec{m}}{\partial t} + \nabla \cdot \mathbf{Q}(\vec{m}) = 0$$

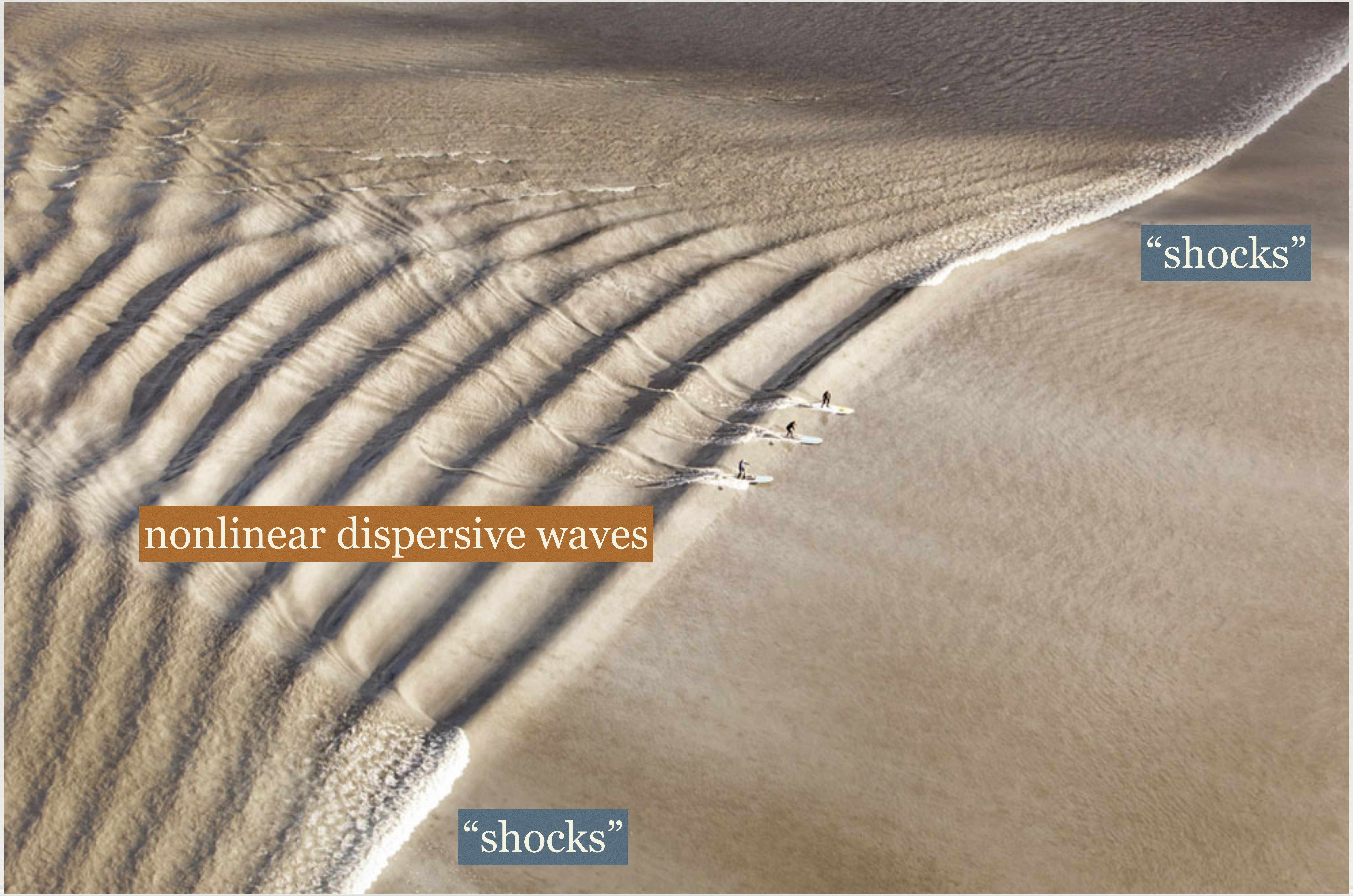
- Integrating would now involve using Gauss' theorem on the divergence to get the flux of m through the boundary.

Some Conclusions

- The inviscid limit is intriguing mathematically, and the fact that a little bit of viscosity gets you useful information about the inviscid limit has been extensively borrowed in many numerical methods.

- We will see how spectral methods do and don't handle shocks in the next slide deck.

- But as a physics inspired bit of foreshadowing consider that the Burgers equation as written explicitly ignored dispersive effects. To consider both dispersion, nonlinearity and diffusion the so-called regularixed longwave equation might be better: $A_t = -cA_x + \alpha A A_x + \beta A_{xxt} + \gamma A_{xx}$


“shocks”
nonlinear dispersive waves
“shocks”