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Computational Physics
Exercise Sheet 01

Code and plots can be found in the zip folder

a)

Newton $f = m\ddot{r}$

Taylor expansion to get next timestep

$$r(t+\Delta t) = r(t) + \dot{r}(t)\Delta t + \frac{1}{2}\ddot{r}(t)\Delta t^2 \quad (1)$$
$$r(t-\Delta t) = r(t) - \dot{r}(t)\Delta t + \frac{1}{2}\ddot{r}(t)\Delta t^2 \quad (2)$$

Adding them

$$r(t+\Delta t) + r(t-\Delta t) = 2r(t) + \ddot{r}(t)\Delta t^2 \leftarrow \text{verlet algorithm}$$

We can show that velocity verlet is the same

$$r(t+\Delta t) = r(t) + \dot{r}(t)\Delta t + \frac{1}{2}\ddot{r}(t)\Delta t^2$$
$$\dot{r}(t+\Delta t) = \dot{r}(t) + \frac{1}{2}\left[\ddot{r}(t) + \ddot{r}(t+\Delta t)\right]\Delta t$$

→ Second order taylor expansion

$$r(t+2\Delta t) = r(t+\Delta t) + \dot{r}(t+\Delta t)\Delta t + \frac{1}{2}\ddot{r}(t+\Delta t)\Delta t^2$$

adding (1)

$$r(t+2\Delta t) + r(t) = 2r(t+\Delta t) + \left[\dot{r}(t+\Delta t) - \dot{r}(t)\right]\Delta t + \frac{1}{2}\left[\ddot{r}(t+\Delta t) - \ddot{r}(t)\right]\Delta t^2$$

use $\dot{r}(t+\Delta t) - \dot{r}(t) = \frac{1}{2}\left[\ddot{r}(t+\Delta t + \ddot{r}(t)\right]\Delta t$

and (2)

$$\Rightarrow r(t+2\Delta t) + r(t) = 2r(t+\Delta t) + \ddot{r}(t+\Delta t)\Delta t^2$$

↳ verlet algorithm

Starting conditions vel.-verlet: r_0 and v_0

Verlet: just r_0 and r_1 no velocity necessary.

b) How can I calculate the positions $\vec{r}$?

I have $\vec{r}_0$ and $\dot{\vec{r}}_0 = \vec{v}_0$.

$$V_{ij}(r_{ij}) = -G \frac{M_i M_j}{r_{ij}}$$

Velocity-Verlet algorithm:

$$\vec{r}_i(t+\Delta t) = \vec{r}_i(t) + \vec{v}_i(t)\Delta t + \frac{\Delta t^2}{2m_i} \cdot \vec{F}_i(t) + \mathcal{O}(\Delta t^3)$$

$$\vec{v}_i(t+\Delta t) = \vec{v}_i(t) + \frac{\Delta t}{2m_i}\left[\vec{F}_i(t) + \vec{F}_i(t+\Delta t)\right] + \mathcal{O}(\Delta t^3)$$

So I need to calculate the force for each t and $t+\Delta t$.

$$\vec{F}(\vec{r}) = -\nabla V(\vec{r}) = -G \frac{M_i M_j}{|r_{ij}|^3} \vec{r}_{ij}$$

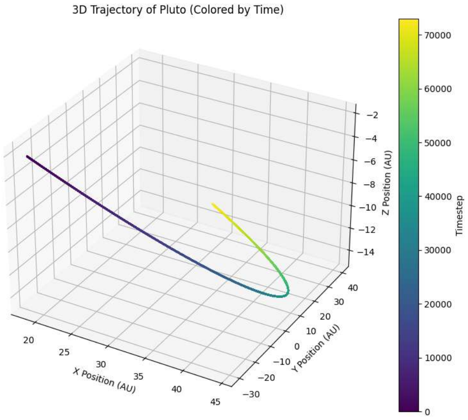
$$\vec{F}_i(\vec{r}(t)) = \sum_{j=1}^{n} -\nabla V_{ij} \quad i \neq j$$

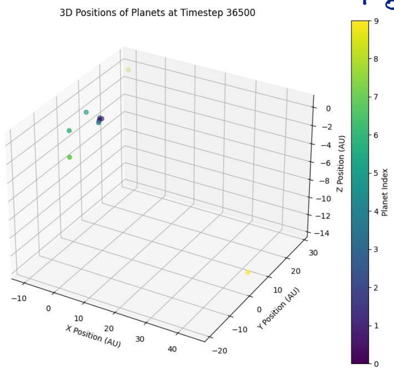
Unit check

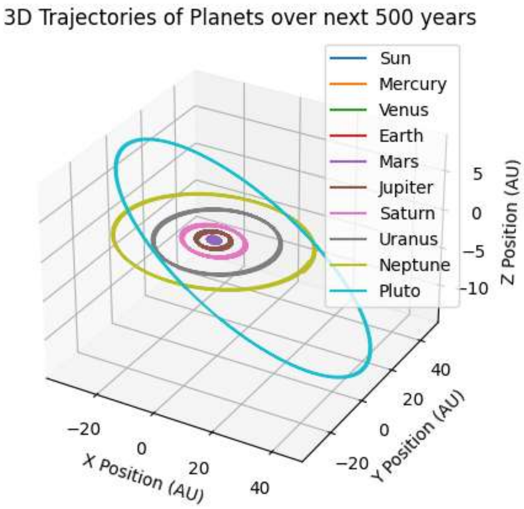
$[v] = \frac{AU}{day}$ $\quad [r] = AU$ $\quad [G] = \frac{m^3}{kg\,s^2}$ $\quad [m] = 10^{29}\,kg$ $\quad 1N = kg\frac{m}{s^2}$

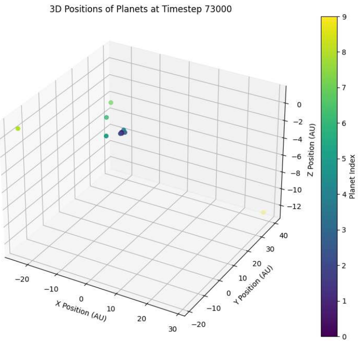
convert $G \Rightarrow \frac{AU^3}{10^{29}kg\ day^2} = \frac{\left(\frac{1}{1{,}5}10^{11}\,AU\right)^3}{\frac{1}{10^{29}}kg\left(\frac{1}{86400}day\right)^2} \cdot 6{,}67\ 10^{-11}$

looks better in Ovito
→ and use radius = 0,1 so they dont overlap.









Plutos orbital period is 256,5 years.

c) Verlet algorithm

Again we have mass, positions t=0 and the potential.

$$f_i(t) = \sum_{i\neq j, j=1}^{n} -\nabla V_{ij}$$

⇒ new_pos : $\vec{r}_i(t+\Delta t) = 2\vec{r}_i(t) - \vec{r}_i(t-\Delta t) + \frac{\Delta t}{m_i}\vec{f}_i(t) + \mathcal{O}(\Delta t^4)$

vel : $\vec{v}_i(t) = \frac{\vec{r}_i(t+\Delta t) - \vec{r}_i(t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^3)$

Start with:

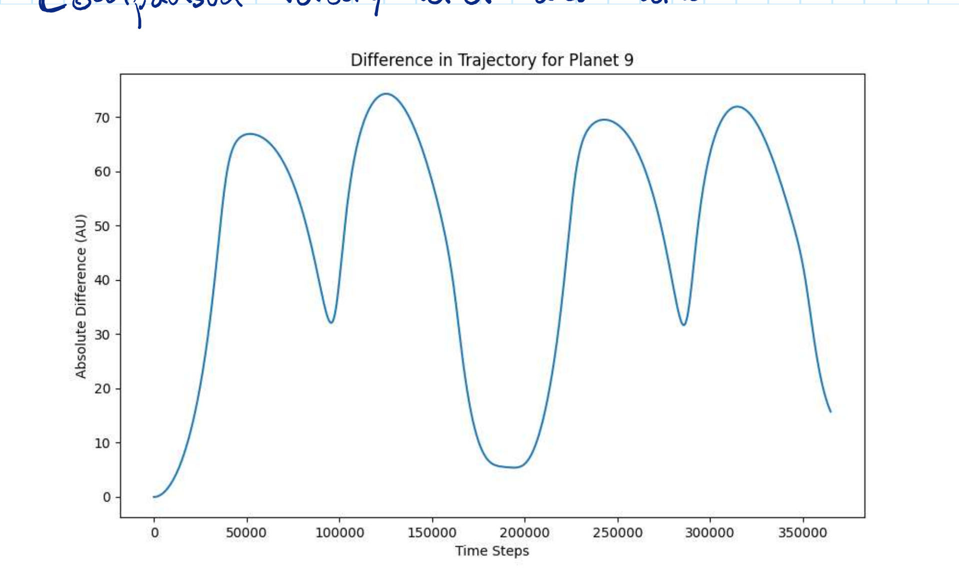
Where do I get the second set of positions?

$$\vec{r}(2) = 2\vec{r}(1) - \vec{r}(0) + \frac{\Delta t}{m_i}\vec{f}(0)$$

We took the calculated pos from the velocity verlet.

$$v(1) = \frac{\vec{r}(2) - \vec{r}(0)}{2\Delta t}$$

Comparison velocity-verlet and Verlet



For each timestep calculated the $|\vec{r}_{vv}(t) - \vec{r}_v(t)|$

The estimated orbital period of Mars is 39,20 years.

d)

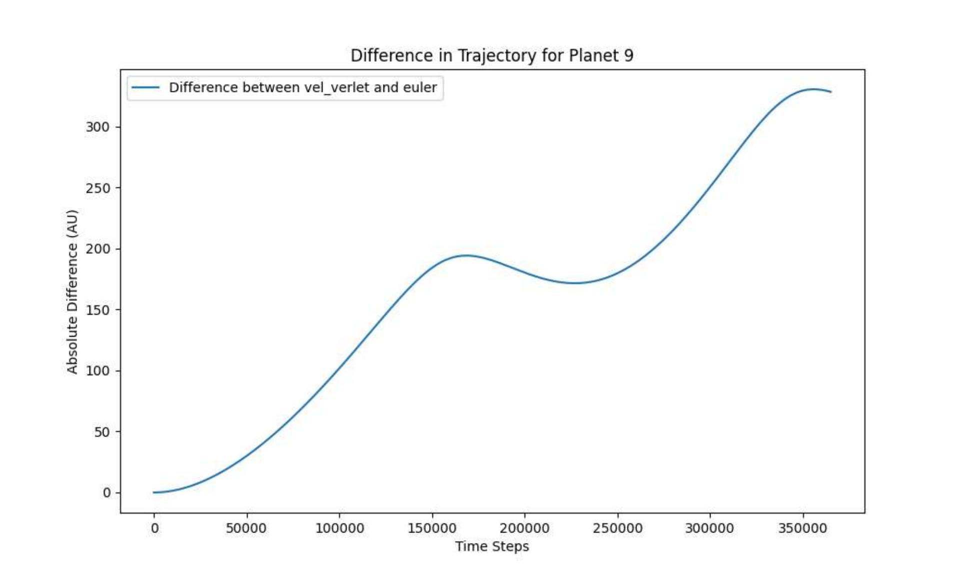
Euler

$$\vec{r}_i(t+\Delta t) = \vec{r}_i(t) + \Delta t \cdot \vec{v}_i(t) + \frac{\Delta t^2}{2m_i} \vec{f}_i(t) + \mathcal{O}(\Delta t^3)$$

← only difference to Velocity Verlet.

$$\vec{v}_i(t+\Delta t) = \vec{v}_i(t) + \frac{\Delta t}{m_i} \vec{f}_i(t) + \mathcal{O}(\Delta t^2)$$

Comparison Euler - Velocity-Verlet

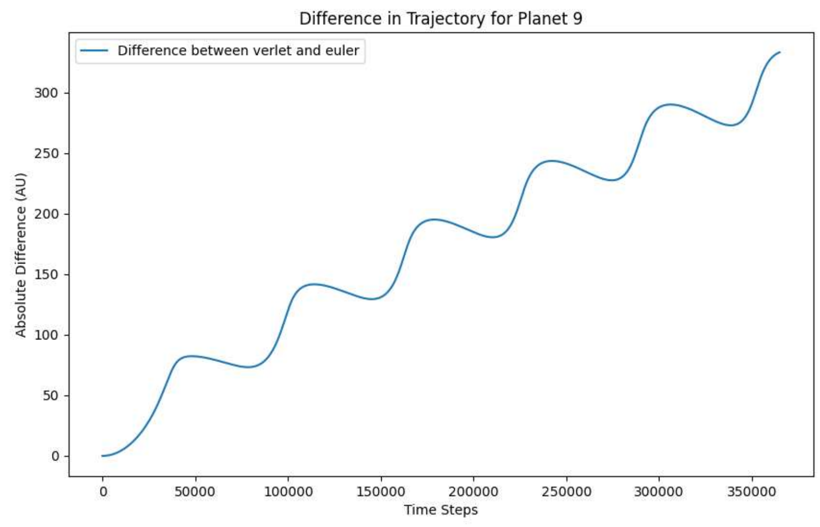


For each timestep calculated the $|\vec{r}_{vv}(t) - \vec{r}_v(t)|$

Accordingly same difference to Verlet. Just like in c)

$$E = \sum_i \frac{1}{2} m_i |\vec{v}_i|^2 + \sum_i \sum_{j, i \neq j} V_{ij}$$

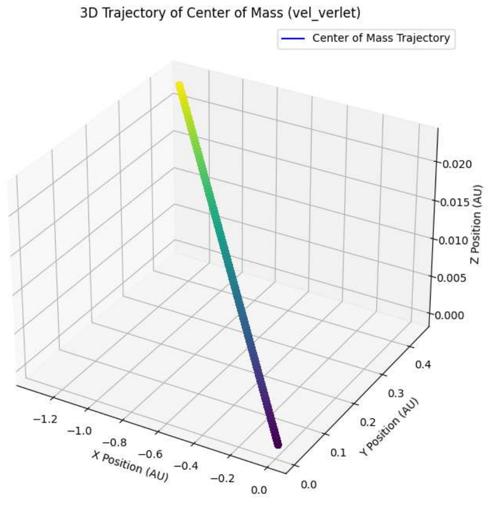
$i \in$ Planets with Sun



e) Center of mass

$$\vec{r}_{com} = \frac{\sum r_i m_i}{\sum m_i}$$

The center of mass is just a straight line. That might cause from the initial dataset. The initial data doesn't come from COM system. Pictures of all simulations can be found in the zip. For all simulations the drift of COM is the same.



f) Not realistic XD

Wikipedia Mercury eccentricity is 0,2056

Simulation eccentricity for each simulation $\varepsilon > 0,96$

But just velocity verlet seems the most realistic because every planet stays in the solar system. In the other simulations planets move on a hyperbolic trajectory.

→ See pictures in zip.