Part III Essays Easter 2020

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Essay Title:

Sutured Floer Homology

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1 Introduction

This essay is meant to serve either as a "guide" or "companion" to Peter Kronheimer and Tomasz Mrowka's paper [1], specifically sections three, four, and parts of six. Their paper explores two algebraic invariants of closed, oriented manifolds. These invariants are homology theories in which the chain groups are generated not by simplices or cells, but by the solutions of certain systems of PDEs defined on the manifold. One of these systems, named after Nathan Seiberg and Edward Witten, results in what is called monopole Floer homology. The other, named after Chen Ning Yang and Robert Mills, results in instanton Floer homology.

We will primarily be interested in monopole Floer homology. Rather than defining it from scratch, which is quite difficult, we will take some basic properties of this theory as axioms, and see what they allow us to prove. Over the course of the essay, we will find that monopole Floer homology behaves nicely with respect to a certain surgery procedure, that there is a further invariant of so-called "sutured manifolds" (to be understood as three-manifolds with a top and a bottom), and that this new invariant also behaves nicely with respect to deleting properly embedded surfaces. The new invariant is called sutured Floer homology.

With only a little more effort than what is presented in the essay, one can show that the sutured Floer homology groups of a knot complement reveal whether the knot has a certain important property—being "fibered." Furthermore, at the end of the essay, we will lean on the work of David Gabai in [2] and that of András Juhász in [3] to prove a nonvanishing theorem for sutured Floer homology.

Since, as we have said, this essay runs parallel to Kronheimer and Mrowka's paper [1], any result or idea which is not explicitly cited can be assumed to come from there. Furthermore, we have tried to use their notation wherever possible to make cross-referencing easier. One important aspect in which this essay differs from [1] is that a standard course in algebraic topology will suffice to understand the vast majority of it. In the next section, we will bridge the gap between what is assumed of the reader and what is required for understanding the rest of the essay.

2 Background Material

Just about every result in this essay involves the cutting apart or gluing together of manifolds. We introduce the machinery for this and some vocabulary pertaining to it in the next subsection. We also discuss the idea of independent curves on a surface before offering some easy ways to recognize them. Finally, we define the main objects of study, sutured manifolds.

Following these topological ideas, we will axiomatize monopole Floer homology. Like the familiar singular, cellular, and simplicial homology theories, monopole Floer homology is a functor. Its domain category, however, is not that of smooth manifolds, but of cobordisms. We will therefore introduce cobordisms, and explain how to view them as morphisms between closed manifolds. Finally, we will discuss relative versions of monopole homology and versions with different coefficients, listing additional axioms in both cases.

2.1 Curves, Surfaces, and Sutured Manifolds

Def. 1 Let M be a manifold and S a codimension-one submanifold satisfying $S \pitchfork \partial M = \partial S$, with both oriented. In this case, recall that a tubular neighborhood of S is diffeomorphic to $S \times I$, where $I = [-1, 1]$. To delete, excise, or cut along S means to take the complement of a tubular neighborhood of S in M. Two codimension-one submanifolds are said to be parallel if the one is a boundary component of a tubular neighborhood of the other. The orientations of the two boundary components of a tubular neighborhood V are said to be parallel if one agrees with the boundary orientation of V and the other does not.

Def. 2 Let S be a closed, oriented surface, and let c_i be a pairwise disjoint collection of oriented simple closed curves in S. We will say that the c_i are homologically independent over a field $\mathbb{F}$ if the singular homology classes $[c_i] \in H_1(S; \mathbb{F})$ are linearly independent. We will also say that a system of curves $\{\eta_i\}_{i=1}^n$ is dual to the $\{c_i\}_{i=1}^n$ if $([\eta_i], [c_j]) = \delta_{ij}$ for all i and j. Here $(-, -)$ denotes the cup product/intersection pairing. Finally, we will say that the c_i are nonseparating if deleting them does not change the number of components of S.

Prop. 3 Let S be a closed, oriented surface, and let c_i be a pairwise disjoint collection of oriented simple closed curves in S. Then the following are equivalent:

1. The c_i are nonseparating.
2. The c_i admit a system of dual curves η_i.
3. The c_i are homologically independent over $\mathbb{F}$.

Proof. $1 \Rightarrow 2$. Let V_i be a tubular neighborhood of c_i such that the V_i are pairwise disjoint. Let $\bar{\eta}_i \subseteq V_i$ be a fiber of the normal bundle of c_i, that is, a tiny segment in S crossing through c_i. Since $S' = S \setminus \bigcup_i V_i$ has the same number of components as S, there is a path $\tilde{\eta}_i$ in S' connecting the endpoints of $\bar{\eta}_i$. Without loss of generality, we can assume $\tilde{\eta}_i$ has no self intersections. Then together, $\eta_i = \bar{\eta}_i \cup \tilde{\eta}_i$ is a curve in S which intersects c_i exactly once, and misses c_j for all $j \neq i$. This is because every c_i is disjoint from S'. Hence η_i is a system of curves dual to c_i.

$2 \Rightarrow 3$. Let η_i be a dual system of curves to c_i, and let $\sum_i x_i[c_i]$ be an $\mathbb{F}$-linear combination of the $[c_i]$. If $\sum_i x_i[c_i] = 0$, then we can see that the $[c_i]$ are linearly independent by taking the cup product with $[\eta_j]$, as follows:

$$x_j = \textstyle\sum_i x_i \delta_{ij} = \sum_i x_i([\eta_j], [c_i]) = ([\eta_j], \sum_i x_i[c_i]) = ([\eta_j], 0) = 0.$$

$\neg 1 \Rightarrow \neg 3$. Assume that cutting the c_i out of S increases the number of components of S. Then the boundary components of the resulting surface S' consist of pairs of curves $+c_i$ and $-c_i$ for all i. For at least one of these pairs, $+c_i$ and $-c_i$ must belong to different components of S', or else we could glue each $+c_i$ to $-c_i$ and get back our original surface S without changing the number of components. So, let Σ be one of these components which only contains one of $+c_i$ and $-c_i$. Triangulating Σ gives a 2-cycle in S whose boundary is the signed sum of the boundary components of Σ. Since Σ only contains one of $+c_i$ and $-c_i$, this sum does not vanish, and indeed gives a nontrivial—but also nullhomologous—linear combination of the $[c_i]$. ■

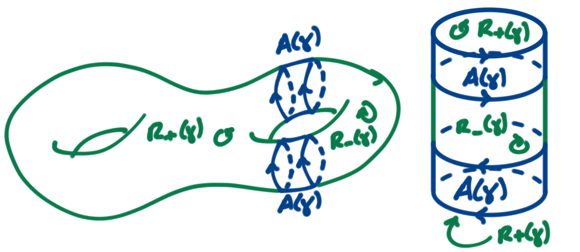


Figure 1: Separately unbalanced sutured manifolds whose union is balanced.

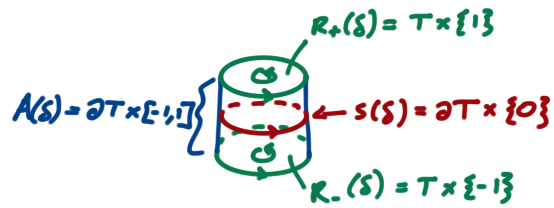


Figure 2: A product sutured manifold $D^2 \times I$.

Def. 4 A sutured manifold, for our purposes, is a compact, oriented 3-manifold M together with a closed, oriented 1-manifold $s(\gamma)$ in ∂M and an orientation of the surface $R(\gamma)$ obtained by cutting along $s(\gamma)$. The boundary orientation on $\partial R(\gamma)$ must be parallel to that of $s(\gamma)$.

This definition is accompanied by some terminology and notation. The components of $s(\gamma)$ are called sutures, and we write $A(\gamma)$ for a tubular neighborhood of the sutures in ∂M. The components of $A(\gamma)$ are called the annuli of the sutured manifold, and they are the complement of $R(\gamma)$ in ∂M. The components of $R(\gamma)$ whose orientations agree with the boundary orientation on ∂M are denoted $R_+(\gamma)$, and the remaining ones $R_-(\gamma)$. Intuitively, one can think of $R_+(\gamma)$ as the top of M, and $R_-(\gamma)$ as the bottom. Some of the later terminology will reflect this. The sutured manifold itself is written (M, γ).

A *balanced* sutured manifold satisfies two additional properties. First, neither M nor $R(\gamma)$ may have any closed components (this implies (M, γ) is completely determined by the sutures). Second, we require that $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Ex. 5 A few examples of sutured manifolds can be found in Figure 1. A large class of further examples are so-called product sutured manifolds: given a compact, oriented surface T with no closed components, there is a canonical way to make $T \times I$ into a balanced sutured manifold. The sutures $s(\delta)$ are $\partial T \times \{0\}$, the annuli $A(\delta)$ are $\partial T \times I$, the top $R_+(\delta)$ is $T \times \{+1\}$, and the bottom $R_-(\delta)$ is $T \times \{-1\}$. The case of $T = D^2$ is shown in Figure 2.

2.2 Monopole Floer Homology

Our goal for the remainder of the introduction is to axiomatize monopole Floer homology and some of its variants. Before we do this, however, we must define what a cobordism is.

Def. 6 A cobordism $X^n \to Y^n$ from one closed, oriented manifold to another is a compact, oriented manifold W^{n+1} whose boundary is the disjoint union $\partial W = Y \sqcup -X$. The category Cob_n has smooth n-manifolds as its objects and cobordisms as its morphisms. The identity cobordism on a manifold X is the product $X \times I$, and composition is defined by gluing $W_1 : X \to Y$ and $W_2 : Y \to Z$ along their boundaries using the orientation-reversing diffeomorphism $\mathrm{id} : Y \to -Y$.

Rmk. If two manifolds X and Y are diffeomorphic (i.e. isomorphic in the smooth category), then they are also isomorphic in the cobordism category: given a diffeomorphism $f : X \to Y$, the mapping cylinder C_f is a cobordism $X \to Y$. It is easy to see that the compositions $-C_f \circ C_f$ and $C_f \circ -C_f$ are the products $X \times I$ and $Y \times I$ respectively.

Rmk. For a fixed manifold M^n, there is a functor $F_M : \mathsf{Cob}_n \to \mathsf{Cob}_n$ which sends an object X to $X \sqcup M$ and a morphism $W : X \to Y$ to $W \sqcup (M \times I) : X \sqcup M \to Y \sqcup M$. There is also an adjunction α_M between F_M and F_{-M} since $\mathrm{Hom}(F_M(X), Y)$ and $\mathrm{Hom}(X, F_{-M}(Y))$ are naturally bijective—they're the same set.

Def. 7 Monopole Floer homology (or just monopole homology—sometimes we omit the "Floer") is a functor $HM_* : \mathsf{Cob}_3 \to \mathsf{Mod}_R$. For the sake of simplicity, we fix $R = \mathbb{F}$ to be a field of characteristic zero throughout the entire essay, although everything can be tailored to $\mathbb{Z}$ coefficients if necessary. We often refer to the homology vector spaces simply as groups. Our axioms for this homology theory are:

1. $HM_*(X \sqcup Y) \cong HM_*(X) \otimes_{\mathbb{F}} HM_*(Y)$, and similarly for morphisms.
2. $HM_*(-X) \cong HM_*(X)^\dagger$, and similarly for morphisms. Here † denotes the dual space.
3. The diagram below commutes, writing X_* for $HM_*(X)$. Replacing $\mathrm{Hom}_{\mathbb{F}}(X_*, Y_* \otimes_{\mathbb{F}} M_*^\dagger)$ with the naturally isomorphic $\mathrm{Hom}_{\mathbb{F}}(X_*, \mathrm{Hom}_{\mathbb{F}}(M_*, Y_*))$, it says that HM_* carries the adjunction in the previous remark to the tensor-hom adjunction β_M.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Cob}_3}(X \sqcup M, Y) \arrow[r, "\alpha_M"] \arrow[d, "HM_*"] & \mathrm{Hom}_{\mathsf{Cob}_3}(X, Y \sqcup -M) \arrow[d, "HM_*"] \\
\mathrm{Hom}_{\mathbb{F}}(X_* \otimes_{\mathbb{F}} M_*, Y_*) \arrow[r, "\beta_M"] & \mathrm{Hom}_{\mathbb{F}}(X_*, Y_* \otimes_{\mathbb{F}} M_*^\dagger)
\end{tikzcd}$$

The definition above captures all the main ideas going forward. For completeness, however, we must wade through a few more details. Specifically, we will build up to definitions of local and relative versions of monopole homology.

Def. 8 Let Cob_n^* be a category in which the objects of Cob_n are augmented with 1-cycles and the morphisms with 2-cycles. If $(W, \nu) : (X, \eta_1) \to (Y, \eta_2)$ is a morphism, we require $\partial \nu = \eta_2 - \eta_1$. In a similar vein, let $\mathsf{Cob}_n^\dagger$ be a category in which both the objects and morphisms of Cob_n are augmented with closed, oriented subsurfaces. If $(W, F) : (X, R) \to (Y, S)$ is a morphism, then we require that $S \sqcup -R \subseteq F$, that each component of X (resp. Y) contain a component of R (resp. S), and that each component of both of R and S have genus at least two.

Def. 9 Throughout this essay, fix a commutative ring $\mathcal{R}$, which has an $\mathbb{F}$-vector space structure as well as several other properties that will not concern us. Given $\mathcal{R}$, local monopole Floer homology is a functor $\mathsf{Cob}_3^* \to \mathsf{Bimod}_{(\mathcal{R},\mathbb{F})}$ which is notated $HM_*(X; \Gamma_\eta)$ and which satisfies axioms analogous to the ones above. One additional axiom relates the local and nonlocal versions of monopole homology:

4. If $\eta \subseteq X$, then there is a simultaneous isomorphism of $\mathbb{F}$-vector spaces and $\mathcal{R}$-modules:

$$HM_*(X \sqcup Y; \Gamma_\eta) \cong HM_*(X; \Gamma_\eta) \otimes_{\mathbb{F}} HM_*(Y).$$

Def. 10 Relative monopole Floer homology is a functor $\mathsf{Cob}_3^\dagger \to \mathsf{Vect}_{\mathbb{F}}$ which is notated $HM_*(X|\Sigma)$ and which satisfies axioms analogous to those already listed. There is a local version of this theory as well, notated $HM_*(X|\Sigma;\Gamma_\eta)$. We will mostly use these relative versions of monopole homology, and they come with a few more axioms used for specific calculations.

5. For any surface Σ of genus at least two, there are $\mathbb{F}$-vector space isomorphisms
$$HM_*(\Sigma \times S^1 \mid \Sigma \times \{*\}) \cong \mathbb{F} \qquad \text{and} \qquad HM_*(\Sigma \times S^1 \mid \Sigma \times \{*\}; \Gamma_\eta) \cong \mathcal{R},$$
where the second equation is also an isomorphism of $\mathcal{R}$-modules, and holds even when Σ merely has genus one. Here, η is any 1-cycle in $\Sigma \times S^1$.
6. For any surface Σ of genus at least two, consider the cobordism $W = \Sigma \times (D^2 \sqcup -D^2)$ augmented with subsurfaces $\Sigma_1 = \Sigma \times \{*\} \subseteq \Sigma \times D^2$ and $\Sigma_2 = \Sigma \times \{*\} \subseteq \Sigma \times -D^2$. Then $(W, \Sigma_1 \cup \Sigma_2)$ induces the identity on $HM_*(\Sigma \times S^1 \mid \Sigma \times \{*\})$.

Def. 11 For us, a spin^c structure $\mathfrak{s}$ on a manifold X is an object with a certain associated cohomology class $c_1(\mathfrak{s})$. There are versions of both ordinary and local monopole homology augmented with spin^c structures, and those defined above are direct sums over these augmented ones:
$$HM_*(X) = \bigoplus\nolimits_{\mathfrak{s}} HM_*(X,\mathfrak{s}) \qquad\qquad HM_*(X;\Gamma_\eta) = \bigoplus\nolimits_{\mathfrak{s}} HM_*(X,\mathfrak{s};\Gamma_\eta)$$
The relative homology groups $HM_*(X|\Sigma)$ and $HM_*(X|\Sigma;\Gamma_\eta)$ are obtained by restricting the direct sums above to those spin^c structures which satisfy $\langle c_1(\mathfrak{s}), [\Sigma_i]\rangle = -\chi(\Sigma_i)$ for every component Σ_i of Σ.

Prop. 12 Let X be a closed, oriented 3-fold with a genus at least one surface F and a spin^c structure $\mathfrak{s}$. If $\langle c_1(\mathfrak{s}), [F]\rangle > -\chi(F)$, then $HM_*(Y,\mathfrak{s}) = 0$. If there are two additional genus at least one surfaces R and S in X such that $[F] = [R] + [S]$ and $\chi(F) = \chi(R) + \chi(S)$, then $HM_*(Y|F)$ is a direct summand of $HM_*(Y|R)$.

Proof. The first statement of this proposition is called the adjunction inequality, and we will not prove it. For the second statement, let $\mathcal{S}(X|F)$ denote the spin^c structures on X satisfying $\langle c_1(\mathfrak{s}), [F]\rangle$. Certainly, if $\mathfrak{s} \in \mathcal{S}(X|R) \cap \mathcal{S}(X|S)$, then $\mathfrak{s} \in \mathcal{S}(X|F)$ since
$$\langle c_1(\mathfrak{s}), [F]\rangle = \langle c_1(\mathfrak{s}), [R]\rangle + \langle c_1(\mathfrak{s}), [S]\rangle = -\chi(R) - \chi(S) = -\chi(F).$$
Now suppose instead that $\mathfrak{s} \in \mathcal{S}(X|F)$. If on the one hand $\langle c_1(\mathfrak{s}), [R]\rangle > -\chi(R)$, then $HM_*(X,\mathfrak{s}) = 0$ by the adjunction inequality. If on the other hand $\langle c_1(\mathfrak{s}), [R]\rangle < -\chi(R)$, then $HM_*(X,\mathfrak{s}) = 0$ by the adjunction inequality again, because now
$$\begin{aligned}\langle c_1(\mathfrak{s}), [S]\rangle &= \langle c_1(\mathfrak{s}), [F]\rangle - \langle c_1(\mathfrak{s}), [R]\rangle = -\chi(F) - \langle c_1(\mathfrak{s}), [R]\rangle \\ &= -\chi(S) - \chi(R) - \langle c_1(\mathfrak{s}), [R]\rangle > -\chi(S).\end{aligned}$$
The above equation also shows that if $\langle c_1(\mathfrak{s}), [R]\rangle = -\chi(R)$, then $\langle c_1(\mathfrak{s}), [S]\rangle = -\chi(S)$, too. We conclude that an element $\mathfrak{s} \in \mathcal{S}(X|F)$ not in $\mathcal{S}(X|R) \cap \mathcal{S}(X|S)$ has $HM_*(X,\mathfrak{s}) = 0$. Therefore $HM_*(X|F)$ is given by summing $HM_*(X,\mathfrak{s})$ over $\mathcal{S}(X|R) \cap \mathcal{S}(X|S)$, which proves the proposition. ∎

3 Geometric Constructions

In this section, we describe the Floer excision procedure, the closure of a sutured manifold, and the notion of a decomposing surface. The first uses cutting and gluing techniques to turn one closed manifold into another; the second is a closed manifold which results from gluing a product sutured manifold to an arbitrary one; the third can be deleted from a given sutured manifold to produce a new one. These geometric constructions form the basis of the essay, so we collect them all in one section.

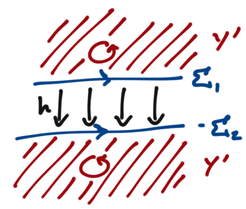
3.1 The Floer Excision Procedure

For the Floer excision procedure, we will need a closed, oriented, three-dimensional manifold Y containing closed, oriented, and connected surfaces Σ_1 and Σ_2 of the same genus. Such surfaces are diffeomorphic, and in fact we will need to fix an orientation-preserving diffeomorphism $h : \Sigma_1 \to \Sigma_2$. The Floer excision procedure can then be used in one of the following two cases. Most applications we will see make use of the disconnected case.

Connected Case	**Disconnected Case**
Y has one component	Y has two components, Y_1 and Y_2
$[\Sigma_1]$ and $[\Sigma_2]$ are independent	$\Sigma_1 \subseteq Y_1$ and $\Sigma_2 \subseteq Y_2$ are nonseparating

For some applications, we will also need a 1-cycle η that passes transversely through each of Σ_1 and Σ_2 exactly once, as well as another surface F that intersects Σ_1 and Σ_2 transversely. If we have this additional data, we will need h to send the point $\eta \cap \Sigma_1$ to the point $\eta \cap \Sigma_2$ and to restrict to an orientation-preserving diffeomorphism $F \cap \Sigma_1 \to F \cap \Sigma_2$.

Given all this data, the Floer excision procedure consists of two steps. First, excise Σ_1 and Σ_2 from Y to obtain a manifold-with-boundary Y'. One of the resulting copies of Σ_1 in $\partial Y'$ will have orientation compatible with that of Y', and one will not. Write Σ_1 for the one that does, and $-\Sigma_1$ for the one that doesn't. Do the same for Σ_2. Second, use h to glue $\Sigma_1 \to -\Sigma_2$ and $-\Sigma_1 \to \Sigma_2$. This is to ensure the orientation of Y' passes to an orientation of the resulting closed, connected surface $\tilde{Y}$ (see inset). We write $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ for the glued-together surfaces $\Sigma_1 \cong -\Sigma_2$ and $\Sigma_2 \cong -\Sigma_1$ in $\tilde{Y}$. If we are given the additional data η and F, the procedure just described applies cutting and gluing to form a new 1-cycle $\tilde{\eta}$ and a new surface $\tilde{F}$ in $\tilde{Y}$.



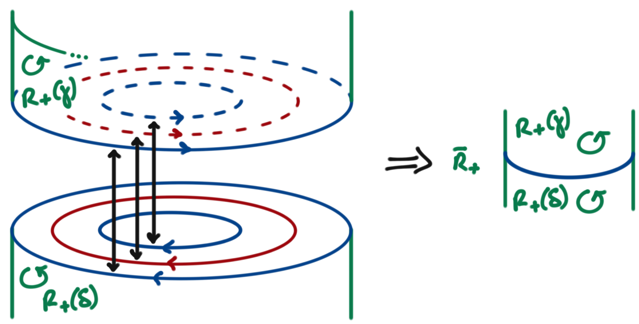
Ex. 1 Suppose Σ is a closed, oriented, and connected surface, and suppose also that we have two diffeomorphisms $f, g : \Sigma \to \Sigma$. Applying the Floer excision procedure to the mapping tori T_f and T_g using the identity to glue two cross-sections together produces another mapping torus, namely, T_{gf}.

3.2 Closures of a Sutured Manifold

Let (M, γ) be a balanced sutured manifold, and let T be a compact, oriented surface (we do not assume T is connected) with as many disks removed as there are sutures $s(\gamma)$ and with no closed components. The product balanced sutured manifold $(T \times I, \delta)$ and (M, γ)

then have the same number of sutures. The goal, as stated earlier, is to produce a closed, connected, and oriented manifold. We will begin to achieve this by gluing $(T \times I, \delta)$ to (M, γ) along their annuli.

Recall that the annuli of a sutured manifold contain three distinguished circles: a suture, which orients the annulus, and two boundary circles, one of which touches $R_+(\gamma)$ and the other of which touches $R_-(\gamma)$. To glue (M, γ) to $(T \times I, \delta)$, we first pair up the annuli in $A(\gamma)$ with those in $A(\delta)$ using a bijection j between the path components of these sets. Then we glue the pairs together such that the distinguished circles of one annulus are sent to the corresponding circles in the pair, but with orientations reversed. The inset shows that the resulting manifold Z inherits a consistent orientation from (M, γ) and $(T \times I, \delta)$ and has two boundary components $\bar{R}_\pm = R_\pm(\gamma) \cup T \times \{\pm 1\}$ which also inherit consistent orientations from $R_\pm(\gamma)$ and $R_\pm(\delta)$. Note that $\bar{R}_+$ has the boundary orientation of the resulting manifold and that $\bar{R}_-$ has negative the boundary orientation. At this point, we impose some conditions on Z, which may restrict the choices of surface T as well as the possible pairings of $A(\gamma)$ with $A(\delta)$:



1. First, we require $\bar{R}_+$ and $\bar{R}_-$ to be connected (because (M, γ) and $(T \times I, \delta)$ are balanced this implies Z is also connected).
2. Second, we require $\bar{R}_+$ and $\bar{R}_-$ to have genus at least two.
3. Third, we require each component T_i of T to have an oriented simple closed curve c_i such that $c_i \times \{\pm 1\}$ represent independent homology classes in $\bar{R}_\pm$ respectively.

In light of the first of these conditions together with the calculation

$$\begin{aligned}\chi(\bar{R}_+) &= \chi(R_+(\gamma) \cup T \times \{+1\}) = \chi(R_+(\gamma)) + \chi(T \times \{+1\}) \\ &= \chi(R_-(\gamma)) + \chi(T \times \{-1\}) = \chi(R_-(\gamma) \cup T \times \{-1\}) = \chi(\bar{R}_-)\end{aligned}$$

we may conclude $\bar{R}_+$ and $\bar{R}_-$ are closed, connected, oriented surfaces of the same genus, and hence diffeomorphic. To complete the process of closing up (M, γ), choose a diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ which sends each $c_i \times \{+1\}$ to $c_i \times \{-1\}$ and which preserves orientations of both the surfaces $\bar{R}_\pm$ and curves $c_i \times \{\pm 1\}$. Using h, glue the two boundary components of Z together. The resulting closed, connected, oriented manifold Y contains a closed, connected, oriented, nonseparating surface $\bar{R}$ of genus at least two arising from the identification of $\bar{R}_+$ and $\bar{R}_-$. In turn, $\bar{R}$ contains oriented simple closed curves $\bar{c}_i$ arising from the identification of $c_i \times \{+1\}$ and $c_i \times \{-1\}$.

We say that Y is a closure of (M, γ), and note that it depends on the choice of T, the choice of pairing $j : \pi_0(A(\gamma)) \to \pi_0(A(\delta))$, and the choice of diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$. We will devote an entire section to proving that these choices do not affect the group $HM_*(Y|\bar{R})$.

Ex. 2 Any closure of a product sutured manifold $(S \times I, \gamma)$ is a mapping torus. This is because the gluing of $(T \times I, \delta)$ to $(S \times I, \gamma)$ produces a product three-fold $Z = G \times I$ whose two boundary components $G \times \{+1\}$ and $G \times \{-1\}$ are glued together with a diffeomorphism $h : G \to G$, yielding the mapping torus of h.

3.3 Decomposing Surfaces

Unlike the rest of this essay, which draws mostly on [1], this subsection is based on [2].

Def. 3 An oriented surface S in a sutured manifold (M, γ) is said to be decomposing if $\partial S = \partial M \pitchfork S$ and every component of $S \cap A(\gamma)$ is homologous to either $* \times I \subseteq A$ or $S^1 \times * \subseteq A$ where $A \cong S^1 \times I$ is an element of $A(\gamma)$.

Cutting along a decomposing surface in a sutured manifold (M, γ) results in another sutured manifold (M', γ'). The idea behind this is as follows. Recall that cutting S out of M produces two parallel copies of S in $\partial M'$. We orient these copies parallel to S, write S_+ for the copy whose orientation agrees with the boundary orientation of M', and write S_- for the other. Now, because of the assumptions on S, each component c of ∂S is without loss of generality in one of three cases:

1. c lies entirely within $R_+(\gamma)$ or $R_-(\gamma)$. This is depicted in Figure 3 and corresponds to when $c \cap A(\gamma)$ is empty. Suppose c lies entirely within $R_+(\gamma)$. Since both S_+ and $R_+(\gamma)$ carry the boundary orientation of M', their orientations will be compatible along the component of their intersection parallel to c. Arguing similarly, the orientations of S_- and $R_+(\gamma)$ will not be compatible along the component d of their intersection parallel c. To account for this, we make d into a new suture $s(\gamma')$ and add a tubular neighborhood of d to $A(\gamma')$.
2. c meets a suture transversely an even number of times. This is depicted in Figure 5 and corresponds to when each component of $c \cap A(\gamma)$ is homologous to $I \times * \subseteq A$. The sutures $s(\gamma)$ divide c into an even number of arcs, which are alternatingly contained in $R_+(\gamma)$ and $R_-(\gamma)$. A parallel copy of each of these arcs is added to an existing suture in $s(\gamma)$, respecting orientations. This has the effect of smoothing out $s(\gamma) \cup c$ near their points of intersection. The annuli $A(\gamma')$ are just chosen to be tubular neighborhoods of the new sutures.
3. c *is* a suture. This is depicted in Figure 4 and corresponds to when $c \cap A(\gamma)$ is homologous to $* \times S^1 \subseteq A$. Cutting out c divides the annulus $A \in A(\gamma)$ into two smaller annuli both of which become elements of $A(\gamma')$. Within these smaller annuli we choose parallel copies of c to be the new sutures in $s(\gamma')$.

In addition to the alterations described above, we include in $s(\gamma')$ and $A(\gamma')$ any sutures/annuli in $s(\gamma)$ and $A(\gamma)$ which did not intersect ∂S. Finally, we set

$$R_+(\gamma') = (R_+(\gamma) \cup S_+) \backslash (V(S) \cup A(\gamma')) \quad \text{and} \quad R_-(\gamma') = (R_-(\gamma) \cup S_-) \backslash (V(S) \cup A(\gamma')).$$

Observe that the orientations on $R_+(\gamma)$ and S_+ are compatible because they are both the boundary orientation on M'. Similarly for $R_-(\gamma)$ and S_-. Hence $R_+(\gamma')$ and $R_-(\gamma')$ are

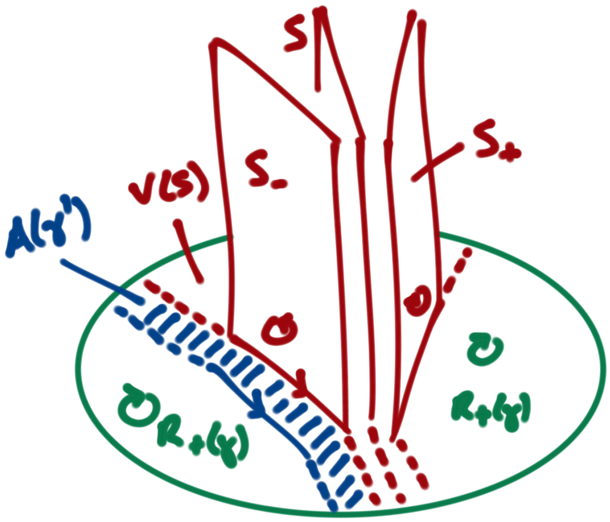


Figure 3: Cutting S out of (M, γ) when c lies in $R_+(\gamma)$.

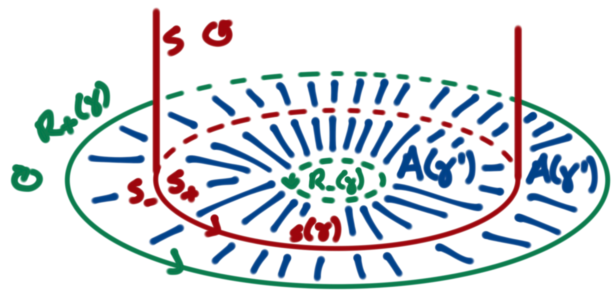


Figure 4: Cutting S out of (M, γ) when c is a suture.

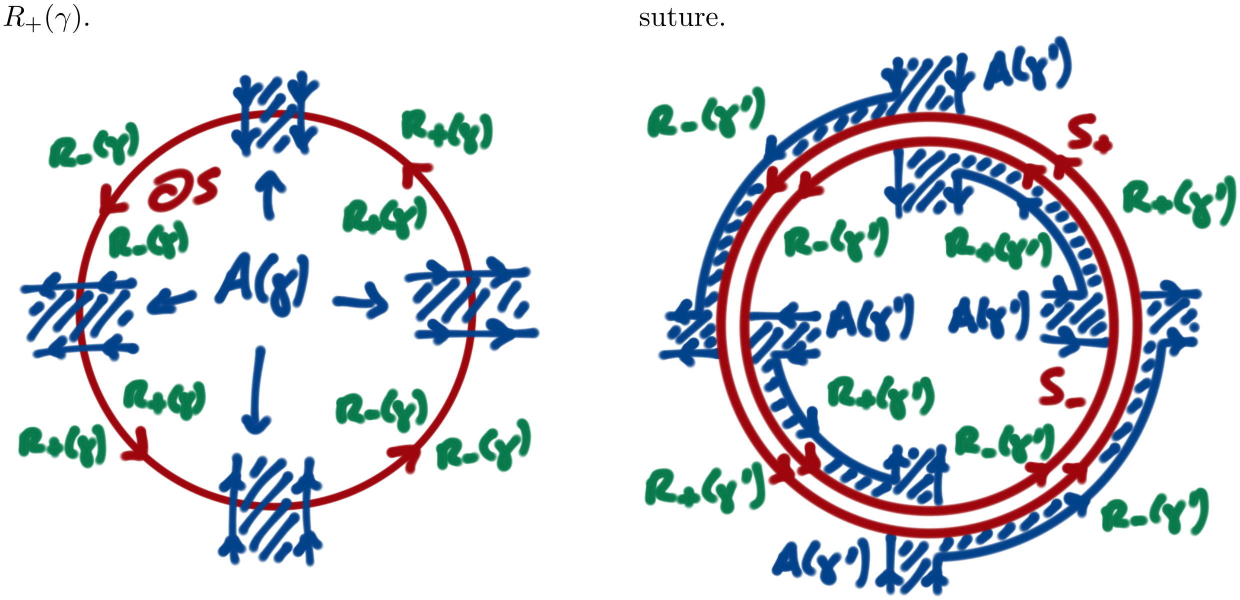


Figure 5: Cutting S out of (M, γ) when c crosses $s(\gamma)$.

oriented. The reason this produces a sutured manifold is essentially the preceding casework, which shows $R_+(\gamma')$ and $R_-(\gamma')$ are disjoint, and have orientations compatible with the sutures $s(\gamma')$. We finish this section with a few related definitions.

Def. 4 We will write $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ for when (M',γ') is obtained from (M,γ) by cutting along a decomposing surface S and both (M,γ) and (M',γ') are balanced sutured manifolds.

Def. 5 A good decomposing surface is one where every component of ∂S meets both $R_+(\gamma)$ and $R_-(\gamma)$. In other words, cases 1 and 3 above are excluded.

Def. 6 This is not standard terminology, but we will refer to a decomposing surface in which $S \cap A(\gamma) = \emptyset$ as a Juhász surface. In other words, cases 2 and 3 above are excluded. An annular Juhász surface whose boundary circles d_+ and d_- lie entirely in $R_+(\gamma)$ and $R_-(\gamma)$ respectively is called a product annulus.

Def. 7 A horizontal surface is a decomposing surface in a balanced sutured manifold which separates (M,γ) into a disjoint union of two smaller balanced sutured manifolds:

$$(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma') = (M_1,\gamma_1) \sqcup (M_2,\gamma_2).$$

We also require that each component of ∂S lie within some annulus in $A(\gamma)$ and be homologous to the corresponding suture. In other words, cases 1 and 2 above are excluded.

4 Excision Theorems

Our first objective is to prove that the Floer excision procedure has no effect on the level of monopole homology. Though the proof will be quite long, it will be even more rewarding, for the excision theorem forms the basis of most of the results we will discuss. In addition to stating some variants of the theorem at the end of the section, we will find a way to relate the ordinary and local versions of monopole homology.

Thm. 1 (Floer Excision) Using the notation of Section 3.1, assume $(\tilde{Y}, \tilde{\Sigma}_1, \tilde{\Sigma}_2)$ results from applying the Floer excision procedure to $(Y, \Sigma_1, \Sigma_2, h)$, where Σ_1 and Σ_2 both have genus at least two. Then $HM_*(Y|\Sigma) \cong HM_*(\tilde{Y}|\tilde{\Sigma})$, where $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$.

Proof. The idea of the proof is construct a cobordism $W : \tilde{Y} \to Y$ such that the composites $W \circ -W = W \cup_{\tilde{Y}} -W$ and $-W \circ W = W \cup_Y -W$, while not themselves the products $Y \times I$ and $\tilde{Y} \times I$, can be made into the identity cobordisms by a cutting and gluing operation that does not change the induced maps on relative homology. Functoriality of HM_* will then make the induced map an isomorphism.

To construct W, we excise Σ_1 and Σ_2 from Y to obtain Y'. The idea is then to glue $Y' \times I$ to $\Sigma_2 \times U$, where U is an octagon viewed as a manifold with corners. To explain how the gluing works, and how it produces the desired cobordism, we must first understand the boundaries of $Y' \times I$ and $\Sigma_2 \times U$. The boundary of $\Sigma_2 \times U$ is of course a union of eight

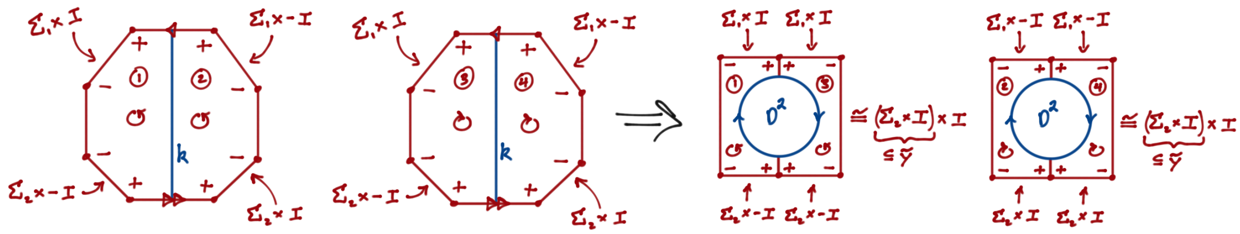


Figure 6: Turning the cobordism X (octagons) into the cobordism X^* (squares).

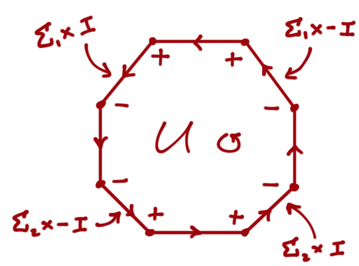
copies of $\Sigma_2 \times I$. The boundary of $Y' \times I$ is computed as follows.

$$\begin{aligned}
\partial(Y' \times I) &= \partial Y' \times I \cup Y' \times \partial I \\
&= (\Sigma_1 \times I) \cup (-\Sigma_1 \times I) \cup (\Sigma_2 \times I) \cup (-\Sigma_2 \times I) \cup (Y' \times \{+,-\}) \\
&= (\Sigma_1 \times I) \cup (\Sigma_1 \times -I) \cup (\Sigma_2 \times I) \cup (\Sigma_2 \times -I) \cup +Y' \cup -Y'
\end{aligned}$$

Let $g : I \to I$ be the orientation-reversing diffeomorphism given by $t \mapsto -t$. Then each of the four attaching maps

$$\begin{aligned}
&h \times g : \Sigma_1 \times I \to \Sigma_2 \times I && h \times \mathrm{id}_I : \Sigma_1 \times -I \to \Sigma_2 \times I \\
&\mathrm{id}_{\Sigma_2} \times g : \Sigma_2 \times I \to \Sigma_2 \times I && \mathrm{id}_{\Sigma_2} \times \mathrm{id}_I : \Sigma_2 \times -I \to \Sigma_2 \times I
\end{aligned}$$

reverses orientation, and can be used to glue $\partial Y' \times I$ to half the boundary of $\Sigma_2 \times U$, as depicted in the inset. After this is done, all that will remain of $\partial(Y' \times I)$ is $+Y'$ and $-Y'$, and these will be touching $\Sigma_2 \times U$ at the points indicated in the inset with a $+$ and $-$. This makes it easy to see that in the resulting manifold W, the Σ_1 and $-\Sigma_1$ components of $\partial(+Y')$ are connected by a product belonging to $U \times \Sigma_2$ which is diffeomorphic to $\Sigma_1 \times I$. Similarly, the Σ_2 and $-\Sigma_2$ components of $\partial(+Y')$ are connected by a product $\Sigma_2 \times I$ belonging to $U \times \Sigma_2$. The resulting manifold W therefore has a boundary component diffeomorphic to Y, since Y is what results when one glues Σ_1 to $-\Sigma_1$ and Σ_2 to $-\Sigma_2$ in Y'. One can similarly argue that the $-Y'$ boundary component of $\partial(Y' \times I)$ turns into $-\tilde{Y}$ upon being attached to $U \times \Sigma_2$. This proves W is a cobordism $\tilde{Y} \to Y$.



Next, we describe a surgery operation that turns the cobordism $X := W \cup_Y -W$ into $\tilde{Y} \times I$. This is best done by picture. We take k to be the circle indicated in Figure 6 (note that the two octagons are glued together as indicated by the arrows), and K to be the three-dimensional product submanifold $\Sigma_2 \times k$ of X. We then cut K out of X to obtain a cobordism $X' : \tilde{Y} \sqcup K \to \tilde{Y} \sqcup K$. Now either we can recover X by gluing $K \times I$ back into X' or we can obtain X^* by gluing two oppositely-oriented copies of $\Sigma_2 \times D^2$ (which have boundary $\partial(\Sigma_2 \times D^2) = \Sigma_2 \times S^1 \cong K$) into the K-shaped boundary components of X'. One can see that X^* is the product $\tilde{Y} \times I$ by referring to Figure 6.

Our next goal is to explain why X and X^* induce the same map on relative homology. To this end, let Z and Z^* be the cobordisms $K \to K$ given by $K \times I$ and $\Sigma_2 \times (D^2 \sqcup -D^2)$ respectively. This gives us access to three cobordisms $\tilde{Y} \sqcup K \to \tilde{Y} \sqcup K$, listed in the first column of the equations below. In the second column are cobordisms which we define to be those corresponding to the ones in the first column under the adjunction discussed before Definition 2.7. Recall that these are the same 4-manifolds, but viewed with different domains and codomains.

$$\begin{aligned} \mathrm{id}_{\tilde{Y}} \sqcup Z &: \tilde{Y} \sqcup K \to \tilde{Y} \sqcup K & \bar{Z} &: \tilde{Y} \to \tilde{Y} \sqcup K \sqcup -K \\ \mathrm{id}_{\tilde{Y}} \sqcup Z^* &: \tilde{Y} \sqcup K \to \tilde{Y} \sqcup K & \bar{Z}^* &: \tilde{Y} \to \tilde{Y} \sqcup K \sqcup -K \\ X' &: \tilde{Y} \sqcup K \to \tilde{Y} \sqcup K & \bar{X}' &: \tilde{Y} \sqcup K \sqcup -K \to \tilde{Y} \end{aligned}$$

The next step is to augment each of these cobordisms with subsurfaces, so as to induce maps on relative homology. For this purpose, single out a cross-section $\bar{\Sigma} = \Sigma_2 \times \{*\} \subseteq K$, and put $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \subseteq \tilde{Y}$. Now augment each occurrence of $\tilde{Y}$ in the cobordisms above with $\tilde{\Sigma}$ and each occurrence of $\pm K$ with $\pm\bar{\Sigma}$. Then, for example, $\bar{Z}^*$ becomes a morphism

$$(\bar{Z}^*, -\tilde{\Sigma} \sqcup \tilde{\Sigma} \sqcup \bar{\Sigma} \sqcup -\bar{\Sigma}) : (\tilde{Y}, \tilde{\Sigma}) \to (\tilde{Y} \sqcup K \sqcup -K, \tilde{\Sigma} \sqcup \bar{\Sigma} \sqcup -\bar{\Sigma}).$$

The rest of the proof will now go very quickly. The augmented versions of $\mathrm{id}_{\tilde{Y}} \sqcup Z$ and $\mathrm{id}_{\tilde{Y}} \sqcup Z^*$ induce the same maps on relative homology by Axiom 6. Since $\bar{Z}$ and $\bar{Z}^*$ are the corresponding cobordisms under the aforementioned adjunction, they too must induce the same maps on relative homology by Axiom 3. But then $X = \bar{X}' \circ \bar{Z}$ and $X^* = \bar{X}' \circ \bar{Z}^*$ also induce the same maps on relative homology.

Since $X^* = \tilde{Y} \times I$ is a product, it induces the identity on $HM_*(\tilde{Y}|\tilde{\Sigma})$. Hence $X = -W \circ W$ also induces the identity. A similar argument applies to $W \circ -W$, which means W induces an isomorphism. ■

The next two theorems are proved similarly, so we just state them. The local version of the last theorem basically allows the genus of Σ_1 and Σ_2 to be one at the cost of using local coefficients and requiring the "extra data" in the Floer excision procedure.

Thm. 2 (Local Floer Excision) Using the notation of Section 3.1 again, assume $(\tilde{Y}, \tilde{\Sigma}_1, \tilde{\Sigma}_2, \tilde{F}, \tilde{\eta})$ results from applying the Floer excision procedure to $(Y, \Sigma_1, \Sigma_2, F, \eta, h)$, where Σ_1 and Σ_2 now both have genus at least one. Then there is an isomorphism $HM_*(Y|F; \Gamma_\eta) \cong HM_*(\tilde{Y}|\tilde{F}; \Gamma_{\tilde{\eta}})$.

Thm. 3 Assume $(\tilde{Y}, \tilde{\Sigma}_1, \tilde{\Sigma}_2)$ results from applying the Floer excision procedure to $(Y, \Sigma_1, \Sigma_2, h)$, where Σ_1 and Σ_2 once again have genus at least two. If η is a 1-cycle in Y disjoint from Σ_1 and Σ_2, then it is unaffected by the procedure and passes unaltered to $\tilde{Y}$ where we call it $\tilde{\eta}$. In this case, $HM_*(Y|\Sigma; \Gamma_\eta) \cong HM_*(\tilde{Y}|\tilde{\Sigma}; \Gamma_{\tilde{\eta}})$ where $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$.

Cor. 4 If $\eta \subseteq \Sigma \subseteq Y$ where η is a 1-cycle, Σ is a closed, oriented surface with components of genus at least two, and Y is a closed, connected, and oriented three-fold, then we can relate the local and nonlocal homology groups simply by $HM_*(Y|\Sigma; \Gamma_\eta) \cong HM_*(Y|\Sigma) \otimes \mathcal{R}$, which is a simultaneous isomorphism of $\mathbb{F}$-vector spaces and $\mathcal{R}$-modules.

Proof. Let $(Y_1, \Sigma_1) = (Y, \Sigma)$, let $(Y_2, \Sigma_2) = (\Sigma \times S^1, \Sigma \times \{*\})$, and let $\eta' = \eta \times \{*'\}$ be a parallel copy of $\eta \times \{*\}$. The key insight is that applying the Floer excision procedure to $(Y_1 \cup Y_2, \Sigma_1, \Sigma_2, x \mapsto (x, *))$ produces a tuple $(\tilde{Y}, \tilde{\Sigma}_1, \tilde{\Sigma}_2)$ such that there is a diffeomorphism

$$(Y, V(\Sigma)) \cong (\tilde{Y}, V(\tilde{\Sigma}_1) \cup V(\tilde{\Sigma}_2)).$$

This is because you obtain $\tilde{Y}$ by cutting out Σ from Y, and then essentially just gluing it back in; the whole procedure leaves the complement of a tubular neighborhood $V(\Sigma)$ untouched. The diffeomorphism can be made to send η to $\tilde{\eta}'$. Since diffeomorphic manifolds are isomorphically cobordant (see the remark after Definition 2.6), this affords the first of the following equalities.

$$\begin{aligned} HM_*(Y|\Sigma;\Gamma_\eta) &\cong HM_*(\tilde{Y}|\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2;\Gamma_{\tilde{\eta}'}) \cong HM_*(Y_1 \cup Y_2 \mid \Sigma_1 \cup \Sigma_2;\Gamma_{\eta'}) \\ &\cong HM_*(Y_1|\Sigma_1) \otimes HM_*(Y_2|\Sigma_2;\Gamma_{\eta'}) \cong HM_*(Y|\Sigma) \otimes \mathcal{R}. \end{aligned}$$

The remaining equalities are afforded by the parent theorem and our axiomatization of monopole Floer homology in Definition 2.7. ■

5 Sutured Monopole Floer Homology

Recall when forming the closure of a sutured manifold (M, γ) that there were, in a sense, three degrees of freedom: the auxiliary surface T, the pairing j of the sutures of $T \times I$ with those of M, and the diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ identifying the boundary components of the manifold obtained by gluing M to $T \times I$.

In this section, we will show in successive stages that the monopole Floer homology groups of the closures of (M, γ) are independent of h, j, and T. But then these groups only depend on (M, γ) itself, from which we will conclude that they are actually invariants of balanced sutured manifolds. Our name for these new invariants carries the section title.

Def. 1 We define the sutured monopole Floer homology groups of a sutured manifold (M, γ) to be the relative monopole Floer homology groups $SHM(M, \gamma, T, j, h) = HM_*(Y|\bar{R})$, where Y, $\bar{R}$, T, j, and h are as in Section 3.2. Upon proving that these groups are independent of T, j, and h, we shall simply write $SHM(M, \gamma)$. Note that we will often refer to them as the nonlocal groups when there is a need to distinguish them from the local groups, defined next.

Def. 2 Recall from constructing closures of (M, γ) that $\bar{R}$ contains homologically independent curves $\bar{c}_i$. By Proposition 2.3, there is a dual system of curves η_i in $\bar{R}$. For any such dual system, let $\eta = \sum_i \eta_i$ be a 1-cycle and define $SHM(M, \gamma, T, j, h; \Gamma_\eta) = HM(Y|\bar{R}; \Gamma_\eta)$ to be the local sutured monopole Floer homology groups. As with the nonlocal groups, we shall soon be able simply to write $SHM(M, \gamma; \Gamma_\eta)$.

Our first proposition cuts the amount of work we will need to do in half. Having proved that the local (nonlocal) sutured monopole Floer homology groups are independent of T, j, or h, it will let us transfer the result to the nonlocal (local) ones.

Prop. 3 The local and nonlocal sutured monopole Floer homology groups are related by the following isomorphism of $\mathcal{R}$-modules: $SHM(M, \gamma, T, j, h; \Gamma_\eta) \cong SHM(M, \gamma, T, j, h) \otimes_{\mathbb{F}} \mathcal{R}$. Moreover, if one of $SHM(M, \gamma, T, j, h)$ and $SHM(M, \gamma, T, j, h; \Gamma_\eta)$ is independent of T, j, or h, then so is the other.

Proof. The isomorphism is immediate from Corollary 4.4. Now suppose the nonlocal groups $SHM(M, \gamma, T, j, h)$ are independent of one of T, j, or h. Then the right hand side of the isomorphism is independent of it. This implies the left is too, which means the local groups are also independent. The other direction is a bit harder. If for example you know that

$$\begin{aligned} SHM(M,\gamma,T_1,j,h)\otimes\mathcal{R} &\cong SHM(M,\gamma,T_1,j,h;\Gamma_\eta)\cong SHM(M,\gamma,T_2,j,h;\Gamma_\eta) \\ &\cong SHM(M,\gamma,T_2,j,h)\otimes\mathcal{R} \end{aligned}$$

for any appropriate choices of T_1 and T_2, then you need a way to cancel the tensor product with $\mathcal{R}$ from the left and right sides of the above equation. We won't go into detail because it is not crucial to understanding the main ideas of the essay, but for this purpose one could choose $\mathcal{R}=\mathbb{F}[\mathbb{R}]$. ■

First, we will show that the sutured monopole Floer homology groups are independent of the diffeomorphism h used to glue $\bar{R}_+$ to $\bar{R}_-$. To this end, we begin with the following proposition.

Prop. 4 The monopole Floer homology of a mapping torus is the base field: if $h:R\to R$ is an orientation-preserving diffeomorphism of a genus at least two surface, then $HM_*(T_h|R)\cong\mathbb{F}$.

Proof. In Example 3.1, we saw that applying the Floer excision procedure to T_h and $T_{h^{-1}}$ yields $T_{h^{-1}h}\cong T_{\mathrm{id}}\cong R\times S^1$. Note also that there is an orientation-reversing diffeomorphism $T_h\to T_{h^{-1}}$ (and hence an orientation-preserving one $-T_h\to T_{h^{-1}}$). By our axiomatization of monopole Floer homology and the Floer excision theorem, we have

$$\begin{aligned} \mathbb{F} &\cong HM_*(R\times S^1|R)\cong HM_*(T_{h^{-1}h}|R)\cong HM_*(T_h\sqcup T_{h^{-1}}|R\sqcup R) \\ &\cong HM_*(T_h|R)\otimes_{\mathbb{F}} HM_*(T_{h^{-1}}|R)\cong HM_*(T_h|R)\otimes_{\mathbb{F}} HM_*(-T_h|R) \\ &\cong HM_*(T_h|R)\otimes_{\mathbb{F}} HM_*(T_h|R)^\dagger \end{aligned}$$

Comparing dimensions of the left and right hand sides, we must have $HM_*(T_h|R)\cong\mathbb{F}$. ■

Cor. 5 The sutured monopole Floer homology groups $SHM(M,\gamma,T,j,h)$ are independent of h.

Proof. Recall that when forming the closure of a sutured manifold (M,γ), there arises a manifold Z whose boundary components $\bar{R}_+$ and $\bar{R}_-$ are glued together with some diffeomorphism. Suppose Y_1 and $\bar{R}_1$ result from gluing with h_1 and suppose Y_2 and $\bar{R}_2$ result from gluing with h_2. If $\bar{R}_2'$ denotes a parallel copy of $\bar{R}_2$ in Y_2 and if we view h_2 as a diffeomorphism $\bar{R}_1\to\bar{R}_1$, then observe that $(Y_2,\bar{R}_2,\bar{R}_2')$ results from applying the Floer excision procedure to $(Y_1\sqcup T_{h_2},\bar{R}_1,\bar{R}_1,\mathrm{id})$. By the Floer excision theorem, our axiomatization of monopole Floer homology, and the parent proposition, we have

$$\begin{aligned} HM_*(Y_2|\bar{R}_2) &\cong HM_*(Y_1\sqcup T_{h_2}\mid \bar{R}_1\sqcup\bar{R}_1)\cong HM_*(Y_1|\bar{R}_1)\otimes_{\mathbb{F}} HM_*(T_{h_2}|\bar{R}_1) \\ &\cong HM_*(Y_1|\bar{R}_1)\otimes_{\mathbb{F}}\mathbb{F}\cong HM_*(Y_1|\bar{R}_1). \end{aligned}$$ ■

Ex. 6 We are ready to do our first (and only) calculation. If $(T\times I,\delta)$ is a product sutured manifold, then $SHM(T\times I,\delta,T',j,h)\cong\mathbb{F}$ and $SHM(T\times I,\delta,T',j,h;\Gamma_\eta)\cong\mathcal{R}$ for any appropriate choices of T', j, and h. The nonlocal calculation is immediate from Example 3.2 and Proposition 4. The local calculation follows from this using Proposition 3.

Prop. 7 The local sutured monopole Floer homology groups $SHM(M,\gamma,T,j,h;\Gamma_\eta)$ are independent of the number of components of T. Hence they are independent of j.

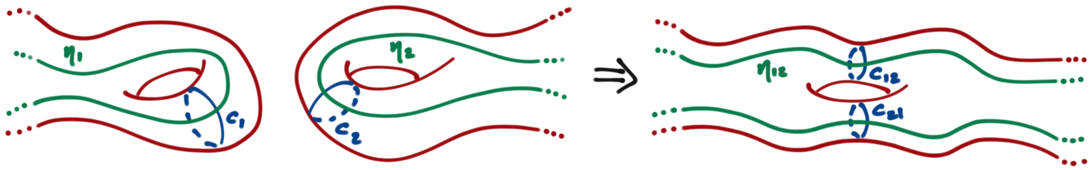


Figure 7: Joining components of the auxiliary surface. This is similar to Figure 5 in [1].

Proof. Recall that each component T_i of T has a curve c_i such that h sends $c_i \times \{+1\}$ to $c_i \times \{-1\}$, preserving orientation. Therefore the cylinders $c_i \times I$ in Z become tori $c_i \times S^1$ in Y. Because we also require that the curves $c_i \times \{\pm 1\}$ be homologically independent in $\bar{R}_\pm$, the tori $c_i \times S^1$ are homologically independent in Y. Let Σ_1 and Σ_2 be two such tori, say $c_1 \times S^1$ and $c_2 \times S^1$. Observe that $\bar{R}$ intersects Σ_1 and Σ_2 transversely, namely in the curves $\bar{c}_1$ and $\bar{c}_2$, and that the system of curves η_i from Definition 2 intersects Σ_1 and Σ_2 exactly once, say at the points $p_1 \in \bar{c}_1$ and $p_2 \in \bar{c}_2$. Finally, let $f : \bar{c}_1 \to \bar{c}_2$ be an orientation-preserving diffeomorphism sending p_1 to p_2, which we extend to an orientation-preserving diffeomorphism $\bar{f} = f \times \mathrm{id}_{S^1} : \Sigma_1 \to \Sigma_2$.

At this point we can apply the Floer excision procedure to $(Y, \Sigma_1, \Sigma_2, \bar{R}, \eta, \bar{f})$ and obtain a data set $(\tilde{Y}, \tilde{\Sigma}_1, \tilde{\Sigma}_2, \tilde{R}, \tilde{\eta})$ such that $HM_*(Y|\bar{R}; \Gamma_\eta) = HM_*(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}})$. The key insight is that we can arrive at $\tilde{Y}$ and $\tilde{R}$ by essentially reversing the order in which we close up (M, γ) and apply the Floer excision procedure. Indeed, suppose that before attaching $(T \times I, \delta)$ to (M, γ), we cut $c_1 \times I$ and $c_2 \times I$ out of $T_1 \times I$ and $T_2 \times I$ and then glue the resulting cylinders together as we would in the Floer excision procedure. This is depicted in Figure 7, and essentially just joins the $T_1 \times I$ and $T_2 \times I$ components of $T \times I$ to make a balanced product sutured manifold $\tilde{T} \times I$ with one less component. The Figure also shows how the curves c_1 and c_2 become a new pair of curves c_{12} and c_{21}.

Let us examine why using $\tilde{T} \times I$ instead of $T \times I$ still results in a valid closure of (M, γ). First, this cutting and gluing leaves the original sutures of T_1 and T_2 completely untouched, so we can attach them to (M, γ) as we did originally. Second, $c_{12} \times \{+1\}$ and $c_{21} \times \{-1\}$ can be joined by a path in $\tilde{R}_+$. This gives a way of connecting any two points in $\tilde{R}_+$ based on whether the path connecting the corresponding points in $\bar{R}_+$ crosses c_1 or c_2. Hence $\tilde{R}_+$ (and similarly $\tilde{R}_-$) is connected. Third, it is easy to check that T and $\tilde{T}$ have the same Euler characteristic, which means that the genera of $\tilde{R}_+$ and $\tilde{R}_-$ will be the same as those of $\bar{R}_+$ and $\bar{R}_-$. In particular, they will be equal and at least two. Fourth, we claim that each component $\tilde{T}_i$ of $\tilde{T}$ contains a curve $\tilde{c}_i$ such that the $\tilde{c}_i \times \{\pm 1\}$ are homologically independent in $\tilde{R}_\pm$. For the component that used to be $T_1 \cup T_2$, we choose the curve c_{12}, though c_{21} would do equally well. Perhaps the easiest way to see that the $\tilde{c}_i$ are independent is to observe that they have a system of dual curves $\tilde{\eta}_i$, which can be obtained from η_i simply by replacing η_1 and η_2 with the curve indicated in Figure 7.

When we close up (M, γ) with $\tilde{T} \times I$ as described above, we get diffeomorphic copies of the $\tilde{Y}$, $\tilde{R}$, and $\tilde{\eta}$ obtained from applying the Floer excision procedure to $(Y, \Sigma_1, \Sigma_2, \bar{R}, \eta, \bar{f})$. The intuition for this is as follows. Instead of just cutting out Σ_1 and Σ_2 from Y, also cut out $\bar{R}$. Instead of just gluing Σ_1 to $-\Sigma_2$ and $-\Sigma_1$ to Σ_2, also glue $\bar{R}$ to $-\bar{R}$ with the identity. Because $\bar{f}$ restricts to a diffeomorphism $\Sigma_1 \cap \bar{R} \to \Sigma_2 \cap \bar{R}$, this additional surgery does not affect the resulting manifold, and we can do the glueing in whatever order we want. Observe that if we glue $\bar{R}$ to $-\bar{R}$ first, then we get the usual Floer excision procedure applied to the closure of (M, γ) with $T \times I$ as the auxiliary surface. However, if we glue $\bar{R}$ to $-\bar{R}$ last, then we instead get the closure of (M, γ) with $\tilde{T} \times I$ as the auxiliary surface.

Since $\tilde{T}$ has one less path component than does T, by repeating the above argument until only one component is left, we can always assume that the auxiliary surface is connected. But if the auxiliary surface T is connected, then the sutures of $T \times I$ cannot be distinguished up to diffeomorphism. The order in which they are attached to (M, γ) is therefore irrelevant. This proves that the groups $SHM(M, \gamma, T, j, h; \Gamma_\eta)$ are independent of j. ■

Prop. 8 The local sutured monopole Floer homology groups $SHM(M, \gamma, T, j, h; \Gamma_\eta)$ are independent of T.

Proof. Thanks to the previous proposition, we can assume that the auxiliary surface T is connected. Since T is by hypothesis oriented and compact, it only remains to show that the homology groups in question are independent of the genus of T. The proof of this very closely resembles the proof of the last proposition, so we only give a brief sketch.

Since T has only one component, we only require that it contain a single curve c such that c passes to a nonseparating curve $\bar{c}$ in $\bar{R}$ with dual curve η_1. We will write Σ_1 for the torus $c \times S^1$ in the closure Y_1 of (M, γ) using T as the auxiliary surface. Next, let S be a genus two surface, let d be a nonseparating curve on S, and let d' be dual to d. Take $Y_2 = S \times S^1$, take $\Sigma_2 = d \times S^1$, take $F = S \times \{*\}$, and take $\eta_2 = d' \times \{*\}$. Applying the Floer excision procedure to $(Y_1 \sqcup Y_2, \Sigma_1, \Sigma_2, \bar{R} \sqcup F, \eta_1 + \eta_2)$ produces a closure $\tilde{Y}$ of (M, γ) using an auxiliary surface $\tilde{T}$ of genus one greater than that of T. By the local Floer excision theorem and Example 6, we have

$$\begin{aligned} HM_*(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}) &= HM_*(Y_1 \sqcup Y_2 \mid \bar{R} \sqcup F; \Gamma_{\eta_1+\eta_2}) \\ &= HM_*(Y_1|\bar{R}; \Gamma_{\eta_1}) \otimes HM_*(S|F; \Gamma_{\eta_2}) \\ &= HM_*(Y_1|\bar{R}; \Gamma_{\eta_1}) \otimes_{\mathbb{F}} \mathcal{R} \cong HM_*(Y_1|\bar{R}; \Gamma_{\eta_1}). \end{aligned}$$ ■

We have now shown that the nonlocal sutured monopole Floer homology groups are independent of h, and that the local ones are independent of T and j. By Proposition 3, both groups are independent of all three of T, j, and h. Therefore we have at last earned the right to write $SHM(M, \gamma)$ and $SHM(M, \gamma; \Gamma_\eta)$.

6 Decomposition Theorems

In this section, we will investigate what happens on the level of sutured monopole Floer homology when we cut various kinds of nicely embedded surfaces out of a sutured manifold. In pursuit of this goal, we start by considering the homology groups of a disconnected sutured manifold.

Prop. 1 If (M_1, γ_1) and (M_2, γ_2) are balanced sutured manifolds, then we have an isomorphism $SHM(M_1 \sqcup M_2, \gamma_1 \sqcup \gamma_2) \cong SHM(M_1, \gamma_1) \otimes_{\mathbb{F}} SHM(M_2, \gamma_2)$.

Proof. Once again the proof mimics that of Proposition 5.7, so we just give a sketch. Say we use auxiliary surfaces T_1 and T_2 to form closures Y_1 and Y_2 of (M_1, γ_1) and (M_2, γ_2).

Choose curves c_1 and c_2 in T_1 and T_2 that pass to nonseparating curves $\bar{c}_1$ and $\bar{c}_2$ in $\bar{R}_1$ and $\bar{R}_2$ with dual curves η_1 and η_2. Let Σ_1 and Σ_2 be the tori $c_1 \times S^1$ and $c_2 \times S^1$ in Y_1 and Y_2. By applying the Floer excision procedure to this data, one obtains a closure $\tilde{Y}$ of $(M_1 \sqcup M_2, \gamma_1 \sqcup \gamma_2)$ using an auxiliary surface $\tilde{T}$ similar to the one in Figure 7. By the local Floer excision theorem and our axiomatization of monopole Floer homology,

$$\begin{aligned} SHM(M_1 \sqcup M_2, \gamma_1 \sqcup \gamma_2; \Gamma_{\tilde{\eta}}) &= HM_*(\tilde{Y}|\tilde{R}; \Gamma_{\tilde{\eta}}) \cong HM_*(Y_1 \sqcup Y_2 \mid \bar{R}_1 \sqcup \bar{R}_2; \Gamma_{\eta_1+\eta_2}) \\ &\cong HM_*(Y_1|\bar{R}_1; \Gamma_{\eta_1}) \otimes_{\mathbb{F}} HM_*(Y_2|\bar{R}_2; \Gamma_{\eta_2}) \\ &= SHM(M_1, \gamma_1; \Gamma_{\eta_1}) \otimes_{\mathbb{F}} SHM(M_2, \gamma_2; \Gamma_{\eta_2}). \end{aligned}$$

Using the techniques in the proof of Proposition 5.3, we obtain the desired isomorphism. ■

Prop. 2 If $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ where S is a horizontal surface, then $SHM(M, \gamma) \cong SHM(M', \gamma')$.

Proof. Recall that excising a horizontal surface separates a balanced sutured manifold into two smaller ones. In our case, $(M', \gamma') = (M_1, \gamma_1) \sqcup (M_2, \gamma_2)$. Furthermore, the equations for $R_\pm(\gamma')$ in Section 3.3 become $R_+(\gamma') = R_+(\gamma) \sqcup S_-$ and $R_-(\gamma') = R_-(\gamma) \sqcup S_+$. Since $R_\pm(\gamma)$ and $S_\mp$ still belong to the same path components of M' after S is excised, we must have

$$R_+(\gamma_1) = R_+(\gamma) \qquad R_-(\gamma_1) = S_- \qquad R_+(\gamma_2) = S_+ \qquad R_-(\gamma_2) = R_-(\gamma).$$

Now consider the construction of a closure of (M', γ'). By the proof of Proposition 1, we can close up (M_1, γ_1) and (M_2, γ_2) separately and still get the correct sutured monopole Floer homology group for (M', γ'). Since (M_1, γ_1) and (M_2, γ_2) have the same number of sutures, we can choose to use the same auxiliary surface T for both. In other words, we attach $(T \times I, \delta)$ to both (M_1, γ_1) and (M_2, γ_2), forming the four surfaces

$$\begin{aligned} R_1^+ &= R_+(\gamma) \cup T \times \{+1\} \qquad & R_2^+ &= S_+ \cup T \times \{+1\} \\ R_1^- &= S_- \cup T \times \{-1\} \qquad & R_2^- &= R_-(\gamma) \cup T \times \{-1\}. \end{aligned}$$

To finish constructing the closures Y_1 and Y_2 of (M_1, γ_1) and (M_2, γ_2), we glue R_1^+ to R_1^- and R_2^+ to R_2^-, thereby forming $\bar{R}_1$ and $\bar{R}_2$. The trick is now to apply the Floer excision procedure to $(Y_1 \sqcup Y_2, \bar{R}_1, \bar{R}_2, h)$ where h, viewed as a map $-R_1^- \to R_2^+$, acts as the natural diffeomorphism $S_- \to S_+$ and as the identity $T \times \{-1\} \to T \times \{+1\}$.

Recall that the Floer excision procedure consists of two steps. First, we cut $\bar{R}_1$ and $\bar{R}_2$ out of $Y_1 \sqcup Y_2$; then, we glue $\bar{R}_1$ to $-\bar{R}_2$ and $-\bar{R}_1$ to $\bar{R}_2$. In our case, the first step simply undoes the gluing of R_1^+ to R_1^- and of R_2^+ to R_2^-. As noted at the end of the last paragraph, the gluing of $-\bar{R}_1$ to $\bar{R}_2$ in the second step can be viewed as a gluing of R_1^- to R_2^+. It reattaches S_+ and S_-, thereby turning M_1 and M_2 back into M, and attaches the two copies of $T \times I$ front-to-end, thereby forming a single elongated copy of $T \times I$. Finally, the gluing of $\bar{R}_1$ to $-\bar{R}_2$ can be viewed as the final step in closing up (M, γ) using the elongated auxiliary surface $T \times I$. Denote this closure by $\tilde{Y}$. From the Floer excision theorem, we conclude

$$SHM(M', \gamma') \cong HM_*(Y_1 \sqcup Y_2 \mid \bar{R}_1 \sqcup \bar{R}_2) \cong HM_*(\tilde{Y}|\tilde{R}) \cong SHM(M, \gamma). \qquad ■$$

Prop. 3 Suppose $(M, \gamma) \overset{A}{\rightsquigarrow} (M', \gamma')$ where A is a product annulus separating (M, γ) into two disjoint balanced sutured manifolds (M_1, γ_1) and (M_2, γ_2). In addition, suppose either that (M_1, γ_1)

is a product sutured manifold where the component of M_1 whose boundary contains A is not $D^2 \times I$, or that the boundary circles of A do not both bound disks on the same side of A. In both cases, $SHM(M, \gamma) \cong SHM(M', \gamma')$.

Proof. First, assume that (M_1, γ_1) is a product sutured manifold $(T \times I, \delta)$, where $T_A \times I$ is the component whose boundary contains A. Assuming T_A is not a disk, we will prove the proposition by showing that (M, γ) and (M_2, γ_2) have a closure in common. In fact, we claim that any closure of (M, γ) is a closure of (M_2, γ_2). To see this, suppose we use an auxiliary surface S to produce a valid closure Y of (M, γ). Recalling Example 3.2, we can view $T \cup S$ as an auxiliary surface for (M_2, γ_2) which produces the same 3-manifold Y. It remains to check that this, too, is a valid closure; in other words, we must show that the three conditions in Section 3.2 are satisfied when using $T \cup S$ to close up (M_2, γ_2).

The same surface $\bar{R} \subseteq Y$ is obtained both when closing up (M, γ) and when closing up (M_2, γ_2), so it is connected and of genus at least two. If T_A meets S, then every component of $T \cup S$ contains a component of S. Therefore, since every component S_i of S contained a curve c_i which passed to a homologically independent family $\bar{c}_i$ in $\bar{R}$, this must be true of the components of $T \cup S$ as well. If T_A does not meet S, then it is a surface of genus at least one with a single disk removed. Hence any nonseparating curve c in T_A passes to a curve $\bar{c}$ in $\bar{R}$ which will be homologically independent from the $\bar{c}_i$. This completes the proof for the first half of the proposition, since by Example 5.6 and Proposition 1, we have

$$\begin{aligned} SHM(M, \gamma) &\cong SHM(M_2, \gamma_2) \cong SHM(M_2, \gamma_2) \otimes_{\mathbb{F}} \mathbb{F} \\ &\cong SHM(M_2, \gamma_2) \otimes_{\mathbb{F}} SHM(M_1, \gamma_1) \cong SHM(M', \gamma'). \end{aligned}$$

To start on the second half of the proposition, let $R_{+,1}$ and $R_{-,2}$ be horizontal surfaces in (M_1, γ_1) and (M_2, γ_2) parallel to $R_+(\gamma_1)$ and $R_-(\gamma_2)$. Since we assume that the boundary circles of A do not both bound disks on the same side of A, we can assume without loss of generality that the components of $R_{+,1}$ and $R_{-,2}$ touching A are not disks. If we cut A out of M and then $R_{+,1}$ and $R_{-,2}$ out of M_1 and M_2, we separate M into the following four pieces.

$$\begin{aligned} &M_1 \backslash V(R_+(\gamma_1)) \cong M_1 && V(R_+(\gamma_1)) \cong R_{+,1} \times I \\ &M_2 \backslash V(R_-(\gamma_2)) \cong M_2 && V(R_-(\gamma_2)) \cong R_{-,2} \times I \end{aligned}$$

We can glue these back together along parts of A as shown in Figure 8 to obtain the two sutured manifolds

$$\begin{aligned} M_1' &= \big(M_1 \backslash V(R_+(\gamma_1))\big) \cup_{A_1} \big(V(R_-(\gamma_2))\big) \cong M_1 \cup R_{-,2} \times I \\ M_2' &= \big(M_2 \backslash V(R_-(\gamma_2))\big) \cup_{A_2} \big(V(R_+(\gamma_1))\big) \cong M_2 \cup R_{+,1} \times I \end{aligned}$$

We claim that M_1' and M_2' are balanced. Recalling that M_1 and M_2 are balanced by hypothesis, the only condition to check which might not be obvious is of the sort

$$\begin{aligned} \chi(R_+(\gamma_1')) &= \chi(R_{+,1} \cup A_3 \cup R_{-,2}) = \chi(R_+(\gamma_1)) + 0 + \chi(R_-(\gamma_2)) \\ &= \chi(R_-(\gamma_1)) + \chi(R_-(\gamma_2)) = \chi(R_-(\gamma_1) \cup_{d_-} R_-(\gamma_2)) = \chi(R_-(\gamma_1')). \end{aligned}$$

By construction then, the surface S separating M_1' and M_2' in M is a horizontal surface. It consists of $R_{+,1}$, $R_{-,2}$, and A_3, as shown in Figure 8. Thus by Propositions 1 and 2,

$$SHM(M, \gamma) \cong SHM(M_1' \sqcup M_2', \gamma_1' \sqcup \gamma_2') \cong SHM(M_1', \gamma_1') \otimes_{\mathbb{F}} SHM(M_2', \gamma_2').$$

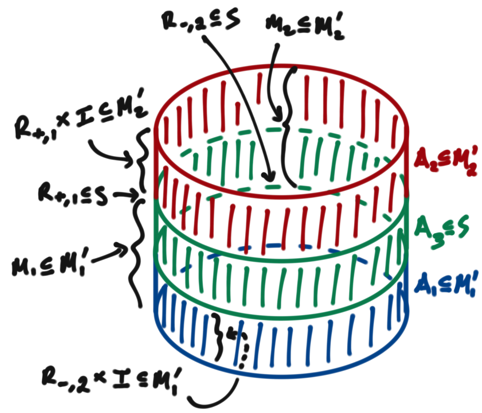


Figure 8: Turning a product annulus into a decomposing surface.

To complete the proof, observe that A_1 (resp. A_2) is a product annulus in M_1' (resp. M_2') satisfying the hypothesis of the first half of the proposition. Thus $SHM(M_1', \gamma_1') \cong SHM(M_1, \gamma_1)$ and $SHM(M_2', \gamma_2') \cong SHM(M_2, \gamma_2)$. Finally, another application of Proposition 1 proves that $SHM(M, \gamma) \cong SHM(M', \gamma')$, and we are done. ∎

Rmk. We will not have further use for the case that the boundary circles of A in the last proposition do not both bound disks on the same side of A. It is just included for interest. In fact, the same is true for Proposition 2.

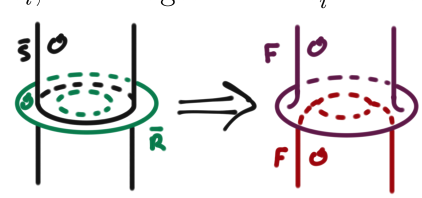
Thm. 4 If $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ where S is a connected Juhász surface whose boundary is a collection of $n > 0$ homologically independent curves c_i^+ in $R_+(\gamma)$ and equally many homologically independent curves c_i^- in $R_-(\gamma)$, then $SHM(M', \gamma')$ is a direct summand of $SHM(M, \gamma)$.

Proof. The idea is to find a surface F in the closure Y of (M, γ) such that

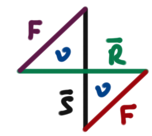
$$SHM(M', \gamma') := HM_*(Y'|\bar{R}') \cong HM_*(Y|F).$$

Moreover, F will have the properties $[F] = [\bar{R}] + [\bar{S}]$ and $\chi(F) = \chi(\bar{R}) + \chi(\bar{S})$ where $\bar{S}$ is another surface in Y. The theorem will then follow from Proposition 2.12 in the introduction, which says that in this case $HM_*(Y|F)$ is a direct summand of $HM_*(Y|\bar{R}) \cong SHM(M, \gamma)$.

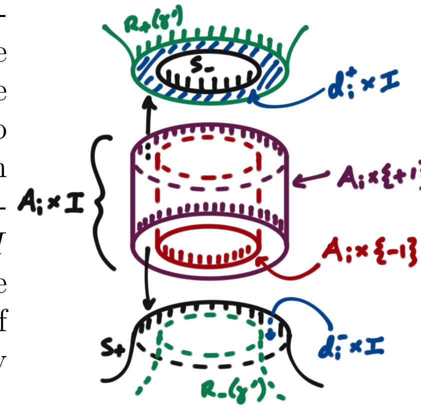
Since the $c_i^\pm$ are homologically independent in $R_\pm(\gamma)$, they will continue to be homologically independent in $\bar{R}_\pm$ (in other words, when the auxiliary product $T \times I$ is attached). We may therefore choose the gluing diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ to send each curve c_i^+ to c_i^-, preserving their orientations as boundary components of S. Since this process glues all the boundary components of S together, S passes to a closed, connected, oriented surface $\bar{S}$ in Y. Note that $\bar{S}$ intersects $\bar{R}$ transversely in the circles $\bar{c}_i$, the images of the $c_i^\pm$. The desired surface F is formed by excising these circles $\bar{c}_i$ from $\bar{R} \cup \bar{S}$ and then reattaching the pieces of $\bar{R}$ and $\bar{S}$ in the unique way which preserves orientation. This process is made explicit by the inset, and in the language of [1] can be thought of as "smoothing out" $\bar{R} \cup \bar{S}$.



Let us briefly argue that F has both of the desired properties $\chi(F) = \chi(\bar{R}) + \chi(\bar{S})$ and $[F] = [\bar{R}] + [\bar{S}]$. If we excise $\bar{c}_i$ from $\bar{R} \cup \bar{S}$, then both $\bar{R} \sqcup \bar{S}$ and F can be obtained from the resulting surface by attaching cylinders. Since attaching cylinders does not affect the Euler characteristic, the first property is proven. To prove the second property, we must argue that $[F] - [\bar{R}] - [\bar{S}]$ is the boundary of a 3-cycle. In this vein, consider a cross section of $F \triangle (\bar{R} \cup \bar{S})$, where $\triangle$ denotes the symmetric difference. This is shown in the inset, and right away we can see a 2-cycle ν that relates the one-dimensional cross sections of F and $\bar{R} \cup \bar{S}$. If we multiply the inset by S^1, divide the product into triangular prisms, and further divide the prisms into 3-simplices, the result is the desired 3-cycle.



Finally, we show that there is a diffeomorphism $(Y, F) \to (Y', \bar{R}')$, which will complete the proof. To this end, we will recall the structure of (M', γ'), and then form a very specific closure of this sutured manifold that can be put in correspondence with Y and F. Since S is a Juhász surface (see Definition 3.6), the sutures $s(\gamma)$ pass unaltered to $s(\gamma')$. The remaining members of $s(\gamma')$ are parallel copies $d_i^{\pm}$ of $c_i^{\pm}$. To form Y', we first attach the auxiliary surface $T \times I$ used in forming Y to the annuli $A(\gamma) \subseteq A(\gamma')$. Then for each i, we attach an auxiliary annulus $A_i \times I$ connecting $d_i^+ \times I$ and $d_i^- \times I$ as shown in the inset. Note that the annuli $A_i \times \{\pm 1\}$ serve as a bridge between corresponding boundary components of $R_\pm(\gamma')$ and $S_\pm$. Before moving on, let us briefly examine why this will result in a valid closure of (M', γ').



The first step is to show that $\bar{R}'_\pm$ are connected. Following the description in the previous paragraph, we see that $\bar{R}'_\pm$ are constructed by gluing four pieces together:

$$\bar{R}'_\pm = (T \times \{\pm 1\}) \cup_{s(\gamma)} R_\pm(\gamma') \cup_{d_i^\pm} (A_i \times \{\pm 1\}) \cup_{d_i^\mp} S_\pm.$$

(Importantly, note that $S_\pm$ is connected to $R_\pm(\gamma')$ not just through the annuli $A_i \times \{\pm 1\}$, but also directly.) The first two pieces of $\bar{R}'_\pm$ taken together are connected because they can be obtained by cutting the homologically independent curves $c_i^\pm$ out of the connected surface $\bar{R}_\pm$. Attaching to these pieces the connected surfaces $A_i \times \{\pm 1\}$ and $S_\pm$ of course keeps everything connected, so the result is a closed, connected, oriented surface $\bar{R}'_\pm$. The next condition to check for forming a valid closure of (M', γ') is that $\bar{R}'_\pm$ have genus at least two. We can ensure this by choosing T to have genus at least two. Finally, we must show that the center circles of each of the annuli $A_i \times \{\pm 1\}$ are homologically independent. Equivalently, we can show that deleting these annuli entirely keeps $\bar{R}'_\pm$ connected. This is true because for every boundary component of $S_\pm$ which is attached to $A_i \times \{\pm 1\}$, the boundary component parallel to $c_i^\pm$ is attached directly to $R_\pm(\gamma')$ (also recall that S is assumed to be connected).

To finish the closing up of (M', γ'), we will paste together the two diffeomorphisms

$$\mathrm{id} : S_+ \cup_i (A_i \times \{+1\}) \to S_- \cup_i (A_i \times \{-1\}) \qquad \text{and} \qquad h : \bar{R}_+ \to \bar{R}_-,$$

where the latter sends $R_+(\gamma') \cup T \times \{+1\}$ to $R_-(\gamma') \cup T \times \{-1\}$ because it sends each c_i^+ to c_i^-. It remains to argue that (Y, F) and $(Y', \bar{R}')$ are diffeomorphic. This is best done by picture. Figure 9 shows how the process of reversing the closing up of (M, γ), cutting out

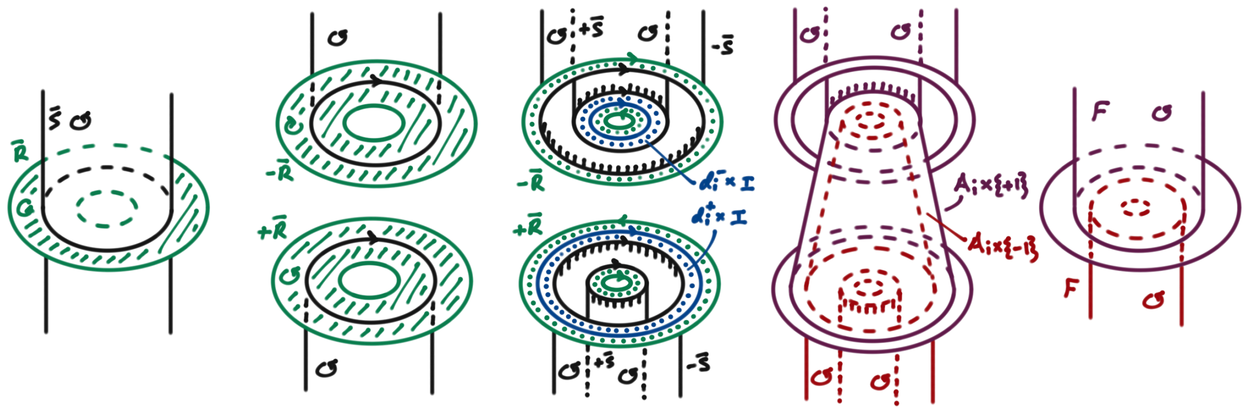


Figure 9: Closing up (M', γ') is equivalent to smoothing out $\bar{R} \cup \bar{S}$.

S to form (M', γ'), and then closing up (M', γ') is actually equivalent to "smoothing out" $\bar{R} \cup \bar{S}$. This completes the proof. ∎

Thm. 5 If $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ where S is a good decomposing surface such that the number of components of $c \cap A(\gamma)$ is 2 mod 4 for each component c of ∂S, then $SHM(M', \gamma')$ is a direct summand of $SHM(M, \gamma)$.

Proof. This proof will consist of two parts. First, we introduce the idea of a product handle, an object which we can attach to a given sutured manifold to produce a new one. We will find that this process induces an isomorphism on sutured monopole Floer homology. Second, we will see that by attaching product handles to (M, γ), the good decomposing surface S will turn into a Juhász surface S_1 in the new sutured manifold (M_1, γ_1) satisfying the hypothesis of Theorem 4. We will find that the sutured manifold (M_1', γ_1') obtained by cutting S_1 out of (M_1, γ_1) can also be obtained by attaching product handles to (M', γ'). In short, there is a commutative diagram

$$\begin{tikzcd} (M, \gamma) \arrow[r, rightsquigarrow, "S"] \arrow[d, rightsquigarrow] & (M', \gamma') \arrow[d, rightsquigarrow] \\ (M_1, \gamma_1) \arrow[r, rightsquigarrow, "S_1"] & (M_1', \gamma_1') \end{tikzcd}$$

where the vertical, coiled arrows indicate the attaching of some product handles. These and the bottom arrow induce isomorphisms, so the top one will also have to.

Let Q be a square $I \times I$, viewed as a surface with four corners. The product sutured manifold $(Q \times I, \delta)$ is called a product handle. Its annulus $A(\delta)$ has four square faces, and if we glue two opposite faces to a sutured manifold (M, γ) as shown in Figure 10, a new sutured manifold (M_1, γ_1) results. Up to diffeomorphism, the annuli of $A(\gamma)$ and $A(\gamma_1)$ are related by a cutting and gluing operation reminiscent of the Floer excision procedure, though this remark will not be important in what follows.

Let us examine why attaching product handles has no effect on the level of sutured monopole Floer homology. For each annulus $A_i' \in A(\gamma)$ in the original sutured manifold (M, γ), let A_i be a parallel copy of A_i' whose boundary circles lie in $R_+(\gamma)$ and $R_-(\gamma)$ and run parallel to the suture of A_i'. These annuli A_i are clearly product annuli, and bound a

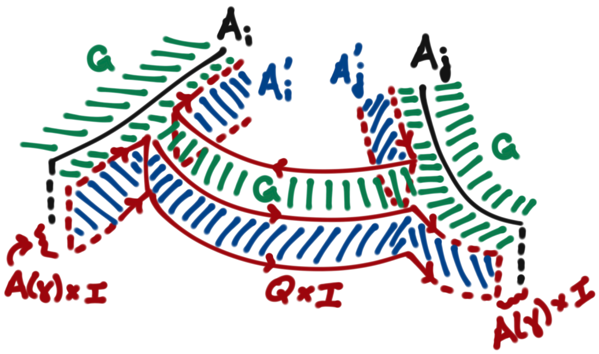


Figure 10: Attaching a product handle $Q \times I$ to a sutured manifold can be reversed by cutting out a collection of product annuli A_i.

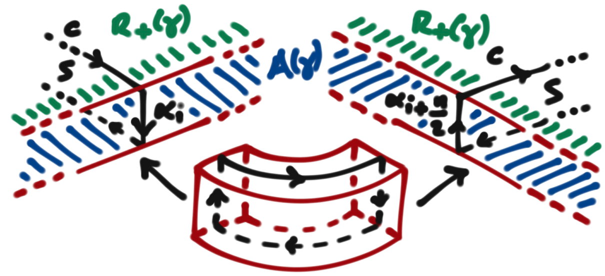


Figure 11: Attaching product handles turns a good decomposing surface S into a Juhász surface by joining segments of ∂S.

tubular neighborhood of $A(\gamma)$. This means they separate (M, γ) into two pieces, the first of which is diffeomorphic to (M, γ), and the second of which is diffeomorphic to $A(\gamma) \times I$. But now notice that attaching product handles only affects the second piece: this is shown in Figure 10. We conclude that the A_i separate (M_1, γ_1) into (M, γ) and a product sutured manifold $(T \times I, \varepsilon)$, with $T \times I = (A(\gamma) \times I) \cup (Q \times I)$ as in the Figure. Since the proof of Proposition 3 goes through unaltered for a collection of multiple product annuli, we have

$$\begin{aligned} SHM(M_1, \gamma_1) &\cong SHM(M \sqcup (T \times I), \gamma \sqcup \varepsilon) \cong SHM(M, \gamma) \otimes_{\mathbb{F}} SHM(T \times I, \varepsilon) \\ &\cong SHM(M, \gamma). \end{aligned}$$

In the second and third steps, we used Proposition 1 and Example 5.6.

Having introduced the idea of product handles, we explain how they will help us reduce this theorem to the previous one. For the sake of brevity, we will replace $A(\gamma)$ and $R_\pm(\gamma)$ with A and $R_\pm$ in this paragraph. Since S is a good decomposing surface, any time that a component c of ∂S meets an annulus $S^1 \times I \in A$, the intersection is homologous to the segment $\{*\} \times I$. Travelling along c in the direction of the boundary orientation on S, let us index the n components of $c \cap A$, oriented along c, with the symbol α_i. Going all the way around c, we end up back where we started, so $\alpha_i = \alpha_{i+n}$. Each time we cross A, we go from $R_\pm$ to $R_\mp$, so if α_i runs from $R_\pm$ to $R_\mp$, then α_{i+1} runs from $R_\mp$ to $R_\pm$. These two properties of α_i imply n is even. But we have assumed more specifically that $n \equiv 2 \pmod 4$, which implies that if α_i runs from $R_\pm$ to $R_\mp$, then $\alpha_{i+n/2}$ runs from $R_\mp$ to $R_\pm$. Thus, for all i, we may glue a product handle to A such that α_i and $\alpha_{i+n/2}$ are connected by a vertical square cross section. This is shown in Figure 11. The effect is that S passes to a new surface in a new sutured manifold where the boundary component c of S has been replaced by two new boundary components, c_+ and c_-, contained entirely within the new R_+ and R_- respectively. If we repeat this procedure for every component of ∂S, the result is a Juhász surface S_1 each of whose components has twice the number of boundary components as the corresponding component of S, divided evenly between $R_+(\gamma_1)$ and $R_-(\gamma_1)$. Since each component of ∂S_1 runs along its own unique product handle(s), the components of ∂S_1 must be homologically independent in $R_+(\gamma_1)$ and $R_-(\gamma_1)$. This proves S_1 satisfies the hypothesis of Theorem 4.

Since the new Juhász surface splits each of the former product handles in half, we can obtain the sutured manifold (M_1', γ_1') simply by attaching twice as many product handles to (M', γ') as we did to (M, γ). This gives us the commutative diagram from above, and completes the proof. ■

Rmk. We end this section by remarking that the previous theorem holds more generally, namely when S is assumed to be any decomposing surface whose boundary components are "boundary coherent" when contained entirely in $R(\gamma)$. Juhász reduces the more general statement to the previous theorem in [3, Lemma 4.5]. For any component c of ∂S, the isotopy in that lemma increases the number of components of $c \cap A(\gamma)$ by two, so as in the previous theorem we can assume that there are 2 mod 4 components of $c \cap A(\gamma)$.

7 Conclusion

Let us review what we have learned over the course of this essay. We started with a functor $\mathsf{Cob}_3 \to \mathsf{Vect}_{\mathbb{F}}$ called monopole Floer homology, about which we assumed some very elementary properties. Our first big result was the Floer excision theorem, which said that monopole homology is invariant under the surgery procedure where we cut two diffeomorphic surfaces out of a closed 3-fold, and glue the resulting boundaries together in the only nontrivial, orientation-preserving way. The Floer excision theorem was the key idea in most of the subsequent proofs. Our next big result was that monopole homology leads to a well-defined invariant of sutured manifolds. The invariant, sutured monopole Floer homology, is obtained by forming any of several possible closures of the sutured manifold, and taking the monopole homology of that closure. Our last big result was that, on the level of homology, cutting a decomposing surface out of a sutured manifold amounts to restricting to a direct summand. Building up to the proof of this fact, we found that cutting horizontal surfaces and certain vertical annuli out of sutured manifolds has no effect on homology.

While we have found many desirable properties of our invariants, we have only performed one computation. This was in Example 5.6, where we found that the sutured monopole Floer homology of any product sutured manifold is a one-dimensional vector space over $\mathbb{F}$. One is naturally led to ask what the homology groups of other sutured manifolds are. To this end, Juhász has found a nonvanishing theorem in [3] for so-called Heegaard Floer homology. Using our decomposition theorems in the previous section, we find that the proof carries over to the monopole case. The ideas are as follows.

Def. 1 Let (M, γ) be a sutured manifold, and let $S = R(\gamma)$. We say that (M, γ) is taut if:

1. Every copy of S^2 in M bounds a copy of D^3 (that is, M is "irreducible").
2. The boundary of any disk D in M satisfying $D \pitchfork S = \partial D$ bounds another disk D' contained in $S \neq S^2$ (that is, S is "incompressible").
3. S minimizes the following summation over all properly embedded surfaces S' in M for which $\partial S' \subseteq A(\gamma)$ and $[S, \partial S] = [S', \partial S'] \in H_2(M, A(\gamma); \mathbb{Z})$.

$$\sum_{\substack{S_i' \text{ a component of } S' \\ \chi(S_i') < 0}} |\chi(S_i')|$$

Def. 2 A sutured manifold (M, γ) is decomposable if, starting with (M, γ), one can make a product sutured manifold $(T \times I, \delta)$ by removing sufficiently many decomposing surfaces. That is, we have a so-called hierarchy of sutured manifolds as in the sequence below.

$$(M, \gamma) = (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \rightsquigarrow (M_{n-1}, \gamma_{n-1}) \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n) = (T \times I, \delta).$$

Thm. 3 If (M, γ) is a taut balanced sutured manifold, then $\dim(SHM(M, \gamma)) \geq 1$.

Proof. (Sketch.) In [2, Theorem 4.2], Gabai shows that a taut balanced sutured manifold is decomposable. In [3, Lemma 4.5], Juhász shows that each decomposing surface S_i in Gabai's hierarchy can be made to satisfy the hypothesis of Theorem 6.5 in this essay. We conclude that $SHM(T \times I, \delta) \cong \mathbb{F}$ is a direct summand of the monopole homology of each sutured manifold (M_i, γ_i) in the hierarchy. In particular, $\mathbb{F}$ is a direct summand of $SHM(M, \gamma)$. ■

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