Exercise 1 : (from L 1)

TITLE : In Poisson's problem, prove that $F(v)$ and $A(u,v)$ satisfy the hypothesis of LAX-MILGRAM Lemma

MODEL: $$A(u,v) = \int_0^1 u'v'\,dx \qquad F(v) = \int_0^1 fv\,dx$$

PART 1: $F(v)$ needs to be continuous, meaning :

$$\exists\, \gamma_F : \| F(v) \| \leq \gamma_F \|v\|_1$$

•) if $v$ is (constant) $v=0$, since $F(v) : H_0^1 \to \mathbb{R}$ and it is linear it is clear the implication.

•) other cases:

$$\| F(v) \| = \left( \int_0^1 |fv|^2 dx \right)^{\frac{1}{2}}$$

Note: $f$ is continuous $C^0$ on $[0,1] = \Omega$ by Poisson. This function must have a maximum $f_{max}$ in $\Omega$.

So, the idea is to bound by considering the maximum in every evaluation of the integral in $\Omega$.

$$\left( \int_0^1 |fv|^2 dx \right)^{\frac{1}{2}} \leq \left( \int_0^1 |f_{max} v|^2 dx \right)^{\frac{1}{2}} = \left( |f_{max}|^2 \int_0^1 |v|^2 dx \right)^{\frac{1}{2}}$$

but now, inside the integral it is possible to see the $\|\cdot\|^2_{L^2}$ norm

so we can re-write:

remove $|\cdot|$ since squared.

$$\|F(v)\| \leq \left( |f_{max}|^2 \|v\|^2_{L^2} \right)^{\frac{1}{2}} = \left( f^2_{max} \|v\|^2_{L^2} \right)^{\frac{1}{2}} = f_{max} \|v\|_{L^2}$$

(Note: $v$ can be normed by $L^2$, since $v \in L^2$)

(Note: we need to prove at this point, that this $f_{max} \|v\|_{L^2}$ can be bounded by $\|\cdot\|_{H^1}$.)

Maybe, it was better to keep this format:

$$\left( f_{max}^2 \|v\|^2_{L^2} \right)^{\frac{1}{2}}$$

again we need to compare $\|\cdot\|_{L^2}$ with $\|\cdot\|_{H^1}$

Look $\|v\|^2_{L^2} = \left( \int_\Omega |v|^2 dx \right)^{\frac{1}{2} \cdot 2} = \int_\Omega |v|^2 dx$

However, this integral can be defined as (using theorem of calculus):

$$\int_\Omega v \, dx = \int_\Omega \left| v(0) + \int_0^x v'(\varphi) \, d\varphi \right| dx$$

which means:

$$\|v\|^2_{L^2} = \int_\Omega \left| v(0) + \int_0^x v'(\varphi) \, d\varphi \right|^2 dx$$

now $\Omega = [0,1]$ and for Homogeneous Dirichlet's condition :

$$\|v\|_{L^2}^2 = \int_0^1 \left| 0 + \int_0^x v'(\varphi)\, d\varphi \right|^2 dx = \int_0^1 \left| \int_0^x v'(\varphi)\, d\varphi \right|^2 dx$$

it can be bouded by moving $|\cdot|$ inside the second integral:

$$\|v\|_{L^2}^2 \le \int_0^1 \int_0^x |v'(\varphi)|^2 d\varphi\, dx$$

it is possible to recognize $\|\cdot\|_1^2$ norm.

we can bound :

$$\|v\|_{L^2}^2 \le \|v\|_1^2$$

Finally :

some constant

$$\|F(v)\| \le \left( f_{max}^2 \|v\|_{L^2}^2 \right)^{\frac{1}{2}} \le \gamma_F \|v\|_1 \qquad \text{(proved).}$$

PART 2: $A(u,v)$ needs to be continuous, meaning:

$$A(u,v) \le \gamma \|u\|_1 \|v\|_1$$

The bi-linear operator is defined as follows:

$$A(u,v) = \int_0^1 u' v'\, dx$$

this can be bound by $|\cdot|$, indeed absolute value $>0$

$$A(u,v) \le \left| \int_0^1 u'v'\, dx \right| = \left( \left| \int_0^1 u'v'\, dx \right|^2 \right)^{\frac{1}{2}}$$

this is something that has been observed before and can be useful.

this inside the square is an inner product, which can be bounded by Schwartz's inequality: $\longrightarrow$ $\boxed{\int_0^1 u'v' dx \cdot \int_0^1 u'v' dx}$

$$A(u,v) \le \left( \int_0^1 u'u' \, dx \cdot \int_0^1 v'v' \, dx \right)^{\frac{1}{2}} = \left( \int_0^1 u'u' \, dx \right)^{\frac{1}{2}} \cdot \left( \int_0^1 v'v' \, dx \right)^{\frac{1}{2}} =$$

$$= \|u\|_1 \cdot \|v\|_1$$

There will always be a constant $\gamma$, such that

$$A(u,v) \le \gamma \, \|u\|_1 \, \|v\|_1 \qquad \text{(proved)}$$

PART 3: $A(u,v)$ needs to be coercive, meaning

$$A(v,v) > \alpha_0 \|v\|_1^2$$

$$A(v,v) = \int_0^1 v'v' \, dx = \int_0^1 (v')^2 \, dx = \|v\|_1^2$$

There will always a constant $\alpha_0$ such that

$$A(v,v) \ge \alpha_0 \|v\|_1^2 \qquad \text{(proved)}$$

Exercise 2 (from L1):

TITLE: Proof "$\Rightarrow$" and "$\Leftarrow$" of the theorem which states that $u$ is weak problem solution $\Leftrightarrow$ $u$ minimize the potential function:

$$J(v) = \frac{1}{2}A(v,v) - F(v)$$

PART 1: "$\Rightarrow$" $u$ is weak problem solution.

Then $\forall v \in V$: Consider a perturbation of potential with true solution

$$J(v+u) = \frac{1}{2}A(v+u, v+u) - F(v+u)$$

thanks to bilinearity and continuity of $A(\cdot,\cdot)$:

$$\frac{1}{2}A(v+u, v+u) = \Big(A(v,v) + A(v,u) + A(u,v) + A(u,u)\Big)\cdot\frac{1}{2} =$$

$$= \frac{A(v,v) + A(u,u)}{2} + A(u,v)$$

thanks to linearity of $F(\cdot)$:

$$F(v+u) = F(v) + F(u)$$

now, putting all together:

$$J(v+u) = \frac{A(v,v) + A(u,u)}{2} + \underline{A(u,v)} - F(u) \underline{- F(v)}$$

since $u$ solves the weak problem $\rightarrow$ $A(u,v) = F(v)$

$$J(v+u) = \frac{A(v,v) + A(u,u)}{2} - F(u)$$

now,

$$J(v+u) = \frac{1}{2} A(v,v) + \frac{1}{2} A(u,u) - F(u)$$

consider the potential with true solution $u$:

$$J(u) = \frac{1}{2} A(u,u) - F(u) \qquad \text{(present in equation above)}$$

$$J(v+u) = \frac{1}{2} A(v,v) + J(u)$$

now, to prove that $u$ minimize $J(\cdot)$, we need to prove that $A(v,v)$ is $\geq 0$

$$A(v,v) = \int_0^1 v'v' \, dx = \|v\|_1^2 \quad \text{by def. a norm always} \geq 0.$$

$$J(v+u) \geq J(u) \qquad \text{(proved)}$$

PART 2: "$\Leftarrow$" $u$ minimize $J(\cdot)$. Prove this means $u$ is weak problem solution.

The technique to do that is calculus of variation, we need to perturb the data:

If $u$ minimize $J(\cdot)$ $\longrightarrow$ $\left.\frac{\partial}{\partial \varepsilon}\left(J(u+\varepsilon v)\right)\right|_{\varepsilon=0} = 0$ with $\varepsilon \ll 1$

Recall: $J(u) = \frac{1}{2} A(u,u) - F(u)$

$$\frac{\partial}{\partial \varepsilon}\left( \frac{1}{2}A(u+\varepsilon v, u+\varepsilon v) - F(u+\varepsilon v)\right)\Big|_{\varepsilon=0} = 0$$

by operation allowed by derivatives:

$$\frac{\partial}{\partial \varepsilon}\left( \frac{1}{2}A(u+\varepsilon v, u+\varepsilon v)\right)\Big|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon}F(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$$

by linearity of $F(\cdot)$ and bilinearity of $A(\cdot,\cdot)$:

$$\frac{\partial}{\partial \varepsilon}\left( \frac{1}{2}\left(A(u,u) + A(\varepsilon v, \varepsilon v) + 2A(u, \varepsilon v)\right)\right)\Big|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon}\left(F(u) + \varepsilon F(v)\right)\Big|_{\varepsilon=0} = 0$$

$$\frac{\partial}{\partial \varepsilon}\left( \frac{A(u,u)}{2} + \frac{\varepsilon^2 A(v,v)}{2} + \varepsilon A(u,v)\right)\Big|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon}\left(F(u) + \varepsilon F(v)\right)\Big|_{\varepsilon=0} = 0$$

now, take the derivative respect to $\varepsilon$:

$$\left(0 + \varepsilon A(v,v) + A(u,v)\right)\Big|_{\varepsilon=0} - \left(0 + F(v)\right)\Big|_{\varepsilon=0} = 0$$

evaluate at $\varepsilon=0$:

$$\left(0 + 0 + A(u,v)\right) - \left(0 + F(v)\right) = 0$$

$$A(u,v) = F(v)$$

This is the weak problem formulation.

(proved)

Exercise 3 (from L 2):

TIME: Prove $\max_{\bar{\Omega}} u = \max_{\Omega} u$ from Strong Maximum Principle (SMP)

The assumption of SMP are:

$u$ is $C^2(\Omega) \cap C(\bar{\Omega})$

$\Delta u = 0 \longrightarrow u$ is harmonic on open $\Omega$

PART 1:
Suppose $x_0 \in \Omega$ and it is the maximum point for which maximum of $u$ is obtained:

$$u(x_0) = \max_{\bar{\Omega}} u = M$$

Since $\bar{\Omega}$ contains also $\partial\Omega$, it follows that $M$ is maximum on $\Omega$ and on $\partial\Omega$.

PART 2:
However, $x_0 \notin \Omega$ and $x_0 \in \partial\Omega$, so that $x_0 \in \bar{\Omega}$.

Since working on elliptic problem: $Lu < 0$ in $\Omega$.

Suppose $x_M \in \Omega$ and $u(x_M) = \max_{\Omega} u$

At $x_M$, the $1^{st}$ derivative of $u$ will be negative. Also, the $2^{nd}$ derivative will be either negative or 0. This means $Lu$ cannot be $<0$. This broke the previous assumption, meaning $x_M$ cannot be on $\Omega$, but only in $\partial\Omega$.

$$\max_{\bar{\Omega}} u = \max_{\Omega} u \qquad \text{(proved)}$$

Exercise 4 ( from 2L)

TITLE: Prove uniqueness of Poisson's problem solution

Consider $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. The problem is

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

To prove the uniqueness of the solution, it will be used the trick of two solutions : $u_1$ and $u_2$.

$$\begin{cases} -\Delta u_1 = f \\ u_1 = g \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = f \\ u_2 = g \end{cases}$$

Suppose we can define $u_3 = u_1 - u_2$ , we get

$$\begin{cases} -\Delta \overbrace{(u_1 - u_2)}^{u_3} = f - f \\ u_1 - u_2 = g - g \end{cases} \xrightarrow[\text{implies}]{} \begin{cases} -\Delta u_3 = 0 & \text{on } \Omega \\ \overbrace{u_1 - u_2}^{u_3} = 0 & \text{on } \partial\Omega \end{cases}$$

The problem has been transformed in homogeneous Dirichlet problem : Now, by Strong Maximum Principle

$$\max_{\Omega} u_3 = \max_{\Omega} u_3$$

$u$ has to be constant on all $\Omega$ .

$$\max_{\bar{\Omega}} u_3 = \max_{\Omega} u_3 = 0 \qquad \text{since} \quad u_3(0) = u_3(1) = 0$$

this implies: $\max_{\bar{\Omega}} (u_1 - u_2) = 0$

meaning $u_1 - u_2 = 0 \iff u_1 = u_2$

Cannot be two solutions! (proved)

Exercise 5 (from 23):

TITLE: Find truncation error bound in FD method for elliptic problem, with $c=0$

$$\begin{cases} Lu := -a u'' + b u' = f & \text{on } \Omega = [0,1] \\ u(0) = u_L \qquad u'(1) = 0 \end{cases}$$

Let's apply the FD method: ($a, b$ constant for simplicity)

$$-a_i \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b_i \frac{U_{i+1} - U_{i-1}}{2h} = f_i$$

$$U_{i+1}\left(-\frac{a_i}{h^2}\right) + U_i\left(\frac{2a_i}{h^2}\right) + U_{i-1}\left(-\frac{a_i}{h^2}\right) + U_{i+1}\left(\frac{b_i}{2h}\right) - U_{i-1}\left(\frac{b_i}{2h}\right) = f_i$$

$$U_{i+1}\left(-\frac{a_i}{h^2} + \frac{b_i}{2h}\right) + U_i\left(\frac{2a_i}{h^2}\right) + U_{i-1}\left(-\frac{a_i}{h^2} - \frac{b_i}{2h}\right) = f_i$$

$$U_{i+1}\left(\frac{-2a_i + b_i \cdot h}{2h^2}\right) + U_i\left(\frac{2a_i}{h^2}\right) + U_{i-1}\left(\frac{-2a_i - b_i \cdot h}{2h^2}\right) = f_i$$

$$\frac{U_{i+1}\left(-a_i + \frac{hb_i}{2}\right) + U_i\left(2a_i\right) + U_{i-1}\left(-a_i - \frac{hb_i}{2}\right)}{h^2} = f_i$$

Now, re-write with correct signs:

$$-U_{i+1}\left(a_i - \frac{hb_i}{2}\right) + U_i(2a_i) - U_{i-1}\left(a_i + \frac{hb_i}{2}\right) = f_i$$

Now, it is desired to have every equation on the same node $U_i$.
Apply Taylor until $4^{th}$ order: $\varepsilon = [x; x+h]$ and $\eta = [x-h; x]$

$$U_{i+1}\left(a_i - \frac{hb_i}{2}\right) \cong \left(a_i - \frac{hb_i}{2}\right)u(x_i) + \left(ha_i - \frac{h^2 b_i}{2}\right)u'(x_i) + \left(\frac{h^2}{2}a_i - \frac{h^3}{4}b_i\right)u''(x_i)$$
$$+ \left(\frac{h^3}{6}a_i - \frac{h^4}{12}b_i\right)u'''(x_i) + \left(\frac{h^4}{24}a_i - \frac{h^5}{48}b_i\right)u^{IV}(\varepsilon)$$

$$U_{i-1}\left(a_i + \frac{h}{2}b_i\right) \cong \left(a_i + \frac{h}{2}b_i\right)u(x_i) + \left(-ha_i - \frac{h^2}{2}b_i\right)u'(x_i) + \left(\frac{h^2}{2}a_i + \frac{h^3}{4}b_i\right)u''(x_i)$$
$$+ \left(-\frac{h^3}{6}a_i - \frac{h^4}{12}b_i\right)u'''(x_i) + \left(\frac{h^4}{24}a_i + \frac{h^5}{48}b_i\right)u^{IV}(\eta)$$

Re-write the FD scheme:

$$-\cancel{a_i u(x_i)} + \cancel{\frac{h}{2}b_i u(x_i)} - \cancel{ha_i u'(x_i)} + \frac{h^2}{2}b_i u'(x_i) - \frac{h^2}{2}a_i u''(x_i) + \cancel{\frac{h^3}{4}b_i u''(x_i)}$$
$$-\cancel{\frac{h^3}{6}a_i u'''(x_i)} + \frac{h^4}{12}b_i u'''(x_i) - \frac{h^4}{24}a_i u^{IV}(\varepsilon) + \cancel{\frac{h^5}{48}b_i u^{IV}(\varepsilon)} + 2\cancel{a_i u(x_i)}$$
$$-\cancel{a_i u(x_i)} - \cancel{\frac{h}{2}b_i u(x_i)} + \cancel{ha_i u'(x_i)} + \frac{h^2}{2}b_i u'(x_i) - \frac{h^2}{2}a_i u''(x_i) - \cancel{\frac{h^3}{4}b_i u''(x_i)}$$
$$+\cancel{\frac{h^3}{6}a_i u'''(x_i)} + \frac{h^4}{12}b_i u'''(x_i) - \frac{h^4}{24}a_i u^{IV}(\eta) - \cancel{\frac{h^5}{48}b_i u^{IV}(\eta)}$$

$$h^2 b_i u'(x_i) - h^2 a_i u''(x_i) + \frac{h^4}{6}b_i u'''(x_i) - \frac{h^4}{24}a_i\left(u^{IV}(\varepsilon) + u^{IV}(\eta)\right)$$

This is the numerator of the discretized $L_h u$:

$$L_h u := \frac{h^2 b_i u'(x_i) - h^2 a_i u''(x_i) + \frac{h^4}{6} b_i u'''(x_i) - \frac{h^4}{24} a_i \left(u^{IV}(\varepsilon) + u^{IV}(\eta)\right)}{h^2}$$

$$L_h u := b_i u'(x_i) - a_i u''(x_i) + \frac{h^2}{6} b_i u'''(x_i) - \frac{h^2}{24} a_i \left(u^{IV}(\varepsilon) + u^{IV}(\eta)\right)$$

Remember that $Lu := -a u''(x_i) + b u'(x_i)$

Define the truncation error:

$$T_i = Lu - L_h u = \cancel{-a u''(x_i)} + \cancel{b u'(x_i)} - \cancel{b_i u'(x_i)} + \cancel{a_i u''(x_i)} - \frac{h^2}{6} b_i u'''(x_i) + \frac{h^2}{24} a_i \left(u^{IV}(\varepsilon) + u^{IV}(\eta)\right)$$

$$T_i = -\frac{h^2}{6} b_i u'''(x_i) + \frac{h^2}{24} a_i \left(u^{IV}(\varepsilon) + u^{IV}(\eta)\right)$$

Now, it can be considered $2u^{IV} = u^{IV}(\varepsilon) + u^{IV}(\eta)$

$$T_i = -\frac{h^2}{6} b_i u'''(x_i) + a_i \frac{h^2}{12} u^{IV}(\cdot)$$

Bound using triangular inequality:

$$|T_i| \leq \left| -\frac{h^2}{6} b_i \right| |u'''(x_i)| + \left| a_i \frac{h^2}{12} \right| |u^{IV}(\cdot)|$$

Now, apply the norm:

$$|T_i| \leq + \frac{h^2}{6} \|b_i\|_C \|u'''\|_C + \frac{h^2}{12} \|a_i\|_C \|u^{IV}\|_C \qquad \text{(proved)}$$

Exercise 6 (from L 3):

TITLE: Set $b=0$, on generalized Poisson's problem, prove that

$$|u(x_i) - u_i| \leq ch^2 \|u^{IV}\|_C$$

TOOL: ·) for $b=0$: $\quad \|U\|_\infty \leq \max\{|U_0|, |U_N|\} + c\,\|L_h U\|_\infty$

$U = \{U_i\}_{i=1}^N$

·) $|T_i| \leq \frac{h^2}{12} \|a\|_C \|u^{IV}\|_C$

·) $\begin{cases} L_h u := -a u'' + cu = f & \text{on } \Omega \\ u_0 = u_L \quad u'(1) = 0 \end{cases}$

Define the error between true solution and approximated one:

$$err_i = u(x_i) - u_i$$

Consider only point in $\Omega$. (For simplicity).

The solution is bounded $\|L_h U\|_\infty$

Now, apply the discrete operator on the error

$$L_h(err_i) = L_h u(x_i) - L_h u_i$$

This is the truncation error: we can bound the absolute value of the error:

$$|err_i| \leq \frac{h^2}{12} \|a\|_C \|u^{IV}\|_C \qquad \text{define } \frac{h^2}{12} \|a\|_C = C$$

$$|err_i| \le C \|u^{IV}\|_C \qquad \text{(proved)}$$

## Exercise 7 (from L3):

**TITLE:** Show the development of this system in the method of non-uniform grids (method of undetermined coefficients)

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} - \gamma h_i = 0 \\ \alpha \dfrac{h_{i+1}^2}{2} + \gamma \dfrac{h_i^2}{2} = 1 \end{cases} \longrightarrow \begin{cases} \alpha = \dfrac{2}{h_i (h_i + h_{i+1})} \\ \beta = -\dfrac{2}{h_i h_{i+1}} \\ \gamma = \dfrac{2}{h_i (h_i + h_{i+1})} \end{cases}$$

This system is defined by 3 equations and 3 variables, so it could be solved.

$$\begin{cases} \beta = -\alpha - \gamma \\ \alpha = \dfrac{\gamma h_i}{h_{i+1}} \\ \dfrac{\gamma h_i}{\cancel{h_{i+1}}} \cdot \dfrac{h_{i+1}^{\cancel{2}}}{2} + \gamma \dfrac{h_i^2}{2} = 1 \end{cases} \longrightarrow \begin{cases} \beta = \dots \\ \alpha = \dots \\ \gamma \left( \dfrac{h_i h_{i+1}}{2} + \dfrac{h_i^2}{2} \right) = 1 \end{cases}$$

$$\begin{cases} \beta = \dots \\ \alpha = \dots \\ \gamma \dfrac{h_i}{2} \left( h_{i+1} + h_i \right) = 1 \end{cases} \longrightarrow \begin{cases} \beta = \dots \\ \alpha = \dots \\ \gamma = \dfrac{2}{h_i (h_i + h_{i+1})} \end{cases} \longrightarrow \begin{cases} \beta = \dots \\ \alpha = \dfrac{\cancel{h_i}}{h_{i+1}} \cdot \dfrac{2}{\cancel{h_i}(h_i + h_{i+1})} \\ \gamma = \dfrac{2}{h_i (h_i + h_{i+1})} \end{cases}$$

$$
\begin{cases}
\beta = -\dfrac{2}{h_{i+1}(h_i+h_{i+1})} - \dfrac{2}{h_i(h_i+h_{i+1})} = -2\left(\dfrac{\cancel{h_i + h_{i+1}}}{\cancel{(h_i+h_{i+1})}\, h_i h_{i+1}}\right) = -\dfrac{2}{h_i h_{i+1}} \\[2ex]
\alpha = \dfrac{2}{h_{i+1}(h_i+h_{i+1})} \\[2ex]
\gamma = \dfrac{2}{h_i(h_i+h_{i+1})}
\end{cases}
$$

(proved)

## Exercise 8 (from 4L):

TITLE: Truncation error on 2D Poisson's problem using FD scheme on 5 points.

TOOL: ·) $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$

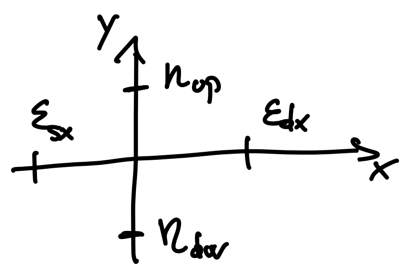
·) Scheme 5 points: $L_h u_{i,j} = -\dfrac{u_{i+1,j} + u_{i,j+1} - 4u_{i,j} + u_{i-1,j} + u_{i,j-1}}{h^2}$

·) $L u_{ij} = -\Delta u_{i,j} = -\left(\partial_{xx} u_{i,j} + \partial_{yy} u_{i,j}\right)$

Define the truncation error:

$$T_{i,j} = L u_{i,j} - L_h u_{i,j}$$

Also, since it is need to do Taylor's expansion until $4^{th}$ order, it is useful to define:




$\xi_{dx} = \begin{pmatrix} x & & x+h \\ y & ; & y \end{pmatrix}$ $\quad \xi_{sx} = \begin{pmatrix} x-h & & x \\ y & ; & y \end{pmatrix}$ $\quad \eta_{up} = \begin{pmatrix} x & & x \\ y & ; & y+h \end{pmatrix}$ $\quad \eta_{dw} = \begin{pmatrix} x & & x \\ y-h & ; & y \end{pmatrix}$

NUMERATOR of $L_h u$: Bring every evaluation at $(i,j)$

$$u_{i+1,j} \cong \cancel{u_{i,j}} + \cancel{h\,\partial_x u_{i,j}} + \frac{h^2}{2}\partial_{xx} u_{i,j} + \cancel{\frac{h^3}{6}\partial_{xxx} u_{i,j}} + \frac{h^4}{24}\partial_{xxxx} u_{\varepsilon_{dx}}$$

$$u_{i-1,j} \cong \cancel{u_{i,j}} - \cancel{h\,\partial_x u_{i,j}} + \frac{h^2}{2}\partial_{xx} u_{i,j} - \cancel{\frac{h^3}{6}\partial_{xxx} u_{i,j}} + \frac{h^4}{24}\partial_{xxxx} u_{\varepsilon_{sx}}$$

$$u_{i,j+1} \cong \cancel{u_{i,j}} + \cancel{h\,\partial_y u_{i,j}} + \frac{h^2}{2}\partial_{yy} u_{i,j} + \cancel{\frac{h^3}{6}\partial_{yyy} u_{i,j}} + \frac{h^4}{24}\partial_{yyyy} u_{\eta_{up}}$$

$$u_{i,j-1} \cong \cancel{u_{i,j}} - \cancel{h\,\partial_y u_{i,j}} + \frac{h^2}{2}\partial_{yy} u_{i,j} - \cancel{\frac{h^3}{6}\partial_{yyy} u_{i,j}} + \frac{h^4}{24}\partial_{yyyy} u_{\eta_{dw}}$$

$$-4\cancel{u_{i,j}}$$

Now, consider whole operator $L_h u$:

$$L_h u \cong -\frac{\cancel{h^2}\partial_{xx} u_{i,j} + \frac{h^{\cancel{4}\,2}}{24}\left(\partial_{xxxx} u_{\varepsilon_{dx}}\ \partial_{xxxx} u_{\varepsilon_{sx}}\right) + \cancel{h^2}\partial_{yy} u_{i,j} + \frac{h^{\cancel{4}\,2}}{24}\left(\partial_{yyyy} u_{\eta_{up}} + \partial_{yyyy} u_{\eta_{dw}}\right)}{\cancel{h^2}}$$

Now, compute the $T_i$, also aggregate $\varepsilon_{dx}, \varepsilon_{sx}$ and $\eta_{up}, \eta_{dw}$.

$$T_{i,j} = -\left(+\cancel{\partial_{yy} u_{i,j}} + \cancel{\partial_{xx} u_{i,j}}\right) - \left(-\left(+\cancel{\partial_{xx} u_{i,j}} + \frac{h^2}{12}\partial_{xxxx} u_\varepsilon + \cancel{\partial_{yy} u_{i,j}} + \frac{h^2}{12}\partial_{yyyy} u_\eta\right)\right) =$$

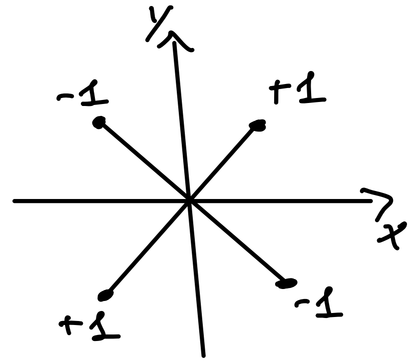
$$T_{i,j} = \frac{h^2}{12}\partial_{xxxx} u_\varepsilon + \frac{h^2}{12}\partial_{yyyy} u_\eta$$

Now, bound and norm:

$$|T_{i,j}| \leq \frac{h^2}{12}\left(\|\partial_{xxxx} u_\varepsilon\|_C + \|\partial_{yyyy} u_\eta\|_C\right) \qquad \text{(proved)}$$

## Exercise 9 (from L4):

**TITLE:** Show the development of FD for mixed term $\delta^y_{2h}\,\delta^x_{2h}\, u_{i,j}$




$$\delta^y_{2h}\,\delta^x_{2h}\, u_{i,j} = \delta^y_{2h}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right) =$$

$$= \frac{1}{2h}\,\delta^y_{2h}\left(u_{i+1,j} - u_{i-1,j}\right) =$$

$$= \frac{1}{2h}\left[\frac{(u_{i+1,j+1} - u_{i+1,j-1})}{2h} - \frac{(u_{i-1,j+1} - u_{i-1,j-1})}{2h}\right] =$$

$$= \frac{1}{4h^2}\left(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}\right)$$

(proved)

## Exercise 10 (from L5):

**TITLE:** Defined the space $H^1_\Gamma$, show that $\|\cdot\|_1$ is a norm

**TOOLS:** 1) $H^1_\Gamma(\Omega) = \left\{ v \in H^1(\Omega) : v|_\Gamma = 0 \right\}$

with $\Gamma \subset \partial\Omega$

2) Poincaré Inequality: Let $\Omega \in \mathbb{R}^d$ (open + bounded) and $\Gamma \subset \partial\Omega$ (regular enough)

$$\exists\, C_\Omega > 0 : \int v^2 dx \le C_\Omega \int |\nabla v|^2 dx \qquad \forall v \in H^1_\Gamma(\Omega)$$

3) $\|v\|_1 = (\nabla v, \nabla v)^{\frac{1}{2}} = \left( \int_\Omega \nabla v \cdot \nabla v \, dx \right)^{\frac{1}{2}}$

It must be shown that $\|v\|_1 = 0$ only when $v = 0$

$\|v\|_1 = 0 \iff \left( \int_\Omega \nabla v \nabla v \, dx \right)^{\frac{1}{2}} = 0$ now square lhs and rhs

$$\int_\Omega \nabla v \nabla v \, dx = 0 \longrightarrow \int_\Omega \nabla v^2 dx = 0$$

now, to have the above equation, it is needed that $\nabla v^2 = 0$.
Indeed, $\nabla v^2 \geq 0$ since squared values and the integral on $\Omega$ of $\nabla v^2$ will be always positive, unless $\nabla v = 0$.

So, $\nabla v^2 = 0 \iff \nabla v = 0$

The gradient is 0, only when $v$ is constant.

Now, using Poincaré's Inequality:

$$\int_\Omega v^2 dx \leq C_\Omega \int_\Omega |\nabla v|^2 dx \qquad \forall v \in H^1_\pi(\Omega)$$

$C_\Omega > 0$

From previous result, $\nabla v = 0$ :

$$\int_\Omega v^2 dx \leq C_\Omega \int_\Omega 0 = 0$$

Here, $v^2$ will always be $\geq 0$, meaning that the integral on $\Omega$

cannot be less than 0. Meaning:

$$\int_\Omega v^2 dx = 0 \quad \text{deriving both side} \quad v^2 = 0$$

this means $v=0$. This means $\|v\|_1$ goes to 0 only when $v=0$. This is a norm.

(proved)

## Exercise 11 (from L6):

TITLE: In general elliptic problem, prove $A$ (not symmetric) is continuous

TOOLS: ·) norm to use $\|v\|_V = \left( \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$

·) $A(u,v) = \int_\Omega A \nabla u \nabla v \, dx - \int_\Omega \vec{b} u \nabla v \, dx + \int_\Omega c u v \, dx$

Since it is desired to prove continuity of $A(u,v)$:

$$\exists \gamma > 0 \quad : \quad A(u,v) \leq \gamma \|u\|_V \|v\|_V$$

it is desired to bound $A(u,v)$ with something greater, it is possible to work with + sign by defining $A^*(u,v)$:

$$A^*(u,v) = \left| \int_\Omega A \nabla u \nabla v \, dx \right| + \left| - \int_\Omega \vec{b} u \nabla v \, dx \right| + \left| \int_\Omega c u v \, dx \right|$$

it is possible to notice that:

$A(u,v) \leq A^*(u,v)$

it is desired now :

$$A^*(u,v) \leq \gamma \|u\|_V \|v\|_V$$

$$\left|\int_\Omega A \nabla u \nabla v \, dx\right| + \left|\int_\Omega -\vec{b}\, u \nabla v \, dx\right| + \left|\int_\Omega c u v \, dx\right| \leq \gamma \|u\|_V \|v\|_V$$

this is what it is desired to prove. look at the lhs :

$$|A| \left|\int_\Omega \nabla u \nabla v\right| + |\vec{b}| \left|\int_\Omega u \nabla v \, dx\right| + |c| \left|\int_\Omega u v \, dx\right|$$

now, using Schwarz on each single term:

$$\|A\| \, \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} + |b| \, \|u\|_{L^2} \|\nabla v\|_{L^2} + |c| \, \|u\|_{L^2} \|v\|_{L^2}$$

again, it is needed to look at each single term and try to bound norm on $L^2$ with $\|\cdot\|_V$.

Look:

$\|\nabla v\|_{L^2}$ $\rightarrow$ it is compared with $\left(\|\nabla v\|_{L^2} + \|v\|_{L^2}\right)^{\frac{1}{2}}$. This last one will be always greater than $\|\nabla v\|_{L^2}$

Same for $\|v\|_{L^2}$.

Bound each single term:

$$|A| \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} + |b| \|\nabla v\|_{L^2} \|u\|_{L^2} + |c| \|u\|_{L^2} \|v\|_{L^2} \leq$$

$$|A| \|u\|_V \|v\|_V + |b| \|v\|_V \|u\|_V + |c| \|u\|_V \|v\|_V = \left(|A| + |b| + |c|\right) \|u\|_V \|v\|_V$$

$(|A| + |b| + |c|) \|u\|_V \|v\|_V$ this shoul be bounded by $\gamma \|u\|_V \|v\|_V$ to prove continuity.

$$(|A| + |b| + |c|) \cancel{\|u\|_V} \cancel{\|v\|_V} \leq \gamma \cancel{\|u\|_V} \cancel{\|v\|_V}$$

Now, there exist a $\gamma \geq (|A| + |b| + |c|)$. Since $A(u,v) \leq A^*(u,v)$ and $A^*(u,v) \leq \gamma \|u\|_V \|v\|_V$

$$A(u,v) \leq \gamma \|u\|_V \|v\|_V$$

$A$ is continuous. (proved)

Exercise 12 (from L6):

TITLE: Show that CEA-LEMMA (with $A$ symmetric) under same requirements of LAX-MILGRAM is:

$$\|u-u_h\|_A = \min_{w_h \in V} \|u - w_h\|_A \quad \text{and} \quad \|u-u_h\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_h \in V} \left\{ \|u-u_h\|_V \right\}$$

TOOLS: ·) it is known that $A$ is continuous and coercive and $F$ is continuous.

·) By Lax-Milgram: $\exists! \, u \in V : A(u,v) = F(v)$

and $\|u\|_V \leq \frac{1}{\alpha_0} \|F(v)\|_V$

·) Cea-lemma: $\|u-u_h\| \leq \frac{\gamma}{\alpha_0} \inf_{v_h \in V} \|u - u_h\|_V$

PART 1: Demonstrate that $\|u-u_h\|_A = \min_{w_h \in V} \|u-w_h\|_A$

If $A$ is symmetric, we know from Galerkin's orthogonality that:

$$A(u-u_h, v_h) = 0 \qquad \forall v_h \in V_h$$

$e = u-u_h \longrightarrow e \perp V_h$ ($u_h$ is projection of $u$ into $V_h$)

Now, it could be defined the ENERGY NORM:

$$\|u\|_A = \sqrt{A(v,v)}$$

following that: $\|e\|_A = \sqrt{A(e,e)}$

$$\|e\|_A^2 = A(e,e) \iff \|u-u_h\|_A^2 = A(u-u_h, u-u_h) \quad \text{by bilinearity:}$$

$$= A(u-u_h, u) - \underbrace{A(u-u_h, u_h)}_{\to 0 \text{ by Galerkin' } \perp}$$

$$= A(u-u_h, u)$$

now, it is possible to add a term that is 0:

$$= A(u-u_h, u) - \underline{A(u-u_h, v_h)}$$

$$= A(u-u_h, u-v_h) \qquad \forall v_h \in V_h$$

Since $A$ is symmetric: (possible to define the norm)

$$\|u-u_h\|_A^{\cancel{2}} \leq A(u-u_h, u-v_h) \qquad \forall v_h \in V_h$$

$$\leq \cancel{\|u-u_h\|_A} \, \|u-v_h\|_A$$

Re-write as:

$$\|u-u_h\|_A \leq \|u-v_h\|_A$$

this implies:

$$\|u-u_h\|_A = \min_{v_h \in V_h} \|u - v_h\|_A \qquad \text{(proved)}$$

PART 2: Prove $\quad \|u-u_h\|_V \le \sqrt{\frac{\gamma}{\alpha_0}} \min_{w_h \in V_h} \left\{ \|u - w_h\|_V \right\}$

For coercitivity: $\quad A(v,v) \ge \alpha_0 \|u\|_V^2$

For continuity: $\quad A(u,v) \le \gamma \|u\|_V \|v\|_V$

Suppose to have:

$$A(u-u_h, u-u_h) \ge \alpha_0 \|u-u_h\|_V^2$$

$$A(u-u_h - u - v_h) \le \gamma \|u-u_h\|_V \|u-v_h\|_V$$

So, it is possible to notice that, it is true the following relationship:

$$A(u-u_h, u-u_h) \le A(u-u_h, u-v_h)$$

considering $u_h$ the best discrete solution by Galerkin's orthogonality.

Then, since $\alpha_0 \|u-u_h\|^2 \le A(v-v_h, u-u_h)$:

$$\alpha_0 \|u-u_h\|_V^2 \le A(u-u_h, u-v_h)$$

at this point, by continuity:

$$\alpha_0 \|u-u_h\|_V^2 \le A(u-u_h, u-u_h) \le A(u-u_h, u-v_h) \le A(u-v_h, u-v_h) \le \gamma \|u-v_h\|_V^2$$

it follows that:

$$\alpha_0 \|u-u_h\|_V^2 \le \gamma \|u-v_h\|_V^2$$

it is possible to re-write as:

$$\|u-u_h\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \; \|u-v_h\|_V$$

this can be bound by the lowest requirement:

$$\|u-u_h\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{w_h \in V_h} \left\{ \|u-w_h\|_V \right\}$$

(proved)

## Exercise B (from L11):

**TITLE: Show orderd of convergence of $\theta$-method**

TOOLS:

$$\cdot) \; T_i^{n+\frac{1}{2}} = \delta_{k,t}^{+} u(x_i, t_{n+\frac{1}{2}}) - \theta \left(\delta_h^{x}\right)^2 u(x_i, t_{n+1}) - (1-\theta)\left(\delta_h^{x}\right)^2 u(x_i, t_n)$$

First, it is needed some adjustment on time difference operator, since it would be nice to evaluate it at $(x_i, t_n)$: Apply Taylor:

$$\delta_{k,t}^{t} u(x_i, t_{n+\frac{1}{2}}) \approx \delta_k^{t} \left( u(x_i,t_n) + \frac{k}{2}(u_t)_i^n + \frac{k^2}{4}(u_{tt})_i^n \right)$$

stopping at second order and group everything in $O(k^2)$:

$$\delta_{k,t}^{t} \left( u(x_i,t_n) + \frac{k}{2}(u_t)_i^n + O(k^2) \right)$$

Recall that: $\delta^t_{k,+} u(x_i, t_n) = \dfrac{u(x_i, t_{n+1}) - u(x_i, t_n)}{k}$

this could be expanded as (using above Taylor's expansion):

$$\frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{k} = \frac{1}{k}\Bigg( \left(u(x_i,t_{n+1}) + \frac{k}{2}(u_t)_i^{n+1} + O(k^2)\right) - \left(u(x_i,t_n) + \frac{k}{2}(u_t)_i^n + O(k^2)\right)\Bigg) =$$

now evaluate at $(x_i, t_n)$, so again Taylor's expansion:

$$= \frac{1}{k}\Bigg( \cancel{u(x_i,t_n)} + k(u_t)_i^n + \cancel{O(k^2)} + \cancel{\frac{k}{2}(u_t)_i^n} + \frac{k^2}{2}(u_{tt})_i^n + O(k^3) - \cancel{u(x_i,t_n)} - \cancel{\frac{k}{2}(u_t)_i^n} - \cancel{O(k^2)}\Bigg)$$

$$= \frac{1}{k}\left( k(u_t)_i^n + \frac{k^2}{2}(u_{tt})_i^n + O(k^3)\right) = (u_t)_i^n + \frac{k}{2}(u_{tt})_i^n + O(k^2)$$

The term $(1-\theta)$ is evaluated at $(x_i, t_n)$, so the only term to adjust is the $\theta$ term: (recall that from previous exercise $(\delta^x_h)^2 u(x_i, t_n) = (u_{xx})_i^n + O(h^2)$

$$(\delta^x_h)^2 u(x_i, t_{n+1}) \approx (u_{xx})_i^{n+1} + O(h^2) = \quad \text{by Taylor}$$

$$= (u_{xx})_i^n + k\left((u_{xx})_t\right)_i^n + O(h^2)$$

Now, let's re-write the scheme with the expansions:

$$\delta_{k,t}^{t}\, u\left(x_i, t_{n+\frac{1}{2}}\right) - \theta\left(\delta_h^x\right)^2 u(x_i, t_{n+1}) - (1-\theta)\left(\delta_h^x\right)^2 u(x_i, t_n) =$$

$$= (u_t)_i^n + \frac{k}{2}(u_{tt})_i^n + O(k^2) - \theta(u_{xx})_i^n$$

$$- \theta k\left((u_{xx})_t\right)_i^n - \theta\, O(h^2) - (1-\theta)(u_{xx})_i^n$$

$$- (1-\theta)\, O(h^2)$$

now group terms $O(h^2)$:

$$(u_t)_i^n + \frac{k}{2}(u_{tt})_i^n - \cancel{\theta(u_{xx})_i^n} - \theta k\left((u_{xx})_t\right)_i^n - (u_{xx})_i^n$$

$$+ \cancel{\theta(u_{xx})_i^n} + O(h^2) + O(k^2)$$

Now, define the truncation error: ($a=1$ for simplicity)

$$T_i^n = \cancel{(u_t)_i^n} + \frac{k}{2}(u_{tt})_i^n - \theta k\left((u_{xx})_t\right)_i^n - \cancel{(u_{xx})_i^n} + O(h^2) + O(k^2)$$

$$\underbrace{-\cancel{(u_t)_i^n} + \cancel{(u_{xx})_i^n}}_{\to \text{ the problem equation}}$$

$$T_i^n = \frac{k}{2}(u_{tt})_i^n - \theta k\left((u_{xx})_t\right)_i^n + O(h^2) + O(k^2)$$

Now, if $\theta \neq \frac{1}{2}$, the truncation error (not bound at the moment) is linear in $k$, so $O(k, h^2)$ (proved)

Otherwise, if $\theta = \frac{1}{2}$ (Crank-Nicholson), then terms in $k$ can be re-written as:

$$\frac{k}{2}\left((u_{tt})_i^n - ((u_{xx})_t)_i^n\right) = \frac{k}{2}\left((u_t)_{t\,i}^{\,n} - (u_{xx})_{t\,i}^{\,n}\right) =$$

$$= \frac{k}{2}\left(\left(\underbrace{(u_t) - (u_{xx})}\right)_t\right)_i^n = 0$$

this is the problem's equation

$$u_t - u_{xx} = 0$$

The $T_i^n$ is $O(k^2, h^2)$. (proved)

Exercise 1h (from 2 11):

TITLE: Given $\tilde{\mu}(1-\theta) \leq \frac{1}{2}$ show $\theta$-method is STABLE in $\ell_\infty$.

The scheme of $\theta$-method is:

$$-\tilde{\mu}\theta U_{i+1}^{n+1} + (1+2\theta\tilde{\mu})U_i^{n+1} - \tilde{\mu}\theta U_{i-1}^{n+1} = \tilde{\mu}(1-\theta)U_{i+1}^n - \left(1 - 2\tilde{\mu}(1-\theta)\right)U_i^n + \tilde{\mu}(1-\theta)U_{i-1}^n$$

now, to prove stability:

$$\|U^n\|_\infty \leq \|U^0\|_\infty$$

Let's start by bounding RHS of scheme: by Triang. Inequality

$$-\tilde{\mu}(1-\theta)U_{i+1}^{n} + \left(1-2\tilde{\mu}(1-\theta)\right)U_i^n - \tilde{\mu}(1-\theta)U_{i-1}^{n} \leq$$

$$\left|-\tilde{\mu}(1-\theta)\right|\left|U_{i+1}^{n}\right| + \left|1-2\tilde{\mu}(1-\theta)\right|\left|U_i^n\right| - \left|\tilde{\mu}(1-\theta)\right|\left|U_{i-1}^{n}\right|$$

·) now $\tilde{\mu}$ is Courant Number, so $>0$.

·) $\theta > 0$ by Scheme definition.

·) Given $\tilde{\mu}\left(1-\theta\right) \leq \frac{1}{2} \longrightarrow 1-2\tilde{\mu}(1-\theta) > 0$

So,

$$\ldots \leq \tilde{\mu}(1-\theta)\left|U_{i+1}^{n}\right| + \left(1-2\tilde{\mu}(1-\theta)\right)\left|U_i^n\right| + \tilde{\mu}(1-\theta)\left|U_{i-1}^{n}\right|$$

So, define $U_{max}^n$, for which it is the maximum $\forall i$, given time n.
It is possible to bound the LHS by this:

$$\ldots \leq \tilde{\mu}(1-\theta)\left|U_{max}^{n}\right| + \left(1-2\tilde{\mu}(1-\theta)\right)\left|U_{max}^n\right| + \tilde{\mu}(1-\theta)\left|U_{max}^{n}\right|$$

$$\leq \left(\cancel{\tilde{\mu}} - \cancel{\tilde{\mu}\theta} + 1 - \cancel{2\tilde{\mu}} + \cancel{2\tilde{\mu}\theta} + \cancel{\tilde{\mu}} - \cancel{\tilde{\mu}\theta}\right)\left|U_{max}^n\right|$$

$\leq \left|U_{max}^n\right|$ this can be bound by the sup. So,

$$\leq \|U^n\|_\infty$$

Now, focus on LHS:

$$-\tilde{\mu}\theta\, U_{i+1}^{n+1} + \left(1+2\tilde{\mu}\theta\right)U_i^{n+1} - \tilde{\mu}\theta\, U_{i-1}^{n+1}$$

This can be bound by abs operator

$$-\tilde{\mu}\theta U_{i+1}^{n+1} + \left(1+2\tilde{\mu}\theta\right) U_i^{n+1} - \tilde{\mu}\theta U_{i-1}^{n+1} \le -\tilde{\mu}\theta \left|U_{i+1}^{n+1}\right| + \left(1+2\tilde{\mu}\theta\right)\left|U_i^{n+1}\right| - \tilde{\mu}\theta\left|U_{i-1}^{n+1}\right|$$

$$\dots \le -\tilde{\mu}\theta \left|U_{i+1}^{n+1}\right| + (1+2\tilde{\mu}\theta)\left|U_i^{n+1}\right| - \tilde{\mu}\theta\left|U_{i-1}^{n+1}\right|$$ now, bound by $\left|U_{max}^{n+1}\right|$

$$\dots \le \cancel{-\tilde{\mu}\theta\left|U_{max}^{n+1}\right|} + 1\left|U_{max}^{n+1}\right| + \cancel{2\tilde{\mu}\theta\left|U_{max}^{n+1}\right|} \cancel{-\tilde{\mu}\theta\left|U_{max}^{n+1}\right|} \le \left|U_{max}^{n+1}\right|$$

which can be bound by sup.

$$\left|U_{max}^{n+1}\right| \le \|U^{n+1}\|_\infty$$

Looking overall

$$\|U^{n+1}\|_\infty \le \|U^n\|_\infty \xrightarrow{\text{recursive}} \|U^{n+1}\|_\infty \le \|U^0\|_\infty$$

(proved)

Exercise is (from L 11):

TITLE: Show for $\theta$-method that defining the mode $U_i^n = (\lambda(J))^n e^{iJih}$ it is possible to get: (STABILITY ANALYSIS):

$$\lambda(J) = \frac{1-4\tilde{\mu}(1-\theta)\sin^2\left(\frac{Jh}{2}\right)}{1+4\tilde{\mu}\theta\sin^2\left(\frac{Jh}{2}\right)}$$

Let's apply a common analysis's trick for Von Neumann's stability analysis:

Define $U_i^0 = e^{iJih}$ as the starting step. Plug-in the scheme:

$$-\tilde{\mu}\theta U_{i+1}^{n+1} + (1+2\tilde{\mu}\theta)U_i^{n+1} - \tilde{\mu}\theta U_{i-1}^{n+1} = -\tilde{\mu}(1-\theta)U_{i+1}^n + \left(1-2\tilde{\mu}(1-\theta)\right)U_i^n - \tilde{\mu}(1-\theta)U_{i-1}^n$$

Of course, it is needed $U_i^1 = \lambda e^{iJih}$

Look LHS: $n=0$

$$-\tilde{\mu}^2\theta\lambda e^{iJ(i+1)h}+(1+2\tilde{\mu}^2\theta)\lambda e^{iJih}-\tilde{\mu}^2\theta\lambda e^{iJ(i-1)h}=$$

$$=\lambda e^{iJih}\left(-\tilde{\mu}^2\theta e^{iJh}+1+2\tilde{\mu}^2\theta-\theta\tilde{\mu}^2 e^{-iJh}\right)=$$

$$=\lambda e^{iJih}\left(1+\tilde{\mu}^2\theta\left(2-e^{iJh}-e^{-iJh}\right)\right)=$$

$$=\lambda e^{iJih}\left(1-\tilde{\mu}^2\theta\left(-2+e^{iJh}+e^{-iJh}\right)\right)=$$

now, recall $\dfrac{e^{iJh}+e^{-iJh}-2}{(2i)^2}=\dfrac{\left(e^{iJ\frac{h}{2}}-e^{-iJ\frac{h}{2}}\right)^2}{(2i)^2}=\sin^2\left(\frac{Jh}{2}\right)$

so, it is needed to : $\dfrac{(2i)^2}{(2i)^2}\tilde{\mu}^2\theta\left(-2e^{iJh}+e^{-iJh}\right)$

$\rightarrow =1$

$$=\lambda e^{iJih}\left(1-(2i)^2\tilde{\mu}^2\theta\sin^2\left(\frac{Jh}{2}\right)\right)=\lambda e^{iJih}\left(1+4\tilde{\mu}^2\theta\sin^2\left(\frac{Jh}{2}\right)\right)$$

Now, look the RHS: $n=0$

$$+\tilde{\mu}^2(1-\theta)U_{i+1}^n+\left(1-2\tilde{\mu}^2(1-\theta)\right)U_i^n+\tilde{\mu}^2(1-\theta)U_{i-1}^n=$$

$$=+\tilde{\mu}^2(1-\theta)e^{iJ(i+1)h}+\left(1-2\tilde{\mu}^2(1-\theta)\right)e^{iJih}+\tilde{\mu}^2(1-\theta)e^{iJ(i-1)h}=$$

$$=e^{iJih}\left(+\tilde{\mu}^2(1-\theta)e^{iJh}+\left(1-2\tilde{\mu}^2(1-\theta)\right)+\tilde{\mu}^2(1-\theta)e^{-iJh}\right)=$$

$$=e^{iJih}\left(1+\tilde{\mu}^2(1-\theta)\left(e^{iJh}-2+e^{-iJh}\right)\right)$$

using same equation and $\frac{(2i)^2}{(2i)^2}$ as before

$$=e^{iJih}\left(1+(2i)^2\tilde{\mu}^2(1-\theta)\sin^2\left(\frac{Jh}{2}\right)\right)=$$

$$=e^{iJih}\left(1-4\tilde{\mu}^2(1-\theta)\sin^2\left(\frac{Jh}{2}\right)\right)$$

Now, comparing LHS and RHS:

$$\lambda \cancel{e^{iJjh}} \left(1 + h\mu^2\theta \sin^2\left(\frac{Jh}{2}\right)\right) = \cancel{e^{iJjh}} \left(1 - h\mu^2(1-\theta) \sin^2\left(\frac{Jh}{2}\right)\right)$$

$$\lambda = \frac{1 - h\mu^2(1-\theta)\sin^2\left(\frac{Jh}{2}\right)}{1 + h\mu^2\theta \sin^2\left(\frac{Jh}{2}\right)} \qquad \text{(proved)}$$

Exercise 16 (from L 14):

TITLE: Show that $u$ and $v$ satisfy $u_{tt} - u_{xx} = 0$, even when the wave equation is written as:

$$\begin{cases} u_x + v_t = 0 \\ u_t + v_x = 0 \end{cases}$$

Suppose $u$ is continuous.

Let's start by the equation: $u_x + v_t = 0$ and derive in space.

$$v_{tx} + u_{xx} = 0 \qquad (1)$$

Now, derive on time the equation $u_t + v_x = 0$:

$$u_{tt} + v_{xt} = 0 \qquad (2)$$

Since both equations (1 and 2) are equal to 0, if the difference is done between 1 and 2 it will be always 0. So,

$$v_{tx} + u_{xx} - u_{tt} - v_{xt} = 0 \qquad \text{by Schwartz's theorem}$$

$$\cancel{v_{xt}} + u_{xx} - u_{tt} - \cancel{v_{xt}} = 0 \qquad \text{changing the sign}$$

$$u_{tt} - u_{xx} = 0 \qquad \text{(proved)}$$

Exercise 17 ( from L 14):

TITLE: Show that $\frac{1}{2}\int_{\mathbb{R}}\left((u_t)^2 + (u_x)^2\right)dx$ is conserved in wave equation $u_{tt} - u_{xx} = 0$

Hint: test for $v = u_t$

Given the wave equation, it is tested with $v = u_t$

$$u_{tt} \cdot v - u_{xx} \cdot v = 0$$ integrate on domain $\mathbb{R}$

$$\int_{\mathbb{R}} u_{tt} \cdot u_t \cdot dx - \int_{\mathbb{R}} u_{xx} \cdot u_t \, dx = 0$$

Consider firstly $\int_{\mathbb{R}} u_{tt} \cdot u_t \, dx$. This integral is this one:

$$\int_{\mathbb{R}} u_{tt} \cdot u_t \cdot dx = \int_{\mathbb{R}} \frac{1}{2} \frac{d}{dt} (u_t)^2 \, dx \qquad (1)$$

Then, consider $\int_{\mathbb{R}} u_{xx} \cdot u_t \, dx$. Defying $f' = u_{xx}$ and $g = u_t$, integrate by parts:

$$\int_{\mathbb{R}} f' g \, dx = fg \Big|_{\mathbb{R}} - \int_{\mathbb{R}} f g' \, dx$$ meaning:

$$\int_{\mathbb{R}} u_{xx} \cdot u_t \, dx = u_x \cdot u_t \Big|_{\mathbb{R}} - \int_{\mathbb{R}} u_x \cdot u_{t_x} \, dx$$

Since $u \in L^2$ it means that $u_x \Big|_{-\infty}^{+\infty} \longrightarrow 0$, so:

$$\int_{\mathbb{R}} u_{xx} \cdot u_t \, dx = - \int_{\mathbb{R}} u_x \cdot u_{tx} \, dx = - \int_{\mathbb{R}} \frac{1}{2} \frac{d}{dt} (u_x)^2 \, dx \qquad (2)$$

Merging the two integrals: (1 and 2)

$$\int_{\mathbb{R}} \frac{1}{2} \frac{d}{dt} (u_t)^2 \, dx + \int_{\mathbb{R}} \frac{1}{2} \frac{d}{dt} (u_x)^2 dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left((u_t)^2 + (u_x)^2\right) dx = 0$$

$$\frac{1}{2} \int_{\mathbb{R}} \left((u_t)^2 + (u_x)^2\right) dx = 0 \qquad \text{(proved)}$$

Exercise 18 ( from L 15 ):

TITLE: Show LW method has consistency

$$|T_i| \le \frac{K^2}{2} M_{ttt} + |a| \frac{h^2}{2} M_{xxx} \quad \text{in problem}$$

$$\begin{cases} u_t + a u_x = 0 \\ u(0, x) = u_0(x) \\ u(t, x_L) = u_L \end{cases}$$

Define the LW scheme:

$$\frac{U_i^{n+1} - U_i^n}{K} + a_i^n \frac{U_{i+1}^n - U_{i-1}^n}{2h} - \frac{K(a_i^n)^2}{2} \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

$a = a_i^n$ for simplicity: Re-write as:

$$\frac{U_i^{n+1}}{K} - \frac{U_i^n}{K} + \frac{a}{2h} U_{i+1}^n - \frac{a}{2h} U_{i-1}^n - \frac{K a^2}{2h^2} U_{i+1}^n + \frac{K a^2}{h^2} U_i^n - \frac{K a^2}{2h^2} U_{i-1}^n = 0$$

remember Courant's number: $r = \frac{k}{h}$. The multiply by $K$.

$$U_i^{n+1} - U_i^n + \frac{ak}{2h} U_{i+1}^n - \frac{ak}{2h} U_{i-1}^n - \frac{K^2 a^2}{2h^2} U_{i+1}^n + \frac{K^2 a^2}{h^2} U_i^n - \frac{K^2 a^2}{2h^2} U_{i-1}^n = 0$$

$$U_i^{n+1} - U_i^n + \frac{ar}{2} U_{i+1}^n - \frac{ar}{2} U_{i-1}^n - \frac{a^2 r^2}{2} U_{i+1}^n + r^2 a^2 U_i^n - \frac{r^2 a^2}{2} U_{i-1}^n = 0$$

$$U_i^{n+1} = U_i^n - \frac{ar}{2} U_{i+1}^n + \frac{ar}{2} U_{i-1}^n + \frac{a^2 r^2}{2} U_{i+1}^n - r^2 a^2 U_i^n + \frac{r^2 a^2}{2} U_{i-1}^n$$

$$U_i^{n+1} = U_{i+1}^n \left( \frac{a^2 r^2}{2} - \frac{ar}{2} \right) + U_i^n \left( 1 - a^2 r^2 \right) + U_{i-1}^n \left( \frac{ar}{2} + \frac{a^2 r^2}{2} \right)$$

$$= \frac{ar}{2} U_{i+1}^n \left( ar - 1 \right) + U_i^n \left( 1 - a^2 r^2 \right) + \frac{ar}{2} U_{i-1}^n \left( 1 + ar \right)$$

At this point, it is needed Taylor's expansion until $O(h^4)$ and $O(K^4)$, since it is required $3^{rd}$ derivative on $\Omega$ and $I$.

$\downarrow$ space. $\hookrightarrow$ time.

Prepare the expansions as follows:

$$U_i^{n+1} = U_i^n + k D_t U_i^n + \frac{k^2}{2} D_{tt} U_i^n + \frac{k^3}{6} D_{ttt} U_i^n + O(k^4)$$

$$U_{i+1}^n = U_i^n + h D_x U_i^n + \frac{h^2}{2} D_{xx} U_i^n + \frac{h^3}{6} D_{xxx} U_i^n + O(h^4)$$

$$U_{i-1}^n = U_i^n - h D_x U_i^n + \frac{h^2}{2} D_{xx} U_i^n - \frac{h^3}{6} D_{xxx} U_i^n + O(h^4)$$

Plug in into the scheme:

$$U_i^n + k D_t U_i^n + \frac{k^2}{2} D_{tt} U_i^n + \frac{k^3}{6} D_{ttt} U_i^n + O(k^4) = \frac{ar}{2}(ar-1)\Big\{ U_i^n + h D_x U_i^n + \frac{h^2}{2} D_{xx} U_i^n + \frac{h^3}{6} D_{xxx} U_i^n + O(h^4) \Big\} + U_i^n (1 - a^2 r^2) + \frac{ar}{2}(ar+1) \Big\{ U_i^n - h D_x U_i^n + \frac{h^2}{2} D_{xx} U_i^n - \frac{h^3}{6} D_{xxx} U_i^n + O(h^4) \Big\}$$

Analyze by elements (excluding derivative on time)

$$U_i^n \left( \cancel{1} - \cancel{\frac{a^2 r^2}{2}} + \cancel{\frac{ar}{2}} - \cancel{1} + \cancel{a^2 r^2} - \cancel{\frac{a^2 r^2}{2}} - \cancel{\frac{ar}{2}} \right) = 0$$

$$D_x U_i^n \left( \cancel{h \frac{a^2 r^2}{2}} - h \frac{ar}{2} - \cancel{h \frac{a^2 r^2}{2}} - h \frac{ar}{2} \right) = -h a r \, D_x U_i^n$$

$$D_{xx} U_i^n \left( \frac{a^2 r^2}{2} \frac{h^2}{2} - \cancel{\frac{ar}{2} \frac{h^2}{2}} + \frac{a^2 r^2}{2} \frac{h^2}{2} + \cancel{\frac{ar}{2} \frac{h^2}{2}} \right) = \frac{a^2 r^2 h^2}{2} D_{xx} U_i^n$$

$$D_{xxx} U_i^n \left( \cancel{\frac{a^2 r^2}{2} \frac{h^3}{6}} - \frac{ar}{2} \frac{h^3}{6} - \cancel{\frac{a^2 r^2}{2} \frac{h^3}{6}} - \frac{ar}{2} \frac{h^3}{6} \right) = - \frac{a r h^3}{6} D_{xxx} U_i^n$$

The scheme will become:

$$k D_t U_i^n + \frac{k^2}{2} D_{tt} U_i^n + \frac{k^3}{6} D_{ttt} U_i^n + O(k^4) = - h a r \, D_x U_i^n + \frac{a^2 r^2 h^2}{2} D_{xx} U_i^n - \frac{a r h^3}{6} D_{xxx} U_i^n$$

Divide by $k$ and explicit $r = \frac{k}{h}$

$$D_t U_i^n + \frac{k}{2} D_{tt} U_i^n + \frac{k^2}{6} D_{ttt} U_i^n = - \cancel{h} \, a \frac{1}{\cancel{h}} D_x U_i^n + \frac{a^2}{2} \frac{k}{\cancel{h^2}} \cancel{h^2} D_{xx} U_i^n - \frac{a h^{\cancel{3}\,2}}{6 \cancel{h}} D_{xxx} U_i^n$$

$$\cancel{D_t U_i^4} + \frac{k}{2} D_{tt} U_i^4 + \frac{k^2}{6} D_{ttt} U_i^4 + \cancel{a D_x U_i^4} - \frac{a^2 k}{2} D_{xx} U_i^4 + \frac{a h^2}{6} D_{xxx} U_i^4$$

This is the discrete operator expansion. (Removed for simplicity $O(h^4)$ and $O(k^4)$). Define the truncation error: to do it, it has to be remove problem eqution $\cancel{D_t U_i^4} + a \cancel{D_x U_i^4} = 0$

$$T_i = \frac{k}{2} D_{tt} U_i^4 + \frac{k^2}{6} D_{ttt} U_i^4 - \frac{a^2 k}{2} D_{xx} U_i^4 + \frac{a h^2}{6} D_{xxx} U_i^4$$

It is required, to prove the statement that:

$$\frac{k}{2} D_{tt} U_i^4 - \frac{a^2 k}{2} D_{xx} U_i^4 = 0$$

$$\frac{k}{2}\left( D_{tt} U_i^4 - a^2 D_{xx} U_i^4 \right) = 0$$

this is similar to

$u_t + a u_x = 0$

Now, work on $u_t + a u_x = 0$.

↳ Take derivative on time: $u_{tt} + a u_{xt} = 0$ (1)

↳ Take derivative on space: $u_{tx} + a u_{xx} = 0$ (2)

It follows that from (2) $u_{tx} = -a u_{xx}$. By Schwartz's theorem:

$$(1) \quad u_{tt} + a u_{tx} \overset{(2)}{=} u_{tt} + a(-a u_{xx}) = u_{tt} - a u_{xx} = 0$$

It follows that:

$$\frac{k}{2}\left( D_{tt} U_i^4 - a^2 D_{xx} U_i^4 \right) = 0$$

So, $T_i = \frac{k^2}{6} D_{ttt} U_i^n + a \frac{h^2}{6} D_{xxx} U_i^n$. Bounding:

$$|T_i| \le \frac{k^2}{6} D_{ttt}^{max} U_i^n + |a| \frac{h^2}{6} D_{xxx}^{max} U_i^n$$

$D_{ttt}^{max} U_i^n = M_{ttt}$

$D_{ttt}^{max} U_i^n = M_{xxx}$

$$|T_i| \le \frac{k}{6} M_{ttt} + |a| \frac{h^2}{6} M_{xxx} \quad \text{(proved)}$$

By divind 3, it is obtain the requirement.

## Exercise 19 (from L15):

### PART 1

TITLE: write wave equation in system of 1$^{st}$ order PDE

$u_{tt} - u_{xx} = 0$, now add and remove $v_{xt} - v_{xt} = 0$

$u_{tt} - v_{xt} - u_{xx} + v_{xt} = 0$ By Schwartz's theorem:

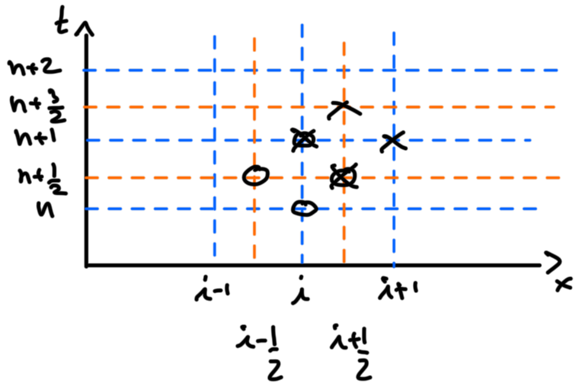
$u_{tt} - v_{tx} - u_{xx} + v_{xt} = 0$

$(u_{tt} + v_{xt}) - (u_{xx} + v_{tx}) = 0$ the difference between those terms has to be 0, meaning:

$$\begin{cases} u_{tt} + v_{xt} = 0 & \text{integrate on time} \rightarrow \\ u_{xx} + v_{tx} = 0 & \text{integrate on space} \rightarrow \end{cases} \quad \begin{cases} u_t + v_x = 0 \\ u_x + v_t = 0 \end{cases} \quad \text{(done)}$$

### PART 2:

TITLE: Discretize by LF scheme




Hint: Use staggered grids

Hint: $v$ is shifted by $\frac{1}{2}$ respect to $U$ (ex $v_{i+\frac{1}{2}}^{n+\frac{1}{2}}$)

CD on both space and time

$O \rightarrow U$ $\quad$ $x \rightarrow V$

Let's apply CD on $u_x + v_t = 0$ :

$$(1)\quad \frac{U^n_{i+\frac{1}{2}} - U^n_{i-\frac{1}{2}}}{h} + \frac{V^{n+1}_{i+\frac{1}{2}} - V^n_{i+\frac{1}{2}}}{k} = 0$$

Now, CD on $u_t + v_x = 0$ ;

$$(2)\quad \frac{U_i^{n+\frac{1}{2}} - U_i^{n-\frac{1}{2}}}{k} + \frac{V_{i+1}^{n+\frac{1}{2}} - V_i^{n+\frac{1}{2}}}{h} = 0$$

(done)

PART 3:

TITLE: Derive the scheme for Leap Frog (LF)

Since it is needed to evaluate at nodes and not in middle point it is necessary to transform those points into node:

4 operations needed:

$$\cdot)\ U^{\cdot}_{i+\frac{1}{2}} = \frac{U^{\cdot}_{i+1} - U^{\cdot}_i}{h} \qquad \cdot)\ U^{\cdot}_{i-\frac{1}{2}} = \frac{U^{\cdot}_i - U^{\cdot}_{i-1}}{h}$$

$$\cdot)\ U_{\cdot}^{n+\frac{1}{2}} = \frac{U_{\cdot}^{n+1} - U_{\cdot}^n}{k} \qquad \cdot)\ U_{\cdot}^{n-\frac{1}{2}} = \frac{U_{\cdot}^n - U_{\cdot}^{n-1}}{k}$$

Let's plug-in into two discretize equations:

$$(1)\quad \frac{U^n_{i+\frac{1}{2}} - U^n_{i-\frac{1}{2}}}{h} + \frac{V^{n+1}_{i+\frac{1}{2}} - V^n_{i+\frac{1}{2}}}{k} = 0 \iff \frac{1}{h}\left(\frac{(U^n_{i+1} - U^n_i)}{h} - \frac{(U^n_i - U^n_{i-1})}{h}\right) + \frac{1}{k}\left(\frac{(V^{n+1}_{i+1} - V^{n+1}_i)}{h} - \frac{(V^n_{i+1} - V^n_i)}{h}\right) = 0$$

$$(2)\quad \frac{U_i^{n+\frac{1}{2}} - U_i^{n-\frac{1}{2}}}{k} + \frac{V_{i+1}^{n+\frac{1}{2}} - V_i^{n+\frac{1}{2}}}{h} = 0 \iff \frac{1}{k}\left(\frac{(U_i^{n+1} - U_i^n)}{k} - \frac{(U_i^n - U_i^{n-1})}{k}\right) + \frac{1}{h}\left(\frac{(V_{i+1}^{n+1} - V_{i+1}^n)}{k} - \frac{(V_i^{n+1} - V_i^n)}{k}\right) = 0$$

Since eq. 1 and 2 are equal to 0, subtraction is also 0, so:

$$\frac{1}{h^2}\left(U^n_{i+1} - U^n_i - U^n_i + U^n_{i-1}\right) + \frac{1}{\cancel{kh}}\left(\cancel{V^{n+1}_{i+1}} - \cancel{V^{n+1}_i} - \cancel{V^n_{i+1}} + \cancel{V^n_i}\right)$$

$$-\frac{1}{k^2}\left(U^{n+1}_i - U^n_i - U^n_i + U^{n-1}_i\right) - \frac{1}{\cancel{kh}}\left(\cancel{V^{n+1}_{i+1}} - \cancel{V^n_{i+1}} - \cancel{V^{n+1}_i} + \cancel{V^n_i}\right) = 0$$

Now, changing the sign:

$$\frac{U^{n+1}_i - 2U^n_i + U^{n-1}_i}{k^2} - \frac{U^n_{i+1} - 2U^n_i + U^n_{i-1}}{h^2} = 0$$

$$\left(\delta^t_k\right)^2 u(x_i, t_n) - \left(\delta^x_h\right)^2 u(x_i, t_n) = 0 \qquad (\text{proved})$$