Analysis of Bazykin-Berezovskaya model

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1 Introduction

The aim of this analysis is the in-depth study of Bazykin-Berezovskaya population model. Two version of this model will be presented: the deterministic and the stochastic one. The model is presented in the paper "*Noise-induced extinction in Bazykin-Berezovskaya population model*" of Bashkirtseva I. and Ryashko L., Eur. Phys. J.B (2016).

The model presented is a nonlinear prey-predator model which exhibits very interesting behaviours, like local "*Andronov-Hopf*" bifurcation which forms a limited cycle or weak forms of coexistence between prey and predator. In this analysis, some of the most interesting analysis presented in the paper have been reproduced, and in certain cases extended.

This report is divided in four sections: the first will introduce the deterministic model, the second will extend the analysis to the stochastic one. The third is about a specific analysis on coexistence, a topic to which I am interested and the last one is the conclusion.

2 Deterministic model

The model is defined as follows:

$$\frac{dx}{dt} = rx(x-l)(k-x) - xy \tag{1}$$

that describes the evolution over time of the prey's density in the environment while

$$\frac{dy}{dt} = y(x-m) \tag{2}$$

describes the predator's density. x and y are the proportion of prey and predators in a closed environment. r is the intrinsic growth, l is the prey survival threshold (Allee effect), k is the carrying capacity of the system and m is the mortality of the predators.

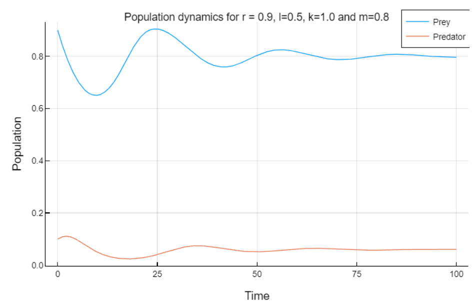

Figure 1: An example of deterministic trajectories of prey and predator over time, given a specific parameters' configuration.

It is important to define the parameters and variables' domain and constraints:

- $r, l, k, m > 0$
- $l < k$
- $x + y \leq 1$
- $x \in [0; 1]$ and $y \in [0; 1]$

Let's begin the analysis of the deterministic model: it is important to understand if there are local or global attractors and to investigate if they are stable or not.// From (2) it is possible to see that for both $y = 0$ and $m = x$, the $\frac{dy}{dt}$ goes to 0. So, setting $y = 0$ in $rx(x-l)(k-x) - xy = 0$, this equation is respected for $x = 0$, $x = l$ and $x = k$, while setting $x = m$ in $rx(x-l)(k-x) - xy = 0$ the solution is $y = r(m-l)(k-m)$. To sum up four solutions have been found out:

- $X_{e1} = (0; 0)$
- $X_{e2} = (l; 0)$
- $X_{e3} = (k; 0)$
- $X_{e4} = (m; r(m-l)(k-m))$

Now, it's time to find out if they are unstable or not, and in the case of stability if they are local or global. Let's define the Jacobian matrix as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} r(-3x^2 + 2x(l+k) - lk) - y & -x \\ y & x - m \end{bmatrix}$$

2.1 Solution X_{e1}

For $X_{e1} = (0; 0)$ we obtain:

$$J(X_{e1}) = \begin{bmatrix} -rlk & 0 \\ 0 & -m \end{bmatrix}$$

from which it is possible to obtain the char. eq.

$$det(J - \lambda\ I) = (rlk + \lambda\ )(m + \lambda\ )$$

so, the eigenvalues are:

$$\begin{cases} \lambda & = -rlk \\ \lambda & = -m \end{cases} \tag{3}$$

From (3) it is possible to notice that both the eigenvalues are always negatives since r,l,k and m are always greater than 0. This means that the $X_{e1} = (0; 0)$ is a stable attractor.

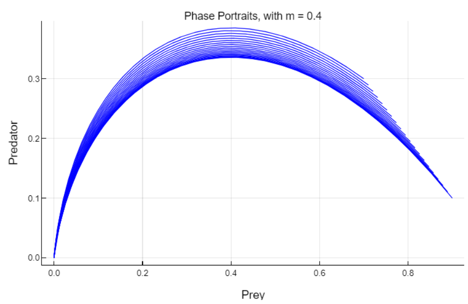

Figure 2: Phase Portraits: $X_{e1} = (0; 0)$

It has been set $r = k = 1$, $l = 0.5$ and $m = 0.4$. As presented in the qualitative analysis of the deterministic system, for these parameters there is only one attractor, also global: $XY_e = (0; 0)$.

2.2 Solution X_{e2}

For $X_{e2} = (l; 0)$ we obtain:

$$J(X_{e2}) = \begin{bmatrix} rl(-l + k) & -l \\ 0 & l - m \end{bmatrix}$$

from which it is possible to obtain the char. eq.

$$det(J - \lambda\ I) = (rl(-l + k) - \lambda\ )(l - m - \lambda\ )$$

so, the eigenvalues are:

$$\begin{cases} \lambda\ = l - m \\ \lambda\ = rl(-l + k) \end{cases} \tag{4}$$

From (4) it is possible to see that $\lambda\ = rl(-l + k) \geq 0$ since by construction $l < k$. This means that $X_{e2} = (l; 0)$ is unstable.

2.3 Solution X_{e3}

For $X_{e3} = (k; 0)$ we obtain:

$$J(X_{e3}) = \begin{bmatrix} rk(-k + l) & -k \\ 0 & k - m \end{bmatrix}$$

from which it is possible to obtain the char. eq.

$$det(J - \lambda\ I) = (rk(-k + l) - \lambda\ )(k - m - \lambda\ )$$

so, the eigenvalues are:

$$\begin{cases} \lambda\ = k - m \\ \lambda\ = rk(-l + k) \end{cases} \tag{5}$$

From (5) it is possible to see that $\lambda\ = rk(-k + l) < 0$ since $k > l$ and $\lambda\ = k - m < 0$ if $k < m$. In conclusion, $X_{e3} = (k; 0)$ is a stable attractor for $m \geq k$.

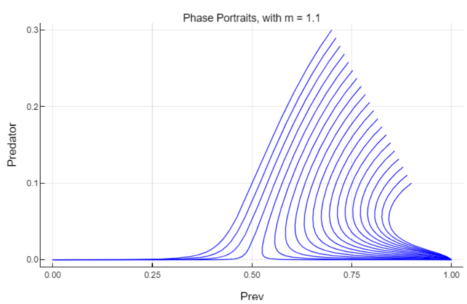

Figure 3: Phase Portraits: $X_{e3} = (k; 0)$

It has been set $r = k = 1$, $l = 0.5$ and $m = 1.1$. As presented in the qualitative analysis of the deterministic system, for these parameters there are two attractors : $XY_e = (0; 0)$ and $X_{e3} = (k; 0)$. In Fig. 3, the x-axis has been truncated to 1.0.

2.4 Solution X_{e4}

For $X_{e4} = (m; r(m-l)(k-m))$ we obtain, assuming that $m \neq l$ and $m \neq k$:

$$J(X_{e4}) = \begin{bmatrix} r(-2m^2 + ml + mk) & -m \\ r(m-l)(k-m) & 0 \end{bmatrix}$$

from which it is possible to obtain the char. eq.

$$det(J - \lambda\ I) = (r(-2m^2 + ml + mk) - \lambda\ )(-\lambda\ ) - (m)(r(m-l)(k-m))$$

so, the eigenvalues are:

$$\begin{cases} \lambda &= \frac{rm(-2m+l+k)+\sqrt{\Delta}}{2} \\ \lambda &= \frac{rm(-2m+l+k)-\sqrt{\Delta}}{2} \end{cases} \tag{6}$$

with $\Delta = r^2(-2m^2 + ml + mk)^2 - 4rm(m-l)(k-m)$.

From (6), to determine if X_{e4} is stable or not, it is necessary to study the real part of the eigenvalues: so, it is studied $\mathbb{R}e(rm(-2m+l+k))$, from which it is possible to understand that X_{e4} will be stable if $m \in]\frac{l+k}{2}\ ; k[$, since $\mathbb{R}e(rm(-2m+l+k))$ is negative. It is an open interval, since if $m = \frac{l+k}{2}$ then the real part is 0, while if $m = k$ the system moves to $X_{e3} = (k; 0)$ (transition of the system's behaviour).

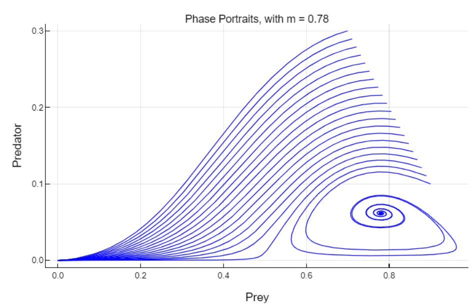

Figure 4: Phase Portraits: $X_{e4} = (m; r(m-l)(k-m))$

It has been set $r = k = 1$, $l = 0.5$ and $m = 0.78$. As presented in the qualitative analysis of the deterministic system, for these parameters there is a local attractor $XY_e = (0; 0)$ and since $m \in [\frac{l+k}{2}; k] = [0.75; 1]$, it will be present also another local attractor $XY_e = (0.78; 0.0336)$. The (6) is the only solution

of the characteristic equation that has a square root, and this means that the eigenvalues could belong to the $\mathbb{C}$. So, it is important to understand if Δ can be negative. Let's start with the associate equation to the inequality:

$$r^2(-2m^2 + ml + mk)^2 - 4rm(m-l)(k-m) = 0 \tag{7}$$

$$r^2(4m^4 - 4m^3(l+k) + m^2(l+k)^2) - 4rm^2k + 4rm^3 + 4rmlk - 4rm^2l = 0 \tag{8}$$

Now, given $r, m > 0$, it is possible to remove from the LHS of (8) r and m:

$$4rm^3 - 4m^2(r(l+k) - 1) + m(r(l+k)^2 - 4(l+k)) + 4lk = 0 \tag{9}$$

The goal is to solve this equation in m, since all the analysis of this model has been conducted on m. r, l and k can be considered in the domain $]0;1]$, since $x, y \in [0;1]$. Otherwise, it would be a nonsense to consider greater parameter. On (9) it is possible to conduct a dimensional analysis, to understand what terms influence more the equation.

- $4rm^3 \to 10^{-4}$
- $4m^2(r(l+k)) \to 10^{-4}$
- $4m^2 \to 10^{-2}$
- $m(r(l+k)^2) \to 10^{-4}$
- $4m(l+k)) \to 10^{-2}$
- $4lk \to 10^{-2}$

To continue the analysis of this equation, the terms of dimension 10^{-4} can be not considered. **Note** : the following results come from an approximation:

$$4m^2 - 4m(l+k) + 4lk = 0 \tag{10}$$

and the solution of (10) is:

$$\begin{cases} m_1 = l \\ m_2 = k \end{cases} \tag{11}$$

So, for every $m \in [l;k]$ the eigenvalues of (6) belongs to $\mathbb{C}$. Since $\lambda \in \mathbb{C}$, then it could also be analyzed if the real part of the eigenvalues changes sign: this will means that there is a limited cycle, due to *Andronov-Hopf* bifurcation. If $m \in [l; \frac{l+k}{2}]$, then there is a limited cycle. To sum up:

- $m \in [\frac{l+k}{2}; k]$ the $X_{e4} = (m; r(m-l)(k-m))$ is a local stable attractor.
- $m = \frac{l+k}{2}$ is the point of *Andronov-Hopf* bifurcation
- $m \in [l; \frac{l+k}{2}]$ then the system has a limited cycle.

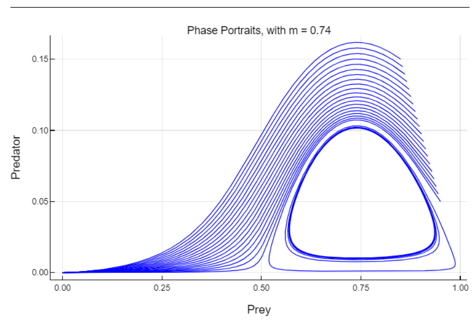

Figure 5: Limited cycle for m = 0.74

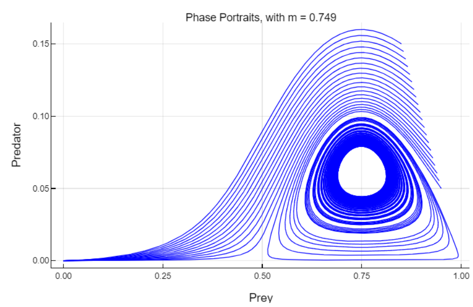

Figure 6: Limited cycle for m = 0.749

It has been set $r = k = 1$, $l = 0.5$ and $m \in [l; \frac{l+k}{2}] = [0.5; 0.75]$, then as presented in the qualitative analysis of the deterministic system, for these parameters there is a local attractor $XY_e = (0; 0)$ and since it will be present a limit cycle due to the presence of Andronov-Hopf bifurcation. Here, to observe this phenomenology, it has been specifically selected a narrow range of initial conditions. Also, this effect is so weak that the range of m to observe it goes from 0.74 to 0.749. Indeed, as the parameter m approach 0.74, the cycle becomes more large and when m goes to 0.739 the cycle disappear.

3 Stochastic model

In the previous section the deterministic model has been presented. In this section it will be introduced the stochastic model, defined as follows:

$$\begin{cases} dx = (rx(x-l)(k-x) - xy)dt + arx(x-k)dW^{(1)} \\ dy = y(x-m)dt - bydW^{(2)} \end{cases} \tag{12}$$

with $a, b > 0$ and $dW^{(1)}, dW^{(2)}$ white Gaussian independent noise. The system (12) is written in Ito's equations.

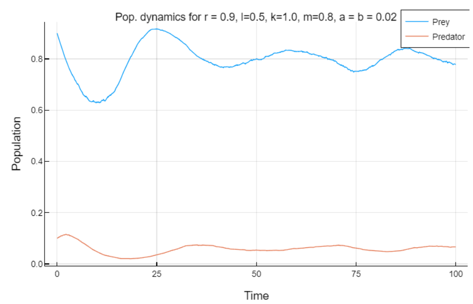

Figure 7: An example of stochastic trajectories of prey and predator over time, given a specific parameters' configuration.

It is interesting to understand how the introduction of noise on parameters l and m modify the deterministic system. Let's start with the case of $m > k$, where the local attractors are $XY_e = (0; 0)$ and $X_{e3} = (k; 0)$.

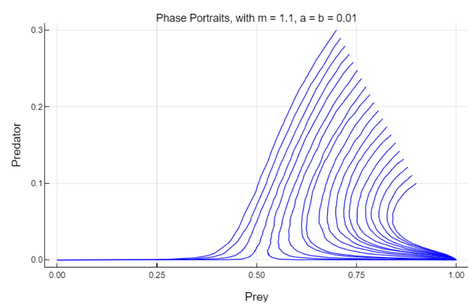

Figure 8: Phase Portraits: $X_{e3} = (k; 0)$ with $a = b = 0.01$

This is a Phase Portraits with the intensities of the white noises fixed. In this particular configuration, it is possible to notice that the noises have not enough impact on the system, so the behaviour is the same of the deterministic one. A way to understand the threshold of the required intensities of a and b to impact on the deterministic behavior can be evaluated by computing the probability of exiting from the basin of attraction of the local attractor X_{e3}.
N simulations are made to compute the trajectories of a point in the basin of attraction of X_{e3}. Then it is computed the number of times that the trajectories ended in X_{e1}. The following analysis will be computed for other cases.

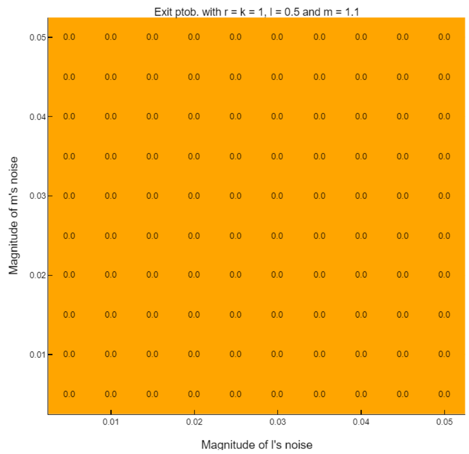

Figure 9: Probability of exiting from X_{e3}'s heatmap

From Fig. 9 it is possible to observe that there are no combinations of $a, b \in [0.005, 0.05]^2$ such to change the deterministic behaviour of the system with $r = k = 1$, $l = 0.5$ and $m = 1.1$.
In Fig. 9 bis, the heatmap represents the probabilities of exiting from X_{e3} if $a, b \in [0.05; 0.5]^2$. The results are similar to the one found in Fig. 9.

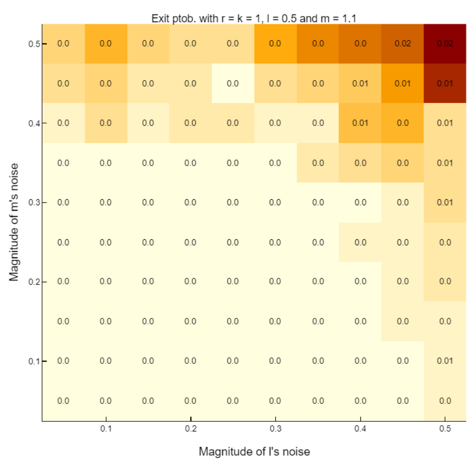

Figure 9-bis: Probability of exiting from X_{e3}'s heatmap, with stronger noises' magnitude

It is possible to conclude that there is not a noise-induced transition from X_{e3} to X_{e1} if the initial conditions are enough close to the basin of attraction of X_{e3}.

Another interesting case is when $m \in [\frac{l+k}{2}; k]$: in this configuration there will be two local attractors, X_{e1} and X_{e4}.

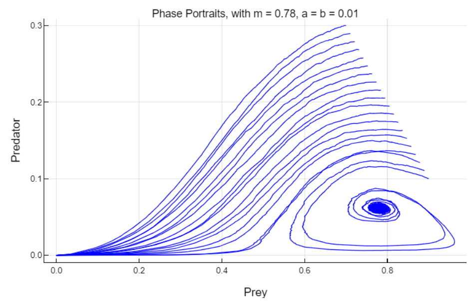

Figure 11: Phase Portraits: X_{e4} with $a = b = 0.01$

As before, it is interesting to evaluate the probability of exiting:

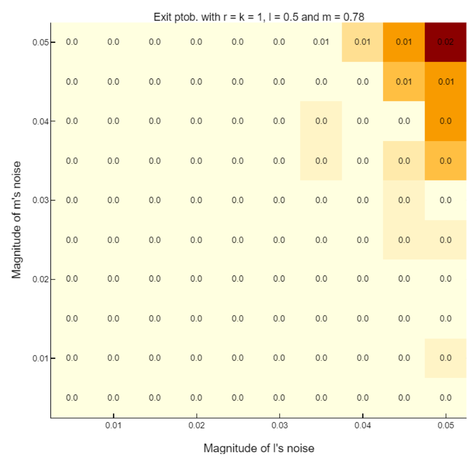

Figure 12: Probability of exiting from X_{e4}'s heatmap

Generally, for combinations of $a, b \in [0.005, 0.05]^2$ there are low probabilities to exit from the X_{e4}. An interesting question would concern a wider interval, like $a, b \in [0.05, 0.5]^2$.

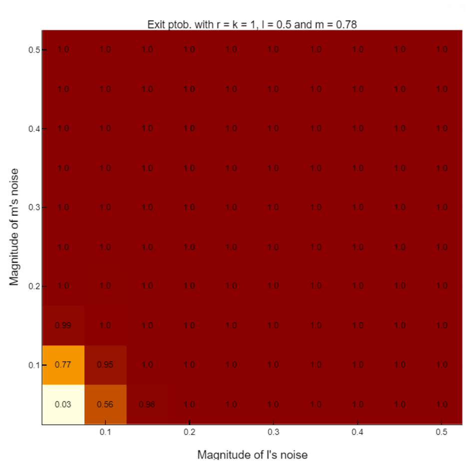

Figure 11-bis: Probability of exiting from X_{e4}'s heatmap, with stronger noises' magnitude

It is possible to observe how the probabilities of exiting from X_{e4} towards X_{e1} dramatically changed when the noises' magnitude increase. So, for a given noises' intensities there is a noise-induced transition from a coexistence regime to a full extinction.
Another interesting case is for $m \in [l; \frac{l+k}{2}]$. In the following Figures, from 12 to 15, there will be represented the phase portraits for the setting $r = k = 1$, $l = 0.5$ and $m = 0.74$. In this plots, the intensities a and b will be $0.005, 0.015, 0.05$ and 0.15.

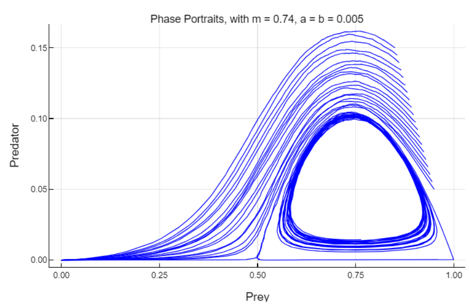

Figure 12: Phase Portraits: limited cycle with $a = b = 0.005$

In this case the noise has not a significant magnitude to perturb the system and so the phase portraits is similar to the deterministic one, however it is not possible to see smooth phase lines anymore.

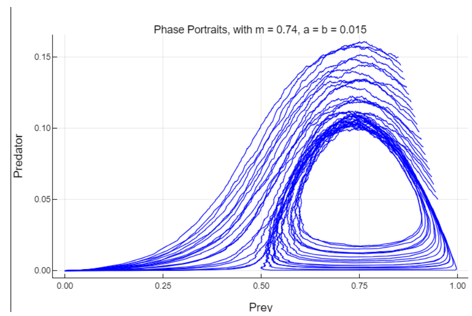

Figure 13: Phase Portraits: limited cycle with $a = b = 0.015$

If the magnitude increases to $a = b = 0.015$, it is possible to see the noise effect on the deterministic system. The behaviour is still similar to the deterministic one, however, the trajectories are perturbed in a substantial way.

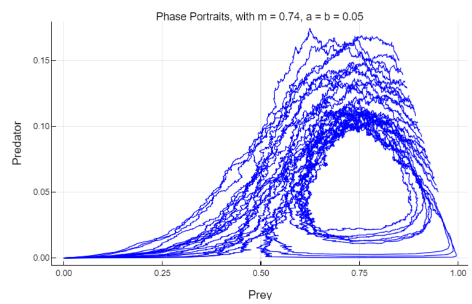

Figure 14: Phase Portraits: limited cycle with $a = b = 0.05$

When the magnitude's noise reaches $a = b = 0.05$ the deterministic system is perturbed and only few cycles remains. Indeed. the noise intensity is enough to makes the trajectories exit from the orbit of the cycles. Once escaped from the cycle, these trajectories are attracted by $X_e = (0; 0)$.

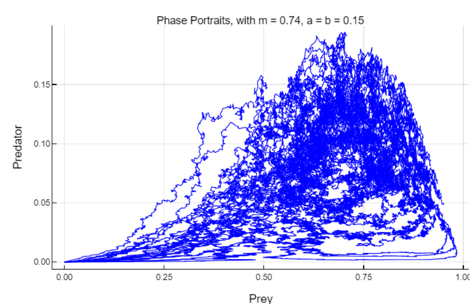

Figure 15: Phase Portraits: limited cycle with $a = b = 0.15$

If $a = b >= 0.15$, then there are no more cycles and there is only a global attractor $X_e = (0; 0)$.
To sum up the insights obtained from these plots, it has been computed the probability of exiting from the basin of attraction of the cycle.

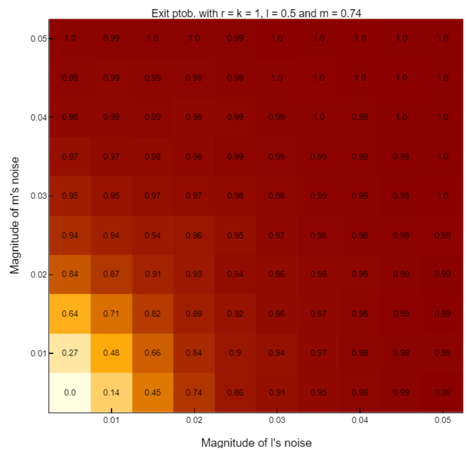

Figure 16: Probability of exiting from the cycle's heatmap

From Fig.16 it is clear that the basin of attraction of the cycle is "weak" and so, only if the magnitude/intensity of the noise is really small the cycle survives. Differently from the previous analysis on the probability of exiting from X_{e3} and X_{e4}, here it is not necessary to show how the probabilities changes when $a, b \in [0.05, 0.5]^2$ since it is sufficient starting from $a = b = 0.02$ to be sure that the limited cycle will disappear and there will be only one global attractor X_{e1}. This means a complete extinction of prey and predator.

4 Coexistence analysis

The coexistence between prey and predator, for my point of view, is the main goal to pursue. This section will analyze more deeply the combination of parameters that leads to the stable X_e4, and the limited cycle, which even if it is weak from a phenomenological point of view, is also an example of coexistence. It is interesting to get a more compact view about how trajectories end after different long times, like $T \in [1000, 1050, 1100]$ with different noises' intensities and different level of m. Figures 17 and 18 represent, respectively the ending states of prey and predators' densities after different long time. Indeed, a simulation has been done by running the system for different a, b, m and T, while keeping fixed as usual $r = k = 1.0$ and $l = 0.5$. The *green* points are those obtained with $a = b = 0.05$, the *blue* with $a = b = 0.01$ and the *red* with $a = b = 0.1$.

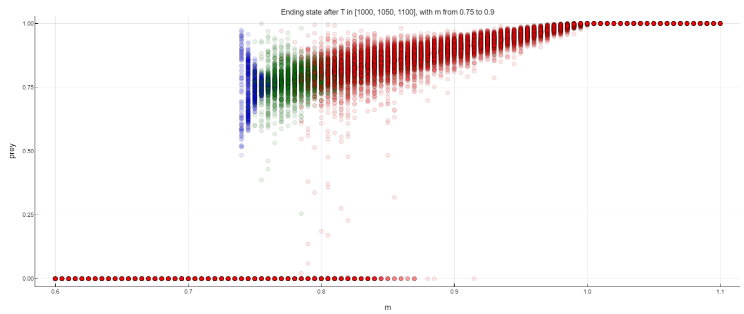

Figure 17: Prey ending state

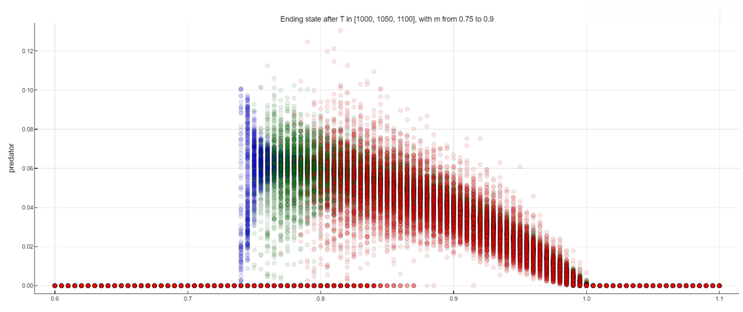

Figure 18: Predator ending state

The key points are:

1. when $m \to 0.74$, which is the case of the limited cycle, then there will be coexistence if and only if the noises' intensities is small, which means around $\to 0.01$.

2. when m is in the range between $[0.75; 8]$ and the limited cycle decays in single point attractor, X_e4, then there is coexistence if the $a = b \leq 0.01$. Indeed, here the mortality rate of predators is not enough high to compete with perturbation caused by the noises.

3. when m increases and also is $m < k$ then the regime of coexistence endures even if the magnitude of noises is high, like 0.1. This means that, according to this model, to pursue coexistence between prey and predators, the mortality rate of the last one should be very high (around 0.8 and 0.9).

4. the above points are true if and only if the intensities of noises are below a certain threshold, like $a = b = 0.05$ for $m \in [0.74; 0.749]$ and $a = b = 0.1$ for $m \in [0.75; 1.0[$.

It can be seen that the coexistence regime does not depend on the growth parameter r (it is assumed that there is always growth). Now, it is useful, in order to understand better the system, to conduct a link analysis on parameter l and m. Here, it is assumed that k is 1, meaning that the system has a fully carrying capacity. The link analysis has been conducted in the following way:

1. For each pair of (l, m) it has been computed some simulations of the system, by varying the initial conditions and keeping fixed $r = k = 1$.

2. Once obtained the normalized distribution (bins simulated) for each pair, it has been computed the total variation distance in its equivalence with the L^1 norm:
$$\delta(P, Q) = \frac{1}{2} \sum_{k=1}^{n} |P(x_k) - Q(x_k)|$$

3. The total variation distances are visualized using an heatmap.

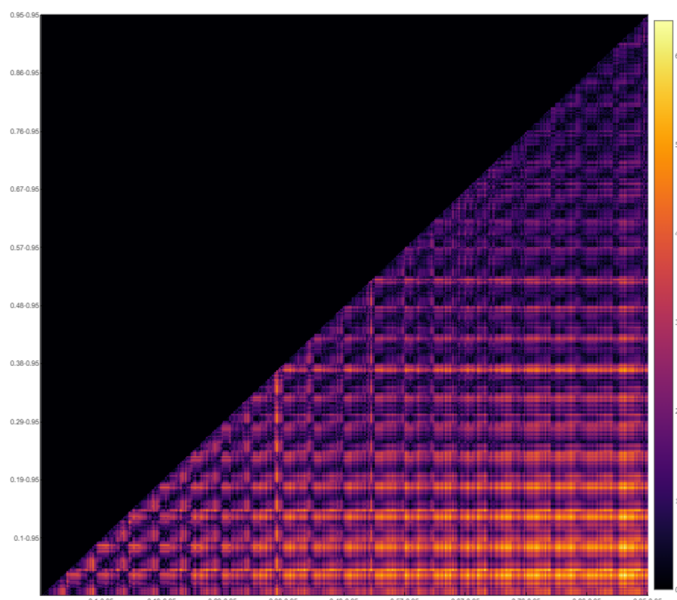

Figure 19: Heatmap for $l, m \in [0.05; 0.95]$

It has been computed only the lower triangular matrix due to symmetry of total variation distance. On the y-axis, the first value corresponds to parameter l, while the second to m, while on the y-axis it is the opposite. For a better visualization purpose, only the last tuple of each interval has been shown (between 0.29-0.95 and 0.38-0.95, it has been evaluated the total variation distance for the tuple from 0.38-0.05 to 0.38-0.95). It is notable that the distance assumes low value when both l and k are greater than 0.57, meaning that the behaviour of the system would not change too much for different values of l and m. There will be a strong coexistence if and only if $m \in [\frac{l+k}{2}; k[$.

5 Conclusion

The Bazykin-Berezovskaya prey-predator population model has been investigated and analyzed both in its deterministic and stochastic form. Only few kind of analyses has been conducted, focusing on the case where $r = k = 1$, a strong assumptions. To conclude it has been proposed a model comparison on the forecasting abilities of the deterministic and stochastic version of the Bazykin-Berezovskaya and a statistical method for forecasting, the ARIMA (*AutoRegressive Integrated Moving Average*) defined as follows:

- p: Order of the autoregressive part
- d: Degree of differencing
- q: Order of the moving average part

The mathematical representation of the ARIMA model can be expressed as:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t$$

Where:

- y_t: The value of the time series at time t.
- ϕ_i: The coefficients of the autoregressive terms (for $i = 1, \ldots, p$).
- θ_i: The coefficients of the moving average terms (for $i = 1, \ldots, q$).
- ϵ_t: The white noise error term at time t, which is assumed to be normally distributed with a mean of zero.

Before applying the AR and MA components, the series is differenced d times to achieve stationarity. The differencing process can be represented as:

$$\Delta^d y_t = y_t - y_{t-1}$$

The dataset used is the population of *Wolves and Moose of Isle Royale* (Vucetich, JA and Peterson RO. 2012. The population biology of Isle Royale wolves and moose: an overview.). To fit the population model a strong assumption and a pre-processing on the raw data has been done:

- The total population of wolfes and moose is considered constant over time.
- To compute the densities of wolves and moose, it has been computed the ratio between the count of wolves or moose in a given year and the sum of two in that year.

The trainig procedure has been the following one:

1. It has been defined a trainig set of 40 years, a validation set of 10 years and a test set of 11 years.
2. Multiple iteration of grid search method have been used to optimize the parameter of both the deterministic model and ARIMA.
3. The RMSE has been computed by summing the RMSE of both prey and predator predictions.

The model that performed better, under the metric of RMSE on the test set, is the stochastic model followed by ARIMA and the deterministic model. The parameter of the stochastic model were: $r = k = 1$, $l = 0.001$ and $m = 0.981579$.

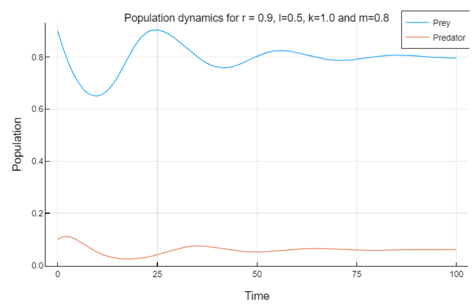

Figure 20: Prediction of deterministic model

The RMSE of deterministic model's prediction is 0.045.

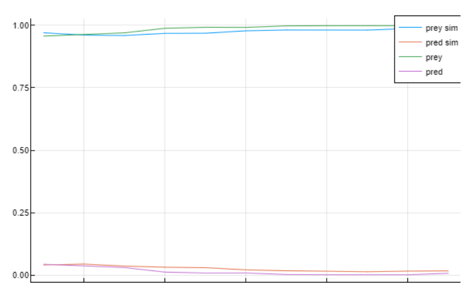

Figure 21: Prediction of stochastic model

while the RMSE of the prediction obtained by the stochastic model is 0.0245.

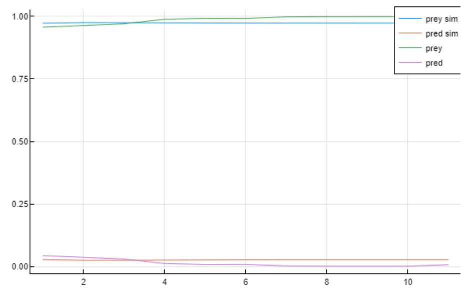

Figure 22: Prediction of ARIMA model

Finally, the RMSE from ARIMA predictions is 0.0395.