Descriptive Complexity for Function Complexity Classes

Marcelo Arenas

PUC Chile & U. of Oxford

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Warming up: Counting complexity classes

NP	#P
SAT: is a propositional formula satisfiable?	COUNTSAT: how many assignments satisfy a propositional formula?

A function $f(\cdot)$ is in $\#\mathrm{P}$ if there exists a polynomial-time non-deterministic TM M such that:

$f(x) =$ number of accepting computations of M with input x

- COUNTSAT is $\#\mathrm{P}$-complete

Warming up: Capturing $\#\mathrm{P}$

Can we characterize $\#\mathrm{P}$ in the description complexity sense?

- What is an appropriate logic for this?

Answer (Saluja, Subrahmanyam &Thakur): $\#\mathrm{FO}$ captures $\#\mathrm{P}$

- One considers FO-formulas of the form $\varphi(\bar{X}, \bar{x})$, where $\bar{X}$, $\bar{x}$ are tuples of FO and SO variables.
- Evaluation of $\varphi(\bar{X}, \bar{x})$ over $\mathfrak{A}$: Number of different instantiations $\bar{R}$ of $\bar{X}$ and $\bar{a}$ of $\bar{x}$ such that $\mathfrak{A} \models \varphi(\bar{R}, \bar{a})$

Warming up: More counting complexity classes

A function $f(\cdot)$ is in $\textsc{spanP}$ if there exists a polynomial-time non-deterministic TM M with output tape such that:

$f(x) =$ number of distinct valid outputs of M with input x

A bit of intuition:

- $\#\mathrm{P}$: Given a graph G, return the number of Hamiltonian cycles of G
- $\textsc{spanP}$: Given a graph G and an integer k, return the number of Hamiltonian subgraphs of G of size k

Warming up: Capturing $\textsc{spanP}$

Can we characterize $\textsc{spanP}$ in the description complexity sense?

Answer (Compton & Gradel): $\#\exists\mathrm{SO}$ captures $\textsc{spanP}$

But this is just the starting point ...

Can we characterize the following classes?

- FP: Class of functions computable in polynomial time
- $\mathrm{FPSPACE}$: Class of functions computable in polynomial space
- It is straightforward to prove that $\mathrm{FP} \subsetneq \mathrm{FPSPACE}$: Consider 2^{2^n}
  - $\natural\mathrm{PSPACE}$: Class of functions $f \in \mathrm{FPSPACE}$ such that $|f(n)|$ is bounded by a polynomial
- $\textsc{gap}\mathrm{P}$: Class of functions $f(\cdot)$ for which there exists a polynomial-time non-deterministic TM M such that

$$\begin{aligned} f(x) \quad &= \quad (\text{number of accepting computations of } M \text{ with input } x) - \\ &\quad\quad (\text{number of rejecting computations of } M \text{ with input } x) \end{aligned}$$

Encoding of finite structures

Assume $\mathbf{R} = \{R_1, \ldots, R_k\}$, $\mathfrak{A} \in \textsc{Struct}[\mathbf{R}]$ with domain $A = \{a_1, \ldots, a_n\}$, and $<$ is the linear order $a_1 < a_2 < \ldots < a_n$.

Encoding of $R_i^{\mathfrak{A}}$:

- Assume $\ell = \mathrm{arity}(R_i)$, and consider an enumeration of the ℓ-tuples over A in the lexicographic order induced by $<$
- $\mathrm{enc}(R_i^{\mathfrak{A}})$: string of length n^ℓ whose i-th bit is 1 if the i-th tuple in the previous enumeration belongs to $R_i^{\mathfrak{A}}$, and 0 otherwise

Encoding of $\mathfrak{A}$:

$$\mathrm{enc}(\mathfrak{A}) \quad = \quad 0^n\, 1\, \mathrm{enc}(R_1^{\mathfrak{A}}) \cdots \mathrm{enc}(R_k^{\mathfrak{A}})$$

Function complexity classes

Let S be a countable set.

- A function $f : \mathbf{R} \to S$ is a total function from $\{\mathrm{enc}(\mathfrak{A}) \mid \mathfrak{A} \in \textsc{Struct}[\mathbf{R}]\}$ to S

Definition

$\mathscr{C}$ is a function complexity class over S if for every $f \in \mathscr{C}$, there exists a relational signature $\mathbf{R}$ such that $f : \mathbf{R} \to S$

Function complexity classes (cont'd)

If $\mathscr{K}$ is a class of finite structures and $f : \mathbf{R} \to S$, we denote by $f|_{\mathscr{K}}$ the restriction of f to $\mathscr{K}$:

- Domain of $f|_{\mathscr{K}}$ is $\{\mathrm{enc}(\mathfrak{A}) \mid \mathfrak{A} \in \textsc{Struct}[\mathbf{R}] \cap \mathscr{K}\}$
- $f|_{\mathscr{K}}(\mathrm{enc}(\mathfrak{A})) = f(\mathrm{enc}(\mathfrak{A}))$ for every $\mathfrak{A} \in \textsc{Struct}[\mathbf{R}] \cap \mathscr{K}$

Capturing function complexity classes: Informal

Fix a set S.

Assume given a logic $\mathscr{F}$ such that for every sentence θ in $\mathscr{F}$:

$$[\![\theta]\!] : \mathbf{R} \to S$$

Let $\mathscr{C}$ a function complexity class over S and $\mathscr{K}$ a class of finite structures. Then $\mathscr{F}$ *captures* $\mathscr{C}$ over $\mathscr{K}$ if:

- for every θ in $\mathscr{F}$, there exists $f \in \mathscr{C}$ such that $\theta|_{\mathscr{K}} = f|_{\mathscr{K}}$
- for every $f \in \mathscr{C}$, there exists θ in $\mathscr{F}$ such that $f|_{\mathscr{K}} = \theta|_{\mathscr{K}}$

Capturing function complexity classes: Informal (cont'd)

Notation

In the previous definition, if $\mathscr{K}$ is the class of all ordered finite structures, then we say that $\mathscr{F}$ captures $\mathscr{C}$ over the class of ordered structures.

Semirings

Structure $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ such that:

- $(S, \oplus, \mathbb{0})$ is a commutative monoid
- $(S, \odot, \mathbb{1})$ is a monoid
- $\odot$ distributes over $\oplus$
- $\mathbb{0} \odot s = s \odot \mathbb{0} = \mathbb{0}$ for every $s \in S$

We consider commutative semirings: $\odot$ is commutative

Quantitative Logic: Syntax

Fix: semiring $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$.

Given a relational signature **R**, the set of $\mathbb{S}$-quantitative second-order logic formulas ($\mathrm{QSO}(\mathbb{S})$-formulas) over **R** is given by the following grammar:

$$\theta \;:=\; \varphi \;\mid\; s \;\mid\; (\theta + \theta) \;\mid\; (\theta \cdot \theta) \;\mid\; \Sigma x\,\theta \;\mid\; \Pi x\,\theta \;\mid\; \Sigma X\,\theta \;\mid\; \Pi X\,\theta$$

where φ is an SO-formula over **R** and $s \in S$

Quantitative Logic: Semantics

Given: relational signature **R**, finite **R**-structure $\mathfrak{A}$ with domain A, first-order variable assignment v for $\mathfrak{A}$, and second-order variable assignment V for $\mathfrak{A}$

Evaluation of $\mathrm{QSO}(\mathbb{S})$-formula θ over $(\mathfrak{A}, v, V)$ is defined as a function $[\![\theta]\!]$ that on input $(\mathfrak{A}, v, V)$ returns a value in S

Formally:

$$
\begin{aligned}
[\![\varphi]\!](\mathfrak{A}, v, V) &= \begin{cases} \mathbb{1} & \text{if } (\mathfrak{A}, v, V) \models \varphi \\ \mathbb{0} & \text{otherwise} \end{cases} \\
[\![s]\!](\mathfrak{A}, v, V) &= s
\end{aligned}
$$

Quantitative Logic: Semantics (cont'd)

$$
\begin{aligned}
[\![\theta_1 + \theta_2]\!](\mathfrak{A}, v, V) &= [\![\theta_1]\!](\mathfrak{A}, v, V) \oplus [\![\theta_2]\!](\mathfrak{A}, v, V) \\
[\![\theta_1 \cdot \theta_2]\!](\mathfrak{A}, v, V) &= [\![\theta_1]\!](\mathfrak{A}, v, V) \odot [\![\theta_2]\!](\mathfrak{A}, v, V) \\
[\![\Sigma x\, \theta]\!](\mathfrak{A}, v, V) &= \bigoplus_{a \in A} [\![\theta]\!](\mathfrak{A}, v[a/x], V) \\
[\![\Pi x\, \theta]\!](\mathfrak{A}, v, V) &= \bigodot_{a \in A} [\![\theta]\!](\mathfrak{A}, v[a/x], V) \\
[\![\Sigma X\, \theta]\!](\mathfrak{A}, v, V) &= \bigoplus_{B \subseteq A^{\mathrm{arity}(X)}} [\![\theta]\!](\mathfrak{A}, v, V[B/X]) \\
[\![\Pi X\, \theta]\!](\mathfrak{A}, v, V) &= \bigodot_{B \subseteq A^{\mathrm{arity}(X)}} [\![\theta]\!](\mathfrak{A}, v, V[B/X])
\end{aligned}
$$

Quantitative Logic: An example

Consider $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ and $\mathbf{R} = \{<\}$

Exercise

Define the function 2^k over the class of ordered structures.

- That is, define formula $\theta(x)$ such that for every $\mathbf{R}$-structure $\mathfrak{A}$:

$$\langle \{a_0, \ldots, a_{n-1}\}, <^{\mathfrak{A}} = \{(a_i, a_j) \mid 0 \leq i < j \leq n-1\} \rangle$$

  and $v(x) = a_i$, it holds that $[\![\theta]\!](\mathfrak{A}, v) = 2^i$

Answer: $\theta(x) \ = \ \Pi y \left[2 \cdot (y < x) \ + \ \neg(y < x) \right]$

Capturing function complexity classes: Formal

Fix $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$

A sentence θ in $\mathrm{QSO}(\mathbb{S})$ denotes a function from **R** to S:

$$\theta(\mathrm{enc}(\mathfrak{A})) \quad = \quad [\![\theta]\!](\mathfrak{A})$$

Definition

Let $\mathscr{F}$ be a fragment of $\mathrm{QSO}(\mathbb{S})$, $\mathscr{C}$ a function complexity class over S and $\mathscr{K}$ a class of finite structures. Then $\mathscr{F}$ *captures* $\mathscr{C}$ over $\mathscr{K}$ if:

- for every θ in $\mathscr{F}$, there exists $f \in \mathscr{C}$ such that $\theta|_{\mathscr{K}} = f|_{\mathscr{K}}$
- for every $f \in \mathscr{C}$, there exists θ in $\mathscr{F}$ such that $f|_{\mathscr{K}} = \theta|_{\mathscr{K}}$

Fragments of $\mathrm{QSO}(\mathbb{S})$

$\mathrm{QFO}(\mathbb{S})$: Quantifiers ΣX , ΠX are not allowed

$\Sigma\mathrm{QSO}(\mathbb{S})$: Quantifier ΠX is not allowed

$\mathrm{QSO}(\mathbb{S}, \mathscr{L})$: $\mathscr{L}$ is a fragment of SO

Known results reformulated

Theorem (Saluja, Subrahmanyam & Thakur)

$\Sigma\mathrm{QSO}(\mathbb{N}, \mathrm{FO})$ *captures* $\#\mathrm{P}$ *over the class of ordered structures.*

Proposition (Compton & Gradel)

$\Sigma\mathrm{QSO}(\mathbb{N}, \mathrm{FO})$ *does not capture* $\#\mathrm{P}$ *over the class of all structures.*

Theorem (Compton & Gradel)

$\Sigma\mathrm{QSO}(\mathbb{N}, \exists\mathrm{SO})$ *captures* $\textsc{span}\mathrm{P}$ *over the class of ordered structures.*

New results

Theorem

$\Sigma\mathrm{QSO}(\mathbb{Z}, \mathrm{FO})$ *captures* $\textsc{gap}\mathrm{P}$ *over the class of ordered structures.*

New results: Polynomial-time computable functions

Theorem

$\mathrm{QFO}(\mathbb{Z}, \mathrm{LFP})$ *captures* FP *over the class of ordered structures.*

Proof: In the whiteboard, let's consider a function $f : \mathbf{R} \to \mathbb{N}$

Theorem

$\mathrm{QFO}(\mathbb{Z}, \mathrm{PFP})$ *captures* $\natural\mathrm{PSPACE}$ *over the class of ordered structures.*

New results: Polynomial-space computable functions

Theorem

$\mathrm{QSO}(\mathbb{Z}, \mathrm{PFP})$ *captures* $\mathrm{FPSPACE}$ *over the class of ordered structures.*

Proof: Consider the following order

$$\varphi(X, Y) = \exists \bar{u} \left(\neg X(\bar{u}) \wedge Y(\bar{u}) \wedge \forall \bar{v}(\bar{u} < \bar{v} \rightarrow (X(\bar{v}) \leftrightarrow Y(\bar{v}))) \right)$$

Ongoing research

- We want to get better characterizations of counting problems with easy decision problems
  - $\#\mathrm{PE}$, $\textsc{TotP}$, $\ldots$
- We also want to study optimization problems: $\textsc{OptP}$
  - Consider the tropical semiring: $(\mathbb{N} \cup \{-\infty\}, \min, +, -\infty, 0)$
- $\ldots$