Counting complexity classes defined by fragments of second-order logic

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1 Some properties of $\#\Sigma_1[\mathrm{FO}]$

We define the vocabulary $\mathcal{L} = \{S_1, \ldots, S_t, \leq\}$, where $S_1, \ldots, S_t$ have arity $b_1, \ldots, b_t$. Let

$$\begin{aligned} \textsc{Struct}[\mathcal{L}] &= \{\mathfrak{A} \mid \mathfrak{A} \text{ is an } \mathcal{L}\text{-structure with a finite domain } A \text{ such that} \\ &\quad\ \leq \text{ is interpreted as a total order for } A\}. \end{aligned}$$

We also define a set of second order variables $\mathcal{X} = \{X_i \mid i \in \mathbb{N}\}$ where X_i has arity a_i, and for every $n \in \mathbb{N}$ there are infinite variables in $\mathcal{X}$ of arity n. A quantifier-free $\mathcal{L}$-formula is defined by the following grammar:

$$\begin{aligned} \varphi \quad ::= \quad & x = y \ \mid \ S_i(x_1, \ldots, x_{b_i}), i \in \{1, \ldots, t\} \ \mid \ x \leq y \ \mid \\ & X_i(x_1, \ldots, x_{a_i}), i \in \mathbb{N} \ \mid \\ & (\neg\varphi) \ \mid \ (\varphi \wedge \varphi) \ \mid \ (\varphi \vee \varphi), \end{aligned}$$

where x, y and x_i are first order variables for every i. We now define an FO-extended quantifier-free $\mathcal{L}$-formula as follows:

$$\begin{aligned} \varphi \quad ::= \quad & \alpha, \alpha \text{ is an FO-formula over } \mathcal{L} \ \mid \\ & X_i(x_1, \ldots, x_{a_i}), i \in \mathbb{N} \ \mid \\ & (\neg\varphi) \ \mid \ (\varphi \wedge \varphi) \ \mid \ (\varphi \vee \varphi). \end{aligned}$$

Let $\bar{Y} = (Y_1, \ldots, Y_q)$ be a tuple of second-order variables of arity $c_1, \ldots, c_q$, and let $\bar{y}$ be a tuple of first order variables. For every $\mathcal{L}$-formula $\psi(\bar{Y}, \bar{y})$, we define the function $f_{\psi(\bar{Y},\bar{y})} : \textsc{Struct}[\mathcal{L}] \to \mathbb{N}$ as follows:

$$f_{\psi(\bar{Y},\bar{y})}(\mathfrak{A}) \quad = \quad |\, \{\langle \bar{P}, \bar{e}\rangle \mid \mathfrak{A} \models \psi(\bar{P}, \bar{e})\} \,|,$$

for every $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle \in \textsc{Struct}[\mathcal{L}]$, where $\bar{P} = (P_1, \ldots, P_q)$ is a tuple of predicates of arity $c_1, \ldots, c_q$, $P_i \subseteq A^{c_i}$ for every $i \in \{1, \ldots, q\}$, and $\bar{e}$ is a tuple of elements from $\mathfrak{A}$.

A function $f : \textsc{Struct}[\mathcal{L}] \to \mathbb{N}$ is in $\#\Sigma_0[\mathrm{FO}]$ if there exists an FO-extended quantifier-free $\mathcal{L}$-formula $\varphi(\bar{Y}, \bar{y})$ such that $f = f_{\varphi(\bar{Y},\bar{y})}$.

Similarly, a function $f : \textsc{Struct}[\mathcal{L}] \to \mathbb{N}$ is in $\#\Sigma_1[\mathrm{FO}]$ if there exists an FO-extended quantifier-free $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{Y}, \bar{y})$ such that $f = f_{\exists\bar{x}\ \varphi(\bar{x},\bar{Y},\bar{y})}$.

Theorem 1. $\#\Sigma_0[\mathrm{FO}] \subseteq \mathrm{FP}$.

Proof. Let $f \in \#\Sigma_0[\mathrm{FO}]$, and let $\varphi(\bar{X}, \bar{x})$ be and FO-extended quantifier-free $\mathcal{L}$-formula such that:

$$f(\mathfrak{A}) \quad = \quad |\{\langle \bar{P}, \bar{e}\rangle \mid \mathfrak{A} \models \varphi(\bar{P}, \bar{e})\}|$$

for each $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, where $\bar{e} \in A^m$ and $\bar{P} = (P_1, \ldots, P_q)$ is a tuple of predicates. We will now show that counting $f(\mathfrak{A})$ can be done in polynomial time.

For each FO-formula $\beta(\bar{x})$ in $\varphi(\bar{X}, \bar{x})$, let R_β be a predicate of arity m. Let $\mathfrak{A}' = \langle A, \bar{S}^{\mathfrak{A}}, R_\beta^{\mathfrak{A}'}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, where $R_\beta^{\mathfrak{A}'} = \{\bar{d} \mid \mathfrak{A} \models \beta(\bar{d})\}$. Note that each $\beta(\bar{x})$ is fixed in $\varphi(\bar{X}, \bar{x})$, and for each $\bar{d} \in A^m$, checking whether $\mathfrak{A} \models \beta(\bar{d})$ can be done in polynomial time. Therefore, generating $R_\beta^{\mathfrak{A}'}$ can also be done in polynomial time.

Let $\psi(\bar{X}, \bar{x})$ be obtained by replacing each FO-formula $\beta(\bar{x})$ in $\varphi(\bar{X}, \bar{x})$ by $R_\beta(\bar{x})$. Also, let $g = f_{\psi(\bar{X}, \bar{x})}$. Note that for each tuple of predicates $\bar{P}$ and each $\bar{e} \in A^m$, $\mathfrak{A} \models \varphi(\bar{P}, \bar{e})$ if and only if $\mathfrak{A}' \models \psi(\bar{P}, \bar{e})$, and so, $g(\mathfrak{A}') = f(\mathfrak{A})$. However, $\psi(\bar{X}, \bar{x})$ is a quantifier-free $\mathcal{L}$-formula, and therefore, $g \in \#\Sigma_0$. Since it is shown in [1] that $\#\Sigma_0 \subseteq \text{FP}$, we conclude that $f(\mathfrak{A})$ can be evaluated in polynomial time.

□

The *decision problem* associated to a function f is defined by the language $L_f = \{\mathfrak{A} \in \textsc{Struct}[\mathcal{L}] \mid f(\mathfrak{A}) > 0\}$.

Theorem 2. *The decision problem associated to a function in* $\#\Sigma_1[\text{FO}]$ *is in* P.

Proof. Let f be a function in $\#\Sigma_1[\text{FO}]$. Then there is an extended quantifier-free $\mathcal{L}-$formula $\varphi(\bar{x}, \bar{X}, \bar{z})$ such that

$$f(\mathfrak{A}) \quad = \quad |\, \{\langle \bar{P}, \bar{e} \rangle \mid \mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P}, \bar{e})\} \,|,$$

where $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, $A = \{a_1, \ldots, a_{|A|}\}$, $\bar{z}$ is an m-tuple of variables and $\bar{x}$ is a k-tuple of variables. As said in [1], a function $h : \textsc{Struct}[\mathcal{L}] \to \mathbb{N}$ is in $\#\Sigma_0$ if there exists a quantifier-free $\mathcal{L}$-formula $\theta(\bar{Y}, \bar{y})$ such that $h = f_{\theta(\bar{Y}, \bar{y})}$. Let $\bar{y} = (\bar{x}, \bar{z})$ and let $\psi(\bar{X}, \bar{y}) = \varphi(\bar{x}, \bar{X}, \bar{z})$. Moreover, let $g = f_{\psi(\bar{X}, \bar{y})}$.

Claim 2.1. *For each* $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$*,* $f(\mathfrak{A}) > 0$ *iff* $g(\mathfrak{A}) > 0$.

Proof. ($\Rightarrow$) Suppose $f(\mathfrak{A}) > 0$. Let $\bar{P}$ and $\bar{e}$ be such that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P}, \bar{e})$. It follows that there is at least one $\bar{d} \in A^k$ such that $\mathfrak{A} \models \varphi(\bar{d}, \bar{P}, \bar{e}) = \psi(\bar{P}, (\bar{d}, \bar{e}))$. Therefore, $g(\mathfrak{A}) > 0$. ($\Leftarrow$) Suppose $g(\mathfrak{A}) > 0$. Let $\bar{Q}$ and $\bar{c} = (\bar{c}_1, \bar{c}_2)$, where $\bar{c}_1$ and $\bar{c}_2$ have k and m elements respectively, be such that $\mathfrak{A} \models \psi(\bar{Q}, \bar{c}) = \varphi(\bar{c}_1, \bar{Q}, \bar{c}_2)$. Therefore, $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{Q}, \bar{c}_2)$, from which we conclude that $f(\mathfrak{A}) > 0$. □

Note that $\psi(\bar{X}, \bar{y})$ is an FO-extended $\mathcal{L}$-formula, so $g \in \#\Sigma_0[\text{FO}]$. As we showed, for each $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$, it holds that $\mathfrak{A} \in L_f$ if and only if $g(\mathfrak{A}) > 0$, but as it is shown in Theorem 1, $g(\mathfrak{A})$ can be evaluated in polynomial time. From this we conclude that the decision version of f is in P. □

For a given pair of functions f, g, we define $f \dot{-} g$ as follows:

$$(f \dot{-} g)(\mathfrak{A}) = \begin{cases} f(\mathfrak{A}) - g(\mathfrak{A}), & \text{if } f(\mathfrak{A}) > g(\mathfrak{A}) \\ 0, & \text{if } f(\mathfrak{A}) \leq g(\mathfrak{A}). \end{cases}$$

for every $\mathcal{L}$-structure $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$. A function class $\mathcal{F}$ is *closed under substraction* if for every pair of functions $f, g \in \mathcal{F}$, it holds that $f \dot{-} g \in \mathcal{F}$.

Theorem 3. *If* $\#\Sigma_1[\text{FO}]$ *is closed under substraction, then* P $=$ NP.

Proof. Suppose that $\#\Sigma_1[\text{FO}]$ is closed under substraction, that is, for each pair of functions $f, g \in \#\Sigma_1[\text{FO}]$, there is an $h \in \#\Sigma_1[\text{FO}]$ such that $(f \dot{-} g)(\mathfrak{A}) = h(\mathfrak{A})$ for each $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$.

Let $\mathfrak{A} = \langle A, S_1^{\mathfrak{A}}, S_2^{\mathfrak{A}}, S_3^{\mathfrak{A}}, S_4^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle$ be an $\mathcal{L}-$structure that represents an instance of a 3DNF formula ψ, where A is the set of variables mentioned in ψ, $S_i^{\mathfrak{A}}$ is a ternary relation described as follows, for each $i \in \{1, 2, 3, 4\}$:

$$\begin{aligned} S_1^{\mathfrak{A}} &= \{(a_1, a_2, a_3) \mid (\neg a_1 \wedge \neg a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \psi\}, \\ S_2^{\mathfrak{A}} &= \{(a_1, a_2, a_3) \mid (a_1 \wedge \neg a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \psi\}, \\ S_3^{\mathfrak{A}} &= \{(a_1, a_2, a_3) \mid (a_1 \wedge a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \psi\}, \\ S_4^{\mathfrak{A}} &= \{(a_1, a_2, a_3) \mid (a_1 \wedge a_2 \wedge a_3) \text{ appears as a disjunct in } \psi\}. \end{aligned}$$

Now let $f_{\#3DNF}$ be a function that counts the satisfying assignments to a 3DNF formula ψ. As shown in [1], $f_{\#3DNF} \in \#\Sigma_1$. Let $f_{all} = f_{\exists x\, \varphi(x,X)}$, where

$$\varphi(x, X) = (X(x) \vee \neg X(x)).$$

Note that f_{all} counts every possible truth assignment (satisfying or not) to a 3DNF formula. As we supposed initially, let $h \in \#\Sigma_1[\mathrm{FO}]$ such that $f_{all} - f_{\#3DNF} = h$. For each structure $\mathfrak{A}$ that represents a 3DNF formula ψ, it holds that $h(\mathfrak{A}) = f_{all}(\mathfrak{A}) - f_{\#3DNF}(\mathfrak{A}) = 0$ if and only if ψ is a tautology, so the decision version L_h of $f_{all} - f_{\#3DNF}$ is co-NP-complete. However, as we showed previously in Theorem 2, $L_h \in \mathrm{P}$. Then, co-NP $\subseteq$ P, from which we conclude that P = NP. □

Theorem 4. $\#\Sigma_1 \subsetneq \#\Sigma_1[\mathrm{FO}]$

Proof. We will show that the $\#\Sigma_1[\mathrm{FO}]$ function f defined by $\varphi(x_1) = (x_1 = x_1) \wedge \forall y\, S(y)$ is not in $\#\Sigma_1$. By contradiction, suppose that it is. Let $\mathfrak{A} = \langle A = \{1\}, S^{\mathfrak{A}} = \{1\}, \leq^{\mathfrak{A}} = \{(1,1)\}\rangle$. Then, $f(\mathfrak{A}) = 1$. Now let $\mathfrak{B} = \langle B = \{1,2\}, S^{\mathfrak{B}} = \{1\}, \leq^{\mathfrak{B}} = \{(1,1),(1,2),(2,2)\}\rangle$. Note that $\mathfrak{A}$ is an induced substructure of $\mathfrak{B}$.

We have that for each function $g \in \#\Sigma_1$ and structures $\mathfrak{A}_1, \mathfrak{A}_2 \in \textsc{Struct}[\mathcal{L}]$, if $\mathfrak{A}_1$ is an induced substructure of $\mathfrak{A}_2$, then $g(\mathfrak{A}_1) \leq g(\mathfrak{A}_2)$ [1]. Therefore, $f(\mathfrak{B}) \geq f(\mathfrak{A}) = 1$. However, there is no assignment $s \in B$ to x such that $\mathfrak{B} \models \varphi(s)$, so $f(\mathfrak{B}) = 0$, which leads to a contradiction. □

For a given function f, we define $f \dot{-} 1$ as follows:

$$f \dot{-} 1(\mathfrak{A}) = \begin{cases} f(\mathfrak{A}) - 1, & \text{if } f(\mathfrak{A}) > 0 \\ 0, & \text{if } f(\mathfrak{A}) = 0. \end{cases}$$

for every $\mathcal{L}$-structure $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$. A function class $\mathcal{F}$ is *closed under substraction by one* if for every function $f \in \mathcal{F}$, it holds that $f \dot{-} 1 \in \mathcal{F}$.

Theorem 5. $\#\Sigma_1[\mathrm{FO}]$ *is closed under substraction by one.*

Proof. Consider an $\mathcal{L}$-formula of the form $\exists \bar{x}\, \varphi(\bar{x}, \bar{X}, \bar{z})$ where $\bar{z} = (z_1, \ldots, z_d)$ and $\bar{X} = (X_1, \ldots, X_r)$. There are three possibilities regarding the size of the tuples of free variables $\bar{z}$ and $\bar{X}$: (1) $d > 0$ and $r = 0$ (2) $d = 0$ and $r > 0$ (3) $d, r > 0$. This separates the proof in three cases:

1. Let $f \in \#\Sigma_1[\mathrm{FO}]$ be defined by an extended quantifier free $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{z})$, where $\bar{z} = (z_1, \ldots, z_d)$. That is,

$$f(\mathfrak{A}) \quad = \quad |\, \{\langle \bar{e}\rangle \mid \mathfrak{A} \models \exists \bar{x}\ \varphi(\bar{x}, \bar{e})\}\, |,$$

for every $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle \in \textsc{Struct}[\mathcal{L}]$, where $\bar{e} \in A^d$. Our goal here is to eliminate the lexicographically smallest sequence of variables, which can be done easily. First, let $\bar{y} = (y_1, \ldots, y_k)$, $\bar{y}\,' = (y'_1, \ldots, y'_k)$ and

$$\varphi_{k,<}(\bar{y}\,', \bar{y}) \quad = \quad \bigvee_{i=1}^{k} \left(\bigwedge_{j=1}^{i-1} y'_j = y_j \wedge y'_i < y_i \right).$$

This formula is true if $\bar{y}\,'$ is lexicographically smaller than $\bar{y}$. Now, let f' be defined by

$$\varphi'(\bar{x}, \bar{z}) \quad = \quad \varphi(\bar{x}, \bar{z}) \wedge \exists \bar{z}\,'(\varphi(\bar{x}, \bar{z}\,') \wedge \varphi_{d,<}(\bar{z}\,', \bar{z})).$$

If $f(\mathfrak{A}) > 0$, then $f'(\mathfrak{A})$ will count exactly one element less than $f(\mathfrak{A})$. Otherwise, if $f(\mathfrak{A}) = 0$, then $\mathfrak{A} \not\models \exists \bar{x}\, \varphi(\bar{x}, \bar{e})$ for every tuple $\bar{e}$ of elements in A, so $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'(\bar{x}, \bar{e})$ for every $\bar{e}$ and, therefore, $f'(\mathfrak{A}) = 0$. Hence, $f' = f \dot{-} 1$, from which we conclude that $f \dot{-} 1 \in \#\Sigma_1[\mathrm{FO}]$.

2. Let $f \in \#\Sigma_1[\mathrm{FO}]$ be defined by an extended quantifier free $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{X})$ where $\bar{x} = (x_1, \ldots, x_d)$ and $\bar{X} = (X_1, \ldots, X_r)$. That is,

$$f(\mathfrak{A}) \quad = \quad |\{\langle \bar{P} \rangle \mid \mathfrak{A} \models \exists \bar{x}\ \varphi(\bar{x}, \bar{P})\}|,$$

for every $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, where $\bar{P} = (P_1, \ldots, P_r)$ and $P_i \subseteq A^{a_i}$ for every $i \in \{1, \ldots, r\}$. For the time being, suppose that

$$\varphi(\bar{x}, \bar{X}) \quad = \quad \left(\bigwedge_{i=1}^{n} X_{\lambda(i)}(\bar{x}_i) \right) \wedge \varphi^{-}(\bar{X}, \bar{y}) \wedge \theta(\bar{x}) \wedge \beta(\bar{x}) \tag{1}$$

where n is the number of times a non-negated variable in $\bar{X}$ is referred to, according to the function $\lambda : \{1, \ldots, n\} \to \{1, \ldots, r\}$, $\bar{y}$ is a p-tuple of variables in $\bar{x}$, $\varphi^{-}(\bar{X}, \bar{y})$ is a conjunction of negated predicates in $\bar{X}$, $\theta(\bar{x})$ defines a total order on a partition of $\bar{x}$, and $\beta(\bar{x})$ is an FO-formula over $\mathcal{L}$ which mentions all variables in $\bar{x}$. Note that $\theta(\bar{x})$ also mentions all variables in $\bar{x}$. We also assume that $(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}) = \bar{x}$. As an example, the following formula is of this form:

$$\begin{aligned} \varphi(\bar{x}, \bar{X}) = {} & X_1(x_1, x_2) \wedge X_3(x_3) \wedge X_2(x_4, x_5) \wedge X_3(x_6) \wedge \neg X_1(x_7, x_8) \wedge \\ & (x_1 < x_2 \wedge x_1 = x_3 \wedge x_1 = x_4 \wedge x_2 = x_8 \wedge x_2 = x_5 \wedge x_8 < x_6 \wedge x_6 = x_7) \wedge \\ & \forall z \big(S_1(x_1, z, x_2) \wedge x_3 = x_3 \wedge x_4 = x_4 \wedge x_5 = x_5 \wedge x_6 = x_6 \wedge x_7 = x_7 \wedge x_8 = x_8 \big), \end{aligned}$$

where $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ and $\bar{X} = (X_1, X_2, X_3)$. Here, $n = 4$, $\lambda(1) = 1$, $\lambda(2) = \lambda(4) = 3$ and $\lambda(3) = 2$, $\bar{x}_1 = (x_1, x_2)$, $\bar{x}_2 = (x_3)$, $\bar{x}_3 = (x_4, x_5)$, $\bar{x}_4 = (x_6)$ and $\bar{y} = (x_7, x_8)$. Moreover, $\varphi^{-}(\bar{X}, \bar{y}) = \neg X_1(x_7, x_8)$, $\theta(\bar{x}) = (x_1 < x_2 \wedge x_1 = x_3 \wedge x_1 = x_4 \wedge x_2 = x_8 \wedge x_2 = x_5 \wedge x_8 < x_6 \wedge x_6 = x_7)$, which defines a total order on the partition of $\bar{x}$ $\{\{x_1, x_3, x_4\}, \{x_2, x_5, x_8\}, \{x_6, x_7\}\}$, and $\beta(\bar{x}) = \forall z \big(S_1(x_1, z, x_2) \wedge x_3 = x_3 \wedge x_4 = x_4 \wedge x_5 = x_5 \wedge x_6 = x_6 \wedge x_7 = x_7 \wedge x_8 = x_8 \big)$.

Similarly to the previous proof, we would like to eliminate the *lexicographically smallest*[1] tuple of predicates that satisfies the formula (1). Let $\bar{u}_i$ be a $a_{\lambda(i)}$-tuple of variables for every $i \in \{1, \ldots, n\}$, and let $m = \sum_{i=1}^{n} a_{\lambda(i)}$ be the number of variables of $(\bar{x}_1, \ldots, \bar{x}_n)$. We now define

$$\begin{aligned} \alpha_{\min}(\bar{u}_1, \ldots, \bar{u}_n) = {} & \exists \bar{y} \Big[\alpha(\bar{u}_1, \ldots, \bar{u}_n, \bar{y}) \wedge \\ & \forall \bar{v}_1 \cdots \forall \bar{v}_n \forall \bar{w} \Big(\big(\alpha(\bar{v}_1, \ldots, \bar{v}_n, \bar{w}) \wedge \bigvee_{i=1}^{n} (\bar{u}_i \neq \bar{v}_i) \big) \to \varphi_{m,<}((\bar{u}_1, \ldots, \bar{u}_n), (\bar{v}_1, \ldots, \bar{v}_n)) \Big) \Big], \end{aligned}$$

where $\alpha(\bar{x}) = \theta(\bar{x}) \wedge \beta(\bar{x})$. Note that $\alpha_{\min}$ is satisfied only by the lexicographically smallest assignment $(\bar{d}_1, \ldots, \bar{d}_n)$ to $(\bar{x}_1, \ldots, \bar{x}_n)$ such that $\mathfrak{A} \models \theta(\bar{d}_1, \ldots, \bar{d}_n, \bar{\ell})$ and $\mathfrak{A} \models \beta(\bar{d}_1, \ldots, \bar{d}_n, \bar{\ell})$ for some $\bar{\ell} \in A^p$. Our new formula is

$$\begin{aligned} \varphi'(\bar{x}, \bar{X}) = {} & \left(\bigwedge_{i=1}^{n} X_{\lambda(i)}(\bar{x}_i) \right) \wedge \varphi^{-}(\bar{X}, \bar{y}) \wedge \theta(\bar{x}) \wedge \beta(\bar{x}) \wedge \\ & \exists \bar{u}_1 \cdots \exists \bar{u}_n \Bigg[\alpha_{\min}(\bar{u}_1, \ldots, \bar{u}_n) \wedge \Bigg(\left(\bigvee_{i=1}^{n} \neg X_{\lambda(i)}(\bar{u}_i) \right) \vee \bigvee_{i=1}^{r} \exists \bar{v} \Big(X_i(\bar{v}) \wedge \bigwedge_{j \in [1,n] :\, \lambda(j) = i} \bar{v} \neq \bar{u}_j \Big) \Bigg) \Bigg]. \end{aligned} \tag{2}$$

We now show a result by which the main proof will follow.

Lemma 5.1. $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})} = f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})} \dot{-} 1$.

[1] We consider the lexicographically smallest tuple of predicates as the one in which its predicates contain the lexicographically smallest tuples and do not contain any more tuples than those

Proof. Let $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$. Consider two cases: assume first that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{R})$ for some assignment $\bar{R}$ to $\bar{X}$. Let $\bar{d} = (\bar{d}_1, \dots, \bar{d}_n, \bar{o})$ be the lexicographically smallest assignment to $\bar{x}$ for which $\mathfrak{A} \models \alpha(\bar{d})$, where $\bar{d}_i$ is the respective assignment to $\bar{x}_i$, for every $i \in \{1, \dots, n\}$, and $\bar{o}$ is an assignment for $\bar{y}$. Consider now the tuple $\bar{P} = (P_1, \dots, P_r)$ where $P_i = \bigcup_{j \in [1,n]\colon \lambda(j)=i} \{\bar{d}_j\}$. We will show that this assignment to $\bar{X}$ is such that (a) $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P})$, (b) $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P})$ and (c) $\bar{P}$ is the only assignment that satisfies (a) and (b).

(a) By contradiction, suppose that $\mathfrak{A} \not\models \exists \bar{x}\, \varphi(\bar{x}, \bar{P})$. That is, there is no assignment $\bar{s}$ to $\bar{x}$ such that $\mathfrak{A} \models \varphi(\bar{s}, \bar{P})$. Since $\bar{d}$ is such that $\mathfrak{A} \models \bigwedge_{i=1}^{n} P_{\lambda(i)}(\bar{d}_i)$ and $\mathfrak{A} \models \alpha(\bar{d})$, it follows that $\mathfrak{A} \not\models \varphi^{-}(\bar{P}, \bar{o})$ (since $\alpha(\bar{d}) = \theta(\bar{d}) \wedge \beta(\bar{d})$). Therefore, there is an $i \in \{1, \dots, n\}$ such that $\neg P_{\lambda(i)}(\bar{d}_i)$ appears in $\varphi^{-}(\bar{P}, \bar{o})$. Then, there is an $a_{\lambda(i)}$-tuple $\bar{z}$ in $\bar{y}$ such that $\neg X_{\lambda(i)}(\bar{z})$ appears in $\varphi^{-}(\bar{X}, \bar{x})$. We know that either (1) $\theta(\bar{x}) \models \bar{z} = \bar{x}_i$, (2) $\theta(\bar{x}) \models \varphi_{<,a_i}(\bar{z}, \bar{x}_i)$ or (3) $\theta(\bar{x}) \models \varphi_{<,a_i}(\bar{x}_i, \bar{z})$. Considering that (2) and (3) are not possible given that both $\bar{z}$ and $\bar{x}_i$ are assigned the value $\bar{d}_i$ and $\mathfrak{A} \models \theta(\bar{d})$, we have that $\theta(\bar{x}) \models \bar{z} = \bar{x}_i$. But if this is the case, then $X_{\lambda(i)}(\bar{x}_i), \neg X_{\lambda(i)}(\bar{z})$ and $\bar{z} = \bar{x}_i$ are all logical consequences of $\varphi(\bar{x}, \bar{X})$, which means $\varphi(\bar{x}, \bar{X})$ is inconsistent. That is, there's no structure $\mathfrak{A}' \in \textsc{Struct}[\mathcal{L}]$ such that $\mathfrak{A}' \models \exists \bar{x}\, \varphi(\bar{x}, \bar{R}')$ for any assignment $\bar{R}'$ to $\bar{X}$. In particular, $\mathfrak{A} \not\models \exists \bar{x}\, \varphi(\bar{x}, \bar{R}')$ for every possible assignment $\bar{R}'$ to $\bar{X}$, which contradicts the initial assumption that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{R})$ for some assignment $\bar{R}$ to $\bar{X}$.

(b) Note that if $\mathfrak{A} \models \alpha_{\min}(\bar{c}_1, \dots, \bar{c}_n)$, then necessarily $\bar{c}_i = \bar{d}_i$ for $i \in \{1, \dots, n\}$. However, by the construction of $\bar{P}$, we see that

$$\mathfrak{A} \not\models \bigvee_{i=1}^{n} \neg P_{\lambda(i)}(\bar{d}_i) \text{ and that } \mathfrak{A} \not\models \bigvee_{i=1}^{r} \exists \bar{v} \Big(P_i(\bar{v}) \wedge \bigwedge_{j \in [1,n]\colon \lambda(j)=i} \bar{v} \neq \bar{d}_j \Big).$$

Then, $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P})$.

(c) By contradiction, let $\bar{P}' \neq \bar{P}$ be such that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P}')$ and $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P}')$. We consider two cases: first, suppose that $\bar{P}'$ is missing a tuple of $\bar{P}$. Let $i \in \{1, \dots, n\}$ such that in $\bar{d}_i$ is not in P'_i. Then $\mathfrak{A} \models \neg P'_i(\bar{d}_i)$, and so,

$$\mathfrak{A} \models \Big(\bigvee_{i=1}^{n} \neg X_{\lambda(i)}(\bar{d}_i) \Big),$$

from which we conclude that $\mathfrak{A} \models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P}')$. Second, suppose there is some predicate $\bar{P}'_i$ in $\bar{P}'$ which has a tuple that P_i does not have. If this is the case, then

$$\mathfrak{A} \models \bigvee_{i=1}^{r} \exists \bar{v} \Big(P_i(\bar{v}) \wedge \bigwedge_{j \in [1,n]\colon \lambda(j)=i} \bar{v} \neq \bar{u}_j \Big),$$

so $\mathfrak{A} \models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P}')$. On both cases, we have a contradiction.

With this, we conclude that for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$ such that $f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) > 0$, we have that $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})}(\mathfrak{A}) = f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) - 1$.

Second, assume that there is no assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{R})$. Let $\bar{P}$ be an arbitrary assignment to $\bar{X}$. Since $\mathfrak{A} \not\models \exists \bar{x}\, \varphi(\bar{x}, \bar{P})$, we see that $\mathfrak{A} \not\models \exists \bar{x}\, (\varphi(\bar{x}, \bar{P}) \wedge \psi(\bar{x}, \bar{P}))$ for any formula $\psi(\bar{x}, \bar{P})$. It follows that there is no assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists \bar{x}\, \varphi'(\bar{x}, \bar{R})$. And so, for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$ such that $f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) = 0$, it holds that $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})}(\mathfrak{A}) = 0$. We conclude that $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})} = f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})} \dot{-} 1$, which was to be shown. □

We now continue with the general case, in which $\varphi(\bar{x}, \bar{X})$ is an arbitrary extended quantifier-free $\mathcal{L}$-formula. By using a standard DNF transformation algorithm and considering FO-formulas over $\mathcal{L}$ as literals, we can find formulas $\gamma_i(\bar{x}, \bar{X})$ with $i \in \{1, \dots, \ell\}$ such that

$$\varphi(\bar{x}, \bar{X}) \quad \equiv \quad \gamma_1(\bar{x}, \bar{X}) \vee \gamma_2(\bar{x}, \bar{X}) \vee \cdots \vee \gamma_\ell(\bar{x}, \bar{X}),$$

where, for every $i \in \{1, \ldots, \ell\}$,

$$\gamma_i(\bar{x}, \bar{X}) \;=\; \left(\bigwedge_{j=1}^{n_i} X_{\lambda_i(j)}(\bar{x}_{i,j})\right) \wedge \gamma_i^-(\bar{X}, \bar{y}_i) \wedge \delta_i(\bar{x}).$$

The function λ_i is defined analogously to λ of the first part of the proof. The tuple $\bar{x}_{i,j}$ has $a_{\lambda_i(j)}$ variables for $j \in \{1, \ldots, n_i\}$, $\bar{y}_i$ has p_i variables and are such that $(\bar{x}_{i,1}, \ldots, \bar{x}_{i,n_i}, \bar{y}_i) = \bar{x}$. The formulas $\gamma_i^-(\bar{X}, \bar{y}_i)$, $\delta_i(\bar{x})$ are defined analogously to $\varphi^-(\bar{X}, \bar{y})$ and $\beta(\bar{x})$ [2] respectively (see (1)). Let g be a function that counts the number of possible orders over partitions on a d-tuple of variables. Let the formulas $\theta^i(\bar{x})$ for $i \in \{1, \ldots, g(d)\}$ represent each of these orders over $\bar{x}$. Note that for every formula $\eta(\bar{x})$,

$$\exists \bar{x}\, \eta(\bar{x}) \;\equiv\; \exists \bar{x}(\eta(\bar{x}) \wedge \theta^1(\bar{x})) \vee \cdots \vee \exists \bar{x}(\eta(\bar{x}) \wedge \theta^{g(d)}(\bar{x})).$$

We define the following formulas $\xi_i(\bar{x}, \bar{X})$, for $i \in \{1, \ldots, m\}$ where $m = \ell \cdot g(d)$, as follows:

$$\begin{aligned}
\xi_1(\bar{x}, \bar{X}) &= \gamma_1(\bar{x}, \bar{X}) \wedge \theta^1(\bar{x}),\\
&\vdots\\
\xi_{g(d)}(\bar{x}, \bar{X}) &= \gamma_1(\bar{x}, \bar{X}) \wedge \theta^{g(d)}(\bar{x}),\\
\xi_{g(d)+1}(\bar{x}, \bar{X}) &= \gamma_2(\bar{x}, \bar{X}) \wedge \theta^1(\bar{x}),\\
&\vdots\\
\xi_{2\cdot g(d)}(\bar{x}, \bar{X}) &= \gamma_2(\bar{x}, \bar{X}) \wedge \theta^{g(d)}(\bar{x}),\\
&\vdots\\
\xi_{(\ell-1)g(d)+1}(\bar{x}, \bar{X}) &= \gamma_\ell(\bar{x}, \bar{X}) \wedge \theta^1(\bar{x}),\\
&\vdots\\
\xi_{\ell\cdot g(d)}(\bar{x}, \bar{X}) &= \gamma_\ell(\bar{x}, \bar{X}) \wedge \theta^{g(d)}(\bar{x}).
\end{aligned}$$

Having every disjunct with a total order allows us to eliminate the ones that are unsatisfiable for every $\mathcal{L}$-structure $\mathfrak{A}$, that is, each $i \in \{1, \ldots, m\}$ such that for every assignment $\bar{s}$ to $\bar{x}$, $f_{\xi_i(\bar{s}, \bar{X})}(\mathfrak{A}) = 0$, for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$. Let k be the number of disjuncts that are left after eliminating the unsatisfiable disjuncts. We use an injective[3] function $\rho : \{1, \ldots, k\} \to \{1, \ldots, m\}$ such that $\varphi_i(\bar{x}, \bar{X}) = \xi_{\rho(i)}(\bar{x}, \bar{X})$ is satisfiable, for every $i \in \{1, \ldots, k\}$.

We can conclude that

$$\varphi(\bar{x}, \bar{X}) \;\equiv\; \varphi_1(\bar{x}, \bar{X}) \vee \varphi_2(\bar{x}, \bar{X}) \vee \cdots \vee \varphi_k(\bar{x}, \bar{X}),$$

and for every $i \in \{1, \ldots, k\}$,

$$\varphi_i(\bar{x}, \bar{X}) \;=\; \left(\bigwedge_{j=1}^{n_i} X_{\lambda_i(j)}(\bar{x}_{i,j})\right) \wedge \varphi_i^-(\bar{X}, \bar{y}_i) \wedge \theta_i(\bar{x}) \wedge \beta_i(\bar{x}),$$

where n_i is the number of times a non-negated variable in $\bar{X}$ is referred to, according to the function $\lambda_i : \{1, \ldots, n_i\} \to \{1, \ldots, r\}$, $\bar{y}_i$ is a p_i-tuple of variables in $\bar{x}$, $\varphi_i^-(\bar{X}, \bar{y}_i)$ is a conjunction of negated predicates in $\bar{X}$, $\theta_i(\bar{x})$ defines a total order on a partition of $\bar{x}$, and $\beta_i(\bar{x})$ is an FO-formula over $\mathcal{L}$ which mentions all variables in $\bar{x}$. Let $\alpha_i(\bar{x}) = \theta_i(\bar{x}) \wedge \beta_i(\bar{x})$.

[2] Note that $\delta_i(\bar{x})$ can include subformulas of the form $(\bar{u} = \bar{u})$.

[3] We need this function to be injective because this way we can assure each one of the k satisfiable disjuncts to be represented

Claim 5.1. *Let $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$ and $i \in \{1, \ldots, k\}$. For every assignment $\bar{s}$ to $\bar{x}$ such that $\mathfrak{A} \models \alpha_i(\bar{s})$, it holds that $f_{\varphi_i(\bar{s},\bar{X})}(\mathfrak{A}) > 0$.*

Proof. Let $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_{n_i}, \bar{t})$ such that $\mathfrak{A} \models \alpha_i(\bar{s})$, and let $\bar{P} = (\bar{P}_1, \ldots, \bar{P}_r)$ where $P_i = \bigcup_{j\in[1,n_i]:\, \lambda(j)=i}\{s_i\}$. We will show that $\mathfrak{A} \models \varphi_i(\bar{s}, \bar{P})$. By contradiction, suppose that $\mathfrak{A} \not\models \exists\bar{x}\, \varphi_i(\bar{x}, \bar{P})$. That is, there is no assignment $\bar{e}$ to $\bar{x}$ such that $\mathfrak{A} \models \varphi(\bar{e}, \bar{P})$. Following the same proof as in case (a) of Lemma 5.1, we conclude that $\varphi_i(\bar{x}, \bar{P})$ is inconsistent. However, as we mentioned previously, all of such disjuncts have been eliminated, which leads to a contradiction. □

Our plan now is to exclude the lexicographically smallest assignment $\bar{P}$ such that $\mathfrak{A} \models \exists\bar{x}\, \varphi_1(\bar{x}, \bar{P})$, and if there is no such $\bar{P}$, exclude the lexicographically smallest $\bar{Q}$ such that $\mathfrak{A} \models \exists\bar{x}\, \varphi_2(\bar{x}, \bar{Q})$, and so on. As we already know how to exclude that assignment in the first disjunct, we will now deal with the next disjuncts. Similarly to the first part of the proof, for each $i \in \{1, \ldots, k\}$ let $m_i = \sum_{j=1}^{n_i} a_{\lambda_i(j)}$ and let

$$\alpha^i_{\min}(\bar{u}_1, \ldots, \bar{u}_{n_i}) = \exists\bar{y}\Big[\alpha_i(\bar{u}_1, \ldots, \bar{u}_{n_i}, \bar{y}) \wedge \\ \forall\bar{v}_1 \cdots \forall\bar{v}_{n_i} \forall\bar{w}\Big(\big(\alpha_i(\bar{v}_1, \ldots, \bar{v}_{n_i}, \bar{w}) \wedge \bigvee_{j=1}^{n_i} (\bar{u}_j \neq \bar{v}_j)\big) \to \varphi_{m_i,<}((\bar{u}_1, \ldots, \bar{u}_{n_i}), (\bar{v}_1, \ldots, \bar{v}_{n_i}))\Big)\Big],$$

where $\bar{u}_j$ and $\bar{v}_j$, have $a_{\lambda_i(j)}$ variables for $j \in \{1, \ldots, n_i\}$, and $\bar{w}$ has p_i variables. Also, let

$$\psi_i(\bar{X}) = \forall\bar{x}\, \neg\alpha_i(\bar{x}) \vee \exists\bar{u}_1 \cdots \exists\bar{u}_{n_i}\Big[\alpha^i_{\min}(\bar{u}_1, \ldots, \bar{u}_{n_i}) \\ \wedge \Big(\Big(\bigvee_{j=1}^{n_i} \neg X_{\lambda_i(j)}(\bar{u}_i)\Big) \vee \bigvee_{j=1}^{r} \exists\bar{v}\Big(X_j(\bar{v}) \wedge \bigwedge_{\ell\in[1,n_i]:\lambda_i(\ell)=j} \bar{v} \neq \bar{u}_\ell\Big)\Big)\Big].$$

Note that $\psi_i(\bar{X})$ excludes only the lexicographically smallest tuple of predicates $\bar{P}$ such that $\mathfrak{A} \models \exists\bar{x}\, \varphi_i(\bar{P}, \bar{x})$, if there is at least one. In other words, every assignment $\bar{P}' \neq \bar{P}$ is such that $\mathfrak{A} \models \psi_i(\bar{P}')$. Our new formula $\varphi'_i(\bar{x}, \bar{X})$ is defined as follows:

$$\varphi'_i(\bar{x}, \bar{X}) = \varphi_i(\bar{x}, \bar{X}) \wedge \psi_1(\bar{X}) \wedge (\exists\bar{v}\, \alpha_1(\bar{v}) \vee \psi_2(\bar{X})) \wedge \cdots \wedge \\ (\exists\bar{v}\, \alpha_1(\bar{v}) \vee \cdots \vee \exists\bar{v}\, \alpha_{i-1}(\bar{v}) \vee \psi_i(\bar{X})). \quad (3)$$

Let $\varphi'(\bar{x}, \bar{X}) = \varphi'_1(\bar{x}, \bar{X}) \vee \cdots \vee \varphi'_k(\bar{x}, \bar{X})$. We will now show that $f_{\exists\bar{x}\varphi'(\bar{x},\bar{X})} = f_{\exists\bar{x}\varphi(\bar{x},\bar{X})} \dot{-} 1$. Assume that $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$. Suppose first that there is at least one assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists\bar{x}\, \varphi(\bar{x}, \bar{R})$. Let q be the least $i \in \{1, \ldots, k\}$ such that there exists at least one assignment $\bar{R}'$ to $\bar{X}$ for which $\mathfrak{A} \models \exists\bar{x}\, \varphi_i(\bar{x}, \bar{R}')$. Let $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_{n_q}, \bar{o})$ be the lexicographically smallest assignment to $\bar{x}$ for which $\mathfrak{A} \models \alpha_q(\bar{d})$, where $\bar{d}_i$ is the corresponding assignment to $\bar{x}_i$, for every $i \in \{1, \ldots, n_q\}$, and $\bar{o}$ is an assignment for $\bar{y}$. Consider now the tuple $\bar{P} = (P_1, \ldots, P_r)$ where $P_i = \bigcup_{j:\lambda_q(j)=i, j\in[1,n_q]}\{\bar{d}_j\}$. As we did in Lemma 5.1 we will show that $\bar{P}$ is such that (a) $\mathfrak{A} \models \exists\bar{x}\, \varphi(\bar{x}, \bar{P})$, (b) $\mathfrak{A} \not\models \exists\bar{x}\, \varphi'(\bar{x}, \bar{P})$, and (c) $\bar{P}$ is the only assignment to $\bar{X}$ that satisfies both (a) and (b)

(a) As we showed in part (a) of Lemma 5.1, if there is at least one assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists\bar{x}\, \varphi_q(\bar{x}, \bar{R})$, then $\mathfrak{A} \models \exists\bar{x}\, \varphi_q(\bar{x}, \bar{P})$ for this particular $\bar{P}$. However, as we showed in Claim 5.1, if there is an assignment $\bar{s}$ to $\bar{x}$ such that $\mathfrak{A} \models \alpha_q(\bar{s})$, then there *is* such an assignment to $\bar{X}$. The assignment $\bar{d}$ to $\bar{x}$ satisfies that $\mathfrak{A} \models \alpha_q(\bar{d})$, so it holds that $\mathfrak{A} \models \exists\bar{x}\, \varphi_q(\bar{x}, \bar{P})$. It immediately follows that $\mathfrak{A} \models \exists\bar{x}\, \varphi(\bar{x}, \bar{P})$.

(b) We will show that $\mathfrak{A} \not\models \exists\bar{x}\, \varphi'_i(\bar{x}, \bar{P})$ for (1) $i \in \{1, \ldots, q-1\}$, and (2) $i \in \{q, \ldots, k\}$. (1) By the choice of q, it holds that $\mathfrak{A} \not\models \exists\bar{x}\, \varphi'_i(\bar{x}, \bar{P})$ for every $i \in \{1, \ldots, q-1\}$ since there is no possible assignment to $\bar{X}$ for any of their sub-formulas $\varphi_i(\bar{x}, \bar{X})$. (2) We can use the proof in Lemma 5.1 to see that $\mathfrak{A} \not\models \psi_q(\bar{P})$. For each $i \in \{q, \ldots, k\}$, the sub-formula

$$\zeta_q(\bar{X}) = (\exists \bar{v}\, \alpha_1(\bar{v}) \vee \cdots \vee \exists \bar{v}\, \alpha_{q-1}(\bar{v}) \vee \psi_q(\bar{X}))$$

appears as a conjunct in $\varphi'_i(\bar{x}, \bar{X})$. However, by the choice of q, there is no $i \in \{1, \ldots, q-1\}$ such that $\mathfrak{A} \models \exists \bar{v}\, \varphi_i(\bar{v}, \bar{P})$, and also $\mathfrak{A} \not\models \psi_q(\bar{P})$. It follows that $\mathfrak{A} \not\models \zeta_q(\bar{P})$, so $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'_i(\bar{x}, \bar{P})$. And so, we conclude that $\mathfrak{A} \not\models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P})$.

(c) Suppose there is an assignment $\bar{P}' \neq \bar{P}$ to $\bar{X}$ that satisfies both (a) and (b). As we deduce from the part (c) of Lemma 5.1, $\bar{P}$ is the only assignment to $\bar{X}$ such that $\mathfrak{A} \not\models \psi_q(\bar{P})$, so necessarily $\mathfrak{A} \models \psi_q(\bar{P}')$. Since $\bar{P}'$ assigned to $\bar{X}$ satisfies (a), then $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P}')$. By the choice of q, every $i \in \{1, \ldots, q-1\}$ is such that $\mathfrak{A} \models \forall \bar{x}\, \neg \alpha_i(\bar{x})$, so $\mathfrak{A} \models \psi_i(\bar{P}')$ for each i. But as we mentioned, also $\mathfrak{A} \models \psi_q(\bar{P}')$, which means that $\mathfrak{A} \models \exists \bar{x}\, \varphi'_q(\bar{x}, \bar{P}')$, and so, $\mathfrak{A} \models \exists \bar{x}\, \varphi'(\bar{x}, \bar{P}')$, which leads to a contradiction.

With this, we conclude that for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$ such that $f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) > 0$, we have that $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})}(\mathfrak{A}) = f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) - 1$.

Second, assume that there is no assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists \bar{x}\, \varphi_i(\bar{x}, \bar{R})$ for any $i \in \{1, \ldots, k\}$. Let $\bar{P}$ be an arbitrary assignment to $\bar{X}$. Since $\mathfrak{A} \not\models \exists \bar{x}\, \varphi_i(\bar{x}, \bar{P})$, for any i, we see that $\mathfrak{A} \not\models \exists \bar{x}\, (\varphi(\bar{x}, \bar{P}) \wedge \chi(\bar{x}, \bar{P}))$ for any formula $\chi(\bar{x}, \bar{P})$. It follows that there is no assignment $\bar{R}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists \bar{x}\, \varphi'_i(\bar{x}, \bar{R})$, for any $i \in \{1, \ldots, k\}$. And so, for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$ such that $f_{\exists \bar{x}\, \varphi(\bar{x}, \bar{X})}(\mathfrak{A}) = 0$, we have that $f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})}(\mathfrak{A}) = 0$. Hence, from the results in the previous paragraph, if $f' = f_{\exists \bar{x}\, \varphi'(\bar{x}, \bar{X})}$, we have that $f' = f \dot{-} 1 \in \#\Sigma_1[\mathrm{FO}]$

3. Let $f \in \#\Sigma_1[\mathrm{FO}]$ be defined by an extended quantifier free $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{X}, \bar{z})$, where $\bar{x} = (x_1, \ldots, x_d)$, $\bar{X} = (X_1, \ldots, X_r)$ and $\bar{z} = (z_1, \ldots, z_p)$. That is,

$$f(\mathfrak{A}) \quad = \quad |\, \{ \langle \bar{P}, \bar{e} \rangle \mid \mathfrak{A} \models \exists \bar{x}\ \varphi(\bar{x}, \bar{P}, \bar{e}) \}\, |,$$

for every $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, where $\bar{P} = (P_1, \ldots, P_r)$, $P_i \subseteq A^{a_i}$ for every $i \in \{1, \ldots, r\}$ and $\bar{e} \in A^p$. In order to prove that $f \dot{-} 1 \in \#\Sigma_1[\mathrm{FO}]$, we define the formulas $\varphi_i(\bar{x}, \bar{X}, \bar{z})$ for $i \in \{1, \ldots, k\}$ in the same way as case 2, where

$$\varphi(\bar{x}, \bar{X}, \bar{z}) \equiv \varphi_1(\bar{x}, \bar{X}, \bar{z}) \vee \cdots \vee \varphi_k(\bar{x}, \bar{X}, \bar{z}),$$

and

$$\varphi_i(\bar{x}, \bar{X}, \bar{z}) \quad = \quad \left(\bigwedge_{j=1}^{n_i} X_{\lambda_i(j)}(\bar{x}_{i,j}, \bar{z}_{i,j}) \right) \wedge \varphi_i^-(\bar{X}, \bar{y}_i, \bar{w}_i) \wedge \theta_i(\bar{x}, \bar{z}) \wedge \beta_i(\bar{x}, \bar{z}),$$

where n_i is the number of times a non-negated variable in $\bar{X}$ is referred to, according to the function $\lambda_i : \{1, \ldots, n_i\} \to \{1, \ldots, r\}$, the tuple $\bar{x}_{i,j}$ has $b_{\lambda_i(j)}$ variables and the tuple $\bar{z}_{i,j}$ has $c_{\lambda_i(j)}$ for $j \in \{1, \ldots, n_i\}$ (note that $b_\ell + c_\ell = a_\ell$ for $\ell \in \{1, \ldots, r\}$), $\bar{y}_i$ has p_i variables, and $\bar{w}_i$ has q_i variables. Furthermore, we have that $(\bar{x}_{i,1}, \ldots, \bar{x}_{i,n_i}, \bar{y}) = \bar{x}$ and $(\bar{z}_{i,1}, \ldots, \bar{z}_{i,n_i}, \bar{w}_i) = \bar{z}$. The formula $\theta(\bar{x}, \bar{z})$ defines a total order, analogously to case 2. The formulas $\varphi_i^-(\bar{X}, \bar{y}_i, \bar{w}_i)$, $\beta_i(\bar{x}, \bar{z})$, are also defined analogously.

In this case, we mix both the strategies in cases 1 and 2. That is, we are going to *isolate* the lexicographically smallest tuple of predicates that satisfies the first satisfiable disjunct, and then *exclude* the lexicographically smallest tuple that satisfies the isolated disjunct.

Let $m_i = \sum_{j=1}^{n_i} a_{\lambda_i(j)}$ and

$$\alpha^i_{\min}(\bar{x}_1, \ldots, \bar{x}_{n_i}, \bar{z}_1, \ldots, \bar{z}_{n_i}) = \exists \bar{y} \exists \bar{w} \Big[\alpha_i(\bar{x}_1, \ldots, \bar{x}_{n_i}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_{n_i}, \bar{w}) \wedge$$
$$\forall \bar{u}_1 \cdots \forall \bar{u}_{n_i} \forall \bar{s} \forall \bar{v}_1 \cdots \forall \bar{v}_{n_i} \forall \bar{t} \Big(\big(\alpha_i(\bar{u}_1, \ldots, \bar{u}_{n_i}, \bar{s}, \bar{v}_1, \ldots, \bar{v}_{n_i}, \bar{t}) \wedge \bigvee_{j=1}^{n_i} (\bar{x}_j \neq \bar{u}_j \vee \bar{z}_j \neq \bar{v}_j)\big) \rightarrow$$
$$\varphi_{m_i,<}((\bar{x}_1, \ldots, \bar{x}_{n_i}, \bar{z}_1, \ldots, \bar{z}_{n_i}), (\bar{u}_1, \ldots, \bar{u}_{n_i}, \bar{v}_1, \ldots, \bar{v}_{n_i})) \Big) \Big],$$

and let

$$\psi_i(\bar{X}, \bar{z}) = \forall \bar{x} \forall \bar{v} \, \neg \alpha_i(\bar{x}, \bar{v}) \vee \exists \bar{u}_1 \cdots \exists \bar{u}_{n_i} \exists \bar{w}_1 \cdots \exists \bar{w}_{n_i} \Big[\alpha^i_{\min}(\bar{u}_1, \ldots, \bar{u}_{n_i}, \bar{w}_1, \ldots, \bar{w}_{n_i})$$
$$\wedge \Big(\Big(\bigvee_{j=1}^{n_i} \neg X_{\lambda_i(j)}(\bar{u}_i, \bar{w}_i) \Big) \vee \bigvee_{j=1}^{r} \exists \bar{s} \exists \bar{t} \Big(X_j(\bar{s}, \bar{t}) \wedge \bigwedge_{\ell \in [1, n_i] : \lambda_i(\ell) = j} \bar{s} \neq \bar{u}_\ell \vee \bar{t} \neq \bar{w}_\ell \Big) \Big) \Big] \vee$$
$$\exists \bar{w} (\exists \bar{u} \, \varphi(\bar{u}, \bar{X}, \bar{w}) \wedge \varphi_{p,<}(\bar{w}, \bar{z})).$$

Let $\bar{P}$ and $\bar{e}$ be assignments to $\bar{X}$ and $\bar{z}$. Note that $\psi_i(\bar{X}, \bar{z})$ excludes $\bar{P}$ and $\bar{e}$ only if $\bar{P}$ is the *lexicographically smallest* tuple of predicates (Same as case 2) such that $\mathfrak{A} \models \exists \bar{x} \, \varphi_i(\bar{x}, \bar{P}, \bar{d})$, for some assignment $\bar{d}$ to $\bar{z}$, and $\bar{e}$ is the lexicographically smallest assignment to $\bar{z}$ such that $\mathfrak{A} \models \exists \bar{x} \, \varphi_i(\bar{x}, \bar{P}, \bar{e})$. We define $\varphi'_i(\bar{x}, \bar{X}, \bar{z})$ in the same way as case 2:

$$\varphi'_i(\bar{x}, \bar{X}, \bar{z}) = \varphi_i(\bar{x}, \bar{X}, \bar{z}) \wedge \psi_1(\bar{X}, \bar{z}) \wedge (\exists \bar{u} \exists \bar{v} \, \alpha_1(\bar{u}, \bar{v}) \vee \psi_2(\bar{X}, \bar{z})) \wedge \cdots \wedge$$
$$(\exists \bar{u} \exists \bar{v} \, \alpha_1(\bar{u}, \bar{v}) \vee \cdots \vee \exists \bar{u} \exists \bar{v} \, \alpha_{i-1}(\bar{u}, \bar{v}) \vee \psi_i(\bar{X}, \bar{z})).$$

Finally, let $\varphi'(\bar{x}, \bar{X}, \bar{z}) = \bigvee_{i=1}^{k} \varphi'_i(\bar{x}, \bar{X}, \bar{z})$.

Let q be the least $i \in \{1, \ldots, k\}$ such that $\mathfrak{A} \models \exists \bar{x} \, \varphi_i(\bar{x}, \bar{R}, \bar{d})$ for some assignment R to $\bar{X}$ and some assignment $\bar{d}$ to $\bar{z}$. Let $\bar{P}$ be the lexicographically smallest tuple of predicates such that $\mathfrak{A} \models \exists \bar{x} \, \varphi_q(\bar{x}, \bar{P}, \bar{d}')$ for some assignment $\bar{d}'$ to $\bar{z}$. Let $\bar{e}$ be the lexicographically smallest assignment to $\bar{z}$ such that $\mathfrak{A} \models \exists \bar{x} \, \varphi_q(\bar{x}, \bar{P}, \bar{e})$. This formula is such that (a) $\mathfrak{A} \models \exists \bar{x} \, \varphi(\bar{x}, \bar{P}, \bar{e})$, (b) $\mathfrak{A} \not\models \exists \bar{x} \, \varphi'(\bar{x}, \bar{P}, \bar{e})$ and $\bar{P}$ and $\bar{e}$ are the only assignments that satisfy (a) and (b). The proof of this is analogous to case 2. Therefore, we conclude that $f_{\exists \bar{x} \, \varphi'(\bar{x}, \bar{X}, \bar{z})} = f_{\exists \bar{x} \, \varphi(\bar{x}, \bar{X}, \bar{z})} \dot{-} 1$.

□

Theorem 6. *#DNF is hard for $\#\Sigma_1[\mathrm{FO}]$ under parsimonious reductions.*

Proof. Let f be an arbitrary function in $\#\Sigma_1[\mathrm{FO}]$ and let f_{DNF} be the function that characterizes #DNF. We will now show a function $h : \textsc{Struct}[\mathcal{L}] \rightarrow L(P)$ such that for every $\mathfrak{A} \in \textsc{Struct}[\mathcal{L}]$, it holds that $f(\mathfrak{A}) = f_{\mathrm{DNF}}(h(\mathfrak{A}))$, and which can be computed in polynomial time. Let ψ be such that $f = f_\psi$. We separate the proof in three cases:

1. All free variables in ψ are of first order. Let $\psi = \exists x \, \varphi(\bar{x}, \bar{z})$ where $\bar{x} = (x_1, \ldots, x_c)$ and $\bar{z} = (z_1, \ldots, z_d)$. Given a structure $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, we generate a propositional formula $\Phi = f(\mathfrak{A})$. First we notice that this case is trivial since we can calculate this function in polynomial time, by checking for each $\bar{e} \in A^d$ if $\mathfrak{A} \models \exists \bar{x} \, \varphi(\bar{x}, \bar{e})$. Let n be the number of such assignments to $\bar{z}$. Let $\bar{p}$ be an n-tuple of propositional variables where $\bar{p} = (p_1, \ldots, p_n)$. The DNF formula we return is

$$\Phi = \bigvee_{i=1}^{n} \neg p_1 \wedge \cdots \wedge \neg p_{i-1} \wedge p_i \wedge \neg p_{i+1} \wedge \cdots \wedge \neg p_n$$

which has exactly n possible assignments.

2. There is only second-order free variables in ψ. Let $\psi = \exists x\, \varphi(\bar{x}, \bar{X})$, where $\bar{x} = (x_1, \dots, x_c)$ and $\bar{X} = (X_1, \dots, X_r)$. Given a structure $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$, we generate a DNF-formula $\Phi = f(\mathfrak{A})$ as follows:

   First, we convert $\varphi(\bar{x}, \bar{X})$ into an equivalent DNF formula $\bigvee_{i=1}^{k} \varphi_i(\bar{x}, \bar{X})$. Let each $\varphi_i(\bar{x}, \bar{X})$ be of the form
   $$\varphi_i(\bar{x}, \bar{X}) = X_{s_1}(\bar{x}_{s_1}) \wedge \cdots \wedge X_{s_m}(\bar{x}_{s_m}) \wedge \neg X_{t_1}(\bar{x}_{t_1}) \wedge \cdots \wedge \neg X_{t_n}(\bar{x}_{t_n}) \wedge \alpha_i(\bar{x}),$$
   where every $\bar{x}_{s_i}$ and $\bar{x}_{t_j}$ have the corresponding number of variables, and $\alpha(\bar{x})$ is a FO-formula.

   Then, let $\Gamma_i = \{\bar{a} \in A^c \mid \mathfrak{A} \models \alpha(\bar{a})\}$. There is exactly $|A|^c$ tuples to check, and each of those checks can be done in polynomial time, so this set can also be generated in polynomial time. We generate a propositional variable $p_{\bar{a}}^{\ell}$ for each $\bar{a} \in \bigcup_{j=1}^{c} A^j$, and for each $\ell \in \{1, \dots, r\}$. Let $\bar{p}$ be the tuple of all such variables. Also, let $\rho_\ell(\bar{a})$ be the tuple that results of reordering $\bar{a}$ using the reordering of the variables of $\bar{x}$ in $\bar{x}_\ell$.

   Finally, for each $\varphi_i(\bar{x}, \bar{X})$ the DNF formula we return is
   $$\Phi_i = \bigvee_{\bar{a} \in \Gamma_i} p^{s_1}_{\rho_{s_1}(\bar{a})} \wedge \cdots \wedge p^{s_m}_{\rho_{s_m}(\bar{a})} \wedge \neg p^{t_1}_{\rho_{t_1}(\bar{a})} \wedge \cdots \wedge \neg p^{t_n}_{\rho_{t_n}(\bar{a})}.$$
   and $\Phi = \bigvee_{i=1}^{k} \Phi_i$.

   Note that for each assignment $\bar{P}$ to $\bar{X}$ such that $\mathfrak{A} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{P})$ there is a corresponding assignment σ to $\bar{p}$ where $\sigma(p_{\bar{a}}^{\ell}) = 1$ if and only if $\bar{a} \in P_\ell$. We can conclude that $f(\mathfrak{A}) = f_{\mathrm{DNF}}(\Phi)$.

3. Let $\psi = \exists x\, \varphi(\bar{x}, \bar{X}, \bar{z})$, where $\bar{x} = (x_1, \dots, x_c)$, $\bar{X} = (X_1, \dots, X_r)$ and $\bar{z} = (z_1, \dots, z_d)$. Given a structure $\mathfrak{A} = \langle A, \bar{S}^{\mathfrak{A}}, \leq^{\mathfrak{A}} \rangle \in \textsc{Struct}[\mathcal{L}]$ we generate a DNF-formula $\Phi = f(\mathfrak{A})$ as follows:

   As in the previous case, we convert $\varphi(\bar{x}, \bar{X}, \bar{z})$ into an equivalent DNF formula $\bigvee_{i=1}^{k} \varphi_i(\bar{x}, \bar{X}, \bar{z})$. Let each $\varphi_i(\bar{x}, \bar{X}, \bar{z})$ be of the form
   $$\varphi_i(\bar{x}, \bar{X}, \bar{z}) = X_{s_1}(\bar{x}_{s_1}, \bar{z}_{s_1}) \wedge \cdots \wedge X_{s_m}(\bar{x}_{s_m}, \bar{z}_{s_m}) \wedge \neg X_{t_1}(\bar{x}_{t_1}) \wedge \cdots \wedge \neg X_{t_n}(\bar{x}_{t_n}) \wedge \alpha_i(\bar{x}),$$

$\square$

2 Extended Horn counting hierarchies

We define syntactically the classes $\#\Sigma_i[\text{FO}]$-HORN and $\#\Pi_i[\text{FO}]$-HORN as following. First we define Horn clauses.

$$\begin{array}{rcl} PL & ::= & X_i(\bar{x}),\ i \in \mathbb{N}, \\ NL & ::= & \neg X_i(\bar{x}),\ i \in \mathbb{N} \ \mid\ \exists x\, NL, \\ NC & ::= & NL \ \mid\ \alpha, \alpha \text{ is an FO-formula over } \mathcal{L} \ \mid\ (NC \vee NC), \\ HC & ::= & NC \ \mid\ (NC \vee PL), \end{array}$$

where $\bar{x}$ is a tuple with the corresponding number of variables. Now we define the syntax of the classes inductively.

1. $\#\Sigma_0[\text{FO}]$-HORN:

$$E_0 \quad ::= \quad HC \ \mid\ E_0 \wedge E_0.$$

2. $\#\Pi_0[\text{FO}]$-HORN:

$$U_0 \quad ::= \quad E_0.$$

3. $\#\Sigma_{i+1}[\text{FO}]$-HORN:

$$E_{i+1} \quad ::= \quad U_i \ \mid\ \exists x\, E_{i+1}.$$

4. $\#\Pi_{i+1}[\text{FO}]$-HORN:

$$U_{i+1} \quad ::= \quad E_i \ \mid\ \forall x\, U_{i+1}.$$

A function f is in $\#\Sigma_i[\text{FO}]$-HORN (resp. $\#\Pi_i[\text{FO}]$-HORN) if there is an $\mathcal{L}$-formula φ defined by the grammar E_i (resp. U_i) such that $f = f_\varphi$.

The class #P-Easy is defined in [?] as $\#\text{P-Easy} = \{f \mid f \in \#P \text{ and its decision version } L_f \in P\}$.

Theorem 7. $\#\Sigma_2[\text{FO}]\text{-HORN} \subseteq \#\text{P-Easy}$

Proof. Let $f = f_{\varphi(\bar{X},\bar{z})}$ such that $\varphi(\bar{X}, \bar{z})$ is defined by the grammar E_2. First we notice that, as stated in [1], for every $\mathcal{L}$-formula φ, $f_\varphi \in \#P$.

We will now prove that there is a polynomial time algorithm that decides L_f. This is equivalent to decide, given $\mathfrak{A} = \langle A, \bar{S}\mathfrak{A}, \leq^{\mathfrak{A}}\rangle \in \text{Struct}[\mathcal{L}]$, whether there exist assignments $\bar{P}, \bar{z}$ such that $\mathfrak{A} \models \varphi(\bar{P}, \bar{z})$. This is the same as $\mathfrak{A} \models \exists\bar{X}\exists\bar{x}\,\varphi(\bar{X}, \bar{x})$. Since $\varphi(\bar{X}, \bar{x})$ is defined by the grammar E_2, there exists $\psi(\bar{X}, \bar{x}, \bar{y}, \bar{u})$ such that $\varphi(\bar{X}, \bar{x}) = \exists\bar{x}\exists\bar{y}\forall\bar{u}\exists\bar{v}\,\psi(\bar{X}, \bar{x}, \bar{y}, \bar{u}, \bar{v})$, where every variable in $\bar{v}$ appears on a negated second-order literal $\neg X_i$, with $i \in \mathbb{N}$.

Given $\mathfrak{A} \in \text{Struct}[\mathcal{L}]$ we generate an equivalent $\mathcal{L}$-formula $\theta(\bar{X}, \bar{x}, \bar{y})$ with a series of operations. First, we replace every instance of $\bar{v}$ by a disjunction of all r-tuples in A^r, where r is the number of variables in $\bar{v}$. The remainder is still a Horn-clause. Second, we replace every instance of $\bar{u}$ by a conjunction of all s-tulples in A^s, where s is the number or variables in $\bar{u}$. The remainder is still a conjunction of Horn-clauses.

Now we notice that $\zeta = \exists\bar{X}\exists\bar{x}\exists\bar{y}\,\theta(\bar{X}, \bar{x}, \bar{y})$ is an existential second-order FO-formula, for which $\mathfrak{A} \models \zeta$ can be decided in polynomial time. $\square$

3 Some examples in extended logic counting classes

3.1 #3-DNF

We will now show that #3-DNF $\in \#\Sigma_1[\text{FO}]$.

Let $\mathcal{L} = \{S_0, S_1, S_2, S_3, \leq\}$, and $\mathfrak{A} = \langle A, S_0^{\mathfrak{A}}, S_1^{\mathfrak{A}}, S_2^{\mathfrak{A}}, S_3^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle$ be an $\mathcal{L}-$structure that represents an instance of a 3-DNF formula Ψ, where A is the set of variables mentioned in Ψ, $S_i^{\mathfrak{A}}$ is a ternary relation described as follows, for each $i \in \{0, 1, 2, 3\}$:

$$\begin{array}{rcl} S_0^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (\neg a_1 \wedge \neg a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \Psi\}, \\ S_1^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (a_1 \wedge \neg a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \Psi\}, \\ S_2^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (a_1 \wedge a_2 \wedge \neg a_3) \text{ appears as a disjunct in } \Psi\}, \\ S_3^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (a_1 \wedge a_2 \wedge a_3) \text{ appears as a disjunct in } \Psi\}. \end{array}$$

Now let T be a unary second order variable, and

$$\begin{aligned} \varphi(T) = {} & \exists x \exists y \exists z\, (S_0(x, y, z) \wedge \neg T(x) \wedge \neg T(y) \wedge \neg T(z)) \vee \\ & \exists x \exists y \exists z\, (S_1(x, y, z) \wedge T(x) \wedge \neg T(y) \wedge \neg T(z)) \vee \\ & \exists x \exists y \exists z\, (S_2(x, y, z) \wedge T(x) \wedge T(y) \wedge \neg T(z)) \vee \\ & \exists x \exists y \exists z\, (S_3(x, y, z) \wedge T(x) \wedge T(y) \wedge T(z)). \end{aligned}$$

It is easily seen that each relation T over $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi(T)$ represents a valid assignment to the variables in Ψ, and therefore, $f_{\varphi(T)}$ represents #3-DNF.

3.2 #3-HORN-SAT

We will now show that #3-HORN-SAT $\in \#\Pi_1[\text{FO}]$-HORN.

Let $\mathcal{L} = \{P_1, P_2, P_3, N_1, N_2, N_3, \leq\}$, and $\mathfrak{A} = \langle A, P_1^{\mathfrak{A}}, P_2^{\mathfrak{A}}, P_3^{\mathfrak{A}}, N_1^{\mathfrak{A}}, N_2^{\mathfrak{A}}, N_3^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle$ be an $\mathcal{L}-$structure that represents an instance of a 3-HORN-SAT formula Ψ, where A is the set of variables mentioned in Ψ, and the relations in $\mathcal{L}$ are interpreted as follows:

$$\begin{array}{rcl} P_1^{\mathfrak{A}} & = & \{a \mid a \text{ appears as a conjunct in } \Psi\}, \\ P_2^{\mathfrak{A}} & = & \{(a_1, a_2) \mid (a_1 \vee \neg a_2) \text{ appears as a conjunct in } \Psi\}, \\ P_3^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (a_1 \vee \neg a_2 \vee \neg a_3) \text{ appears as a conjunct in } \Psi\}, \\ N_1^{\mathfrak{A}} & = & \{a \mid (\neg a) \text{ appears as a conjunct in } \Psi\}, \\ N_2^{\mathfrak{A}} & = & \{(a_1, a_2) \mid (\neg a_1 \vee \neg a_2) \text{ appears as a conjunct in } \Psi\}, \\ N_3^{\mathfrak{A}} & = & \{(a_1, a_2, a_3) \mid (\neg a_1 \vee \neg a_2 \vee \neg a_3) \text{ appears as a conjunct in } \Psi\}. \end{array}$$

Now let T be a unary second order variable, and

$$\begin{aligned} \varphi(T) = {} & \forall x\, (\neg P_1(x) \vee T(x)) \wedge \\ & \forall x \forall y\, (\neg P_2(x, y) \vee T(x) \vee \neg T(y)) \wedge \\ & \forall x \forall y \forall z\, (\neg P_3(x, y, z) \vee T(x) \vee \neg T(y) \vee \neg T(z)) \wedge \\ & \forall x\, (\neg N_1(x) \vee \neg T(x)) \wedge \\ & \forall x \forall y\, (\neg N_2(x, y) \vee \neg T(x) \vee \neg T(y)) \wedge \\ & \forall x \forall y \forall z\, (\neg N_3(x, y, z) \vee \neg T(x) \vee \neg T(y) \vee \neg T(z)) \wedge \end{aligned}$$

We can easily reorder $\varphi(T)$ as $\forall x \forall y \forall z\, \theta(x, y, z, T)$ where $\theta(x, y, z, T)$ is an FO-extended quantifier-free $\mathcal{L}$-formula such that any second order variable only appears in a Horn clause. It is easily seen that each relation T over $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x \forall y \forall z\, \theta(x, y, z, T)$ represents a valid assignment to the variables in Ψ, and therefore, $f_{\forall x \forall y \forall z\, \theta(x,y,z,T)}$ represents #3-HORN-SAT.

3.3 #DNF

We will now show that $\#\text{DNF} \in \#\Sigma_2[\text{FO}]\text{-HORN}$.

Let $\mathcal{L} = \{P, N, \leq\}$, and $\mathfrak{A} = \langle A, P^{\mathfrak{A}}, N^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle$ be an $\mathcal{L}-$structure that represents an instance of a DNF formula Ψ, where A is the set of variables mentioned in Ψ, P and N is a binary relations described as follows:

$$\begin{aligned} P^{\mathfrak{A}} &= \{(c, \ell) \mid \ell \text{ appears in the } c\text{-th clause of } \Psi\}, \\ N^{\mathfrak{A}} &= \{(c, \ell) \mid \neg\ell \text{ appears in the } c\text{-th clause of } \Psi\}. \end{aligned}$$

Now let T be a unary second order variable, and

$$\varphi(T) \quad = \quad \exists c \Big(\forall \ell_1(\neg P(c, \ell_1) \vee T(\ell_1)) \wedge \forall \ell_2(\neg N(c, \ell_2) \vee \neg T(\ell_2))\Big).$$

We reorder $\varphi(T)$ as $\exists c \forall \ell_1 \forall \ell_2\, \theta(c, \ell_1, \ell_2, T)$ where $\theta(c, \ell_1, \ell_2, T)$ is an FO-extended quantifier-free $\mathcal{L}$-formula such that any second order variable only appears in a Horn clause. It is easily seen that each relation T over $\mathfrak{A}$ such that $\mathfrak{A} \models \exists c \forall \ell_1 \forall \ell_2\, \theta(c, \ell_1, \ell_2, T)$ represents a valid assignment to the variables in Ψ, and therefore, $f_{\exists c \forall \ell_1 \forall \ell_2\, \theta(c, \ell_1, \ell_2, T)}$ represents #DNF.

By extension, $\#\text{3-DNF} \in \#\Sigma_2[\text{FO}]\text{-HORN}$.

3.4 #HORN-SAT

We will now show that $\#\text{HORN-SAT} \in \#\Pi_1[\text{FO}]\text{-HORN}$.

Let $\mathcal{L} = \{P, N, \leq\}$, and $\mathfrak{A} = \langle A, P^{\mathfrak{A}}, N^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle$ be an $\mathcal{L}-$structure that represents an instance of a HORN-SAT formula Ψ, where A, P and N are defined as in the previous case.

Now let T be a unary second-order variable, and

$$\begin{aligned} \psi(T) &= \forall c \forall \ell\, [(P(c, \ell) \wedge \forall \ell'(N(c, \ell') \to T(\ell'))) \to T(\ell)] \\ &\equiv \forall c \forall \ell\, [\neg P(c, \ell) \vee \exists \ell'(N(c, \ell') \wedge \neg T(\ell')) \vee T(\ell)]. \end{aligned}$$

Let $\neg X(c, \ell')$ be defined as $N(c, \ell') \wedge \neg T(\ell)$. That is,

$$\begin{aligned} & \forall c \forall \ell'\, [(N(c, \ell') \wedge \neg T(\ell')) \leftrightarrow \neg X(c, \ell')] \\ \equiv\ & \forall c \forall \ell'\, [(\neg N(c, \ell') \vee T(\ell') \vee \neg X(c, \ell')) \wedge (X(c, \ell') \vee \neg T(\ell')) \wedge (X(c, \ell') \vee N(c, \ell'))]. \end{aligned}$$

Finally, let

$$\begin{aligned} \varphi(T, X) = \forall c \forall \ell'\, [(\neg N(c, \ell') \vee T(\ell') \vee \neg X(c, \ell')) \wedge (X(c, \ell') \vee \neg T(\ell')) \wedge (X(c, \ell') \vee N(c, \ell'))] \wedge \\ \forall c \forall \ell\, [\neg P(c, \ell) \vee \exists \ell' \neg X(c, \ell') \vee T(\ell)]. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \varphi'(T, X) = \forall c \forall \ell \exists \ell'\, [(\neg N(c, \ell) \vee T(\ell) \vee \neg X(c, \ell)) \wedge (X(c, \ell) \vee \neg T(\ell)) \wedge (X(c, \ell) \vee N(c, \ell)) \wedge \\ (\neg P(c, \ell) \vee \neg X(c, \ell') \vee T(\ell))]. \end{aligned}$$

Let $\theta(c, \ell, \ell', T, X)$ be such that $\varphi'(T, X) = \forall c \forall \ell \exists \ell'\, \theta(c, \ell, \ell', T, X)$. Note that $\theta(c, \ell, \ell', T, X)$ is an FO-extended quantifier-free $\mathcal{L}$-formula such that any second order variable only appears in a Horn clause. First, note that each relation T over $\mathfrak{A}$ such that $\mathfrak{A} \models \psi(T)$ represents a valid assignment to the variables in Ψ, and that in $\theta(c, \ell, \ell', T, X)$, X is completely defined by the assignment to T. Therefore $f_{\psi(T)}(\mathfrak{A}') = f_{\varphi'(T,X)}(\mathfrak{A}')$ for each $\mathfrak{A}' \in \text{STRUCT}[\mathcal{L}]$, from which we conclude that $f_{\forall c \forall \ell \exists \ell'\, \theta(c, \ell, \ell', T, X)}$ represents #HORN-SAT.

References

[1] Sanjeev Saluja, K. V. Subrahmanyam, and Madhukar N. Thakur. Descriptive Complexity of #P Functions. *J. Comput. Syst. Sci.*, 50(3):493–505, 1995.