KOPERNIC team

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INRIA Internship Report

ON THE TOPIC

Inverse Gaussian and Gaussian distributions in real-time systems

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Contents

1 Real-time systems

This work will be directly related to the theory of the real-time systems, so before proceeding further, it is necessary to define the phrase " Real-time system " more precisely for further understanding.

1.1 What are the Real-time systems?

All of the interpretations of the exact nature of a real-time system have in common the notion of response time - the time taken for the system to generate output from some associated input .

To illustrate the various ways in which " real-time " systems are defined, two further definitions will be given. [1] Young (1982) defines a real - time system to be:

> any information processing activity or system which has to respond to externally generated input stimuli within a finite and specified period.

[1] The PDCS (Predictably Dependable Computer Systems) project gives the following definition :

> A real - time system is a system that is required to react to stimuli from the environment (including the passage of physical time) within time intervals dictated by the environment.

In their most general sense , all these definitions cover a very wide range of computer activities. There are types of systems, which are characterised by the fact, that it is usually not a disaster if the response is not forthcoming. They can be distinguished from those where failure to respond can be considered just as bad as a wrong response. Indeed, for some, it is this aspect that distinguishes a real-time system from others where response time is important but not crucial. Consequently, the correctness of a real-time system depends not only on the logical result of the computation, but also on the time at which the results are produced.

1.2 Our case

Real-time systems are usually used for embedded systems with small energy and computing resources, they have a specific design with a micro-controller architecture and a set of programs (or tasks) running on it. An important part of this design, is to make correspond a processing unit (a CPU for example) neither over nor under dimensioned for a given set of programs. This set must be by construction *schedulable*, meaning that their must exist an order (or a schedule) in which programs should execute in a way that they respect timing requirements that we call *deadlines* in the given processing unit.

In this work we consider *periodic tasks*, meaning that an instance of each tasks is periodically released at a given rate, and a *single-core system*, i.e. only one task is processed at a time. The schedule studied in this paper is called the *fixed-priority* *scheduling policy*, meaning that each task has its own priority and that if a task is released it preempts all tasks with a lower priority and execute until it is finished or preempted by a higher priority task. The deadlines are *implicit*, meaning that an instance of a task should be over before the next instance of the same task in order to respect its requirements.

Let's consider a single-core processor real-time system composed of a finite task set of n tasks $\Gamma = \{\tau_1 > \cdots > \tau_n\}$, ordered by decreasing priority order and scheduled with a fixed-priority preemptive policy, that means, that task τ_1 will be executed primarily, then τ_2, etc, and the τ_n will be executed last.

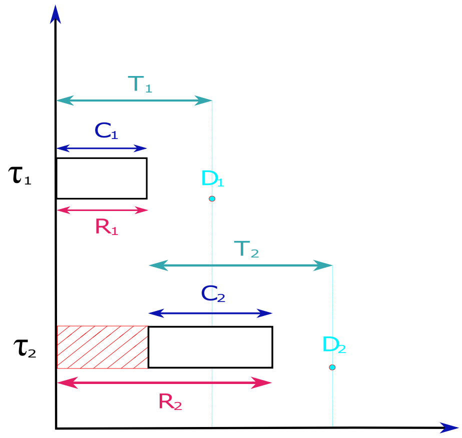


Figure 1: Real-time system

In this picture we denote:

- C_i - execution time of τ_i;
- D_i - a deadline of τ_i;
- R_i - a response time, which consists of the waiting time and the execution time;
- T_i - is a period, which denotes the time from the start of the work to a deadline.

In real-time scheduling, the systems need to be by construction such that no job misses its deadline. In order to do so, the task sets are built for the worst-case scenario, i.e. the context providing the largest response time has to satisfy the timing constraints. From our assumptions response times R_i are independent random variables, which can be expressed as first-passage time of Brownian motions. As it wold be explained a bit later, in such case $R_i \sim IG$, it means, that each response time has an Inverse Gaussian distribution.

Having such assumptions, it is easy to use Inverse Gaussian distribution to discover this real-time system. But what if we don't necessary have independent response times, and at the same time we know, that the vector of the response times $\vec{R} = (R_1, \ldots, R_n)$ has a Multivariate Inverse Gaussian distribution? What, then, can we say about the R_i's? Can we say, that they all are IG distributed? Under what conditions can we say, that? What can we have if it is true ? Further, in my work, we will try to answer these questions.

2 Research work

Firstly, we did a research on Inverse Gaussian, and then, on Multivariate Inverse Gaussian distributions. It was helpful to practice to do a research and to dive-in the topic we are working on.

2.1 Inverse Gaussian Distribution

In probability theory, the Inverse Gaussian distribution (also known as the Wald distribution) is a two-parameter family of continuous probability distributions with support on $(0, \infty)$.

Definition 2.1. A random variable X has an Inverse Gaussian distribution with the positive parameters μ and λ if it has a density function:

$$f(x; \mu, \lambda) = \begin{cases} \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), & x > 0 \\ 0, & x < 0 \end{cases} \tag{1}$$

where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter.

The fact, that X is an Inverse Gaussian distributed random variable with mean μ and shape parameter λ will be denoted by:

$$X \sim IG(\mu, \lambda)$$

In this form, the mean and variance of the distribution are equal.

2.1.1 Some properties

1. **Summation**
   If X_i has an $IG(\mu_0 w_i, \lambda_0 w_i^2)$ distribution for $i = 1, 2, ..., n$ and all X_i are independent, then
   $$\sum_{i=1}^{n} X_i \sim IG\left(\mu_0 \sum w_i, \lambda_0 \left(\sum w_i\right)^2\right)$$
2. **Scaling**
   For any $t > 0$ it holds that
   $$X \sim IG(\mu, \lambda) \Rightarrow tX \sim IG(t\mu, t\lambda).$$
3. If X_i has an $IG(\mu, \lambda)$ distribution for $i = 1, 2, ..., n$, then $\bar{X} \sim IG(\mu, n\lambda)$.

4. The distribution is unimodal, with its mode at

$$\theta = \mu \left(1 + \frac{9\mu^2}{4\lambda^2}\right)^{\frac{1}{2}} - \frac{3\mu^2}{2\lambda}.$$

5. All moments exist. In particular, if $X \sim IG(\mu, \lambda)$, then

$$E[X] = \mu$$

and

$$\mathrm{Var}(X) = \frac{\mu^3}{\lambda}.$$

6. The density curve is positively skewed, and λ is a shape parameter.

2.2 Relationship of an Inverse Gaussian distribution with Brownian motion

Definition 2.2. The Brownian motion process W_t is characterised by the following properties:

- $W_0 = 0$;
- W has independent increments: for every $0 \leq s \leq t$, $W_t - W_s$ are independent of the past $\{W_\tau\}_{\tau \leq s}$;
- W has Gaussian increments: the increment $W_t - W_s \sim N(0, t-s)$ is normally distributed with mean 0 and variance $t - s$;
- W has continuous paths: W_t is continuous in t.

Liu and Layland, followed by Zagalo *et al.*, prove that in the context of implicit deadlines, when $\bar{u_i} = E[W_i(t)/t] < 1$ there exists a Brownian motion W_i of drift $\bar{u_i}$ and deviation σ_i such that

$$R_i = \inf \left\{ t > 0 : W_i(t) = t - \sum_{j=1}^{i} C_j \right\} \tag{2}$$

and that conditionally to $\sum_{j=1}^{i} C_j = x$, R_i is distributed according to an inverse-Gaussian distribution

$$R_i \sim IG\left(\frac{x}{1 - \bar{u_i}}, \frac{x^2}{\sigma_i^2}\right) \tag{3}$$

2.3 Sampling from an Inverse-Gaussian distribution

Inverse Gaussian sampling — that is, generating samples from an Inverse Gaussian distribution. We will use it further in our work. But, deep down, how does a computer know how to generate Gaussian samples?

The standard form for the density of the inverse Gaussian distribution is given by 1. The cumulative distribution function as given by [2] Chhikara and Folks is expressed in terms of cumulatives of the standard normal and is not easily inverted.
Following [5] Shuster we may write:

$$g(X) = \frac{\lambda(X-\mu)^2}{\mu^2 X} \sim \chi^2_{(1)}. \tag{4}$$

Observations from $\chi^2_{(1)}$ are easily generated as the squares of standard normals, so first, we generate a random variate from a normal distribution with mean 0 and standard deviation equal 1

$$\nu \sim N(0,1)$$

and square the value

$$v_0 = \nu^2.$$

For each chi-square variate, v_0, we must solve 4 for x to obtain a corresponding observation from the inverse Gaussian distribution. For any $v_0 > 0$ there are exactly two roots of the associated quadratic equation which can always be expressed as

$$x_1 = \mu + \frac{\mu^2 v_0}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda v_0 + \mu^2 v_0^2} \tag{5}$$

and

$$x_2 = \frac{\mu^2}{x_1} \tag{6}$$

since the relationship which exists between the roots of any quadratic equation implies here that $x_1 x_2 = \mu^2$. The difficulty in generating observations with the desired distribution now lies in choosing between the two roots. In [3] it was shown, that x_1 should be chosen with the probability:

$$p_1(v_0) = \frac{\mu}{\mu + x_1}. \tag{7}$$

So for each random observation from a chi-square distribution with one degree-of-freedom, v_0, the smaller root is calculated. Then, we generate another random variate, this time sampled from a uniform distribution between 0 and 1

$$z \sim U(0,1).$$

If $z \leq \frac{\mu}{\mu+x}$ then return 5, else return 6.

2.4 Multivariate inverse Gaussian distribution

Definition 2.3. A random vector $\vec{X} = (X_1, \ldots, X_n)^T$ is multivariate inverse Gaussian distributed, if there exists a non-singular matrix P, of n rows and n columns, such that $\vec{Z} = PX$ is a vector of independently distributed Inverse Gaussian random variables. In symbols, one writes: $\vec{X} \sim MVIG$.

Let's formulate the Lemma, which we will need for the proof of the following Theorem 2.5.

Lemma 2.4. Let $X_1, \ldots, X_n$ be independent random variables with $X_j \sim IG(\mu_j, \lambda_j),\ j = 1, \ldots, n$. Then a necessary and sufficient condition that $X_0 = \sum_{j=1}^{n} X_j$ be an Inverse Gaussian random variable is that $\frac{\mu_j^2}{\lambda_j}$ does not depend on $j,\ j = 1, \ldots, n$.

The following theorem gives necessary and sufficient conditions that a Multivariate Inverse Gaussian random vector have aIl its univariate components Inverse Gaussian distributed.

Theorem 2.5. *Let $\vec{Z} = (Z_1, \ldots, Z_n)^T$ be a vector of independent random variables with $Z_j \sim IG(\mu_j, \lambda_j), j = 1, \ldots, n$. Let P be an $n \times n$ non-singular matrix, and let $\vec{X} = (X_1, \ldots, X_n)^T$ be the random vector satisfying a $\vec{Z} = PX$.*

Denote P^{-1} as $Q = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}$.

Then necessary and sufficient conditions that for a fixed $i, i = 1, \ldots, n$, that X_i be Inverse Gaussian distributed, is that for each j such that $q_{ij} \neq 0$, the following holds:

- $q_{ij} > 0$*;*
- $\frac{\mu_j^2 q_{ij}}{\lambda_j}$ *does not depend on j.*

Proof. **Necessity of (1):** We assume $X_i \sim IG(\cdot, \cdot)$ and shall conclude $q_{ij} \geq 0$. Let's suppose, that it is possible that $X_i \sim IG(\cdot, \cdot)$ and $q_{ij} < 0$, the independence and non-negativity of z_j give: $P(X_i < 0) > 0$. This cannot happen for Inverse Gaussian random variables, which are positive with probability 1.

The assumption that some q_{ij} could be negative, with X_i Inverse Gaussian distributed cannot be compatible.

That is, the condition for q_{ij} is indeed necessary.

Necessity and sufficiency of (2): Let $\{q_{ij_1}, \ldots, q_{ij_r}\}$ be the set of positive q_{ij} for a fixed i. This set is non-empty, since P is non-singular.

$$X_i = \sum_{k=1}^{r} q_{ij_k} z_{j_k} = \sum_{k=1}^{r} w_{j_k},$$

where $w_{j_k} = q_{ij_k} z_{j_k},\ k = 1, \ldots r$ are independent random variables, such that: by the Scaling property 2,

$$w_{j_k} \sim IG(q_{ij_k} \mu_{j_k}, q_{ij_k} \lambda_{j_k}), \quad \text{where } k = 1, \ldots, r.$$

Hence, by Lemma 2.4 ,

$$X_i \sim IG(\cdot,\cdot) \quad \text{iff} \quad \frac{q_{ij_k}\mu_{j_k}^2}{\lambda_{j_k}},\ k = 1,\dots,r$$

does not depend on k.
In view of the definition of $q_{ij_k},\ k = 1,\dots,r$, this completes the proof. □

After the research we mostly focused on working with the Theorem 2.5. Firstly, we will check the statement of it empirically, using Python. Let's dive into the code more precisely.

Hence, we will construct all the random variables, respecting the conditions of the Theorem 2.5. Let's consider the 3-dimensional space, where $\vec{Z} = (Z_1, Z_2, Z_3)$ is the same, as in the Theorem 2.5. Here we randomly choose the parameters $(\mu_j, \lambda_j), j = 1, 2, 3$, for the Inversely Gaussian distributed variables $Z_j \sim IG(\mu_j, \lambda_j)$.

Lets' denote

$$\gamma_j = \frac{\mu_j^2}{\lambda_j} \tag{8}$$

and

$$\rho_i = q_{ij}\gamma_j \tag{9}$$

where q_{ij} are the elements of previously defined matrix Q.

We are willing to find the elements of the matrix Q. Eventually,

$$q_{ij} = \frac{\rho_i}{\gamma_j}. \tag{10}$$

It has to be noted, that here we are building a triangular matrix Q:

$$Q = \begin{pmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

Otherwise, if we build all of the coefficients q_{ij}, using the the formula (9), the determinant of such matrix will be always zero, which can be easily proved. We will show it for 3×3 matrix. For the other dimensions the proof is similar.

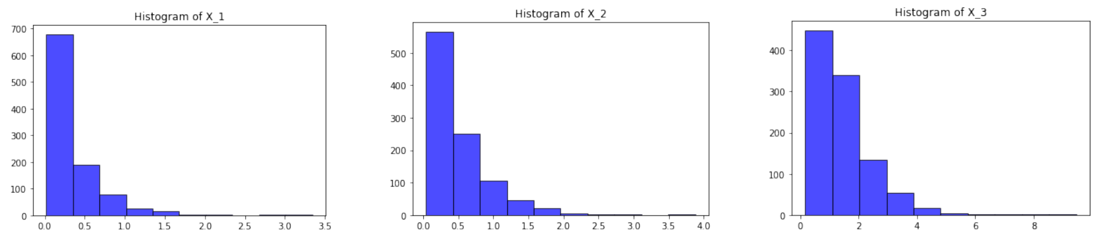
$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} \frac{\rho_1}{\gamma_1} & \frac{\rho_1}{\gamma_2} & \frac{\rho_1}{\gamma_3} \\ \frac{\rho_2}{\gamma_1} & \frac{\rho_2}{\gamma_2} & \frac{\rho_2}{\gamma_3} \\ \frac{\rho_3}{\gamma_1} & \frac{\rho_3}{\gamma_2} & \frac{\rho_3}{\gamma_3} \end{pmatrix}$$
$$\Rightarrow \det Q = \frac{\rho_1\rho_2\rho_3}{\gamma_1\gamma_2\gamma_3} + \frac{\rho_1\rho_2\rho_3}{\gamma_1\gamma_2\gamma_3} - \frac{\rho_1\rho_2\rho_3}{\gamma_1\gamma_2\gamma_3} - \frac{\rho_1\rho_2\rho_3}{\gamma_1\gamma_2\gamma_3} = 0$$

By the conditions of the Theorem 2.5, matrix Q has to be a non-singular matrix, because its inverse is P and it exists. Should be noted, that the triangularity is not a necessary condition for matrix to be non-singular, but it suits our case, so we will stick to using this.

Then here, using the sampling method, described in the Chapter 2.3, we generate our vector $\vec{Z} = (Z_1, Z_2, Z_3)$ such that each component $Z_j \sim IG(\mu_j, \lambda_j)$.

Now, we can easily find a $\vec{X} = (X_1, X_2, X_3)$ vector, that, according to the Theorem2.5 has an inverse Gaussian distribution, and moreover, its each component also has an IG distribution.

Now, by generating the vector $\vec{Z}$ a lot of times and finding from it $\vec{X}$, we can build a histogram of X_1, X_2 and X_3 and see, that each of them turns out to have a distribution really close to an Inverse Gaussian one.



(a) (b) (c)

Figure 2: Histogram of every component of vector $\vec{X}$

From now on, we will stick to the conditions of the Theorem 2.5 and work further, respecting them.

In this part of work, we want to find out, what are the parameters of the distribution of vector $\vec{X} \sim MVIG(\nu, \Lambda)$, knowing the parameters of the Inverse Gaussian distribution of every vector's $\vec{X} = (X_1, ..., X_n)$.

Let's say, that we have as an input two vectors of parameters, of the distribution of $Z_j \sim IG(\mu_j, \lambda_j)$, $j = 1, \ldots, n$, meaning, we have $(\mu_1, ..., \mu_n)$ and $(\lambda_1, \ldots, \lambda_n)$. It is known, that the new components are equal to

$$\nu_i = (Q\mu)_i = \sum_{j=1}^{n} q_{ij}\mu_j$$

and

$$\Lambda_{ij} = \mathrm{Cov}(X_i, X_j)$$

Now we have to find out, what the covariance between the components of $\vec{X}$ is equal to in our notations.

2.5 Computing Covariance

$$
\begin{aligned}
\mathrm{Cov}(X_i, X_j) &= E\left[(X_i - EX_i)(X_j - EX_j)\right] \\
&= E\left[\left(\sum_{k=1}^{n} q_{ik} Z_k - E\left[\sum_{k=1}^{n} q_{ik} Z_k\right]\right)\left(\sum_{l=1}^{n} q_{jl} Z_l - E\left[\sum_{l=1}^{n} q_{jl} Z_l\right]\right)\right] \\
&= E\left[\left(\sum_{k=1}^{n} q_{ik} Z_k - \sum_{k=1}^{n} q_{ik} E\left[Z_k\right]\right)\left(\sum_{l=1}^{n} q_{jl} Z_l - \sum_{l=1}^{n} q_{jl} E\left[Z_l\right]\right)\right] \\
&= E\left[\left(\sum_{k=1}^{n} q_{ik} Z_k - \sum_{k=1}^{n} q_{ik} \mu_k\right)\left(\sum_{l=1}^{n} q_{jl} Z_l - \sum_{l=1}^{n} q_{jl} \mu_l\right)\right] \\
&= E\left[\left(\sum_{k=1}^{n} q_{ik} \left(Z_k - \mu_k\right)\right)\left(\sum_{l=1}^{n} q_{jl} \left(Z_l - \mu_l\right)\right)\right] \\
&= E\left[\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ik} q_{jl} \left(Z_k - \mu_k\right)\left(Z_l - \mu_l\right)\right] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} q_{ik} q_{jl} \mathrm{Cov}(Z_k . Z_l)
\end{aligned}
$$

As the components of vector $\vec{Z}$ are independent by the conditions of the Theorem2.5, when $k \neq l$ we have

$$\mathrm{Cov}(Z_k . Z_l) = 0.$$

In the case, when $k = l$, we have

$$
\begin{aligned}
E\left[\sum_{k=1}^{n} q_{ik} q_{jk} \left(Z_k - \mu_k\right)^2\right] &= \sum_{k=1}^{n} q_{ik} q_{jk} E\left[\left(Z_k - \mu_k\right)^2\right] \\
&= \sum_{k=1}^{n} q_{ik} q_{jk} \frac{\mu_k^3}{\lambda_k} \\
&= \rho_i \rho_j \sum_{k=1}^{\min(i,j)} \frac{\lambda_k}{\mu_k},
\end{aligned}
$$

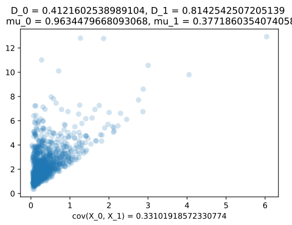
where ρ_i, as defined before in (9). Hence,

$$\Lambda_{ij} = \rho_i \rho_j \sum_{k=1}^{\min(i,j)} \frac{\lambda_k}{\mu_k} \tag{11}$$

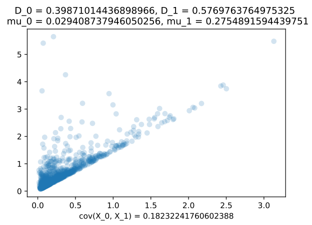
Furthermore, we have $\mathrm{Var}(X_i) = \rho_i^2 \sum_{k=1}^{i} \frac{\lambda_k}{\mu_k}$.

2.6 Simulations

Here you can see, how with the changes of the values of ρ_i, ρ_j or μ_k, λ_k the covariance (11) between X_0 and X_1 changes.
We can see is that the value of the covariance (11) is always non-negative, as all of the component by definition are non-negative.



(a)



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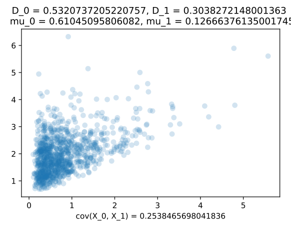

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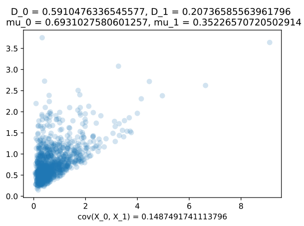

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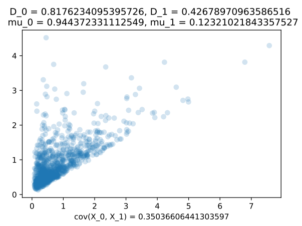
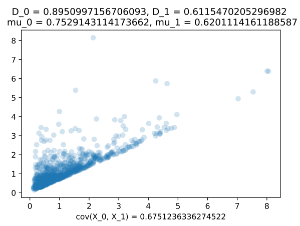


(e)



(f)

Figure 3: Covariance between X_0 and X_1

Let $X, X^{(1)}, X^{(2)}, \dots$ be and i.i.d. sequence and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X^{(i)}$. We build the estimation of the covariance (11) to check our theoretical result. Let

$$\Lambda^{(n)} = \frac{1}{n-1}\sum_{i=1}^{n}(X^{(i)} - \bar{X})(X^{(i)} - \bar{X})^T.$$

be the empirical covariance matrix.

Next step is to compute the distance between the real matrix of covariance and its estimation. Let's consider two different distances. The first one is the maximum element of the element-wise subtraction of two matrices:

$$\parallel \Lambda^{(n)} - \Lambda \parallel_\infty = \sup_{1\le i,j\le n} |\Lambda_{ij}^{(n)} - \Lambda_{ij}| \tag{12}$$

And the result with the distance (12) on 10 000 samples is satisfying for us, because it is easy to see, that with an increase of the sample, the distance goes to zero.

Let's consider another norm of distance between matrices:

$$\parallel \Lambda^{(n)} - \Lambda \parallel = \mathrm{Tr}\left((\Lambda^{(n)} - \Lambda)(\Lambda^{(n)} - \Lambda)^T\right) \tag{13}$$

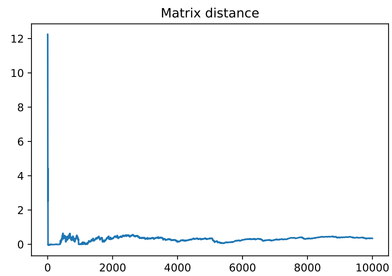


Figure 4: Convergence of the real covariance to its estimation with distance (12)

As we can see, with the distance (13) the result is even more satisfying.

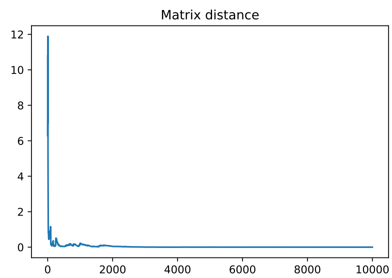


Figure 5: Convergence of the real covariance to its estimation with distance (13)

Therefore, we can be sure, that we computed the covariance correctly, because it coincides with its estimation, and the new components of the MVIG distribution of the vector $\vec{X}$ can be computed as:

$$\nu_i = (Q\vec{\mu})_i = \sum_{i=1}^{n} q_{ij}\mu_j$$

and

$$\Lambda_{ij} = \rho_i \rho_j \sum_{k=1}^{\min(i,j)} \frac{\lambda_k}{\mu_k}.$$

Furthermore, we can compute the correlation coefficient defined by

$$\begin{aligned} \mathrm{Corr}(X_i, X_j) &= \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}} \\ &= \frac{\sum_{k=1}^{\min(i,j)} \frac{\lambda_k}{\mu_k}}{\sqrt{\sum_{k=1}^{i} \frac{\lambda_k}{\mu_k}}\sqrt{\sum_{k=1}^{j} \frac{\lambda_k}{\mu_k}}} \end{aligned}$$

3 Related work

Here we will consider two articles, that stand in a good stead for us, while doing this work. The first one is [6], where the author mostly explores the IG and MVIG properties and the link between them. The second one is [4], where they propose a new multivariate extension of the IG distribution and explore multivariate inverse relationship with a multivariate normal distribution.

3.1 Jonathan Shuster. Properties of the Inverse Gaussian distributions.

This dissertation is divided into two sections.
The first, that interested us mostly, consists of Chapters 4 to 9, concentrates on the distribution theory of the Inverse Gaussian distribution. The author develops a general limit theorem for convergence in law to this distribution. This result is of interest to nonparametric statisticians. Other results from this section include: a convenient method of obtaining percentage points; Bayes estimates of parameters; and four original characterizations of the distribution. A definition of the Multivariate Inverse Gaussian distribution (MVIG) is given. Necessary and sufficient conditions that all the marginal distributions of the MVIG be Inverse Gaussian are developed.
The second section, Chapters 10 to 12, deals with a general family of stochastic processes, of which the separable Inverse Gaussian and Poisson processes are members. The topic is treated from a measure theoretic point of view. Properties of the sample functions and Stieltjes stochastic integrals are treated in detail. Illustrative examples of how ones intuition, regarding path properties, can lead him astray, are included.

3.2 . A Multivariate Extension of Inverse Gaussian Distribution Derived from Inverse Relationship

In their work they propose a new multivariate extension of the inverse Gaussian distribution derived from a certain multivariate inverse relationship. First they define a multivariate extension of the inverse relationship between two sets of multivariate distributions, then define a reduced inverse relationship between two multivariate distributions. They derive the multivariate continuous distribution that has the reduced multivariate inverse relationship with a multivariate normal distribution and call it a multivariate inverse Gaussian distribution. The distribution is also characterised as the distribution of the location of a multivariate Brownian motion at some stopping time. The marginal distribution in one direction is the inverse Gaussian distribution, and the conditional distribution in the space perpendicular to this direction is a multivariate normal distribution. Mean, variance and higher order cumulants are derived from the multivariate inverse relationship with a multivariate normal distribution. Other properties such as reproductivity and infinite divisibility are also given.

References

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