ANALYSIS OF TAPE DRIVES

by

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A.K.S.

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ABSTRACT

A variable speed device may be constructed of two reels connected by a tape. The input motion of the drive reel is transmitted to the driven reel through the tape. As the reels turn, the tape winds from one reel to the other and the resulting relative motions continually change. For a constant angular velocity input, it is possible to change the output by a variation of the tape thickness as a function of its length. The relative kinematics of the two reels are of major importance, and the outputs were investigated for the cases of (1) constant tape thickness, (2) tape thickness varied to produce a constant acceleration, and (3) tape thickness varied to produce a log function. The input in all cases was a constant angular velocity. Analytical solutions of all the important parameters were obtained and then verified experimentally. Dynamics were not considered. The outputs obtained have in general continually increasing displacements, velocities, and accelerations. They are both predictable and controllable to a high degree of accuracy. By the use of the derived equations and graphically presented results, it is possible to completely design any one of the units investigated.

NOMENCLATURE

a	angular acceleration (rad./$sec.^2$)
A	angular displacement of reel 1 at start (rad.)
b	tape thickness per radian (in./rad.)
B	slope to base e ($rad.^{-1}$)
C	A B product
θ	angular displacement (rad.)
L	slope to base 10 ($rad.^{-1}$)
N	number of decades
r	radius (in.)
s	tape length (in.)
t	time (sec.)
T	tape thickness (in.)
w	angular velocity (rad./sec.)

Subscripts

1	reel 1
2	reel 2
o	value at θ_1 equals zero
m	value at θ_1 equals maximum
obs	observed value
I	reel 1 value at θ_{1obs} equals zero
II	reel 2 value at θ_{1obs} equals zero

ANALYSIS OF TAPE DRIVES

1. Introduction and Summary

1-1. General Background, Idea of Tape Drive. In the field of mechanics and kinematics there has been an almost continual desire to be able to change and control velocity outputs of mechanical devices. Depending upon the application and the power source available, this change of velocity output has been accomplished in a large number of ways. Gears form a basic method, but they prove relatively unsatisfactory if the application requires a continually varying velocity ratio. Hydraulic drives and friction drives solve this problem, but produce drawbacks of their own. Hydraulic drives often require energy dissipation as a means of varying output, and friction drives are severely limited as to the power they may transmit.

One device, which has the characteristic of a continually varying velocity output, consists of two reels connected by a tape. Such a device is shown in Fig. 1. If reel 1 is driven by a power source and the output is taken from reel 2, as the tape is wound from reel 2 onto reel 1, the velocity of the output is required to increase. Thus, here is a device with continually varying velocity output that requires neither energy dissipation nor friction for its operation.

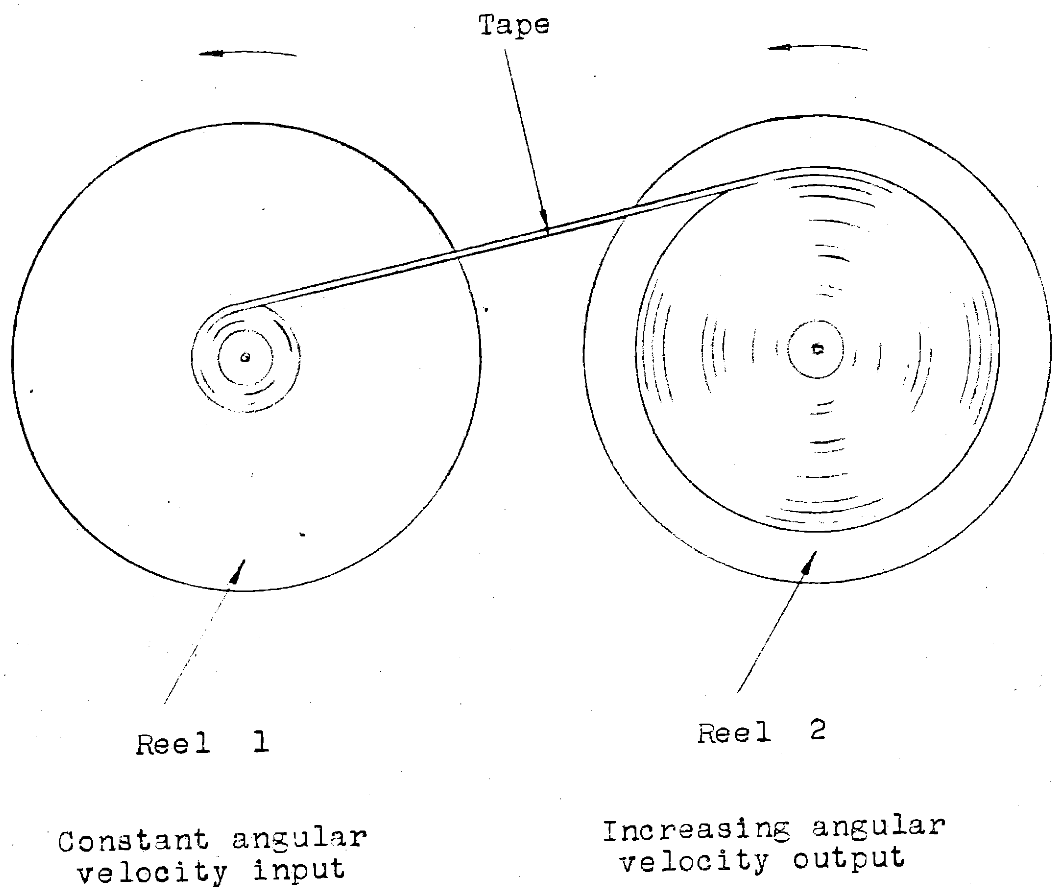

Fig. 1. Basic components of tape drive

Some of the general characteristics are immediately observed. The device is transient in that it can run only until all the tape has unwound from reel 2. The torque is transmitted by the tape in tension and is thus limited only by the tensile strength of the tape. The mechanical advantage starts at a large value and decreases as the ratio of the tape decreases. In the limit as the radius of reel 2 approaches zero the angular velocity and accelerations of reel 2 will become extremely large. A certain amount of variation or control of the output is obtainable if the tape thickness is not a constant, but is varied as a function of its length.

1-2. <u>Scope of Problem, Method of Investigation.</u> As noted from the above observation, the kinematics of this device are very important and must be understood before an attempt is made to deal with the dynamics. The following is thus a general investigation into the kinematic characteristics of a reel that is driven by another reel when the two are connected by a tape. Throughout the investigation the input angular velocity of the drive reel will be taken as being constant, i.e., the angular acceleration of reel 1 is zero.

Two methods of investigation were used. The first and most extensive investigation was completely analytical. The second method consisted of an experimental verification of the analytically derived equations. Using both methods, three separate kinematic problems were solved and verified. The first constitutes a solution to the kinematic relationships between the two reels for a constant tape thickness. The second solves the tape thickness variation as well as the kinematic relations required to produce a constant acceleration of reel 2. The third solves the relations required to produce a log function displacement of reel 2 in relation to reel 1, by a variation in tape thickness.

It might be noted that during the running of the constant thickness tape, observations were made of the dynamic characteristics and that no limiting problems were encountered.

1-3. Results. The results show that the kinematics are in general completely solvable and predictable within a high degree of accuracy. The solutions readily lend themselves to nondimensional form and thus all characteristics may be graphed considering only relative sizes. Figures 3, 4, and 5 in the text give the general results for the constant tape thickness and constant acceleration cases. By a variation of the tape thickness as a function of the tape length, the output may be easily controlled within a range limited only by acceptable tape thickness variations.

For the constant tape thickness case, as the radius of reel 2 approaches zero, the angular displacement approaches a maximum value, and both the angular velocity and acceleration approach infinite values. For a device that has a tape variation that gives a constant acceleration output, no theoretical limits are encountered, but after an angular displacement of reel 2 has been reached, which is approximately equal to the maximum displacement obtainable by a similar reel with a constant thickness tape, the practical range of tape thickness limits the device. The log function producing device, though theoretically unlimited, is found to have practical limitations which allow a log function to be produced only across about three decades.

In practice, it is possible to design any of the three types of units considered here by use of the derived equations and graphs if the proper transformations are used to compensate for various end conditions. The required transforms are given in the Appendix.

Part A

ANALYTICAL SOLUTION

2. Introduction, Basic Equations, and Units

2-1. Introduction. In this section, the analytical solutions are derived for three different cases: (1) constant tape thickness device, (2) constant acceleration device, (3) and log function producing device. The input in all cases is a constant angular velocity. The solutions contain the kinematic equations giving the displacement characteristics as well as graphical representations of the results. The controlling dimensionless groups are found. Where appropriate, the variation of tape thickness as a function of tape length is developed.

2-2. Analytical Model. For the purposes of this investigation, the analytical model as shown in Fig. 2 will be used. Reel 1 is the drive reel, and reel 2 is the driven reel. Both reels turn in a counterclockwise direction as shown, and the tape is therefore unwound from reel 2 and onto reel 1. The tape thickness will be represented by T, its displacement by s, and the angular displacement, velocity, and acceleration, by θ, w, and a, respectively. Subscripts 1 and 2 refer to reels 1 and 2, and subscripts o and m refer to values at θ_1 equals zero and θ_1 equals its maximum value, respectively. The values of the radii (r_1, r_2) are taken as the distances from the center of the reels to the center line of the tape at its point of tangency on the respective reels.

2-3. Basic Equations, Units. Two basic equations are seen to apply to the above model. One is the differential

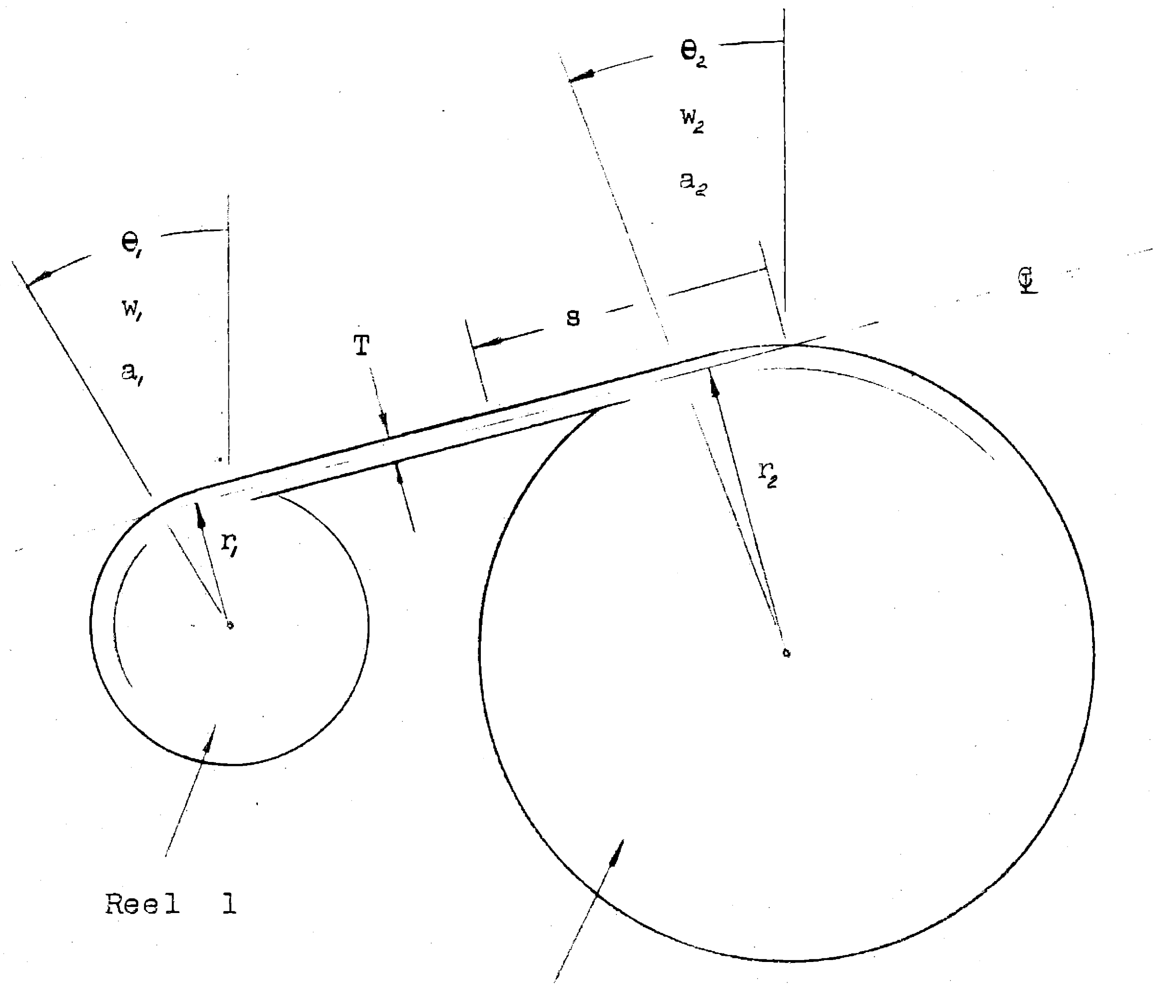

Fig. 2. Analytical model

form of the equation relating the tangential displacement s to the radius r and angular displacement θ.

$$ds = r\,d\theta \tag{1}$$

Here ds and r have the units of inches and $d\theta$ is in radians. The other equation comes from the fact that the radius changes by a factor equal to the tape thickness for each revolution of the reel. Or written as a differential equation

$$dr = T\,d\theta \tag{2}$$

where dr and T are given in inches and $d\theta$ is in revolutions. A more useful form of Eq. (2) is obtained by converting $d\theta$ into radians. If the right-hand term of Eq. (2) is divided by 2π and a new tape thickness b is defined as

$$b \equiv T/2\pi \quad \text{or} \quad T = 2b\pi \tag{3}$$

the result is

$$dr = b\,d\theta \tag{4}$$

where dr is given in inches, b is in inches per radian, and $d\theta$ is in radians.

If it is assumed that $ds_1 = ds_2$, then the motion of the two reels may be connected analytically and by use of

Eqs. (1) and (4) their relative kinematics obtained. Strictly speaking, ds_1 does not equal ds_2. The length of tape suspended between the tangent point on each reel varies, being a maximum when one reel has all the tape on it and the other one is at zero radius, and being a minimum when the two reel radii are equal. Thus, part of the ds of one reel goes into either shortening or lengthening the amount of tape between reels. In practice, for the size of reels being dealt with here, this variation is completely negligible and thus will not be considered.

3. Constant Tape Thickness Analysis

3-1. Derivation of Kinematic Equations. Consider now the normal case of having a tape of constant thickness wound on reels 1 and 2. Taking Eq. (4) and writing it for reel 1 gives

$$dr_1 = b\, d\theta_1$$

Integrating this between the limits, (r_{01}, r_1) and (θ_{01}, θ_1)

$$\int_{r_{01}}^{r_1} dr = \int_{\theta_{01}}^{\theta_1} b\, d\theta$$

Thus

$$r_1 - r_{01} = b(\theta_1 - \theta_{01}) \qquad (5)$$

But θ_{o1} is by definition equal to zero, and if it is assumed that at $\theta_1 = 0$ all the tape is wound on reel 2, then

$$r_{o1} = 0$$

Equation (6) thus becomes

$$r_1 = b\,\theta_1 \tag{6}$$

Using a similar approach on reel 2, assuming that at $\theta_1 = 0$

$$\theta_{2o} = 0$$

and noting that dr is now negative, gives

$$r_2 - r_{o2} = -b\,\theta_2$$

Or

$$r_2 = r_{o2} - b\,\theta_2 \tag{7}$$

Taking Eq. (1) and writing it for reel 1 gives

$$ds_1 = r_1\,d\theta_1$$

Substituting the value of r_1 from Eq. (6)

$$ds_1 = b\,\theta_1\,d\theta_1$$

and integrating between the limits $(0, s_1)$ and $(0, \theta_1)$ gives the result that

$$s_1 = b\theta_1^2/2 \tag{8}$$

Writing Eq. (1) for reel 2, substituting in the value of r as given by Eq. (7) and integrating, gives a corresponding expression for s_2.

$$s_2 = r_{02}\theta_2 - b\theta_2^2/2 \tag{9}$$

Recalling the assumption that $ds_1 = ds_2$ or in the integrated case where $s_{01} = s_{02} = 0$

$$s_1 = s_2 \tag{10}$$

Thus equating s_1 and s_2 of Eqs. (8) and (9)

$$b\theta_1^2/2 = r_{02}\theta_2 - b\theta_2^2/2$$

Rearranging and solving for θ_2 by use of the quadratic formula results in

$$\theta_2 = r_{02}/b - \sqrt{r_{02}^2/b^2 - \theta_1^2}$$

Dividing both sides by r_{02}/b gives the nondimensional form

$$\theta_2 b/r_{02} = 1 - \sqrt{1 - (\theta_1 b/r_{02})^2} \tag{11}$$

Looking back to Eq. (6) and realizing that when θ_1 has reached its maximum value, i.e., $\theta_1 = \theta_{1m}$, all of the tape will have wound on to reel 1 and its radius will now equal r_{02}, it is seen that

$$\theta_{1m} = r_{02}/b \tag{12}$$

Replacing b/r_{02} in the right-hand term of Eq. (11) by this value, results in a nondimensional equation expressing θ_2 as a function of b, r_{02}, and the ratio θ_1/θ_{1m}.

$$\theta_2 \, b/r_{02} = 1 - \sqrt{1 - (\theta_1/\theta_{1m})^2} \tag{13}$$

Differentiating Eq. (13) with respect to time, noting that b, r_{02} and θ_{1m} are all constant

$$\frac{1}{\theta_{1m}} \frac{d\theta_2}{dt} = \frac{\theta_1}{\theta_{1m}} \frac{d\theta_1}{dt} \left[1 - \left(\frac{\theta_1}{\theta_{1m}}\right)^2\right]^{-\frac{1}{2}} \tag{14}$$

Substituting in the fact that $d\theta_2/dt \equiv w_2$ and $d\theta_1/dt = w_1$ and rearranging

$$\frac{w_2}{w_1} = \frac{\theta_1}{\theta_{1m}} \left[1 - \left(\frac{\theta_1}{\theta_{1m}}\right)^2\right]^{-\frac{1}{2}} \tag{15}$$

Here w_2 is found to depend only upon w_1 and the ratio θ_1/θ_{1m}.

Before differentiating Eq. (15), it proves advantageous to combine the right-hand terms to give

$$\frac{w_2}{w_1} = \left[\left(\frac{\theta_{1m}}{\theta_1}\right)^2 - 1\right]^{-\frac{1}{2}} \tag{16}$$

Differentiating Eq. (16) with w_1 and θ_{1m} constant produces the result

$$\frac{1}{w_1}\frac{dw_2}{dt} = \left(\frac{\theta_1}{\theta_{1m}}\right)^{-3}\frac{d\theta_1}{dt}\left[\left(\frac{\theta_{1m}}{\theta_1}\right)^2 - 1\right]^{-\frac{3}{2}}$$

Substituting in $dw_2/dt = a_2$, $d\theta_1/dt = w_1$, and $\theta_{1m} = r_{02}/b$, combining and rearranging terms gives

$$\frac{a_2 \ r_{02}}{bw_1^2} = \left[1 - \left(\frac{\theta_1}{\theta_{1m}}\right)^2\right]^{\frac{3}{2}} \tag{17}$$

The acceleration of the output a_2 is then seen to be a function of r_{02}, b, w_1, and the ratio θ_1/θ_{1m}.

3-2. Presentation of Results. The results of the analysis may be summerized by the three equations relating θ_2, w_2, and a_2 to the variables r_{02}, b, w_1, and the ratio θ_1/θ_{1m} as given in Eqs. (13), (15), and (17) above. To better visualize the actual characteristics of the device, the nondimensional forms of θ_2, w_2, and a_2 are plotted vs. θ_1/θ_{1m} in Figs. 3, 4, and 5 respectively. Numerical plots of the values are given in the Appendix in Table 1.

From Fig. 3 the angular displacement θ_2 is found to increase steadily, starting from a value of 0 and ending at a maximum value of r_{02}/b. From Fig. 4 the angular velocity w_2 starts at a value of zero, increases almost linearly over the initial two thirds of operation, and then changes

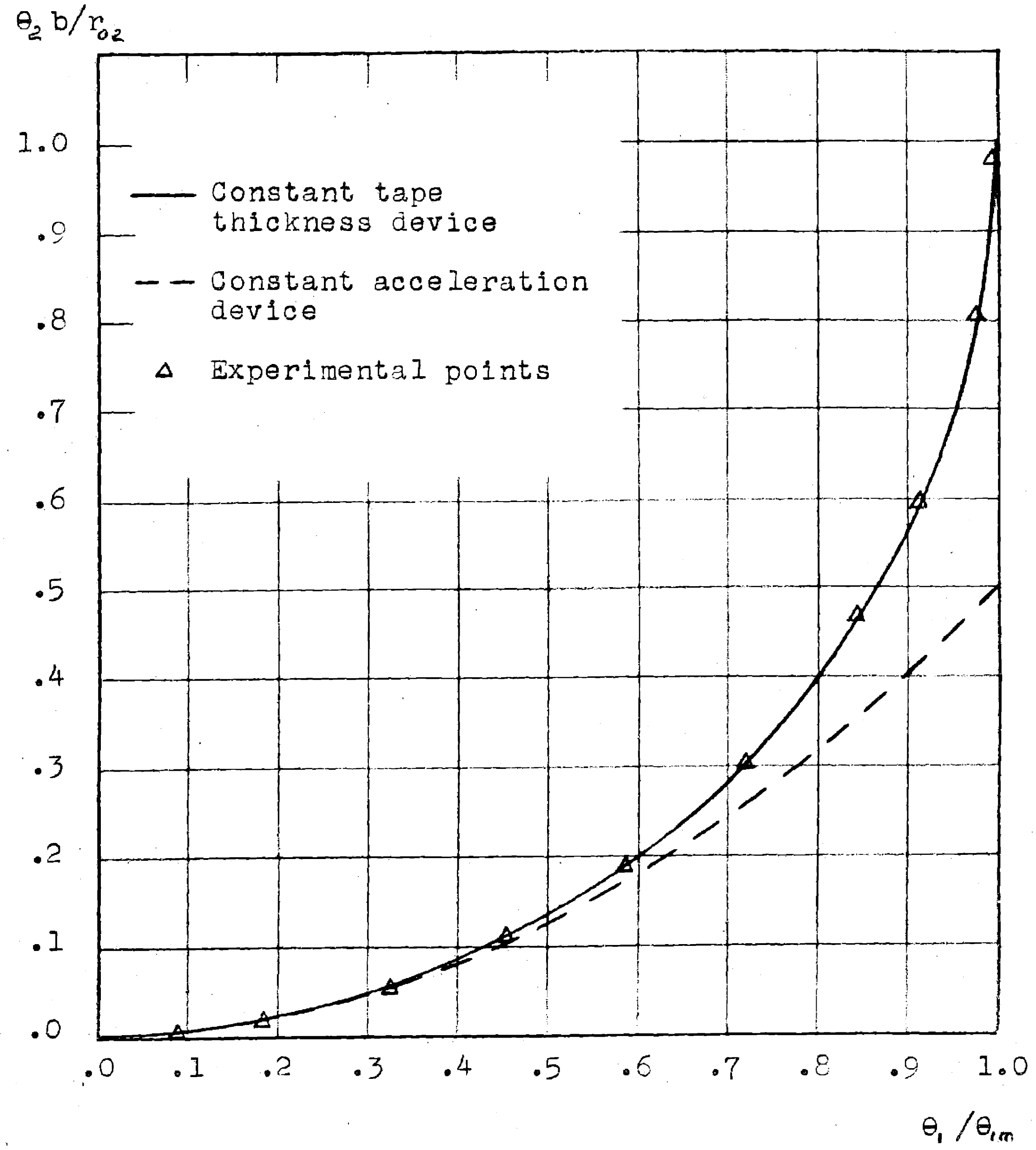

Fig. 3. Nondimensional plot of output angular displacement vs. input angular displacement for constant tape thickness, and constant acceleration devices.

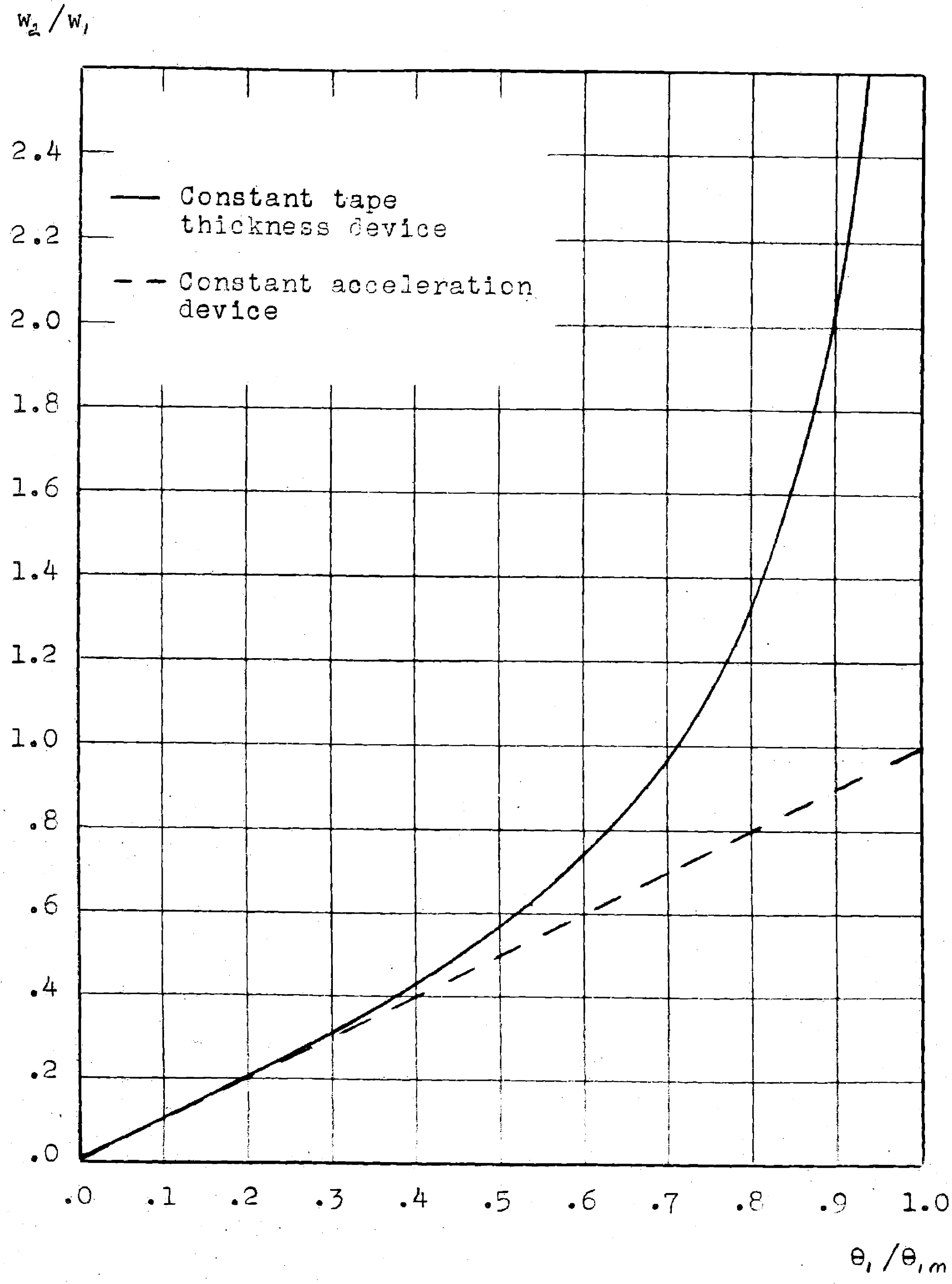

Fig. 4. Nondimensional plot of output angular velocity vs. input angular displacement for constant tape thickness, and constant acceleration devices.

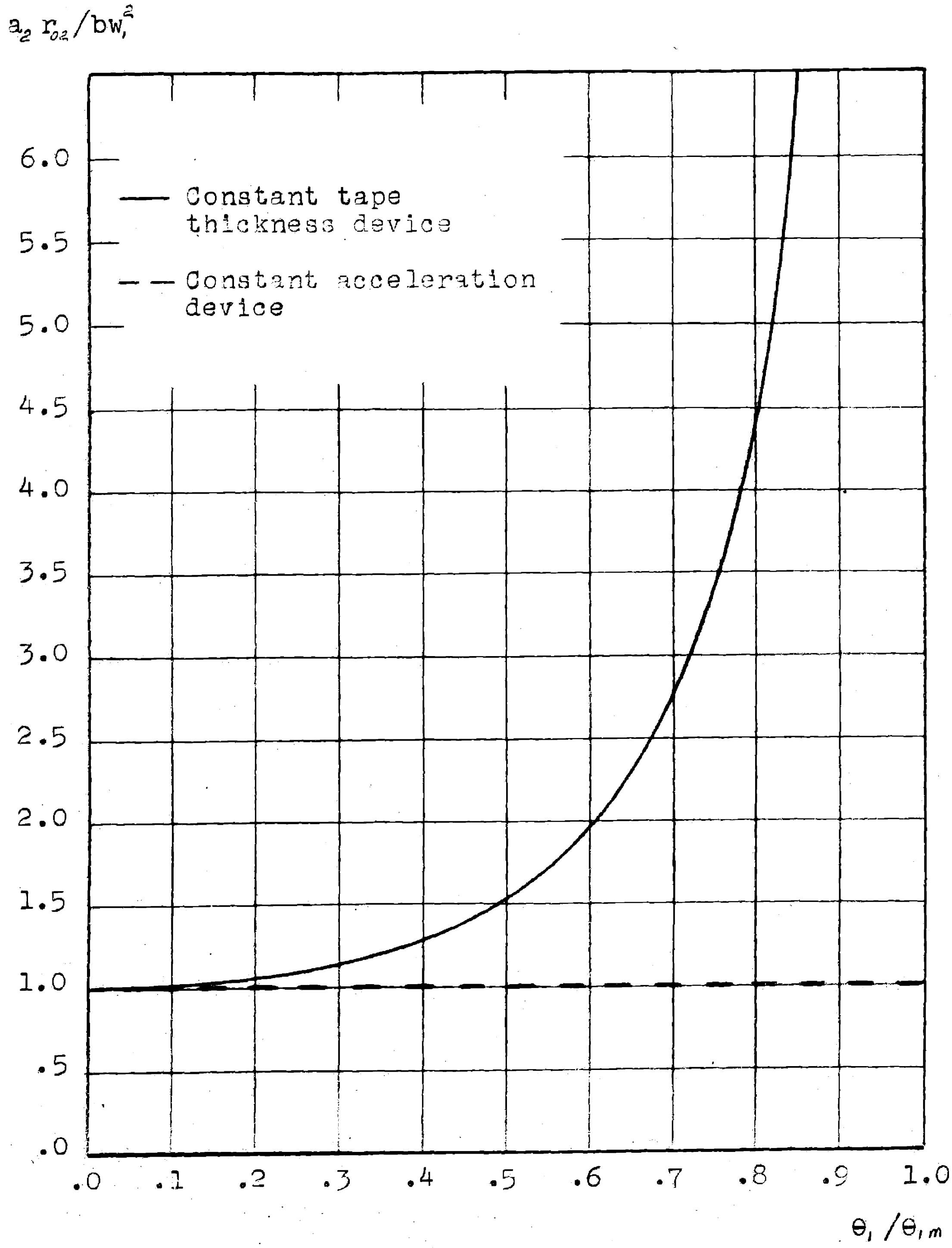

Fig. 5. Nondimensional plot of output angular acceleration vs. input angular displacement for constant tape thickness, and constant acceleration devices.

rapidly and goes to infinity as the radius r_2 goes to zero. It is interesting to note that the ratio of the two radii r_1 and r_2, varies inversely as the angular velocity ratio w_2/w_1. The mechanical advantage of the device may therefore be found by taking the inverse of the angular velocity ratio. This is seen to go from infinity at $\theta_1 = 0$ to zero at $\theta_1 = \theta_{1m}$.

The nondimensional angular acceleration as given in Fig. 5 starts out with an initial value of 1. By the time two thirds of the over all operation of the device is completed, the nondimensional acceleration has increased only by a factor of about 2.5, which compares very favorable to the fact that when $\theta_1 = \theta_{1m}$ the acceleration becomes infinite. This would suggest the possibility of varying the tape thickness as a function of its length to produce a theoretically constant acceleration over the entire range of operation of the device, with the fairly good assumption that an actual constant acceleration device could be built to operate over at least two thirds of the normal range.

If the nondimensional displacement is plotted on semi log paper as shown in Fig. 6, the results are found to vary only moderately from a straight line over approximately three decades. From this observation comes the idea that

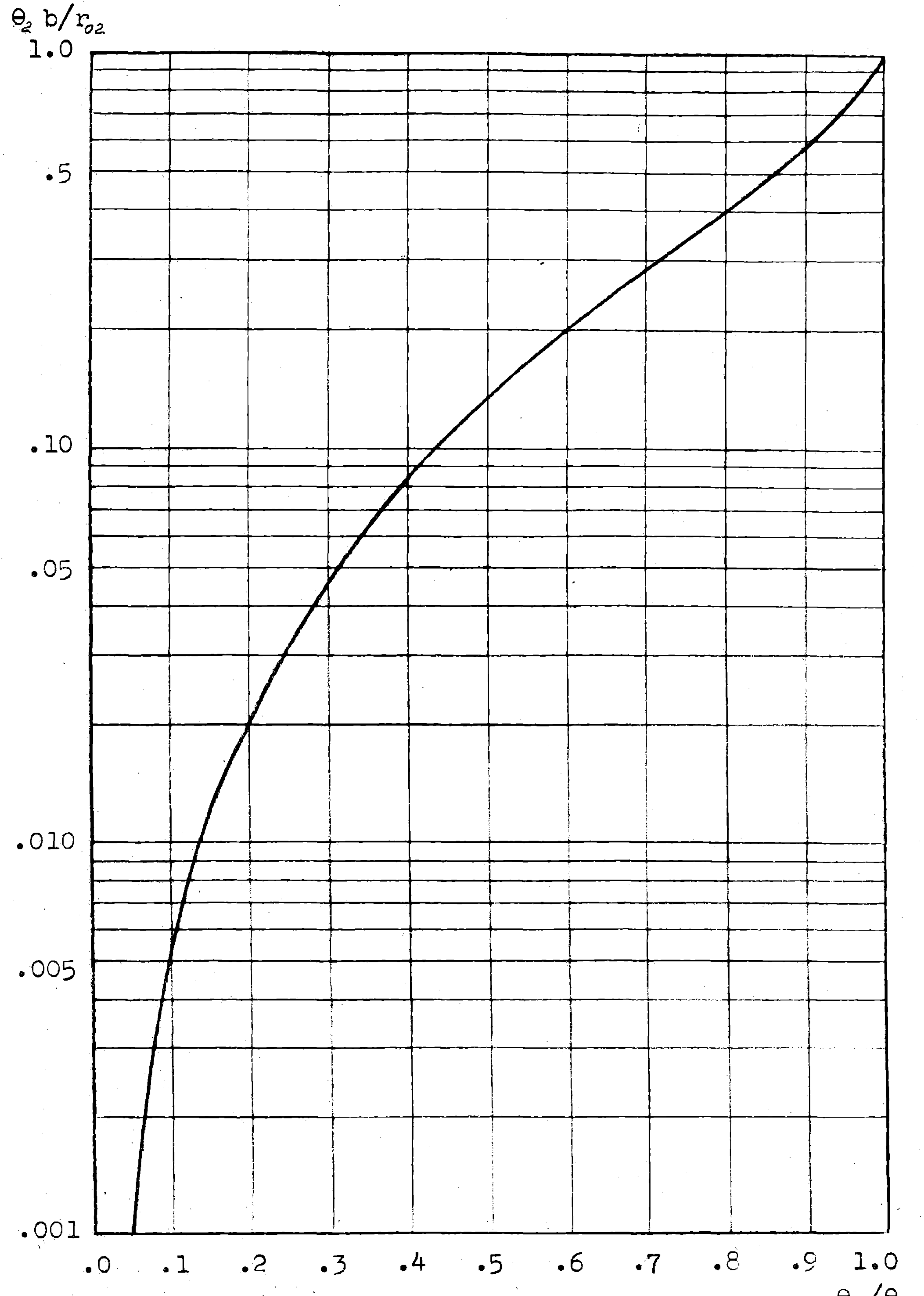

Fig. 6. Nondimensional semi-log plot of output angular displacement vs. input angular displacement for constant tape thickness device.

it would be possible to develop the device into a log function producer by proper variation of the tape thickness.

4. Analysis of Constant Acceleration Device

4-1. General Problem. As noted above, the ability to produce a constant acceleration output depends upon the proper variation in tape thickness. It is easily reasoned that the tape must decrease in thickness, for in the limit if a tape were made of approximately zero thickness, the acceleration would approach zero. The solution required must lie somewhere between the broad limits of the present constant thickness tape and a zero thickness tape.

By defining the output acceleration as equaling a constant value, the values of w_2 and θ_2 are easily found from the familiar equations

$$\begin{aligned} a_2 &= d^2\theta_2/dt^2 = \text{constant} \\ w_2 &= d\theta_2/dt = a_2 t \\ \theta_2 &= a_2 t^2/2 \end{aligned} \tag{18}$$

The value of a_2 being set at the initial acceleration found for the constant tape thickness case, i.e.,

$$a_2 = w_1^2 b_0/r_{02} \tag{19}$$

Thus the analysis of the constant acceleration device requires mainly the determination of the tape thickness as a function of the tape length.

4-2. <u>Derivation of Tape Equation.</u> As in the previous analysis it proves advantageous to make the assumption that at $\theta_1 = 0$, all the tape is wound on reel 2, i.e., $r_{01} = 0$. Also, the conditions that w_1 is a constant and that $ds_1 = ds_2$ will be used. The new restriction that is placed upon the system is that $a_2 = d^2\theta_2/dt^2 = \text{constant}$. The Eqs. (1) and (4) as presented above may now be used to produce the desired tape equation. Due to the relationship of Eqs. (1) and (4), the process of solving for the tape equation required first the finding of the tape thickness as a function of radius, and then the tape length as a function of radius. By an elimination of the radius from the two equations, the tape equation may be found giving the required tape thickness as a function of the tape length.

Writing Eqs. (1) and (4)

$$ds = r\,d\theta \qquad (1)$$

$$dr = b\,d\theta \qquad (4)$$

for reel 1, and then eliminating $d\theta_1$ from the two equations, gives

$$ds_1/r_1 = dr_1/b_1$$

Or

$$ds_1\ b_1 \ = \ r_1\ dr_1 \tag{20}$$

A similar procedure used for reel 2, results in

$$ds_2\ b_2 \ = \ -\ r_2\ dr_2 \tag{21}$$

But $ds_1 = ds_2$, and if the reels are assumed to be close together so that a point on the tape may be considered as leaving reel 2 and arriving at reel 1 simultaneously, then

$$b_2 \ = \ b_1 \ = \ b \tag{22}$$

and Eqs. (20) and (21) combine to give

$$r_1\ dr_1 \ = \ -\ r_2\ dr_2 \tag{23}$$

Integrating this between the limits of (r_{01}, r_1) and (r_{02}, r_2) and multiplying both sides by 2 produces

$$r_1^2 - r_{01}^2 \ = \ r_{02}^2 - r_2^2 \tag{24}$$

But as assumed for this case $r_{01} = 0$, therefore

$$r_1^2 \ = \ r_{02}^2 - r_2^2 \tag{25}$$

which is equivalent to saying that the area of the edge of the tape is constant.

Writing Eq. (1) for reels 1 and 2 and equating the two equations by the relation ds = ds gives the result that

$$r_1 \; d\theta_1 \; = \; r_2 \; d\theta_2 \tag{26}$$

Rearranging and substituting in the value for r_2 as defined by Eq. (25) shows that

$$\frac{d\theta_2}{d\theta_1} = \frac{r_1}{(r_{02}^2 - r_1^2)^{\frac{1}{2}}} \tag{27}$$

But from Eq. (18) and the definition of w

$$\frac{d\theta_2}{d\theta_1} = \frac{d\theta_2}{dt} \; \frac{dt}{d\theta_1} = \frac{a_2 t}{w_1} \tag{28}$$

Inserting this value into Eq. (27) and combining tems results in

$$\frac{a_2}{w_1} \; t = \left[\left(\frac{r_{02}}{r_1} \right)^2 - 1 \right]^{-\frac{1}{2}} \tag{29}$$

Differentiating this with respect to time, simplifying and solving for dr /dt gives

$$\frac{dr_1}{dt} = \frac{a_2}{w_1} \; r_{02} \left[1 - \left(\frac{r_1}{r_{02}} \right)^2 \right]^{\frac{3}{2}} \tag{30}$$

But dr /dt is easily found by differentiating Eq. (5) to be

$$\frac{dr_1}{dt} = b \; \frac{d\theta_1}{dt} = b \; w_1 \tag{31}$$

Substituting this into Eq. (30) and recalling from Eq. (19) that $a_2 = w_1^2 b_0 / r_{02}$ gives

$$b = b_0 \left[1 - \left(\frac{r_1}{r_{02}} \right)^2 \right]^{\frac{3}{2}} \tag{32}$$

The tape thickness b is therefore found as a function of the variable r .

If Eq. (20) is next rewritten as

$$ds_1 = r_1 / b \; dr_1$$

the value of $b = f(r_1)$ from Eq. (32) may be substituted in directly. Integrating between the limits of (0, s) and (0, r_1) respectively, produces the result that

$$b_0 s = r_{02}^2 \left[1 - \left(\frac{r_1}{r_{02}} \right)^2 \right]^{-\frac{1}{2}} - r_{02}^2$$

Rearranging gives the required relation between r and s

$$\left[1 - \left(\frac{r_1}{r_{02}} \right)^2 \right]^{\frac{1}{2}} = \frac{r_{02}}{b_0 s + r_{02}^2} \tag{33}$$

The left-hand term of this expression has the same form as the right-hand term of Eq. (32). Cubing Eq. (33) the result may be directly substituted into Eq. (32) to produce the final form of the tape equation which upon rearranging is

$$b = b_0 \left[\frac{1}{\frac{b_0 s}{r_{02}^2} + 1} \right]^3 \qquad (34)$$

4-3. <u>Presentation of Results.</u> To help produce a better understanding of the variation in tape thickness, a graphical plot of this nondimensional tape thickness b/b_0 vs. the nondimensional tape length $b_0 s/r_{02}^2$ is given in Fig. 7. A numerical plot of these values appears in the Appendix in Table 2. The kinematic characteristics are compared to the characteristics of the constant tape thickness device in Figs. 3, 4, and 5. Although θ_{1m} actually has no real meaning for the constant acceleration device, as θ_1 does not reach a maximum value, it does prove a convenient number to use in nondimensionalizing the results obtained.

Putting the kinematic outputs in terms of θ_1/θ_{1m} may be easily done by use of Eqs. (18) and (19) and the relation that $\theta_1 = w_1 t$, where $\theta_{1m} \equiv r_{02}/b_0$. Solving for the nondimensional forms results in

$$\begin{aligned} a_2 r_{02}/b_0 w_1^2 &= 1 \\ w_2/w_1 &= \theta_2/\theta_{1m} \\ \theta_2 b_0/r_{02} &= \tfrac{1}{2} (\theta_1/\theta_{1m})^2 \end{aligned} \qquad (35)$$

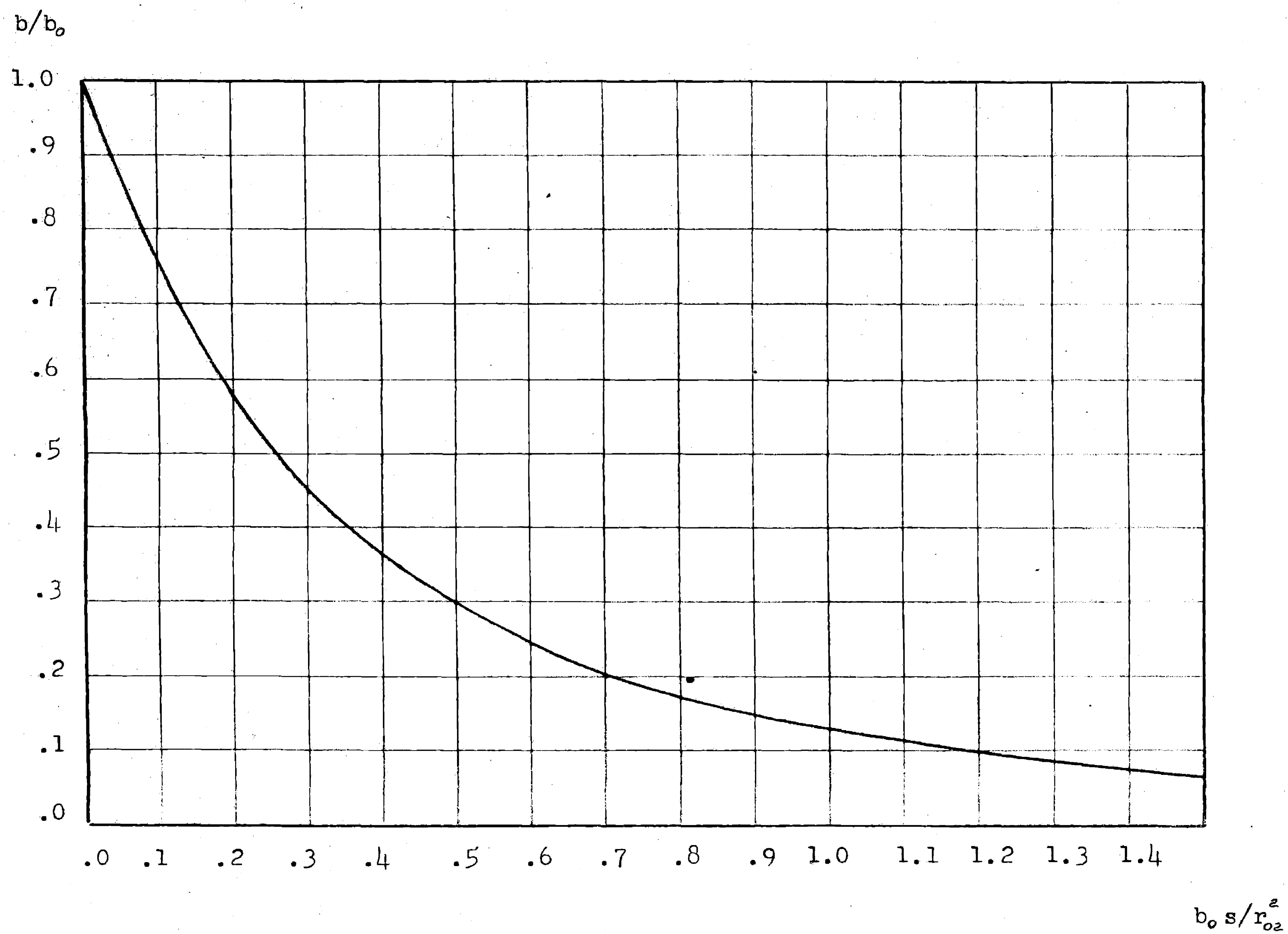

Fig. 7 Nondimensional tape thickness vs. tape length for constant acceleration device.

Numberical plots of these functions appear in the Appendix in Table 3.

In both the constant tape thickness and the constant acceleration devices analyzed above, it proved theoretically convenient to set $r_{o1} = 0$. In actual practice, it is impossible to achieve such a condition, as any physical reel must start with a finite radius. To use the above derived equations and graphs in such a case requires merely that they be modified to allow for the new starting conditions. The required transforms are given in the Appendix.

5. Analysis of Log Function Producing Device.

5-1. General Problem and Equations. As pointed out previously, the possibility suggests itself that by a proper variation of the tape thickness, various functions could be produced from the relative displacements of the two reels. Figure 6 specifically suggests the production of a log function. The main problem is to derive the tape equation which will produce a log displacement output, while satisfying the required end conditions. Also of interest will be the parameters which control the motion and nondimensional groups which characterize the output.

Starting with the log relationship shown in Fig. 8 it is first necessary to define various terms. Using similar notation as in the previous investigations, and recalling that the slope is given by the rise divided by the run, a slope may easily be defined. Let L equal the slope to the base 10. Therefore from Fig. 8

$$L = \frac{\log \theta_{2m} - \log \theta_2}{\theta_{1m} - \theta_1} \tag{36}$$

where log is used to refer to logarithms to the base 10. Rearranging, this becomes

$$\log (\theta_2 / \theta_{2m}) = L (\theta_1 - \theta_{1m}) \tag{37}$$

Changing from a base 10 to the natural base e produces

$$\ln (\theta_2 / \theta_{2m}) = B (\theta_1 - \theta_{1m}) \tag{38}$$

where ln is used to refer to logarithms of the base e, and B is the slope to the base e. B and L are thus related by the equation

$$B = 2.3026 \; L \tag{39}$$

Eq. (38) may be changed to exponential form, which upon multiplying through by θ gives

$$\theta_2 = \theta_{2m} e^{B(\theta_1 - \theta_{1m})} \tag{40}$$

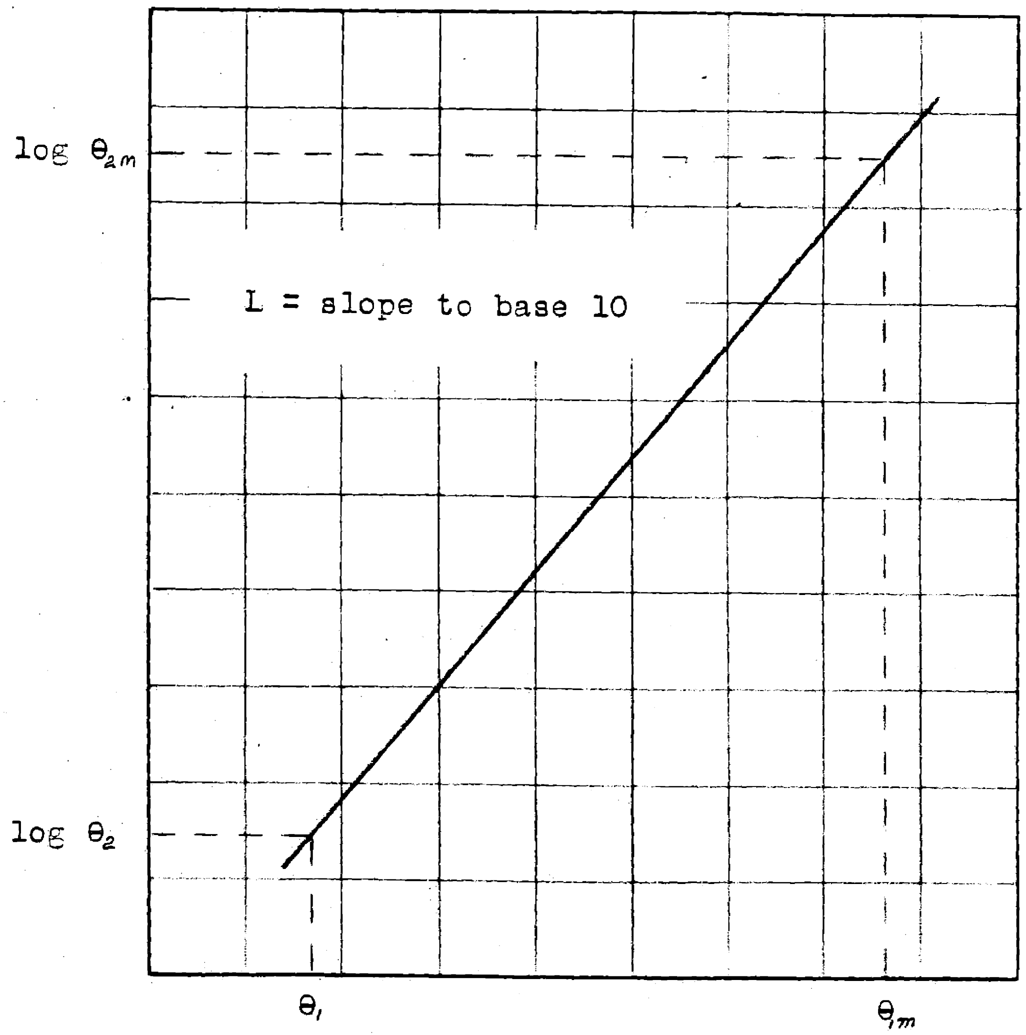

Fig. 8. Basic log relationship.

If a new constant A is defined by the equation

$$A \equiv \theta_{2m} e^{-\beta \theta_{1m}} \tag{41}$$

Eq. (40) reduces to

$$\theta_2 = A e^{\beta \theta_1} \tag{42}$$

This is the basic equation relating θ_2 to θ_1 and it must be satisfied at all times. When $\theta_1 = 0$ this reduces to

$$A = \theta_{20} \tag{43}$$

so that even when reel 1 has not moved, there must be an initial displacement of reel 2. This initial displacement is equal to A.

Before attempting the solution of the tape equation, several useful relationships are found. If N is defined by the equation

$$N = \log \theta_{2m}/\theta_{20} \tag{44}$$

i.e., the number of decades over which the log function is produced, then from Eq. (37) evaluated at the initial conditions of $\theta_1 = 0$ and $\theta_2 = \theta_{20}$, comes the result that

$$L = N/\theta_{1m} \tag{45}$$

It then directly follows that

$$B = 2.3026\ N/\theta_{1m} \tag{46}$$

By using the above results and applying them with the Eqs. (1) and (4), the solution to the tape equation may be found. The method of solution that must be followed is similar to that used in the constant acceleration case. The thickness b is first found as a function of θ_2, then the displacement s is found as a function of θ_2, and the two results equated to give $b = f(s)$. It is found advisable to find b and s as functions of θ_2 rather than θ_1, as the use of θ_1 requires the carrying along of an extra exponential form throughout the solution.

5-2. Derivation of the Tape Equation. Starting with Eq. (42)

$$\theta_2 = A\, e^{B\theta_1}$$

and differentiating with respect to θ_1 gives

$$d\theta_2/d\theta_1 = A\, B\, e^{B\theta_1} = B\, \theta_2 \tag{47}$$

But from Eq. (1) written for both reels, and the fact that $ds_1 = ds_2$, is still valid comes the relation

$$r_1 d\theta_1 = r_2 d\theta_2$$

Or

$$d\theta_2/d\theta_1 = r_1/r_2 \tag{48}$$

Thus Eq. (47) becomes

$$r_1/r_2 = B\,\theta_2 \tag{49}$$

If it is then assumed that $b_1 = b_2$, Eq. (24) is seen to be valid and upon rearranging gives

$$r_1 = (r_{01}^2 + r_{02}^2 - r_2^2)^{1/2} \tag{50}$$

Putting this value into Eq. (49) and squaring, results in

$$(r_{01}^2 + r_{02}^2 - r_2^2)/r_2^2 = B^2\,\theta_2^2$$

Solving for r_2

$$r_2 = (r_{01}^2 + r_{02}^2)^{1/2}(B^2\,\theta_2^2 + 1)^{-1/2} \tag{51}$$

and then differentiating with respect to θ_2

$$\frac{dr_2}{d\theta_2} = -\frac{(r_{01}^2 + r_{02}^2)^{1/2}\,B^2\,\theta_2}{(B^2\,\theta_2^2 + 1)^{3/2}} \tag{52}$$

From Eq. (4) written for reel 2

$$dr_2/d\theta_2 = -\,b$$

Therefore combining this with Eq. (52) yields

$$b = \frac{(r_{01}^2 + r_{02}^2)^{1/2}\,B^2\,\theta_2}{(B^2\,\theta_2^2 + 1)^{3/2}} \tag{53}$$

which presents b as a function of the variable θ_2.

Looking back at Eq. (51) and recalling that at $\theta_1 = 0$; $r_2 = r_{02}$, $\theta_2 = A$ and defining a new constant C by the equation

$$C = A\,B \tag{54}$$

the result is that

$$C = r_{01}/r_{02} \tag{55}$$

The tape length s may now be found as a function of θ_2 by substituting the value of r_2 as found in Eq. (51) into Eq. (1) written for reel 2

$$ds = \frac{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}}{(B^2\theta_2^2 + 1)^{\frac{1}{2}}}\,d\theta_2 \tag{56}$$

To integrate this, it is noted that it is of the form

$$\frac{du}{\sqrt{u^2 + a^2}} \tag{57}$$

By letting u = a sinh t, the integral of Eq. (57) is found to be

$$\int \frac{a \cosh t\, dt}{(a^2 \sinh^2 t + a^2)^{\frac{1}{2}}} = \int dt$$

Thus when Eq. (56) is integrated between the limits of $(0,\ s\ )$ and (θ_{20}, θ_2) the result is

$$s = \frac{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}}{B}\left[\sinh^{-1}\theta_2 B - \sinh^{-1}\theta_{20} B\right] \tag{58}$$

giving s as a function of θ_2 , or solving for $\theta_2 B$

$$\theta_2 B = \sinh\left[\frac{B\ s}{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}} + \sinh^{-1}\theta_{20} B\right] \tag{59}$$

If Eq. (59) is then substituted directly into Eq. (53) the tape thickness b is found as a function of the tape length s, which in nondimensional form is

$$\frac{b}{B(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}} = \frac{\sinh\left[\frac{B\ s}{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}} + \sinh^{-1}\theta_{20} B\right]}{\left[\sinh\left[\frac{B\ s}{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}} + \sinh^{-1}\theta_{02} B\right] + 1\right]^{3/2}} \tag{60}$$

5-3. <u>Presentation of Results.</u> Looking at the tape equation as given inEq. (60) the important parameters are seen to be the starting radii r_{01} and r_{02}, the starting displacement of θ_2 , i.e., θ_{20}, and the slope B to the base e. To completely specify a device that is being designed to produce a log function over a definite range, one more parameter is required. This may be either s, θ_{1m}, θ_{2m}, N, or r_{2m}.

As the tape thickness is not a function of only one dimensionaless group, a single graph will not suffice to show all possible cases. In Fig. 9 however, typical plots of the tape thickness vs. θ_2/θ_{2m} are given at various values of B for $N = 3$, $A = .1$. It will be noted that variations in A do not effect the nondimensional tape thickness.

Some useful results for calculation of various quantities may be found in the following manner.

Substituting Eqs. (41), (46) and (54) into Eq. (55) and rearranging results in

$$\frac{\theta}{\theta} = \frac{2.3026\ N\ r_{02}}{e^{2.3026\ N}\ r_{01}} \qquad (61)$$

To check on the smallest value of r obtained when $\theta_2 = \theta_{2m}$ use Eqs. (49), (50), (54) and (55). Combining these and solving for r_{2m} produces

$$r = \sqrt{\frac{r_{01}^2 + r_{02}^2}{1 + \left(\frac{r_{01}\ \theta_{1m}}{r_{02}\ A}\right)^2}} \qquad (62)$$

If $r_{01} = r_{2m}$ as it might be in a practical application,

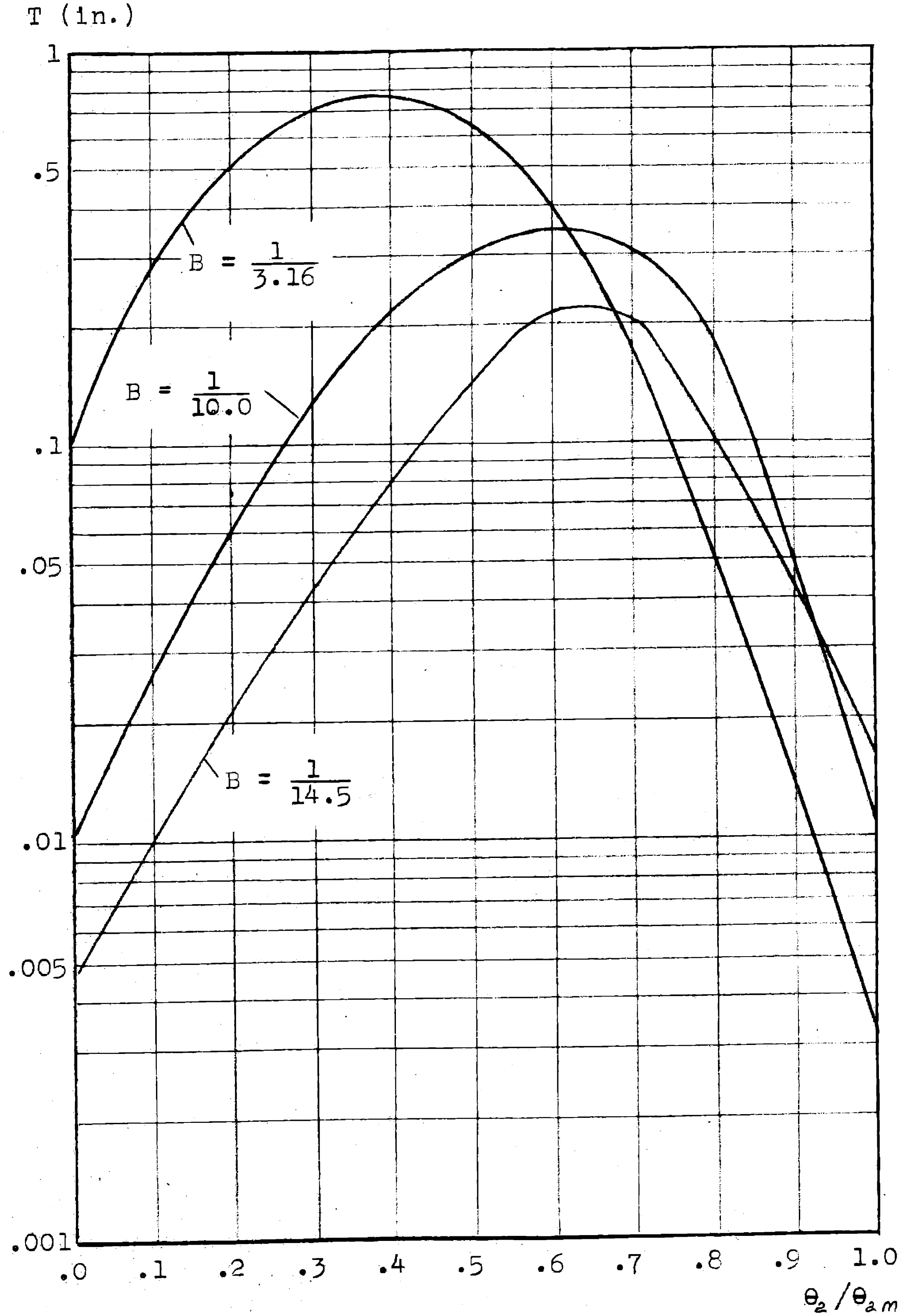

Fig. 9. Tape thickness vs. nondimensional output angular displacement at various values of B, for N = 3, and A = .1 .

from Eqs. (47) and (49) comes the result that

$$C^{2} = \frac{1}{e^{2.3026\ N}} \tag{63}$$

which when introduced into Eq. 61 gives

$$\frac{\theta_{1m}}{\theta_{2m}} = 2.3\ N \frac{r_{o1}}{r_{o2}} \tag{64}$$

To further help in the actual calculations required to construct a log function producing device, additional design criteria appear in the Appendix.

Part B

EXPERIMENTAL VERIFICATION

6. Introduction

6-1. <u>Scope of Investigation.</u> This section of the investigation forms an experimental check of the main analytical results obtained in Part A. As in the analysis, the kinematics of the device are of primary interest and only visual observations are made of the dynamics. The main aim is not a complete check of all the variables, but rather an experimental verification of the important parameters for each case studied.

6-2. <u>Experimental Apparatus.</u> The device constructed to carry out the experiments consisted basically of two reels connected to separate electric motors, a tape similar to the type used in recording machines, and two mechanical counters. The actual working model is shown in Fig. 10.

The reels were made of lucite and fastened directly to the shafts of the respective motors by means of a small hub. The minimum radius was variable down to a value of .05 inches, and the maximum allowable radius was 3 inches. The electric motors used were both of the variable speed type and were controlled by separate variacs. The drive motor to reel 1 was mounted directly to the base as shown in Fig. 10 while the motor fastened to reel 2 was suspended to form a dynamometer. The angle of deflection of reel 2 being an indication of the force being transmitted.

In the actual tests, the motors were run counter to each other to allow the operator to vary the load and direction of motion at will. The tape thus could be wound from one reel to the other and back again, merely by changing the relative power produced by each motor. To check the displacements, mechanical counters were fastened directly to the reels. They were capable of running in

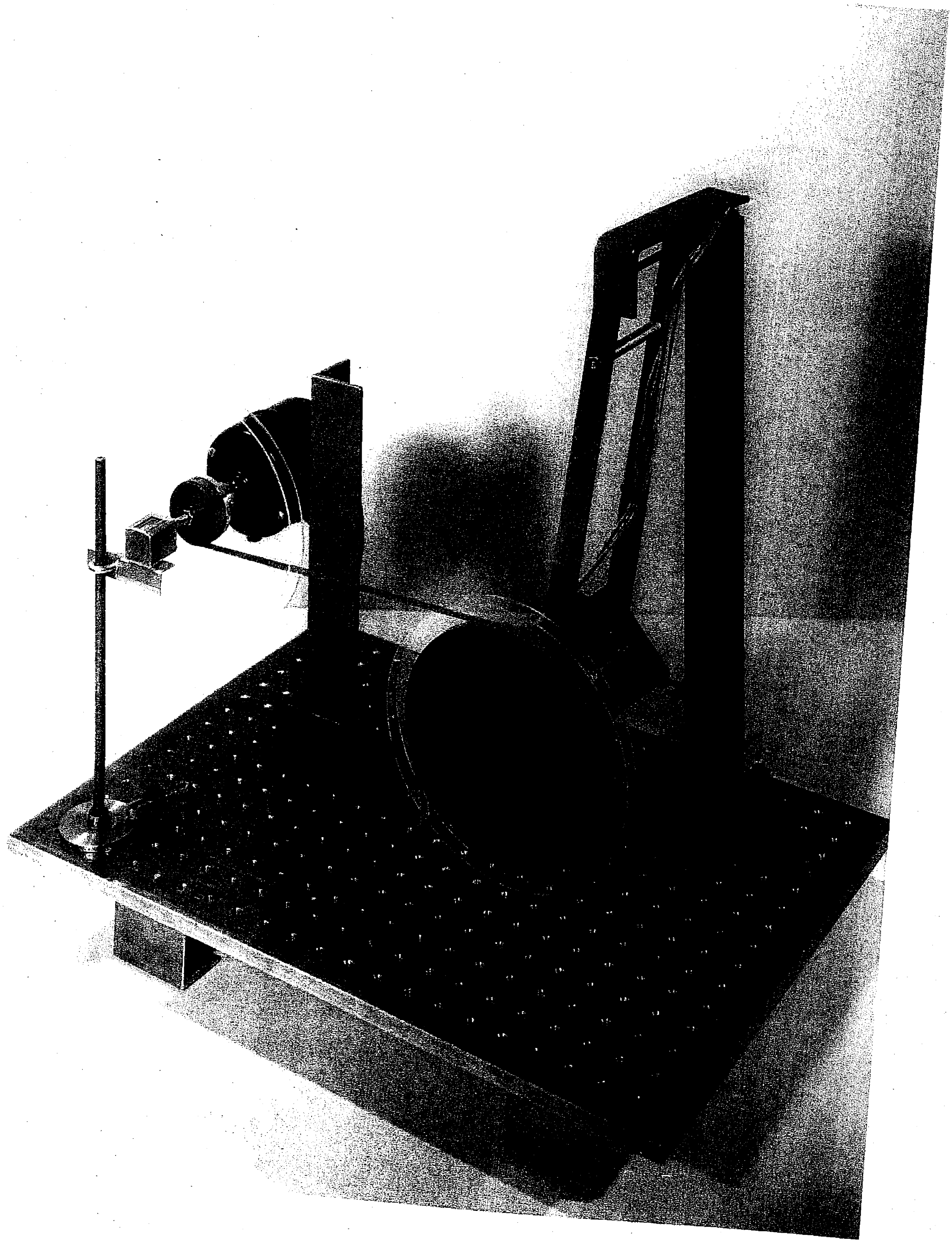

Fig. 10. Experimental model of tape drive device.

either direction and subdivided to read down to one-tenth of a revolution.

7. Verification of Constant Tape Thickness Analysis

7-1. Design of Device. Before any knowledge can be gained by running the experimental apparatus, it is necessary to know what the expected results should be. If a check of the displacement is contemplated, the maximum number of turns along with the compensation required because of the fact that $r_{01} \neq 0$, must be calculated. In the analysis of the constant thickness tape performed in Part A it was assumed that the starting radius of reel 1 was zero. In the actual case this is impossible and so the proper transformations as given in the Appendix must be used.

In this particular case, the starting radius of reel 1 and the ending radius of reel 2 were $r_I = r_{2m} = .16$ in. and the starting radius for reel 2 was $r_{II} = 2.95$ inches. Using the transform

$$r_{02} = (r_I^2 + r_{II}^2)^{\frac{1}{2}} \qquad (65)$$

and the above values gives

$$r_{02} = (.16^2 + 2.95^2)^{\frac{1}{2}} = 2.954$$

If the condition $r_{o1} = 0$ did exist the total number of turns would be

$$\theta_{2m} = \frac{r_{o2}}{T} = \frac{2.954}{.002} = 1477 \text{ rev.}$$

As $r_I = .16$ inches in this case, the angular displacement of θ_2 that would be expected to be observed would be the total theoretically possible minus the sum of those lost due to the fact that r_{o1} and r_{2m} do not equal zero. Or, in equation form as shown in the Appendix.

$$\theta_{2m\,obs} = \theta_{2m} - \frac{r_{o2} - r_{I}}{T} - \frac{r_{2m}}{T} \qquad (66)$$

Therefore

$$\theta_{2m\,obs} = 1477 - \frac{2.954 - 2.950}{.002} - \frac{.16}{.002}$$

$$\theta_{2m\,obs} = 1477 - 2 - 80 = 1395 \text{ rev.}$$

As the device is symmetrical, the value of $\theta_{1m\,obs}$ would also be expected to have a maximum value of

$$\theta_{1\,obs} = 1395 \text{ rev.}$$

7-2. Results of Test. In the actual run made on the constant tape thickness device, the values of $\theta_{2m\,obs}$ and $\theta_{1m\,obs}$ obtained were 1429.7 and 1436.7 revolutions respectively. The maximum variation from the calculated value being less than 3%. The error between $\theta_{2m\,obs}$ and $\theta_{1m\,obs}$ is approximately .5%. The variation in the first case is well within the error involved in measuring the tape thickness, with the error between the maximum observed values of displacement coming mainly from slippage.

The observed data may now be transformed into the form plotted in Fig. 3 by adding 80 to the observed value of $\theta_{1\,obs}$ and then dividing this number by $(\theta_{1m,obs} + 82)$. Similarly, $\theta_{2\,obs}$ is transformed by adding 2 to its observed value and then dividing by $(\theta_{1m,obs} + 82)$. The transformed points are plotted in Fig. 3 where they give a direct comparison with the predicted results. The experimental displacement characteristics are thus seen to verify the analytic solution for this particular case.

7-3. <u>Dynamic Observations.</u> For the constant tape thickness case, several dynamic observations were made which are worthy of note. The stability of the tape in passing from one reel to another does not seem to present any problems. As reel 2 is continually being accelerated, there is always some load on the tape and this effect, though slight, is enough to insure the general stability of the tape.

Slippage of the tape on the reel does not appear to be a problem. Although small amounts of it were observed in the actual tests, this was due mainly to the lack of a large load on the tape to cause it to wind tightly. The only time slippage had noticeable effects was when there was a large variation in load applied.

An extremely notable effect was the large and rapid change in mechanical advantage. Unless the load is continually decreased or the power input increased, it is impossible to continue the input angular velocity at a constant value.

8. Experimental Check of Constant Acceleration Device.

8-1. <u>Design of Constant Acceleration Tape.</u> An experimental check of the analytical results for a constant acceleration device may be carried out in a similar manner to that used in varifying the constant tape thickness solution. However, now the main problem of producing a working device lies in the design of the tape to be used.

From a physical standpoint, it was found convenient to start with the radius of reel 1 equal to .16 inches. It is seen from Fig. 7 of b/b_0 vs. $b_0 s/(r_I^2 + r_{II}^2)$ that for a maximum value of $b_0 s/(r_I^2 + r_{II}^2) = 1$, the tape thickness must decrease by a factor of approximately 10. The Appendix shows that the nondimensional tape length is given by

$$\frac{b_0 s}{r_I^2 + r_{II}^2} = \frac{(r_I^2 + r_{II}^2)^{\frac{1}{2}}}{r_{Im}} - \frac{(r_I^2 + r_{II}^2)^{\frac{1}{2}}}{r_{II}} \qquad (67)$$

where r_{2m} should be approximately $r_{II}/2$.

To keep the tape a convenient length to handle physically, the value of $r_{II} = .75$ inches was taken. From the above, the value of r_{2m} used was therefore .375 inches. As given in the Appendix, the transform for the displacement axis of Fig. 7 is

$$\frac{b_o s}{r_{o2}^2} = \frac{b_o s_{obs}}{r_I^2 + r_{II}^2} + \frac{(r_{II}^2 + r_I^2)^{\frac{1}{2}}}{r_{II}} - 1 \tag{68}$$

The new zero point on Fig. 7 may therefore be found by substituting in the values given above and setting $s_{obs} = 0$. This results in

$$\frac{b_o s}{r_{o2}^2} = \frac{(.75^2 + .375^2)^{\frac{1}{2}}}{.75} - 1 = .02$$

Equation (67), when evaluated shows that

$$\frac{b_o s}{r_I^2 + r_{II}^2} = 1.02$$

Therefore

$$b_o s = (1.02)(r_I^2 + r_{II}^2) = .599 \text{ in.}^2$$

But recalling that b is defined as

$$b \equiv T/2\pi$$

which for initial theoretical conditions is

$$b_o = T_o/2\pi$$

gives the result that

$$T_o s = 2\pi(.599)\ \text{in.}^2 = 3.76\ \text{in.}^2 \qquad (69)$$

It would be convenient to have the observed starting value of T equal to .020 inches. Therefore, looking at the new zero point found on Fig. 7 the value of b/b_o is seen to equal .95. The initial theoretical thickness must thus be

$$T_o = (.020)\ (1/.95) = .0211\ \text{in.}$$

Putting this result into Eq. (69) gives the required tape length

$$s = \frac{3.76\ \text{in.}^2}{.0211\ \text{in.}} = 178.5\ \text{in.} = 14.9\ \text{ft.}$$

Using the displacement axis transformation of Eq. (68), the nondimensional tape thickness may be taken directly from Fig. 7. As shown above, this transform merely moves the observed zero point. Putting in the actual values for the case being considered and plotting the results as, observed tape thickness T_{obs} vs. length s, a graph of the required tape is obtained. This is shown in Fig. 11.

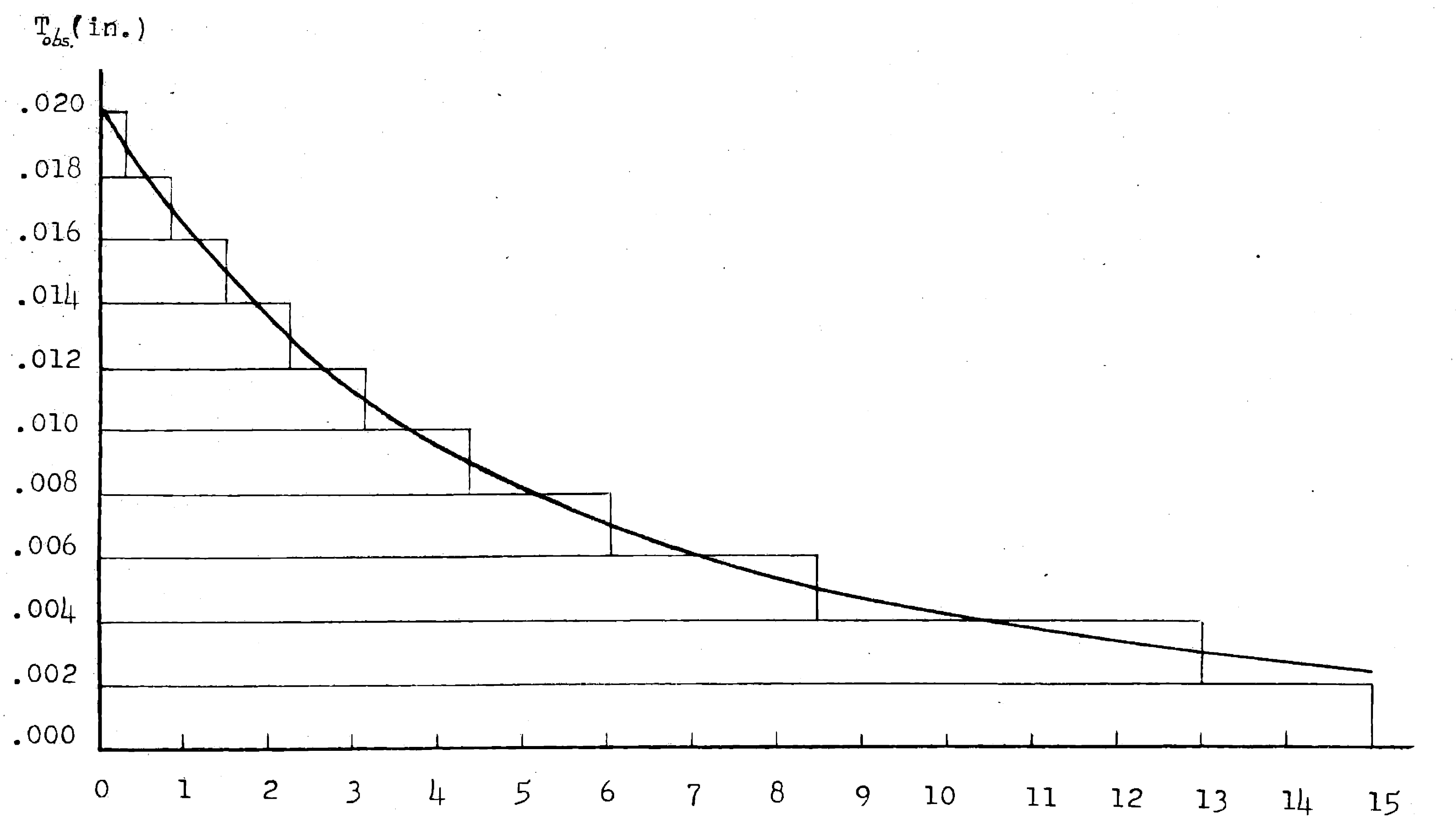

Fig. 11 Tape thickness vs. tape length for constant acceleration device, with multilayer tape approximation shown.

In the actual production of such a tape, it was found practical to use step approximations to the thickness variations. This was accomplished by placing various lengths of constant thickness tape on top of each other. The approximate tape produced is also represented on Fig. 11. It proved impossible to fasten the several layers together, as the presence of any binding agent caused thickness variations in the order of magnitude of those trying to be produced.

8-2. Results of Test. The tests made to check the validity of the derived tape equation consisted of a displacement check between the two reels. During this check, the respective radii were also measured. The data taken appears in the Appendix. No serious attempt was made at a direct velocity check due to the composite nature of the tape.

Displacement readings of θ_2 were taken at equal increments in θ_1. If the change in θ_2 over each increment in θ_1 is plotted vs. θ_1, the result should be a straight line. The actual experimental plot of this appears in Fig. 12 and is seen to confirm the expected results.

9. Experimental Check of Log Function Device

9-1. Design of Tape. As in the case of the constant

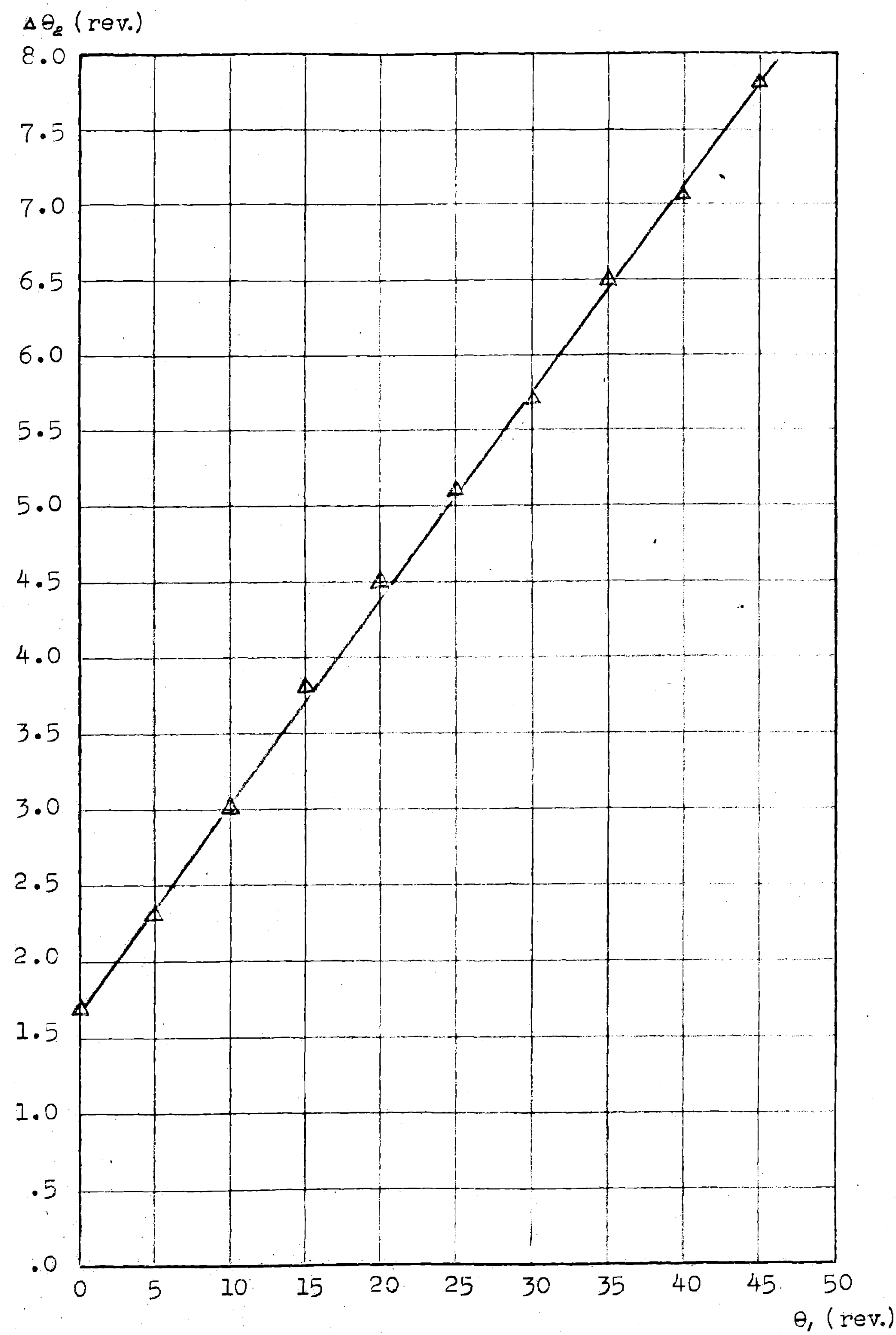

Fig. 12. Experimental plot of the change in output angular displacement vs. input angular displacement for constant acceleration device.

acceleration, the main problem in developing a log function device is found in the design of the tape. Going back to the typical log tape profiles shown in Fig. 9, it is noted that the tapes start at a given thickness, increase to a maximum and then decrease again to some ending thickness. In general the starting and ending thicknesses are not the same, but in the case where they are, the maximum tape variation required is a minimum. In the design of a log tape, it thus becomes important to see that the starting and ending thicknesses are equal.

As shown in the Appendix, this may be accomplished by having

$$\theta_{2m} B = \sinh (x + \theta_{20} B) \qquad (70)$$

where x vs. $\theta_{20} B$ is given in Table 4 in the Appendix.

Assume the following values of initial radii, initial displacement of reel 2, and slope to the base e.

$$r_{01} = .10 \text{ in.}$$

$$r_{02} = 1.0 \text{ in.}$$

$$\theta_{20} = 10 \text{ rad.}$$

$$B = .01 \text{ rad.}^{-1}$$

The product

$$\theta_{20} B = (10)(.01) = .10$$

when used in conjunction with Table 4 gives the value

$$x = 1.69$$

Together, these results when substituted into Eq. (70) give the $\theta_{2m}B$ product as

$$\theta_{2m} B = \sinh 1.79 = 2.91$$

The value of θ_{2m} is then found to be

$$\theta_{2m} = 2.91/B = 2.91/.01 = 291 \text{ rad.}$$

From Eq. (44)

$$N = \log \theta_{2m}/\theta_{20} = \log 291/10 = 1.464$$

Thus this particular design will produce a log function over 1.464 decades.

The maximum displacement θ_{1m} of reel 1 is found by using Eq. (46)

$$B = 2.3\ N/\theta_{1m}$$

Therefore

$$\theta_{1m} = 2.3\ N/B = 337 \text{ rad.}$$

The length of tape needed is found from Eq. (58)

$$s = \frac{(r_{01}^2 + r_{02}^2)^{\frac{1}{2}}}{B}\left[\sinh^{-1}\theta_2 B - \sinh^{-1}\theta_{20} B\right]$$

which upon substituting in the proper values yields

$$s = 169 \text{ in.} = 14.1 \text{ ft.}$$

The radius at the end of operation is found from Eq. (62) and is seen to be

$$r_{2m} = .325 \text{ in.}$$

With the value of all the constants known, it now remains only to substitute them into the log tape equation, i.e., Eq. (60) and vary the value of s from 0 to 169 inches. When this is done, the profile of the required tape may be plotted, and is shown in Fig. 13. As previously done, this shape may be approximated by placing the proper lengths of constant thickness tape on top of each other.

9-2. Results of Test. In the test that was made, the displacement θ_2 was found as a function of θ_1. The results of this are shown on a semi log plot in Fig. 14. It is seen that the points taken lie essentially on a straight line over the entire range of operation and thus confirm the log tape solution.

A comparison of the calculated to experimentally found displacements gives similarly good results.

$$\theta_2 \text{ calculated} = 291 \text{ rad.}$$
$$\theta_2 \text{ observed} = 289 \text{ rad.}$$

$$\theta_1 \text{ calculated} = 337 \text{ rad.}$$
$$\theta_1 \text{ observed} = 328 \text{ rad.}$$

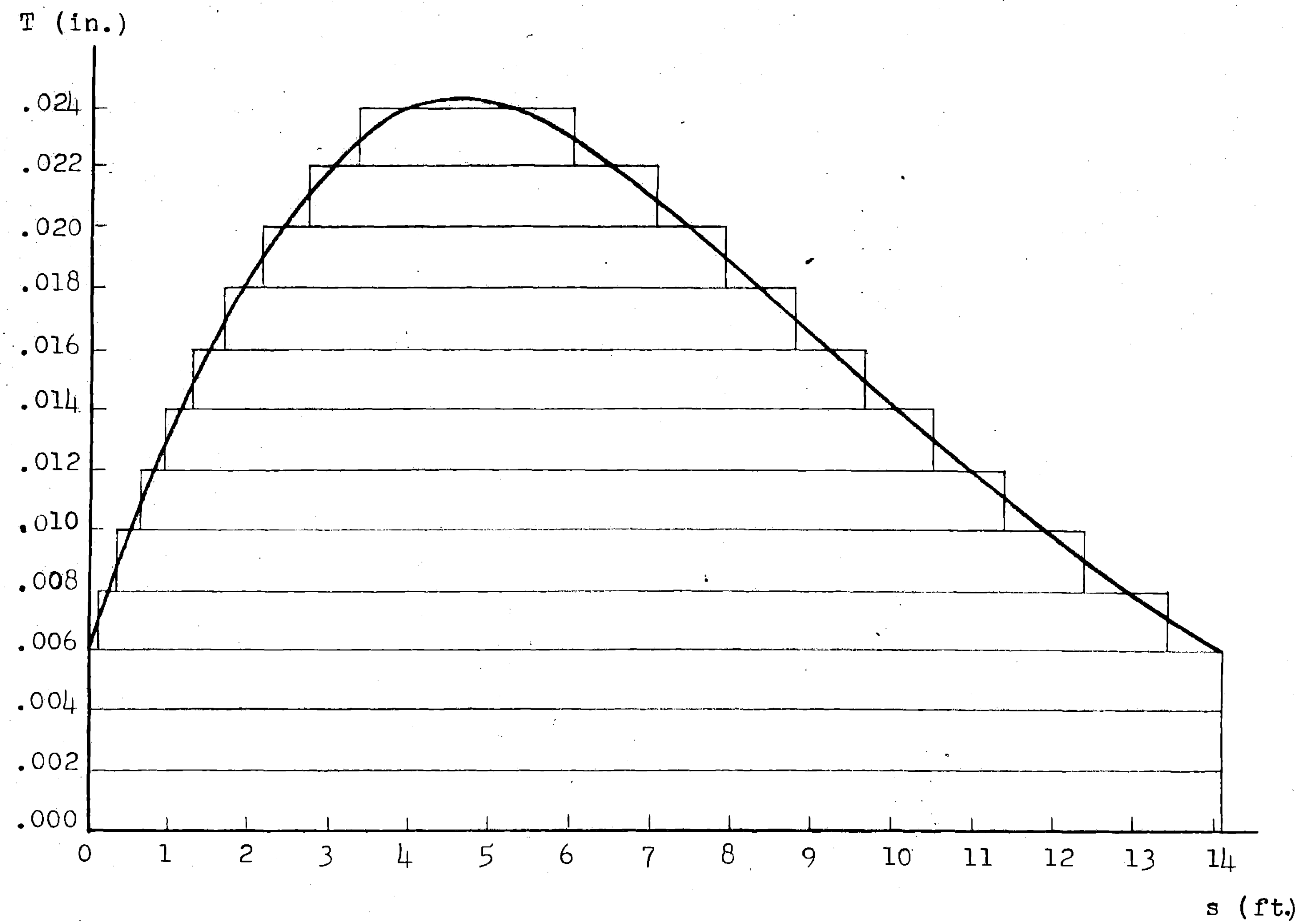

Fig. 13. Tape thickness vs. tape length for log function producing device, with multilayer tape approximation shown.

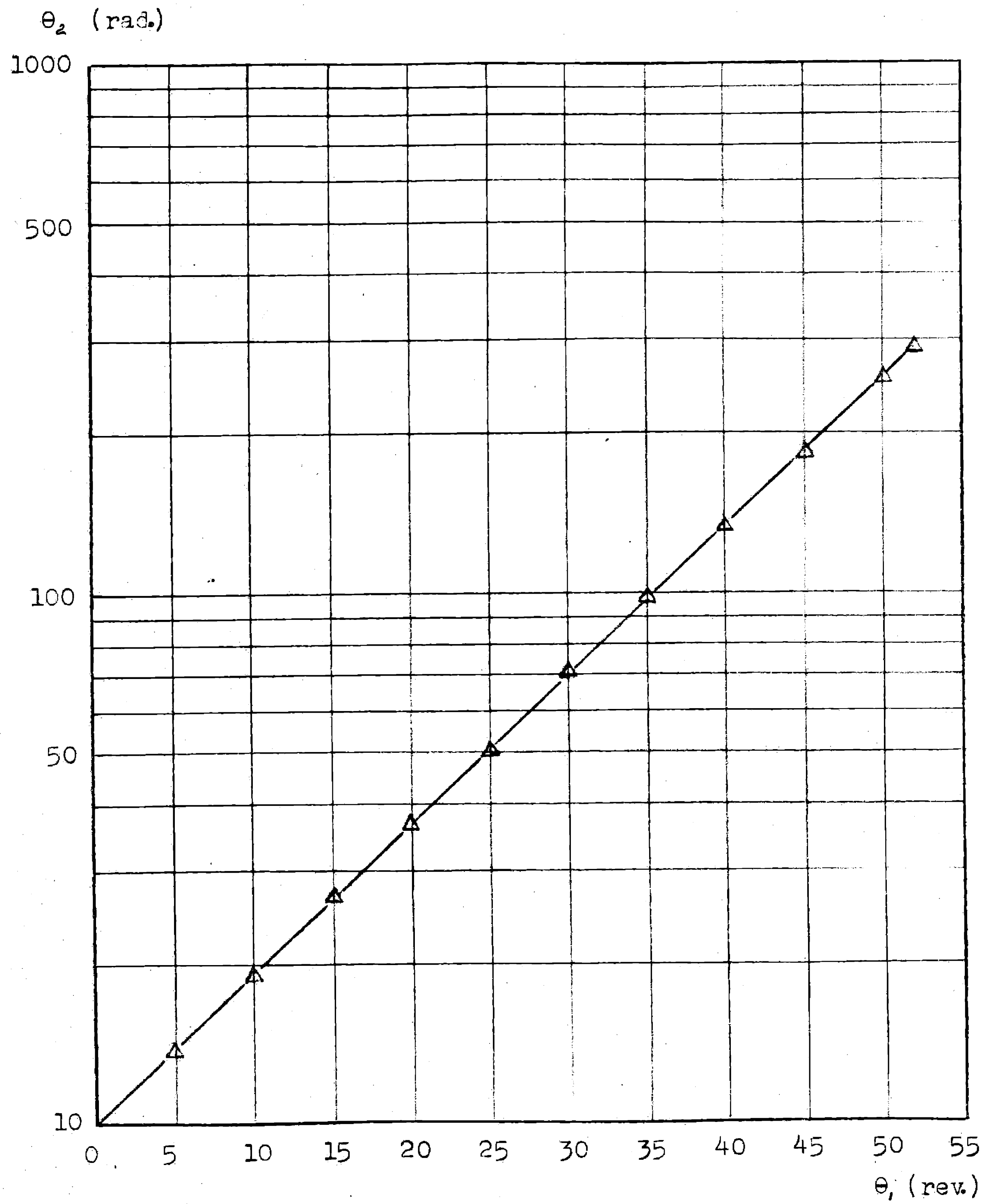

Fig. 14. Experimental plot of output angular displacement vs. input angular displacement for log function producing device.

10. Remarks

10-1. Possible Applications. Once the kinematic characteristics of this device are known to be easily controllable, possible direct applications of it become of interest. It would appear that these applications would fall into two general regions, (1) the production of functions and, (2) the transmission of power.

The production of functions is rather self explanatory. Within a range, dictated roughly by acceptable tape sizes and variations in thickness, the device could be used to produce functions in terms of actual physical outputs. To go beyond this range would require a variable input. This would suggest the possibility of cascading two or more units in series.

In the transmission of power, several points are worth noting. It has already been pointed out that the device will operate only until the end of the tape is reached, and that after this point some steady state drive must be used. The inability of the device to transmit negative torques is also an important consideration.

One extremely useful application would appear to be found in using the device in conjunction with a small power source to start up high rotary inertia machinery. As was noted earlier, the mechanical advantage decreases as the output velocity increases, but the possibility exists of balancing out this effect with the transient torque characteristic of the power source.

10-2. Future Investigation. Both types of possible application referred to above suggest the need of expanding the present solution to include an input acceleration. Although this does not require much more mathematics, it does add another independant variable which does not allow a single graphical representation as before. However, once the input acceleration is given, the over-all solution should prove much the same as that found above.

If a serious need should arise that requires the production of any function within the range of the device, graphical or numerical methods of developing a tape equation would certainly prove useful. Along this line of solution appears to be the problem encountered when the assumption that $ds_1 = ds_2$ is no longer valid, as in the case when two cams are connected by a tape.

As a further suggestion for future investigation, the field of the dynamics of a tape drive seem fairly open. Encountered here are not only the normal dynamic problems of the device, but also the associated problems dealing with the design of an entire unit, including matching of torque characteristics and transition to direct drive.

10-3. Conclusion. The forgoing is considered as an initial step in the understanding of tape drives. In dealing with the kinematics of this type of device, it clearly shows that it is possible to predict within a high degree of accuracy the output to be expected for a given geometry. It is also possible to accurately control this output by varying the thickness of the tape as a function of its length. Although in practice the derivation of the tape equation may be rather awkward, it is in general not difficult.

The actual design of a desired drive may be carried out mathematically or by use of the graphical results.

APPENDIX

I Numerical Plots

Table 1. Numerical plot of nondimensional angular displacement, velocity, and acceleration output for constant tape thickness device

input displacement θ_1/θ_{1m}	output displacement $\theta_2 b/r_{o2}$	output velocity w_2/w_1	output acceleration $a_2 r_{o2}/bw_1^2$
.00	.0000	.0000	1.000
.05	.0012	.0501	1.003
.10	.0051	.1005	1.015
.15	.0126	.1519	1.039
.20	.0204	.2042	1.064
.25	.0318	.2582	1.101
.30	.0461	.3145	1.152
.35	.0632	.3736	1.216
.40	.0835	.4364	1.299
.45	.1070	.5039	1.404
.50	.1340	.5770	1.538
.55	.1648	.6585	1.716
.60	.2000	.7500	1.953
.65	.2401	.8554	2.279
.70	.2859	.9800	2.746
.75	.3386	1.1339	3.456
.80	.4000	1.336	4.629
.85	.4732	1.6136	6.840
.90	.5641	2.0646	12.072
.95	.6877	3.0419	32.829
.99	.8909	9.0742	770.0
1.00	1.0000	∞	∞

Table 2. Nondimensional numerical plot of tape thickness vs. tape length for constant acceleration device

tape length $b_0 s/r_{02}^2$	tape thickness b/b_0
.0	1.000
.1	.751
.2	.578
.3	.454
.4	.363
.5	.296
.6	.244
.7	.203
.8	.171
.9	.145
1.0	.125
2.0	.0369
5.0	.00466

Table 3. Nondimensional numerical plot of angular displacement output vs. angular displacement input for constant acceleration device

input displacement θ_1/θ_{1m}	output displacement $\theta_2 b_0/r_{02}$
.0	.000
.1	.005
.2	.020
.3	.045
.4	.080
.5	.125
.6	.180
.7	.245
.8	.320
.9	.405
1.0	.500
1.2	.720
1.4	.980
1.6	1.28
1.8	1.62
2.0	2.00

II Transformation Equations and Design Criteria

Constant Tape Thickness Device

For the general case of having a device starting with the initial conditions that

$$\theta_1 = \theta_2 = 0$$

$$r_1 = r_I > 0$$

$$r_2 = r_{II} > 0$$

and ending with the condition

$$r_2 = r_{2m} > 0$$

the Eqs. (13), (15) and (17) derived for the displacement, velocity, and acceleration, respectively, are still valid if the following substitutions are made.

Replace r_{o2} by $(r_I^2 + r_{II}^2)^{\frac{1}{2}}$

Replace θ_2 by $\theta_{2obs} \pm \left[(r_I^2 + r_{II}^2)^{\frac{1}{2}} - r_I\right]/b$

Replace θ_1 by $\theta_{1obs} + r_I/b$

Replace θ_{1m} by $(r_I^2 + r_{II}^2)^{\frac{1}{2}}/b$

The tape length required is given by the equation

$$s = r_I^2 - r_{2m}^2/2b$$

The graphical results given in Figs. 3, 4, and 5 may similarly be used if the axes are redefined in the following manner

Replace $\theta_2 b/r_{o2}$ by $\left[b\theta_{2obs} - r_{II}\right]/(r_I^2 + r_{II}^2)^{\frac{1}{2}} + 1$

Replace $a_2 r_{o2}/bw_1^2$ by $a_2 (r_I^2 + r_{II}^2)^{\frac{1}{2}}/bw_1^2$

Replace θ_1/θ_{1m} by $\left[b\theta_{1obs} + r_I\right]/(r_I^2 + r_{II}^2)^{\frac{1}{2}}$

All other axes remain the same.

Constant Acceleration Device

For the general case of having a unit starting with the initial conditions that

$$\theta_1 = \theta_2 = 0$$
$$r_1 = r_I > 0$$
$$r_2 = r_{II} > 0$$

and ending with the condition

$$r_2 = r_{2m} > 0$$

the general kinematic equations are found to be

$$a_2 = w_1^2 b_o / r_{o2}$$
$$w_2 = b_o w_1 \theta_{1obs} / r_{o2} + w_1 r_I / r_{II}$$
$$\theta_{2obs} = \tfrac{1}{2}(b_o/r_{o2})\, \theta_{1obs}^2 + \theta_{1obs} r_I / r_{II} + \tfrac{1}{2}\, r_{o2} b_o (r_I / r_{II})^2$$

The graphical results given in Figs. 3, 4, 5, and 7 may be used if the axes are redefined in the following manner.

Replace $\theta_2 b_0/r_{02}$ by $\theta_{1obs} a_2/w_1^2 + \frac{1}{2}(r_I/r_{II})^2$

Replace θ_1/θ_{1m} by $\theta_{1obs} a_2/w_1^2 + r_I/r_{II}$

Replace $b_0 s/r_{02}^2$ by $b_0 s/(r_I^2 + r_{II}^2) + \left[(r_{II}^2 + r_I^2)^{\frac{1}{2}} - r_{II}\right]/r_{II}$

All other axes remain the same.

From Eqs. (25) and (33) comes the result

$$b_0 s/r_{02}^2 = r_{02}/r_2 - 1$$

which when evaluated at the start and end of operation gives the required actual tape length

$$\frac{b_0 s}{r_I^2 + r_{II}^2} = \frac{(r_I^2 + r_{II}^2)^{\frac{1}{2}}}{r_{2m}} - \frac{(r_I^2 + r_{II}^2)^{\frac{1}{2}}}{r_I} \tag{71}$$

A check of b/b_0 vs. $b_0 s/(r_I^2 + r_{II}^2)$ shows that in practice $b_0 s/(r_I^2 + r_{II}^2) \leq 2$, for at the value 2, b/b_0 has decreased by a factor of approximately 30. If this is arbitrarily set as the maximum thickness change allowable, and r is approximately zero, Eq. (71) gives

$$2 = r_{II}/r_{2m} - 1$$

Therefore

$$r_{II}/3 \leq r_{2m} < r_{II}$$

as the other limit of r_{2m} is obviously r_{II}.

If $r_{1m} = r_{II}/2$, b/b_0 is approximately 10.

Log Function Device

To keep the over-all tape thickness change to a minimum, the tape thickness must start and end with the same value. The requirements needed to be fulfilled to achieve this condition may be found by the use of Eq. (60). For b at s = 0 to equal b at s = s_m, Eq. (60) shows that

$$\theta_{10} B = \left[\frac{\sinh\left[x + \sinh^{-1}\theta_{01}B\right]}{\sinh^2\left[x + \sinh^{-1}\theta_{10}B\right] + 1}\right]^{1/2} \qquad (72)$$

where

$$x \equiv \frac{B\ s_m}{(r_{01}^2 + r_{02}^2)^{1/2}}$$

Equation (72) is satisfied at various values of $\theta_{01}B$ as shown in Table 4.

Table 4

$\theta_{01}B$	x
.1	1.69
.01	2.99
.001	4.1

Substituting the definition of x into Eq. (58) shows that

$$x = \sinh^{-1}\theta_{1m}B - \sinh^{-1}\theta_{10}B \qquad (73)$$

But for small values of $\theta_{2o}B$,

$$\sinh^{-1}\theta_{2o}B \approx \theta_{2o}B$$

Thus Eq. (73) reduces to

$$\theta_{2m}B \approx \sinh\,(x + \theta_{2o}B) \tag{74}$$

Therefore if this relation is fulfilled, the tape produced will be of approximate equal thickness on both ends.

III Data

Run of Constant Tape Thickness

4-14-60

r = .16 in.
r = 2.95 in.

θ_{1obs}	θ_{2obs}
0.0 (rev.)	0.0 (rev.)
100.0	8.8
200.0	24.6
300.0	47.4
400.0	77.5
500.0	115.5
600.0	161.8
700.0	217.5
800.0	283.9
900.0	362.9
1000.0	457.4
1050.0	511.5
1100.0	571.5
1150.0	638.4
1200.0	713.9
1250.0	800.1
1300.0	902.5
1350.0	1030.8
1400.0	1215.8
1429.7	1436.7

Run of Constant Acceleration Tape

4-30-60

s = 14.9 ft.
r = .16 in.
r = .75 in.

$\theta_{1\,obs}$	$\theta_{2\,obs}$
0 (rev.)	0 (rev.)
5	1.7
10	4.0
15	7.0
20	10.8
25	15.2
30	20.4
35	26.1
40	32.6
45	39.7
50	47.5
55	56.0

Run of Log Tape

5-1-60

r = .1 in.
r = .99 in.
s = 14.1 ft.

θ_1	θ_2
0 (rev.)	1.6 (rev.)
5	2.2
10	3.1
15	4.3
20	6.0
25	8.3
30	11.5
35	15.7
40	21.6
45	29.6
50	40.4
52.1	46.1