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Summary

Let $\mathcal{L}$ be a central simple Lie algebra over a field k of characteristic zero. Let $\mathfrak{J}$ be a maximal split toral subalgebra of $\mathcal{L}$. Let $\overline{\overline{\Sigma}}$ be the set of non-zero restricted roots with respect to $\mathfrak{J}$. Let

$$\mathcal{L} = \mathcal{L}_o \oplus \sum_{\gamma \in \overline{\overline{\Sigma}}} \mathcal{L}_\gamma$$

be the restricted root decomposition of $\mathcal{L}$ with respect to $\mathfrak{J}$, where $\mathcal{L}_o$ is the centralizer of $\mathfrak{J}$ in $\mathcal{L}$. Suppose $[\mathcal{L}_o, \mathcal{L}_o] \neq (0)$. Let $\overline{\overline{\Pi}}$ be a fundamental system for $\overline{\overline{\Sigma}}$. In Chapter 4, we prove some results about the structure of the anisotropic kernel $[\mathcal{L}_o, \mathcal{L}_o]$ of $(\mathcal{L}, \mathfrak{J})$ for particular restricted root diagrams $\overline{\overline{\Pi}}$. These results may be obtained as consequences of the classification of admissible indices given in Tits [9]. The proofs given here however are independent of this classification. For $\gamma \in \overline{\overline{\Sigma}}$, we regard $\mathcal{L}_\gamma$ as an $\mathcal{L}_o$ module with respect to the adjoint action. In Chapter 5, we prove that with one specified exception $(\mathcal{L}, \mathfrak{J})$ is determined up to isomorphism by the diagram $\overline{\overline{\Pi}}$, the algebra $\mathcal{L}_o$, and the action of $\mathcal{L}_o$ on $\mathcal{L}_{\gamma_{\overline{\overline{\Pi}}}}$, where $\gamma_{\overline{\overline{\Pi}}}$ is a distinguished element of $\overline{\overline{\Pi}}$ depending only on the diagram $\overline{\overline{\Pi}}$.

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Table of Contents

Chapter 1

Introduction

In this chapter, we introduce some notation and recall some results about finite dimensional semi-simple Lie algebras over a field of characteristic zero. For the most part, proofs will be omitted. All of the results of this chapter are stated and proved in Seligman [7]. There are two main theorems in this chapter. Theorem 1, on the conjugacy of maximal split toral subalgebras, was presented in a seminar at Yale by T. Tamagawa and J. Humphreys and was attributed to G.D. Mostow. Theorem 2, the Witt type isomorphism theorem, is due independently to Tits [9] and Satake [5].

Let $\mathcal{L}$ be a semi-simple Lie algebra over a field k of characteristic zero. A <u>toral subalgebra</u> of $\mathcal{L}$ is a commutative subalgebra $\mathcal{K}$ of $\mathcal{L}$ such that $\mathrm{ad}_{\mathcal{L}}(L)$ is semi-simple for all $L \in \mathcal{K}$. If $\mathfrak{h}$ is a subalgebra of $\mathcal{L}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathcal{L}$ if and only if $\mathfrak{h}$ is a maximal toral subalgebra of $\mathcal{L}$ (see Prop.1 of Seligman [7] or Chapter VI, §4, Thm. 3 and Prop. 18 of Chevalley [2]). A toral subalgebra $\mathfrak{J}$ of $\mathcal{L}$ is said to be <u>split</u> if $\mathrm{ad}_{\mathcal{L}}(L)$ has all its characteristic roots in k for all $L \in \mathfrak{J}$.

Let $\mathfrak{J}$ be a maximal split toral subalgebra of $\mathcal{L}$. Then, $\mathcal{L} = \oplus\sum_{\gamma} \mathcal{L}_{\gamma}$, where γ runs over all linear functions γ on $\mathfrak{J}$ and $\mathcal{L}_{\gamma} = \{L \in \mathcal{L} : [L,T] = \gamma(T)L\}$. The linear functions γ such that $\mathcal{L}_{\gamma} \neq (0)$ are called <u>restricted roots relative to $\mathfrak{J}$</u> (or roots of $(\mathcal{L},\mathfrak{J})$). Let Σ be the set of non-zero restricted roots relative to $\mathfrak{J}$. Then,

$$\mathcal{L} = \mathcal{L}_0 \oplus (\sum_{\gamma\in\overline{\overline{\Sigma}}} \mathcal{L}_\gamma)$$

and $\mathfrak{J} \subseteq \mathcal{L}_0$ (since $\mathfrak{J}$ is commutative). Since $\mathrm{ad}_{\mathcal{L}}(T)$ is semi-simple for all $T \in \mathfrak{J}$, $\mathrm{ad}_{\mathcal{L}}(\mathfrak{J})$ is completely reducible and hence $\mathrm{ad}_{\mathcal{L}}(\mathcal{L}_0)$, the centralizer of $\mathrm{ad}_{\mathcal{L}}(\mathfrak{J})$ in $\mathrm{ad}_{\mathcal{L}}(\mathcal{L})$, is completely reducible (see Thm. 3.10 and Thm. 3.18 of Jacobson [3]). Thus, $\mathcal{L}_0 = \mathrm{center}(\mathcal{L}_0) \oplus [\mathcal{L}_0, \mathcal{L}_0]$, where $[\mathcal{L}_0, \mathcal{L}_0]$ is semi-simple and $\mathrm{ad}_{\mathcal{L}}(L)$ is semi-simple for all $L \in \mathrm{center}(\mathcal{L}_0)$. $[\mathcal{L}_0, \mathcal{L}_0]$ is called the (semi-simple) <u>anisotropic kernel</u> of $(\mathcal{L}, \mathfrak{J})$. If $\mathfrak{J} = (0)$ (and hence $\mathcal{L} = [\mathcal{L}_0, \mathcal{L}_0]$), we say $\mathcal{L}$ is <u>anisotropic</u>.

The Killing form of $\mathcal{L}$ will be denoted by $(\ ,\ )$. Then, for $\gamma \in \overline{\overline{\Sigma}}$, $\mathcal{L}_\gamma$ and $\mathcal{L}_{-\gamma}$ are totally isotropic and dual to each other. The restriction of $(\ ,\ )$ to $\mathcal{L}_0$ and to $\mathfrak{J}$ is non-degenerate. Thus, we obtain a corresponding form on the dual space $\mathfrak{J}^*$ in the usual way. Now, $\overline{\overline{\Sigma}}$ generates $\mathfrak{J}^*$ as a vector space over k and we denote by $\mathfrak{X}_{\overline{\overline{\Sigma}}}$ the $\mathbb{Q}$-space generated by $\overline{\overline{\Sigma}}$ in $\mathfrak{J}^*$. Then, $(\ ,\ )$ induces a positive definite symmetric form on the $\mathbb{Q}$-space $\mathfrak{X}_{\overline{\overline{\Sigma}}}$. Moreover, we have that

(I) $\quad 2\dfrac{(\gamma,\delta)}{(\delta,\delta)}$ is an integer for $\gamma, \delta \in \overline{\overline{\Sigma}}$

and

(II) $\quad \gamma - 2\dfrac{(\gamma,\delta)}{(\delta,\delta)}\delta \in \overline{\overline{\Sigma}}$ for $\gamma, \delta \in \overline{\overline{\Sigma}}$.

i.e. $\overline{\overline{\Sigma}}$ is a <u>system of roots</u>. If $\gamma \in \overline{\overline{\Sigma}}$, $\lambda \in \mathbb{Q}$, and $\lambda\gamma \in \overline{\overline{\Sigma}}$, then $\lambda \in \{-2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2\}$. A fundamental system for $\overline{\overline{\Sigma}}$ is a subset $\overline{\overline{\Pi}}$ of $\overline{\overline{\Sigma}}$ such that the elements of $\overline{\overline{\Pi}}$ are linearly independent over $\mathbb{Q}$ and, for $\gamma \in \overline{\overline{\Sigma}}$, we may write either γ or $-\gamma$ in the form $\sum_{\delta\in\overline{\overline{\Pi}}} m_\delta \delta$, where the m_γ are non-negative integers. The number of elements r in such a fundamental system $\overline{\Pi}$ is called the <u>rank</u> of $\mathfrak{L}$ (or the rank of $\overline{\Pi}$) and we have $r = \dim_k \mathfrak{h}$. We say $\overline{\Sigma}$ is <u>reduced</u> (or $\overline{\Pi}$ is reduced) if $2\gamma \notin \overline{\Sigma}$ for $\gamma \in \overline{\Sigma}$. If $\overline{\Sigma}$ is connected and not reduced and $\overline{\Pi}$ is a fundamental system for $\overline{\Sigma}$, then the Dynkin diagram for $\overline{\Pi}$ is $\underset{\gamma_1}{\bullet}\!-\!\!-\!\underset{\gamma_2}{\bullet}\!-\!\!-\!\bullet \cdots \bullet\!-\!\!-\!\underset{\gamma_{r-1}}{\bullet}\!\Rightarrow\!\underset{\gamma_r}{\boxed{\bullet}}$ if $r \geq 2$ or $\underset{\gamma_1}{\boxed{\bullet}}$ if $r = 1$, where the square around the vertex γ_r indicates that $2\gamma_r \in \overline{\Sigma}$.

For $\gamma \in \overline{\Sigma}$, define $\mathcal{H}_{\overline{\Sigma}} \xrightarrow{\overline{w}_\gamma} \mathcal{H}_{\overline{\Sigma}}$ by $\varepsilon^{\overline{w}_\gamma} = \varepsilon - 2\frac{(\varepsilon, \gamma)}{(\gamma, \gamma)}\gamma$, $\varepsilon \in \mathcal{H}_{\overline{\Sigma}}$. Then, for $\gamma \in \overline{\Sigma}$, $\overline{w}_\gamma$ is in the orthogonal group of our form and $\overline{\Sigma}^{\overline{w}_\gamma} = \overline{\Sigma}$. The group $\overline{W}$ generated by $\{\overline{w}_\gamma\}_{\gamma \in \overline{\Sigma}}$ is called the <u>restricted Weyl group</u> of $(\mathfrak{L}, \mathfrak{h})$ and we have that $\overline{W}$ is generated by $\{\overline{w}_\gamma\}_{\gamma \in \overline{\Pi}}$ for any fundamental system $\overline{\Pi}$.

Now, for $\gamma, \delta \in \overline{\Sigma} \cup \{0\}$, we have $[\mathfrak{L}_\gamma, \mathfrak{L}_\delta] \subseteq \mathfrak{L}_{\gamma+\delta}$. In particular, $[\mathfrak{L}_\gamma, \mathfrak{L}_0] \subseteq \mathfrak{L}_\gamma$ for $\gamma \in \overline{\Sigma}$. Thus, under the adjoint action, $\mathfrak{L}_\gamma$ is an $\mathfrak{L}_0$ module for $\gamma \in \overline{\Sigma}$. But for $\gamma \in \overline{\Sigma}$ and $X_\gamma \in \mathfrak{L}_\gamma - \{0\}$ we have $[X_\gamma, \mathfrak{L}_0] = \mathfrak{L}_\gamma$ (see lemma 7 of Seligman [7]). Thus, under the adjoint action, $\mathfrak{L}_\gamma$ is an irreducible $\mathfrak{L}_0$ module for $\gamma \in \overline{\Sigma}$.

Suppose $\gamma \in \overline{\Sigma}$. Then, there exists a unique element T_γ of $\mathfrak{h}$ such that $T_\gamma \in [\mathfrak{L}_\gamma, \mathfrak{L}_{-\gamma}]$ and $\gamma(T_\gamma) = 2$. Moreover, for $X_\gamma \in \mathfrak{L}_\gamma$, there exists an element $X_{-\gamma}$ of $\mathfrak{L}_{-\gamma}$ such that $[X_\gamma, X_{-\gamma}] = T_\gamma$. For these choices, the Lie algebra with k-basis $\{T_\gamma, X_\gamma, X_{-\gamma}\}$ is the three dimensional split simple Lie algebra with multiplication table $[X_\gamma, X_{-\gamma}] = T_\gamma$, $[X_\gamma, T_\gamma] = 2X_\gamma$, and $[X_{-\gamma}, T_\gamma] = -2X_{-\gamma}$. (See

§1.1 in Seligman [7] for this information.)

The origin of the following theorem was discussed at the beginning of this chapter. A proof is found in §1.3 of Seligman [7].

Theorem 1: Let $\mathfrak{T}'$ be a second maximal split toral subalgebra of $\mathfrak{L}$. Then, there exists $\varphi \in \mathrm{Aut}(\mathfrak{L})$ such that φ is a product of elements of $\{\exp(\mathrm{ad}_{\mathfrak{L}}(N)): N \in \mathfrak{L}, \mathrm{ad}_{\mathfrak{L}}(N)$ nilpotent$\}$ and $\mathfrak{T}' = \mathfrak{T}^{\varphi}$.

Theorem 1 tells us, among other things, that in order to classify semi-simple algebras $\mathfrak{L}$ up to isomorphism it suffices to classify pairs $(\mathfrak{L}, \mathfrak{T})$ up to isomorphism. A second consequence of Theorem 1 is the known result that any two splitting Cartan subalgebras of a semi-simple Lie algebra are conjugate by an automorphism of the above form (see Steinberg [8]).

Let G be the group of all automorphisms φ of $\mathfrak{L}$ such that φ is a product of elements of $\{\exp(\mathrm{ad}_{\mathfrak{L}}(N)): N \in \mathfrak{L}, \mathrm{ad}_{\mathfrak{L}}(N)$ nilpotent$\}$ and $\mathfrak{T}^{\varphi} = \mathfrak{T}$. For $\varphi \in G$, we define $\varphi^* \in \mathfrak{X}_{\overline{\Sigma}}$ by $\varphi^* = ((\varphi | \mathfrak{T})^t)^{-1}$ (i.e. $\mathcal{E}^{\varphi^*}(T^{\varphi}) = \mathcal{E}(T)$ for $\mathcal{E} \in \mathfrak{X}_{\overline{\Sigma}}$ and $T \in \mathfrak{T}$). Then, for $\varphi \in G$, $\overline{\Sigma}^{\varphi^*} = \overline{\Sigma}$ and $\mathfrak{L}_{\gamma}^{\varphi} = \mathfrak{L}_{\gamma\varphi^*}$ for $\gamma \in \overline{\Sigma}$. The map $\varphi \longrightarrow \varphi^*$ is a group homomorphism of G onto $\overline{\mathfrak{W}}$ with kernel $\{\varphi \in G: T^{\varphi} = T$ for $T \in \mathfrak{T}\}$.

It follows from Theorem 1 that there exists a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{L}$ if and only if $\mathfrak{T}$ is a splitting Cartan subalgebra for $\mathfrak{L}$. The last of these statements is equivalent to $\mathfrak{T} = \mathfrak{L}_0$, and so we say $\mathfrak{L}$ is <u>split</u> if $\mathfrak{T} = \mathfrak{L}_0$. If $[\mathfrak{L}_0, \mathfrak{L}_0] = (0)$, we say $\mathfrak{L}$ is <u>quasi-split</u>. By the above remark and Theorem 1, it is clear that these two definitions are independent of our choice of $\mathfrak{T}$.

Now if $\mathfrak{L}$ is split, the classical theory tells us that $(\mathfrak{L}, \mathfrak{T})$

is determined up to isomorphism by the fundamental system $\overline{\Pi}$. The main theorem of this work is an attempt to generalize this result to our more general situation. It is easy to see that $\overline{\Pi}$ does not in general determine $(\mathcal{L}, \mathfrak{J})$ up to isomorphism (see §1.2 of Seligman [7]). The question which then arises is whether or not $\overline{\Pi}$ and $\mathcal{L}_0$ determine $(\mathcal{L}, \mathfrak{J})$ up to isomorphism. I.e., suppose $\mathcal{L}'$ is another semi-simple algebra over k with maximal split toral subalgebra $\mathfrak{J}'$, and $\mathcal{L}_0'$ and $\overline{\Pi}'$ are defined as above. Suppose that $\mathcal{L}_0$ and $\mathcal{L}_0'$ are isomorphic and $\overline{\Pi}$ and $\overline{\Pi}'$ are isomorphic. Then, is it necessarily the case that $(\mathcal{L}, \mathfrak{J})$ and $(\mathcal{L}', \mathfrak{J}')$ are isomorphic? We now give an example which answers this question in the negative.

Let D be a finite dimensional central division algebra over k of degree $d > 1$. Suppose J is an involution in D of first kind and suppose the anti-symmetric elements of D with respect to J have dimension $\frac{1}{2}d(d-1)$ over k. Let V be a 2r dimensional left vector space over D. We assume some D-basis for V is fixed. Let h be the non-degenerate hermitian form on V with matrix

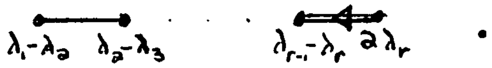
$$\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} .$$

Let $\mathcal{L}$ be the set of D-linear transformations of V which are skew with respect to the form h. Then, $\mathcal{L}$ is a central simple Lie algebra over k. The elements of $\mathcal{L}$ are matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^{Jt} \end{pmatrix} ,$$

where A_{11}, A_{12}, and A_{21} are $r \times r$ matrices with coefficients in D,

$A_{12}^{Jt} = -A_{12}$, and $A_{21}^{Jt} = -A_{21}$. Let $\mathcal{T}$ be the subalgebra of $\mathcal{L}$ consisting of elements of $\mathcal{L}$ which are diagonal with entries in k. Then, $\mathcal{T}$ is a maximal split toral subalgebra of $\mathcal{L}$. Let e_{ij}, $1 \leq i, j \leq 2r$, be the matrix units. Define $\lambda_i : \mathcal{T} \longrightarrow k$ by $\lambda_i(\sum_{j=1}^{r} a_j(e_{jj} - e_{r+j,r+j})) = a_i$, $i=1,\ldots,r$ (where the a_j are in k). Then, the restricted roots of $\mathcal{T}$ are $\pm(\lambda_i \pm \lambda_j)$, $1 \leq i < j \leq r$ and $\pm 2\lambda_i$, $1 \leq i \leq r$. A fundamental system $\overline{\overline{\Pi}}$ for these roots is $\{\lambda_1 - \lambda_2, \ldots, \lambda_{r-1} - \lambda_r, 2\lambda_r\}$ and has diagram



The centralizer $\mathcal{L}_0$ of $\mathcal{T}$ in $\mathcal{L}$ consists of the elements of $\mathcal{L}$ which are diagonal with entries in D. Hence, $\mathcal{L}_0$ as a Lie algebra is isomorphic to r copies of D. Now, if y is a non-zero anti-symmetric element of D with respect to J, then the map $x \xrightarrow{K} y^{-1}x^{J}y$ is an involution of the first kind such that the anti-symmetric elements of D with respect to K have dimension $\frac{1}{2}d(d+1)$ over k. Moreover, the above discussion holds word for word with J replaced by K. Thus, we obtain two different algebras (one using J and the other using K) with the same $\overline{\overline{\Pi}}$ and $\mathcal{L}_0$. However, it is well known (see for example §10.6 of Jacobson [3]) that the two algebras are not isomorphic, since the anti-symmetric elements with respect to J and K do not have the same dimension over k. We note before leaving this example that, whether we are dealing with J or K, the restricted root space $\mathcal{L}_{2\lambda_r}$ corresponding to the long root in the diagram $\overline{\overline{\Pi}}$ is $D_{-}e_{r,r+1}$, where D_{-} is the space of anti-symmetric elements with respect to the involution. Hence, the $\mathcal{L}_0$ module is different in the two cases i.e. the two $\mathcal{L}_0$ modules in question are not isomorphic (indeed they have different dimensions over k).

The above example motivates the isomorphism theorem of Chapter 5 which states (roughly) that if $\mathcal{L}$ is central simple, then $(\mathcal{L}, \mathfrak{J})$ is determined up to isomorphism by $\mathcal{L}_0$, $\overline{\overline{\Pi}}$, and the action of $\mathcal{L}_0$ on $\mathcal{L}_{\gamma_{\overline{\overline{\Pi}}}}$, where $\gamma_{\overline{\overline{\Pi}}}$ is a distinguished element of $\overline{\overline{\Pi}}$ depending only on the root system $\overline{\overline{\Sigma}}$. This isomorphism theorem is rational in the sense that it can be stated without reference to a splitting extension for $\mathcal{L}$. Its proof is based upon applications of Theorem 2 of this chapter. Theorem 2 is also an isomorphism theorem but it is non-rational (in the above sense). In order to state Theorem 2, we must introduce some "splitting data" for $\mathcal{L}$, and in particular, we must introduce the index of $\mathcal{L}$.

We may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathcal{L}$ such that $\mathfrak{J} \subseteq \mathfrak{h}$. We may also choose a finite dimensional Galois extension K/k such that K/k splits the adjoint action of $\mathfrak{h}$ on $\mathcal{L}$. Let $\mathcal{G} = \mathrm{Gal}(K/k)$.

Let Σ be the set of non-zero roots of $(\mathcal{L}_K, \mathfrak{h}_K)$. Now,

$$(\mathcal{L}_\gamma)_K = \sum_{\alpha \in \Sigma, \alpha|\mathfrak{J}_K = \gamma} (\mathcal{L}_K)_\alpha \quad \text{for} \quad \gamma \in \overline{\overline{\Sigma}}, \text{ and } (\mathcal{L}_0)_K = \mathfrak{h}_K + \sum_{\alpha \in \Sigma, \alpha|\mathfrak{J}_K = 0} (\mathcal{L}_K)_\alpha .$$

Thus, $\overline{\overline{\Sigma}}$ is the set of non-zero restrictions of elements of Σ to $\mathfrak{J}_K$.

We may regard (,) as a form on $\mathcal{L}_K$. (,) is non-degenerate when restricted to $\mathfrak{h}_K$. As usual, we may transfer (,) to a form (,) on the dual space $\mathfrak{h}_K^*$ of $\mathfrak{h}_K$. We regard $\mathfrak{J}_K^*$ as a K-subspace of $\mathfrak{h}_K^*$ by identifying $\varepsilon \in \mathfrak{J}_K^*$ with the element of $\mathfrak{h}_K^*$ which is equal to ε on $\mathfrak{J}_K$ and zero on the orthogonal complement of $\mathfrak{J}_K$ in $\mathfrak{h}_K$. The restriction of (,) to $\mathfrak{J}_K^*$ is exactly the form

$(\ ,\ )$ defined on $\mathfrak{T}^*$ previously (and extended to $\mathfrak{T}_K^*$).

Now, Σ generates $\mathfrak{h}_K^*$ as a vector space over K. Let $\mathfrak{X}_\Sigma$ be the $\mathbb{Q}$-vector space generated by Σ in $\mathfrak{h}_K^*$. Then, $(\ ,\ )$ induces a positive definite symmetric form on the $\mathbb{Q}$-space $\mathfrak{X}_\Sigma$. For $\alpha \in \Sigma$, we will use the notation $\hat{\alpha} = 2\frac{\alpha}{(\alpha,\alpha)}$.

Now, $\mathfrak{G}$ acts on $\mathfrak{L}_K$ fixing the elements of $\mathfrak{L}$. This action fixes the elements of $\mathfrak{h}$ and hence stabilizes $\mathfrak{h}_K$. Thus, we obtain an action of $\mathfrak{G}$ on $\mathfrak{h}_K^*$ as follows: $\mathcal{E}^\sigma(H^\sigma) = \mathcal{E}(H)^\sigma$, $\mathcal{E} \in \mathfrak{h}_K^*$, $H \in \mathfrak{h}_K$, $\sigma \in \mathfrak{G}$. This action stabilizes Σ and we have $(\mathfrak{L}_{K\alpha})^\sigma = (\mathfrak{L}_K)_{\alpha^\sigma}$ for $\alpha \in \Sigma$, $\sigma \in \mathfrak{G}$. Thus, $\mathfrak{X}_\Sigma^\sigma = \mathfrak{X}_\Sigma$ for $\sigma \in \mathfrak{G}$, and we have an action of $\mathfrak{G}$ on $\mathfrak{X}_\Sigma$.

Write $\mathfrak{h}_K = \mathfrak{T}_K \oplus \mathfrak{a}_K$, where $\mathfrak{a}$ is the orthogonal complement of $\mathfrak{T}$ in $\mathfrak{h}$. We have $\mathfrak{X}_\Sigma = \mathfrak{X}_\mathfrak{G} \oplus \mathfrak{X}_0$, where

$$\mathfrak{X}_\mathfrak{G} = \{\mathcal{E} \in \mathfrak{X}_\Sigma : \ \mathcal{E}^\sigma = \mathcal{E} \text{ for all } \sigma \in \mathfrak{G}\} \quad \text{and}$$

$$\mathfrak{X}_0 = \{\mathcal{E} \in \mathfrak{X}_\Sigma : \ \sum_{\sigma \in \mathfrak{G}} \mathcal{E}^\sigma = 0\}.$$

We have $(\mathcal{E}_1, \mathcal{E}_2)^\sigma = (\mathcal{E}_1^\sigma, \mathcal{E}_2^\sigma)$ for $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{h}_K^*$ and $\sigma \in \mathfrak{G}$. Therefore, $(\mathcal{E}_1^\sigma, \mathcal{E}_2^\sigma) = (\mathcal{E}_1, \mathcal{E}_2)$ for $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{X}_\Sigma$ and $\sigma \in \mathfrak{G}$. Thus, $\mathfrak{X}_\mathfrak{G}$ and $\mathfrak{X}_0$ are orthogonal. Indeed using the fact that $\mathfrak{T}$ is a maximal split toral subalgebra of $\mathfrak{L}$, we have that

$$\mathfrak{X}_\mathfrak{G} = \{\mathcal{E} \in \mathfrak{X}_\Sigma : \ \mathcal{E}(H) = 0 \text{ for all } H \in \mathfrak{a}_K\} \text{ and}$$

$$\mathfrak{X}_0 = \{\mathcal{E} \in \mathfrak{X}_\Sigma : \ \mathcal{E}(H) = 0 \text{ for all } H \in \mathfrak{T}_K\}.$$

For $\mathcal{E} \in \mathfrak{X}_\Sigma$, define $\bar{\mathcal{E}} = \frac{1}{|\mathfrak{G}|}\sum_{\sigma \in \mathfrak{G}} \mathcal{E}^\sigma$. Then, $\mathcal{E} \longrightarrow \bar{\mathcal{E}}$ is the projection of $\mathfrak{X}_\Sigma$ onto the first factor in $\mathfrak{X}_\mathfrak{G} \oplus \mathfrak{X}_0$. By the above remarks it follows that for $\mathcal{E} \in \mathfrak{X}_\Sigma$,

$$\bar{\varepsilon}(H) = \begin{cases} \varepsilon(H) & \text{if } H \in \mathfrak{I}_K \\ 0 & \text{if } H \in \mathfrak{a}_K \end{cases}$$

i.e. $\bar{\varepsilon}$ is the restriction of ε to $\mathfrak{I}_K$ (regarded as an element of $\mathfrak{h}_K^*$ by our previous identification). Thus, for $\varepsilon \in \mathcal{X}_\Sigma$, the restriction of ε to $\mathfrak{I}_K$ is an element of $\mathcal{X}_\Sigma$. But $\bar{\Sigma}$ is the set of non-zero restrictions of elements of Σ to $\mathfrak{I}_K$. ($\bar{\Sigma}$ here is the set of restricted roots and not the image of Σ under the bar mapping.) Thus, $\bar{\Sigma} \subseteq \mathcal{X}_\Sigma$. Indeed, $\bar{\Sigma} \subseteq \mathcal{X}_{\mathfrak{b}}$ and, from the dimensions of $\mathcal{X}_{\bar{\Sigma}}$ and $\mathcal{X}_{\mathfrak{b}}$, it follows that $\mathcal{X}_{\mathfrak{b}} = \mathcal{X}_{\bar{\Sigma}}$.

Let $\Sigma_0 = \Sigma \cap \mathcal{X}_0$ i.e. Σ_0 is the set of non-zero roots of $(\mathcal{L}_K, \mathfrak{h}_K)$ which are zero on $\mathfrak{I}_K$. Thus, Σ_0 is the set of non-zero roots of $(\mathcal{L}_K, \mathfrak{h}_K)$ whose root vectors are elements of $(\mathcal{L}_0)_K$. But $\mathcal{L}_0 = \text{center}(\mathcal{L}_0) \oplus [\mathcal{L}_0, \mathcal{L}_0]$ and $\text{center}(\mathcal{L}_0) \subseteq \mathfrak{h}$. Thus, Σ_0 can be identified with the set of non-zero roots of $([\mathcal{L}_0, \mathcal{L}_0]_K, (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K)$.

We may regard $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K^*$ as a subspace of $\mathfrak{h}_K^*$ by identifying $\varepsilon \in (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K^*$ with the element of $\mathfrak{h}_K^*$ which is equal to ε on $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K$ and is zero on the orthogonal complement of $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K$ in $\mathfrak{h}_K$. Let $\mathcal{X}_{\Sigma_0}$ be the $\mathbb{Q}$-space generated by Σ_0 in $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K^*$ (or equivalently in $\mathfrak{h}_K^*$). Then, $\mathcal{X}_{\Sigma_0} \subseteq \mathcal{X}_0$.

A fundamental system Π for Σ is said to be <u>admissible</u> if the following condition holds:

$$\alpha \in \Sigma - \Sigma_0, \ \alpha \underset{\Pi}{>} 0 \implies \alpha^\sigma \underset{\Pi}{>} 0 \text{ for all } \sigma \in \mathfrak{G},$$

where $\alpha \underset{\Pi}{>} 0$ indicates that α is a positive root with respect to the fundamental system Π. If Π is an admissible fundamental system, then $\Pi_0 = \Pi \cap \mathfrak{X}_0$ is a fundamental system for Σ_0 and the set $\bar{\Pi}$ of non-zero restrictions of Π to $\mathfrak{J}_K$ is a fundamental system for $\bar{\Sigma}$. Conversely, suppose Π_0 is a fundamental system for Σ_0 and let $\bar{\Pi}$ be a fundamental system for $\bar{\Sigma}$. Then,

$$P = \{\alpha \in \Sigma - \Sigma_0 : \bar{\alpha} \underset{\bar{\Pi}}{\geq} 0\} \cup \{\alpha \in \Sigma_0 : \alpha \underset{\Pi_0}{\geq} 0\}$$

is a positive system for Σ i.e. $\Sigma = P \cup (-P)$ and $P \cap (-P) = \phi$. The fundamental system Π for Σ corresponding to P is admissible. The two processes described here are inverse to one another and we have a 1-1 correspondence between admissible fundamental systems Π for Σ and pairs $(\bar{\Pi}, \Pi_0)$, where $\bar{\Pi}$ is a fundamental system for $\bar{\Sigma}$ and Π_0 is a fundamental system for Σ_0.

Let $\mathfrak{W}$ be the Weyl group of $(\mathfrak{L}_K, \mathfrak{h}_K)$ acting in $\mathfrak{X}_\Sigma$. Let $\mathfrak{W}_1$ be the subgroup of $\mathfrak{W}$ stabilizing $\mathfrak{X}_0$. Let $\mathfrak{W}_0$ be the subgroup of $\mathfrak{W}$ generated by the reflections $\{w_\beta\}_{\beta \in \Sigma_0}$. Then, $\mathfrak{W}_1$ stabilizes $\Sigma_0 = \Sigma \cap \mathfrak{X}_0$ and hence $\mathfrak{W}_0$ is a normal subgroup of $\mathfrak{W}_1$. Now, $\mathfrak{W}_1$ stabilizes $\mathfrak{X}_0$ and hence it stabilizes the orthogonal complement $\mathfrak{X}_{\bar{\Sigma}}$ of $\mathfrak{X}_0$ in $\mathfrak{X}_\Sigma$. For $w_1 \in \mathfrak{W}_1$, let $\bar{w}_1$ be the restriction of w_1 to $\mathfrak{X}_{\bar{\Sigma}}$. Since $\mathfrak{W}_1$ stabilizes Σ, $\bar{\Sigma}^{w_1} = \bar{\Sigma}$ for $w_1 \in \mathfrak{W}_1$. Thus, we have a homomorphism $w_1 \longrightarrow \bar{w}_1$ of $\mathfrak{W}_1$ into the automorphisms of $\mathfrak{X}_{\bar{\Sigma}}$ which stabilize $\bar{\Sigma}$. The kernel of this map is $\mathfrak{W}_0$ and the image is the Weyl group $\bar{\mathfrak{W}}$ of $\bar{\Sigma}$. (See lemma 22 and Prop. 6 of Seligman [7].) Thus, the restriction map $w_1 \longrightarrow \bar{w}_1$ induces an isomorphism $\mathfrak{W}_1/\mathfrak{W}_0 \longrightarrow \bar{\mathfrak{W}}$.

We now choose an admissible fundamental system Π for Σ. Let $\Pi_0 = \Pi \cap \mathfrak{X}_0$ and let $\overline{\Pi}$ be the set of non-zero restrictions of Π to $\mathfrak{J}$. Π, Π_0, and $\overline{\Pi}$ will remain fixed throughout the remainder of this chapter.

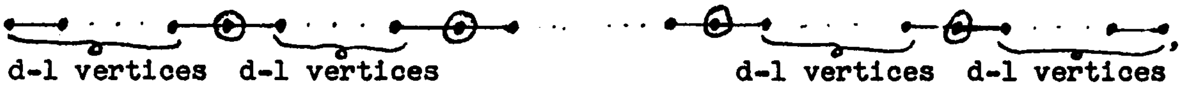
Now, for $\sigma \in \mathcal{G}$, Π^{σ} is a fundamental system for Σ. Thus, for $\sigma \in \mathcal{G}$, there exists a unique $w_\sigma \in W$ such that $\Pi^{\sigma w_\sigma} = \Pi$. We have, in fact, that $w_\sigma \in W_0$ for $\sigma \in \mathcal{G}$. Thus, for $\sigma \in \mathcal{G}$, we may define σ^* in the orthogonal group of the form (,) on $\mathfrak{X}_\Sigma$ by $\xi^{\sigma^*} = \xi^{\sigma w_\sigma}$, $\xi \in \mathfrak{X}_\Sigma$. We have then that $\Pi^{\sigma^*} = \Pi$ and $\Pi_0^{\sigma^*} = \Pi_0$ for $\sigma \in \mathcal{G}$. We have also that $(\sigma \tau)^* = \sigma^* \tau^*$ for $\sigma, \tau \in \mathcal{G}$. The homomorphism $\sigma \xrightarrow{*} \sigma^*$ is called the <u>*-action of $\mathcal{G}$</u>. The triple $(\Pi, \Pi_0, *)$ is called the <u>index of $\mathcal{L}$</u>. For simplicity, we usually refer to the pair (Π, Π_0) as the index of $\mathcal{L}$ (omitting reference to the *-action). If $\mathcal{L}$ is anisotropic, we have $\Pi = \Pi_0$ and in this case we simply refer to Π as the index of $\mathcal{L}$. An orbit in Π of the *-action of $\mathcal{G}$ is called a <u>*-orbit</u>.

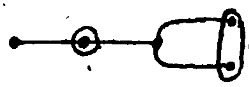
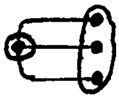
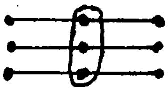
The index of $\mathcal{L}$ is independent of our choice of $\mathfrak{J}$, $\mathfrak{g}$, and Π. Indeed, if $\mathfrak{J}'$, $\mathfrak{g}'$, and Π' are another choice and Π_0' is defined as above, then there exists an isomorphism $(\Pi, \Pi_0) \longrightarrow (\Pi', \Pi_0')$ of diagrams which preserves the *-action.

Let $\gamma \in \overline{\Pi}$. Put $\mho_\gamma = \{\alpha \in \Pi : \bar{\alpha} = \gamma\}$. Then, $\mho_\gamma^{\sigma^*} = \mho_\gamma$ for $\sigma \in \mathcal{G}$. Suppose $\alpha \in \Sigma$ such that $\bar{\alpha} = \gamma$. Then, we may write $\alpha = \alpha_1 + \sum_{\beta \in \Pi_0} m_\beta \beta$ for some $\alpha_1 \in \mho_\gamma$ and non-negative integers m_β. Moreover, if $\sigma \in \mathcal{G}$, then $\overline{\alpha^\sigma} = \gamma$ and the element of $\mho_\gamma$ appearing in a similar sum for α^σ is $\alpha_1^{\sigma^*}$.

Let $\gamma \in \overline{\Pi}$. We show that $\mathcal{O}_\gamma$ is a *-orbit of Π. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two disjoint non-empty *-orbits of Π contained in $\mathcal{O}_\gamma$. Let $V_i = \sum(\mathcal{L}_K)_\alpha$, where the sum runs over all $\alpha \in \Sigma$ such that $\bar{\alpha} = \gamma$ and the element of $\mathcal{O}_\gamma$ appearing in the above sum for α lies in $\mathcal{O}_i$, i=1,2. Then, V_i is a non-zero $(\mathcal{L}_0)_K$ submodule of $(\mathcal{L}_\gamma)_K$ such that $V_i^\sigma = V_i$ for $\sigma \in \mathcal{G}$, i=1,2. Moreover, $V_1 \cap V_2 = (0)$. This contradicts the irreducibility of $\mathcal{L}_\gamma$ as an $\mathcal{L}_0$ module. Thus, $\mathcal{O}_\gamma$ is a *-orbit of $\overline{\Pi}$.

Following Tits [9], we may introduce the following diagrammatic representation of the index of $\mathcal{L}$: The Dynkin diagram of Π is drawn in such a way that elements in the same *-orbit are close together. The *-orbits $\mathcal{O}_\gamma$, $\gamma \in \overline{\Pi}$, are circled and are referred to as <u>distinguished orbits</u>. For example, we have the following index in which the *-action is trivial and there are r distinguished orbits:



where $d \geq 1$. This is in fact the index of the derived algebra of the algebra of $(r+1)\times(r+1)$ matrices with coefficients in a central division algebra of degree d over k. Now, in general, the diagrammatic representation described above indicates only the orbits of the *-action and not the *-action itself. For example, consider the diagram: . In order to determine the *-action from that diagram, we must know which elements of $\mathcal{G}$ act non-trivially on Π in the *-action. In other cases, even this additional information is not enough to determine the *-action. (Consider for example the diagram  or the non-connected diagram .) Thus, when we represent an index by a diagram $\mathcal{D}$ as above, we say only that the index is of the form $\mathcal{D}$.

Suppose for the remainder of this chapter, that $\mathcal{L}'$ is another semi-simple Lie algebra over $\mathfrak{k}$. Suppose $\mathfrak{J}'$ is a maximal split toral subalgebra for $\mathcal{L}'$ and suppose $\mathfrak{h}'$ is a Cartan subalgebra for such that $\mathfrak{J}' \subseteq \mathfrak{h}'$. We assume that the extension $K/\mathfrak{k}$ is also a splitting extension for the adjoint action of $\mathfrak{h}'$ on $\mathcal{L}'$. Define $\bar{\Sigma}'$, $\mathcal{X}'_{\bar{\Sigma}'}$, Σ', $\mathcal{X}'_{\Sigma'}$, $\mathcal{X}'_{\mathcal{G}}$, $\mathcal{X}'_{o}$, Σ'_{o}, and $\mathcal{X}'_{\Sigma'_o}$ as above. Let Π' be an admissible fundamental system for Σ'. Define $\bar{\Pi}'$ and Π'_o as above. Define the *-action of $\mathcal{G}$ on (Π', Π'_o) as above.

A *-isomorphism $(\Pi, \Pi_o) \xrightarrow{f} (\Pi', \Pi'_o)$ is an isomorphism $\Pi \xrightarrow{f} \Pi'$ of Dynkin diagrams such that $\Pi_o^f = \Pi'_o$ and f preserves the *-action of $\mathcal{G}$.

Suppose that we have an isomorphism $(\mathcal{L}, \mathfrak{h}) \xrightarrow{\varphi} (\mathcal{L}', \mathfrak{h}')$. Then, $\mathfrak{J}^{\varphi} = \mathfrak{J}'$. Define $(\mathfrak{h}_K)^* \xrightarrow{\varphi^*} (\mathfrak{h}'_K)^*$ by $\mathcal{E}^{\varphi^*}(H^{\varphi}) = \mathcal{E}(H)$ for $\mathcal{E} \in (\mathfrak{h}_K)^*$ and $H \in \mathfrak{h}_K$ i.e. $\varphi^* = ((\varphi | \mathfrak{h}_K)^t)^{-1}$. Then, $\Sigma^{\varphi^*} = \Sigma'$ and $\Sigma \xrightarrow{\varphi^*} \Sigma'$ is called the map $\Sigma \longrightarrow \Sigma'$ induced by φ. Π^{φ^*} is a fundamental system for Σ' and hence there exists $w' \in W'$ (the Weyl group of Σ') such that $\Pi^{\varphi^* w'} = \Pi'$. Put $f = (\varphi^* w') | \Pi$. Then, $\Pi^f = \Pi'$, $\Pi_o^f = \Pi'_o$, and f preserves the *-action of $\mathcal{G}$. f is called the *-isomorphism $(\Pi, \Pi_o) \longrightarrow (\Pi', \Pi'_o)$ induced by φ. We note that if $\Pi^{\varphi^*} = \Pi'$, then the *-isomorphism $(\Pi, \Pi_o) \longrightarrow (\Pi', \Pi'_o)$ induced by φ is the map $\Sigma \longrightarrow \Sigma'$ induced by φ restricted to Π.

If we have an isomorphism

$$([\mathcal{L}_0, \mathcal{L}_0], \mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0]) \xrightarrow{\varphi_0} ([\mathcal{L}_0', \mathcal{L}_0'], \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']),$$

we may apply the above considerations to obtain a map $\Sigma_0 \xrightarrow{\varphi_0^*} \Sigma_0'$ induced by φ_0 and a $*$-isomorphism $\Pi_0 \xrightarrow{f_0} \Pi_0'$ induced by φ_0.

We are now ready to state the Witt type isomorphism theorem. It is due independently to Tits [9] and Satake [5]. A proof is found in §2.2 of Seligman [7].

<u>Theorem 2</u>: Suppose $\mathcal{L}$, $\mathfrak{I}$, $\mathfrak{h}$, Π, $\mathcal{L}'$, $\mathfrak{I}'$, $\mathfrak{h}'$, and Π' are as above and all other notation is as above. Suppose we have an isomorphism

$$([\mathcal{L}_0, \mathcal{L}_0], \mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0]) \xrightarrow{\varphi_0} ([\mathcal{L}_0', \mathcal{L}_0'], \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']),$$

Let $\Pi_0 \xrightarrow{f_0} \Pi_0'$ be the $*$-isomorphism induced by φ_0. Suppose we have a $*$-isomorphism $(\Pi, \Pi_0) \xrightarrow{f} (\Pi', \Pi_0')$. Suppose there exists $w_1 \in W_1$ such that $\Pi_0^{w_1} = \Pi_0$ and $f \mid \Pi_0 = (w_1 \mid \Pi_0) \cdot f_0$. Then, there exists an isomorphism $(\mathcal{L}, \mathfrak{h}, \mathfrak{I}) \longrightarrow (\mathcal{L}', \mathfrak{h}', \mathfrak{I}')$ such that $\varphi \mid [\mathcal{L}_0, \mathcal{L}_0] = \varphi_0$ and the $*$-isomorphism $(\Pi, \Pi_0) \longrightarrow (\Pi', \Pi_0')$ induced by φ is f.

Chapter 2

Some Generalities About Indices

In this chapter, we introduce some general notions which will enable us to study the relationship between the restricted diagram and the index of an algebra. We will assume throughout the chapter that $\mathcal{L}$ is a semi-simple algebra over k and that $\mathfrak{I}$, $\mathfrak{h}$, K, $\mathfrak{k}$, Π, Π_0, etc. are as in Chapter 1.

We are interested in the way in which the index (Π, Π_0) of $\mathcal{L}$ depends on the restricted diagram $\overline{\overline{\Pi}}$. Borel and Tits [1, Thm. 6.13] have dealt with the converse question and have shown that the diagram $\overline{\Pi}$ is uniquely determined by the index (Π, Π_0). Their method of verifying this fact suggests the importance of certain subalgebras of $\mathcal{L}$ of (restricted) rank 1 (together with their indices). Satake's discussion (see Chapter II, §3.1 of Satake [6]) of the classification of admissible diagrams (i.e. diagrams which arise as indices of semi-simple algebras) also suggests the consideration of these same rank 1 subalgebras.

We are led then to define subalgebras $\mathfrak{g}_\gamma$ of $\mathcal{L}$ for $\gamma \in \overline{\overline{\Sigma}}$ as follows: Suppose $\gamma \in \overline{\Sigma}$. Define

$$\mathfrak{g}_\gamma = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \oplus \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \mathcal{L}_{m\gamma}.$$

Now, if $2\gamma \in \overline{\overline{\Sigma}}$, it is easily seen (see lemma 7 of Seligman [7]) that $\mathcal{L}_{2\gamma} = [\mathcal{L}_\gamma, \mathcal{L}_\gamma]$ and hence $[\mathcal{L}_{2\gamma}, \mathcal{L}_{-2\gamma}] \subseteq [\mathcal{L}_\gamma, \mathcal{L}_\gamma]$. Thus, $\mathfrak{g}_\gamma$ is a subalgebra of $\mathcal{L}$. Now, $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ is an ideal of $\mathcal{L}_0$ and hence, since $\mathcal{L}_0$ is reductive, we have

$$[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] = \text{center}([\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]) \oplus [[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]],$$

$$\text{center}([\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]) = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \cap \text{center}(\mathcal{L}_0),$$

and

$$[[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]] = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \cap [\mathcal{L}_0, \mathcal{L}_0].$$

We define

$$\mathcal{L}_{0,\gamma} = [[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]] = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \cap [\mathcal{L}_0, \mathcal{L}_0] .$$

We then have:

Prop. 1: Let $\gamma \in \Sigma$. Then, $\mathfrak{g}_\gamma$ is a simple subalgebra of $\mathcal{L}$ with maximal split toral subalgebra $k\,T_\gamma$, where $T_\gamma \in \mathfrak{J}$ is defined as in Chapter 1. The centralizer of kT_γ in $\mathfrak{g}_\gamma$ is $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ and the anisotropic kernel of $\mathfrak{g}_\gamma$ (with respect to kT_γ) is $\mathcal{L}_{0,\gamma}$. $\mathfrak{h} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ is a Cartan subalgebra of $\mathfrak{g}_\gamma$ containing kT_γ.

Proof: We show that $\mathfrak{g}_\gamma$ is simple. The rest of the proposition is clear. Let $\mathfrak{b}$ be a proper ideal of $\mathfrak{g}_\gamma$. Since T_γ normalizes $\mathfrak{b}$, we have $\mathfrak{b} = \mathfrak{b} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \oplus \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \mathfrak{b} \cap \mathcal{L}_{m\gamma}$. Suppose for contradiction that $\mathfrak{b} \cap \mathcal{L}_\gamma \neq (0)$. Choose $X_\gamma \in \mathfrak{b} \cap \mathcal{L}_\gamma - \{0\}$. Choose $X_{-\gamma} \in \mathcal{L}_{-\gamma}$ so that $\mathfrak{s}_\gamma = kT_\gamma + kX_\gamma + kX_{-\gamma}$ is the three dimensional split simple algebra as in Chapter 1. Regarding $\mathfrak{g}_\gamma$ as an $\mathfrak{s}_\gamma$ module, it follows easily from the representation theory for $\mathfrak{s}_\gamma$ that $[[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], X_\gamma] = \mathcal{L}_\gamma$. Therefore, $\mathcal{L}_\gamma \subseteq \mathfrak{b}$. But $\mathcal{L}_{2\gamma} = [\mathcal{L}_\gamma, \mathcal{L}_\gamma]$ and hence $\mathcal{L}_{2\gamma} \subseteq \mathfrak{b}$. Also, $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \subseteq \mathfrak{b}$. Therefore, $T_\gamma \in \mathfrak{b}$. Thus, $\mathcal{L}_{-\gamma} \cup \mathcal{L}_{-2\gamma} \subseteq \mathfrak{b}$. Therefore, $\mathfrak{b} = \mathfrak{g}_\gamma$ and we have a contradiction. Thus, $\mathfrak{b} \cap \mathcal{L}_\gamma = (0)$. But if $X_\gamma \in \mathcal{L}_\gamma - \{0\}$ and we define $\mathfrak{s}_\gamma$ as above, it follows from the representation theory for $\mathcal{S}_\gamma$ applied to the $\mathcal{S}_\gamma$ module $\mathcal{B}$ that $[\mathcal{B}\cap\mathcal{L}_\gamma, X_\gamma] = \mathcal{B}\cap\mathcal{L}_{2\gamma}$. Therefore, $\mathcal{B}\cap\mathcal{L}_{2\gamma} = (0)$. Similarly, $\mathcal{B}\cap\mathcal{L}_{-\gamma} = (0)$ and $\mathcal{B}\cap\mathcal{L}_{-2\gamma} = (0)$. Thus, $\mathcal{B} \subseteq [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$. But then $[\mathcal{L}_\gamma, \mathcal{B}] \subseteq \mathcal{B}\cap\mathcal{L}_\gamma = (0)$ and hence $[\mathcal{L}_\gamma, \mathcal{B}] = (0)$. Similarly, $[\mathcal{L}_{-\gamma}, \mathcal{B}] = (0)$. Thus, $[[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], \mathcal{B}] = (0)$ and hence $\mathcal{B} \subseteq \text{center}([\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]) \subseteq \text{center}(\mathcal{L}_0) \subseteq \mathfrak{h}$. Now, if $H \in \mathcal{B}$, we have $\alpha(H) = 0$ for all $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$ (since $[\mathcal{L}_\gamma, \mathcal{B}] = (0)$). Thus, to complete the proof that $\mathcal{B}$ is zero, it suffices to show that the form (,) restricted to the K-space generated by $\{\alpha \in \Sigma : \overline{\alpha} = \gamma\}$ is non-degenerate. This follows immediately from the fact that the form (,) restricted to the $\mathbb{Q}$-space generated by $\{\alpha \in \Sigma : \overline{\alpha} = \gamma\}$ is non-degenerate. q.e.d.

For $\gamma \in \overline{\Sigma}$, define

$$\mathfrak{a}_\gamma = \{X_0 \in \mathcal{L}_0 : [\mathcal{L}_\gamma, X_0] = (0)\}.$$

I.e. $\mathfrak{a}_\gamma$ is the annihilator in $\mathcal{L}_0$ of the $\mathcal{L}_0$ module $\mathcal{L}_\gamma$ (under the adjoint action).

<u>Prop. 2</u>: Let $\gamma \in \overline{\Sigma}$. Then, $\mathfrak{a}_\gamma$ is orthogonal to $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ with respect to the Killing form,

$$\mathcal{L}_0 = \mathfrak{a}_\gamma \oplus [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}],$$

$\mathfrak{a}_{-\gamma} = \mathfrak{a}_\gamma$, and $\mathfrak{a}_\gamma$ centralizes $\mathfrak{g}_\gamma$.

<u>Proof</u>: We first show that $\mathfrak{a}_\gamma = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]^\perp$, where $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]^\perp$ denotes the k-vector space of elements of $\mathcal{L}_0$ orthogonal to $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$. Let $X_0 \in \mathfrak{a}_\gamma$. Then, for $X_\gamma \in \mathcal{L}_\gamma$ and $X_{-\gamma} \in \mathcal{L}_{-\gamma}$, $(X_0, [X_\gamma, X_{-\gamma}]) = ([X_0, X_\gamma], X_{-\gamma}) = (0, X_{-\gamma}) = 0$. Therefore,

$X_0 \in [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]^\perp$. Conversely, suppose $X_0 \in [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]^\perp$. Let $X_\gamma \in \mathcal{L}_\gamma$. Then, for $X_{-\gamma} \in \mathcal{L}_{-\gamma}$, $([X_0, X_\gamma], X_{-\gamma}) = (X_0, [X_\gamma, X_{-\gamma}]) = 0$. Therefore, $[X_0, X_\gamma]$ is orthogonal to $\mathcal{L}_{-\gamma}$. But $[X_0, X_\gamma] \in \mathcal{L}_\gamma$ and hence $[X_0, X_\gamma]$ is orthogonal to $\mathcal{L}$. Therefore, $[X_0, X_\gamma] = 0$. Thus, $X_0 \in \mathfrak{a}_\gamma$.

For the proof of (1), it remains to show that (,) restricted to $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ is non-degenerate. But (,) restricted to $\mathfrak{g}_\gamma$ is the trace form of the representation $\mathfrak{g}_\gamma \xrightarrow{\rho = \mathrm{ad}_{\mathcal{L}}} \mathrm{End}_{\mathbb{R}}(\mathcal{L})$. But $\mathfrak{g}_\gamma$ is simple and ρ is not zero (since $\mathcal{L}$ is semi-simple). Thus, by Cartan's criterion for semi-simplicity, (,) restricted to $\mathfrak{g}_\gamma$ is non-degenerate.

Since $\mathfrak{a}_\gamma = [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]^\perp$, $\mathfrak{a}_\gamma = \mathfrak{a}_{-\gamma}$.

It remains to show that $\mathfrak{a}_\gamma$ centralizes $\mathfrak{g}_\gamma$. By definition, $\mathfrak{a}_\gamma$ centralizes $\mathcal{L}_\gamma$. By (1), $\mathfrak{a}_\gamma$ centralizes $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$. Since $\mathfrak{a}_\gamma = \mathfrak{a}_{-\gamma}$, $\mathfrak{a}_\gamma$ centralizes $\mathcal{L}_{-\gamma}$. But as we have remarked previously, $\mathcal{L}_{2\gamma} = [\mathcal{L}_\gamma, \mathcal{L}_\gamma]$ and hence $\mathfrak{a}_\gamma$ centralizes $\mathcal{L}_{2\gamma}$. Similarly, $\mathfrak{a}_\gamma$ centralizes $\mathcal{L}_{-2\gamma}$. Therefore, $\mathfrak{a}_\gamma$ centralizes $\mathfrak{g}_\gamma$. q.e.d.

<u>Corollary</u>: Let $\gamma \in \overline{\overline{\Sigma}}$. Let $\mathcal{L}_{0,1}$ be a simple summand of $[\mathcal{L}_0, \mathcal{L}_0]$. Then, $\mathcal{L}_{0,1}$ acts non-trivially on $\mathcal{L}_\gamma$ if and only if $\mathcal{L}_{0,1} \subseteq \mathcal{L}_{0,\gamma}$.

We will be interested in certain subsets of Π associated with the algebras $\mathfrak{g}_\gamma$ and $\mathcal{L}_{0,\gamma}$ but first we require some definitions.

Suppose P is a subset of Π. We say P is <u>*-connected</u> if for each $\alpha_1, \alpha_2 \in$ P there exists $\beta_1, \ldots, \beta_m \in$ P such that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_m$, and for i=1,...,m-1 either $(\beta_i, \beta_{i+1}) \neq 0$ or β_{i+1} is an image under the *-action of $\mathcal{B}_i$. Maximal *-connected subsets of P are called <u>*-components</u> of P and it is clear that P is the disjoint union of its *-components.

Now, since $\sigma^* = \sigma w_\sigma$ for $\sigma \in \mathcal{B}$, where $w_\sigma \in W_0$, it follows that the actions $\alpha \longrightarrow \alpha^\sigma$ and $\alpha \longrightarrow \alpha^{\sigma^*}$ of $\mathcal{B}$ permute the components of Σ in exactly the same way. It is clear then that Π is *-connected if and only if $\mathcal{L}$ is simple.

Applying this last paragraph to the algebra $[\mathcal{L}_0, \mathcal{L}_0]$ with index Π_0, it follows that there exists a 1-1 correspondence between simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ and non-empty *-components of Π_0. Indeed, if $\mathcal{L}_{0,1}$ is a simple summand of $[\mathcal{L}_0, \mathcal{L}_0]$ and $\Pi_{0,1}$ is a *-component of Π_0, then $\mathcal{L}_{0,1}$ and $\Pi_{0,1}$ correspond if and only if $(\mathcal{L}_{0,1})_K$ is the subalgebra generated by $\bigcup_{\alpha \in \Pi_{0,1}} ((\mathcal{L}_K)_\alpha \cup (\mathcal{L}_K)_{-\alpha})$. We say that <u>$\mathcal{L}_{0,1}$ corresponds to $\Pi_{0,1}$</u> or <u>$\Pi_{0,1}$ corresponds to $\mathcal{L}_{0,1}$</u>.

Suppose next that P is a non-empty *-connected subset of Π. Let P_1 be a component of P. It is clear that P is the disjoint union of the images of P_1 under the *-action. If the connected fundamental system P_1 is of type X, we say P is of <u>*-type X</u>. Let $\mathcal{B}_1 = \{\sigma \in \mathcal{B} : P^{\sigma^*} = P\}$ and $\mathcal{B}_{0,1} = \{\sigma \in \mathcal{B} : \alpha^{\sigma^*} = \alpha$ for all $\alpha \in P_1\}$. Let $\mathcal{R}$ be the Roman numeral representing $[\mathcal{B}_1 : \mathcal{B}_{0,1}]$. Then, if P_1 is of type X, we say <u>P is of *-type $X_{\mathcal{R}}$</u>. (e.g. P is of *-type D_{4III}.) Both definitions are clearly independent of our choice of P_1.

Suppose once again that $\gamma \in \bar{\bar{\Sigma}}$. We define

$$\Sigma_{o,\gamma} = \{ \beta \in \Sigma_o : (\mathcal{L}_K)_\beta \subseteq (\mathcal{L}_{o,\gamma})_K \}.$$

Equivalently, we may put

$$\Sigma_{o,\gamma} = \{ \beta \in \Sigma_o : \beta = \alpha_1 - \alpha_2 \text{ for some } \alpha_1, \alpha_2 \in \Sigma \ni \bar{\alpha}_1 = \bar{\alpha}_2 = \gamma \}.$$

Then, $\Sigma_{o,\gamma}$ can be identified with the roots of $(\mathcal{L}_{o,\gamma})_K$ with respect to $(\mathfrak{h} \cap \mathcal{L}_{o,\gamma})_K$. We put

$$\Pi_{o,\gamma} = \Sigma_{o,\gamma} \cap \Pi_o.$$

$\Pi_{o,\gamma}$ can be identified with the index of the anisotropic Lie algebra $\mathcal{L}_{o,\gamma}$. Since $\mathcal{L}_{o,\gamma}$ is an ideal of $[\mathcal{L}_o, \mathcal{L}_o]$, it is clear that $\Pi_{o,\gamma}$ is the union of *-components of Π_o. We note that in view of the Corollary to Prop. 2, if $\Pi_{o,1}$ is a *-component of Π_o with corresponding simple summand $\mathcal{L}_{o,1}$ of $[\mathcal{L}_o, \mathcal{L}_o]$, then $\Pi_{o,1} \subseteq \Pi_{o,\gamma}$ if and only if $\mathcal{L}_{o,1}$ acts non-trivially on $\mathcal{L}_\gamma$ in the adjoint action. We note also that $\mathfrak{g}_\gamma = \mathfrak{g}_{-\gamma}$ and $\mathcal{L}_{o,\gamma} = \mathcal{L}_{o,-\gamma}$, and hence $\Sigma_{o,\gamma} = \Sigma_{o,-\gamma}$ and $\Pi_{o,\gamma} = \Pi_{o,-\gamma}$.

Suppose now that $\gamma \in \bar{\Pi}$. We recall that

$$\sigma_\gamma = \{ \alpha \in \Pi : \bar{\alpha} = \gamma \}$$

is a *-orbit of Π. We put

$$\Pi_\gamma = \sigma_\gamma \cup \Pi_{o,\gamma}.$$

Then, we can identify $(\Pi_\gamma, \Pi_{o,\gamma})$ with the index of $\mathfrak{g}_\gamma$ (with respect to the subalgebras $\mathfrak{h}\, T_\gamma$ and $\mathfrak{h} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$).

We wish next to give an interpretation of Π_γ and $\Pi_{o,\gamma}$ for $\gamma \in \bar{\Pi}$ in terms only of the index (Π, Π_o). But first we require the following two lemmas:

Lemma 1: Let $P = \{\alpha_1, \beta_1, \ldots, \beta_n\}$ be a fundamental system of roots and suppose $\alpha = \alpha_1 + \sum_{i=1}^{n} m_i \beta_i$ is a root where the m_i are non-negative integers not all zero. Then, $\alpha - \beta_i$ is a root for some i.

Proof: We may assume $n > 1$ and all the m_i are positive. Suppose for contradiction that $\alpha - \beta_j$ is not a root for all j. Then, $\alpha - \alpha_1 = \sum_{i=1}^{n} m_i \beta_i$ is a root and hence $\{\beta_1, \ldots, \beta_n\}$ is connected. Thus, there exists a unique element of $\{\beta_1, \ldots, \beta_n\}$, say β_1, which is connected to α_1. Now, if $\alpha - \alpha_1 - \beta_j$ is a root for some $j > 1$, we have $(\alpha - \alpha_1 - \beta_j, \alpha_1) = (\alpha - \alpha_1, \alpha_1) = (\sum_{i=1}^{n} m_i \beta_i, \alpha_1) < 0$ and hence $\alpha - \beta_j$ is a root. This gives a contradiction and so we may assume that $\alpha - \alpha_1 - \beta_j$ is not a root for $j > 1$. Thus, $\alpha - \alpha_1 - \beta_1$ is a root. Then, $(\alpha - \alpha_1 - \beta_1, \alpha_1) = (\sum_{i=2}^{n} m_i \beta_i + (m_1 - 1)\beta_1, \alpha_1) = (m_1 - 1)(\beta_1, \alpha_1)$. But $\alpha - \beta_1$ is not a root and hence $(\alpha - \alpha_1 - \beta_1, \alpha_1) \geqslant 0$. Therefore, since $(\beta_1, \alpha_1) < 0$, we have $m_1 = 1$. But $\alpha - \alpha_1 = \beta_1 + \sum_{i=2}^{n} m_i \beta_i$ and hence by induction on the number of elements in the fundamental system, $\alpha - \alpha_1 - \beta_j$ is a root for some $j > 1$. We have a contradiction. q.e.d.

Lemma 2: Let $\gamma \in \overline{\Pi}$ and $\alpha \in \Sigma$. Then, $\bar{\alpha} = \gamma$ if and only if $\alpha = \alpha_1 + \sum_{\beta \in \Pi_{0,\gamma}} m_\beta \beta$ for some $\alpha_1 \in \mho_\gamma$ and some non-negative integers m_β.

Proof: Suppose $\bar{\alpha} = \gamma$. Then, $\alpha = \alpha_1 + \sum_{\beta \in \Pi_0} m_\beta \beta$ for some $\alpha_1 \in \mho_\gamma$ and non-negative integers m_β. By lemma 1, we may successively subtract elements of Π_0 from this sum, each time producing an element of Σ, until we reach α_1. Thus, any $\beta \in \Pi_0$ for which $m_\beta > 0$ is an element of $\Pi_{0,\gamma}$. Therefore, $\alpha = \alpha_1 + \sum_{\beta \in \Pi_{0,\gamma}} m_\beta \beta$.

The converse is clear. q.e.d.

Corollary: Let $\gamma \in \overline{\overline{\Pi}}$. Then, γ is in the Q-space generated by Π_γ.

Proof: Choose $\alpha \in \mho_\gamma$. Then, $\gamma = \frac{1}{|\mathcal{B}|}\sum_{\sigma \in \mathcal{B}} \alpha^\sigma$. But $\overline{\alpha^\sigma} = \gamma$ for all $\sigma \in \mathcal{B}$ and the result follows from lemma 2. q.e.d.

We can now prove:

Prop. 3: Let $\gamma \in \overline{\overline{\Pi}}$ and $\beta_1 \in \Pi_0$. Then, $\beta_1 \in \Pi_{0,\gamma}$ if and only if there exists $\alpha_1 \in \mho_\gamma$ and a subset P_0 of Π_0 such that $\beta_1 \in P_0$ and $\{\alpha_1\} \cup P_0$ is connected.

Proof: Suppose $\beta_1 \in \Pi_{0,\gamma}$. Then, there exists $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$ and $\alpha - \beta_1 \in \Sigma$. Write $\alpha = \alpha_1 + \sum_{\beta \in \Pi_{0,\gamma}} m_\beta \beta$ as in lemma 2. Then, $\{\alpha_1\} \cup \{\beta \in \Pi_{0,\gamma} : m_\beta > 0\}$ is connected and $\beta_1 \in \{\beta \in \Pi_{0,\gamma} : m_\beta > 0\}$.

For the converse, put $\alpha = \alpha_1 + \sum_{\beta \in P_0} \beta$ and apply lemma 2. q.e.d.

Corollary: Let $\gamma \in \overline{\overline{\Pi}}$. Then, Π_γ is the *-component of $\Pi_0 \cup \mho_\gamma$ containing $\mho_\gamma$.

We now present two lemmas which concern themselves with the relationship between the restricted root system and the index. The first of these lemmas is Prop. 6.15 of Borel and Tits [1]. However, the proof is easy and so we present it.

Lemma 3: Let γ and δ be distinct elements of $\overline{\overline{\Pi}}$ and suppose $\alpha_1 \in \mho_\gamma$. Then, $(\gamma, \delta) < 0$ if and only if there exists $\alpha_2 \in \mho_\delta$ and a subset P_0 of Π_0 such that $\{\alpha_1\} \cup P_0 \cup \{\alpha_2\}$ is connected.

Proof: Suppose $(\gamma, \delta) < 0$. Then, $\gamma + \delta \in \overline{\Sigma}$. Thus, there exists $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma + \delta$. Then, $\alpha = \alpha_3 + \alpha_4 + \sum_{\beta \in \Pi_0} m_\beta \beta$ for some $\alpha_3 \in \mho_\gamma$, $\alpha_4 \in \mho_\delta$, and non-negative integers m_β. Therefore,

$\{\alpha_3, \alpha_4\} \cup \{\beta \in \Pi_0 : m_\beta > 0\}$ is connected. Choose $\sigma \in \mathcal{L}$ such that $\alpha_1 = \alpha_3^{\sigma^*}$. Then, $\{\alpha_1, \alpha_4^{\sigma^*}\} \cup \{\beta \in \Pi_0 : m_\beta > 0\}^{\sigma^*}$ is connected.

Conversely, suppose $\alpha_2 \in \mathcal{O}_\delta$ and $\{\alpha_1\} \cup P_0 \cup \{\alpha_2\}$ is connected. Then, $\alpha = \alpha_1 + \alpha_2 + \sum_{\beta \in P_0} \beta \in \Sigma$. Thus, $\bar{\alpha} = \gamma_1 + \gamma_2 \in \bar{\Sigma}$. q.e.d.

If P is a connected fundamental system of roots we denote by μ_P the dominant root for P i.e. the uniquely determined root with the property that $\mu_P + \alpha$ is not a root for all $\alpha \in P$.

Lemma 4: Let T be a non-empty connected subset of $\bar{\bar{\Pi}}$. Put $\Pi_T = \bigcup_{\gamma \in T} \Pi_\gamma$. Let P be a connected component of Π_T. Then, $\overline{\mu_P} = \mu_T$.

Proof: It clearly suffices to show that there exists $\alpha_0 \in \Sigma$ such that $\bar{\alpha}_0 = \mu_T$ and α_0 is a positive integral sum of elements in P. Now, there exists $\alpha_1 \in \Sigma$ such that $\bar{\alpha}_1 = \mu_T$. We may write $\alpha_1 = \sum_{\alpha \in \Pi} m_\alpha \alpha$, with non-negative integers m_α. Then, $Q = \{\alpha \in \Pi : m_\alpha > 0\}$ is a connected set such that $Q \subseteq \Pi_T$ and hence Q is contained in some component P_1 of Π_T. But $P = P_1^{\sigma^*}$ for some $\sigma \in \mathcal{L}$. Then, $\alpha_0 = \alpha_1^{\sigma^*}$ is the required element. q.e.d.

Lemma 4 with $T = \{\gamma\}$ a singleton is Thm. 6.13(ii) of Borel and Tits [1].

If $\mathcal{L}$ is simple, the action of $\mathcal{L}_0$ on certain restricted root spaces determines the number of simple summands of $\mathcal{L}_K$. We have:

Prop. 4: Suppose $\mathcal{L}$ is simple and $\gamma \in \overline{\Sigma}$ is of maximum length in $\overline{\Sigma}$. Then, the number of simple summands of $\mathcal{L}_K$ is equal to the number of irreducible summands in the decomposition of $(\mathcal{L}_\gamma)_K$ as an $(\mathcal{L}_0)_K$ module. In particular, $\mathcal{L}$ is central if and only if $\mathcal{L}_\gamma$ is an absolutely irreducible $\mathcal{L}_0$ module.

Proof: By rechoosing $\overline{\Pi}$ if necessary, we may assume that γ is the dominant root for $\overline{\Pi}$. Write $\mathcal{L}_K = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m$, where the $\mathcal{L}_i$ are the simple summands of $\mathcal{L}_K$ and write $(\mathcal{L}_\gamma)_K = V_1 \oplus \cdots \oplus V_n$, where the V_i are irreducible $(\mathcal{L}_0)_K$ modules.

For $i=1,\ldots,n$, let $\mathcal{M}_i$ be the K-subspace of $\mathcal{L}_K$ generated by $V_i \cup \{ X \operatorname{ad}_{\mathcal{L}_K}(X_1)\cdots\operatorname{ad}_{\mathcal{L}_K}(X_t) : X \in V_i,\ X_1,\ldots,X_t \in \bigcup_{\delta\in\overline{\Pi}} (\mathcal{L}_{-\delta})_K \}$. It is easy to see that $\mathcal{M}_i$ is an ideal of $\mathcal{L}_K$ (using the facts that $[V_i,(\mathcal{L}_0)_K] \subseteq V_i$ and $[V_i,(\mathcal{L}_\delta)_K] = (0)$ for $\delta \in \overline{\Pi}$), $i=1,\ldots,n$. But from the definition of $\mathcal{M}_i$, it is immediate that $\mathcal{M}_i \cap (\mathcal{L}_\gamma)_K = V_i$, $i=1,\ldots,n$. Thus, since V_i generates $\mathcal{M}_i$ as an ideal for $i=1,\ldots,n$, it follows that $\mathcal{M}_i \cap \mathcal{M}_j = (0)$ for $1 \leq i,j \leq n$, $i \neq j$. Therefore, $n \leq m$.

Since $\mathfrak{J}$ normalizes $\mathcal{L}_i$, $i=1,\ldots,m$, it follows that $(\mathcal{L}_\gamma)_K = ((\mathcal{L}_\gamma)_K \cap \mathcal{L}_1) \oplus \cdots \oplus ((\mathcal{L}_\gamma)_K \cap \mathcal{L}_m)$. But since $\mathcal{L}$ is simple, the $\mathcal{L}$ module generated by $\mathcal{L}_\gamma$ is $\mathcal{L}$. Thus, the $\mathcal{L}_i$ module generated by $(\mathcal{L}_\gamma)_K \cap \mathcal{L}_i$ is $\mathcal{L}_i$, $i=1,\ldots,m$. Therefore, $(\mathcal{L}_\gamma)_K \cap \mathcal{L}_i$ is a non-zero $(\mathcal{L}_0)_K$ submodule of $(\mathcal{L}_\gamma)_K$. Thus, $m \leq n$.

Now K splits $\mathcal{L}$ and hence $\mathcal{L}$ is central if and only if $m = 1$. But for the same reason, $\mathcal{L}_\gamma$ is an absolutely irreducible $\mathcal{L}_0$ module if and only if $n = 1$. The final statement of the proposition is then immediate. q.e.d.

<u>Corollary</u>: Suppose $\mathcal{L}$ is simple, $\gamma \in \overline{\Pi}$, and $m\gamma$ is of maximal length in $\overline{\Sigma}$ for some positive integer m. Then, the intersection of Π_γ with each component of Π is a non-empty connected set.

<u>Proof</u>: Let P be a connected component of Π and let $P_\gamma = P \cap \Pi_\gamma$. Now, Π is the disjoint union of images under the $*$-action of P. Thus, Π_γ is the disjoint union of the same number of images under the $*$-action of P_γ . Thus, to prove the Corollary, it suffices to show that the number of components of Π_γ is the same as the number of components of Π.

By the proposition, the number of components of Π is equal to the number of irreducible summands of the $(\mathcal{L}_0)_K$ module $(\mathcal{L}_{m\gamma})_K$. But $[\mathcal{L}_{m\gamma}, \mathcal{L}_{-m\gamma}] \subseteq [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ and $\mathcal{L}_0 = \mathfrak{a}_{m\gamma} \oplus [\mathcal{L}_{m\gamma}, \mathcal{L}_{-m\gamma}]$. Thus, the number of irreducible summands of the $(\mathcal{L}_0)_K$ module $(\mathcal{L}_{m\gamma})_K$ is equal to the number of irreducible summands of the $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]_K$ module $(\mathcal{L}_{m\gamma})_K$. Applying the proposition to the algebra $\mathfrak{g}_\gamma$, we have that the number of irreducible summands of the $[\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]_K$ module $(\mathcal{L}_{m\gamma})_K$ is equal to the number of components of Π_γ. q.e.d.

We close this chapter by introducing an action of the restricted Weyl group $\overline{W}$ on Π_0 and investigating some of its properties.

We need:

<u>Lemma 5</u>: Let $\overline{w} \in \overline{W}$. Then, there exists a unique element w of W_1 such that $\Pi_0^{w} = \Pi_0$ and $\overline{\alpha^{w}} = \overline{\alpha}^{\overline{w}}$ for $\alpha \in \Sigma$.

<u>Proof</u>: Since the map $W_1 \longrightarrow \overline{W}$ is onto, there exists $w_1 \in W_1$ such that $\overline{\alpha^{w_1}} = \overline{\alpha}^{\overline{w}}$ for $\alpha \in \Sigma$. Now, Π_0^{w} is a fundamental

system for $\sum_0$. Therefore, there exists $w_0 \in W_0$ such that $\Pi_0^{w_1 w_0} = \Pi_0$. Then, putting $w = w_1 w_0$, we have $w \in W_1$, $\overline{\alpha^w} = \overline{\alpha^{w_1 w_0}} = \overline{\alpha^{w_1}} = \bar{\alpha}^{\bar{w}}$ for $\alpha \in \sum$, and $\Pi_0^w = \Pi_0^{w_1 w_0} = \Pi_0$. Suppose that w_2 is a second element of W_1 satisfying the conclusion of the lemma. Then, for $\alpha \in \sum$, $\overline{\alpha^{w_2^{-1} w}} = \overline{\alpha}^{w_2^{-1} \bar{w}} = \overline{\alpha^{w_2^{-1} w_2}} = \bar{\alpha}$. Therefore, for $\alpha \in \Pi - \Pi_0$, $\alpha^{w_2^{-1} w} \underset{\Pi}{>} 0$. But $\Pi_0^{w_2^{-1} w} = \Pi_0$. Thus, for $\alpha \in \Pi$, $\alpha^{w_2^{-1} w} \underset{\Pi}{>} 0$. Therefore, $w = w_2$. q.e.d.

It is clear that the map $\bar{w} \xrightarrow{j} w$ (with w as in lemma 3) is a group monomorphism of $\overline{W}$ into W_1. We identify the elements of $\overline{W}$ with their images under j. We then have

$$\Pi_0^{\bar{w}} = \Pi_0 \text{ and } \overline{\alpha^{\bar{w}}} = \bar{\alpha}^{\bar{w}} \text{ for } \alpha \in \sum \text{ and } \bar{w} \in \overline{W}.$$

This identification will be used throughout the rest of this work. It clearly depends on our choice of Π_0. However, this dependence will not result in any confusion.

<u>Prop. 5</u>: W_1 is the semi-direct product of $\overline{W}$ and W_0. For $\bar{w} \in \overline{W}$ and $\sigma \in \mathcal{A}$, we have $\sigma^* \bar{w} = \bar{w}\, \sigma^*$ and in particular $\bar{w} \mid \Pi_0$ is a $*$-automorphism of Π_0.

<u>Proof</u>: W_0 is normal in W_1 as we have remarked in Chapter 1. Let $w_1 \in W_1$ and let $\bar{w}$ be the image of w_1 under the map $W_1 \longrightarrow \overline{W}$. Choose $w_0 \in W_0$ such that $\Pi_0^{w_1} = \Pi_0^{w_0}$. Then, $\Pi_0^{w_1 w_0^{-1}} = \Pi_0$ and $\overline{\alpha^{w_1 w_0^{-1}}} = \overline{\alpha^{w_1}} = \bar{\alpha}^{\bar{w}}$ for $\alpha \in \sum$. By lemma 5, $w_1 w_0^{-1} = \bar{w}$. Thus, $w_1 = \bar{w}\, w_0$. Therefore, $W_1 = \overline{W} W_0$. But if $w_0 \in W_0 \cap \overline{W}$, we have $\Pi_0^{w_0} = \Pi_0$ and hence $w_0 = 1$. Therefore, W_1 is the semi-direct product of W_0 and $\overline{W}$.

Let $\overline{w} \in \overline{W}$ and $\sigma \in \mathcal{G}$. Then, $\Pi_0^{\sigma^* \overline{w} \sigma^{*-1}} = \Pi_0$ and $\overline{\alpha^{\sigma^* \overline{w} \sigma^{*-1}}} = \overline{\alpha}^{\overline{w}}$ for $\alpha \in \Sigma$. But $\sigma^* \overline{w} \sigma^{*-1} \in W_1$. Thus, $\sigma^* \overline{w} \sigma^{*-1} = \overline{w}$. Therefore, $\sigma^* \overline{w} = \overline{w} \sigma^*$. q.e.d.

We also have:

<u>Prop. 6</u>: Let $\gamma \in \overline{\Sigma}$ and $\overline{w} \in \overline{W}$. Then, $\Pi_{0,\gamma}^{\overline{w}} = \Pi_{0,\gamma\overline{w}}$. Moreover, if $\gamma \in \overline{\Pi}$ and $\gamma^{\overline{w}} \in \overline{\Pi}$, we have $\mho_\gamma^{\overline{w}} = \mho_{\gamma\overline{w}}$ and $\Pi_\gamma^{\overline{w}} = \Pi_{\gamma\overline{w}}$.

<u>Proof</u>: Put $\Sigma_\delta = \{\alpha \in \Sigma : \overline{\alpha} = \delta\}$ for $\delta \in \overline{\Sigma}$. Now, $\Pi_0^{\overline{w}} = \Pi_0$, $\mho_\gamma = \{\alpha \in \Sigma_\gamma : \alpha - \beta \notin \Sigma$ for $\beta \in \Pi_0\}$ if $\gamma \in \overline{\Pi}$, and $\Pi_{0,\gamma} = \{\beta \in \Pi_0 : \beta = \alpha_1 - \alpha_2$ for some $\alpha_1, \alpha_2 \in \Sigma_\gamma\}$. Thus, it suffices to show that $\Sigma_\gamma^{\overline{w}} = \Sigma_{\gamma\overline{w}}$. Let $\alpha \in \Sigma_\gamma$. Then, $\overline{\alpha^{\overline{w}}} = \overline{\alpha}^{\overline{w}} = \gamma^{\overline{w}}$ and hence $\alpha^{\overline{w}} \in \Sigma_{\gamma\overline{w}}$. Therefore, $\Sigma_\gamma^{\overline{w}} \subseteq \Sigma_{\gamma\overline{w}}$. Applying this to $\gamma^{\overline{w}}$ and $\overline{w}^{-1}$, we obtain $\Sigma_{\gamma\overline{w}}^{\overline{w}^{-1}} \subseteq \Sigma_\gamma$ and hence $\Sigma_{\gamma\overline{w}} \subseteq \Sigma_\gamma^{\overline{w}}$. Thus, $\Sigma_\gamma^{\overline{w}} = \Sigma_{\gamma\overline{w}}$. q.e.d.

By restriction, we have an action of $\overline{W}$ on Π_0 which commutes with the $*$-action. We wish now to explicitly calculate this action for the reflections $\overline{w}_\gamma$, $\gamma \in \overline{\Pi}$. For this calculation, we will need the following two lemmas:

<u>Lemma 6</u>: Let $\gamma \in \overline{\Sigma}$. Then, $\overline{w}_\gamma$ is a product of reflections corresponding to roots in $\{\alpha \in \Sigma : \overline{\alpha} = \gamma\} \cup \Pi_{0,\gamma}$ and $\overline{w}_\gamma$ fixes the elements of $\Pi_0 - \Pi_{0,\gamma}$.

<u>Proof</u>: The roots of $(\mathfrak{h}_\gamma \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}])_K$ in $(\mathfrak{g}_\gamma)_K$ can be identified with $\{\alpha \in \Sigma : \overline{\alpha} \neq 0, \overline{\alpha} \in \mathbb{Z}\gamma\} \cup \Sigma_{0,\gamma}$, and a fundamental system for these roots is contained in $\{\alpha \in \Sigma : \overline{\alpha} = \gamma\} \cup \Pi_{0,\gamma}$. Applying lemma 5 to the algebra $\mathfrak{g}_\gamma$ and the only non-trivial element of its restricted Weyl group, it follows that there exists a product w of

reflections corresponding to roots in $\{\alpha\in\Sigma : \bar{\alpha}=\gamma\}\cup\Pi_{o,\gamma}$ such that $\Pi_{o,\gamma}^{\,w}=\Pi_{o,\gamma}$ and $\{\alpha\in\Sigma : \bar{\alpha}=\gamma\}^{w}=\{\alpha\in\Sigma : \bar{\alpha}=-\gamma\}$. Now, if $\beta\in\Pi_o-\Pi_{o,\gamma}$ and $\alpha\in\Sigma$ such that $\bar{\alpha}=\gamma$, we have $\alpha+\beta\notin\Sigma$ and $\alpha-\beta\notin\Sigma$, and hence $\beta^{w_\alpha}=\beta$. But the elements of $\Pi_o-\Pi_{o,\gamma}$ are fixed by reflections corresponding to elements of $\Pi_{o,\gamma}$. Therefore, the elements of $\Pi_o-\Pi_{o,\gamma}$ are fixed by w.

It remains to show that $w=\bar{w}_\gamma$. Since $\Pi_{o,\gamma}^{\,w}=\Pi_{o,\gamma}$, we have $\Pi_o^{\,w}=\Pi_o$. Since w is a product of elements of $\{\alpha\in\Sigma : \bar{\alpha}=\gamma\}\cup\Pi_{o,\gamma}$, it follows that $\varepsilon^{w}=\varepsilon$ for all $\varepsilon\in\mathfrak{X}_{\bar{\Sigma}}$ such that $(\varepsilon,\gamma)=0$, and $\gamma^{w}\in\mathbb{Q}\gamma$. Put $\gamma^{w}=q\gamma$, $q\in\mathbb{Q}$. Then, for $\alpha\in\Sigma$ such that $\bar{\alpha}=\gamma$, we have $-\gamma=\overline{\alpha^{w}}=\overline{q\gamma}=q\gamma$. Therefore, $q=-1$. Thus, w stabilizes $\mathfrak{X}_{\bar{\Sigma}}$ and $w\,|\,\mathfrak{X}_{\bar{\Sigma}}=\bar{w}_\gamma\,|\,\mathfrak{X}_{\bar{\Sigma}}$. Therefore, $\overline{\alpha^{w}}=\bar{\alpha}^{w}=\bar{\alpha}^{\bar{w}_\gamma}$ for $\alpha\in\Sigma$. Thus, $w=\bar{w}_\gamma$. q.e.d.

We recall a definition. Let P be a fundamental system of roots (not necessarily reduced). Let w be the unique element of the Weyl group of P such that $P^{w}=-P$. The map $\alpha\longrightarrow-\alpha^{w}$ is an automorphism of P and is called the <u>opposition involution</u> of P. It is denoted by i_P. If P_1 is a component of P, then $P_1^{\,i_P}=P_1$ and $i_P\,|\,P_1=i_{P_1}$. If P is connected, then i_P is trivial unless P is reduced and of type $A_n (n\geq 2)$, E_6, or $D_n (n>4$, n odd) in which case i_P is the unique non-trivial automorphism of P.

<u>Lemma 7</u>: Let T be a non-empty subset of $\bar{\Pi}$. Put $\Pi_T=\bigcup_{\gamma\in T}\Pi_\gamma$ and $\Pi_{To}=\bigcup_{\gamma\in T}\Pi_{o\gamma}$. Let $\bar{w}$ be the uniquely determined product of elements of $\{\bar{w}_\gamma : \gamma\in T\}$ such that $T^{\bar{w}}=-T$. Then,

i_{Π_T} and $\bar{w}$ stabilize Π_{T0}, and $\bar{w}|\Pi_{T0} = (i_{\Pi_T}|\Pi_{T0})\circ i_{\Pi_{T0}}$. If $\gamma \in T$, we have $\mho_\gamma^{i_{\Pi_T}} = \mho_{\gamma i_T}$, $\Pi_{0,\gamma}^{i_{\Pi_T}} = \Pi_{0,\gamma i_T}$, and $\Pi_\gamma^{i_{\Pi_T}} = \Pi_{\gamma i_T}$.

Proof: Let $\gamma \in T$. Then, $\bar{w}_\gamma$ is a product of reflections corresponding to elements of $\{\alpha \in \Sigma : \bar{\alpha} = \gamma\} \cup \Pi_{0,\gamma}$ and $\bar{w}_\gamma$ fixes the elements of $\Pi_{0,\gamma}$. But Π_γ is a fundamental system for the system of roots $\{\alpha \in \Sigma : \bar{\alpha} \neq 0, \bar{\alpha} \in Z\gamma\} \cup \Sigma_{0,\gamma}$. Therefore, $\bar{w}_\gamma$ is a product of reflections corresponding to elements of Π_γ.

By the above, $\bar{w}$ is a product of reflections corresponding to elements of Π_T and $\bar{w}$ fixes the elements of $\Pi_0 - \Pi_{T0}$. Therefore, $\bar{w}$ stabilizes Π_{T0}. Let w_0 be the uniquely determined product of reflections corresponding to elements of Π_{T0} such that $\Pi_{T0}^{w_0} = -\Pi_{T0}$. For $\alpha \in \Pi_T - \Pi_{T0}$, $\overline{\alpha^{\bar{w} w_0}} = \overline{\alpha^{\bar{w}}} = \bar{\alpha}^{\bar{w}}$ and hence $\alpha^{\bar{w} w_0} \underset{\Pi}{<} 0$. But for $\beta \in \Pi_{T0}$, $\beta^{\bar{w} w_0} \underset{\Pi}{<} 0$. Therefore, $\Pi_T^{\bar{w} w_0} = -\Pi_T$. Since $\bar{w} w_0$ is a product of reflections corresponding to elements of Π_T, we have $\bar{w} w_0|\Pi_T = -i_{\Pi_T}$. Therefore, $\bar{w}|\Pi_{T0} = (i_{\Pi_T}|\Pi_{T0})\circ i_{\Pi_{T0}}$.

Let $\Sigma_\gamma = \{\alpha \in \Sigma : \bar{\alpha} = \gamma\}$ for $\gamma \in \bar{\Sigma}$. Then, for $\gamma \in T$, $\Sigma_\gamma^{i_{\Pi_T}} = \Sigma_\gamma^{(-\bar{w})w_0} = \Sigma_{\gamma i_T}^{w_0} = \Sigma_{\gamma i_T}$. The last statement of the lemma is then clear. q.e.d.

We can now prove:

Prop. 7: Let $\gamma \in \bar{\Sigma}$. Then, $\bar{w}_\gamma$ fixes the elements of $\Pi_0 - \Pi_{0,\gamma}$. If $\gamma \in \bar{\Pi}$, then i_{Π_γ} stabilizes $\Pi_{0,\gamma}$ and $\mho_\gamma$, and

$$\bar{w}_\gamma|\Pi_{0,\gamma} = (i_{\Pi_\gamma}|\Pi_{0,\gamma})\circ i_{\Pi_{0,\gamma}}.$$

Proof: The first statement is part of lemma 6. The second statement is a consequence of lemma 7 with $T = \{\gamma\}$. q.e.d.

If $\gamma \in \overline{\overline{\Pi}}$, we can calculate the action of $[\mathcal{L}_0, \mathcal{L}_0]_K$ on $(\mathcal{L}_\gamma)_K$ using our action of $\overline{W}$ on Π_0.

<u>Prop 8</u>: Let $\gamma \in \overline{\overline{\Pi}}$. Write $\sigma_\gamma = \{\alpha_1, \ldots, \alpha_f\}$. For $\beta \in \Pi_0$, let λ_β be the fundamental dominant integral weight of $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K$ corresponding to β. Let $\lambda_i = \sum_{\beta \in \Pi_0} -(\alpha_i, \hat{\beta})\lambda_\beta^{\overline{W}_\gamma}$, $i=1,\ldots,f$. Then, as an $[\mathcal{L}_0, \mathcal{L}_0]_K$ module, $(\mathcal{L}_\gamma)_K$ is isomorphic to the direct sum of the f irreducible $[\mathcal{L}_0, \mathcal{L}_0]_K$ modules with highest weights $\lambda_1, \ldots, \lambda_f$.

<u>Proof</u>: The weights of the representation of $[\mathcal{L}_0, \mathcal{L}_0]_K$ in $(\mathcal{L}_\gamma)_K$ are $\{\alpha | (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K : \alpha \in \Sigma \text{ and } \overline{\alpha} = \gamma\}$. Thus, the highest weights of the irreducible summands of this representation are $\{\alpha | (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K : \alpha \in \Sigma, \overline{\alpha} = \gamma, \text{ and } \alpha + \beta \notin \Sigma \text{ for } \beta \in \Pi_0\}$. But $\overline{\alpha^{\overline{W}_\gamma}} = -\gamma$ for $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$ and $\Pi_0^{\overline{W}_\gamma} = \Pi_0$. Thus, $\overline{W}_\gamma$ takes these highest weights onto

$\{\alpha | (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K : \alpha \in \Sigma, \overline{\alpha} = -\gamma, \text{ and } \alpha + \beta \notin \Sigma \text{ for } \beta \in \Pi_0\}$.

But (by lemma 1) this last set is $\{-\alpha | (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K : \alpha \in \sigma_\gamma\}$. Therefore, the highest weights of the irreducible summands of the representation of $[\mathcal{L}_0, \mathcal{L}_0]_K$ in $(\mathcal{L}_\gamma)_K$ are

$\{-\alpha_i^{\overline{W}_\gamma} | (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K : i=1,\ldots,f\}$. But

$(-\alpha_i^{\overline{W}_\gamma}, \widehat{\beta^{\overline{W}_\gamma}}) = -(\alpha_i, \hat{\beta}) = (\lambda_i, \widehat{\beta^{\overline{W}_\gamma}})$ for $\beta \in \Pi_0$ and $i=1,\ldots,f$. q.e.d.

Chapter 3

The Restricted Diagram and the Index

In this chapter, we study the relationship between the restricted diagram and the index of an algebra. We will assume throughout the chapter that $\mathcal{L}$ is a simple algebra over k and that $\mathfrak{I}$, $\mathfrak{h}$, K, $\mathfrak{g}$, Π, Π_0, etc. are as in Chapter 1. We also use the notation of Chapter 2.

Our method of deducing some facts about the index (Π, Π_0) will be to first prove a result about the indices $(\Pi_\gamma, \Pi_{0,\gamma})$, $\gamma \in \Pi$, and then concern ourselves with how these indices fit together. In order to prove the result about the indices $(\Pi_\gamma, \Pi_{0,\gamma})$, $\gamma \in \overline{\Pi}$, we will need the following lemma:

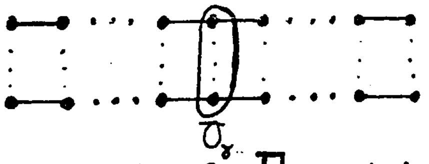
<u>Lemma 1</u>: Let P be a connected fundamental system of roots. Let $\alpha \in$ P and suppose α has coefficient 1 in the dominant root for P. Then, either P is of type A or P$-\{\alpha\}$ is connected.

<u>Proof</u>: We may assume P$-\{\alpha\} \neq \phi$. Let $\mu_P = \alpha + \sum_{\beta \in P-\{\alpha\}} m_\beta \beta$ be the dominant root for P. Let $P_1, \ldots, P_t$ be the components of P$-\{\alpha\}$. Then, there exists a unique element β_i of P_i such that α and β_i are connected, i=1,...,t. But then $0 \leq (\mu_P, \hat{\alpha}) = 2 + \sum_{i=1}^{t} m_{\beta_i} (\beta_i, \hat{\alpha})$ and hence $\sum_{i=1}^{t} m_{\beta_i} (-(\beta_i, \hat{\alpha})) \leq 2$. If t=1, then P$-\{\alpha\}$ is connected and we are done. Suppose $t \geq 2$. Since $-(\beta_i, \hat{\alpha})$ is a positive integer for i=1,...,t, we have t=2, $m_{\beta_1} = m_{\beta_2} = 1$, and $-(\beta_1, \hat{\alpha}) = -(\beta_2, \hat{\alpha}) = 1$. Now, $\alpha - (\alpha, \hat{\beta_i})\beta_i$ is a root and hence $-(\alpha, \hat{\beta_i}) \leq m_{\beta_i}$, i=1,2. Therefore, $-(\alpha, \hat{\beta_i}) = 1$, i=1,2. Thus, $\{\alpha, \beta_i\}$ is a segment of

type A_2, i=1,2. Now, the coefficient of β_i in the dominant root for $P_i \cup \{\alpha\}$ is less than or equal to m_{β_i} and hence is 1, i=1,2. Thus, by induction on the number of elements in P, either $P_i \cup \{\alpha\}$ is of type A or $(P_i \cup \{\alpha\}) - \{\beta_i\}$ is connected, i=1,2. But if $(P_i \cup \{\alpha\}) - \{\beta_i\}$ is connected, it follows that $P_i = \{\beta_i\}$ and hence $P_i \cup \{\alpha\} = \{\alpha, \beta_i\}$ is of type A_2, i=1,2. Thus, in any case $P_i \cup \{\alpha\}$ is of type A , i=1,2. Thus, P is of type A. q.e.d.

We can now prove:

<u>Prop. 1</u>: Let $\gamma \in \Pi$ and suppose $2\gamma \notin \Sigma$. Then, either $\Pi_{0,\gamma}$ is $*$-connected or $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form:



(2) ,

where the two $*$-components of $\Pi_{0,\gamma}$ contain the same number of elements.

<u>Proof</u>: Let P be a connected component of Π_γ. Put $P_0 = \Pi_{0,\gamma} \cap P$ and $P_\gamma = \mathfrak{O}_\gamma \cap P$. Then, Π_γ (resp. $\Pi_{0,\gamma}$; $\mathfrak{O}_\gamma$) is the disjoint union of images of P (resp. P_0 ; P_γ) under the $*$-action. Let μ_P be the dominant root for P. Then, by lemma 2.4, $\bar{\mu}_P = \gamma$. Hence, P_γ is a singleton $\{\alpha\}$ and the coefficient of α in μ_P is 1. Then, $P_0 = P - \{\alpha\}$ and, by lemma 1, either P is of type A or P_0 is connected. If P_0 is connected, then $\Pi_{0,\gamma}$ is $*$-connected. Suppose P is of type A_n. But, by Prop. 2.7, i_{Π_γ} stabilizes $\mathfrak{O}_\gamma$ and hence, since $i_{\Pi_\gamma} | P = i_P$, $\alpha^{i_P} = \alpha$. Therefore, n is odd and α is the middle root of P. If $P_0 \neq \phi$ and there exists $\sigma \in \mathcal{A}$ such that $P^{\sigma^*} = P$ and σ^* exchanges the two components of P_0 , then Π_γ is of $*$-type A_{nII} and $\Pi_{0,\gamma}$ is $*$-connected. Otherwise, $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form (2). q.e.d.

We concern ourselves now with how the Π_γ, $\gamma \in \overline{\overline{\Pi}}$, fit together. We begin by proving four lemmas which will be useful in this regard. The first of these deals with the intersections of the $\Pi_{0,\gamma}$, $\gamma \in \overline{\overline{\Pi}}$, with each other.

<u>Lemma 2</u>: Let γ and δ be distinct elements of $\overline{\overline{\Sigma}}$.

(i) If $\gamma, \delta \in \overline{\overline{\Pi}}$, then $\Pi_{0,\gamma} \cap \Pi_{0,\delta}$ is *-connected.

(ii) If $\gamma + \delta \notin \overline{\overline{\Sigma}}$ and $\gamma - \delta \notin \overline{\overline{\Sigma}}$, then $\Pi_{0,\gamma} \cap \Pi_{0,\delta} = \phi$.

<u>Proof</u>: Suppose $\gamma, \delta \in \overline{\overline{\Pi}}$. Suppose for contradiction that $\Pi_{0,\gamma} \cap \Pi_{0,\delta}$ contains two distinct non-empty *-components $\Pi_{0,1}$ and $\Pi_{0,2}$. It follows immediately from Prop. 2.3 that there exists $\alpha_1 \in \mho_\gamma$, $\alpha_2 \in \mho_\delta$, and a subset $P_{0,1}$ of $\Pi_{0,1}$ such that $\{\alpha_1\} \cup P_{0,1} \cup \{\alpha_2\}$ is connected. Similarly, there exists $\alpha_3 \in \mho_\gamma$, $\alpha_4 \in \mho_\delta$, and a subset $P_{0,2}$ of $\Pi_{0,2}$ such that $\{\alpha_3\} \cup P_{0,2} \cup \{\alpha_4\}$ is connected. Replacing α_3, $P_{0,2}$, and α_4 by their images under σ^*, where $\sigma \in \mathcal{L}$ is chosen so that $\alpha_3^{\sigma^*} = \alpha_1$, we may assume that $\alpha_3 = \alpha_1$. Since $P_{0,1} \subseteq \Pi_{0,1}$ and $P_{0,2} \subseteq \Pi_{0,2}$, we have $\alpha_2 \neq \alpha_4$. Then, $\alpha = \alpha_1 + \alpha_2 + \alpha_4 + \sum_{\beta \in P_{0,1} \cup P_{0,2}} \beta \in \Sigma$ and $\overline{\overline{\alpha}} = \gamma + 2\delta$. Thus, $\gamma + 2\delta \in \overline{\overline{\Sigma}}$. Similarly, $\delta + 2\gamma \in \overline{\overline{\Sigma}}$ and we have a contradiction. (i) is proved.

Suppose $\gamma + \delta \notin \overline{\overline{\Sigma}}$ and $\gamma - \delta \notin \overline{\overline{\Sigma}}$. Suppose for contradiction that there exists $\beta \in \Pi_{0,\gamma} \cap \Pi_{0,\delta}$. Then, there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma$ such that $\beta = \alpha_1 - \alpha_3 = \alpha_2 - \alpha_4$, $\overline{\overline{\alpha_1}} = \overline{\overline{\alpha_3}} = \gamma$, and $\overline{\overline{\alpha_2}} = \overline{\overline{\alpha_4}} = \delta$. Then, $\alpha_1 + \alpha_2 \notin \Sigma$ and $\alpha_1 - \alpha_2 \notin \Sigma$ and hence $(\alpha_1, \alpha_2) = 0$. Similarly, $(\alpha_1, \alpha_4) = 0$. Therefore, $(\alpha_1, \beta) = 0$. Similarly, $(\alpha_3, \beta) = 0$. Thus, $(\beta, \beta) = 0$ and we have a contradiction. (ii) is proved. q.e.d.

Lemma 3: Suppose γ and δ are distinct elements of $\overline{\Sigma}$ such that $(\gamma, \delta) < 0$. Then,

$$(\Pi_{0,\gamma} - \Pi_{0,\delta}) \cup (\Pi_{0,\delta} - \Pi_{0,\gamma}) \subseteq \Pi_{0,\gamma+\delta} .$$

Proof: It suffices to show that $\Pi_{0,\gamma} - \Pi_{0,\delta} \subseteq \Pi_{0,\gamma+\delta}$. Let $\beta \in \Pi_{0,\gamma} - \Pi_{0,\delta}$. Then, there exists $\alpha_1 \in \Sigma$ such that $\overline{\alpha_1} = \gamma$ and $\alpha_1 + \beta \in \Sigma$. By replacing α_1 with the element of least height in the β-chain containing α_1, we may assume $\alpha_1 - \beta \notin \Sigma$. Therefore, $(\alpha_1, \beta) < 0$. Choose $\alpha_2 \in \Sigma$ such that $\overline{\alpha_2} = \delta$. Then,
$0 > (\gamma, \delta) = (\frac{1}{|\mathcal{S}|}\sum_{\sigma\in\mathcal{S}}\alpha_1^{\sigma}, \delta) = \frac{1}{|\mathcal{S}|}\sum_{\sigma\in\mathcal{S}}(\alpha_1, \delta^{\sigma^{-1}}) = (\alpha_1, \delta) = \frac{1}{|\mathcal{S}|}\sum_{\sigma\in\mathcal{S}}(\alpha_1, \alpha_2^{\sigma})$.
Thus, $(\alpha_1, \alpha_2^{\sigma}) < 0$ for some $\sigma \in \mathcal{S}$. Replacing α_2 by α_2^{σ}, we may assume $(\alpha_1, \alpha_2) < 0$. Since $\beta \notin \Pi_{0,\delta}$, we have $\alpha_2 - \beta \notin \Sigma$ and $\alpha_2 + \beta \notin \Sigma$. Therefore, $(\alpha_2, \beta) = 0$. But $\alpha_1 + \alpha_2 \in \Sigma$, $(\alpha_1 + \alpha_2, \beta) = (\alpha_1, \beta) < 0$, and hence $\alpha_1 + \alpha_2 + \beta \in \Sigma$. Therefore, $\beta \in \Pi_{0,\gamma+\delta}$. q.e.d.

Lemma 4: Let $\Pi_{0,1}$ and $\Pi_{0,2}$ be *-components of Π_0. Then, there exists $\gamma \in \overline{\Sigma}$ such that $\Pi_{0,1} \cup \Pi_{0,2} \subseteq \Pi_{0,\gamma}$.

Proof: If $\Pi_{0,1} \cup \Pi_{0,2} \subseteq \Pi_{0,\gamma}$ for some $\gamma \in \overline{\Pi}$, we are done. Suppose not. Then, since $\overline{\Pi}$ is connected, there exists distinct $\gamma_1, \ldots, \gamma_t \in \overline{\Pi}$ such that $t \geq 2$, $(\gamma_i, \gamma_{i+1}) < 0$ for $i=1,\ldots,t-1$, $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1}$, $\Pi_{0,1} \nsubseteq \Pi_{0,\gamma_i}$ for $i=2,\ldots,t$, and $\Pi_{0,2} \subseteq \Pi_{0,\gamma_t}$. We show by induction on i that $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_i}$ for $i=1,\ldots,t$. This is clear for $i=1$. If $1 < i \leq t$ and $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_{i-1}}$, then $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_{i-1}} - \Pi_{0,\gamma_i}$ and hence, by lemma 3, $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_i}$. Hence, in particular, $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_{t-1}}$ and $\Pi_{0,1} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_t}$. If $\Pi_{0,2} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_{t-1}}$, putting $\gamma = \gamma_1 + \ldots + \gamma_{t-1}$ we are done. Otherwise, $\Pi_{0,2} \subseteq \Pi_{0,\gamma_t} - \Pi_{0,\gamma_1+\cdots+\gamma_{t-1}}$ and hence, by lemma 3, $\Pi_{0,2} \subseteq \Pi_{0,\gamma_1+\cdots+\gamma_{t-1}+\gamma_t}$. Putting $\gamma = \gamma_1 + \ldots + \gamma_t$, we are done. q.e.d.

For the purposes of the following lemma, we put $A_{\gamma,\delta} = 2\frac{(\gamma,\delta)}{(\delta,\delta)}$ for $\gamma, \delta \in \Sigma$.

Lemma 5: Suppose $\gamma, \delta \in \Sigma$. Suppose $\gamma - \delta \notin \overline{\Sigma}$ and $2\gamma \notin \overline{\Sigma}$.

Then:

(i) $A_{\gamma,\delta} = A_{\delta,\gamma} = 0 \Longrightarrow \Pi_{o,\gamma} \cap \Pi_{o,\delta} = \phi$.

(ii) $A_{\gamma,\delta} = A_{\delta,\gamma} = -1 \Longrightarrow \Pi_{o,\gamma} \cap \Pi_{o,\delta} \neq \phi$ whenever $\Pi_{o,\gamma} \cup \Pi_{o,\delta} \neq \phi$.

(iii) $A_{\gamma,\delta} = -1,\ A_{\delta,\gamma} = -2 \Longrightarrow \Pi_{o,\delta} \subseteq \Pi_{o,\gamma}$ and, if $\Pi_{o,\gamma}$ is *-connected, $\Pi_{o,\delta} = \phi$.

(iv) $A_{\gamma,\delta} = -1,\ A_{\delta,\gamma} = -3 \Longrightarrow \Pi_{o,\delta} = \phi$.

Proof: (i) follows from lemma 2(ii). To prove (ii), (iii), and (iv), we may assume $A_{\gamma,\delta} = -1$ and $A_{\delta,\gamma} = -p$, where p=1,2, or 3. By lemma 3, $\Pi_{o,\delta} - \Pi_{o,\gamma} \subseteq \Pi_{o,\gamma+\delta} \cap \Pi_{o,\delta}$. But by Prop. 2.7, $\overline{w}_\gamma$ fixes the elements of $\Pi_{o,\delta} - \Pi_{o,\gamma}$. Hence, operating on both sides of $\Pi_{o,\delta} - \Pi_{o,\gamma} \subseteq \Pi_{o,\gamma+\delta} \cap \Pi_{o,\delta}$ with $\overline{w}_\gamma$, we obtain $\Pi_{o,\delta} - \Pi_{o,\gamma} \subseteq \Pi_{o,(p-1)\gamma+\delta} \cap \Pi_{o,p\gamma+\delta}$. Since p = 1,2, or 3, combining our two inclusions we obtain

$$\Pi_{o,\delta} - \Pi_{o,\gamma} \subseteq \bigcap_{i=o}^{p} \Pi_{o,i\gamma+\delta}. \tag{3}$$

Suppose p = 1. Suppose for contradiction that $\Pi_{o,\gamma} \cap \Pi_{o,\delta} = \phi$ and $\Pi_{o,\gamma} \cup \Pi_{o,\delta} \neq \phi$. By (3), $\Pi_{o,\delta} \subseteq \Pi_{o,\gamma+\delta}$. Operating on both sides of this inclusion with $\overline{w}_\delta$, we obtain $\Pi_{o,\delta} \subseteq \Pi_{o,\gamma}$. Similarly, $\Pi_{o,\gamma} \subseteq \Pi_{o,\delta}$. Therefore, $\Pi_{o,\gamma} = \Pi_{o,\delta}$ and we have a contradiction since $\Pi_{o,\gamma} \cap \Pi_{o,\delta} = \Pi_{o,\gamma} \cup \Pi_{o,\delta}$. (ii) is proved.

Suppose $p \geq 2$. Now, $\gamma + \delta = \gamma^{\overline{w}_\delta}$ and hence $2\gamma + 2\delta \notin \overline{\Sigma}$. Thus, $(2\gamma + \delta) + \delta \notin \overline{\Sigma}$ and $(2\gamma + \delta) - \delta \notin \overline{\Sigma}$. Thus, by lemma 2(ii), $\Pi_{o,2\gamma+\delta} \cap \Pi_{o,\delta} = \phi$. Thus, by (3), $\Pi_{o,\delta} - \Pi_{o,\gamma} = \phi$. Therefore, $\Pi_{o,\delta} \subseteq \Pi_{o,\gamma}$.

Suppose p = 2 and $\Pi_{0,\gamma}$ is *-connected. Then, $\Pi_{0,2\gamma+\delta} = \Pi_{0,\delta}^{\bar{w}_\gamma} \subseteq \Pi_{0,\gamma}^{\bar{w}_\gamma} = \Pi_{0,\gamma}$. Therefore, $\Pi_{0,2\gamma+\delta}$ and $\Pi_{0,\delta}$ are two disjoint subsets of $\Pi_{0,\gamma}$. Thus, $\Pi_{0,\delta} = \phi$ or $\Pi_{0,2\gamma+\delta} = \phi$. But $\Pi_{0,2\gamma+\delta} = \Pi_{0,\delta}^{\bar{w}_\gamma}$. Therefore, $\Pi_{0,\delta} = \phi$. (iii) is proved.

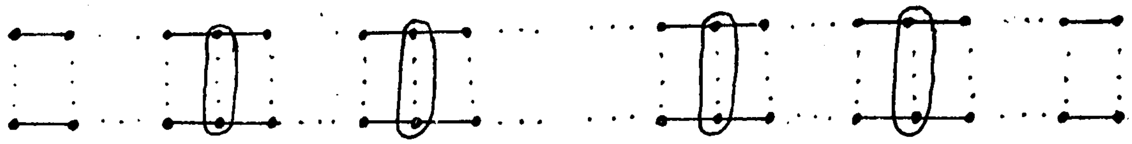
Suppose p = 3. Now, by Prop. 2.7, $\Pi_{0,\gamma}^{\bar{w}_\delta} = \Pi_{0,\gamma}$ (since $\Pi_{0,\delta} \subseteq \Pi_{0,\gamma}$). Thus, $\Pi_{0,2\gamma+\delta} = \Pi_{0,\gamma}^{\bar{w}_\delta \bar{w}_\gamma} = \Pi_{0,\gamma}^{\bar{w}_\gamma} = \Pi_{0,\gamma}$. Thus, $\phi = \Pi_{0,2\gamma+\delta} \cap \Pi_{0,\delta} = \Pi_{0,\gamma} \cap \Pi_{0,\delta} = \Pi_{0,\delta}$. (iv) is proved. q.e.d.

The first consequence of these lemmas is:

<u>Prop. 2</u>: Suppose $\bar{\Pi}$ is reduced. Let $\Pi_{0,1}$ and $\Pi_{0,2}$ be two *-components of Π_0. Then, there exists $\bar{w} \in \bar{W}$ such that $\Pi_{0,1}^{\bar{w}} = \Pi_{0,2}$.

<u>Proof</u>: We may assume $\Pi_{0,1} \neq \Pi_{0,2}$. By lemma 4, there exists $\gamma \in \bar{\Sigma}$ such that $\Pi_{0,1} \cup \Pi_{0,2} \subseteq \Pi_{0,\gamma}$. But there exists $\bar{w}_1 \in \bar{W}$ such that $\gamma^{\bar{w}_1} \in \bar{\Pi}$. Put $\delta = \gamma^{\bar{w}_1}$. Then, $\Pi_{0,1}^{\bar{w}_1}$ and $\Pi_{0,2}^{\bar{w}_1}$ are distinct *-components of $\Pi_{0,\delta}$ and hence, by Prop. 1, $(\Pi_\delta, \Pi_{0,\delta})$ is of the form (2). But $\bar{w}_\delta | \Pi_{0,\delta} = (i_{\Pi_\delta} | \Pi_{0,\delta}) \circ i_{\Pi_{0,\delta}}$ and hence $\bar{w}_\delta$ interchanges $\Pi_{0,1}^{\bar{w}_1}$ and $\Pi_{0,2}^{\bar{w}_1}$. Putting $\bar{w} = \bar{w}_1 \bar{w}_\delta \bar{w}_1^{-1}$ we are done. q.e.d.

We will refer in what follows to indices of the form:



(4)

where all the *-components of the anisotropic part of this index contain the same number of elements. If we wish to specify the number t of distinguished orbits, we say that <u>the index is of the form $(4)_t$</u>.

The following lemma will be useful in studying subsets T of $\overline{\Pi}$ of type A.

<u>Lemma 6</u>: Let T be a non-empty connected subset of $\overline{\Pi}$ of type A_t. Put $\Pi_T = \bigcup_{\gamma \in T} \Pi_\gamma$ and $\Pi_{TO} = \bigcup_{\gamma \in T} \Pi_{o,\gamma}$. Then, exactly one of the following holds:

(a) Π_{TO} is *-connected $\neq \phi$.

(b) (Π_T, Π_{TO}) is of the form (4).

Moreover, if (a) holds, then t=1 or 2 and $\Pi_{TO} = \Pi_{o,\gamma}$ for all $\gamma \in$ T.

<u>Proof</u>: Let Σ_T be the set of restricted roots which are positive or negative integral combinations of elements of T. Label the roots of T as follows: $\gamma_1 - \gamma_2 \cdots \gamma_{t-1} - \gamma_t$.

Suppose first of all that (a) holds and $t > 1$. Then, $\Pi_{o,\gamma} \neq \phi$ for some $\gamma \in T$. Since all elements of T are conjugate under $\overline{W}$, $\Pi_{o,\gamma} \neq \phi$ for all $\gamma \in$ T. Thus, $\Pi_{o,\gamma} = \Pi_{TO}$ for all $\gamma \in$ T. Then, for $\gamma \neq \delta \in$ T, $\Pi_{o,\gamma} \cap \Pi_{o,\delta} \neq \phi$ and hence, by lemma 2(ii), $\gamma + \delta \in \overline{\Sigma}$. Thus, t=2.

It remains to show that if Π_{TO} is empty or not *-connected, then (Π_T, Π_{TO}) is of the form (4). If $\Pi_{TO} = \phi$, this is clear. Suppose Π_{TO} is not *-connected. Now, from the proof of lemma 4, it follows that any two *-components of Π_{TO} are contained in $\Pi_{o,\gamma}$ for some $\gamma \in \Sigma_T$. Since all elements of Σ_T are conjugate under $\overline{W}$, $\Pi_{o,\gamma}$ contains two non-empty *-components for all $\gamma \in \Sigma_T$. By Prop. 1, $(\Pi_\gamma, \Pi_{o,\gamma})$ is of the form (2) for all $\gamma \in$ T. We prove by induction on i that $(\Pi_{\gamma_1} \cup \ldots \cup \Pi_{\gamma_i}, \Pi_{o,\gamma_1} \cup \ldots \cup \Pi_{o,\gamma_i})$ is of the form $(4)_i$ for i=1,...,t. We have this for i=1. Suppose $1 < i \leq t$ and

$(\Pi_{\gamma_1} \cup \dots \cup \Pi_{\gamma_{i-1}}, \Pi_{0,\gamma_1} \cup \dots \cup \Pi_{0,\gamma_{i-1}})$ is of the form $(4)_{i-1}$. By lemma 5(ii), $\Pi_{0,\gamma_{i-1}} \cap \Pi_{0,\gamma_i} \neq \phi$ and, by lemma 2(i), $\Pi_{0,\gamma_{i-1}} \cap \Pi_{0,\gamma_i}$ is *-connected. But if $i > 2$, $\Pi_{0,\gamma_j} \cap \Pi_{0,\gamma_i} = \phi$ for $1 \le j \le i-2$ (since $\gamma_j + \gamma_i \notin \overline{\Sigma}$). Thus, $\Pi_{0,\gamma_{i-1}} \cap \Pi_{0,\gamma_i}$ is one of the *-components of $\Pi_{0,\gamma_{i-1}}$ and, if $i > 2$, $\Pi_{0,\gamma_i} \cap \Pi_{0,\gamma_{i-1}}$ is not one of the *-components of $\Pi_{0,\gamma_1} \cup \dots \cup \Pi_{0,\gamma_{i-2}}$. Since $(\Pi_{\gamma_i}, \Pi_{0,\gamma_i})$ is of the form (2), the result is then clear. q.e.d.

We can now prove some results about the relationship between the restricted diagram $\overline{\overline{\Pi}}$ and the index (Π, Π_0). We will consider successively the cases $\overline{\overline{\Pi}}$ of type A, $\overline{\overline{\Pi}}$ of type D or E, $\overline{\overline{\Pi}}$ of type B,C,F, or G, and $\overline{\overline{\Pi}}$ not reduced.

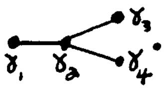
Prop. 3: Suppose $\overline{\overline{\Pi}}$ is of type A_r $(r \geq 1)$. Then, exactly one of the following holds:

(a) $r=1$ or 2 and Π_0 is *-connected $\neq \phi$.

(b) (Π, Π_0) is of the form (4).

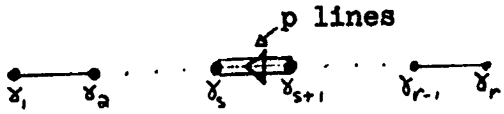
Proof: This follows from lemma 6 with $T = \overline{\overline{\Pi}}$. q.e.d.

Prop. 4: Suppose $\overline{\overline{\Pi}}$ is of type D_r $(r \geq 4)$ or E_r $(r=6,7,$ or $8)$. Then, $\Pi_0 = \phi$.

Proof: We choose a subdiagram of $\overline{\overline{\Pi}}$ of type D_4, labelled 

Suppose for contradiction that $\Pi_0 \neq \phi$. Since all the roots of $\overline{\overline{\Pi}}$ are conjugate under $\overline{W}$, we have $\Pi_{0,\gamma_2} \neq \phi$. Put $T_3 = \{\gamma_1, \gamma_2, \gamma_3\}$ and $T_4 = \{\gamma_1, \gamma_2, \gamma_4\}$ and apply lemma 6. Since T_3 and T_4 contain three elements each, we have that $(\Pi_{\gamma_1} \cup \Pi_{\gamma_2} \cup \Pi_{\gamma_3}, \Pi_{0,\gamma_1} \cup \Pi_{0,\gamma_2} \cup \Pi_{0,\gamma_3})$ and $(\Pi_{\gamma_1} \cup \Pi_{\gamma_2} \cup \Pi_{\gamma_4}, \Pi_{0,\gamma_1} \cup \Pi_{0,\gamma_2} \cup \Pi_{0,\gamma_4})$ are of the form $(4)_3$. But then, since $\Pi_{0,\gamma_2} \neq \phi$, it is clear that $\Pi_{\gamma_1} \cup \Pi_{\gamma_2} \cup \Pi_{\gamma_3} \cup \Pi_{\gamma_4}$ is not the Dynkin diagram of a root system. q.e.d.

We now consider the case when $\overline{\Pi}$ is of type B, C, F, or G. Label the elements of $\overline{\Pi}$ as follows:



, (5)

where p=2 or 3 and $1 \leq s < r$. Let S be the set of short roots of $\overline{\Pi}$ and let L be the set of long roots of $\overline{\Pi}$ i.e. $S = \{\gamma_1, \ldots, \gamma_s\}$ and $L = \{\gamma_{s+1}, \ldots, \gamma_r\}$. Put $\Pi_S = \bigcup_{\gamma \in S} \Pi_\gamma$, $\Pi_{S0} = \bigcup_{\gamma \in S} \Pi_{0,\gamma}$, $\Pi_L = \bigcup_{\gamma \in L} \Pi_\gamma$, and $\Pi_{L0} = \bigcup_{\gamma \in L} \Pi_{0,\gamma}$. We have:

<u>Lemma 7</u>: Π_{L0} is *-connected, $\Pi_{L0} \subseteq \Pi_{0,\gamma_s}$, and $\Pi_{L0} = \Pi_{0,\gamma_{s+1}} = \ldots = \Pi_{0,\gamma_r}$.

<u>Proof</u>: By lemma 5 ((iii) and (iv)), $\Pi_{0,\gamma_{s+1}} \subseteq \Pi_{0,\gamma_s}$. Suppose for contradiction that Π_{L0} is not *-connected. By lemma 6 (applied to T = L), (Π_L, Π_{L0}) is of the form (4). But $\Pi_{L0} \neq \phi$, and hence $\Pi_{0,\gamma_{s+1}}$ is not *-connected. But $\Pi_{0,\gamma_{s+1}} = \Pi_{0,\gamma_s} \cap \Pi_{0,\gamma_{s+1}}$ and we have a contradiction by lemma 2(i). Therefore, Π_{L0} is *-connected.

By lemma 6, $\Pi_{L0} = \Pi_{0,\gamma_{s+1}} = \ldots = \Pi_{0,\gamma_r}$. But $\Pi_{0,\gamma_{s+1}} \subseteq \Pi_{0,\gamma_s}$. Therefore, $\Pi_{L0} \subseteq \Pi_{0,\gamma_s}$. q.e.d.

We may now prove:

<u>Prop. 5</u>: Suppose $\overline{\Pi}$ is of type B, C, F, or G. Suppose $\Pi_0 \neq \phi$. Label the roots of $\overline{\Pi}$ as in (5). Put $\Pi_S = \Pi_{\gamma_1} \cup \ldots \cup \Pi_{\gamma_s}$. Then, $\Pi_0 \subseteq \Pi_S$ and exactly one of the following holds:

(a) s=1 or 2 and Π_0 is *-connected.

(b) (Π_S, Π_0) is of the form (4).

Proof: $\Pi_0 = \Pi_{S0} \cup \Pi_{L0}$ and, by lemma 7, $\Pi_{L0} \subseteq \Pi_{0,\gamma_s} \subseteq \Pi_{S0}$. Therefore, $\Pi_0 = \Pi_{S0} \subseteq \Pi_S$. The remaining conclusion follows from lemma 6 with T = S. q.e.d.

We also have:

Prop. 6: Same assumptions and notation as in Prop. 5. Then, exactly one of the following holds:

(a) $\Pi_{0,\gamma_{s+1}} = \dots = \Pi_{0,\gamma_r} = \phi$.

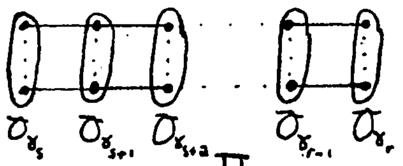
(b) Π is of type C_r $(r \geq 2)$, (Π_S, Π_0) is of the form (4), and $\Pi_{0,\gamma_{s+1}}$ is one of the *-components of Π_{0,γ_s}.

Proof: By lemma 7, $\Pi_{L0} \subseteq \Pi_{0,\gamma_s}$ and $\Pi_{L0} = \Pi_{0,\gamma_{s+1}} = \dots = \Pi_{0,\gamma_r}$. Thus, if $\Pi_{0,\gamma_{s+1}} = \phi$, (a) holds.

Suppose $\Pi_{0,\gamma_{s+1}} \neq \phi$. By lemma 7, $\Pi_{0,\gamma_r} \cap \Pi_{0,\gamma_s} \neq \phi$. Thus, by lemma 2(ii), r=s+1. By lemma 5 ((iii) and (iv)), p=2 and Π_{0,γ_s} is not *-connected. Then, by Prop. 5, (Π_S, Π_0) is of the form (4). Since $\Pi_{0,\gamma_{s+1}} = \Pi_{0,\gamma_s} \cap \Pi_{0,\gamma_{s+1}}$ is *-connected, (b) holds. q.e.d.

In case (a) of Prop. 6, we have more information:

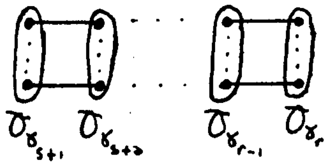
Prop. 7: Same assumptions and notation as in Prop. 5. Suppose $\Pi_{0,\gamma_{s+1}} = \dots = \Pi_{0,\gamma_r} = \phi$. Then, $\sigma_{\gamma_s} \cup \Pi_{\gamma_{s+1}} \cup \dots \cup \Pi_{\gamma_r}$ is of the form



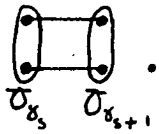
. (6)

If Π is connected, then Π_S is connected and (6) is connected.

Proof: The last statement is a consequence of the first statement. Thus, we need only prove that $\sigma_{\gamma_s} \cup \Pi_{\gamma_{s+1}} \cup \dots \cup \Pi_{\gamma_r}$ is of the form (6). Now, $\Pi_{L0} = \phi$ and hence (by lemma 6) (Π_L, Π_{L0}) is of the form



.

Thus, it suffices to show that $\mho_{\gamma_s} \cup \mho_{\gamma_{s+1}}$ is of the form 

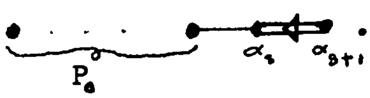
.

Let P be a connected component of $\Pi_{\gamma_s} \cup \mho_{\gamma_{s+1}}$. Let $P_{s+1} = P \cap \mho_{\gamma_{s+1}}$, $P_s = P \cap \mho_{\gamma_s}$, and $P_o = P \cap \Pi_{o,\gamma_s}$. It suffices to show that $P_s \cup P_{s+1}$ is connected of type A_2. Now, by the Corollary to Prop. 2.4, the intersection of $\Pi_{\gamma_{s+1}} = \mho_{\gamma_{s+1}}$ with each component of Π is connected. Hence, the intersection of $\mho_{\gamma_{s+1}}$ with each component of Π is a singleton (since $2\gamma_{s+1} \notin \overline{\overline{\Sigma}}$). But P is contained in a component of Π. Therefore, P_{s+1} is a singleton $\{\alpha_{s+1}\}$. Then, every element of P_s is connected to P_{s+1} (since $\Pi_{o,\gamma_{s+1}} = \phi$). Thus, $P_s \cup P_{s+1}$ is connected. Let μ be the dominant root for $P_s \cup P_{s+1}$. Write $\mu = \sum_{\alpha \in P_s} m_\alpha \alpha + m_{\alpha_{s+1}} \alpha_{s+1}$. Let $\alpha_s \in P_s$. Then, since $\Pi_{o,\gamma_s} \neq \phi$ (by Prop. 5), there exists $\beta \in P_o$ such that $(\alpha_s, \beta) < 0$. Then, $(\beta, \mu) \leq m_{\alpha_s}(\alpha_s, \beta) < 0$. Thus, $\beta \in \Pi_{o,\bar{\mu}}$. Therefore, $\Pi_{o,\bar{\mu}} \neq \phi$. Therefore, $\bar{\mu}$ is not an image under the Weyl group of γ_{s+1}.

Suppose p=2. Then, $\bar{\mu} \neq 2\gamma_s + \gamma_{s+1}$. Thus, $\bar{\mu} = \gamma_s + \gamma_{s+1}$ and hence P_s is a singleton $\{\alpha_s\}$, $m_{\alpha_s} = 1$, and $m_{\alpha_{s+1}} = 1$. Thus, $P_s \cup P_{s+1}$ is connected of type A_2.

Suppose p=3. If $\bar{\mu} = \gamma_s + \gamma_{s+1}$, we are done as above. Suppose for contradiction that $\bar{\mu} \neq \gamma_s + \gamma_{s+1}$. But $\bar{\mu} \neq 3\gamma_s + \gamma_{s+1}$ and $\bar{\mu} \neq 3\gamma_s + 2\gamma_{s+1}$. Therefore, $\bar{\mu} = 2\gamma_s + \gamma_{s+1}$. Then, $m_{\alpha_{s+1}} = 1$ and $\sum_{\alpha \in P_s} m_\alpha \alpha = 2$. Let μ_P be the dominant root for P. Write

$\mu_P = \sum_{\alpha\in P} n_\alpha \alpha$. Now, for $\alpha_s, \alpha_s' \in P_s$, there exists $\sigma \in \mathcal{G}$ such that $P^{\sigma^*} = P$ and $\alpha_s^{\sigma^*} = \alpha_s'$, and hence $n_{\alpha_s} = n_{\alpha_s'}$. Thus, for $\alpha_s \in P_s$, $\bar{\mu}_P = (\sum_{\alpha\in P_s} n_\alpha)\gamma_s + n_{\alpha_{s+1}}\alpha_{s+1} = |P_s|\, n_{\alpha_s}\gamma_s + n_{\alpha_{s+1}}\alpha_{s+1}$.
By lemma 2.4, $\bar{\mu}_P = 3\gamma_s + 2\gamma_{s+1}$. Thus, $|P_s|\, n_{\alpha_s} = 3$ for $\alpha_s \in P_s$.
But since $\sum_{\alpha\in P_s} m_\alpha \alpha = 2$, $|P_s| \leq 2$. Thus, P_s is a singleton and $m_{\alpha_s} = 2$. Thus, $P_s \cup P_{s+1}$ is of type B_2. Therefore, P is of the form



. Then, $\alpha_s{}^{1}\pi_{\gamma_s} = \alpha_s{}^{1}P_0 \cup \{\alpha_s\} \in P_0$ and we have a contradiction (by Prop. 2.7). q.e.d.

Examples: We give here examples of the possible forms which indices may take under the assumptions of Prop. 5. For simplicity, we assume that Π is connected (i.e. $\mathcal{X}$ is central). One can actually show (using, for example, Table II of Tits [9]) that the examples given here are the only possible ones. Combining Prop. 5 and 6, it follows that there are only four cases to be considered, namely:

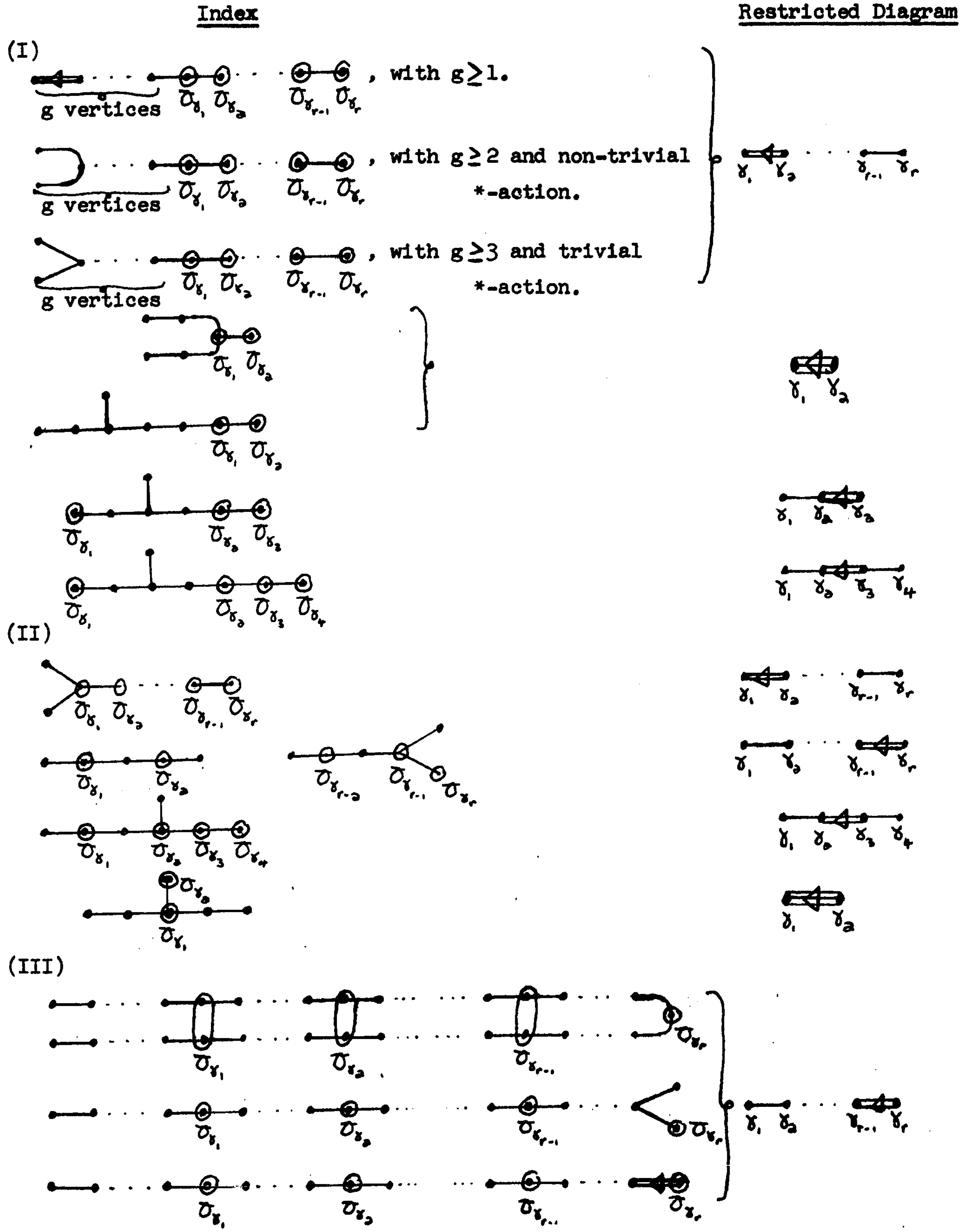
(I) s=1 or 2, Π_0 is *-connected, and $\Pi_{0,\gamma_{s+1}} = \cdots = \Pi_{0,\gamma_r} = \phi$.

(II) (Π_S, Π_0) is of the form (4) and $\Pi_{0,\gamma_{s+1}} = \cdots = \Pi_{0,\gamma_r} = \phi$.

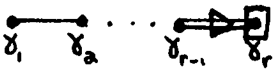
(III) (Π_S, Π_0) is of the form (4), $\bar{\bar{\Pi}}$ is of type C_r $(r \geq 2)$, and Π_{γ_r} is one of the two *-components of $\Pi_{0,\gamma_{r-1}}$.

We note that in cases (I) and (II) $\sigma_{\gamma_s} \cup \Pi_{\gamma_{s+1}} \cup \cdots \cup \Pi_{\gamma_r}$ is of the form

(by Prop. 7). The examples then are:


Index
Restricted Diagram
(I)
, with g ≥ 1.
g vertices
, with g ≥ 2 and non-trivial *-action.
g vertices
, with g ≥ 3 and trivial *-action.
g vertices
(II)
(III)

To complete this chapter, we consider the case when $\overline{\Pi}$ is not reduced. There seems to be little one can say about the form of (Π, Π_0) when $\overline{\Pi}$ has rank 1 and so we deal only with the case when $\mathrm{rank}(\overline{\Pi}) > 1$.

<u>Prop. 8</u>: Suppose $\overline{\Pi}$ is not reduced and $r = \mathrm{rank}(\overline{\Pi}) > 1$. Label the roots of $\overline{\Pi}$ as follows: 

.

Put $\Pi_S = \Pi_{\gamma_1} \cup \dots \cup \Pi_{\gamma_{r-1}}$ and $\Pi_{S0} = \Pi_{0,\gamma_1} \cup \dots \cup \Pi_{0,\gamma_{r-1}}$. Then, exactly one of the following holds:

(a) $r=2$ and Π_{S0} is $*$-connected $\neq \phi$.

(b) (Π_S, Π_{S0}) is of the form (4).

Moreover, if $\Pi_{S0} \neq \phi$, we have $\Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r} \neq \phi$.

<u>Proof</u>: We prove the second statement first. Suppose $\Pi_{S0} \neq \phi$ and suppose for contradiction that $\Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r} = \phi$. Now, since $\Pi_{S0} \neq \phi$ and $\gamma_1, \dots, \gamma_{r-1}$ are conjugate under $\overline{W}$, it follows that $\Pi_{0,\gamma_{r-1}} \neq \phi$. Now, $\Pi_{0,\gamma_{r-1}} = \Pi_{0,\gamma_{r-1}} - \Pi_{0,\gamma_r}$ and hence, by lemma 3, $\Pi_{0,\gamma_{r-1}} \subseteq \Pi_{0,\gamma_{r-1}+\gamma_r}$. Operating on both sides of this inclusion with $\overline{w}_{\gamma_{r-1}}$, we obtain $\Pi_{0,\gamma_{r-1}} \subseteq \Pi_{0,\gamma_r}$. Thus, $\Pi_{0,\gamma_{r-1}} = \Pi_{0,\gamma_r} \cap \Pi_{0,\gamma_{r-1}} = \phi$ and we have a contradiction.

We now prove the first statement. Suppose (b) does not hold. By lemma 6, Π_{S0} is $*$-connected $\neq \phi$. But by the above, $\Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r} \neq \phi$ and, by lemma 6, $\Pi_{S0} = \Pi_{0,\gamma_1} = \dots = \Pi_{0,\gamma_{r-1}}$. Thus, $\Pi_{0,\gamma_1} \cap \Pi_{0,\gamma_r} \neq \phi$. Therefore, by lemma 2(ii), $\gamma_1 + \gamma_r \in \overline{\Sigma}$. Thus, $r=2$. q.e.d.

We have the following additional information about $\Pi_{o,2\gamma_r}$:

Prop. 9: Same assumptions and notation as in Prop. 8. Then, $\Pi_{o,2\gamma_r}$ is $*$-connected, $\Pi_{o,2\gamma_r} \subseteq \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$, and, if Π_{So} is $*$-connected, $\Pi_{o,2\gamma_r} = \phi$.

Proof: We apply lemma 5(iii) with $\gamma = \gamma_{r-1}$ and $\delta = 2\gamma_r$. Thus, $\Pi_{o,2\gamma_r} \subseteq \Pi_{o,\gamma_{r-1}}$ and, if $\Pi_{o,\gamma_{r-1}}$ is $*$-connected, $\Pi_{o,2\gamma_r} = \phi$. Thus, $\Pi_{o,2\gamma_r} \subseteq \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ and, since $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ is $*$-connected, $\Pi_{o,2\gamma_r}$ is $*$-connected. If Π_{So} is $*$-connected, $\Pi_{o,\gamma_{r-1}}$ is $*$-connected and hence $\Pi_{o,2\gamma_r} = \phi$. q.e.d.

Corollary: Same assumptions and notation as in Prop. 8. Then, $(\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r})^{\overline{w}_{\gamma_r}} = \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ and Π_{So} is stabilized by $\overline{W}$.

Proof: It suffices to prove the first statement since $\overline{w}_{\gamma_1}, \ldots, \overline{w}_{\gamma_{r-1}}$ stabilize Π_{So}, and $\Pi_{So} \cap \Pi_{o,\gamma_r} = \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$.

If $\Pi_{o,2\gamma_r} \neq \phi$, we have $\Pi_{o,2\gamma_r} = \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ by Prop. 9 (since both sets are $*$-components of $\Pi_{\bullet}$) and the statement is immediate. Suppose $\Pi_{o,2\gamma_r} = \phi$. Then, by Prop. 2.7, $\overline{w}_{\gamma_r} = \overline{w}_{2\gamma_r}$ fixes the elements of Π_o. q.e.d.

If (a) holds in Prop. 8, we also have:

Prop. 10: Same assumptions and notation as in Prop. 8. If (a) holds in Prop. 8 and Π is connected, then Π_{γ_1} is connected.

Proof: It suffices to show that σ_{γ_1} is a singleton. Suppose the contrary. Let μ_{Π} be the dominant root for Π. By lemma 2.4, $\overline{\mu_{\Pi}} = 2\gamma_1 + 2\gamma_2$. Hence, σ_{γ_1} is a doubleton $\{\alpha_1, \alpha_2\}$. Now, by

Prop. 9, $\Pi_{o,2\gamma_2} = \phi$. Thus, $\Pi_{o,2\gamma_1+2\gamma_2} = \Pi_{o,2\gamma_2}^{\overline{W}_{\gamma_1}} = \phi$. Therefore, $\mu_\Pi - \beta \notin \Sigma$ for $\beta \in \Pi_o$. But since $2\gamma_1 + \gamma_2 \notin \overline{\Sigma}$, $\mu_\Pi - \alpha \notin \Sigma$ for $\alpha \in \mathcal{O}_{\gamma_2}$. Thus, $\mu_\Pi - \alpha_1 \in \Sigma$ or $\mu_\Pi - \alpha_2 \in \Sigma$. But the *-action of $\mathcal{J}$ interchanges α_1 and α_2, and fixes μ_Π. Thus, $\mu_\Pi - \alpha_1 \in \Sigma$ and $\mu_\Pi - \alpha_2 \in \Sigma$. But $(\alpha_1, \alpha_2) = 0$, since $2\gamma_1 \notin \overline{\Sigma}$. Therefore, $\mu_\Pi - \alpha_1 - \alpha_2 \in \Sigma$. Since $\Pi_{S0} = \Pi_{o,\gamma_1} \neq \phi$, there exists $\beta_1 \in \Pi_{o,\gamma_1}$ such that $(\alpha_1, \beta_1) < 0$. Then,
$(\mu_\Pi - \alpha_1 - \alpha_2, \beta_1) = -(\alpha_1, \beta_1) - (\alpha_2, \beta_1) \geq -(\alpha_1, \beta_1) > 0$. Therefore, $\beta_1 \in \Pi_{o,2\gamma_2}$ and we have a contradiction. q.e.d.

<u>Examples</u>: We give here examples of the possible forms which indices may take under the assumptions of Prop. 8. We assume then that $\overline{\Pi}$ is not reduced and has rank $r > 1$. For simplicity, we assume Π is connected. As in the previous set of examples, one can show that the examples given here are the only possible ones. By Prop. 8, it follows that there are only two cases to be considered, namely:

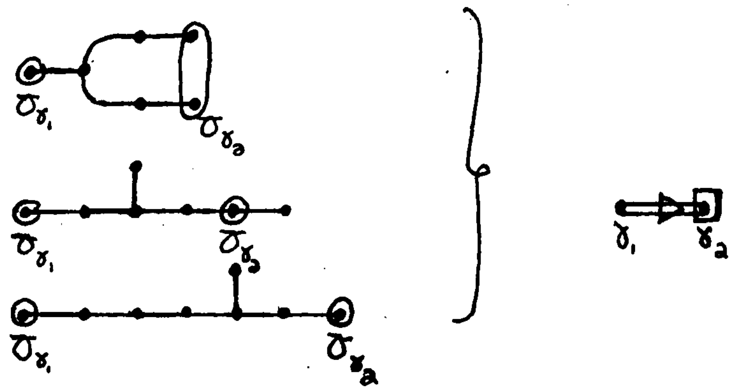
(I) $r=2$ and Π_{S0} is *-connected $\neq \phi$.

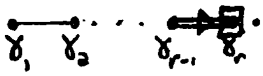
(II) (Π_S, Π_{S0}) is of the form (4).

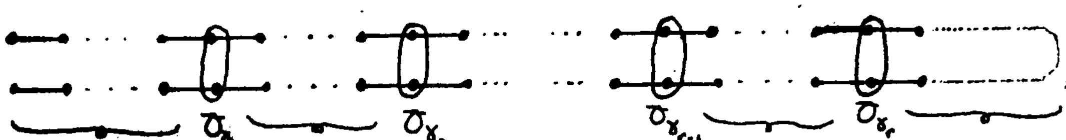
The examples then are:

Index	Restricted Diagram

(I)

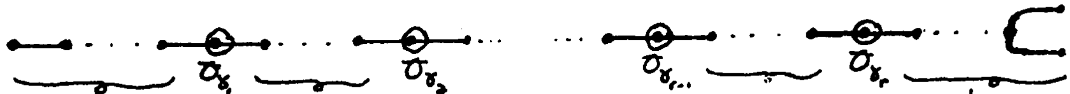


(II) The restricted diagram for all of the examples in (II) is 



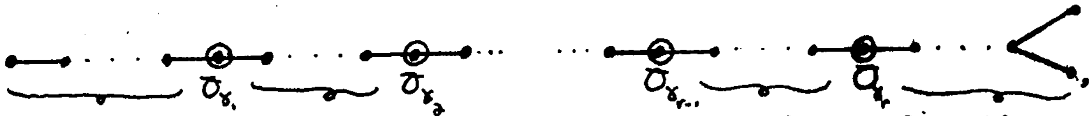
, with g,f ≥ 0.

g vertices g vertices g vertices f vertices



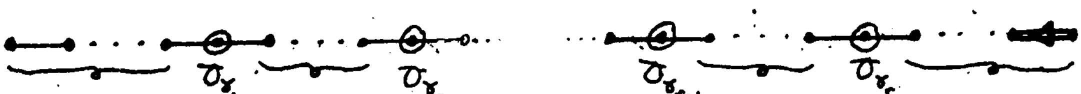
, with g ≥ 1, f ≥ 2, and non-trivial *-action.

g vertices g vertices g vertices f vertices



, with g ≥ 1, f ≥ 2, and trivial *-action.

g vertices g vertices g vertices f vertices



, with g ≥ 0, f ≥ 1.

g vertices g vertices g vertices f vertices

Chapter 4

The Anisotropic Kernel

In this chapter, we interpret the propositions of Chapter 3 to give us information about the anisotropic kernel. We will assume throughout the chapter that $\mathcal{L}$ is a simple Lie algebra over k and that $\mathfrak{I}$, $\mathfrak{h}$, K, $\mathfrak{S}$, Π, Π_0, etc. are as in Chapter 1. We also use the notation of Chapter 2.

We begin by defining some ideals of $[\mathcal{L}_0, \mathcal{L}_0]$ which will be of particular interest when $\overline{\overline{\Pi}}$ is not reduced.

Suppose $\gamma \in \overline{\overline{\Sigma}}$. If all of the elements of $\overline{\overline{\Sigma}}$ have the same length, we put $\mathfrak{n}_\gamma = (0)$. Otherwise, we define

$$\mathfrak{n}_\gamma = \bigcap_{\substack{\delta \in \overline{\overline{\Sigma}} \\ (\delta, \delta) \neq (\gamma, \gamma)}} \mathfrak{a}_\delta \cap [\mathcal{L}_0, \mathcal{L}_0].$$

Now, $\mathfrak{a}_\delta$ is an ideal of $\mathcal{L}_0$ and hence $\mathfrak{a}_\delta \cap [\mathcal{L}_0, \mathcal{L}_0]$ is an ideal of $[\mathcal{L}_0, \mathcal{L}_0]$, $\delta \in \overline{\overline{\Sigma}}$. Thus, $\mathfrak{n}_\gamma$ is an ideal of $[\mathcal{L}_0, \mathcal{L}_0]$. It should also be noted that $\mathfrak{n}_\gamma$ depends only on the length of γ. i.e. If $\delta \in \overline{\overline{\Sigma}}$ such that $(\delta, \delta) = (\gamma, \gamma)$, then $\mathfrak{n}_\gamma = \mathfrak{n}_\delta$.

<u>Prop. 1</u>: Let $\gamma \in \overline{\overline{\Pi}}$ and suppose $m\gamma$ is of maximal length in $\overline{\overline{\Sigma}}$ for some positive integer m. Then:

(i) If $\overline{\overline{\Pi}}$ is reduced, $\mathfrak{n}_\gamma = (0)$.

(ii) If $\overline{\overline{\Pi}}$ is not reduced and has rank 1, then $\overline{\overline{\Pi}} = \{\gamma\}$ and $[\mathcal{L}_0, \mathcal{L}_0] = \mathcal{L}_{0,a\gamma} \oplus \mathfrak{n}_\gamma$.

(iii) If $\overline{\overline{\Pi}}$ is not reduced and has rank > 1, then $\gamma = \gamma_r$ and $\mathcal{L}_{0,\gamma_r} = (\mathcal{L}_{0,\gamma_{r-1}} \cap \mathcal{L}_{0,\gamma_r}) \oplus \mathfrak{n}_{\gamma_r}$, where the roots of $\overline{\overline{\Pi}}$ are labelled as follows:  .

<u>Proof</u>: (i) Suppose $\overline{\overline{\Pi}}$ is reduced. We may assume $[\mathcal{L}_0, \mathcal{L}_0] \neq (0)$ and the elements of $\overline{\overline{\Pi}}$ take on more than one length. Hence, the assumptions of Prop. 3.5 are satisfied. We use the notation of that proposition. Then, $\gamma = \gamma_j$ for some $j \in \{s+1, \ldots, r\}$ and $\mathfrak{n}_\gamma \subseteq \bigcap_{i=1}^{s} \mathfrak{a}_{\gamma_i} \cap [\mathcal{L}_0, \mathcal{L}_0]$. But $\Pi_0 = \Pi_{0,\gamma_1} \cup \ldots \cup \Pi_{0,\gamma_s}$ and hence every simple summand of $[\mathcal{L}_0, \mathcal{L}_0]$ acts non-trivially on one of $\mathcal{L}_{\gamma_1}, \ldots, \mathcal{L}_{\gamma_s}$. Thus, $\bigcap_{i=1}^{s} \mathfrak{a}_{\gamma_i} \cap [\mathcal{L}_0, \mathcal{L}_0] = (0)$ and hence $\mathfrak{n}_\gamma = (0)$.

(ii) Suppose $\overline{\overline{\Pi}}$ is not reduced and has rank 1. Then, $\overline{\overline{\Pi}} = \{\gamma\}$ and $\mathfrak{n}_\gamma = \mathfrak{a}_{2\gamma} \cap \mathfrak{a}_{-2\gamma} \cap [\mathcal{L}_0, \mathcal{L}_0]$. Since $\mathfrak{a}_{2\gamma} = \mathfrak{a}_{-2\gamma}$ (by Prop. 2.2), we have $\mathfrak{n}_\gamma = \mathfrak{a}_{2\gamma} \cap [\mathcal{L}_0, \mathcal{L}_0]$. But $\mathcal{L}_0 = \mathfrak{a}_{2\gamma} \oplus [\mathcal{L}_{2\gamma}, \mathcal{L}_{-2\gamma}]$ (by Prop. 2.2) and both summands are ideals of $\mathcal{L}_0$. Thus,

$$[\mathcal{L}_0, \mathcal{L}_0] = (\mathfrak{a}_{2\gamma} \cap [\mathcal{L}_0, \mathcal{L}_0]) \oplus ([\mathcal{L}_{2\gamma}, \mathcal{L}_{-2\gamma}] \cap [\mathcal{L}_0, \mathcal{L}_0]) = \mathfrak{n}_\gamma \oplus \mathcal{L}_{0,2\gamma}.$$

(iii) Suppose $\overline{\overline{\Pi}}$ is not reduced and has rank > 1. We use the notation of Prop. 3.8. Then, $\gamma = \gamma_r$ and $\mathfrak{n}_\gamma \subseteq \bigcap_{i=1}^{r-1} \mathfrak{a}_{\gamma_i} \cap [\mathcal{L}_0, \mathcal{L}_0]$. Now, every simple summand of $[\mathcal{L}_0, \mathcal{L}_0]$ must act non-trivially on one of $\mathcal{L}_{\gamma_1}, \ldots, \mathcal{L}_{\gamma_r}$ (since $\Pi_0 = \Pi_{0,\gamma_1} \cup \ldots \cup \Pi_{0,\gamma_r}$). But every simple summand of $\mathfrak{n}_\gamma$ acts trivially on $\mathcal{L}_{\gamma_i}$, $i=1,\ldots,r-1$. Thus, $\mathfrak{n}_{\gamma_r} \subseteq \mathcal{L}_{0,\gamma_r}$ and $(\mathcal{L}_{0,\gamma_{r-1}} \cap \mathcal{L}_{0,\gamma_r}) \cap \mathfrak{n}_{\gamma_r} = \phi$ (by the Corollary to Prop. 2.2). To complete the proof of (iii), it suffices to show that if $\mathcal{L}_{0,1}$ is a simple summand of $\mathcal{L}_{0,\gamma_r}$ such that $\mathcal{L}_{0,1} \nsubseteq \mathfrak{n}_{\gamma_r}$, then $\mathcal{L}_{0,1} \subseteq \mathcal{L}_{0,\gamma_{r-1}} \cap \mathcal{L}_{0,\gamma_r}$. Let $\mathcal{L}_{0,1}$ be such a simple summand and let $\Pi_{0,1}$ be the corresponding $*$-component of Π_{0,γ_r}. Since $\mathcal{L}_{0,1} \nsubseteq \mathfrak{n}_{\gamma_r}$, $\mathcal{L}_{0,1}$ acts non-trivially on $\mathcal{L}_\delta$ for some $\delta \in \Sigma$ such that $(\delta, \delta) \neq (\gamma_r, \gamma_r)$. Then, $\Pi_{0,1} \subseteq \Pi_{0,\delta}$. But, since $(\delta, \delta) \neq (\gamma_r, \gamma_r)$, there exists $\overline{w} \in \overline{W}$ such that $\delta = (2\gamma_r)^{\overline{w}}$ or $\delta = (\gamma_{r-1})^{\overline{w}}$. Thus, $\Pi_{0,1} \subseteq \Pi_{0,2\gamma_r}^{\overline{w}}$ or $\Pi_{0,1} \subseteq \Pi_{0,\gamma_{r-1}}^{\overline{w}}$. But, by Prop. 3.9, $\Pi_{0,2\gamma_r} \subseteq \Pi_{0,\gamma_{r-1}}$. Thus,

$\Pi_{o,1} \subseteq \Pi_{o,\gamma_{r-1}}^{\overline{w}}$. But, by the Corollary to Prop. 3.9, $\Pi_{SO}^{\overline{w}} = \Pi_{SO}$. Thus, $\Pi_{o,1} \subseteq \Pi_{SO}$. But $\Pi_{o,1} \subseteq \Pi_{o,\gamma_r}$ and hence $\Pi_{o,1} \subseteq \Pi_{SO} \cap \Pi_{o,\gamma_r} = \Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$. Thus, $\mathcal{L}_{o,1} \subseteq \mathcal{L}_{o,\gamma_{r-1}} \cap \mathcal{L}_{o,\gamma_r}$. q.e.d.

In the remainder of this chapter, we denote by G the group of automorphisms φ of $\mathcal{L}$ such that $\mathfrak{J}^{\varphi} = \mathfrak{J}$ and φ is a product of elements of $\{\exp(\mathrm{ad}_{\mathcal{L}}(X)) : X \in \mathcal{L}$, $\mathrm{ad}_{\mathcal{L}}(X)$ nilpotent$\}$. The elements of G stabilize $\mathcal{L}_o$ and hence stabilize center($\mathcal{L}_o$) and permute the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$. As we have remarked in Chapter 1, for $\overline{w} \in \overline{W}$ there exists $\varphi \in G$ such that $\mathcal{L}_{\gamma}{}^{\varphi} = \mathcal{L}_{\gamma\overline{w}}$ for $\gamma \in \overline{\Sigma}$. We are interested then in how such an autormorphism φ permutes the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$.

Prop. 2: Let $\overline{w} \in \overline{W}$. Suppose $\varphi \in G$ such that $\mathcal{L}_{\gamma}{}^{\varphi} = \mathcal{L}_{\gamma\overline{w}}$ for $\gamma \in \overline{\Sigma}$. Let $\mathcal{L}_{o,1}, \ldots, \mathcal{L}_{o,t}$ be the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$. Let $\Pi_{o,1}, \ldots, \Pi_{o,t}$ (resp.) be the corresponding $*$-components of Π_o. For $i \in \{1,\ldots,t\}$, define $\tilde{i} \in \{1,\ldots,t\}$ by $\mathcal{L}_{o,i}{}^{\varphi} = \mathcal{L}_{o,\tilde{i}}$. Suppose $j \in \{1,\ldots,t\}$. Then, $\Pi_{o,j}^{\overline{w}} = \Pi_{o,\tilde{j}}$. Moreover, if $(\mathfrak{h} \cap \mathcal{L}_{o,j})^{\varphi} = \mathfrak{h} \cap \mathcal{L}_{o,\tilde{j}}$ and $(\varphi | \mathcal{L}_{o,j})^*$ maps $\Pi_{o,j}$ onto $\Pi_{o,\tilde{j}}$, then $(\varphi | \mathcal{L}_{o,j})^* = \overline{w} | \Pi_{o,j}$.

Proof: Suppose $i \in \{1,\ldots,t\}$. Let G_i be the group of automorphisms of $(\mathcal{L}_{o,i})_K$ which are products of elements of $\{\exp(\mathrm{ad}_{(\mathcal{L}_{o,i})_K}(X)) : X \in (\mathcal{L}_{o,i})_K$, $\mathrm{ad}_{(\mathcal{L}_{o,i})_K}(X)$ nilpotent$\}$. Now, if $X \in (\mathcal{L}_{o,i})_K$ and $\mathrm{ad}_{(\mathcal{L}_{o,i})_K}(X)$ is nilpotent, then $\rho(X)$ is nilpotent for every finite dimensional representation ρ of $(\mathcal{L}_{o,i})_K$, and in particular $\mathrm{ad}_{\mathcal{L}_K}(X)$ is nilpotent. Thus, for such X, $\exp(\mathrm{ad}_{(\mathcal{L}_{o,i})_K}(X))$ extends to the automorphism $\exp(\mathrm{ad}_{\mathcal{L}_K}(X))$ of $\mathcal{L}_K$. Hence, we may regard the elements of G_i as automorphisms of $\mathcal{L}_K$ which fix the elements of $(\text{center}(\mathcal{L}_o))_K$ and fix the elements of the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$ other than $\mathcal{L}_{o,i}$.

Suppose $i \in \{1,\dots,t\}$. Then, there exists $\psi_i \in G_{\tilde{i}}$ such that $(\mathcal{L}_{o,i} \cap \mathfrak{h})_K^{\varphi\psi_i} = (\mathcal{L}_{o,\tilde{i}} \cap \mathfrak{h})_K$ and $(\varphi\psi_i | (\mathcal{L}_{o,i})_K)^*$ maps $\Pi_{o,i}$ onto $\Pi_{o,\tilde{i}}$. Moreover, if $(\mathcal{L}_{o,i} \cap \mathfrak{h})^{\varphi} = \mathcal{L}_{o,\tilde{i}} \cap \mathfrak{h}$ and $(\varphi | \mathcal{L}_{o,i})^*$ maps $\Pi_{o,i}$ onto $\Pi_{o,\tilde{i}}$, then we may choose $\psi_i = 1$.

Put $\psi = \psi_1 \dots \psi_t$. Then, ψ fixes the elements of $(\text{center}(\mathcal{L}_o))_K$ and stabilizes $(\mathcal{L}_{o,i})_K$, $i=1,\dots,t$. Moreover, $([\mathcal{L}_o, \mathcal{L}_o] \cap \mathfrak{h})_K^{\varphi\psi} = ([\mathcal{L}_o, \mathcal{L}_o] \cap \mathfrak{h})_K$ and $((\varphi\psi) | [\mathcal{L}_o, \mathcal{L}_o]_K)^*$ maps Π_o onto Π_o. But φ stabilizes center$(\mathcal{L}_o)$. Thus, $\mathfrak{h}_K^{\varphi\psi} = \mathfrak{h}_K$, $\Pi_o^{(\varphi\psi)^*} = \Pi_o$, and $(\varphi\psi)^* | \mathfrak{X}_\Sigma = \bar{w} | \mathfrak{X}_\Sigma$. But $(\varphi\psi)^* \in \mathcal{W}$ and hence $(\varphi\psi)^* = \bar{w}$ (regarding $\bar{w}$ as an element of $\mathcal{W}$).

Suppose $j \in \{1,\dots,t\}$. Then, $(\mathcal{L}_{o,j})_K^{\varphi\psi} = (\mathcal{L}_{o,\tilde{j}})_K$ and hence, since $(\varphi\psi)^* = \bar{w}$, $\Pi_{o,j}^{\bar{w}} = \Pi_{o,\tilde{j}}$. Suppose $(\mathfrak{h} \cap \mathcal{L}_{o,j})^{\varphi} = \mathfrak{h} \cap \mathcal{L}_{o,\tilde{j}}$ and $(\varphi | \mathcal{L}_{o,j})^*$ maps $\Pi_{o,j}$ onto $\Pi_{o,\tilde{j}}$. Then, $\psi_j = 1$. Hence, $(\varphi | \mathcal{L}_{o,j})^* = (\varphi\psi_j | \mathcal{L}_{o,j})^* = (\varphi\psi | \mathcal{L}_{o,j})^* = \bar{w} | \Pi_{o,j}$. q.e.d.

In view of Prop. 2, we have the following interpretation of Prop. 3.2:

<u>Theorem 1</u>: Suppose $\overline{\overline{\Pi}}$ is reduced. Then, the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$ are conjugate under G. i.e. If $\mathcal{L}_{o,1}$ and $\mathcal{L}_{o,2}$ are simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$, then there exists $\varphi \in G$ such that $\mathcal{L}_{o,2}^{\varphi} = \mathcal{L}_{o,1}$.

<u>Proof</u>: Let $\Pi_{o,1}$ and $\Pi_{o,2}$ be the $*$-components of Π_o corresponding to $\mathcal{L}_{o,1}$ and $\mathcal{L}_{o,2}$ respectively. By Prop. 3.2, we may choose $\bar{w} \in \overline{\mathcal{W}}$ such that $\Pi_{o,2}^{\bar{w}} = \Pi_{o,1}$. Choose $\varphi \in G$ such that $\mathcal{L}_\gamma^{\varphi} = \mathcal{L}_{\gamma^{\bar{w}}}$ for $\gamma \in \overline{\overline{\Sigma}}$.

By Prop. 2, the $*$-component of Π_0 corresponding to $\mathcal{L}_{0,a}^{\varphi}$ is $\Pi_{0,a}^{W} = \Pi_{0,1}$. Therefore, $\mathcal{L}_{0,a}^{\varphi} = \mathcal{L}_{0,1}$. q.e.d.

We now prove three theorems about the structure of $[\mathcal{L}_0, \mathcal{L}_0]$ and its action on certain restricted root spaces. If $[\mathcal{L}_0, \mathcal{L}_0] \neq (0)$, we cannot have Π of type D or E and hence we are interested only in the following three cases:

(I) $\overline{\Pi}$ is of type A_r $(r \geq 1)$.

(II) $\overline{\Pi}$ is of type B_r $(r \geq 3)$, C_r $(r \geq 2)$, F_4, or G_2.

(III) $\overline{\Pi}$ is not reduced.

The three theorems will treat these three cases separately and are "rational" interpretations of Prop. 3.3, 3.5, 3.6, 3.8 and 3.9.

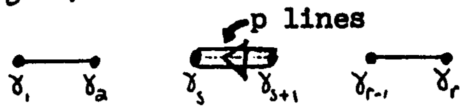
<u>Theorem 2</u>: Suppose $\overline{\Pi}$ is of type A_r $(r \geq 1)$ and $[\mathcal{L}_0, \mathcal{L}_0] \neq (0)$. Then, $[\mathcal{L}_0, \mathcal{L}_0]$ acts non-trivially on $\mathcal{L}_\gamma$ for all $\gamma \in \overline{\Sigma}$ and exactly one of the following holds:

(a) $[\mathcal{L}_0, \mathcal{L}_0]$ is simple and r=1 or 2.

(b) $[\mathcal{L}_0, \mathcal{L}_0]$ is the direct sum of r+1 isomorphic simple algebras.

<u>Proof</u>: Now, $\Pi_{0,\gamma} \neq \phi$ for some $\gamma \in \overline{\Sigma}$. But all elements of $\overline{\Sigma}$ are conjugate under $\overline{W}$. Therefore, $\Pi_{0,\gamma} \neq \phi$ for all $\gamma \in \overline{\Sigma}$. By the Corollary to Prop. 2.2, the first statement follows. By Prop. 3.3, either (Π, Π_0) is of the form (4) or Π_0 is $*$-connected and r=1 or 2. In the latter case, (a) holds. If (Π, Π_0) is of the form (4), $[\mathcal{L}_0, \mathcal{L}_0]$ has r+1 simple summands which (by Thm. 1) are isomorphic. q.e.d.

<u>Theorem 3</u>: Suppose $\overline{\overline{\Pi}}$ is of type B_r $(r \geq 3)$, C_r $(r \geq 2)$, F_4, or G_2. Suppose $[\mathcal{L}_o, \mathcal{L}_o] \neq (0)$. Label the roots of $\overline{\overline{\Pi}}$ as follows:



where p=2 or 3 and $1 \leq s < r$. Then, exactly one of the following holds:

(a) $[\mathcal{L}_o, \mathcal{L}_o]$ is simple and s=1 or s=2.

(b) $[\mathcal{L}_o, \mathcal{L}_o]$ is the direct sum of s+1 isomorphic simple algebras.

Moreover, the adjoint action of $[\mathcal{L}_o, \mathcal{L}_o]$ on $\mathcal{L}_{\gamma_{s+1}}, \ldots, \mathcal{L}_{\gamma_r}$ is trivial unless (b) holds and $\overline{\overline{\Pi}}$ is of type C_r $(r \geq 2)$, in which case at most one of the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$ acts non-trivially on $\mathcal{L}_{\gamma_r}$.

<u>Proof</u>: We use the notation of Prop. 3.5. Since (by Thm. 1) the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$ are isomorphic, the first statement of the theorem follows immediately from Prop. 3.5.

Suppose now that $[\mathcal{L}_o, \mathcal{L}_o]$ acts non-trivially on $\mathcal{L}_{\gamma_j}$ for some $j \in \{s+1, \ldots, r\}$. Let $\mathcal{L}_{o,1}, \ldots, \mathcal{L}_{o,t}$ be the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$ which act non-trivially on $\mathcal{L}_{\gamma_j}$. Let $\Pi_{o,1}, \ldots, \Pi_{o,t}$ (resp.) be the corresponding *-components of Π_o. By the Corollary to Prop. 2.2, we have $\Pi_{o,1}, \ldots, \Pi_{o,t} \subseteq \Pi_{o,\gamma_j}$. In particular, $\Pi_{o,\gamma_j} \neq \phi$. By Prop. 3.6, $\overline{\overline{\Pi}}$ is of type C_r $(r \geq 2)$, (Π_S, Π_o) is of the form (4), and Π_{o,γ_r} is one of the two *-components of $\Pi_{o,\gamma_{r-1}}$. Hence, $r = j = s+1$ and $t = 1$. Since (Π_S, Π_o) is of the form (4), (b) holds. q.e.d.

In considering the final case ($\overline{\Pi}$ not reduced), we assume in this chapter (as in the last) that $\overline{\Pi}$ has rank $r > 1$.

<u>Theorem 4</u>: Suppose $\overline{\Pi}$ is not reduced and has rank $r > 1$. Label the roots of $\overline{\Pi}$ as follows:  . Put $\mathcal{M}_o = \mathcal{L}_{o,\gamma_{r-1}} \cap \mathcal{L}_{o,\gamma_r}$. Then, $\mathcal{M}_o$ is simple or (0), $\mathcal{L}_{o,\gamma_r} = \mathcal{M}_o \oplus \mathcal{N}_{\gamma_r}$, and exactly one of the following holds:

(a) $r=2$, $\mathcal{M}_o \neq (0)$, and $[\mathcal{L}_o, \mathcal{L}_o] = \mathcal{L}_{o,\gamma_r}$.

(b) $[\mathcal{L}_o, \mathcal{L}_o]$ is the direct sum of $\mathcal{L}_{o,\gamma_r}$ and $r-1$ ideals of $[\mathcal{L}_o, \mathcal{L}_o]$ isomorphic to $\mathcal{M}_o$.

<u>Proof</u>: Put $\Pi_S = \Pi_{\gamma_1} \cup \ldots \cup \Pi_{\gamma_{r-1}}$ and $\Pi_{SO} = \Pi_{o,\gamma_1} \cup \ldots \cup \Pi_{o,\gamma_{r-1}}$. By Prop. 1 (iii), $\mathcal{L}_{o,\gamma_r} = \mathcal{M}_o \oplus \mathcal{N}_{\gamma_r}$. Since $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ is *-connected, $\mathcal{M}_o$ is simple or (0).

By Prop. 3.8, either $r = 2$ and Π_{SO} is *-connected $\neq \phi$ or (Π_S, Π_{SO}) is of the form (4).

Suppose $r = 2$ and Π_{SO} is *-connected $\neq \phi$. By Prop. 3.8, $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} \neq \phi$. Therefore, $\mathcal{M}_o \neq (0)$. Since Π_{SO} is *-connected and $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} \subseteq \Pi_{SO}$, $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} = \Pi_{SO}$. But $\Pi_o = \Pi_{SO} \cup \Pi_{o,\gamma_r}$. Therefore, $\Pi_o = \Pi_{o,\gamma_r}$. Therefore, $[\mathcal{L}_o, \mathcal{L}_o] = \mathcal{L}_{o,\gamma_r}$.

Suppose (Π_S, Π_{SO}) is of the form (4). If $\Pi_{SO} = \phi$, then $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} = \phi$ and $\Pi_o = \Pi_{o,\gamma_r}$. Therefore, if $\Pi_{SO} = \phi$, $\mathcal{M}_o = (0)$ and $[\mathcal{L}_o, \mathcal{L}_o] = \mathcal{L}_{o,\gamma_r}$. Thus, (b) holds in this case. Suppose $\Pi_{SO} \neq \phi$. By Prop. 3.8, $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} \neq \phi$. Since $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r}$ is *-connected and (Π_S, Π_{SO}) is of the form (4), it is immediate that (b) holds provided we know that the simple summands corresponding to *-components of Π_{S_0} are isomorphic. This follows from Prop. 2 if the *-components of Π_{S_0} are conjugate under $\overline{W}$. But W_{γ_i} interchanges the two *-components of Π_{0,γ_i}, i=1,...,r-1, and we are done. q.e.d.

Under the assumptions of Theorem 4, we may interpret Prop. 3.9 to obtain information about the action of $[\mathcal{L}_0, \mathcal{L}_0]$ on $\mathcal{L}_{2\gamma_r}$. In particular, if (a) holds, this action is trivial. On the other hand, if (b) holds, all simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ which act non-trivially on $\mathcal{L}_{2\gamma_r}$ are contained in $\mathcal{M}_0$ (and hence there is at most one such summand). We do not need this information and so we omit the verification.

Chapter 5

A Rational Isomorphism Theorem

In this chapter, we apply the propositions of Chapter 3 to prove a rational isomorphism theorem for central simple algebras over k. Throughout the chapter, $\mathcal{L}$ is a simple algebra over k with maximal split toral subalgebra $\mathfrak{J}$. $\overline{\Sigma}$ is the restricted root system for $(\mathcal{L}, \mathfrak{J})$ and $\overline{\Pi}$ is a fundamental system for $\overline{\Sigma}$. In the isomorphism theorem and the preparatory lemmas, $\mathcal{L}'$ is a second simple algebra over k and $\mathfrak{J}'$, $\overline{\Sigma}'$, and $\overline{\Pi}'$ are chosen as above. We assume K/k is a Galois splitting extension for both $(\mathcal{L}, \mathfrak{J})$ and $(\mathcal{L}', \mathfrak{J}')$ i.e. K/k splits $\mathfrak{h}$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathcal{L}$ containing $\mathfrak{J}$ and K/k splits $\mathfrak{h}'$ for some Cartan subalgebra $\mathfrak{h}'$ of $\mathcal{L}'$ containing $\mathfrak{J}'$. We put $\mathcal{G} = \mathrm{Gal}(K/k)$.

Our isomorphism theorem is stated without reference to a Cartan subalgebra of $\mathcal{L}$ or to the data Π_o, Σ_o, Π, and Σ discussed in earlier chapters. Thus, in the proof of this theorem, we are free to choose for our convenience any Cartan subalgebra $\mathfrak{h}$ which contains $\mathfrak{J}$ and is split by K. Once such an $\mathfrak{h}$ is chosen, we are free to choose any fundamental system Π_o for the roots of $([\mathcal{L}_o, \mathcal{L}_o]_K, (\mathfrak{h} \cap [\mathcal{L}_o, \mathcal{L}_o])_K)$. Σ_o, Π, and Σ are then uniquely determined. For simplicity, we simply say that we have <u>chosen $\mathfrak{h}$ and Π_o</u>.

Suppose for the moment that $\mathcal{L}_{o,1}, \ldots, \mathcal{L}_{o,t}$ are the simple summands of $[\mathcal{L}_o, \mathcal{L}_o]$. Suppose that $\mathfrak{h}_{o,1}, \ldots\ldots, \mathfrak{h}_{o,t}$ are Cartan subalgebras of $\mathcal{L}_{o,1}, \ldots, \mathcal{L}_{o,t}$ respectively which are split by K. Then, there is a unique choice of $\mathfrak{h}$ such that $\mathfrak{h} \cap \mathcal{L}_{o,i} = \mathfrak{h}_{o,i}$, i=1,...t, namely center$(\mathcal{L}_o) \oplus \mathfrak{h}_{o,1} \oplus \cdots \oplus \mathfrak{h}_{o,t}$. We say that $\mathfrak{h}$ is <u>the Cartan subalgebra determined by $\mathfrak{h}_{o,1},\ldots,\mathfrak{h}_{o,t}$</u>. Conversely, of course, if $\mathfrak{h}$ is chosen, $\mathfrak{h} \cap \mathcal{L}_{o,i}$ is a Cartan subalgebra of $\mathcal{L}_{o,i}$ split by K.

Suppose now that $\mathcal{L}_{o,1},\ldots,\mathcal{L}_{o,t}$ are as above and $\mathfrak{h}$ is chosen. Suppose that $\Pi_{o,i}$ is a fundamental system for the roots of $((\mathcal{L}_{o,i})_K, (\mathcal{L}_{o,i} \cap \mathfrak{h})_K)$, i=1,...,t. Then, there exists a unique choice of Π_o such that $\Pi_{o,i}$ is the *-component of Π_o corresponding to $\mathcal{L}_{o,i}$, i=1,...,t. We say that <u>Π_o is determined by $\Pi_{o,1},\ldots,\Pi_{o,t}$</u>.

Our method of proof of the rational isomorphism theorem involves applications of Thm. 1.2. Thus, we are interested in the problem of extending certain *-isomorphisms. We now discuss this problem in certain particular situations.

Suppose that T is a subset of $\overline{\Pi}$ of type A_q $(q \geq 1)$. Label the roots of T as follows: $\underset{\gamma_1}{\bullet}\!\!-\!\!\underset{\gamma_2}{\bullet}\ \cdots\ \underset{\gamma_{q-1}}{\bullet}\!\!-\!\!\underset{\gamma_q}{\bullet}$. Define

$$\mathcal{L}_{TO} = \sum_{i=1}^{q} \mathcal{L}_{o,\gamma_i} .$$

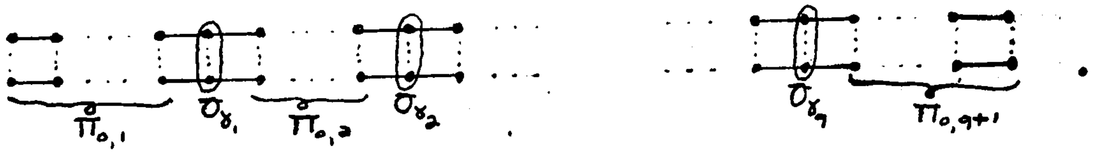
For any choice of $\mathfrak{h}$ and Π_o, define Π_T and Π_{TO} as in lemma 3.6. Let $\overline{W}_T$ be the subgroup of $\overline{W}$ generated by $\{\overline{w}_\gamma\}_{\gamma \in T}$.

We assume to begin with that <u>$\mathcal{L}_{TO}$ is non-zero and not simple</u>. By lemma 3.6, it follows that for any $\mathfrak{h}$ and Π_o, (Π_T, Π_{TO}) is of the form (4). Thus, $\mathcal{L}_{TO}$ is the sum of q+1 simple summands which we may label $\mathcal{L}_{o,1},\ldots,\mathcal{L}_{o,q+1}$ in such a way that

$$\mathcal{L}_{o,\gamma_j} = \mathcal{L}_{o,j} \oplus \mathcal{L}_{o,j+1} , \tag{7}$$

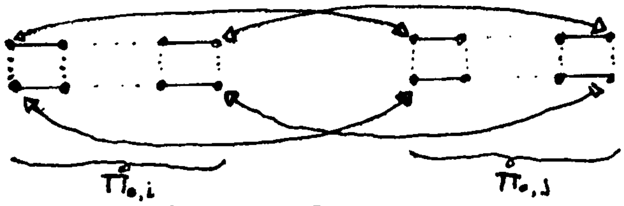
j=1,...,q. (If q=1, this labelling may be accomplished in two ways but at any rate we assume such a labelling is given and fixed.)

For any choice of $\mathfrak{h}$ and Π_o, let $\Pi_{o,i}$ be the $*$-component of Π_o corresponding to $\mathcal{L}_{o,i}$, $i=1,\dots,q+1$, in which case we have the following diagram for (Π_T, Π_{To}):



. (8)

Now, $\overline{W}_T$ is isomorphic to the full permutation group of $\{1,\dots,q+1\}$ in such a way that if $\chi \longrightarrow \overline{w}_\chi$ denotes the isomorphism, then $\overline{w}_{(j,j+1)} = \overline{w}_{\gamma_j}$, $j=1,\dots,q$. But, for any $\mathfrak{h}$ and Π_o, $\overline{w}_{\gamma_j}$ fixes the elements of $\Pi_{To} - \Pi_{o,\gamma_j}$ and $\overline{w}_{\gamma_j} | \Pi_{o,\gamma_j} = (i_{\Pi_{\gamma_j}} | \Pi_{o,\gamma_j}) \cdot i_{\Pi_{o,\gamma_j}}$, $j=1,\dots,q$. Thus, for any $\mathfrak{h}$ and Π_o and for $1 \le i < j \le q+1$, $\overline{w}_{(i,j)}$ fixes the elements of $\Pi_{To} - (\Pi_{o,i} \cup \Pi_{o,j})$ and $\overline{w}_{(i,j)} | \Pi_{o,i} \cup \Pi_{o,j}$ is the map:



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where in this diagram $\Pi_{o,i}$ and $\Pi_{o,j}$ are isolated from (8). Suppose $\overline{w}_-$ is the unique element of $\overline{W}_T$ such that $T^{\overline{w}_-} = -T$. It is then easy to see that $\overline{w}_-$ is the product of the $[\frac{q+1}{2}]$ elements $\overline{w}_{(1,q+1)}, \dots, \overline{w}_{([\frac{q+1}{2}],\, q+2-[\frac{q+1}{2}])}$ of $\overline{W}_T$ which commute with one another, where $[\frac{q+1}{2}]$ is the greatest integer in $\frac{q+1}{2}$. Thus, $\overline{w}_- | (\Pi_{o,i} \cup \Pi_{o,q+2-i}) = \overline{w}_{(i,q+2-i)} | (\Pi_{o,i} \cup \Pi_{o,q+2-i})$, $i=1,\dots,q+1$.

Suppose now that T' is a subset of $\overline{\Pi}'$ of type A_q $(q \ge 1)$ and $\gamma_i \longrightarrow \gamma_i'$ is an isomorphism of T onto T'. Define $\mathcal{L}'_{T'o}$ as above and assume $\mathcal{L}'_{T'o}$ is non-zero and not simple. The above discussion holds for T' and we assume we have all the above notation (with primes added) for T'. We refer to the diagram for T' corresponding to (8) as $(8)'$. Now, for any $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' and any $1 \leq i, i' \leq q+1$, we say a *-isomorphism $\Pi_{0,i} \xrightarrow{f_0} \Pi'_{0,i'}$ is <u>extendable</u> if f_0 takes the leftmost *-orbit of $\Pi_{0,i}$ in (8) onto the leftmost *-orbit of $\Pi'_{0,i'}$ in the diagram $(8)'$. We then have:

<u>Lemma 1</u>: Suppose $1 \leq i, i' \leq q+1$ and suppose we have an isomorphism $\mathcal{L}_{0,i} \xrightarrow{\varphi_0} \mathcal{L}'_{0,i'}$. Then, we may choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' so that $(\mathfrak{h} \cap \mathcal{L}_{0,i})^{\varphi_0} = \mathfrak{h}' \cap \mathcal{L}'_{0,i'}$, $\Pi_{0,i}^{\varphi_0^*} = \Pi'_{0,i'}$ and the following condition holds: Suppose $\overline{w} \in \overline{W}$ such that $\overline{w}$ stabilizes Π_{T0} and fixes the elements of $\Pi_{T0} - \Pi_{0,i}$. Put $\overline{w}_e = \overline{w}_{(1,i)}\ \overline{w}\ \overline{w}_{(i,1)} \cdots \overline{w}_{(q+1,i)}\ \overline{w}\ \overline{w}_{(i,q+1)}$. Put
$$\overline{w}_1 = \begin{cases} \overline{w}_e\ \overline{w}_{(i,i')} & \text{if } (\overline{w} \mid \Pi_{0,i}) \circ \varphi_0^* \text{ is extendable} \\ \overline{w}_{-}\ \overline{w}_e\ \overline{w}_{(i,i')} & \text{otherwise}. \end{cases}$$
Then, there exists an isomorphism $(\mathcal{L}_{T0}, \mathfrak{h} \cap \mathcal{L}_{T0}) \xrightarrow{\psi_0} (\mathcal{L}'_{T'0}, \mathfrak{h}' \cap \mathcal{L}'_{T'0})$ and a *-isomorphism $(\Pi_T, \Pi_{T0}) \xrightarrow{f} (\Pi'_{T'}, \Pi'_{T'0})$ such that $\psi_0 \mid \mathcal{L}_{0,i} = \varphi_0$, $\Pi_{T0}^{\psi_0^*} = \Pi'_{T'0}$, $f \mid \Pi_{T0} = (\overline{w}_1 \mid \Pi_{T0}) \circ \psi_0^*$, and, if $(\overline{w} \mid \Pi_{0,i}) \circ \varphi_0^*$ is extendable, $\mathcal{O}_{\gamma_j}^{\ f} = \mathcal{O}_{\gamma'_j}$ for $j = 1, \ldots, q$.

<u>Proof</u>: Let $\mathfrak{h}_{0,i}$ be some Cartan subalgebra of $\mathcal{L}_{0,i}$ split by K and let $\Pi_{0,i}$ be some fundamental system for the roots of $((\mathcal{L}_{0,i})_K, (\mathfrak{h}_{0,i})_K)$. For $1 \leq j \leq q+1$, $j \neq i$, choose $\varphi_j \in G$ such that $\mathcal{L}_\gamma^{\varphi_j} = \mathcal{L}_\gamma \overline{w}_{(i,j)}$. It follows from the first part of Prop. 4.2, that $\mathcal{L}_{0,i}^{\varphi_j} = \mathcal{L}_{0,j}$, $1 \leq j \leq q+1$, $j \neq i$. Then, for $1 \leq j \leq q+1$, $j \neq i$, put $\varphi_{0,j} = \varphi_j \mid \mathcal{L}_{0,i}$, $\mathfrak{h}_{0,j} = \mathfrak{h}_{0,i}^{\varphi_{0,j}}$, and $\Pi_{0,j} = \Pi_{0,i}^{(\varphi_{0,j})^*}$. Let $\mathfrak{h}$ be the Cartan subalgebra determined by $\mathfrak{h}_{0,1}, \ldots, \mathfrak{h}_{0,q+1}$ (and any Cartan subalgebras split by K for the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ not contained in $\mathcal{L}_{T0}$). Let Π_0 be the fundamental system for the roots of $([\mathcal{L}_0, \mathcal{L}_0]_K, (\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])_K)$ determined by $\Pi_{0,1}, \ldots, \Pi_{0,q+1}$ (and any choice of a fundamental system

for the simple summands of $[\mathcal{L}_o,\mathcal{L}_o]$ not contained in $\mathcal{L}_{T0}$). Then, by the second part of Prop. 4.2, we have $\varphi_{o,j}^* = \bar{w}_{(i,j)}|\Pi_{o,i}$, $1 \le j \le q+1$, $j \ne i$.

Let $\mathfrak{h}'_{o,i'} = \mathfrak{h}_{o,i}^{\varphi_o}$ and let $\Pi'_{o,i'} = \Pi_{o,i}^{\varphi_o^*}$. Choose $\mathfrak{h}'$, Π'_o, and $\mathcal{L}'_{o,i'} \xrightarrow{\varphi'_{o,j'}} \mathcal{L}'_{o,j'}$ for $1 \le j' \le q+1$, $j' \ne i'$, as above.

Define $(\mathcal{L}_{T0}, \mathfrak{h} \cap \mathcal{L}_{T0}) \xrightarrow{\psi_o} (\mathcal{L}'_{T'0}, \mathfrak{h}' \cap \mathcal{L}'_{T'0})$ as follows:

$$\psi_o|(\mathcal{L}_{o,j}) = \begin{cases} \varphi_{o,j}^{-1} \circ \varphi_o \circ \varphi'_{o,j} & \text{for } 1 \le j \le q+1,\ j \ne i, i' \\ \varphi_o & \text{for } j = i \\ \varphi_{o,i'}^{-1} \circ \varphi_o \circ \varphi'_{o,i} & \text{for } j = i',\ \text{if } i \ne i'. \end{cases}$$

Then, $\Pi_{T0}^{\psi_o^*} = \Pi'_{T'0}$ and

$$\psi_o^*|\Pi_{o,j} = \begin{cases} (\bar{w}_{(j,i)}|\Pi_{o,j}) \circ \varphi_o^* \circ (\bar{w}'_{(i',j)}|\Pi'_{o,i'}) & \text{for } 1 \le j \le q+1,\ j \ne i, i' \\ \varphi_o^* & \text{for } j = i \\ (\bar{w}_{(i',i)}|\Pi_{o,i'}) \circ \varphi_o^* \circ (\bar{w}'_{(i',i)}|\Pi'_{o,i'}) & \text{for } j = i',\ \text{if } i \ne i'. \end{cases}$$

Hence, $(\bar{w}_{(i,i')}\psi_o^*)|\Pi_{o,j} = (\bar{w}_{(j,i)}|\Pi_{o,j}) \circ \varphi_o^* \circ (\bar{w}'_{(i',j)}|\Pi'_{o,i'})$ for $1 \le j \le q+1$. Now, $\bar{w}_e|\Pi_{o,j} = (\bar{w}_{(j,i)}|\Pi_{o,j}) \circ (\bar{w}|\Pi_{o,i}) \circ (\bar{w}_{(i,j)}|\Pi_{o,i})$ for $1 \le j \le q+1$. Thus, $(\bar{w}_e \bar{w}_{(i,i')}\psi_o^*)|\Pi_{o,j} = (\bar{w}_{(j,i)}|\Pi_{o,j}) \circ (\bar{w}|\Pi_{o,i}) \circ \varphi_o^* \circ (\bar{w}'_{(i',j)}|\Pi'_{o,i'})$ for $1 \le j \le q+1$. If $(\bar{w}|\Pi_{o,i}) \circ \varphi_o^*$ is extendable, we have $\bar{w}_1 = \bar{w}_e \bar{w}_{(i,i')}$ and it is then clear that $(\bar{w}_1|\Pi_{T0}) \circ \psi_o^*$ extends to a *-isomorphism $(\Pi_T, \Pi_{T0}) \xrightarrow{f} (\Pi'_{T'}, \Pi'_{T'0})$ such that $\mho_{\gamma_j}^f = \mho_{\gamma'_j}$ for $1 \le j \le q$. Suppose $(\bar{w}|\Pi_{o,i}) \circ \varphi_o^*$ is not extendable. But then

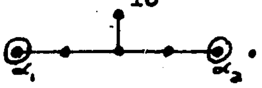
$$\begin{aligned}(\bar{w}_1|\Pi_{T0}) \circ \psi_o^*|\Pi_{o,j} &= (\bar{w}_{(j,q+2-j)}|\Pi_{o,j}) \circ ((\bar{w}_e \bar{w}_{(i,i')}\psi_o^*)|\Pi_{o,q+2-j}) \\ &= (\bar{w}_{(j,i)}|\Pi_{o,j}) \circ (\bar{w}|\Pi_{o,i}) \circ \varphi_o^* \circ (\bar{w}'_{(i',q+2-j)}|\Pi'_{o,i'}),\end{aligned}$$

$j = 1, \ldots, q+1$. It is then clear that $(\bar{w}_1|\Pi_{T0}) \circ \psi_o^*$ extends to a *-isomorphism $(\Pi_T, \Pi_{T0}) \xrightarrow{f} (\Pi'_{T'}, \Pi'_{T'0})$ such that $\mho_{\gamma_j}^f = \mho_{\gamma_{q+1-j}}$, $1 \le j \le q$. q.e.d.

It will be convenient in subsequent discussion to fix some choice of $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0'.

We now suppose that $\underline{\mathcal{L}_{T0}}$ __is simple__. By lemma 3.6, T contains 1 or 2 elements. We deal here only with the case when $T = \{\gamma_1, \gamma_2\}$ is a doubleton. Suppose that T' is a subset of $\overline{\Pi}'$ of type A_2, $\gamma_i \longrightarrow \gamma_i'$ is an isomorphism of T onto T', and $\underline{\mathcal{L}'_{T'0}}$ __is simple.__

__Lemma 2__: Suppose Π_T and $\Pi'_{T'}$ are connected. Suppose we have a $*$-isomorphism $\Pi_{T0} \xrightarrow{f_0} \Pi'_{T'0}$. Then, there exists $\overline{w} \in \overline{W}_T$ and a $*$-isomorphism $(\Pi_T, \Pi_{T0}) \xrightarrow{f} (\Pi'_{T'}, \Pi'_{T'0})$ such that $f \mid \Pi_{T0} = (\overline{w} \mid \Pi_{T0}) \cdot f_0$ and $\mathcal{O}_{\gamma_j}^{f} = \mathcal{O}_{\gamma_j'}$ for $j=1,2$.

__Proof__: By lemma 2.4, $\mu_{\overline{\Pi_T}} = \gamma_1 + \gamma_2$. Thus, $\mathcal{O}_{\gamma_i}$ is a singleton $\{\alpha_i\}$, $i=1,2$. It follows from Prop..2.3 that if P_0 is a component of Π_{T0}, then $\{\alpha_1\} \cup P_0 \cup \{\alpha_2\}$ is connected. Thus, Π_{T0} is connected. By lemma 2.7, $\alpha_1{}^{i_{\Pi_T}} = \alpha_2$. Thus, i_{Π_T} is not the identity map. Thus, P is of type A_n $(n \geq 2)$, D_n (n odd), or E_6. Since Π_{T0} is connected and $\alpha_1{}^{i_{\Pi_T}} = \alpha_2$, it follows in the first two cases that Π_{γ_1} is connected of type A_{n-1} with end root α_1 and hence, since $\Pi_{T0} \neq \phi$, $\alpha_1{}^{i_{\Pi_{\gamma_1}}} \neq \alpha_1$, contradicting Prop.2.7. Hence, Π_{T0} is of type E_6. Since Π_{T0} is connected and $\alpha_1{}^{i_{\Pi_T}} = \alpha_2$, (Π_T, Π_{T0}) is of the form . The same remarks hold for $(\Pi'_{T'}, \Pi'_{T'0})$. By Prop. 2.7, $\overline{w}_{\gamma_1} \mid \Pi_{T0}$ is the map  and $\overline{w}_{\gamma_2} \mid \Pi_{T0}$ is the map . Hence, the full automorphism group of Π_{T0} is obtained by restricting elements of $\overline{W}_T$ to Π_{T0}. The result is then clear. q.e.d.

We now prove some lemmas about the extension of *-isomorphisms $\Pi_{o,\gamma} \longrightarrow \Pi'_{o,\gamma'}$ to *-isomorphisms $\Pi_\gamma \longrightarrow \Pi'_{\gamma'}$, where $\gamma \in \overline{\overline{\Pi}}$, $\gamma' \in \overline{\overline{\Pi}}'$.

<u>Lemma 3</u>: Let $\gamma \in \overline{\overline{\Pi}}$, $\gamma' \in \overline{\overline{\Pi}}'$. Suppose that Π_γ and $\Pi'_{\gamma'}$ are connected. Suppose we have an isomorphism

$$([\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}], \mathfrak{h} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]) \xrightarrow{\chi_o} ([\mathcal{L}'_{\gamma'}, \mathcal{L}'_{-\gamma'}], \mathfrak{h}' \cap [\mathcal{L}'_{\gamma'}, \mathcal{L}'_{-\gamma'}]).$$

Put $\varphi_o = \chi_o | \mathcal{L}_{o,\gamma}$ and assume that $\Pi_{o,\gamma}^{\varphi_o^*} = \Pi'_{o,\gamma'}$. Assume also that one of the following holds:

(a) $2\gamma \notin \overline{\overline{\Sigma}}$ and $2\gamma' \notin \overline{\overline{\Sigma}}'$.

(b) $2\gamma \in \overline{\overline{\Sigma}}$, $2\gamma' \in \overline{\overline{\Sigma}}'$, and either $\Pi_{o,2\gamma}^{\varphi_o^*} \subseteq \Pi'_{o,2\gamma'}$ or $\Pi'_{o,2\gamma'} \subseteq \Pi_{o,2\gamma}^{\varphi_o^*}$.

Suppose we have a linear bijection $\mathcal{L}_\gamma \xrightarrow{\rho} \mathcal{L}'_{\gamma'}$ such that $[X_\gamma, X_o]^\rho = [X_\gamma^\rho, X_o^{\chi_o}]$ for $X_\gamma \in \mathcal{L}_\gamma$, $X_o \in [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$. Then, there exists a *-isomorphism $(\Pi_\gamma, \Pi_{o,\gamma}) \xrightarrow{g} (\Pi'_{\gamma'}, \Pi'_{o,\gamma'})$ such that $g | \Pi_{o,\gamma} = \varphi_o^*$.

<u>Proof</u>: Now, $\mathcal{L}_o = \mathfrak{a}_\gamma \oplus [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}]$ and center$(\mathcal{L}_o) \subseteq \mathfrak{h}$. Hence, $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{a}_\gamma) \oplus (\mathfrak{h} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}])$. From this and the corresponding fact for $\mathcal{L}'$, it follows that ρ induces a map

$$\{\alpha \in \Sigma : \overline{\alpha} = \gamma\} \xrightarrow{\rho^*} \{\alpha' \in \Sigma' : \overline{\alpha'} = \gamma'\}$$

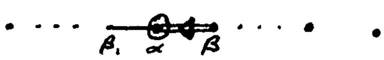
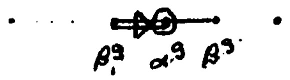
such that $(\mathcal{L}_K)_\alpha^{\rho} = (\mathcal{L}'_K)_{\alpha^{\rho^*}}$ for $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$. Since ρ is a linear bijection, ρ^* is a bijection.

Now, for $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$ and $\beta \in \Pi_{o,\gamma}$, we have

$$\alpha - \beta \in \Sigma \iff [(\mathcal{L}_K)_\alpha, (\mathcal{L}_K)_{-\beta}] \neq (0) \iff [(\mathcal{L}_K)_\alpha, (\mathcal{L}_K)_{-\beta}]^\rho \neq (0)$$
$$\iff [(\mathcal{L}'_K)_{\alpha^{\rho^*}}, (\mathcal{L}'_K)_{-\beta^{\varphi_o^*}}] \neq (0) \iff \alpha^{\rho^*} - \beta^{\varphi_o^*} \in \Sigma',$$

and similarly, $\alpha + \beta \in \Sigma \iff \alpha^{\rho^*} + \beta^{\varphi_o^*} \in \Sigma'$. But $\mho_\gamma$ is the set of $\alpha \in \Sigma$ such that $\overline{\alpha} = \gamma$ and $\alpha - \beta \notin \Sigma$ for $\beta \in \Pi_{o,\gamma}$. Thus, $\mho_\gamma^{\rho^*} = \mho_{\gamma'}$. Define $(\Pi_\gamma, \Pi_{o,\gamma}) \xrightarrow{g} (\Pi'_{\gamma'}, \Pi'_{o,\gamma'})$ by $g | \Pi_{o,\gamma} = \varphi_o^*$

and $g|\mho_\gamma = \rho^*|\mho_\gamma$.

We show first of all that g is an isomorphism of Dynkin diagrams. From the above, it follows that $(\alpha^g, \widehat{\beta^g}) = (\alpha, \hat{\beta})$ for $\alpha \in \mho_\gamma$ and $\beta \in \Pi_{o,\gamma}$. But $(\beta_1{}^g, \widehat{\beta_2{}^g}) = (\beta_1, \hat{\beta_2})$ for $\beta_1, \beta_2 \in \Pi_{o,\gamma}$. Hence, it suffices to show that if $(\beta, \hat{\alpha})$ and $(\beta^g, \widehat{\alpha^g})$ are negative then they are equal ($\alpha \in \mho_\gamma$, $\beta \in \Pi_{o,\gamma}$), and that if α_1, α_2 are distinct elements of $\mho_\gamma$, then $(\alpha_1, \hat{\alpha_2}) = (\alpha_1{}^g, \widehat{\alpha_2{}^g})$. If (a) holds, these statements are immediate since in the first case we must have $(\beta, \hat{\alpha}) = (\beta^g, \widehat{\alpha^g}) = -1$ and the second case cannot occur (since $\mho_\gamma$ and $\mho_{\gamma'}$ are necessarily singletons). Suppose (b) holds. Suppose α_1, α_2 are distinct elements of $\mho_\gamma$. Now $(\alpha_1, \hat{\alpha_2})$ and $(\alpha_1{}^g, \widehat{\alpha_2{}^g})$ lie in $\{0,-1\}$ and hence it suffices to show they are zero together. But if $(\alpha_1, \hat{\alpha_2}) = 0$, there exists a non-empty subset P_o of $\Pi_{o,\gamma}$ such that $\{\alpha_1\} \cup P_o \{\alpha_2\}$ is connected, hence $\{\alpha_1{}^g\} \cup P_o^g \cup \{\alpha_2{}^g\}$ is connected, and thus (since $P_o^g \neq \phi$) $(\alpha_1{}^g, \widehat{\alpha_2{}^g}) = 0$. The converse is similar. Suppose $\alpha \in \mho_\gamma$, $\beta \in \Pi_{o,\gamma}$, and $(\beta, \hat{\alpha})$ and $(\beta^g, \widehat{\alpha^g})$ are negative. Both quantities lie in $\{-1,-2\}$ and hence it suffices to show that $(\beta, \hat{\alpha}) = -2$ if and only if $(\beta^g, \widehat{\alpha^g}) = -2$. Suppose for contradiction that $(\beta, \hat{\alpha}) = -2$ and $(\beta^g, \widehat{\alpha^g}) = -1$. Since $(\beta, \hat{\alpha}) = -2$ and Π_γ is connected, we have $\mho_\gamma = \{\alpha\}$. Therefore, $\mho_{\gamma'} = \{\alpha^{\rho^*}\}$. $(\Pi_\gamma, \Pi_{o,\gamma})$ is of the form  If Π_γ is of type B, then $\Pi'_{\gamma'}$ is of type A (since $(\beta^g, \widehat{\alpha^g}) = -1$) and hence $2\gamma' \notin \Sigma'$, a contradiction. Thus, $(\Pi_\gamma, \Pi_{o,\gamma})$ is of the form  . If there exists $\beta_2 \in \Pi_{o,\gamma}$ such that $(\beta, \beta_2) < 0$, then $2\beta_1 + 3\alpha + 2\beta + \beta_2 \in \Sigma$ and hence $3\gamma \in \Sigma$, a contradiction. Thus, $(\Pi_\gamma, \Pi_{o,\gamma})$ is of the form  . But $(\Pi'_{\gamma'}, \Pi'_{o,\gamma'})$ is not for type A and hence $(\Pi'_{\gamma'}, \Pi'_{o,\gamma'})$ is of the form 

Since $3\gamma' \in \overline{\Sigma}'$, we have as above that $(\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$ is of the form , and hence $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form . Then, $\beta_1 \in \Pi_{0,\partial\gamma}$ and $\beta_1{}^g \notin \Pi'_{0,\partial\gamma'}$. Moreover, $\beta \notin \Pi_{0,\partial\gamma}$ and $\beta^g \in \Pi'_{0,\partial\gamma'}$. This contradicts (b). The converse is similar. Therefore, g is an isomorphism of Dynkin diagrams.

It remains to show that g commutes with the *-action. For this we may assume that $\overline{\Pi} = \{\gamma\}$ and $\overline{\Pi}' = \{\gamma'\}$. Since ρ commutes with the actions of $\mathcal{G}$ on $(\mathcal{L}_\gamma)_K$ and $(\mathcal{L}'_{\gamma'})_K$ and φ_0 commutes with the actions of $\mathcal{G}$ on $[\mathcal{L}_0, \mathcal{L}_0]_K$ and $[\mathcal{L}'_0, \mathcal{L}'_0]_K$, it follows that $\rho^* \sigma = \sigma \rho^*$ and $\varphi_0^* \sigma = \sigma \varphi_0^*$ for $\sigma \in \mathcal{G}$. This together with the fact that conjugation by g takes the Weyl group of Π_γ onto the Weyl group of $\Pi'_{\gamma'}$, implies that g commutes with the *-action. q.e.d.

<u>Lemma 4</u>: Let $\gamma \in \overline{\Pi}$ and suppose $2\gamma \notin \overline{\Sigma}$. Suppose Π_γ is connected and $\Pi_{0,\gamma} \neq \phi$. Then, $\mho_\gamma$ is a singleton $\{\alpha\}$ and we may write
$\mu_{\Pi_\gamma} = \alpha + \sum_{\beta \in \Pi_{0,\gamma}} m_\beta \beta$ and $\gamma = \alpha + \sum_{\beta \in \Pi_{0,\gamma}} q_\beta \beta$ for some positive integers m_β and rational q_β. Moreover, $\alpha^{\overline{w}_\gamma} = -\mu_{\Pi_\gamma}$, $(\alpha, \mu_{\Pi_\gamma}) = 0$, and $q_\beta + q_{\beta\overline{w}_\gamma} = m_\beta = m_{\beta\overline{w}_\gamma}$ for $\beta \in \Pi_{0,\gamma}$.

<u>Proof</u>: The first statement follows from the facts that $2\gamma \notin \overline{\Sigma}$, $\overline{\mu_{\Pi_\gamma}} = \gamma$, and (by the corollary to lemma 2.2) γ is in the $\mathbb{Q}$-space generated by Π_γ.

Now, $-\alpha^{\overline{w}_\gamma} + \alpha \notin \Sigma$ (since $2\gamma \notin \Sigma$) and $-\alpha^{\overline{w}_\gamma} + \beta \notin \Sigma$ for $\beta \in \Pi_{0,\gamma}$ (since $\alpha - \beta^{\overline{w}_\gamma} \notin \Sigma$). Thus, $\alpha^{\overline{w}_\gamma} = -\mu_{\Pi_\gamma}$. But

$$\gamma^{\overline{w}_\gamma} = -\mu_{\Pi_\gamma} + \sum_{\beta \in \Pi_{0,\gamma}} q_\beta \beta^{\overline{w}_\gamma} = -\alpha + \sum_{\beta \in \Pi_{0,\gamma}} (q_{\beta\overline{w}_\gamma} - m_\beta)\beta \quad \text{and}$$

$$\gamma^{\overline{w}_\gamma} = -\gamma = -\alpha - \sum_{\beta \in \Pi_{0,\gamma}} q_\beta .$$

Thus, $q_\beta + q_{\beta\overline{w}_\gamma} = m_\beta$, $\beta \in \Pi_{0,\gamma}$. Since

$\overline{w}_\gamma^2 = 1$, we have $m_\beta = m_{\beta^{\overline{w}_\gamma}}$ for $\beta \in \Pi_{0,\gamma}$.

Suppose for contradiction that $(\alpha, \mu_{\Pi_\gamma}) \neq 0$. Then, $(\mu_{\Pi_\gamma}, \alpha) > 0$. Therefore, $0 < (\mu_{\Pi_\gamma}, \hat{\alpha}) = 2 + \sum_{\beta \in \Pi_{0,\gamma}} m_\beta (\beta, \hat{\alpha})$. Thus, $-\sum_{\beta \in \Pi_{0,\gamma}} m_\beta (\beta, \hat{\alpha}) < 2$. Therefore, there exists a unique element $\beta_1 \in \Pi_{0,\gamma}$ such that $(\beta_1, \alpha) < 0$, and then $m_{\beta_1} = 1$. Thus, $\Pi_{0,\gamma}$ is connected. Since $\Pi_{0,\gamma} \neq \phi$ and $\alpha^{1_{\Pi_\gamma}} = \alpha$, Π_γ is not of type A. Thus, by lemma 3.1, $\Pi_\gamma - \{\beta_1\}$ is connected. Thus, $\Pi_\gamma = \{\alpha, \beta_1\}$. Therefore, $\mu_{\Pi_\gamma} = \alpha + \beta_1$. Thus, Π_γ is of type A and we have a contradiction. q.e.d.

<u>Corollary</u>: Let $\gamma \in \overline{\Pi}$ such that $2\gamma \notin \overline{\Sigma}$. Suppose Π_γ is connected and $\Pi_{0,\gamma}$ is connected $\neq \phi$. Let $\mho_\gamma = \{\alpha\}$, let β_1 be the element of $\Pi_{0,\gamma}$ connected to α, and let $\beta_2 = \beta_1^{\overline{w}_\gamma}$. Let P_0 be the smallest connected subset of $\Pi_{0,\gamma}$ containing β_1 and β_2, and put $\beta_0 = \sum_{\beta \in P_0} \beta$. Then, $\beta_2 = \beta_1^{1_{\Pi_{0,\gamma}}}$, the coefficients of β_1 and β_2 in $\mu_{\Pi_{0,\gamma}}$ are 1, P_0 is of type A, and $\mu_{\Pi_{0,\gamma}} = -\beta_0 + q(\lambda_{\beta_1} + \lambda_{\beta_2})$, where λ_{β_i} is the fundamental dominant integral weight on $(\mathfrak{h} \cap \mathcal{L}_{0,\gamma})_K$ corresponding to β_i, i=1,2, and $q = -(\alpha, \hat{\beta_1})$.

<u>Proof</u>: Now, $\alpha^{1_{\Pi_\gamma}} = \alpha$. Thus, $\beta_1^{1_{\Pi_\gamma}} = \beta_1$. But $\overline{w}_\gamma | \Pi_{0,\gamma} = (1_{\Pi_\gamma} | \Pi_{0,\gamma}) \cdot 1_{\Pi_{0,\gamma}}$. Therefore, $\beta_2 = \beta_1^{1_{\Pi_{0,\gamma}}}$. Now, if the coefficient of β_1 in $\mu_{\Pi_{0,\gamma}}$ is > 1, we have $2\alpha + \mu_{\Pi_{0,\gamma}} \in \Sigma$ and hence $2\gamma \in \overline{\Sigma}$. Therefore, the coefficient of β_1 (and hence of β_2) in $\mu_{\Pi_{0,\gamma}}$ is 1. Thus, P_0 is of type A. Now, there exists a non-empty sequence $\alpha_1, \ldots, \alpha_m$ of elements of Σ such that $\overline{\alpha_i} = \alpha$, i=1,...,m, $\alpha_i - \alpha_{i+1} \in \Pi_{0,\gamma}$ for i=1,...,m-1, $\alpha_1 = \mu_{\Pi_\gamma}$, and $\alpha_m - \alpha = \mu_{\Pi_{0,\gamma}}$. Moreover, $\mu_{\Pi_{0,\gamma}}$ is the root of greatest height with this property. But $(\mu_{\Pi_\gamma}, \alpha) = 0$, β_1

is the unique element of $\Pi_{0,\gamma}$ such that $(\alpha, \beta_1) < 0$, and, since $\mu_{\Pi_\gamma} = -\alpha^{\bar{w}_\gamma}$, β_2 is the unique element of $\Pi_{0,\gamma}$ such that $(\mu_{\Pi_\gamma}, \beta_2) > 0$. Thus, $\mu_{\Pi_{0,\gamma}} = \mu_{\Pi_\gamma} - \beta_0 - \alpha$. Then, for $\beta \in \Pi_{0,\gamma}$,

$$\begin{aligned}(\mu_{\Pi_{0,\gamma}} + \beta_0, \hat{\beta}) &= (\mu_{\Pi_\gamma}, \hat{\beta}) - (\alpha, \hat{\beta}) = -(\alpha, \widehat{\beta^{\bar{w}_\gamma}}) - (\alpha, \hat{\beta}) \\ &= q\,\lambda_{\beta_2}(\hat{\beta}) + q\,\lambda_{\beta_1}(\hat{\beta}).\end{aligned}$$

Therefore, $\mu_{\Pi_{0,\gamma}} = -\beta_0 + q(\lambda_{\beta_1} + \lambda_{\beta_2})$. q.e.d.

<u>Lemma 5</u>: Let $\gamma \in \bar{\Pi}$ and $\gamma' \in \bar{\Pi}'$ and suppose $2\gamma \notin \bar{\Sigma}$ and $2\gamma' \notin \bar{\Sigma}'$. Suppose Π_γ and $\Pi'_{\gamma'}$ are connected and suppose $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are $*$-connected. Suppose $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are not of $*$-type D_{4I}. Suppose one of the following holds:

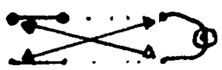
(a) $\bar{w}_\gamma \mid \Pi_{0,\gamma}$ extends to a $*$-automorphism of Π_γ and $\bar{w}'_{\gamma'} \mid \Pi'_{0,\gamma'}$ extends to a $*$-automorphism of $\Pi'_{\gamma'}$.

(b) There exists at most one $\alpha_0 \in \Sigma$ such that $\bar{\alpha}_0 = \gamma$ and $(\alpha_0, \alpha) = (\alpha_0, \alpha^{\bar{w}_\gamma}) = 0$; and there ~~exists~~ at most one $\alpha'_0 \in \Sigma'$ such that $\bar{\alpha}'_0 = \gamma'$ and $(\alpha'_0, \alpha') = (\alpha'_0, \alpha'^{\bar{w}'_{\gamma'}}) = 0$, where $\mho_\gamma = \{\alpha\}$ and $\mho_{\gamma'} = \{\alpha'\}$.

Suppose we have a $*$-isomorphism $\Pi_{0,\gamma} \xrightarrow{g_0} \Pi'_{0,\gamma'}$. Then, either g_0 or $(\bar{w}_\gamma \mid \Pi_{0,\gamma}) \circ g_0$ extends to a $*$-isomorphism $(\Pi_\gamma, \Pi_{0,\gamma}) \longrightarrow (\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$. In particular, if (a) holds, g_0 extends to a $*$-isomorphism $(\Pi_\gamma, \Pi_{0,\gamma}) \longrightarrow (\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$.

<u>Proof</u>: We may assume $\Pi_{0,\gamma} \neq \phi$ and $\Pi'_{0,\gamma'} \neq \phi$.

Suppose $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are not connected. Then, by lemma 3.1, Π_γ and $\Pi'_{\gamma'}$ are of type A and hence $(\Pi_\gamma, \Pi_{0,\gamma})$ and $(\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$ are of the form  (since $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are $*$-connected, $\alpha^{i_{\Pi_\gamma}} = \alpha$, and $\alpha'^{i_{\Pi'_{\gamma'}}} = \alpha'$). By Prop. 2.7, $\bar{w}_\gamma \mid \Pi_{0,\gamma}$ is the map

 and the result is clear.

Suppose $\Pi_{a,\gamma}$ and $\Pi'_{b,\gamma'}$ are connected. We use the notation of the previous corollary and similar primed notation for $(\Pi'_{\gamma'}, \Pi'_{b,\gamma'})$. It suffices then to show that $q=q'$, and $\beta_1^{g_0} = \beta_1'$ or $\beta_1^{g_0} = \beta_2'$. We note that if (a) holds, $\beta_1 = \beta_2$ and $\beta_1' = \beta_2'$.

We put $\beta_1'' = \beta_1'^{g_0^{-1}}$, $\beta_2'' = \beta_2'^{g_0^{-1}}$, $P_o'' = P_o'^{g_0^{-1}}$, $\beta_o'' = \beta_o'^{g_0^{-1}}$, and $q'' = q'$. Then, $\{\beta_1, \beta_2\}$ (resp. $\{\beta_1'', \beta_2''\}$) is an orbit of $i_{\Pi_{b,\gamma}}$, P_o (resp. P_o'') is the smallest connected subset of $\Pi_{b,\gamma}$ containing β_1 and β_2 (resp. β_1'' and β_2''), P_o (resp P_o'') is of type A, the coefficients of β_1 and β_2 (resp. β_1'' and β_2'') in $\mu_{\Pi_{b,\gamma}}$ are 1, and

$$-\beta_o + q(\lambda_{\beta_1} + \lambda_{\beta_2}) = -\beta_o'' + q''(\lambda_{\beta_1''} + \lambda_{\beta_2''}) . \qquad (9)$$

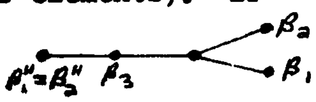
By (9), it suffices to show that $\{\beta_1, \beta_2\} = \{\beta_1'', \beta_2''\}$ (since then $q=q'$ and $\{\beta_1, \beta_2\} = \{\beta_1', \beta_2'\}^{g_0^{-1}}$). We note that if (a) holds, $\beta_1 = \beta_2$ and $\beta_1'' = \beta_2''$.

Suppose for contradiction that $\{\beta_1, \beta_2\} \neq \{\beta_1'', \beta_2''\}$. Then, $\{\beta_1, \beta_2\} \cap \{\beta_1'', \beta_2''\} = \phi$. Thus, from the nature of P_o and P_o'', it follows that $P_o'' \subsetneqq P_o$, $P_o \subsetneqq P_o''$, or $P_o'' \cap P_o = \phi$.

Suppose $P_o'' \subsetneqq P_o$. Taking the Killing form of (9) with $\widehat{\beta_o}$, we obtain $-2+2q = -(\beta_o'', \widehat{\beta_o})+2q''$. Therefore, $(\beta_o'', \widehat{\beta_o}) = -2(q-q''-1)$. Taking the Killing form of (9) with $\widehat{\beta_o''}$, we obtain $-(\beta_o, \widehat{\beta_o''})+0 = -2+2q''$. Thus, $(\beta_o, \widehat{\beta_o''}) = -2(q''-1)$. But, since β_o and β_o'' are not proportional, $(\beta_o, \widehat{\beta_o''})(\beta_o'', \widehat{\beta_o}) \in \{0,1,2,3\}$ and $(\beta_o, \widehat{\beta_o''})(\beta_o'', \widehat{\beta_o}) = 4(q''-1)(q-q''-1)$. Thus, $(\beta_o, \widehat{\beta_o''})(\beta_o'', \widehat{\beta_o}) = 0$. Therefore, $(\beta_o, \widehat{\beta_o''}) = (\beta_o'', \widehat{\beta_o}) = 0$. Thus, $(\beta_o, \beta_o'')=0$, $q'' = 1$, and $q = 2$. Since $P_o'' \subsetneqq P_o$, P_o is of type A_n $(n \geq 3)$. Therefore, there exists $\beta \in P_o$ such that $(\beta_1, \beta) < 0$

and $\beta \neq \beta_2$. Since $\beta_1 \neq \beta_2$, (b) holds. Therefore, there exists at most one $\alpha_0 \in \Sigma$ such that $\overline{\alpha_0} = \gamma$ and $(\alpha_0, \alpha) = (\alpha_0, \alpha^{W_\gamma}) = 0$. But both $\alpha + 2\beta_1$ and $\alpha + 2\beta_1 + \beta$ have this property.

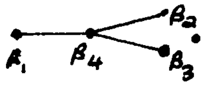
The case $P_0 \subsetneqq P_0''$ is disposed of similarly.

Suppose $P_0 \cap P_0'' = \phi$. Taking the Killing form of (9) with $\widehat{\beta_0}$, we obtain $-2+2q = -(\beta_0'', \widehat{\beta_0})+0$. Hence, $(\beta_0'', \widehat{\beta_0}) = -2(q-1)$. Similarly, $(\beta_0, \widehat{\beta_0''}) = -2(q''-1)$ and hence, as above, $(\beta_0'', \beta_0) = 0$, $q=1$, and $q''=1$. Therefore, no element of P_0 is connected to an element of P_0''. But, by (9), $(\beta_0, \widehat{\beta}) = (\beta_0'', \widehat{\beta})$ for $\beta \in \Pi_{0,\gamma} - \{\beta_1, \beta_2, \beta_1'', \beta_2''\}$. Then, there exists a unique $\beta_3 \in \Pi_{0,\gamma}$ such that $(\beta_0, \beta_3) < 0$ and $(\beta_0'', \beta_3) < 0$. (Such an element exists since $\Pi_{0,\gamma}$ is connected and is unique since $\Pi_{0,\gamma}$ contains no loops.) But $\beta_3^{\,1_{\Pi_{0,\gamma}}}$ also has this property. Therefore, $\beta_3^{\,1_{\Pi_{0,\gamma}}} = \beta_3$. Thus, β_3 is connected to the middle roots of P_0 and P_0'' (and each of these sets contains an odd number of elements). If $\beta_1 \neq \beta_2$, it is easy to see that $\Pi_{0,\gamma}$ is of the form 

and hence $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form 

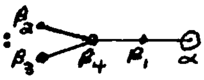
, in which case $\alpha^{1_{\Pi_\gamma}} \neq \alpha$, a contradiction. Thus, $\beta_2 = \beta_1$. Similarly, $\beta_2'' = \beta_1''$. But then $\beta_0 = \beta_1 = \beta_2$, $\beta_0'' = \beta_1'' = \beta_2''$, and thus $(\beta_1, \beta_2) = 0$ and $(\beta_1, \widehat{\beta}) = (\beta_1'', \widehat{\beta})$ for $\beta \in \Pi_{0,\gamma} - \{\beta_1, \beta_1''\}$. But since β_1 and β_1'' have coefficient 1 in $\mu_{\Pi_{0,\gamma}}$, this implies that we also have $(\beta, \widehat{\beta_1}) = (\beta, \widehat{\beta_1''})$ for $\beta \in \Pi_{0,\gamma} - \{\beta_1, \beta_1''\}$ (since both quantities must lie in $\{0,-1\}$). Thus, the map which fixes $\Pi_{0,\gamma} - \{\beta_1, \beta_1''\}$ and interchanges β_1 and β_1'' is a non-trivial automorphism of Dynkin diagrams which is not the opposition involution. Therefore, $\Pi_{0,\gamma}$ is of type D_n (n even) and β_1 and β_1'' are the two roots interchanged by the automorphism group of $\Pi_{0,\gamma}$. Since α is connected to β_1, we must have n=4 or 6. Suppose n=4. Then, $\Pi_{0,\gamma}$ is not of *-type D_{4I}. But, since α and α' are fixed by the *-action of $\mathfrak{L}$, β_1 and β_1'' are fixed by the *-action of $\mathfrak{L}$. Thus, $\beta_1 = \beta_1''$ and we have a contradiction. Suppose n=6. Then, $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form 

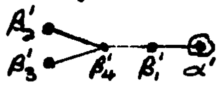
. But the coefficient of α in μ_{Π_γ} is then 2 and we have a contradiction. q.e.d.

Because lemma 5 does not apply when $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are connected of *-type D_{4I}, it seems necessary to include some remarks which will enable us to deal with this case (at least some of the time). Let $\mathfrak{M}_0$ be an anisotropic central simple Lie algebra over $\mathfrak{k}$ with Cartan subalgebra $\mathfrak{h}_{\mathfrak{M}_0}$ split by K and index $\Pi_{\mathfrak{M}_0}$ of *-type D_{4I}. Label the roots of $\Pi_{\mathfrak{M}_0}$ as follows: 

. Let λ_{β_i} be the fundamental dominant integral weight of $(\mathfrak{h}_{\mathfrak{M}_0})_K$ corresponding to β_i, i=1,2,3. Let $\mathfrak{M}_{0K} \xrightarrow{\rho_i} End_K(V_i)$ be the K-representation of $\mathfrak{M}_{0K}$ with highest weight λ_{β_i}, i=1,2,3. For i=1,2,3, we say <u>ρ_i is defined over $\mathfrak{k}$</u> if there exists a $\mathfrak{k}$-form W_i of V_i such that $W_i\mathfrak{M}_0 \subseteq W_i$. It is shown in Jacobson [4] that if ρ_1 is defined over $\mathfrak{k}$, then ρ_2 is defined over $\mathfrak{k}$ if and only if ρ_3 is defined over $\mathfrak{k}$. It is also shown that ρ_1, ρ_2, and ρ_3 are all defined over $\mathfrak{k}$ if and only if $\mathfrak{M}_0$ is isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra.

<u>Lemma 6</u>: Let $\gamma \in \overline{\Pi}$ and $\gamma' \in \overline{\Pi}'$ and suppose $2\gamma \notin \overline{\Pi}$ and $2\gamma' \notin \overline{\Pi}'$. Suppose Π_γ and $\Pi'_{\gamma'}$ are connected and suppose $\Pi_{0,\gamma}$ and $\Pi'_{0,\gamma'}$ are connected of *-type D_{4I}. Suppose $\mathcal{L}_{0,\gamma}$ is not isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra and suppose the same for $\mathcal{L}'_{0,\gamma'}$. Suppose we have an isomorphism $(\mathcal{L}_{0,\gamma}, \mathfrak{h} \cap \mathcal{L}_{0,\gamma}) \xrightarrow{\varphi_0} (\mathcal{L}'_{0,\gamma'}, \mathfrak{h}' \cap \mathcal{L}'_{0,\gamma'})$ such that $\Pi_{0,\gamma}^{\varphi_0^*} = \Pi'_{0,\gamma'}$. Then, there exists a *-isomorphism $(\Pi_\gamma, \Pi_{0,\gamma}) \xrightarrow{f} (\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$ such that $f \mid \Pi_{0,\gamma} = \varphi_0^*$.

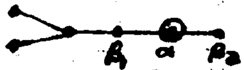
<u>Proof</u>: $(\Pi_\gamma, \Pi_{0,\gamma})$ and $(\Pi'_{\gamma'}, \Pi'_{0,\gamma'})$ are of the form . Label the roots of Π_γ as follows: 

. Label the roots of $\Pi_{0,\gamma}$ as follows: 

. It suffices to show that $\beta_1^{\varphi_0^*} = \beta_1'$. Let λ_{β_i} be the dominant integral weight of $(\mathfrak{h} \cap \mathcal{L}_{0,\gamma})_K$ corresponding to β_i, i=1,2,3. Similarily define $\lambda_{\beta_i'}$, i=1,2,3. By Prop.2.8, $(\mathcal{L}_\gamma)_K$ is an irreducible $(\mathcal{L}_{0,\gamma})_K$ module with highest weight $\lambda_{\beta_1 W_\gamma} = \lambda_{\beta_1}$. Similarily, $(\mathcal{L}'_{\gamma'})_K$ is an irreducible $(\mathcal{L}'_{0,\gamma'})_K$ module with highest weight $\lambda_{\beta_1'}$. This latter representation preceded by φ_0 is an irreducible representation of $(\mathcal{L}_{0,\gamma})_K$ with highest weight $\lambda_{\beta_1' \circ \varphi_0^{*-1}}$. Thus, the irreducible representations of $(\mathcal{L}_{0,\gamma})_K$ with highest weights λ_{β_1} and $\lambda_{\beta_1' \varphi_0^{*-1}}$ are defined over $\mathfrak{k}$. Therefore, since $\mathcal{L}_{0,\gamma}$ is not isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra, $\beta_1 = \beta_1'^{\varphi_0^{*-1}}$. Therefore, $\beta_1^{\varphi_0^*} = \beta_1'$. q.e.d.

We also need:

<u>Lemma 7</u>: Let $\gamma \in \overline{\Pi}$ such that $2\gamma \in \Sigma$. Suppose Π_γ is connected and $\Pi_{0,2\gamma} = \Phi$. Then, $\mathcal{L}_{0,\gamma}$ does not contain a simple summand which is isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra over $\mathfrak{k}$.

<u>Proof</u>: Let $\mathcal{L}_{o,1}$ be a simple summand of $\mathcal{L}_{o,\gamma}$ which is isomorphic to the skew trasformations with respect to the trace form in a Cayley division algebra over k. Let $\Pi_{o,1}$ be the corresponding *-component of $\Pi_{o,\gamma}$. Then, $\Pi_{o,1}$ is connected of *-type D_{4I}. If $\mathcal{O}_{\gamma}$ contains two elements, then each of these elements is connected to $\Pi_{o,1}$ and hence it is easy to see that $\Pi_{o,2\gamma} \cap \Pi_{o,1} \neq \phi$. Thus, $\mathcal{O}_{\gamma}$ is a singleton $\{\alpha\}$. It is then clear that Π_{γ} is of *-type D_{nI} $(n \geq 5)$ and $(\Pi_{\gamma}, \Pi_{o,\gamma})$ is of the form:  . . . . If $n \neq 6$, it follows from Prop. 2.7 that $\bar{W}_{\gamma} | \Pi_{o,\gamma}$ is non-trivial. But, since $\Pi_{o,2\gamma} = \phi$, we have by the same proposition that $\bar{W}_{\gamma} | \Pi_{o,\gamma} = \bar{W}_{2\gamma} | \Pi_{o,\gamma}$ is trivial. Thus, n=6 and $(\Pi_{\gamma}, \Pi_{o,\gamma})$ is of the form:  , where $\Pi_{o,2} = \{\beta_2\}$ is the other *-component of $\Pi_{o,\gamma}$. Let $\mathcal{L}_{o,2}$ be the simple summand of $\mathcal{L}_{o,\gamma}$ corresponding to $\Pi_{o,2}$. Let $(\mathcal{L}_{o,\gamma})_K \xrightarrow{\rho} \mathrm{End}_K((\mathcal{L}_{\gamma})_K)$ be the adjoint representation of $(\mathcal{L}_{o,\gamma})_K$ in $(\mathcal{L}_{\gamma})_K$. By Prop. 2.8, ρ is equivalent to the representation of $(\mathcal{L}_{o,\gamma})_K$ with highest weight the sum of the fundamental dominant weights corresponding to β_1 and β_2. Let $\lambda_{\beta_i} : (\mathfrak{h} \cap \mathcal{L}_{o,i})_K \longrightarrow K$ be the fundamental dominant weight corresponding to β_i and let $(\mathcal{L}_{o,i})_K \longrightarrow \mathrm{End}_K(V_i)$ be the corresponding representation, i=1,2. Then, as an $(\mathcal{L}_{o,\gamma})_K = (\mathcal{L}_{o,1})_K \oplus (\mathcal{L}_{o,2})_K$ module, we have $(\mathcal{L}_{\gamma})_K \cong V_1 \otimes V_2$. Thus, as an $(\mathcal{L}_{o,1})_K$ module, we have $(\mathcal{L}_{\gamma})_K \cong V_1 \oplus V_1$. Thus, the centralizer of $\rho((\mathcal{L}_{o,1})_K)$ in $\mathrm{End}_K((\mathcal{L}_{\gamma})_K)$ is isomorphic to the 2×2 matrices $M_2(K)$ with coefficients in K. But the centralizer $\mathcal{C}$ of $\rho(\mathcal{L}_{o,1})$ in $\mathrm{End}_k(\mathcal{L}_{\gamma})$ is a form of this algebra and hence $\mathcal{C}$ is either a division algebra or isomorphic to $M_2(k)$. But $\rho(\mathcal{L}_{o,2}) \subsetneq \mathcal{C}$ and

hence $\rho(\mathcal{L}_{0,2}) = [\rho(\mathcal{L}_{0,2}), \rho(\mathcal{L}_{0,2})] \subseteq [\mathcal{C},\mathcal{C}]$. Thus, since $\rho(\mathcal{L}_{0,2})$ and $[\mathcal{C},\mathcal{C}]$ have dimension 3, we have $\rho(\mathcal{L}_{0,2}) = [\mathcal{C},\mathcal{C}]$. But $\rho(\mathcal{L}_{0,2})$ is anisotropic and hence $\mathcal{C}$ is not isomorphic to $M_2(k)$. Therefore, $\mathcal{C}$ is a division algebra. Thus, $\mathcal{L}_\gamma$ is an irreducible $\mathcal{L}_{0,1}$ module. But $(\mathcal{L}_\gamma)_K \cong V_1 \oplus V_1$ as an $(\mathcal{L}_{0,1})_K$ module and ρ_1 is defined over k, since $\mathcal{L}_{0,1}$ is isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra over k. Therefore, $\mathcal{L}_\gamma$ is not irreducible as an $\mathcal{L}_{0,1}$ module and we have a contradiction. q.e.d.

In order to apply lemma 5 to algebras of restricted type G_2, we will need the following:

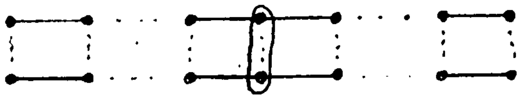
Lemma 8: Suppose $\overline{\Pi}$ is of type G_2, Π is connected and $\Pi_0 \neq \phi$. Label the roots of $\overline{\Pi}$ as follows: $\underset{\gamma_1}{\circ}\!\Rrightarrow\!\underset{\gamma_2}{\circ}$. Then, $\Pi_{0,\gamma_2} = \phi$, Π_{γ_1} is connected, $\mho_{\gamma_1}$ is a singleton $\{\alpha_1\}$, and there exists at most one $\alpha_0 \in \Sigma$ such that $\bar{\alpha}_0 = \gamma_1$ and $(\alpha_0, \alpha_1) = (\alpha_0, \alpha_1^{\overline{W}_{\gamma_1}}) = 0$.

Proof: The first three statements follow from Prop. 3.6 and Prop. 3.7. By Prop. 3.7, $\mho_{\gamma_1} \cup \Pi_{\gamma_2}$ is of the form $\underset{\alpha_1}{\circ}\!-\!\underset{\alpha_2}{\circ}$, where $\mho_{\gamma_1} = \{\alpha_1\}$ and $\mho_{\gamma_2} = \{\alpha_2\}$. Suppose $\alpha_0 \in \Sigma$ such that $\bar{\alpha}_0 = \gamma_1$ and $(\alpha_0, \alpha_1) = (\alpha_0, \alpha_1^{\overline{W}_{\gamma_1}}) = 0$. We show that $\alpha_0 = (\alpha_1 + \alpha_2)^{\overline{W}_{\gamma_1}} - (\alpha_1 + \alpha_2)$. Now, α_0 is of the form $\alpha_1 + \sum_{\beta \in \Pi_0} m_\beta \beta$ with $m_\beta \in \mathbb{Z}$. Hence, since $(\alpha_2, \beta) = 0$ for $\beta \in \Pi_0$, $(\alpha_2, \alpha_0) = (\alpha_2, \alpha_1)$. Similarly, $(\alpha_2, \alpha_0^{\overline{W}_{\gamma_1}}) = -(\alpha_2, \alpha_1)$. Therefore, $(\alpha_1 + \alpha_2, \alpha_0) = (\alpha_2, \alpha_1) < 0$ and $((\alpha_1 + \alpha_2)^{\overline{W}_{\gamma_1}}, \alpha_0) = -(\alpha_2, \alpha_1) > 0$. Thus, $\alpha_1 + \alpha_2 + \alpha_0 \in \Sigma$ and $(\alpha_1 + \alpha_2)^{\overline{W}_{\gamma_1}} - \alpha_0 \in \Sigma$. But $\overline{(\alpha_1 + \alpha_2)^{\overline{W}_{\gamma_1}} - \alpha_0} = \gamma_1 + \gamma_2$ and $\overline{(\alpha_1 + \alpha_2 + \alpha_0)^{\overline{W}_{\gamma_1}}} = \gamma_1 + \gamma_2$. Therefore, we may write $(\alpha_1 + \alpha_2)^{\overline{W}_{\gamma_1}} - \alpha_0 = \alpha_1 + \alpha_2 + \omega_0$ and $(\alpha_1 + \alpha_2 + \alpha_0)^{\overline{W}_{\gamma_1}} = \alpha_1 + \alpha_2 + \nu_0$,

where ω_0 and ν_0 are non-negative integral sums of elements of $\Pi_{0,\gamma}$.
Therefore, $\alpha_0 = (\alpha_1+\alpha_2)^{W_{\gamma_1}} - \alpha_1 - \alpha_2 - \omega_0 = (\alpha_1+\alpha_2)^{W_{\gamma_1}} - \alpha_1 - \alpha_2 + \nu_0^{W_{\gamma_1}}$.
Thus, $-\omega_0 = \nu_0^{W_{\gamma_1}}$. Thus, $\omega_0 = \nu_0 = 0$ and $\alpha_0 = (\alpha_1+\alpha_2)^{W_{\gamma_1}} - (\alpha_1+\alpha_2)$. q.e.d.

We will need the following calculation:

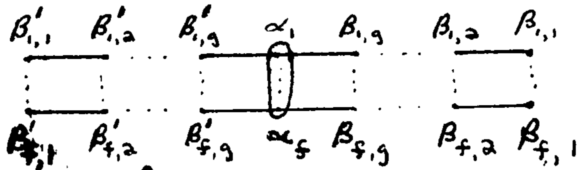
<u>Lemma 9</u>: Let $\gamma \in \overline{\Pi}$ and suppose $(\Pi_\gamma, \Pi_{0,\gamma})$ is of the form



,

with f components of type A_{2g+1}. Then, $(\alpha,\alpha) = f(g+1)(\gamma,\gamma)$ for $\alpha \in \mathcal{O}_\gamma$.

<u>Proof</u>: Label the roots of Π_γ as indicated in the following diagram:



.

Put $\lambda_i = \frac{g+1}{2}\alpha_i + \sum_{j=1}^{g} \frac{j}{2}(\beta_{i,j} + \beta'_{i,j})$ for i=1,...,f. Then,
$(\lambda_i, \hat{\alpha}_i) = 1$ for i=1,...,f, $(\lambda_i, \alpha_j) = 0$ for $1 \le i,j \le f$, $i \ne j$, and
$(\lambda_i, \beta_{j,k}) = (\lambda_i, \beta'_{j,k}) = 0$ for $1 \le i,j \le f$ and $1 \le k \le g$. Put
$\lambda = \sum_{i=1}^{f} \frac{2}{(\alpha_i,\alpha_i)} \lambda_i$. Then, $(\lambda, \hat{\alpha}_j) = 1$ for $1 \le j \le f$ and λ is
orthogonal to all elements of $\Pi_{0,\gamma}$. But

$$\left(\frac{1}{(\gamma,\gamma)}\gamma, \alpha_j\right) = \frac{(\gamma,\alpha_j)}{(\gamma,\gamma)} = \frac{(\gamma,\gamma)}{(\gamma,\gamma)} = 1$$

for $1 \le j \le f$ and $\frac{1}{(\gamma,\gamma)}\gamma$ is orthogonal to all elements of $\Pi_{0,\gamma}$.
Thus, $\lambda - \frac{1}{(\gamma,\gamma)}\gamma$ is orthogonal to all elements of Π_γ. But by the
corollary to lemma 2.2, $\lambda - \frac{1}{(\gamma,\gamma)}\gamma$ is in the $\mathbb{Q}$-space generated by
Π_γ. Therefore, $\lambda = \frac{1}{(\gamma,\gamma)}\gamma$. Thus, $\gamma = 2(\gamma,\gamma)\sum_{i=1}^{f} \frac{\lambda_i}{(\alpha_i,\alpha_i)}$. But
$(\alpha_i,\alpha_i) = (\alpha_j,\alpha_j)$ for $1 \le i,j \le f$. Thus, if $\alpha \in \mathcal{O}_\gamma$, $\gamma = 2\frac{(\gamma,\gamma)}{(\alpha,\alpha)}\sum_{i=1}^{f} \lambda_i$.
Now, $\overline{\lambda_i} = \frac{g+1}{2}\overline{\alpha_i} = \frac{g+1}{2}\gamma$, $1 \le i \le f$. Thus, for $\alpha \in \mathcal{O}_\gamma$,
$\gamma = \overline{\gamma} = 2\frac{(\gamma,\gamma)}{(\alpha,\alpha)}\sum_{i=1}^{f}\frac{g+1}{2}\gamma = \frac{(\gamma,\gamma)}{(\alpha,\alpha)} f\,(g+1)\gamma$ and hence
$(\alpha,\alpha) = (\gamma,\gamma)\, f\,(g+1)$. q.e.d.

We are now ready to prove the rational isomorphism theorem. This theorem states roughly that $\mathcal{L}$ is determined up to isomorphism by $\mathcal{L}_0$, $\overline{\Pi}$, and the action of $\mathcal{L}_0$ on $\mathcal{L}_{\gamma_{\overline{\Pi}}}$, where $\gamma_{\overline{\Pi}}$ is a distinguished element of $\overline{\Pi}$ depending only on $\overline{\Pi}$. Indeed we define $\gamma_{\overline{\Pi}}$ as follows:

(I) If $\overline{\Pi}$ is of type A_1, $\gamma_{\overline{\Pi}}$ is the unique element of $\overline{\Pi}$.

(II) If $\overline{\Pi}$ is of type C_r $(r \geq 2)$, $\gamma_{\overline{\Pi}}$ is the long root of $\overline{\Pi}$. (10)

(III) If $\overline{\Pi}$ is not reduced, $\gamma_{\overline{\Pi}}$ is the unique element of $\overline{\Pi}$ such that $2\gamma_{\overline{\Pi}} \in \overline{\Sigma}$.

(IV) In all other cases, we do not define $\gamma_{\overline{\Pi}}$.

It is not necessary to define $\gamma_{\overline{\Pi}}$ in the cases covered by (IV) since, as we shall see, in these cases $\mathcal{L}$ is determined up to isomorphism by $\mathcal{L}_0$ and $\overline{\Pi}$. We note that in the cases covered by (I), (II), and (III), $\gamma_{\overline{\Pi}}$ is the unique element of $\overline{\Pi}$ such that $m\gamma_{\overline{\Pi}}$ is of maximum length in $\overline{\Sigma}$ for some positive integer m. Conversely, if $\gamma \in \overline{\Pi}$ has this last property, then $\overline{\Pi}$ is covered by cases (I), (II), and (III) or $\overline{\Pi}$ is of type G_2. We need not define $\gamma_{\overline{\Pi}}$ for $\overline{\Pi}$ of type G_2 since, by Thm. 4.3, $[\mathcal{L}_0, \mathcal{L}_0]$ acts trivially on the long root space in this case. We define $\gamma'_{\overline{\Pi}'} \in \overline{\Pi}'$ similarly.

In the theorem, we are interested in extending isomorphisms $\mathcal{L}_0 \longrightarrow \mathcal{L}_0'$ to isomorphisms $(\mathcal{L}, \mathfrak{J}) \longrightarrow (\mathcal{L}', \mathfrak{J}')$. We have the following necessary condition for an isomorphism $\mathcal{L}_0 \longrightarrow \mathcal{L}_0'$ to have such an extension:

<u>Prop. 1</u>: Let $\mathcal{L}_o \xrightarrow{\varphi_o} \mathcal{L}_o'$ be an isomorphism which extends to an isomorphism $(\mathcal{L},\mathfrak{J}) \xrightarrow{\varphi} (\mathcal{L}',\mathfrak{J}')$. Let $\gamma \in \Pi$ such that $m\gamma$ is of maximum length in $\overline{\Sigma}$ for some positive integer m and let $\gamma' \in \overline{\Pi}'$ such that $m'\gamma'$ is of maximum length in $\overline{\Pi}'$ for some positive integer m'. Then, $\mathfrak{N}_\gamma^{\varphi_o} = \mathfrak{N}'_{\gamma'}$.

<u>Proof</u>: It is clear that $(\gamma^{\varphi^*}, \gamma^{\varphi^*}) = (\gamma',\gamma')$. Thus, $\mathfrak{N}'_{\gamma'} = \mathfrak{N}'_{\gamma^{\varphi^*}}$. Therefore, it suffices to show that $\mathfrak{N}^{\varphi_o} = \mathfrak{N}'_{\gamma^{\varphi^*}}$. This however is immediate since $\mathfrak{O}_\delta^{\varphi} = \mathfrak{O}'_{\delta^{\varphi^*}}$ for $\delta \in \overline{\Sigma}$. q.e.d.

<u>Corollary</u>: Suppose $\gamma_{\overline{\Pi}}$ and $\gamma'_{\overline{\Pi}'}$ are defined. Then, $\mathfrak{N}_{\gamma_{\overline{\Pi}}} \subseteq \mathcal{L}_{o,\gamma_{\overline{\Pi}}}$, $\mathfrak{N}'_{\gamma'_{\overline{\Pi}'}} \subseteq \mathcal{L}'_{o,\gamma'_{\overline{\Pi}'}}$, and $\mathfrak{N}_{\gamma_{\overline{\Pi}}}^{\varphi_o} = \mathfrak{N}'_{\gamma'_{\overline{\Pi}'}}$ for any isomorphism $\mathcal{L}_{o,\gamma_{\overline{\Pi}}} \xrightarrow{\varphi_o} \mathcal{L}_{o,\gamma'_{\overline{\Pi}'}}$ which extends to an isomorphism $(\mathcal{L},\mathfrak{J}) \longrightarrow (\mathcal{L}',\mathfrak{J}')$.

<u>Proof</u>: The two inclusions follow from Prop. 4.1. q.e.d.

If $\overline{\Pi}$ and $\overline{\Pi}'$ are both of type A_1 or C_r $(r \geq 2)$, then $\mathfrak{N}_{\gamma_{\overline{\Pi}}} = (0)$ and $\mathfrak{N}'_{\gamma'_{\overline{\Pi}'}} = (0)$, and so the corollary is not of interest in these cases. However, it is of interest when $\overline{\Pi}$ and $\overline{\Pi}'$ are not reduced.

If $\mathfrak{M}_i$ is a Lie algebra over k and V_i is an $\mathfrak{M}_i$-module for i=1,2, an <u>equivalence of representations</u> from $(\mathfrak{M}_1, V_1)$ onto $(\mathfrak{M}_2, V_2)$ is a pair (φ, ρ) such that $\mathfrak{M}_1 \xrightarrow{\varphi} \mathfrak{M}_2$ is a Lie algebra isomorphism, $V_1 \xrightarrow{\rho} V_2$ is a linear bijection, and $(v_1 M_1)^\rho = v_1^\rho M_1^\varphi$ for $v_1 \in V_1$ and $M_1 \in \mathfrak{M}_1$.

We now state and prove the main isomorphism theorem:

Theorem 1: Let $\mathcal{L}$ be a central simple Lie algebra over k with maximal split toral subalgebra $\mathfrak{J}$ and restricted root decomposition $\mathcal{L} = \mathcal{L}_0 \oplus \sum_{\gamma \in \Sigma} \mathcal{L}_\gamma$. Let Π be a fundamental system for Σ and define a distinguished root $\gamma_\Pi \in \Pi$ as in (10). Whenever γ_Π is defined, put $\mathcal{N}_{\gamma_\Pi} = \bigcap_{\substack{\gamma \in \Sigma \\ (\gamma,\gamma)=(\gamma_\Pi,\gamma_\Pi)}} \mathfrak{a}_\gamma \cap [\mathcal{L}_0, \mathcal{L}_0]$, where $\mathfrak{a}_\gamma$ is the annihilator of $\mathcal{L}_\gamma$ in $\mathcal{L}_0$ for $\gamma \in \Sigma$. Assume $[\mathcal{L}_0, \mathcal{L}_0] \neq (0)$ and, if Π is of type B_r $(r \geq 2)$, assume $[\mathcal{L}_0, \mathcal{L}_0]$ is not isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra over k. Make the same assumptions and definitions for $\mathcal{L}'$, $\mathfrak{J}'$, Σ', Π', $\gamma'_{\Pi'}$, $\mathcal{N}'_{\gamma'_{\Pi'}}$, and $[\mathcal{L}'_0, \mathcal{L}'_0]$. Suppose that Π and Π' are isomorphic, $[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}'_0, \mathcal{L}'_0]$ contain the same number of simple summands, and one of the following holds:

(a) γ_Π and $\gamma'_{\Pi'}$ are not defined and we have an isomorphism φ_0 of one of the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ onto one of the simple summands of $[\mathcal{L}'_0, \mathcal{L}'_0]$.

(b) γ_Π and $\gamma'_{\Pi'}$ are defined, $[\mathcal{L}_0, \mathcal{L}_0]$ acts trivially on $\mathcal{L}_{\gamma_\Pi}$, $[\mathcal{L}'_0, \mathcal{L}'_0]$ acts trivially on $\mathcal{L}'_{\gamma'_{\Pi'}}$, and we have an isomorphism φ_0 of one of the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ onto one of the simple summands of $[\mathcal{L}'_0, \mathcal{L}'_0]$.

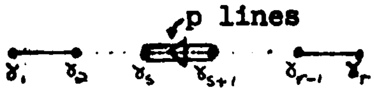
(c) γ_Π and $\gamma'_{\Pi'}$ are defined, $[\mathcal{L}_0, \mathcal{L}_0]$ acts non-trivially on $\mathcal{L}_{\gamma_\Pi}$, $[\mathcal{L}'_0, \mathcal{L}'_0]$ acts non-trivially on $\mathcal{L}'_{\gamma'_{\Pi'}}$, and we have an equivalence of representations (χ_0, ρ) from $([\mathcal{L}_{\gamma_\Pi}, \mathcal{L}_{-\gamma_\Pi}], \mathcal{L}_{\gamma_\Pi})$ onto $([\mathcal{L}'_{\gamma'_{\Pi'}}, \mathcal{L}'_{-\gamma'_{\Pi'}}], \mathcal{L}'_{\gamma'_{\Pi'}})$ such that $\mathcal{N}_{\gamma_\Pi}^{\chi_0} = \mathcal{N}'_{\gamma'_{\Pi'}}$. In this case, put $\varphi_0 = \chi_0 | \mathcal{L}_{0,\gamma_\Pi}$, where $\mathcal{L}_{0,\gamma_\Pi} = [\mathcal{L}_{\gamma_\Pi}, \mathcal{L}_{-\gamma_\Pi}] \cap [\mathcal{L}_0, \mathcal{L}_0]$.

Then, there exists an isomorphism $(\mathcal{L}, \mathfrak{J}) \xrightarrow{\varphi} (\mathcal{L}', \mathfrak{J}')$ which extends φ_0.

<u>Proof</u>: Choose K and $\mathcal{B}$ as at the beginning of this chapter. We will be free to choose any $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, or Π_0'. We may assume that $\mathfrak{J} \neq (0)$.

If $\overline{\Pi}$ and $\overline{\Pi}'$ are of type A_1 or are not reduced, it follows from Thm. 4.2 and Thm. 4.4 that the actions of $[\mathcal{L}_0, \mathcal{L}_0]$ on $\mathcal{L}_{\gamma_{\overline{\Pi}}}$ and $[\mathcal{L}_0', \mathcal{L}_0']$ on $\mathcal{L}'_{\gamma_{\overline{\Pi}'}}$ are non-trivial and hence neither (a) nor (b) holds. But $\overline{\Pi}$ is not of type D or E, by Prop. 3.4. Thus, (a) or (b) holds if and only if one of the following holds:

(d) $\overline{\Pi}$ and $\overline{\Pi}'$ are of type A_r $(r \geq 2)$.

(e) We may label the roots of $\overline{\Pi}$ as follows: p lines  ,
where $1 \leq s < r$, p=2 or 3, and, if $\gamma_i \longrightarrow \gamma_i'$ is the isomorphism of $\overline{\Pi}$ onto $\overline{\Pi}'$, the actions of $[\mathcal{L}_0, \mathcal{L}_0]$ on $\mathcal{L}_{\gamma_{s+1}}$ and $[\mathcal{L}_0', \mathcal{L}_0']$ on $\mathcal{L}'_{\gamma'_{s+1}}$ are trivial.

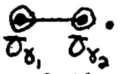
Then, in these cases φ_0 is an isomorphism of one of the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ onto one of the simple summands of $[\mathcal{L}_0', \mathcal{L}_0']$.

Suppose first of all that <u>(d) holds</u>. Label the roots of $\overline{\Pi}$ as follows:  . Let $\gamma_i \longrightarrow \gamma_i'$ be the isomorphism of $\overline{\Pi}$ onto $\overline{\Pi}'$. Putting $T = \overline{\Pi}$ and $T' = \overline{\Pi}'$, we may apply the discussion at the beginning of this chapter. We have $\mathcal{L}_{T0} = [\mathcal{L}_0, \mathcal{L}_0]$ and $\mathcal{L}'_{T'0} = [\mathcal{L}_0', \mathcal{L}_0']$. Suppose <u>$[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are simple</u>. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' so that $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])^{\varphi_0} = \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']$ and $\Pi_0^{\varphi_0^*} = \Pi_0'$. Then, by lemma 2, there exists a *-isomorphism $(\Pi, \Pi_0) \xrightarrow{f} (\Pi', \Pi_0')$ and $\overline{w} \in \overline{W}$ such that $f \mid \Pi_0 = (\overline{w} \mid \Pi_0) \circ \varphi_0^*$. By Thm. 1.2, we are done in this case. Suppose <u>$[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are not simple</u>. Label the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ as in (7). Choose $i, i' \in \{1, \ldots, r+1\}$ such that φ_0 is an isomorphism of $\mathcal{L}_{o,i}$ onto $\mathcal{L}'_{o,i'}$. Choose $\mathfrak{h}$, Π_o, $\mathfrak{h}'$, and Π_o' as in lemma 1. Then, by lemma 1 (with $W = 1$), we obtain an isomorphism $([\mathcal{L}_o,\mathcal{L}_o], \mathfrak{h}\cap[\mathcal{L}_o,\mathcal{L}_o]) \xrightarrow{\psi_o} ([\mathcal{L}_o',\mathcal{L}_o'], \mathfrak{h}'\cap[\mathcal{L}_o',\mathcal{L}_o'])$ such that $\psi_o|\mathcal{L}_{o,i} = \varphi_o$ and, by Thm. 1.2, ψ_o extends to an isomorphism $(\mathcal{L},\mathfrak{I}) \longrightarrow (\mathcal{L}',\mathfrak{I}')$.

Suppose next that <u>(e) holds</u>. Putting $T = \{\gamma_1,\dots,\gamma_s\}$ and $T' = \{\gamma_1',\dots,\gamma_s'\}$, we may apply the discussion at the beginning of this chapter. For any choice of $\mathfrak{h}$ and Π_o, we have $\Pi_{T0} = \Pi_o$ (by Prop. 3.5), $\Pi_{o,\gamma_{s+1}} = \phi$ (by the corollary to Prop. 2.2), $\Pi_{o,\gamma_{s+1}} = \dots = \Pi_{o,\gamma_r} = \phi$ (by Prop. 3.6), Π_T is connected (by Prop. 3.7), and $\mathcal{O}_{\gamma_s}\cup\Pi_{\gamma_{s+1}}\cup\dots\cup\Pi_{\gamma_r}$ is of the form $\underset{\mathcal{O}_{\gamma_s}}{\ominus}-\underset{\mathcal{O}_{\gamma_{s+1}}}{\ominus}\cdots\underset{\mathcal{O}_{\gamma_{r-1}}}{\ominus}-\underset{\mathcal{O}_{\gamma_r}}{\ominus}$ (by Prop. 3.7). A similar remark holds for $\mathcal{L}'$. Therefore, $\mathcal{L}_{T0} = [\mathcal{L}_o,\mathcal{L}_o]$ and $\mathcal{L}_{T'0} = [\mathcal{L}_o',\mathcal{L}_o']$. Moreover, for any choice of $\mathfrak{h}$, Π_o, $\mathfrak{h}'$, and Π_o', we have that any $*$-isomorphism $(\Pi_T, \Pi_o) \xrightarrow{f} (\Pi'_{T'}, \Pi_o')$ such that $\mathcal{O}_{\gamma_s}{}^f = \mathcal{O}_{\gamma_s'}$ extends to a $*$-isomorphism $(\Pi,\Pi_o) \longrightarrow (\Pi',\Pi_o')$. Thus, by Thm. 1.2, it suffices to choose $\mathfrak{h}$, Π_o, $\mathfrak{h}'$, and Π_o' with the following property: There exists an isomorphism $([\mathcal{L}_o,\mathcal{L}_o], \mathfrak{h}\cap[\mathcal{L}_o,\mathcal{L}_o]) \longrightarrow ([\mathcal{L}_o',\mathcal{L}_o'], \mathfrak{h}'\cap[\mathcal{L}_o',\mathcal{L}_o'])$, a $*$-isomorphism $(\Pi_T,\Pi_o) \xrightarrow{f} (\Pi'_{T'},\Pi_o')$, and $\bar{w}_1 \in \overline{W}$ such that ψ_o extends φ_o, $\Pi_o^{\psi_o^*} = \Pi_o'$, and $f|\Pi_o = (\bar{w}_1|\Pi_o)\circ\psi_o^*$. We consider the cases $p=2$ and $p=3$ separately.

Suppose <u>$p=2$</u>. We first make a calculation which holds for all $\mathfrak{h}$ and Π_o. Now, $\{\gamma_s,\gamma_{s+1}\}^{(\bar{w}_{\gamma_s}\bar{w}_{\gamma_{s+1}})^2} = \{-\gamma_s,-\gamma_{s+1}\}$ and $\Pi_{o,\gamma_s} = \Pi_{o,\gamma_s}\cup\Pi_{o,\gamma_{s+1}}$. Thus, by lemma 2.7, $1_{\Pi_{\gamma_s}\cup\Pi_{\gamma_{s+1}}}|\Pi_{o,\gamma_s} = (\bar{w}_{\gamma_s}\bar{w}_{\gamma_{s+1}})^2|\Pi_{o,\gamma_s}\circ 1_{\Pi_{o,\gamma_s}}$. But $\bar{w}_{\gamma_{s+1}}$ fixes the elements of Π_{0,γ_s} (since $\Pi_{0,\gamma_{s+1}} = \phi$) and hence $i_{\Pi_{\gamma_s} \cup \Pi_{\gamma_{s+1}}} | \Pi_{0,\gamma_s} = i_{\Pi_{0,\gamma_s}}$. But $\gamma_s(\overline{w}_{\gamma_s} \overline{w}_{\gamma_{s+1}})^{a} = -\gamma_s$ and hence, by lemma 2.7, $i_{\Pi_{\gamma_s} \cup \Pi_{\gamma_{s+1}}}$ stabilizes $\mathcal{D}_{\gamma_s}$. Thus, $i_{\Pi_{\gamma_s} \cup \Pi_{\gamma_{s+1}}}$ stabilizes Π_{γ_s}. But $i_{\Pi_{\gamma_s} \cup \Pi_{\gamma_{s+1}}}$ commutes with any automorphism of $\Pi_{\gamma_s} \cup \Pi_{\gamma_{s+1}}$ and hence with the $*$-action. Thus, $i_{\Pi_{0,\gamma_s}}$ extends to a $*$-automorphism of Π_s. But $\overline{w}_{\gamma_s} | \Pi_{0,\gamma_s} = (i_{\Pi_{\gamma_s}} | \Pi_{0,\gamma_s}) \circ i_{\Pi_{0,\gamma_s}}$. Thus, $\overline{w}_{\gamma_s} | \Pi_{0,\gamma_s}$ extends to a $*$-automorphism of Π_{γ_s}. A similar result holds for $\mathcal{L}'$. Suppose now that <u>$[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are simple</u>. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' so that $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])^{\varphi_0} = \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']$ and $\Pi_0^{\varphi_0^*} = \Pi_0'$. Then, Π_0 and Π_0' are $*$-connected and hence (by Prop. 3.5) s=1 or 2. If s=2, we are done by lemma 2. Suppose <u>s=1</u>. If Π_0 and Π_0' are $*$-connected of $*$-type D_{4I}, we are done by lemma 5 (applied to $g_0 = \varphi_0^*$, $\gamma = \gamma_1$, and $\gamma' = \gamma_1'$). On the other hand, if Π_0 and Π_0' are not $*$-connected of $*$-type D_{4I}, we are done by lemma 6 (applied to $\gamma = \gamma_1$ and $\gamma' = \gamma_1'$). Suppose then that <u>$[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are not simple.</u> Label the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ as in (7). Choose $i, i' \in \{1, \ldots, s+1\}$ such that φ_0 is an isomorphism of $\mathcal{L}_{0,i}$ onto $\mathcal{L}_{0,i'}'$. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$ and Π_0' as in lemma 1. Applying this lemma (with $\overline{w} = 1$), we are done provided φ_0^* is extendable (since we need $\mathcal{D}_{\gamma_s}{}^{f} = \mathcal{D}_{\gamma_s'}$). But $i_{\Pi_{0,\gamma_s}}$ extends to a $*$-automorphism of Π_{γ_s} and hence, since (Π_T, Π_0) if of the form (8), the $*$-components of Π_{γ_s} are all of $*$-type A_1 and φ_0^* is extendable.

Suppose <u>p=3</u>. Then, s=1 and r=2. Suppose <u>$[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are simple</u>. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' so that $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])^{\varphi_0} = \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']$ and $\Pi_0^{\varphi_0^*} = \Pi_0'$. Then,

$\Pi_0 = \Pi_{0,\gamma_1}$ is *-connected, $\Pi_{0,\gamma_2} = \phi$, $\Pi_T = \Pi_{\gamma_1}$ is connected, and $\sigma_{\gamma_1} \cup \Pi_{\gamma_2}$ is of the form . Therefore, Π_{0,γ_1} is not of *-type D_{4I} (since otherwise Π is of the form  and hence $\mu_{\Pi} = 2\gamma_1 + \gamma_2$, contradicting lemma 2.4). Similar remarks hold for $\mathcal{L}'$. By lemma 8, the hypothesis (b) of lemma 5 is satisfied (with $g_0 = \varphi_0^*$, $\gamma = \gamma_1$, and $\gamma' = \gamma_1'$) and hence, by lemma 5, we are done. Suppose $[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ are not simple. Label the simple summands of $[\mathcal{L}_0, \mathcal{L}_0]$ and $[\mathcal{L}_0', \mathcal{L}_0']$ as in (7) and choose $i, i' \in \{1,2\}$ so that φ_0 maps $\mathcal{L}_{0,i}$ onto $\mathcal{L}_{0,i'}'$. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' as in lemma 1. Applying lemma 1 (with $\overline{w} = 1$), we are done in this case.

Suppose next that (c) holds. Then, we have one of the following:

(f) $\overline{\Pi}$ and $\overline{\Pi}'$ have rank 1.

(g) $\overline{\Pi}$ and $\overline{\Pi}'$ are of type C_r ($r \geq 2$), $[\mathcal{L}_0, \mathcal{L}_0]$ acts non-trivially on $\mathcal{L}_{\gamma_{\overline{\Pi}}}$, and $[\mathcal{L}_0', \mathcal{L}_0']$ acts non-trivially on $\mathcal{L}'_{\gamma'_{\overline{\Pi}'}}$.

(h) $\overline{\Pi}$ and $\overline{\Pi}'$ are not reduced and have rank > 1.

In each of these cases, we have an isomorphism $[\mathcal{L}_{\gamma_{\overline{\Pi}}}, \mathcal{L}_{-\gamma_{\overline{\Pi}}}] \xrightarrow{\chi_0} [\mathcal{L}'_{\gamma'_{\overline{\Pi}'}}, \mathcal{L}'_{-\gamma'_{\overline{\Pi}'}}]$ and a linear bijection $\mathcal{L}_{\gamma_{\overline{\Pi}}} \xrightarrow{\ f\ } \mathcal{L}'_{\gamma'_{\overline{\Pi}'}}$ such that $\Pi_{\gamma_{\overline{\Pi}}}^{\chi_0} = \Pi'_{\gamma'_{\overline{\Pi}'}}$ and $[X, X_0]^f = [X^f, X_0^{\chi_0}]$ for $X \in \mathcal{L}_{\gamma_{\overline{\Pi}}}$ and $X_0 \in [\mathcal{L}_{\gamma_{\overline{\Pi}}}, \mathcal{L}_{-\gamma_{\overline{\Pi}}}]$. Put $\varphi_0 = \chi_0 | \mathcal{L}_{0,\gamma_{\overline{\Pi}}}$.

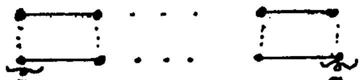
Suppose then that (f) holds. Then, $\overline{\Pi} = \{\gamma_{\overline{\Pi}}\}$ and $\overline{\Pi}' = \{\gamma'_{\overline{\Pi}'}\}$. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' so that $(\mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0])^{\varphi_0} = \mathfrak{h}' \cap [\mathcal{L}_0', \mathcal{L}_0']$ and $\Pi_0^{\varphi_0^*} = \Pi_0'$. Now, if $\overline{\Pi}$ and $\overline{\Pi}'$ are not reduced, we have (by Prop. 4.1(ii)) $\mathcal{L}_{0,2\gamma_{\overline{\Pi}}}^{\varphi_0} = \mathcal{L}'_{0,2\gamma'_{\overline{\Pi}'}}$ and hence $\Pi_{0,2\gamma_{\overline{\Pi}}}^{\varphi_0^*} = \Pi'_{0,2\gamma'_{\overline{\Pi}'}}$. In any case, by lemma 3, we have a *-isomorphism $(\Pi, \Pi_0) \xrightarrow{\ g\ } (\Pi', \Pi_0')$ such that $g | \Pi_0 = \varphi_0^*$ and hence, by Thm. 1.2, we are done in this case.

Suppose next that <u>(g) holds</u>. Label the roots of $\overline{\Pi}$ as follows:

. Suppose that $\gamma_i \longrightarrow \gamma_i'$ is the isomorphism of $\overline{\Pi}$ onto $\overline{\Pi}'$. Then, $\gamma_{\overline{\Pi}} = \gamma_r$ and $\gamma'_{\overline{\Pi}'} = \gamma_r'$. Put T = $\{\gamma_1, \ldots, \gamma_{r-1}\}$ and T' = $\{\gamma_1', \ldots, \gamma_{r-1}'\}$. We may apply the discussion at the beginning of this chapter. For any choice of $\mathfrak{h}$ and Π_o, we have $\Pi_{TO} = \Pi_o$ (by Prop. 3.5), $\Pi_{o,\gamma_r} \neq \phi$ (by the corollary to Prop. 2.2), (Π_T, Π_o) is of the form (4) (by Prop. 3.6), Π_{γ_r} is connected (by the corollary to Prop. 2.4), and Π_{o,γ_r} is one of the two $*$-components of $\Pi_{o,\gamma_{r-1}}$ (by Prop. 3.6). Then, $\mathcal{L}_{TO} = [\mathcal{L}_o, \mathcal{L}_o]$ is not simple and we may label the simple summands $\mathcal{L}_{o,1}, \ldots, \mathcal{L}_{o,r}$ of $[\mathcal{L}_o, \mathcal{L}_o]$ as in (7) and so that $\mathcal{L}_{o,r} = \mathcal{L}_{o,\gamma_r}$. Similarly, $\mathcal{L}_{T'O} = [\mathcal{L}_o', \mathcal{L}_o']$ is not simple and we may label the simple summands $\mathcal{L}_{o,1}', \ldots, \mathcal{L}_{o,r}'$ of $[\mathcal{L}_o', \mathcal{L}_o']$ as in (7) and so that $\mathcal{L}'_{o,\gamma_r'} = \mathcal{L}'_{o,r}$. Then, φ_o is an isomorphism of $\mathcal{L}_{o,r}$ onto $\mathcal{L}'_{o,r}$. Choose $\mathfrak{h}$, Π_o, $\mathfrak{h}'$, and Π_o' as in lemma 1. If $\varphi_o{}^*$ is extendable, we have (by lemma 1 with $\overline{w} = 1$) an isomorphism

$$([\mathcal{L}_o, \mathcal{L}_o], \mathfrak{h} \cap [\mathcal{L}_o, \mathcal{L}_o]) \xrightarrow{\psi_o} ([\mathcal{L}_o', \mathcal{L}_o'], \mathfrak{h}' \cap [\mathcal{L}_o', \mathcal{L}_o'])$$

and a $*$-isomorphism $(\Pi_T, \Pi_o) \xrightarrow{f} (\Pi_{T'}', \Pi_o')$ such that ψ_o extends φ_o, $\Pi_o^{\psi_o^*} = \Pi_o'$, $f \mid \Pi_o = \psi_o{}^*$, and $\sigma_{\gamma_j}{}^f = \sigma_{\gamma_j'}$, j=1,...,r-1. But, by lemma 3, there exists a $*$-isomorphism $(\Pi_{\gamma_r}, \Pi_{o,\gamma_r}) \xrightarrow{g} (\Pi'_{\gamma_r'}, \Pi'_{o,\gamma_r'})$ such that $g \mid \Pi_{o,\gamma_r} = \varphi_o{}^*$. Hence, putting $h \mid \Pi_{\gamma_r} = g$ and $h \mid \Pi_T = f$, we have a $*$-isomorphism $(\Pi, \Pi_o) \xrightarrow{h} (\Pi', \Pi_o')$ such that $h \mid \Pi_o = \psi_o{}^*$. Thus, by Thm. 1.2, we are done if $\varphi_o{}^*$ is extendable. Suppose <u>$\varphi_o{}^*$ is not extendable</u>. We may isolate $\Pi_{o,r}$ from (8) and define $*$-orbits C_L and C_R of $\Pi_{o,r}$ as follows:



C_L $\qquad$ C_R .

Similarly , define $*$-orbits C'_L and C'_R of $\Pi_{o,r}$. Then, $C_L^{\varphi_o^*} = C'_R$ and $C_R^{\varphi_o^*} = C'_L$. By lemma 4, $\mho_{\gamma_r}$ is a singleton $\{\alpha_r\}$, $\mho_{\gamma'_r}$ is a singleton $\{\alpha'_r\}$, $\gamma_r = \alpha_r + \sum_{\beta \in \Pi_{o,\gamma_r}} q_\beta \beta$, and $\gamma'_r = \alpha'_r + \sum_{\beta' \in \Pi'_{o,\gamma'_r}} q'_{\beta'} \beta'$, where the q_β and $q'_{\beta'}$ are rational. Let $\beta \in C_L$. Choose $\alpha_{r-1} \in \mho_{\gamma_{r-1}}$ such that $(\beta, \alpha_{r-1}) < 0$. Then, by lemma 9,

$$q_\beta = -2\frac{(\gamma_r, \alpha_{r-1})}{(\alpha_{r-1}, \alpha_{r-1})} = -2\frac{(\gamma_r, \gamma_{r-1})}{(\alpha_{r-1}, \alpha_{r-1})} = 2\frac{(\gamma_{r-1}, \gamma_{r-1})}{(\alpha_{r-1}, \alpha_{r-1})} = \frac{2}{f(g+1)} ,$$

where $\Pi_{o,r}$ contains f components of type A_g. Thus, $q_\beta = \frac{2}{f(g+1)}$ for $\beta \in C_L$. Similarly , $q'_{\beta'} = \frac{2}{f(g+1)}$ for $\beta' \in C'_L$. Now, by lemma 3, there exists a $*$-isomorphism $(\Pi_{\gamma_r}, \Pi_{o,\gamma_r}) \xrightarrow{g} (\Pi'_{\gamma'_r}, \Pi'_{o,\gamma'_r})$ such that $g \mid \Pi_{o,\gamma_r} = \varphi_o^*$. Then, $C_L'^{\,g^{-1}} = C_L'^{\,\varphi_o^{*-1}} = C_R$. But $q_{\beta'} g^{-1} = q'_{\beta'}$ for $\beta' \in C'_L$ (since $\gamma_r'^{\,g} = \gamma_r$) and hence $q_\beta = \frac{2}{f(g+1)}$ for $\beta \in C_R$. Therefore, $q_\beta = \frac{2}{f(g+1)}$ for $\beta \in C_L \cup C_R$. Write $\mu_{\Pi_{\gamma_r}} = \alpha_r + \sum_{\beta \in \Pi_{o,\gamma_r}} m_\beta \beta$, where the m_β are positive integers. By lemma 4, $q_\beta + q_\beta w_{\gamma_r} = m_\beta$ for $\beta \in C_L \cup C_R$. But $(C_L \cup C_R)^{\overline{w}_{\gamma_r}} = C_L \cup C_R$. Thus, $m_\beta = \frac{4}{f(g+1)}$ for $\beta \in C_L \cup C_R$. But, since φ_o^* is not extendable, $g > 1$. Therefore, $f=1$, $g=3$, and $m_\beta = 1$ for $\beta \in C_L \cup C_R$. Therefore, Π_{o,γ_r} is connected of type A_3. If $\beta \in C_L \cup C_R$, we have $(\beta, \alpha_r) = 0$, since otherwise $(\beta, \hat{\alpha}_r) = -1$ and hence $(\mu_{\Pi_{\gamma_r}}, \hat{\alpha}_r) = 2 - m_\beta = 1$, contradicting lemma 4. Thus, α_r is connected to the middle root of Π_{o,γ_r}. Therefore, $(\Pi_{\gamma_r}, \Pi_{o,\gamma_r})$ and $(\Pi'_{\gamma'_r}, \Pi'_{o,\gamma'_r})$ are of the form  and $\overline{w}_{\gamma_r} \mid \Pi_{o,\gamma_r}$ is the map  . Thus, $(\overline{w}_{\gamma_r} \mid \Pi_{o,\gamma_r}) \circ \varphi_o^*$ is extendable. By lemma 1 (with $\overline{W} = \overline{w}_{\gamma_r}$), there exists an isomorphism

$([\mathcal{L}_0,\mathcal{L}_0], \mathfrak{h}\cap[\mathcal{L}_0,\mathcal{L}_0]) \xrightarrow{\psi_0} ([\mathcal{L}_0',\mathcal{L}_0'], \mathfrak{h}'\cap[\mathcal{L}_0',\mathcal{L}_0'])$, a *-isomorphism $(\Pi_T, \Pi_0) \xrightarrow{f} (\Pi'_{T'}, \Pi'_0)$, and $\bar{w}_1 \in \bar{W}$ such that ψ_0 extends φ_0, $\Pi_0^{\psi_0^*} = \Pi_0'$, $f|\Pi_0 = (\bar{w}_1|\Pi_0)\circ\psi_0^*$, $\sigma_{\gamma_j}^{f} = \sigma_{\gamma_j'}$ for j=1,...,r-1, and $\bar{w}_1|\Pi_{0,\gamma_r} = w_{\gamma_r}|\Pi_{0,\gamma_r}$. Define $(\Pi,\Pi_0)\xrightarrow{h}(\Pi',\Pi_0')$ by $h|\Pi_T = f$ and $h|\Pi_{0,\gamma_r} = h_0\circ g$, where h_0 is the unique *-automorphism of Π_{0,γ_r} extending $w_{\gamma_r}|\Pi_{0,\gamma_r}$. Then, h is a well defined *-isomorphism such that $h|\Pi_0 = (\bar{w}_1|\Pi_0)\circ\psi_0^*$. By Thm. 1.2, we are done in this case.

It remains to consider the case when (h) holds i.e. when $\overline{\overline{\Pi}}$ and $\overline{\overline{\Pi}}'$ are not reduced and have rank $r>1$. Label the roots of $\overline{\overline{\Pi}}$ as follows: $\underset{\gamma_1}{\circ}\!-\!\underset{\gamma_2}{\circ}\cdots\underset{\gamma_{r-1}}{\circ}\!\Rightarrow\!\underset{\gamma_r}{\boxtimes}$. Suppose that $\gamma_i \longrightarrow \gamma_i'$ is the isomorphism of $\overline{\overline{\Pi}}$ onto $\overline{\overline{\Pi}}'$. Then, $\gamma_{\overline{\overline{\Pi}}} = \gamma_r$ and $\gamma'_{\overline{\overline{\Pi}}'} = \gamma_r'$. Put $\mathfrak{M}_0 = \mathcal{L}_{0,\gamma_{r-1}}\cap\mathcal{L}_{0,\gamma_r}$ and $\mathfrak{M}_0' = \mathcal{L}'_{0,\gamma'_{r-1}}\cap\mathcal{L}'_{0,\gamma'_r}$. By Prop. 4.1(iii), $\mathfrak{M}_0^{\varphi_0} = \mathfrak{M}_0'$. By Thm.4.4, $\mathcal{L}_{0,\gamma_r} = \mathfrak{M}_0\oplus\mathfrak{N}_{\gamma_r}$ and exactly one of the following holds:

(i) r=2, $\mathfrak{M}_0 \neq (0)$ and $[\mathcal{L}_0,\mathcal{L}_0] = \mathcal{L}_{0,\gamma_r}$.

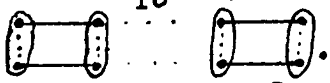
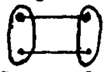
(j) $[\mathcal{L}_0,\mathcal{L}_0]$ is the direct sum of $\mathfrak{N}_{\gamma_r}$, $\mathfrak{M}_0$, and r-1 ideals of $[\mathcal{L}_0,\mathcal{L}_0]$ isomorphic to $\mathfrak{M}_0$.

We have a similar statement for $\mathcal{L}'$ and it is clear that (i) holds for $\mathcal{L}$ if and only if it holds for $\mathcal{L}'$.

Suppose then that <u>(i) holds for $\mathcal{L}$ and $\mathcal{L}'$</u>. Choose $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' such that $(\mathfrak{h}\cap[\mathcal{L}_0,\mathcal{L}_0])^{\varphi_0} = \mathfrak{h}'\cap[\mathcal{L}_0',\mathcal{L}_0']$ and $\Pi_0^{\varphi_0^*} = \Pi_0'$. Then, $\Pi_0 = \Pi_{0,\gamma_2}$, $\Pi_0' = \Pi'_{0,\gamma_2'}$, $\Pi_{0,\gamma_2}\cap\Pi_{0,\gamma_2} = \Pi_{0,\gamma_1}$, and $\Pi_{0,\gamma_1'}\cap\Pi_{0,\gamma_2'} = \Pi_{0,\gamma_1'}$. Now, by the corollary to Prop. 2.4, Π_{γ_2} and $\Pi'_{\gamma_2'}$ are connected and, by Prop. 3.9, $\Pi_{0,2\gamma_2} = \phi$ and $\Pi'_{0,2\gamma_2'} = \phi$. Thus, by lemma 3, there exists a *-isomorphism $(\Pi_{\gamma_2},\Pi_0)\xrightarrow{g}(\Pi'_{\gamma_2'},\Pi_0')$ such that $g|\Pi_0 = \varphi_0^*$.

But, since $\mathfrak{M}_o^{\varphi_o} = \mathfrak{M}_o'$, $\Pi_{o,\gamma_1}{}^g = \Pi'_{o,\gamma_1'}$. Thus, to complete this case, it suffices to show that $g \mid \Pi_{o,\gamma_1}$ extends to a *-isomorphism $\Pi_{\gamma_1} \longrightarrow \Pi'_{\gamma_1'}$ (since then g extends to a *-isomorphism $(\Pi, \Pi_o) \longrightarrow (\Pi', \Pi_o')$ and we are done by Thm. 1.2). But Π_{γ_1} and $\Pi'_{\gamma_1'}$ are connected (by Prop. 3.10) and $[\mathcal{L}_o, \mathcal{L}_o]$ does not contain a simple summand which is isomorphic to the skew transformations with respect to the trace form in a Cayley division algebra over k and the same holds for $[\mathcal{L}_o', \mathcal{L}_o']$ (by lemma 7). Thus, by lemma 5 and 6, we can accomplish the required extension provided that $\overline{w}_{\gamma_1} \mid \Pi_{o,\gamma_1}$ extends to a *-automorphism of Π_{γ_1} (and $\mathcal{L}'$ has the analogous property). But $\overline{\Pi}^{(\overline{w}_{\gamma_1} \overline{w}_{\gamma_2})^2} = -\overline{\Pi}$ and hence, by lemma 2.7, $i_\Pi \mid \Pi_o = ((\overline{w}_{\gamma_1} \overline{w}_{\gamma_2})^2 \mid \Pi_o) \circ i_{\Pi_o}$. But $\Pi_{o,2\gamma_2} = \phi$ and hence $\overline{w}_{\gamma_2} = \overline{w}_{2\gamma_2}$ fixes the elements of Π_o. Hence, $i_\Pi \mid \Pi_o = i_{\Pi_o}$. Thus, i_Π stabilizes Π_{o,γ_1} and $i_\Pi \mid \Pi_{o,\gamma_1} = i_{\Pi_{o,\gamma_1}}$. But $\sigma_{\gamma_1}{}^{i_\Pi} = \sigma_{\gamma_1 i_\Pi} = \sigma_{\gamma_1}$ and hence i_Π stabilizes Π_{γ_1}. Therefore, $i_{\Pi_{o,\gamma_1}}$ extends to a *-automorphism of Π_{γ_1}. But $\overline{w}_{\gamma_1} \mid \Pi_{o,\gamma_1} = (i_{\Pi_{\gamma_1}} \mid \Pi_{o,\gamma_1}) \circ i_{\Pi_{o,\gamma_1}}$. Therefore, $\overline{w}_{\gamma_1} \mid \Pi_{o,\gamma_1}$ extends to a *-automorphism of Π_{γ_1}.

Suppose now that <u>(j) holds for $\mathcal{L}$ and $\mathcal{L}'$</u>. Put $T = \{\gamma_1, \ldots, \gamma_{r-1}\}$ and $T' = \{\gamma_1', \ldots, \gamma_{r-1}'\}$. Then, for any $\mathfrak{h}$, Π_o, $\mathfrak{h}'$, and Π_o', (Π_T, Π_{To}) and $(\Pi'_{T'}, \Pi'_{T'o})$ are of the form (4) (by Prop. 3.8) and Π_{γ_r} and $\Pi'_{\gamma_r'}$ are connected (by the corollary to Prop. 2.2). Suppose that <u>$\mathfrak{M}_o = (0)$ and $\mathfrak{M}_o' = (0)$</u>. Then, $[\mathcal{L}_o, \mathcal{L}_o] = \mathcal{L}_{o,\gamma_r}$ and $[\mathcal{L}_o', \mathcal{L}_o'] = \mathcal{L}'_{o,\gamma_r'}$. Choose $\mathfrak{h}, \Pi_o, \mathfrak{h}', \Pi_o'$ so that $(\mathfrak{h} \cap [\mathcal{L}_o, \mathcal{L}_o])^{\varphi_o} = \mathfrak{h}' \cap [\mathcal{L}_o', \mathcal{L}_o']$ and $\Pi_o^{\varphi_o^*} = \Pi_o'$. Then, $\Pi_{o,\gamma_{r-1}} \cap \Pi_{o,\gamma_r} = \phi$ and $\Pi'_{o,\gamma_{r-1}'} \cap \Pi'_{o,\gamma_r'} = \phi$. Therefore, $\Pi_{o,2\gamma_2} = \phi$ and $\Pi'_{o,2\gamma_r'} = \phi$. Also, by Prop. 3.8, $\Pi_{o,\gamma_{r-1}} = \phi$ and hence $\Pi_{o,\gamma_{r-1}+2\gamma_r} = \Pi_{o,\gamma_{r-1}}^{\overline{w}_{\gamma_r}} = \phi$. Similarily, $\Pi'_{o,\gamma_{r-1}'+2\gamma_r'} = \phi$. Now, by lemma 3, there exists a *-isomorphism $(\Pi_{\gamma_r}, \Pi_o) \xrightarrow{g} (\Pi'_{\gamma_r'}, \Pi_o')$ such that $g \mid \Pi_o = \varphi_o^*$. By Thm. 1.2, it

suffices to show that g extends to a *-isomorphism $(\Pi, \Pi_0) \longrightarrow (\Pi', \Pi_0')$.
But by Prop. 3.8, $\Pi_{T0} = \phi$ and $\Pi'_{T'0} = \phi$. Hence, Π_T and $\Pi'_{T'}$ are
of the form . Thus, it suffices to show that
$\mho_{\gamma_{r-1}} \cup \mho_{\gamma_r}$ is of the form  (and an analogous result for $\mathcal{L}'$).
$\mho_{\gamma_{r-1}}$ $\mho_{\gamma_r}$

But since $\Pi_{\gamma_{r-1}} = \phi$, every element of $\mho_{\gamma_{r-1}}$ is connected to an element of $\mho_{\gamma_r}$ and every element of $\mho_{\gamma_r}$ is connected to an element of $\mho_{\gamma_{r-1}}$. No element of $\mho_{\gamma_{r-1}}$ is connected to two elements of $\mho_{\gamma_r}$ (since otherwise any element of Π_0 connected to one of those two elements of $\mho_{\gamma_r}$ would be an element of $\Pi_{0,\gamma_{r-1}+2\gamma_r}$) and no element of $\mho_{\gamma_r}$ is connected to two elements of $\mho_{\gamma_{r-1}}$ (since $2\gamma_{r-1} + \gamma_r \notin \Sigma$). Therefore, it remains to show that if $\alpha_{r-1} \in \mho_{\gamma_{r-1}}$ and $\alpha_r \in \mho_{\gamma_r}$ are connected then $(\alpha_{r-1}, \hat{\alpha}_r) = -1$ and $(\alpha_r, \widehat{\alpha_{r-1}}) = -1$. The first equation holds since otherwise any element of Π_0 connected to α_r would be an element of $\Pi_{0,\gamma_{r-1}+2\gamma_r}$ and the second equation holds since $2\gamma_{r-1} + \gamma_r \notin \Sigma$. Suppose finally that $\underline{\mathfrak{M}_0 \neq (0)}$ and $\underline{\mathfrak{M}_0' \neq (0)}$. Then, $\mathcal{L}_{T0}$ and $\mathcal{L}'_{T'0}$ are non-zero and non-simple. Label the simple summands $\mathcal{L}_{0,1}, \dots, \mathcal{L}_{0,r}$ of $\mathcal{L}_{T0}$ as in (7) and so that $\mathcal{L}_{0,r} = \mathfrak{M}_0$. Label the simple summands $\mathcal{L}'_{0,1}, \dots, \mathcal{L}'_{0,r}$ of $\mathcal{L}'_{T'0}$ as in (7) and so that $\mathcal{L}'_{0,r} = \mathfrak{M}_0'$. We apply lemma 1 to $\mathcal{L}_{0,r} \xrightarrow{\varphi_0 | \mathcal{L}_{0,r}} \mathcal{L}'_{0,r}$. It is clear from the proof of lemma 1 that the choice of $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π_0' can be made so that the conclusion of the lemma holds and so that $(\mathfrak{h} \cap \mathfrak{n}_{\gamma_r})^{\varphi_0} = \mathfrak{h}' \cap \mathfrak{n}'_{\gamma_r'}$ and $\Pi_{0,\mathfrak{n}_{\gamma_r}}^{\varphi_0^*} = \Pi'_{0,\mathfrak{n}'_{\gamma_r'}}$, where for the moment $\Pi_{0,\mathfrak{n}_{\gamma_r}}$ denotes the union of the *-components corresponding to simple summands of $\mathfrak{n}_{\gamma_r}$ and $\Pi'_{0,\mathfrak{n}'_{\gamma_r'}}$ is defined similarly. Then, since $\Pi_{0,\gamma_r} = (\Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r}) \cup \Pi_{0,\mathfrak{n}_{\gamma_r}}$ and $\Pi'_{0,\gamma_r'} = (\Pi'_{0,\gamma'_{r-1}} \cap \Pi'_{0,\gamma'_r}) \cup \Pi'_{0,\mathfrak{n}'_{\gamma'_r}}$, we have $(\mathfrak{h} \cap \mathcal{L}_{0,\gamma_r})^{\varphi_0} = \mathfrak{h}' \cap \mathcal{L}'_{0,\gamma'_r}$ and $\Pi_{0,\gamma_r}^{\varphi_0^*} = \Pi'_{0,\gamma'_r}$. Now, $\Pi_{0,2\gamma_r} \subseteq \Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r}$, $\Pi'_{0,2\gamma'_r} \subseteq \Pi'_{0,\gamma'_{r-1}} \cap \Pi'_{0,\gamma'_r}$, and the right hand side of these inclusions is *-connected. But, since $\mathfrak{M}_0^{\varphi_0} = \mathfrak{M}_0'$, $(\Pi_{0,\gamma_{r-1}} \cap \Pi_{0,\gamma_r})^{\varphi_0^*} = \Pi'_{0,\gamma'_{r-1}} \cap \Pi'_{0,\gamma'_r}$. Hence, either $\Pi_{0,\partial\gamma_r}^{\varphi_0^*} \subseteq \Pi'_{0,\partial\gamma'_r}$ or $\Pi'_{0,\partial\gamma'_r} \subseteq \Pi_{0,\partial\gamma_r}^{\varphi_0^*}$. Therefore, by lemma 3, there exists a *-isomorphism $(\Pi_{\gamma_r}, \Pi_{0,\gamma_r}) \xrightarrow{g} (\Pi'_{\gamma'_r}, \Pi'_{0,\gamma'_r})$ such that $g \mid \Pi_{0,\gamma_r} = \varphi_0^*$. Define the right *-orbit C_R of $\Pi_{0,r}$ and the right *-orbit C'_R of $\Pi'_{0,r}$ as previously. Now, since $\Pi_{0,r} \subseteq \Pi_{0,\gamma_r}$, the set C of elements of $\Pi_{0,r}$ which are connected to elements of $\mho_{\gamma_r}$ is a *-orbit of $\Pi_{0,r}$. Similarly, the set C$'$ of elements of $\Pi'_{0,r}$ which are connected to elements of $\mho_{\gamma'_r}$ is a *-orbit of $\Pi'_{0,r}$. But if $C \neq C_R$, it is easy to see that $2\gamma_{r-1} + \gamma_r \in \overline{\Sigma}$, a contradiction. Thus, $C = C_R$. Similarly, $C' = C'_R$. But, since $(\Pi_{\gamma_r}, \Pi_{0,\gamma_r}) \xrightarrow{g} (\Pi'_{\gamma'_r}, \Pi'_{0,\gamma'_r})$ is a *-isomorphism such that $\mho_{\gamma_r}{}^{g} = \mho_{\gamma'_r}$ and $\Pi_{0,r}{}^{g} = \Pi'_{0,r}$, we have $C^g = C'$. Therefore, $C_R{}^{\varphi_0^*} = C_R{}^{g} = C^g = C' = C'_R$. Therefore, φ_0^* is extendable. Therefore, by lemma 1 (with $\mathfrak{W} = 1$), $\mathfrak{h}$, Π_0, $\mathfrak{h}'$, and Π'_0 have the following property: There exists an isomorphism $(\mathcal{L}_{T0}, \mathfrak{h} \cap \mathcal{L}_{T0}) \xrightarrow{\psi_0} (\mathcal{L}'_{T'0}, \mathfrak{h}' \cap \mathcal{L}'_{T'0})$ and a *-isomorphism $(\Pi_T, \Pi_{T0}) \xrightarrow{f} (\Pi'_{T'}, \Pi'_{T'0})$ such that $\psi_0 \mid \mathcal{L}_{0,r} = \varphi_0 \mid \mathcal{L}_{0,r}$, $\Pi_{T0}^{\psi_0^*} = \Pi'_{T'0}$, $f \mid \Pi_{T0} = \psi_0^*$, and $\mho_{\gamma_j}^{f} = \mho_{\gamma'_j}$ for $j=1,\dots,r-1$. Define $(\Pi, \Pi_0) \xrightarrow{h} (\Pi', \Pi'_0)$ by $h \mid \Pi_T = f$ and $h \mid \Pi_{\gamma_r} = g$. Define $([\mathcal{L}_0, \mathcal{L}_0], \mathfrak{h} \cap [\mathcal{L}_0, \mathcal{L}_0]) \xrightarrow{\theta_0} ([\mathcal{L}'_0, \mathcal{L}'_0], \mathfrak{h}' \cap [\mathcal{L}'_0, \mathcal{L}'_0])$ by $\theta_0 \mid \mathcal{L}_{T0} = \psi_0$ and $\theta_0 \mid \mathcal{L}_{0,\gamma_r} = \varphi_0$. Then, h is a *-isomorphism such that $h \mid \Pi_0 = \theta_0^*$. By Thm. 1.2, we are done. q.e.d.

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