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AN ANALYSIS OF THE QUALITY CHARACTERISTICS OF MARKOVIAN SYSTEMS

by

Ronald Lessing

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APPROVAL SHEET

Title of Thesis: An Analysis of the Quality Characteristics of Markovian Systems

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ABSTRACT

Title of Dissertation: An Analysis of the Quality Characteristics of Markovian Systems

Ronald Lessing, Doctor of Business Administration, 1970

Dissertation directed by: William B. Widhelm, Ph.D.
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This dissertation develops a structured Markovian analysis of the quality characteristics of a large class of systems. Being Markovian, the proposed models require the use of state variables to describe the quality levels at various points in the system and transition matrices to describe the error-producing and error-detecting characteristics of the servers and machines that compose the system. Under these conditions, the models can predict the error-level distributions of the various systems. Among the applications of this analysis are the design of systems for specified quality objectives, the selection of the optimal means of producing the same output when there are several means available, and the establishment of quality criteria that can be used to evaluate system performance.

The models can also be used to predict the effects on the error-level distribution due to changes in the quality of the incoming products, the performance characteristics of the operators or machines, and the quality reviewers. Ultimately, the model may be used to optimize the system's design by indicating the optimal number, type, and location

of quality reviewers. In a sense, this dissertation can be considered an application of the principles of reliability engineering to systems design because it enables the analyst to measure the effects on the system's performance as the parameters are ranged.

The basic model is applied to linear, branching, loop, and merging systems so that the analyst can model each portion of a complex system separately and then combine the results to describe the entire system. An algebra will be described that facilitates the prediction of transient and steady-state error-level distributions throughout the class of feedback systems. Furthermore, the basic feedback model is described in terms of its sensitivity to the characteristics of the operator and the quality levels of the system's inputs. Finally, the basic models are applied to systems that include sequential, nested, and interlocking loop systems in cases where the loops are feedforward and feedback.

To

PERRY, GARY, AND JOEL

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This dissertation marks the culmination of an extended effort to earn a Doctor's degree. It is, therefore, an especially appropriate time to acknowledge those who have contributed to the success of this endeavor.

Credit for the motivation to obtain the degree goes largely to my parents, Samuel and Sophie Lessing, who taught me to respect education and hard work. For her continuous cooperation and encouragement throughout this effort, I offer my genuine appreciation to my wife Gloria. Additionally, I am appreciative of the continuous interest and encouragement of her parents, Louis and Sarah Epstein.

Scholastically, I have benefited greatly from the influence of Dr. John P. Young of The Johns Hopkins University, Dr. Robert Schellenberger of Southern Illinois University, and Dr. Rudolph P. Lamone of the University of Maryland. It is a special pleasure to acknowledge my indebtedness to Dr. William B. Widhelm for his contributions as advisor to this dissertation.

I am proud to acknowledge my appreciation for support in obtaining this degree to the Social Security Administration in general and to Mr. Jack S. Futterman in particular. It should be noted that the inspiration for the basic model developed in this dissertation was derived from a problem in the simulation of the quality characteristics of the social security claims process. Fortunately, the Administration provided not only the problem but the opportunity to thoroughly investigate a solution.

Additionally I would like to thank Mrs. Mary Mihaltian for her conscientiousness and good cheer in typing this document.

Ronald Lessing

Baltimore, Maryland
February, 1970

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION AND LITERATURE SURVEY

A. INTRODUCTION TO THE MODEL

During the past decade, reliability engineering has come of age. Techniques are now available that facilitate the design of products with predictable quality. However, there is still a need for an analytical model to facilitate the design of systems capable of producing these products. The objective of this dissertation is to expand the scope of Markovian models to meet this need as well as the development of models to investigate the effects of changes in existing systems without interfering with on-going operations.

The models developed in this thesis require the following assumptions:

First, the systems to be modeled must consist of a finite set of operations through which flow a uniform stream of items. A stream of items will be said to flow uniformly if no items are created, destroyed, or stored in the system.

Second, the error level in every item can be measured quantitatively and expressed as a non-negative integer.

Third, the error-level distribution in the stream of items at any point in the system can be described by a state variable in the form of a row vector, w, with elements, w_k, equal to the probability of finding an item in error level k. Being a probability distribution, it is necessary that all $w_k \geq 0$ and $\sum_k w_k = 1$.

Fourth, every operator in the system both generates and removes errors. Let p_{rk} equal the probability that an item will enter an operation with r errors and exit with k errors, P being the matrix of p_{rk}'s. Hence, P is a stochastic matrix, i.e., $P_{rk} \geq 0$ for all r and k and $\Sigma_k P_{rk} = 1$ for all r.

Fifth, the conditional probability of being in state k in stage n+1 given that the item is in state r in stage n is independent of the previous states of the item. In other words, the systems have "no memory" of the previous error levels of the items upon which it is currently operating. Mathematically, this basic assumption of Markovian systems can be expressed as:

$$\Pr(x_{n+1}=c_{n+1} \mid x_n=c_n)$$

$$= \Pr(x_{n+1}=c_{n+1} \mid x_n=c_n, x_{n-1}=c_{n-1}, x_{n-2}=c_{n-2}, \ldots, x_0=c_0)$$

Sixth, each operation in the system can be described by an error-level transition matrix P which is independent of time, i.e., the quality characteristics of the system's operations are stationary over time.

Seventh, the error-level distribution of the system's input, w^0, is known.

The above are the basic assumptions of a finite-state stationary Markov chain. This model implies that one can compute the value of the state variable w^s for each state, s, in the systems using the following relationship:

$w_k^s = \sum_r w_r^{s-1} p_{rk}^s$, for all k, where w_k^s is the k th element of the row vector w^s, and p_{rk}^s is an element of the error-level transition matrix P^s that describes the s th operation in the system. In the development to follow, the superscripts of w_k^s and p_{rk}^s will be omitted whenever it is not necessary for clarity. The previous equation can also be expressed as the matrix equation $w^s = w^{s-1}P^s$.

B. LITERATURE SURVEY

The term "Markov process" is applied to a very large and important class of stochastic processes with both continuous- and discrete-time parameters. However, the term "Markov chain" is reserved for discrete-time Markov processes (Feller, 1962, p. 569).

According to Chung (1960), Markov processes are named after A.A. Markov who introduced the concept in 1907 with a discrete parameter and a finite number of states. The denumerable case was launched by Kolmogorov in 1936, followed closely by Doeblin, whose contributions pervade all parts of Markov theory. Fundamental work on continuous parameter chains was done by Doob (1953) and others. For a history of the early development of the theory of Markov processes, see Frechet (1938) and Feller (1962, p. 375).

Also included in the literature are many applications of Markov models to various phenomena. One classical application presented by Kemeny and Snell (1959, pp. 167-176) is to the Ehrenfest diffusion problem. This model assumes that a volume of k molecules of gas is divided between two containers. The state variable of the Markov model is the number of molecules of gas in one of the two containers. At each instant of time, a molecule chosen at random from one container is moved to the other container. Thus at each instant, the state variable either increases or decreases by one. The model is used to predict how long it will take the system to reach equilibrium, given various starting states.

In 1944, Kakutani observed that there were basic similarities between the theory of random walks and potential theory. This relationship has been well developed in the literature by Kemeny, Snell, and Knapp (1966, pp. 166-322).

Shannon (1949, pp. 11-18) used a Markovian model to simulate a source of discrete information in successive symbols chosen so that their probabilities depend on the preceding letter. Later, in order to increase the similarity between the simulated language and English, he introduced a second-order Markov process in which the letter generated depended on the preceding two letters but not on the message before that point.

Barlow and Proschan (1965, pp. 119-161) used the Markov model extensively to describe systems in which the components must be inspected, repaired, and replaced. They stated, "...the deterioration law of the system will usually be assumed to be Markovian; that is, the future course of the system depends only on its state at the present time and not on its past history." They used the Markovian model because it applies to systems in which each component has approximately an exponential failure law and because there are many systems in which a knowledge of the system's history is of no predictive value.

Barlow and Proschan referenced an important application of Markov models in the determination of inspection-maintenance-replacement schedules by Klein (1962) which used linear programming to determine the optimal inspection, maintenance, and replacement schedules. Also in 1962, Derman applied dynamic programming to the analysis of a similar model.

In a similar vein, Bovaird (1961) used a Markovian model to minimize the sum of the inspection, preventive repair, and failure costs of a system with deteriorating machines. His optimization technique was simply an exhaustive search in the space defined by the number of repairmen and the period between inspections. Additionally, Flehinger (1962) used a Markovian model to describe a system with deteriorating machines. She used a discrete state variable ranging from 0 to m to represent the condition of each machine, with state m indicating machine failure.

Howard (1960) wrote a text on applications of Markov models that involved a reward which is received when the state variable passes through each state. His models involved sequential decisions and hence were optimized by dynamic programming.

Kemeny, Snell, and Thompson (1956, pp. 334-337) applied a Markov analysis to a learning model proposed by Estes (1955). The objective of the learning model was to predict, for given behavior of the experimenter, how the subject's guesses would change in the long run.

Bharucha-Reid (1960) wrote a text in which he devoted nine chapters to applications of Markov processes. He presented applications of Markov models to genetics, epidemics, radioactive particles, fluctuations in the brightness of the milky way, chemical reaction kinetics, and queueing theory.

Beebe, Beightler, and Stark (1968) formulated a multistage transistor production process as a finite Markov decision process with rewards. Their model considered returns that were deterministic since their values were fixed by the decisions, input states, and output states. The model was solved for a set of optimal decisions by means of Howard's "Value Iteration" process (Howard, 1960, pp. 28-31).

Springer, Herlihy, Mall, and Beggs (1968, pp. 82-100) applied an absorbing-state Markov model to the problem of aging accounts receivable. Here an account was described as being payable, paid, or in default. Transition from the payable state to another payable state, the paid state, or the default state took place at the end of each month. The solution to the absorbing-state model was used to determine the expected age distribution of the accounts receivable, the average collection period, and the expected losses due to defaulted accounts.

Harary and Lipstein (1962, pp. 19-40) studied consumer behavior by means of a Markov model in which the state variable represented the particular brand of cigarettes purchased and the Markovian transition matrix the probability that a consumer would switch from brand i to brand j. An ergodic transition matrix gave the probabilities of switching from any brand to another brand or to "no brand" and was solved for the equilibrium market share associated with each brand. Additionally, they used an absorbing-state transition matrix to study the expected period until various smokers purchased a particular brand of cigarettes. Finally, Harary and Lipstein analyzed the various periods subsequent to the introduction of a new brand into an existing market. The objective of this effort was to provide the manufacturer with a means of establishing the expected market share of a new brand in terms of its proportion of first-time triers, the loyalty of its customers, and the loyalty of the customers of competing brands.

The application of Markov models to be presented in this dissertation is unique in that the state variable is the quality level at various points in a general system composed of branches, loops, and nodes. Further, the characteristics of the systems operators are described by transition matrices and the characteristics of the systems quality reviewers are described by "filter" matrices. Finally, it provides a means of determining the transient states of feedback systems and the sensitivity of a system's quality levels to the quality of its inputs, the level of performance of its operators, and its configuration.

Chapter II contains a lexicon of basic systems models and a mathematical analysis of the properties of the models. Examples are developed for linear, branching, merging, feedforward loops, and feedback loops. A comparison of the analysis of transient states by eigen-values and generating functions is given, as well as an analytical description of the sensitivity of the model to its parameters.

Chapter III contains an extension of the model to systems that contain sequential, nested, and interlocking feedback loops. Here, a rule is developed for modeling complex feedback systems and the transient analysis of Chapter II is extended by means of a "profile" matrix.

Chapter IV contains an extension of the model to systems that contain sequential, nested, and interlocking feedforward loops, while Chapter V presents models of systems with all arrangements of a feedforward and feedback loop.

Chapter VI discusses extensions of the model by relaxing the basic assumptions, extensions of the model into the area of cost and timeliness, and means of facilitating the application of the model.

The main contribution of this thesis is the development of a simple, easily analyzed structure which is directly applicable to a large class of probabilistic sequencing problems.

CHAPTER II

DEVELOPMENT OF THE BASIC MODEL

A. INTRODUCTION

In order to efficiently model any large system, a set of basic subsystems must be developed which can then be synthesized to describe the larger system. Here, linear sequential systems will be described in terms of branching, merging, feedforward and feedback subsystems.

The purpose of this chapter is to develop these subsystems conceptually and mathematically, starting with a detailed discussion of the assumptions and applications of the general linear model. Then the four basic subsystems, branching, merging, feedforward and feedback, will be developed in that order. Pertinent examples will be given and appropriate sensitivity, transient and steady-state analyses presented.

B. LINEAR MODELS

Consider a linear system with a series of n ordered operations as shown in Figure 1. As discussed in the first section of Chapter I, let w^0 be the incoming or initial error-level distribution, w^s be the output error-level distribution from the s th operation, and P^s be the error-level transition matrix associated with s th operation. Since P^s is a stochastic matrix and the error-level distributions w^s are represented by row vectors with components

w_k^s, the following relationships hold:

$$\sum_k w_k^s = 1, \text{ where } w_k^s \geq 0, \text{ for all } k, \text{ and}$$

$$w_k^s = \sum_r w_r^{s-1} p_{rk}^s, \text{ for all } k, \text{ or} \tag{2-1}$$

$$w^s = w^{s-1} P^s.$$

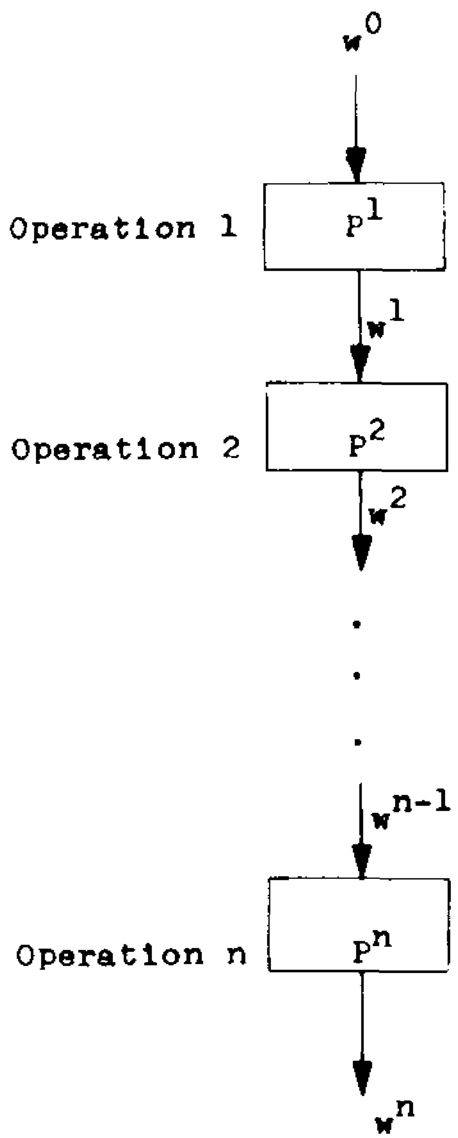


Figure 1. General form of a linear system with n operations.

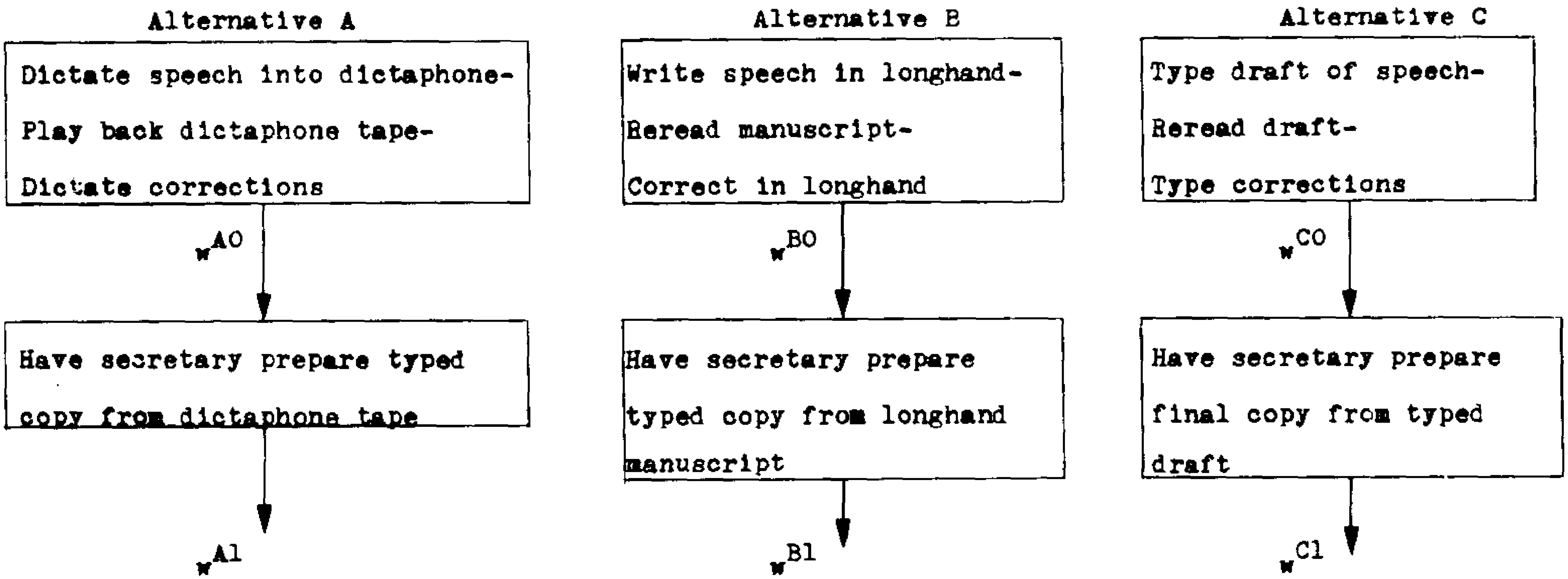


Figure 2. Three alternatives for transcribing a speech.

This linear model can now be used to describe, for example, the problem of the author of a speech who is interested in selecting from the three alternatives in Figure 2 the one that will give him the best expected output quality level. In this context, errors are misused words, misspelled words, or grammatical defects. It is assumed that they can be measured quantitatively. Errors in logic, judgment, or fact not readily quantifiable are ignored.

Assume the following hypothetical data describe the alternatives of Figure 2:

$$w^{AO} = \frac{1}{8}\,(0,\ 2,\ 2,\ 1,\ 3)$$

$$w^{BO} = \frac{1}{8}\,(2,\ 2,\ 2,\ 2)$$

$$w^{CO} = \frac{1}{8}\,(0,\ 0,\ 2,\ 2,\ 2,\ 1,\ 1).$$

Note that the dimension of each initial error-level distribution vector is determined by the number of error levels considered. Similarly, the dimensions of the error-level transition matrices are determined by the dimension of the incoming error-level distribution vector and the number of outgoing error levels considered. Let the following transition matrices describe the error-producing and error-detecting characteristics of the three speech transcription systems:

$$P^{A1}=\frac{1}{8}\ \begin{array}{c|c} & \text{Outgoing Error Level} \\ & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \text{Incoming Error Level} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 4 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 \end{bmatrix} \end{array}$$

$$P^{B1}=\frac{1}{8}\ \begin{array}{c|c} & \text{Outgoing Error Level} \\ & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \text{Incoming Error Level} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \end{array}$$

$$P^{C1}=\frac{1}{8}\ \begin{array}{c|c} & \text{Outgoing Error Level} \\ & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \text{Incoming Error Level} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 4 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} \end{array} .$$

It is now possible to use the matrices corresponding to the various alternatives to evaluate the expected outgoing error-level distributions using the relationships expressed in equations (2-1).

The model then predicts the following output error-level distributions for the three alternatives:

		Outgoing Error Level							
		0	1	2	3	4	5	6	7
w^{A1}	$= \frac{1}{64}$	(4,	11,	10,	11,	12,	9,	4,	3)
w^{B1}	$= \frac{1}{64}$	(16,	16,	14,	12,	4,	2)		
w^{C1}	$= \frac{1}{64}$	(15,	12,	12,	12,	6,	5,	2).	

With the further assumption that the penalty or regret associated with having some errors in the final typed version of the speech is proportional to the number of errors, it is possible to select the best alternative by simply computing the expected value of the outgoing error-level distributions using the following formula:

$$\text{Expected Value of } w = \sum_k k\, w_k.$$

The expected error levels are:

Alternative	Error Level
A	3.16
B	1.66
C	2.08

so that alternative B is clearly the best.[1]

---

[1] If the penalty or regret associated with k errors is equal to some function R_k instead of k, it is still possible to select the best alternative using the following criteria: Expected Regret $= \sum_k R_k\, w_k$.

C. BRANCHING MODELS

Consider a branching system in which a batch of n_0 items with error-level distribution w^0 is divided into two batches of size n_1 and n_2 with error-level distributions w^1 and w^2, respectively. As shown in Figure 3, the error-level distribution of batch 1 and the incoming items is described by the "filter" matrix F, which gives the probability that an item with i errors will go into batch 1. Hence, F must be a diagonal matrix with nonnegative elements less than or equal to 1. The relationship between the error-level distribution of batch 2 and the incoming items is, therefore, described by the filter matrix (I - F), where I is the identity matrix, which gives the probability that an item with error level i will go into batch 2. Algebraically, the filter F divides the incoming stream of n_0 items with error-level distribution w^0 into two streams of items so that:

$$n_1 w^1 = n_0 w^0 F, \text{ and}$$
$$n_2 w^2 = n_0 w^0 (I - F). \tag{2-2}$$

where I is the identity matrix.

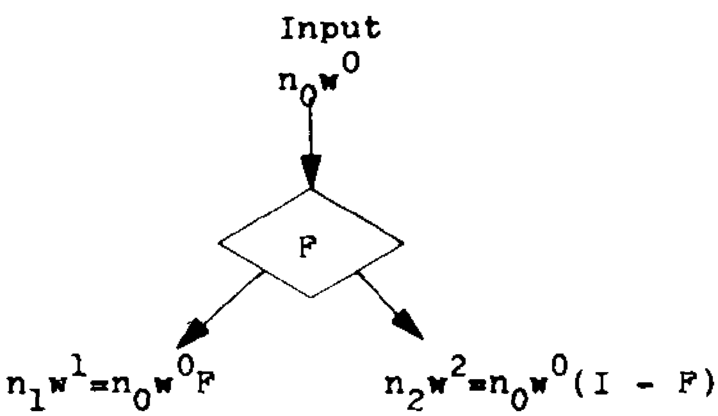


Figure 3. Branching model with two outputs.

This branching model can be used to describe, say, the effect on the distribution of blemishes on apples due to sorting the apples into two batches on the basis of size. Further, this model can be used to describe a system operating under risk in which there are two parallel procedures available to convert wage records into punched cards. If one procedure uses an optical scanner which is very rapid but requires a high quality input while the other procedure is manual keypunching which is slow but accommodates inputs of much lower quality, the branching decision must be made on the basis of the quality of the incoming wage records.

In order to investigate the sensitivity of the outgoing error-level distribution to the incoming error-level distribution and the filter matrix, assume that equation (2-2) describes a system in which there are only two error levels, good (0) and bad (1). Then:

$$n\ (w_0^1,\ w_1^1) = (w_0^0,\ w_1^0)\ \begin{bmatrix} f_{00} & 0 \\ 0 & f_{11} \end{bmatrix}$$

where $n = n_1/n_0 \geq 1$.

This matrix equation can be expressed by the following two equations:

$$nw_0^1 = w_0^0\ f_{00} \quad \text{and} \quad nw_1^1 = w_1^0\ f_{11},$$

which when added, yield

$n = w_0^0\ f_{00} + w_1^0\ f_{11}$, since $w_0^1 + w_1^1 = 1$ from equation (2-1).

One can now express the sensitivities of the outgoing error-level distribution by the following partial derivatives:

$$\frac{\partial w_0^1}{\partial w_0^0} = - \frac{\partial w_0^1}{\partial w_1^0} = c\ f_{00}f_{11}, \text{ and}$$

$$\frac{\partial w_0^1}{\partial f_{00}} = c\ w_0^0\ w_1^0\ f_{11}, \text{ and}$$

$$\frac{\partial w_0^1}{\partial f_{11}} = c\ w_0^0(f_{00} - f_{11}) + f_{11})^{-2}.$$

Clearly, the partials of w_1^1 are the negatives of those of w_0^1, since $w_0^1 + w_1^1 = 1$. Note that the filter concept can be extended to m batches where $F_1 + F_2 + F_3 + \ldots + F_m = I$.

D. MERGING MODELS

Figure 4 illustrates a model in which two batches of n_1 and n_2 items with error-level distributions w^1 and w^2, respectively, are merged into one batch of size n_3 with error-level distribution w^3. In this model, n_3 and w^3 can be determined from the following equations:

$$n_3 = n_1 + n_2, \text{ and}$$

$$n_3w^3 = n_1w^1 + n_2w^2.$$

The extension of this model to m batches is analogous to the two batch case. The sensitivity of the output to the input is, of course, linear.

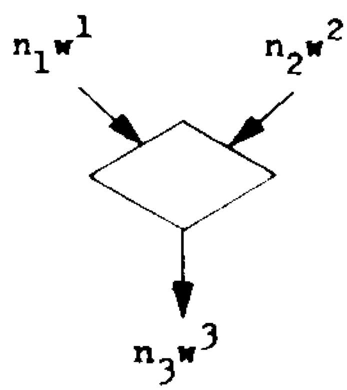


Figure 4. Merging model with two inputs.

To combine the concepts of filtering and merging, note that there are at least two ways in which a quality inspector may operate in a system. In the first case, the quality inspector may be treated as another operator who reviews the work he receives, corrects the errors that he detects, occasionally adds errors, and then forwards the items to the next operator. Here the quality reviewer's effect on the quality of the items he inspects is described completely by his error-level transition matrix.

In the second case, the quality inspector may operate in conjunction with a correction operator. Here the quality reviewer acts as a filter who forwards a portion of the items that he receives directly into output and the remainder with identified errors to a correction operator. Mathematically, this quality reviewer's operations are described in Figure 5 where $(I - F)$ is a diagonal matrix that gives the probability that an item with i errors will be forwarded to the next operation in the system, and F is a diagonal matrix that gives the probability that an item with i errors will be fed back to a correction operator. In this model the correction operator performs in a manner described by the correction matrix P, and then forwards the items for another quality inspection.

From Figure 5 it is clear that the output from a system with an input of w^0 and with a quality reviewer-correction operator team is the sum of a series of $(k+1)$ terms of the form $w^0\ (FP)^i(I - F)$, $i=0, \ldots, k$. Furthermore, for a system in which all items are processed until they

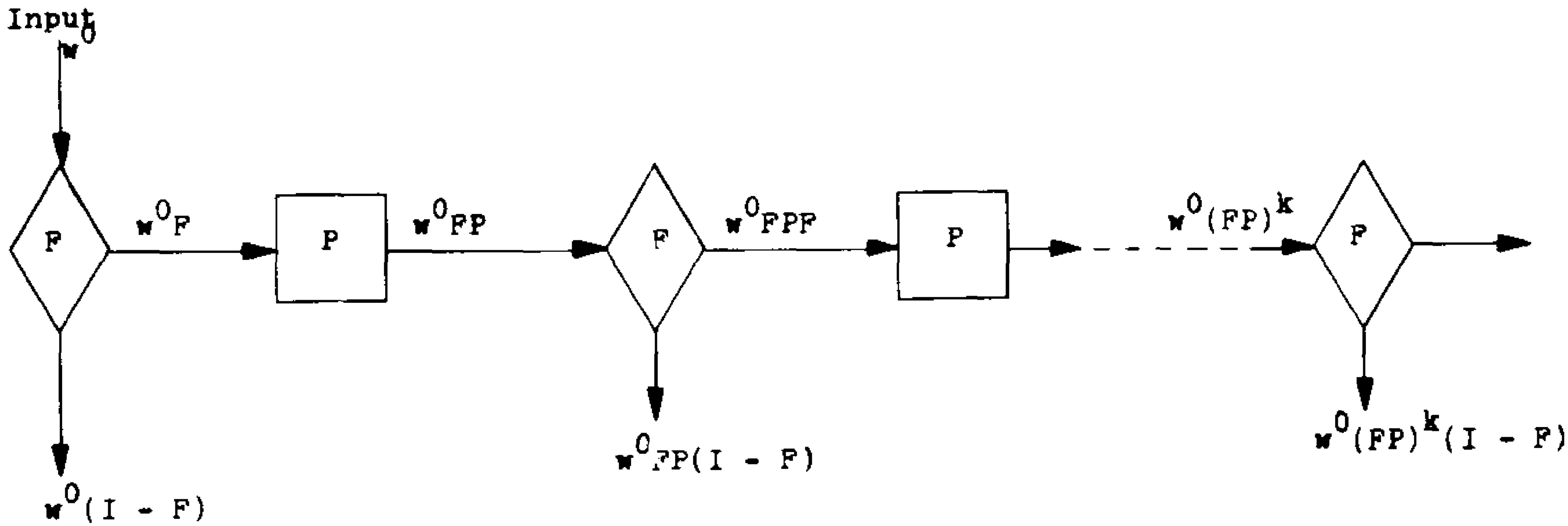


Figure 5. Quality reviewer model with correction operator.

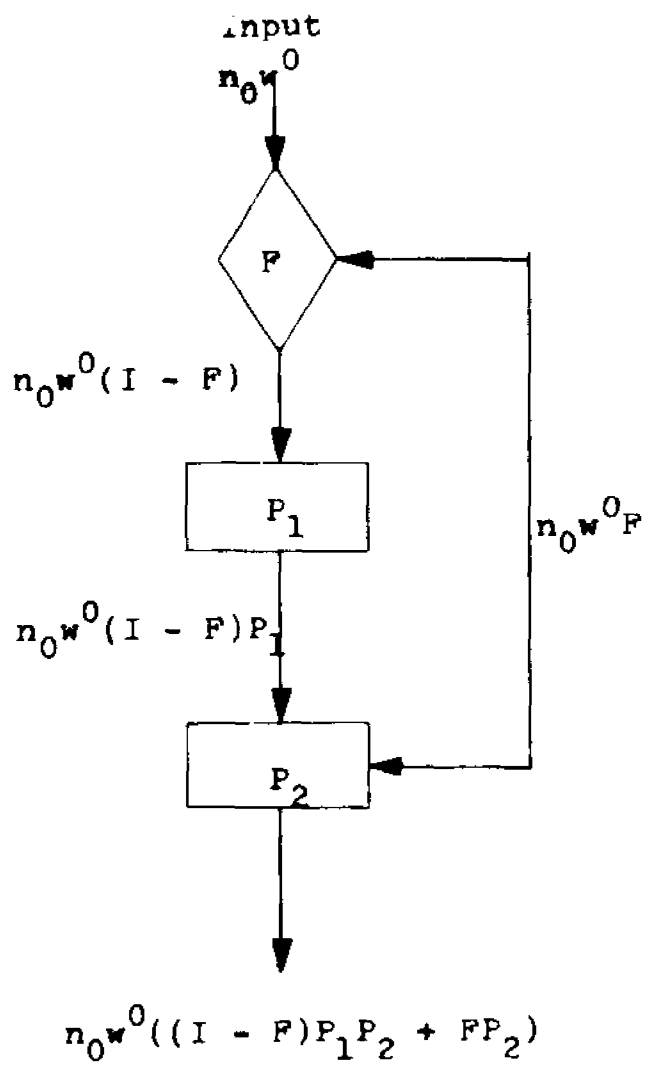


Figure 6. Equilibrium model for a feedforward system.

pass inspection and equilibrium is reached, the following matrix identity is valid:

$$\lim_{k \to \infty} \sum_{i=0}^{k} (FP)^i = (I - FP)^{-1}, \text{ where } (FP)^0 = I.$$

More will be said about the existence of $(I-FP)^{-1}$ in subsection F(5). It is now possible to express the limiting or steady-state output quality vector w^{out} of this team in terms of the incoming error-level distribution w^0, the filter matrix F, and the correction matrix P as:

$$w^{out} = w^0(I - FP)^{-1}(I - F).$$

Here, the sensitivity of w^{out} to w^0 follows directly. The sensitivity of w^{out} to F and P is fairly complicated but can be handled in the same manner as will be illustrated for feedback systems in section F.

E. FEEDFORWARD MODELS

An important class of cyclic system is the feedforward case which consists of an inspector who selects a portion of the items that he inspects to bypass the next step in the process. Figure 6 depicts a feedforward system in which n_0 represents the number of items in a production run, w^0 represents the error-level distribution of the incoming items, and P^1 is the operation that may be bypassed.

As an example of a feedforward system, consider a metal products factory. Let n_0 be the number of items in a production run, and let w^0 represent the distribution of burrs, scratches, and impurities on the surface of the items entering the electroplating department. The filter matrix F describes the inspection that is carried out to direct those items that are ready to be plated to operation P^2 and those items that require additional treatment to operation P^1. There are at least two reasons why such an inspection may occur. First, it may be advantageous to inspect the items at this point to save the cost of plating those items that would be rejected after plating. Second, the savings due to operation P^1 may offset the cost of the inspection at F.

As shown in Figure 6, the outgoing error-level of a feedforward system is given by the following equation:

$$n_{out}w^{out} = n_0w^0((I - F)P^1P^2 + FP^2).$$

Since no buildup is allowed in the system,

$$n_0w^0c = n_0w^0((I - F)P^1P^2 + FP^2)c$$

where c is a column vector with all elements equal to one. Again, the sensitivity of w^{out} to w^0 follows directly whereas the more complicated dependency on F, P^1 and P^2 can be handled analogously to that illustrated in the next section.

F. FEEDBACK MODELS

To introduce the concept of the basic feedback model, consider the example of a new textbook that enters the collection of a public school. Assume that every textbook is loaned out each semester to a student for the entire semester. Let the state vector w represent the distribution of extraneous markings in the books and assume that all books are received new with no extraneous markings. Further, assume that every person who borrows a book may insert markings or remove existing markings. Let the probability that a book will be loaned out with i markings and returned with j markings be represented by p_{ij} which is a function of i and j only and not the semester loaned. These conditions are sufficient to make this a time-homogenous Markov process.

It is reasonable to assume that there is a maximum number of markings that will be tolerated. This implies that the transition matrices are finite and that there is either a barrier limiting the number of markings in a book or that a book is removed from circulation when the number of markings reaches a certain level.

1. <u>Non-Absorbing Barrier Model</u>

Consider the case in which there is a barrier at a certain level of markings, say 4. Let the matrix P represent the transition probabilities associated with the number of markings in a book. Assume also that P is ergodic, that is, it is possible to go from any state i to any state j in a finite number of steps. A P matrix having these characteristics is shown below.

$$P = \frac{1}{8}\ \begin{array}{c} \text{Number of} \\ \text{Markings in} \\ \text{Loaned Book} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{Number of Markings} \\ \text{In a Returned Book} \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4 & 2 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \end{array}$$

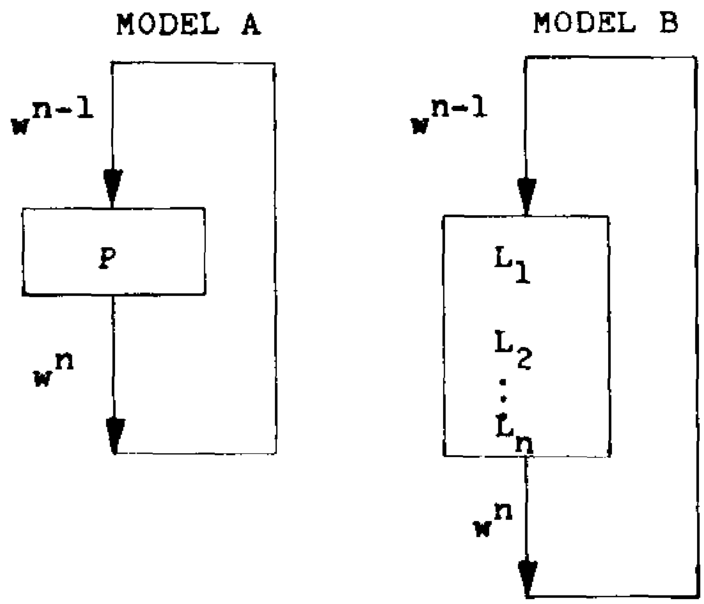


Figure 7. Model A is a simple cyclic model with one operation while model B is cyclic with m operations.

Graphically, the textbook model with a barrier can be described by model A of Figure 7, and it can be generalized to represent model B of Figure 7 when the matrix $P = L^1L^2\ldots L^m$, that is, the book sequentially passes through m students, each with a transition matrix L^1. Since each semester corresponds to one circulation of a book and each circulation of a book corresponds to a cycling of the w vector through the P matrix, $w^n = w^{n-1}P$. Using standard Markovian analysis (see, for example, Feller 1962, Chapters 15 and 16), it is known that, after a large number of circulations, w will reach an equilibrium $w^e = w^eP$, since P is assumed to be ergodic. This matrix equation can now be expressed in the form of four simultaneous linear equations:

$$8\ w_0^e = 4w_0^e + 2w_1^e + 1w_2^e + 2w_3^e$$

$$8\ w_1^e = 2w_0^e + 4w_1^e + 2w_2^e + 2w_3^e$$

$$8\ w_2^e = 1w_0^e + 1w_1^e + 4w_2^e + 2w_3^e$$

$$8\ w_3^e = 1w_0^e + 1w_1^e + 1w_2^e + 2w_3^e; \quad \text{along with}$$

$$1 = w_0^e + w_1^e + w_2^e + w_3^e \quad \text{and}$$

$$w_i^e \geq 0,\ i = 0, \ldots, 3.$$

The solution to the set of simultaneous equations indicates that the steady-state distribution of markings is:

$$\begin{array}{cccccc} & & \multicolumn{4}{c}{\text{Number of Markings}} \\ & & 0 & 1 & 2 & 3 \\ w^e & = & (.30, & .33, & .23, & .14). \end{array}$$

The number of markings in a book will thus approach a mean of 1.21, independent of the initial distribution of markings and will remain at this level indefinitely. Moreover, the mean interval between occurrences of state k is $(w_k^e)^{-1}$, here, 3.33, 3.00, 4.35 and 7.14 for k=0, 1, 2, and 3, respectively.

2. Absorbing Barrier Model with Perfect Inspector

Consider now a second textbook model, as shown in Figure 8, in which the upper limit on the number of markings is now controlled by an absorbing state at a fixed number of markings, say 4. This is equivalent to assuming that the books are inspected each time that they are returned to the school by a perfect inspector who removes any book with 4 or more markings. Let the matrix F describe the characteristics of the perfect inspector, i.e.,

Number of Markings
In Returned Book

Number of Markings
In Loaned Book

$$F = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (2\text{-}3)$$

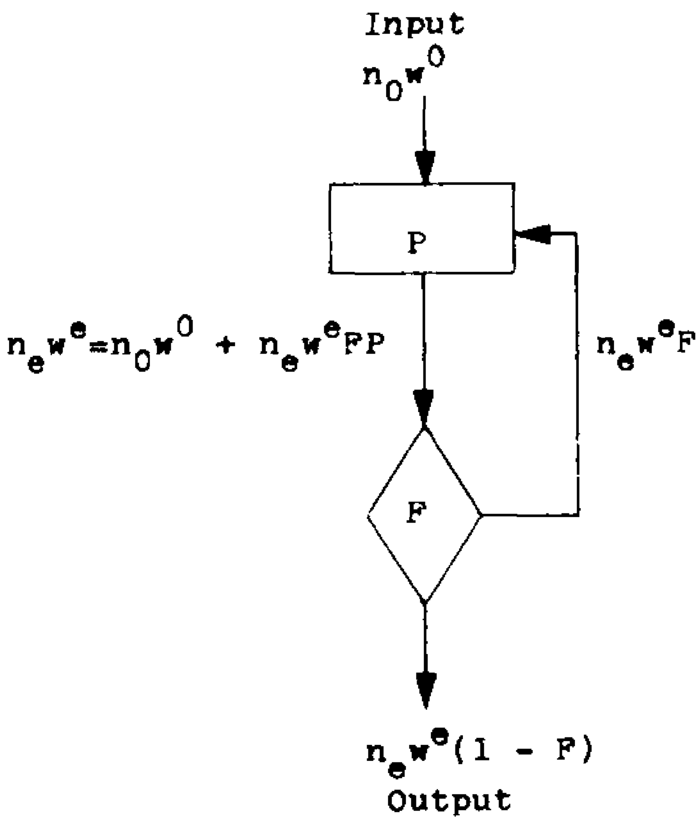


Figure 8. Equilibrium model for cyclic system with feedback.

In order to achieve an equilibrium, the school must, of course, replace every book that is removed by the inspector and destroyed. This basic assumption of the model is referred to as the conservation condition. An analysis of this model leads to the following questions:

a. On the average, how many times will a book circulate in each state before it reaches the level of four markings?

b. On the average, how many circulations will take place before a new book must be destroyed?

In order to answer these questions and to demonstrate some of the properties of a Markov process with absorbing states, assume that the transition matrix for the number of extraneous markings in a book is as follows:

Number of Markings In Returned Book (columns); Number of Markings In Loaned Book (rows)

$$P = \frac{1}{8} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 2 & 4 & 1 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix} \end{array}$$

In general, it is possible to rearrange the rows and columns of a transition matrix with absorbing states to isolate the transitory and absorbing states (Kemeny and Snell, 1959, p.44). This rearrangement will lead to a transition matrix of the form:

$$P = \begin{vmatrix} I & 0 \\ R & Q \end{vmatrix}$$

where I is the identity matrix,[2] 0 is a matrix with all elements equal to zero, Q is a square matrix of transitory states, and R is a matrix describing transitions between transitory and absorbing states.

By multiplying the matrix P by itself n times, it can be shown that the transition matrix corresponding to n transitions (circulations) is of the form:

$$P^n = \begin{vmatrix} I & 0 \\ NR & Q^n \end{vmatrix}$$

where N equals the matrix sum $I + Q + Q^2 + \ldots + Q^{n-1}$. Here the superscripts refer to a power and not to a state as previously done. As $n \to \infty$, each element n_{ij} of the N matrix represents the mean number of passage times through state j before absorption if the book starts in state i. N is easily obtained as $(I - Q)^{-1}$ (see, for example, Feller, 1962, Chapters 15 and 16).[3] For this particular numerical example:

$$R = \frac{1}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and } Q = \frac{1}{8}\begin{bmatrix} 4 & 2 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 0 & 1 & 3 & 3 \end{bmatrix} .$$

---

[2]In order to obtain I, it is necessary to redefine any set of absorbing states into a single state, with a corresponding adjustment in R.

[3]More will be said on the existence of $(I-Q)^{-1}$ in subsection 5.

It is now possible to evaluate N by letting $N = (I - Q)^{-1}$. The result of this computation is:

$$
N = \begin{array}{c c} & \text{Number of Markings} \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \text{Number of Markings} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 5.64 & 4.82 & 4.92 & 3.08 \\ 4.31 & 6.15 & 4.92 & 3.08 \\ 3.18 & 3.59 & 5.54 & 2.46 \\ 2.77 & 3.38 & 4.31 & 3.69 \end{bmatrix} \end{array}
$$

For example, row 1 of the N matrix can be used to answer question a) since it indicates that a book that begins with zero extraneous markings will make, on the average, 5.64 circulations with zero markings, 4.82 circulations with 1 marking, 4.92 circulations with 2 markings, and 3.08 circulations with 3 markings. Finally, in response to question b), if a book begins with no extraneous markings, it will make, on the average, 5.64 + 4.82 + 4.92 + 3.08 = 18.46 circulations before destruction.

From Figure 8, it is apparent that:

$n_e w^e = n_0 w^0 P + n_e w^e FP$, which yields, upon rearrangement:

$$n_e w^e = n_0 w^0 P(I - FP)^{-1}. \qquad (2\text{-}4)$$

The independent variables of the model are n_0, the number of books being added to the system each period, the diagonal elements of the filter matrix F, and the marking-level transition matrix P. The dependent variables are w^e, the equilibrium distribution of markings in the books, and n_e, the equilibrium number of books circulating. It is more convenient to define the ratio of the equilibrium number of books in the system to the number of books added each period as n, i.e.,

$$n = \frac{n_e}{n_0} \geq 1$$

then equation (2-4) can be expressed as

$$nw^e = w^0 P(I - FP)^{-1}. \qquad (2\text{-}5)$$

This matrix equation along with the condition $\sum_k w_k^e = 1$ can be solved uniquely for n and w^e. A good check on the calculations is to note that at equilibrium, the number of books leaving the system must equal the number entering, that is,

$$n_0 = n_e w^e (I - F)c \quad \text{or} \quad 1 = nw^e(I - F)c, \qquad (2\text{-}6)$$

where c is a column vector in which each element equals 1.

3. Numerical Example with Perfect and Imperfect Inspectors

Thus far, the assumption has been made that the school is depicted by a perfect filter that removes every book with more than three markings and never removes a book with three or less markings, as reflected in the chosen value of F in equation (2-3). To generalize the model, it is only necessary to substitute a realistic filter matrix for the perfect filter matrix. Note that the only constraint on a filter matrix is that it be a diagonal matrix with all elements non-negative and not greater than 1.

In order to demonstrate the solution technique and the applicability of this model in investigating the effect on the equilibrium error-level distribution due to the ability of the quality reviewer, two numerical systems differing only in their filter matrices will be presented. First, consider the case in which there is a perfect quality reviewer represented by the matrix:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{where} \qquad P = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Then:

$$I - FP = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad (I - FP)^{-1} = \frac{1}{3}\begin{bmatrix} 8 & 4 & 3 \\ 4 & 8 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Furthermore, assume that the incoming product is error-free so that $w^0 = (1, 0, 0)$ and $w^0P = \frac{1}{4}(2, 1, 1)$.

From equation (2-5), the equilibrium relationship for the feedback loop model can be expressed in component form as:

$$n\,(w_0^e, w_1^e, w_2^e) = (w_0^0, w_1^0, w_2^0)P(I - FP)^{-1}.$$

On substitution,

$$n(w_0^e, w_1^e, w_2^e) = \frac{1}{4}(2, 1, 1)\,\frac{1}{3}\begin{bmatrix} 8 & 4 & 3 \\ 4 & 8 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \frac{1}{3}(5, 4, 3).$$

Therefore,

$$n\,w_0^e = \frac{5}{3}, \quad n\,w_1^e = \frac{4}{3}, \text{ and } n\,w_2^e = \frac{3}{3}. \tag{2-7}$$

Since $w_0^e + w_1^e + w_2^e = 1$, $n = \frac{5}{3} + \frac{4}{3} + \frac{3}{3} = 4$.

From relations(2-7), it is apparent that

$$w^e = \frac{1}{12}(5, 4, 3).$$

Using equation (2-6) as a check

$$1 = 4\,\frac{1}{12}(5, 4, 3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Second, consider the case in which the quality reviewer is described by an imperfect filter. Specifically letting,

$$F = \frac{1}{8}\begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{then} \quad FP = \begin{bmatrix} 14 & 7 & 7 \\ 6 & 12 & 6 \\ 0 & 2 & 2 \end{bmatrix},$$

$$(I - FP) = \frac{1}{32}\begin{bmatrix} 18 & -7 & -7 \\ -6 & 20 & -6 \\ 0 & -2 & 30 \end{bmatrix},$$

and

$$(I - FP)^{-1} = \begin{bmatrix} 2.03636 & .775758 & .630303 \\ .62338 & 1.87013 & .519481 \\ .04156 & .124675 & 1.10130 \end{bmatrix} \quad (2\text{-}8)$$

Again, assume that the incoming product is perfect so that $w^0 = (1, 0, 0)$ and $w^0P = \frac{1}{4}\,(2, 1, 1)$. Equations (2-1) and (2-5) yield:

$n = 2.79$ and $w^e = \frac{1}{806}\,(432, 256, 208)$. Using equation (2-6) as a check,

$$1 = 2.79\left(\frac{1}{806}\right)(342, 256, 208)\;\left(\frac{1}{8}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that the system with the imperfect inspector is less efficient in that it only circulates 2.79 times the period demand for replacements while the system with the perfect inspector circulates 4 times the period demand for replacements. However, w^e is virtually unchanged in value. The sensitivity of w^e and n with respect to the components of F and w^0 will be investigated next.

4. <u>Sensitivity Analysis of a Feedback System</u>

From the equilibrium equation for the feedback loop (2-5), it is possible to evaluate the effects of small changes in the incoming error-level distribution(w^0), the transition matrix (P), and the filter matrix (F) on the equilibrium ratio (n) and the equilibrium error-level distribution (w^e).

Consider the following case in which there are only two states

$$F = \begin{bmatrix} f_{00} & 0 \\ 0 & f_{11} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

Noting that[4]

$$(P(I - FP)^{-1})^{-1} = (I - FP)P^{-1} = P^{-1} - FPP^{-1} = P^{-1} - F,$$

it is clear that

$$P(I - FP)^{-1} = (P^{-1} - F)^{-1}.$$

Here,

$$P^{-1} - F = \frac{1}{|P|} \begin{bmatrix} p_{11} - |P| f_{00} & -p_{01} \\ -p_{10} & p_{00} - |P| f_{11} \end{bmatrix}$$

Hence,

$$(P^{-1} - F)^{-1} = \frac{|P|}{d'} \begin{bmatrix} p_{00} - |P| f_{11} & p_{01} \\ p_{10} & p_{11} - |P| f_{00} \end{bmatrix}$$

where

$$d' = (p_{11} - |P| f_{00})(p_{00} - |P| f_{11}) - p_{01}p_{10},$$

which reduces to

$$d' = |P| (1 - p_{00}f_{00} - p_{11}f_{11} + |P| f_{00}f_{11}).$$

---

[4]The author would like to express his appreciation to Dr. William B. Widhelm for this development of the sensitivity of a feedback system.

Hence, the two component equations from equation (2-5) are:

$$n\, w_0^e = \frac{1}{d}\,(w_0^0\, p_{00} - w_0^0\,|P|\,f_{11} + w_1^0\, p_{10}) \text{ and}$$

$$n\, w_1^e = \frac{1}{d}\,(w_0^0\, p_{01} + w_1^0\, p_{11} - w_1^0\,|P|\,f_{00}), \text{ where}$$

$$d = (1 - p_{00}f_{00} - p_{11}f_{11} + |P|\,f_{00}f_{11}).$$

Since $w_0^e + w_1^e = 1$,

$$n = \frac{1}{d}\,(w_0^0 + w_1^0 - f_{00}w_1^0\, P - f_{11}w_0^0\,|P|\,), \text{ which reduces to}$$

$$n = \frac{1}{d}\,(1 - (w_0^0 f_{11} + w_1^0 f_{00})\; P\;).$$

Therefore,

$$w_0^e = \frac{1}{c}\,(w_0^0 p_{00} + w_1^0 p_{10} - w_0^0 f_{11}\,|P|\,)$$

$$w_1^e = \frac{1}{c}\,(w_0^0 p_{01} + w_1^0 p_{11} - w_1^0 f_{00}\,|P|\,)$$

where $c = 1 - (w_0^0 f_{11} + w_1^0 f_{00})\,|P|\,)$.

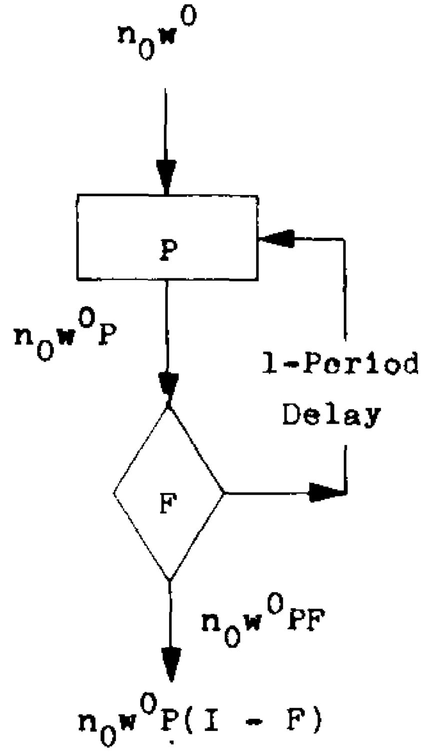
The above equations can be used to study the sensitivity of n and w_0^e with respect to w_0^0, f_{00}, and f_{11}. The sensitivity of w_1^e follows directly from the equation $w_1^e = 1 - w_0^e$ and the sensitivity with respect to w_1^0 follows directly from the equation $w_1^0 = 1 - w_0^0$. A similar analysis could be run on any system.

5. <u>Transient Analysis of a Feedback System</u>

The analysis of the transient states using this matrix model follows from the recurrent properties of the matrix description of the system. Assume that an operating system begins at period i=0 with an input of n_0 items with an error-level distribution w^0. If there is no feedback in the system and the assumptions of continuous flow hold, then equilibrium is reached in the first period. On the other hand, if there is feedback, then a series of transient states results that will approach equilibrium in most realistic situations. The analysis of a simple feedback system leads to a system "profile" matrix that can be generalized to a large class of systems composed of combinations of feedforward and feedback loops in various configurations.

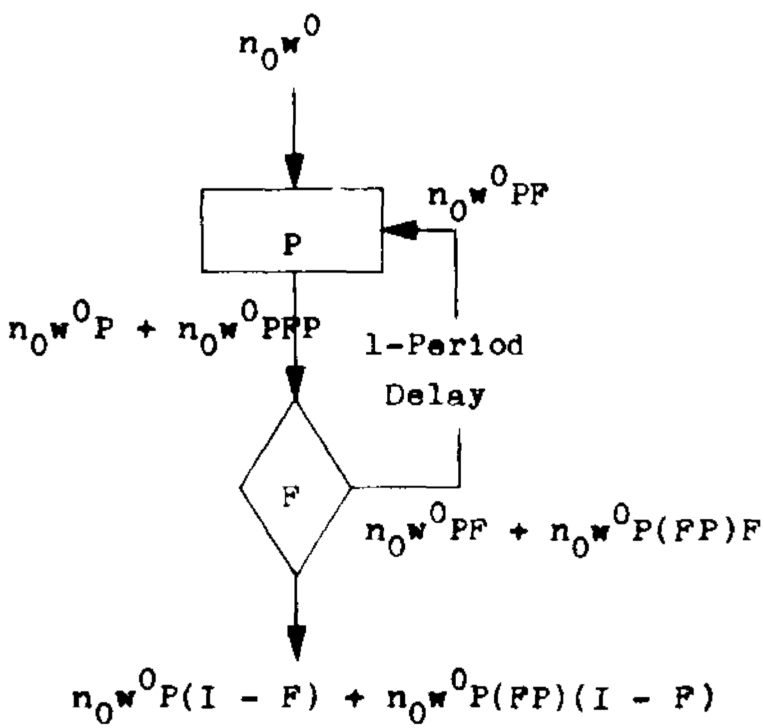
Figures 9 through 12 describe four periods in the operation of a simple feedback system. During each period, there is an input of n_0 items with an error-level distribution of w^0. Due to the feedback, however, there is an additional input for each period from the previous period, except, of course, for the first one. The input from the previous period is indicated to the immediate right of the P operation in Figures 10, 11, and 12. It is seen in Figure 12 that the general form of this input for the n th period is $n_0 w^0 P \sum_{i=0}^{n-1} (FP)^i F$, where $(FP)^0 = I$, as before. To generalize this model, (FP) will be designated the "profile" matrix, since it in a sense defines the profile of the feedback system. Note the general form of the output of the P operation for the n th period is $n_0 w^0 P \sum_{i=0}^{n} (FP)^i$, and the outputs of the filter F during the n th period are $n_0 w^0 P \sum_{i=0}^{n} (FP)^i F$ and $n_0 w^0 P \sum_{i=0}^{n} (FP)^i (I-F)$, as indicated. It is clear that the determination of the error-level distribution at the various points in the system as a function of the period is obtained from the sum of the series $(FP)^i$ for i going from zero to n.

A good method to evaluate this series is to use a diagonal similarity transformation of the system profile matrix (FP). This involves the standard eigenvalue, eigenvector method of real matrices (see, for example, Goldberg, 1967, pp. 207-241). The n roots or eigenvalues of the



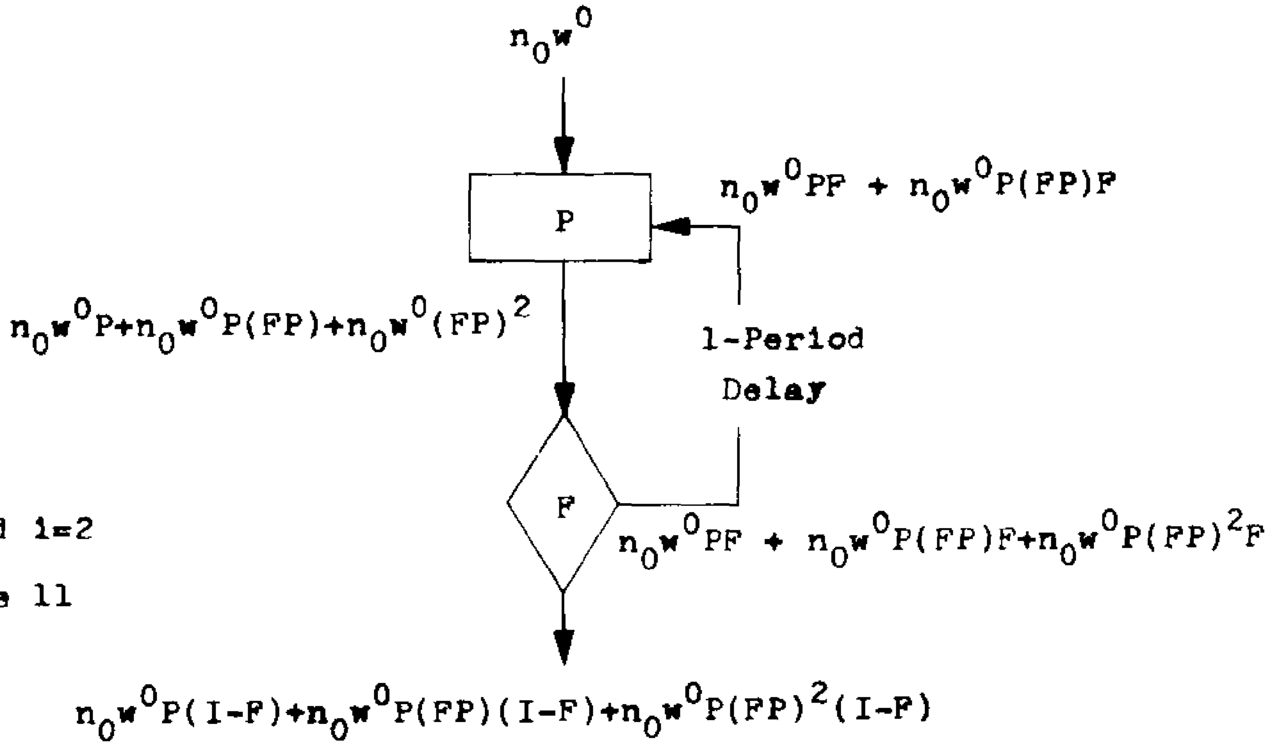
Period i=0

Figure 9



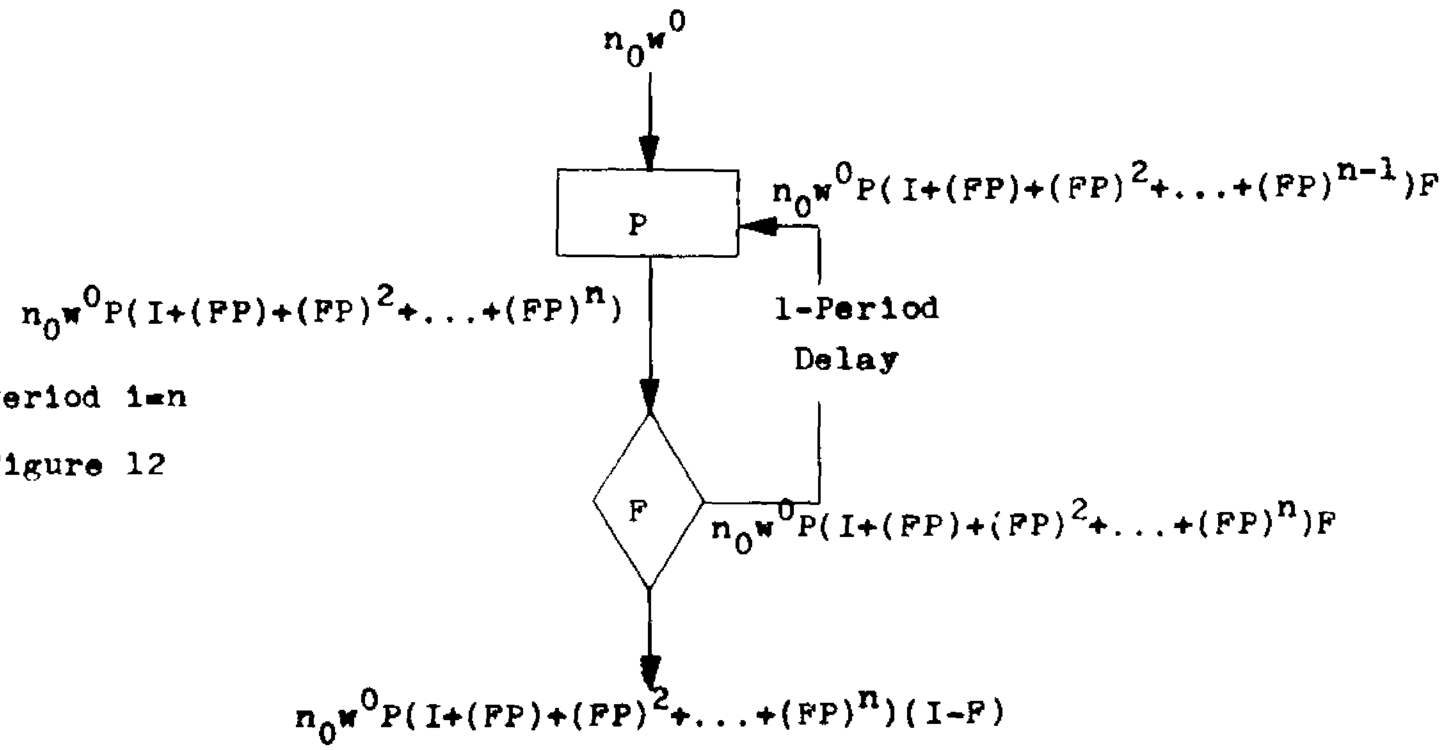
Period i=1

Figure 10



Period i=2

Figure 11



Period i=n

Figure 12

characteristic equation $|FP - \lambda \quad I| = 0$ are obtained, from which it is possible to determine a matrix B of eigenvectors. The end result is the similarity transformation $B^{-1}(FP)B=D$, where D is the diagonal matrix of eigenvalues. The purpose of this operation is clear once it is noted that $(FP)^i = BD^iB^{-1}$. The i th power of (FP) can now be obtained from D^i, a trivial calculation. For example, for a two by two matrix, $D^i = \begin{bmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{bmatrix}$, where λ_1 and λ_2 are the eigenvalues of the matrix (FP). Certain complications arise if two or more eigenvalues are equal. Here, however, the likelihood of having two or more identical λ's is zero because they are empirical values determined by empirical probabilities.

Consider the series $\sum_{i=0}^{n}(FP)^i = I + (FP) + (FP)^2 + \ldots + (FP)^n$. Premultiplying both sides of the equation by (FP) yields: $(FP)\sum_{i=0}^{n}(FP)^i = (FP) + (FP)^2 + (FP)^3 + \ldots + (FP)^n$. Subtracting the second equation from the first results in:

$$(I - FP)\sum_{i=0}^{n}(FP)^i = I - (FP)^{n+1}.$$

If $(I - FP)^{-1}$ exists, then

$$\sum_{i=0}^{n}(FP)^i = (I - FP)^{-1}(I - (FP)^{n+1}). \qquad (2\text{-}9)$$

The existence of $(I - FP)^{-1}$ is guaranteed if the system does indeed reach equilibrium. The singularity of $(I - FP)$ means that $|I - FP| = |FP - I| = 0$ and hence an eigenvalue of (FP) is 1, indicating that the system will not reach equilibrium but has in infinite buildup.

Mathematically, this requires that (FP) be sub-stochastic, i.e., at least one row sum be strictly less than 1, since a stochastic matrix always has at least one eigenvalue of 1, which is the maximum absolute value of any of its eigenvalues. Since P is stochastic, it is necessary that $F \neq I$ for (FP) to be sub-stochastic. However, a sub-stochastic matrix can have an eigenvalue of 1. For example, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix}$$

has eigenvalues of 3/4 and 1, although obviously sub-stochastic. A sufficient condition for a sub-stochastic matrix to have a maximum eigenvalue less than 1 is that all of its elements be strictly positive (Cox and Miller, 1965, pp. 118-132). Care should therefore be taken in any analysis to insure that (I - FP) is non-singular so that the modeled system does indeed reach equilibrium.

Assuming that $(I - FP)^{-1}$ does exist, to complete the analysis of equation (2-9), consider the 2 by 2 problem:

$$(FP)^{n+1} = B\, D^{n+1}\, B^{-1} = B \begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & 0 \end{bmatrix} B^{-1} + B \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2^{n+1} \end{bmatrix} B^{-1}.$$

Since $\lambda < 1$, $\lim\limits_{n\to\infty} \lambda_i^{n+1} = 0$, $i=1, 2$, and therefore $\lim\limits_{n\to\infty} (FP)^{n+1} \to 0$, where 0 is a matrix in which all elements are equal to zero. Now from equation (2-9), the equilibrium solution can be obtained from

$$\lim_{n\to\infty} \sum_{i=0}^{n} (FP)^i = (I - FP)^{-1} \qquad (2\text{-}10)$$

by pre-multiplying by $n_0 w^0 P$. In like manner, the transient solution can be obtained from

$$\sum_{i=0}^{n}(FP)^i = (I - FP)^{-1} B \begin{bmatrix} 1-\lambda_1^{n+1} & 0 \\ 0 & 1-\lambda_2^{n+1} \end{bmatrix} B^{-1}.$$

As an alternative derivation of the above, consider the following equations:

$$\begin{aligned}
\sum_{i=0}^{n}(FP)^i &= I + (FP) + (FP)^2 + \ldots + (FP)^n \\
&= BIB^{-1} + BDB^{-1} + BD^2B^{-1} + \ldots + BD^nB^{-1} \\
&= B \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B^{-1} + B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} B^{-1} + \ldots + B \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} B^{-1} \\
&= B \begin{bmatrix} 1+\lambda_1+\lambda_1^2+\ldots+\lambda_1^n & 0 \\ 0 & 1+\lambda_2+\lambda_2^2+\ldots+\lambda_2^n \end{bmatrix} B^{-1}.
\end{aligned}$$

It follows from the formula for the sum of a geometric series with a ratio $\lambda < 1$ that

$$\sum_{i=0}^{n}(FP)^i = B \begin{bmatrix} \dfrac{1-\lambda_1^{n+1}}{1-\lambda_1} & 0 \\ 0 & \dfrac{1-\lambda_2^{n+1}}{1-\lambda_2} \end{bmatrix} B^{-1}.$$

Furthermore, from the limiting case of the formula for an infinite sum of geometric terms,

$$\lim_{n\to\infty} \sum_{i=0}^{n}(FP)^i = B \begin{bmatrix} \dfrac{1}{1-\lambda_1} & 0 \\ 0 & \dfrac{1}{1-\lambda_2} \end{bmatrix} B^{-1},$$

and by comparison with equation (2-10),

$$(I - FP)^{-1} = B \begin{bmatrix} \dfrac{1}{1-\lambda_1} & 0 \\ 0 & \dfrac{1}{1-\lambda_2} \end{bmatrix} B^{-1}.$$

In order to compare the analysis of the transient state solution of the same system by two different techniques, consider the following numerical treatment of the imperfect inspector example presented previously in subsection 3. Again, the profile matrix is:

$$FP = \frac{1}{32}\begin{bmatrix} 14 & 7 & 7 \\ 6 & 12 & 6 \\ 0 & 2 & 2 \end{bmatrix}$$ and the characteristic equation is:

$$FP-\lambda I = \frac{1}{32}\begin{vmatrix} 14-y & 7 & 7 \\ 6 & 12-y & 6 \\ 0 & 2 & 2-y \end{vmatrix} = 0, \text{ where } y=32\lambda.$$

The determinant yields the following cubic equation:

$-y^3 + 28y^2 - 166y + 168 = 0$ (2-11), which has roots

$y_1 = 1.27294$, $y_2 = 6.53668$, and $y_3 = 20.1904$.

Note that all $\lambda_i = \frac{y_i}{32}$ are less than 1, guaranteeing equilibrium.

Since every eigenvalue satisfies its characteristic equation, the three eigenvectors of the matrix B are obtained as follows:

For λ_1, $(FP - \lambda_1 I)b_1 = 0$, i.e.,

$$\frac{1}{32}\begin{bmatrix} 14-1.27294 & 7 & 7 \\ 6 & 12-1.27294 & 6 \\ 0 & 2 & 2-1.27294 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to this matrix equation is:

$b_{11} = .963$, $b_{21} = 1.000$, and $b_{31} = -2.751$.

In a like manner, the second eigenvalue leads to:

$$b_{21} = -1.351,\ b_{22} = 1.000,\ \text{and}\ b_{23} = .44085.$$

Finally, the third eigenvalue leads to:

$$b_{31} = 1.25506,\ b_{32} = 1.000,\ \text{and}\ b_{33} = .109948.$$

Thus, the similarity transformation matrix $\underline{B}$ is:

$$B = \begin{bmatrix} .963 & -1.351 & 1.25506 \\ 1.000 & 1.000 & 1.000 \\ -2.751 & .44085 & .109948 \end{bmatrix}$$

and the inverse of the B matrix is:

$$B^{-1} = \begin{bmatrix} -.043813 & .092928 & -.345062 \\ -.378811 & .471179 & .038671 \\ .422625 & .435894 & .306391 \end{bmatrix}$$

A check on the calculations is provided by the fact that B^{-1} (FP)B should equal the diagonal matrix D of the eigenvalues of (FP). The specific D obtained in this manner was:

$$D = \frac{1}{32}\begin{bmatrix} 1.27294 & .000091 & -.000047 \\ -.000129 & 6.53668 & -.000284 \\ -.000808 & .00232612 & 20.1904 \end{bmatrix} .$$

Here, due to roundoff error, the resulting matrix is not perfectly diagonal, but the diagonal values are exactly equal to the eigenvalues of (FP) to six significant figures and the magnitude of the off-diagonal elements no greater than .003.

Analytically, it is useful to express the D matrix as the sum of the three matrices:

$$D = \frac{1.27294}{32}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 6.53668\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 20.1904\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then: $\sum_{i=0}^{n} (FP)^i = \sum_{i=0}^{n} B\, D^i\, B^{-1}$ so that:

$$= \sum_{i=0}^{n}\left(\frac{1.27294}{32}\right)^i B\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} B^{-1}$$

$$+\sum_{i=0}^{n}\left(\frac{6.53668}{32}\right)^i B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} B^{-1}$$

$$+\sum_{i=0}^{n}\left(\frac{20.1904}{32}\right)^i B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B^{-1}$$

Substitution in for B and B^{-1} yields

$$\sum_{i=0}^{n} (FP)^i = \sum_{i=0}^{n}\left(\frac{1.27294}{32}\right)^i \begin{bmatrix} -.042191 & .089490 & -.332295 \\ -.043813 & .092928 & -.345062 \\ .120530 & -.255645 & .949095 \end{bmatrix}$$

$$+\sum_{i=0}^{n}\left(\frac{6.53668}{32}\right)^i \begin{bmatrix} .51174 & -.636563 & -.052245 \\ -.378811 & .471179 & .038671 \\ -.166999 & .207719 & .017048 \end{bmatrix} \quad (2\text{-}12)$$

$$+\sum_{i=0}^{n}\left(\frac{20.1904}{32}\right)^{i}\begin{bmatrix} .530420 & .547073 & .384539 \\ .422625 & .435894 & .306391 \\ .046467 & .047926 & .033687 \end{bmatrix}$$

The transient probability distribution of the system for the n th state can now be evaluated as $n_0 w^0 P \sum_{i=0}^{n}(FP)^i$, using the relationship:

$$\sum_{i=0}^{n}\lambda^i = \frac{1-\lambda^{n+1}}{1-\lambda}$$

so long as $\lambda^2 < 1$, which is true for λ_1, λ_2 and λ_3. The steady-state solution can also be evaluated by letting n approach infinity, since then

$$\lim_{n\to\infty}\sum_{i=0}^{n}\lambda^i = \frac{1}{1-\lambda}, \text{ for } \lambda^2 < 1.$$

Note that the equilibrium is more simply obtained directly from $(I - FP)^{-1}$, as discussed earlier in subsection 2. However, as a check, note that here

$$\lim_{n\to\infty}\sum_{i=0}^{n}(FP)^i = \begin{bmatrix} -.043941 & .093197 & -.346061 \\ -.045629 & .096778 & -.359357 \\ .125525 & -.266235 & .988590 \end{bmatrix}$$

$$+\begin{bmatrix} .653151 & -.799974 & -.065656 \\ -.476055 & .592135 & .048598 \\ -.209869 & .261043 & .021425 \end{bmatrix}$$

$$+\begin{bmatrix} 1.43726 & 1.48238 & 1.04197 \\ 1.14517 & 1.81811 & .83021 \\ .12591 & .12986 & .09128 \end{bmatrix}.$$

which, when summed, yields

$$\lim_{n\to\infty} \sum_{i=0}^{n} (FP)^i = \begin{bmatrix} 2.03647 & .77560 & .63025 \\ .62348 & 1.87004 & .51946 \\ .04157 & .12467 & 1.10129 \end{bmatrix}$$

Comparison of this result with equation (2-8) in subsection 3 indicates four-place agreement, well within the accuracy of the calculations.

An alternative and standard approach to the transient analysis of a Markov process involves generating functions, and the z-transform (Howard, 1960, pp. 7-16). Here, the z-transform for f(n) is defined by $t(z) = \sum_{n=0}^{\infty} f(n)z^n$, and f(n+1) has a z-transform equal to $\frac{t(z) - f(0)}{z}$.

Since the key to the transient analysis is the evaluation of $(FP)^n$, the appropriate matrix difference equation is:

$$(FP)^{n+1} = (FP)^n(FP).$$

Letting $f(n) = (FP)^n$, the z-transform of this matrix system is:

$$\frac{t(z) - f(0)}{z} = t(z)\ (FP),$$

where $f(0) = (FP)^0 = I$, as before. Solving for t(z) yields:

$$t(z) = (I - FPz)^{-1}.$$

It is interesting to compare the transient solution using this method on the same problem that was just used to demonstrate the method of eigenvalues. Again, letting

$$FP = \frac{1}{32} \begin{bmatrix} 14 & 7 & 7 \\ 6 & 12 & 6 \\ 0 & 2 & 2 \end{bmatrix} ,$$

then

$$\left|I - FPz\right| = \begin{vmatrix} 1 - \frac{14z}{32} & -\frac{7z}{32} & -\frac{7z}{32} \\ -\frac{6z}{32} & 1 - \frac{12z}{32} & -\frac{6z}{32} \\ 0 & -\frac{2z}{32} & 1 - \frac{2z}{32} \end{vmatrix} .$$

The determinant of the z-transform of the (FP) matrix is $(32)^3 - 28\,(32)^2 z + 166\,(32) z^2 - 168\,z^3$. Note that the coefficients of this expression are the same as those of the characteristic equation of the eigenvalue example (see 2-11), neglecting the "32" terms, but the powers of the variables are ascending instead of descending. It is therefore possible to express the determinant of the z-transform of the (FP) matrix as the product of three terms in which the coefficients of z are the three eigenvalues of the previous example, that is,

$$\left|I - FPz\right| = \left(1 - \frac{1.27294z}{32}\right) \cdot \left(1 - \frac{6.53668z}{32}\right) \cdot \left(1 - \frac{20.1904z}{32}\right) .$$

Then, $(I - FPz)^{-1} =$

$$\frac{\begin{bmatrix} \left(12 \left(\frac{z}{32}\right)^2 - 14 \frac{z}{32} + 1\right) & \left(7 \frac{z}{32}\right) & \left(-42 \left(\frac{z}{32}\right)^2 + 7 \frac{z}{32}\right) \\ \left(-12 \frac{z}{32} + 6 \frac{z}{32}\right) & \left(28 \left(\frac{z}{32}\right)^2 - 16 \frac{z}{32} + 1\right) & \left(-42 \left(\frac{z}{32}\right)^2 + 6 \frac{z}{32}\right) \\ \left(12 \frac{z}{32}\right) & \left(-28 \left(\frac{z}{32}\right)^2 + 2 \frac{z}{32}\right) & \left(126 \left(\frac{z}{32}\right)^2 - 26 \frac{z}{32} + 1\right) \end{bmatrix}}{\left|I - FPz\right|} .$$

In order to recover $(FP)^n = f(n)$ from $\mathfrak{F}(z)$, it is convenient to use a partial fraction expansion in which each of the terms of the previous matrix may be set equal to an expression of the form

$$\frac{A}{1-\frac{1.27294\,z}{32}} + \frac{B}{1-\frac{6.53668\,z}{32}} + \frac{C}{1-\frac{20.1904\,z}{32}}\,.$$

Since the z-transform of a^n is $\frac{1}{1-az}$, $f(n) = A\left(\frac{1.27294}{32}\right)^n +$

$B\left(\frac{6.53668}{32}\right)^n + C\left(\frac{20.1904}{32}\right)^n$.

The particular set of A, B, and C that are needed for each term can be determined through the use of the following set of three simultaneous linear equations:

$$\text{Coefficient of } z^2 = \frac{131.97818\,A}{(32)^2} + \frac{25.70117\,B}{(32)^2} + \frac{8.32080\,C}{(32)^2}$$

$$\text{Coefficient of } z = \frac{-26.72708\,A}{32} - \frac{21.46334\,B}{32} - \frac{7.80962\,C}{32}$$

$$\text{Constant term} = A + B + C\,.$$

This set of three simultaneous linear equations were solved on the CEIR random access computer using the SIMEX1 program for each of the nine elements of the matrix. This resulted in

$$(FP)^n = f(n) = \left(\frac{1.27294}{32}\right)^n \begin{bmatrix} -.042186 & .089485 & -.332301 \\ -.043809 & .092927 & -.345085 \\ .12051 & -.255624 & .949259 \end{bmatrix}$$

$$+\left(\frac{6.53668}{32}\right)^n \begin{bmatrix} .511834 & -.636663 & -.052272 \\ -.378742 & .471113 & .038680 \\ -.166969 & .207691 & .017052 \end{bmatrix}$$

$$+\left(\frac{20.1904}{32}\right)^n \begin{bmatrix} .530352 & .547179 & .384573 \\ .422552 & .435961 & .306405 \\ .046459 & .047933 & .033690 \end{bmatrix}$$

Comparison of this with (2-12), which is the expression for $\sum_{i=0}^{n} (FP)^n$ obtained from the eigenvalue method, indicates that the only difference is due to small roundoff errors. Moreover, since the transient solution developed previously converged to the steady-state solution $n_o w^o P \cdot (I - FP)^{-1}$, it is clear that this transient solution will likewise converge.

CHAPTER III

EXTENSIONS OF THE BASIC MODEL TO MULTIPLE FEEDBACK LOOPS

To demonstrate the applicability of the model to systems with more than one feedback loop, this chapter develops the cases in which the loops are sequential, nested and interlocking. Each system will be described graphically and algebraically. In all cases, it is possible to solve for the equilibrium number of items, n, and the equilibrium distribution of errors, w^e. Furthermore, for each case, a general profile matrix will emerge which can then be used for a transient analysis similar to that of the last subsection of the previous chapter. Again the only requirement is that no items can be created nor destroyed in the system and all items flow uniformly. This condition simply means that the algebraic sum of the items into a node must equal the algebraic sum of the items leaving the node. Also, as a check, the number (volume) of items leaving a system must equal the number of items entering the system when the system is in equilibrium. These conditions correspond to Kirchoff's laws of electricity which state that the sum of the currents into a node must equal the sum of the currents leaving the node since there can be no buildup or generation of electrons in the node.

A. SEQUENTIAL FEEDBACK LOOPS

Consider Figure 13 in which there are two feedback loops in sequence. Here, an algebraic description of the situation results from applying the following feedback rule twice: whenever there is a node with more than one input, define the output of the node as $n_e w^e$. If there is more than one node with more than one input, sequence the outputs as $n_{e1} w^{e1}$. Therefore, the output of operation P^1 is $n_{e1} w^{e1}$ and the output of P^2 is $n_{e2} w^{e2}$. By applying the conservation law to node P^1, the following equilibrium equation is obtained:

$$n_{e1} w^{e1} = n_0 w^0 P^1 + n_{e1} w^{e1} F^1 P^1. \tag{3-1}$$

Applying the conservation law to node P^2 yields

$$n_{e2} w^{e2} = n_{e1} w^{e1} (I - F^1) P^2 + n_{e2} w^{e2} F^2 P^2. \tag{3-2}$$

In these equations, all quantities are assumed known except the equilibrium volumes and error-level distributions. Equation (3-1) can be rearranged to compute n_{e1} and w^{e1} from

$$n_{e1} w^{e1} = n_0 w^0 P^1 (I - F^1 P^1)^{-1}$$

along with $\sum\limits_k w_k = 1$.

Once n_{e1} and w^{e1} are determined, equation (3-2) can be rearranged to determine n_{e2} and w^{e2} from:

$$n_{e2} w^{e2} = n_{e1} w^{e1} (I - F^1) P^2 (I - F^2 P^2)^{-1}. \tag{3-3}$$

$(F^1 P^1)$ and $(F^2 P^2)$ are recognized as profile matrices as were discussed in Chapter II, subsection F(5). Note that there is no additional complexity to the model when two feedback loops appear in sequence except for the term

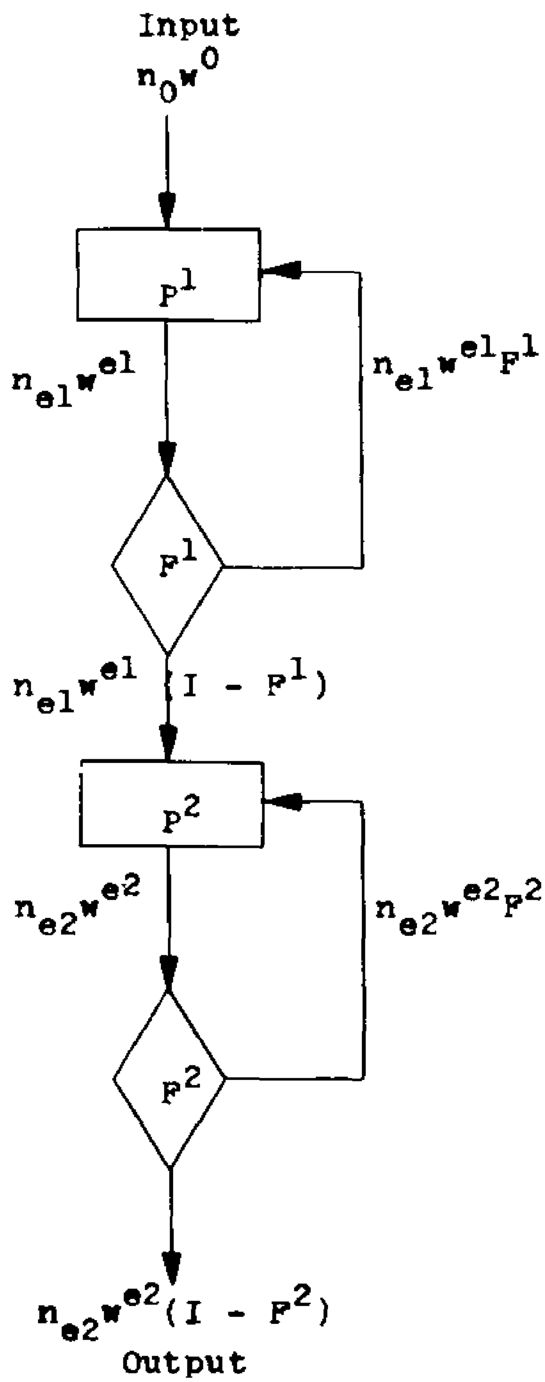


Figure 13. Two feedback loops in sequence.

$(I-F^1)$ in (3-3). Since the conservation law applies to the entire system, the following equation can be used as a check on the equilibrium values derived from the above equations. Then:

$$n_0 w^0 c = n_{e2} w^{e2}(I - F^2)c$$

where c is a column vector with all elements equal to 1.

B. NESTED FEEDBACK LOOPS

Figure 14 illustrates a situation in which one feedback loop occurs entirely within another feedback loop. Again by applying the sequencing rule, four unknown quantities are introduced. n_{e1} and n_{e2} are the equilibrium volumes and w^{e1} and w^{e2} are the equilibrium error-level distributions. The conservation law for node P^1 yields the following equation:

$$n_{e1} w^{e1} = n_0 w^0 P^1 + n_{e2} w^{e2}(I - F^2)F^1 P^1 \ . \qquad (3\text{-}4)$$

Similarly, for node P^2,

$$n_{e2} w^{e2} = n_{e1} w^{e1} P^2 + n_{e2} w^{e2} F^2 P^2. \qquad (3\text{-}5)$$

The solution to $n_{e2} w^{e2}$ is obtained by substitution of equation (3-4) into equation (3-5) so that:

$$n_{e2} w^{e2} = n_0 w^0 P^1 P^2 + n_{e2} w^{e2}(I - F^2)F^1 P^1 P^2 + n_{e2} w^{e2} F^2 P^2;$$

which yields

$$n_{e2} w^{e2} = n_0 w^0 P^1 P^2 (I - (I - F^2)F^1 P^1 P^2 - F^2 P^2)^{-1}. \qquad (3\text{-}6)$$

Again the conservation laws for the system provide the check:

$$n^0 w^0 c = n_{e2} w^{e2} \ (I - F^1) \ (I - F^2)c,$$

where c is a column vector with all elements equal to 1.

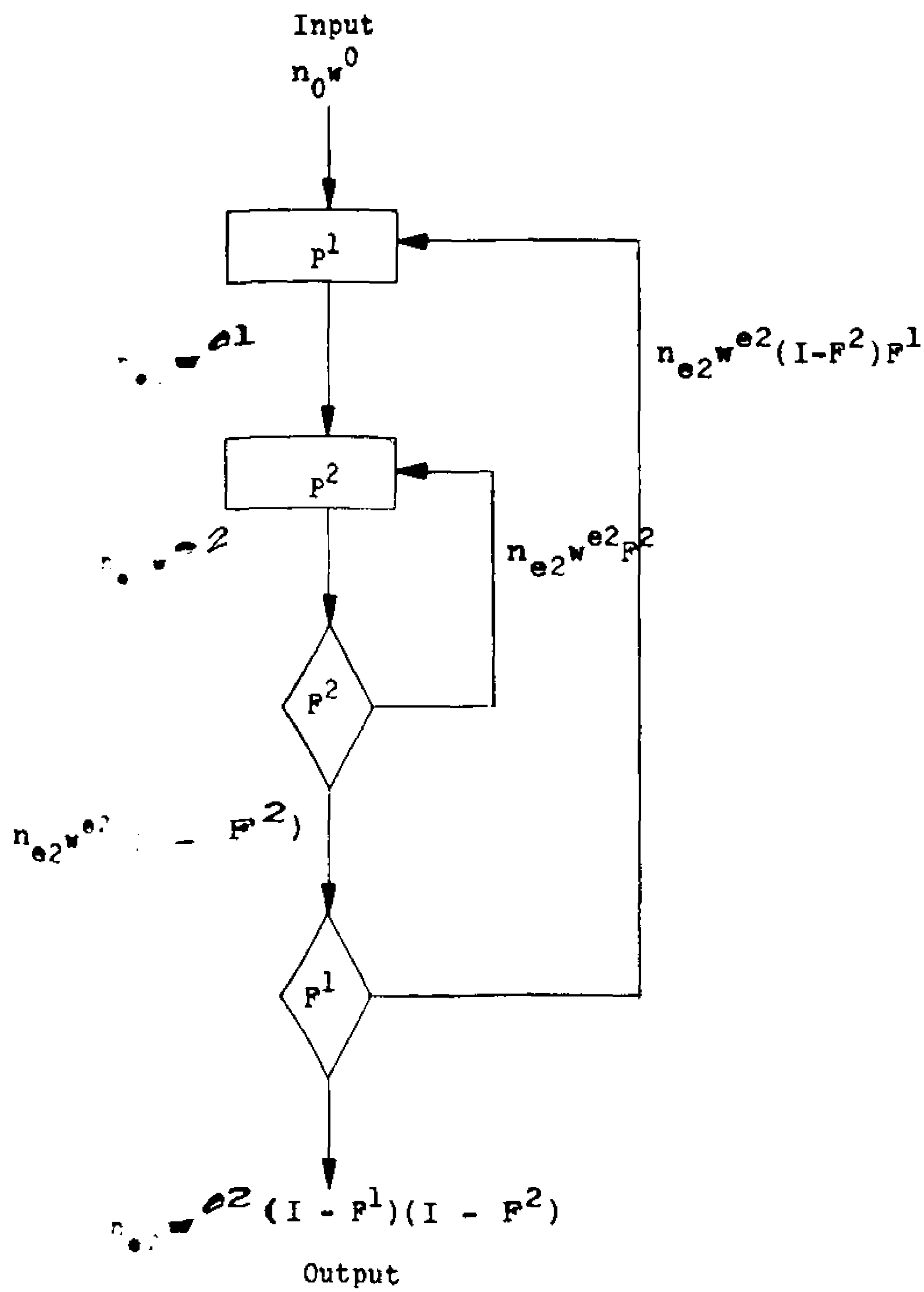


Figure 14. A nested set of feedback loops.

In order to analyze the transient states of this model with feedback, it is possible to draw an analogy with the equilibrium equation of Figure 8, Chapter II, section 2, and the analysis of Chapter II, section 5. Figure 15 describes the basic feedback model and its corresponding "profile" model. By analogy with equation (3-6), $n_0w^0P^1$ is the "profile input", P^2 is the "profile operation", and $(I - F^2)F^1P^1P^2 + F^2P^2$ is the "profile matrix." In future models the only element of the "profile" model that will be stated is the "profile matrix" since it is the only element that is required to carry out the computations of the transient states as in Chapter II, section 5.

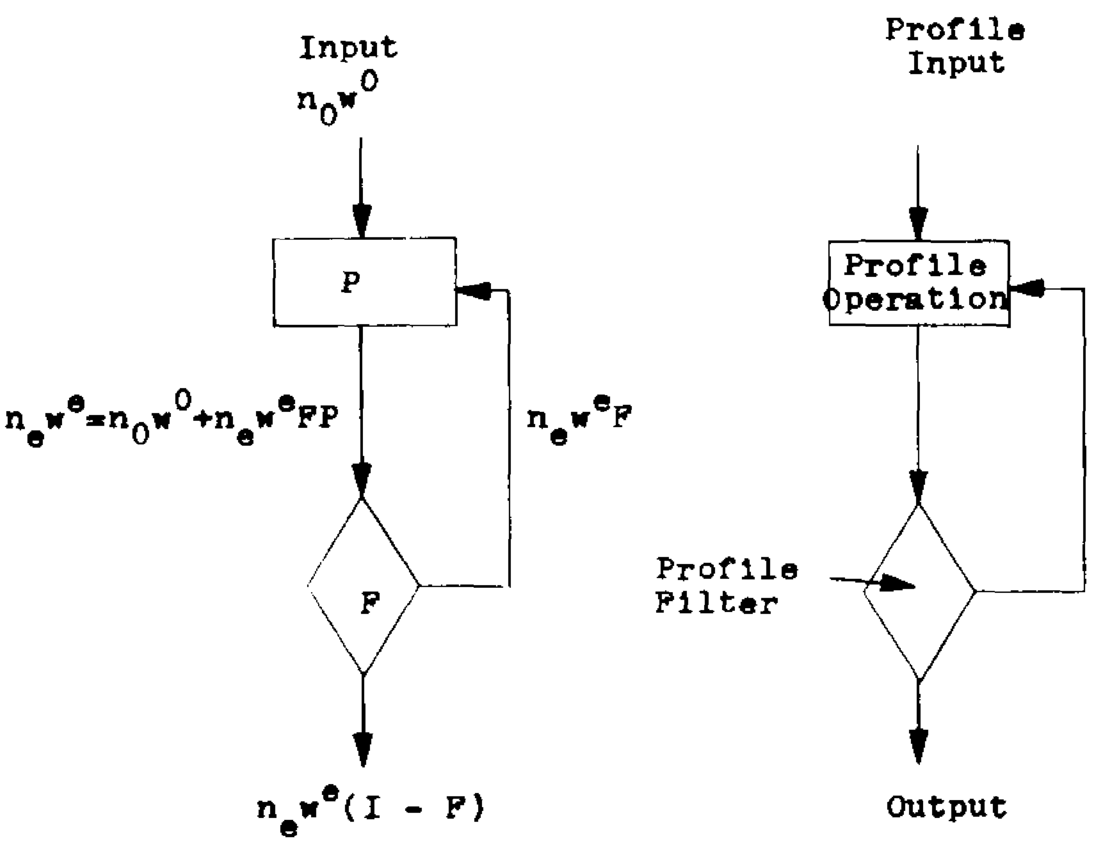


Figure 15. Feedback model with corresponding "profile" model.

C. INTERLOCKING FEEDBACK LOOPS

Figure 16 illustrates a situation in which an element of feedback loop 1 is contained within feedback loop 2 and an element of feedback loop 2 is contained within feedback loop 1. Again, in applying the sequencing rule, $n_{e1}w^{e1}$ and $n_{e2}w^{e2}$ are defined. Also by applying the conservation law to node P^1 the following equation is obtained:

$$n_{e1}w^{e1} = n_0w^0P^1 + n_{e2}w^{e2}F^1P^1. \tag{3-7}$$

From node P^2, the following equation must be valid:

$$n_{e2}w^{e2} = n_{e1}w^{e1}P^2 + n_{e2}w^{e2}(I - F^1)F^2P^2. \tag{3-8}$$

Elimination of $n_{e1}w^{e1}$ from equation (3-8) by substitution of (3-7) yields:

$$n_{e2}w^{e2} = n_0w^0P^1P^2 + n_{e2}w^{e2}F^1P^1P^2 + n_{e2}w^{e2}(I - F^1)F^2P^2.$$

On rearrangement this equation can be expressed:

$$n_{e2}w^{e2} = n_0w^0P^1P^2(I - F^1P^1P^2 - (I - F^1)F^2P^2)^{-1},$$

where $F^1P^1P^2 + (I - F^1)F^2P^2$ is recognized as the profile matrix for the whole system. Once $n_{e2}w^{e2}$ has been determined, it can be used in conjunction with equation (3-7) and equation (3-3) to compute $n_{e1}w^{e1}$. Since the conservation law must apply for the entire system, the following equation holds:

$$n_0w^0c = n_{e2}w^{e2}(I-F^1)(I-F^2)c \ ,$$

where c is a column vector with all elements equal to 1.

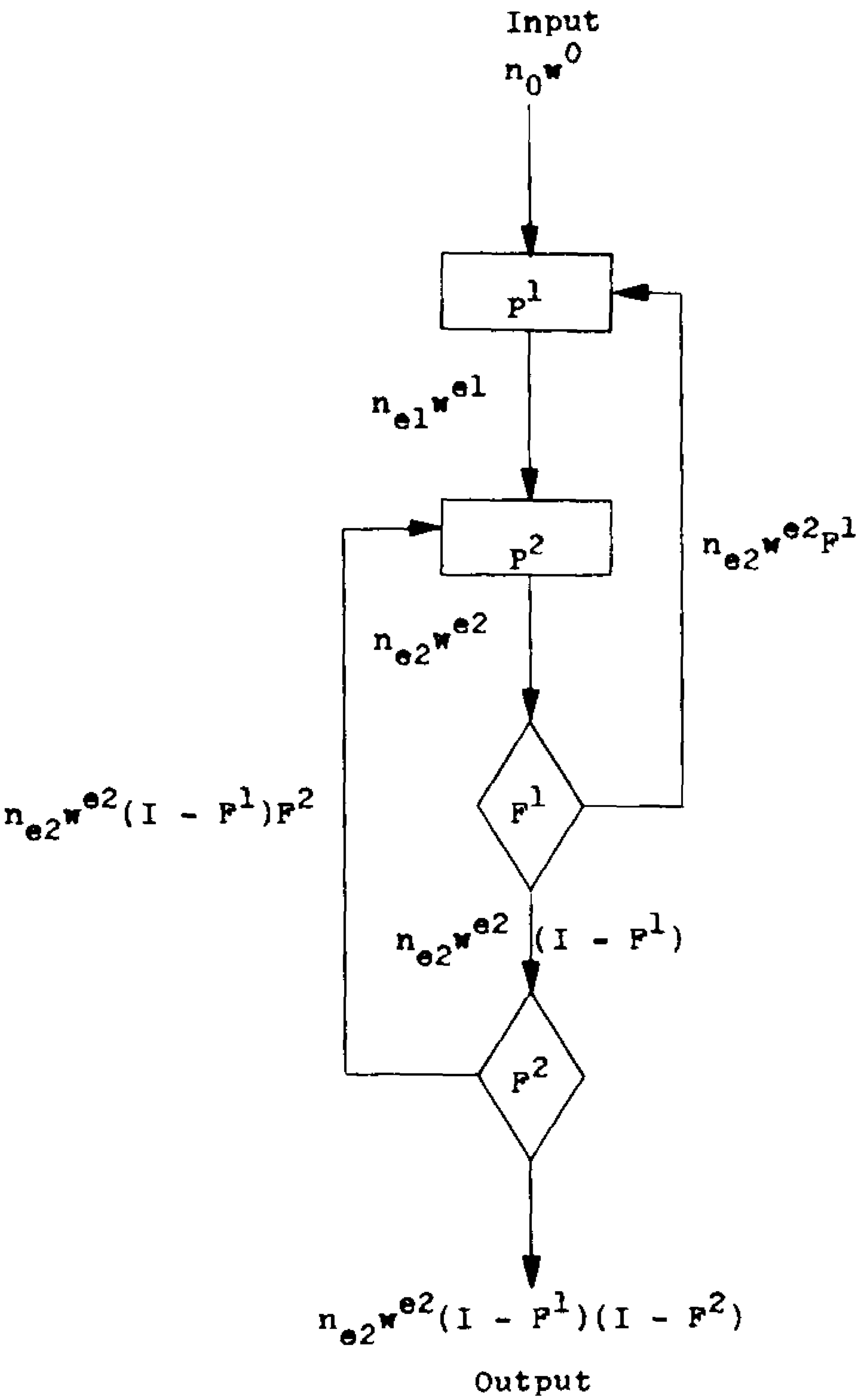


Figure 16. An interlocking set of feedback loops.

CHAPTER IV

EXTENSIONS OF THE BASIC MODEL TO MULTIPLE FEEDFORWARD LOOPS

As in the case of feedback loops, the basic model can be applied to cases in which the feedforward loops are sequential, nested, and interlocking. In this chapter, each of these cases will be described both algebraically and graphically. This time, however, there is no need to use the "profile matrix" because feedforward systems reach equilibrium in one period. Again the only requirement is that no items can be created or destroyed in the system and all items must flow uniformly. The conservation law is, of course, valid at all times since the system achieves equilibrium during the first period.

A. SEQUENTIAL FEEDFORWARD LOOPS

Figure 17 describes a system in which there are two feedforward loops in sequence. For nomenclatural purposes, a bypassed distribution is labeled S instead of P. As before, the analysis is applied to nodes with more than one input, here P^1 and P^2. Due to the feedforward, the incoming vectors to P^1 and P^2 are merely postmultiplied by P^1 and P^2, respectively, and added, as indicated in Figure 17. A check on the equilibrium values is

$$n_0 w^0 c = n_2 w^2 c,$$

where c is a column vector with all elements equal to 1.

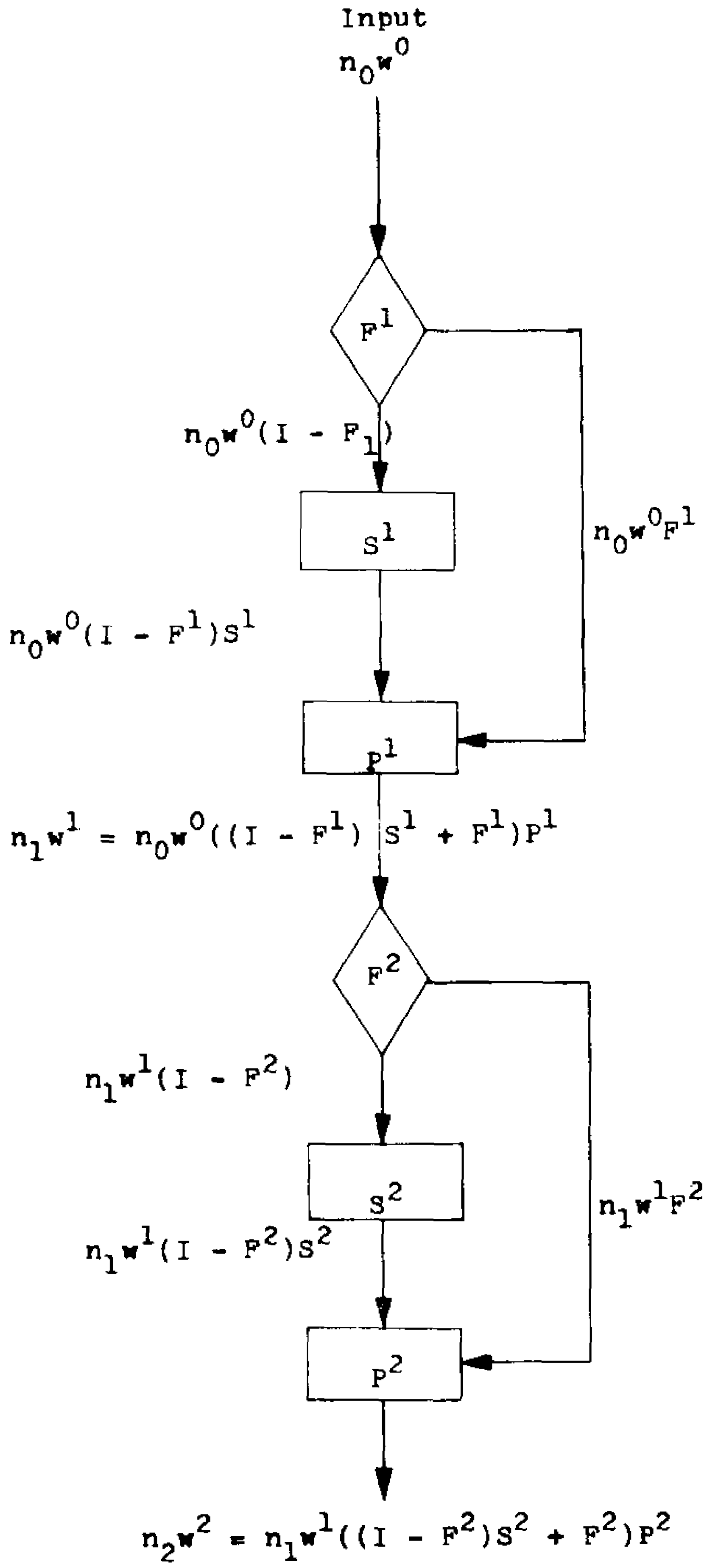


Figure 17. Two sequential feedforward loops.

B. NESTED FEEDFORWARD LOOPS

The application of the basic model to nested feedforward loops in a system presents no problem since each error-level distribution can be computed from the previous one by post-multiplying the error-level distribution vector by the matrix that describes the preceding node. Of course, multiple inputs must be summed if there are, as in nodes P^1 and P^2 of Figure 18, more than one input to a node. Once again, a check on the calculated equilibrium values is

$$n_0 w^0 c = n_1 w^1 c,$$

where c is a column vector with all elements equal to 1.

C. INTERLOCKING FEEDFORWARD LOOPS

Again the application of the basic model to the system with two interlocking feedforward loops is simple. While the algebraic description of the various error-level distributions becomes somewhat lengthy, in practice there is no additional complexity due to the multiplicity of operations because the distributions can be computed sequentially, and it is not necessary to do more than one multiplication of a matrix by a vector and one summing of distributions for each branch value. Figure 19 illustrates the application of the matrix model to a system with two interlocking feedforward loops with the appropriate equations again indicated.

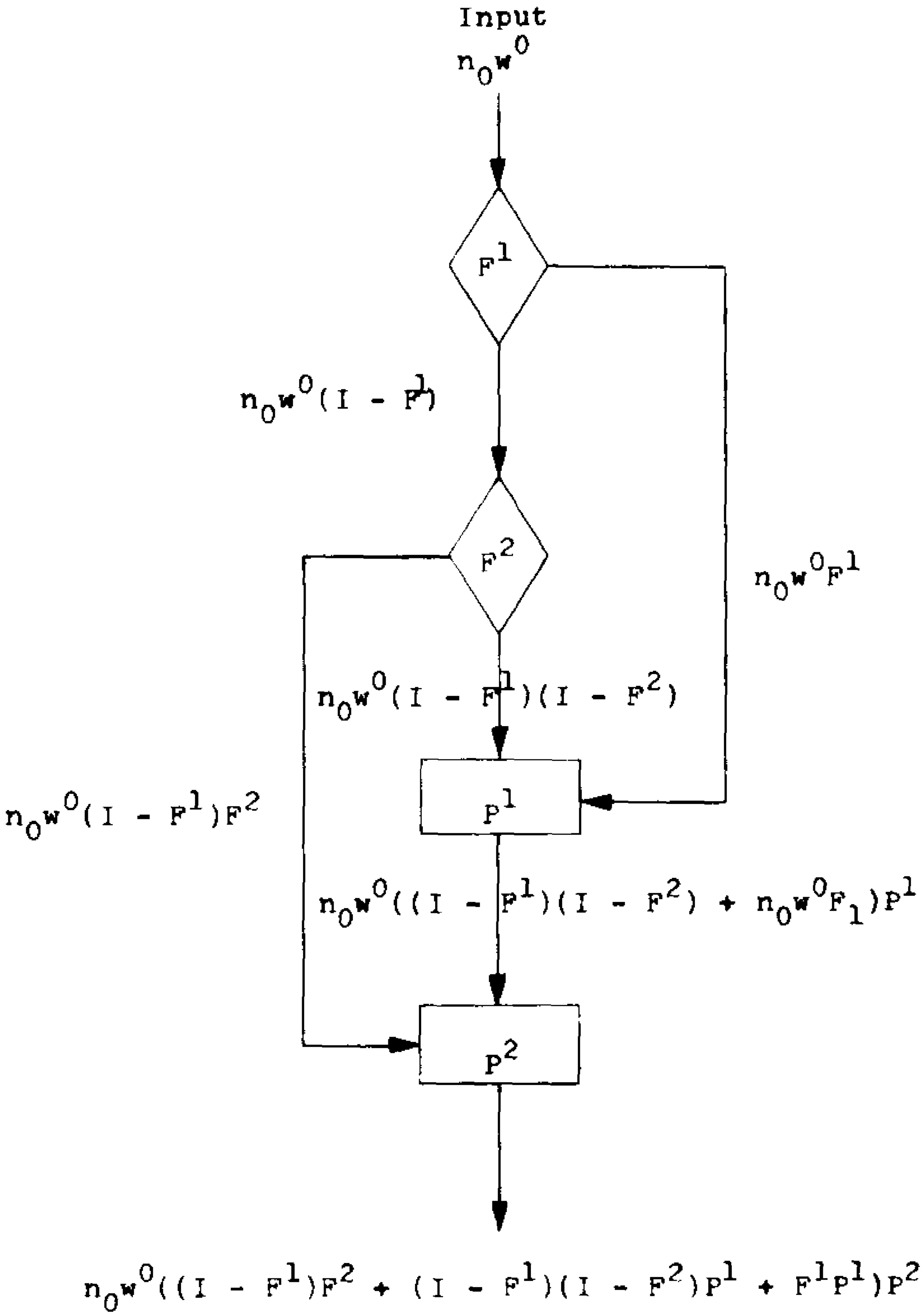


Figure 18. Two interlocking feedforward loops.

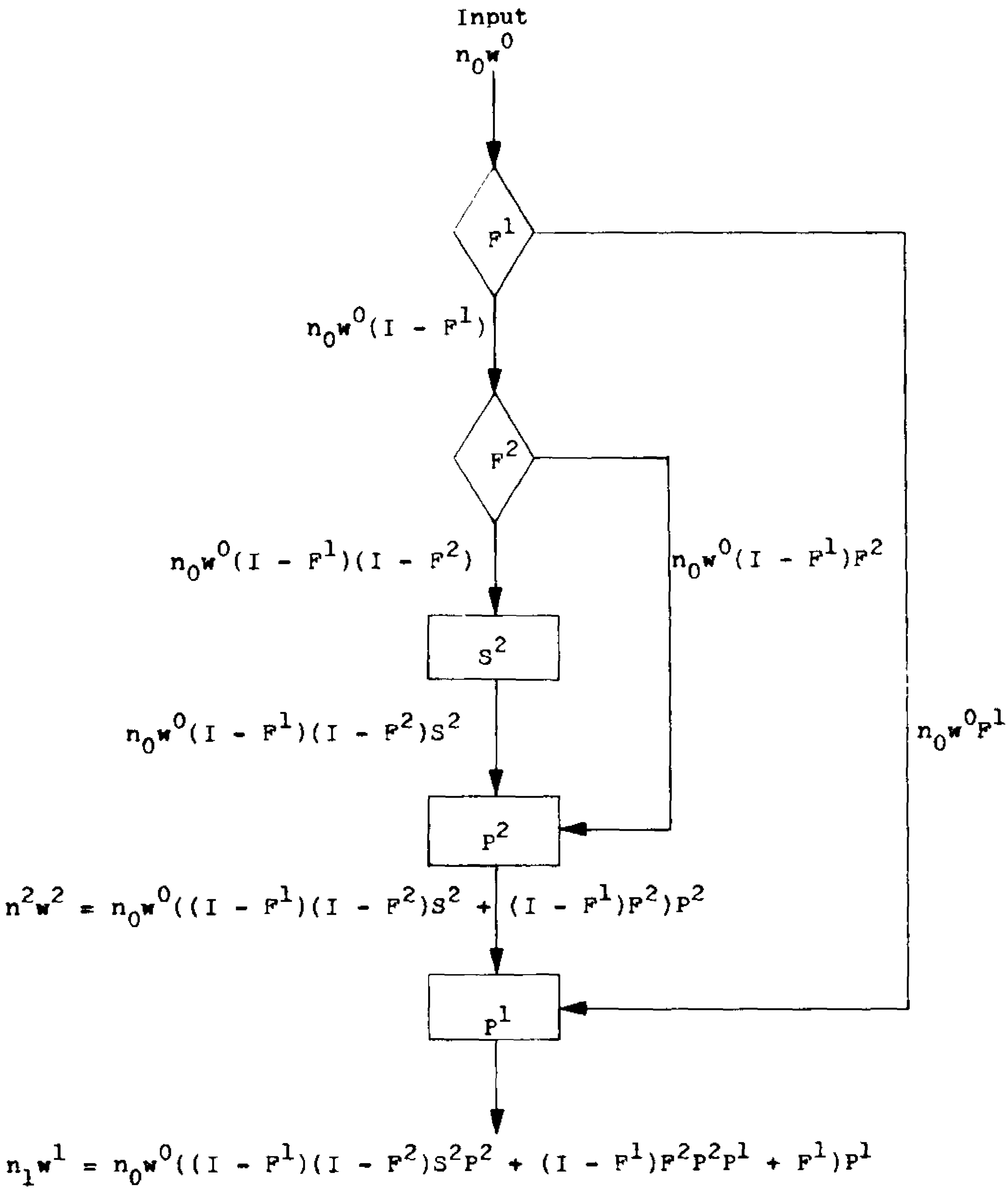


Figure 19. A set of nested feedforward loops.

CHAPTER V

EXTENSIONS OF THE BASIC MODEL TO SYSTEMS WITH BOTH FEEDFORWARD AND FEEDBACK LOOPS

To complete the development of the basic model, this chapter includes the algebraic and graphic descriptions of cases in which both feedforward and feedback loops are sequential, nested and interlocking. In each case there will be two models -- one in which the feedback loop precedes the feedforward loop and one in which the feedforward loop precedes the feedback loop. The solution technique is basically the same as has been used previously and appropriate profile matrices will be indicated. Again, the conservation rule holds at equilibrium.

A. SEQUENTIAL LOOPS

Figure 20 illustrates a system in which a feedback loop is followed by a feedforward loop. Here, the analysis of the feedback loop is essentially identical to the treatment in Chapter II, section F, and the equilibrium equation is: $n_e w^e = n_0 w^0 (I - F^1 P^1)^{-1}$ where $(F^1 P^1)$ is the profile matrix. Similarly, the treatment of the feedforward loop is essentially identical to the treatment of the feedforward loop in Chapter II, section E, except the input to the loop is $n_e w^e (I - F^1)$ instead of $n_0 w^0$.

From the conservation law, the following relationship holds for these sequential loops:

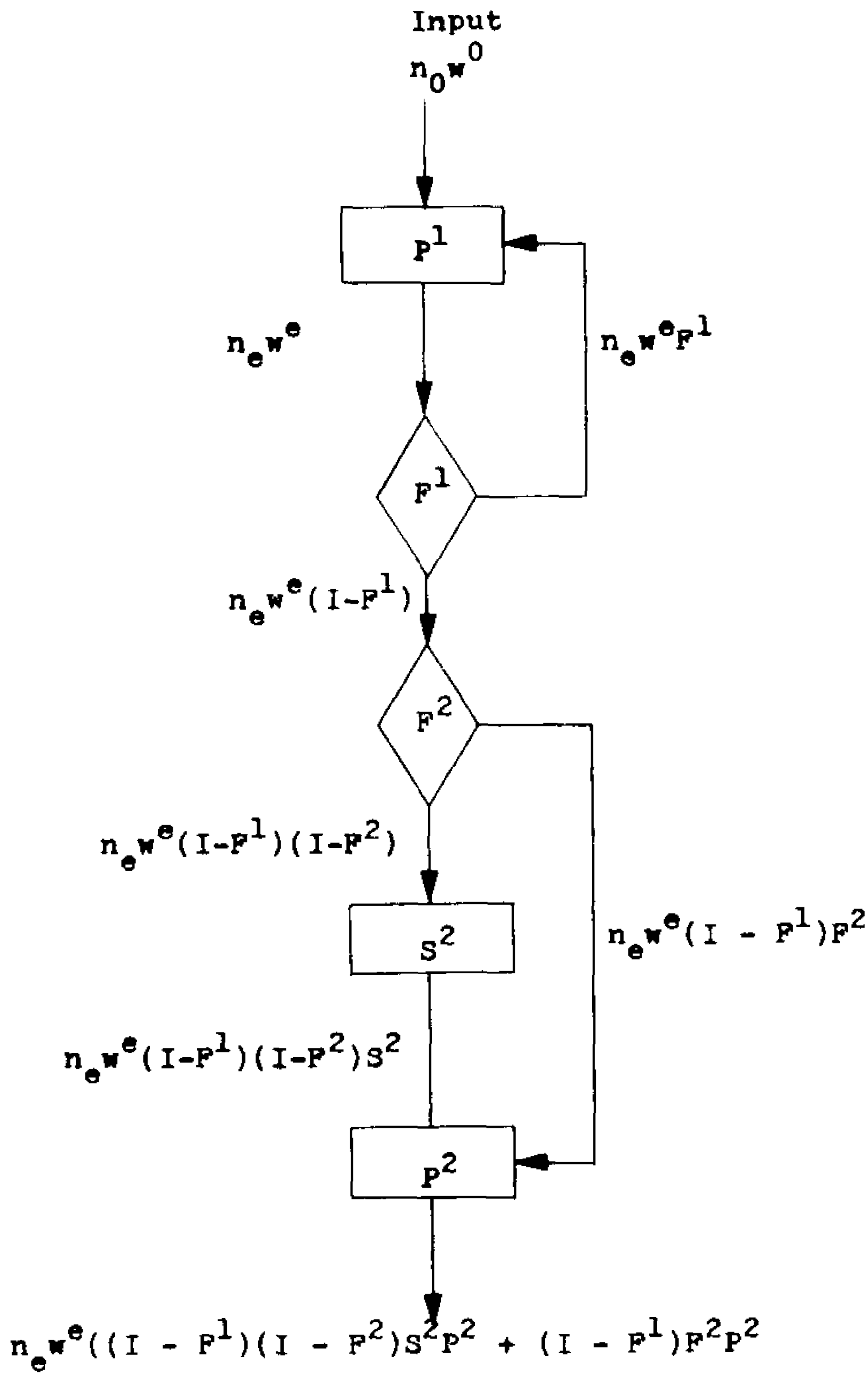


Figure 20. A feedback loop followed by a feedforward loop.

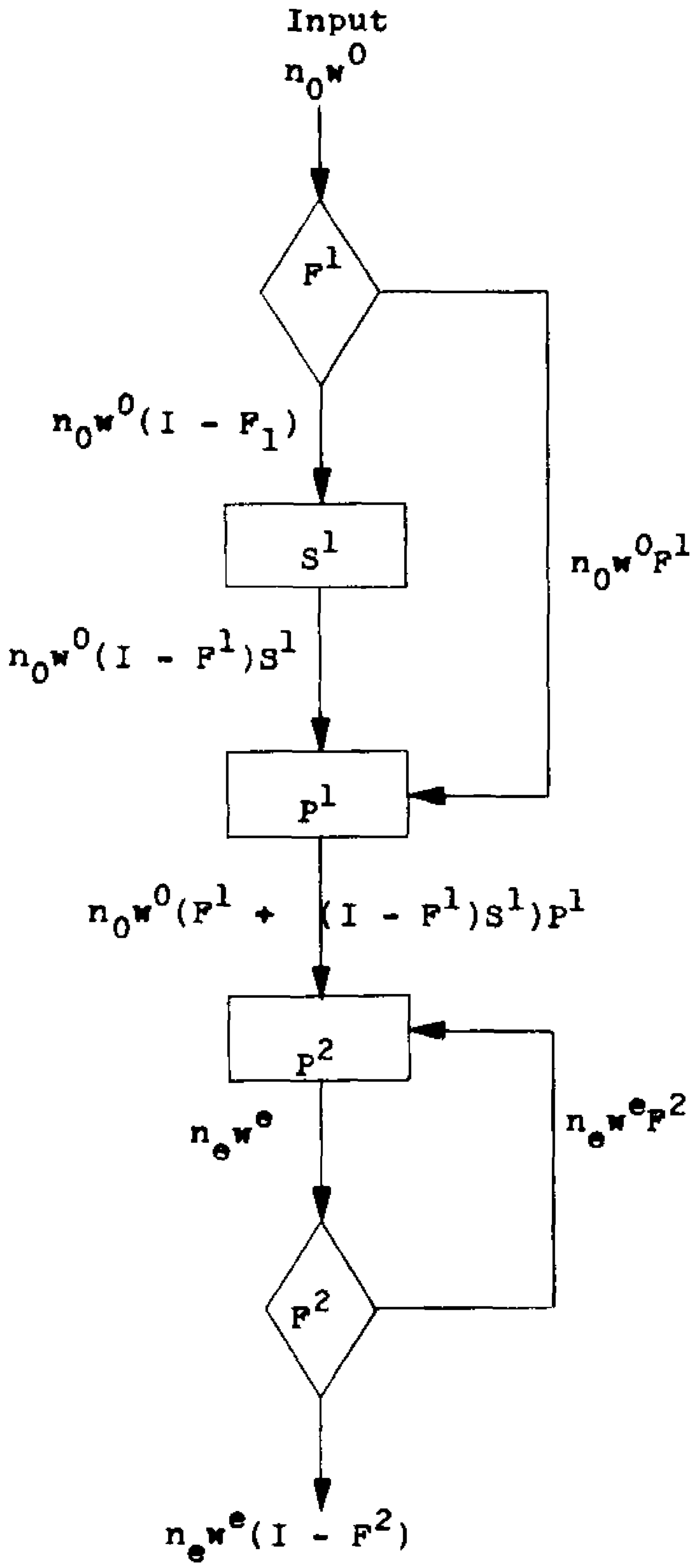


Figure 21. A feedforward loop followed by a feedback loop.

$$n_0 w^0 c = n_e w^e (I - F^1)(F^2 + (I - F^2)S^2)P^2 c$$

and after substitution for $n_e w^e$:

$$n_0 w^0 c = n_0 w^0 (I - F^1 P^1)^{-1}(I - F^1)(F^2 + (I - F^2)S^2)P^2 c,$$

where c is a column vector with all elements equal to 1.

Figure 21 describes a system in which there are also two loops in sequence. This time, however, there is a feedforward loop followed by a feedback loop. As in the previous case, the analysis of the first loop is identical to its treatment in Chapter II, section E.

Application of the feedback rule to node P^2 then yields

$$n_e w^e = n_e w^e F^2 P^2 + n_0 w^0 (F^1 + (I - F^1)S^1)P^1 P^2,$$

which on rearrangement gives

$$n_e w^e = n_0 w^0 (F^1 + (I - F^1)S^1)P^1 P^2 (I - F^2 P^2)^{-1}$$

where $F^2 P^2$ is the profile matrix. Application of the conservation law leads to:

$$n_0 w^0 c = n_e w^e (I - F^2)c$$

and substitution for $n_e w^e$:

$$n_0 w^0 c = n_0 w^0 (F^1 + (I - F^1)S^1)P^1 P^2 (I-F^2 P^2)^{-1}(I-F^2)c,$$

where c is a column vector with all elements equal to 1.

B. NESTED LOOPS

Figure 22 describes a system in which a feedback loop is nested within a feedforward loop. Here, it is necessary to first apply the feedback rule to node P^1. When the rule has been applied and the output of P^1 has been designated $n_e w^e$, the algebraic treatment of the feedforward loop follows the pattern established in Chapter II, section E.

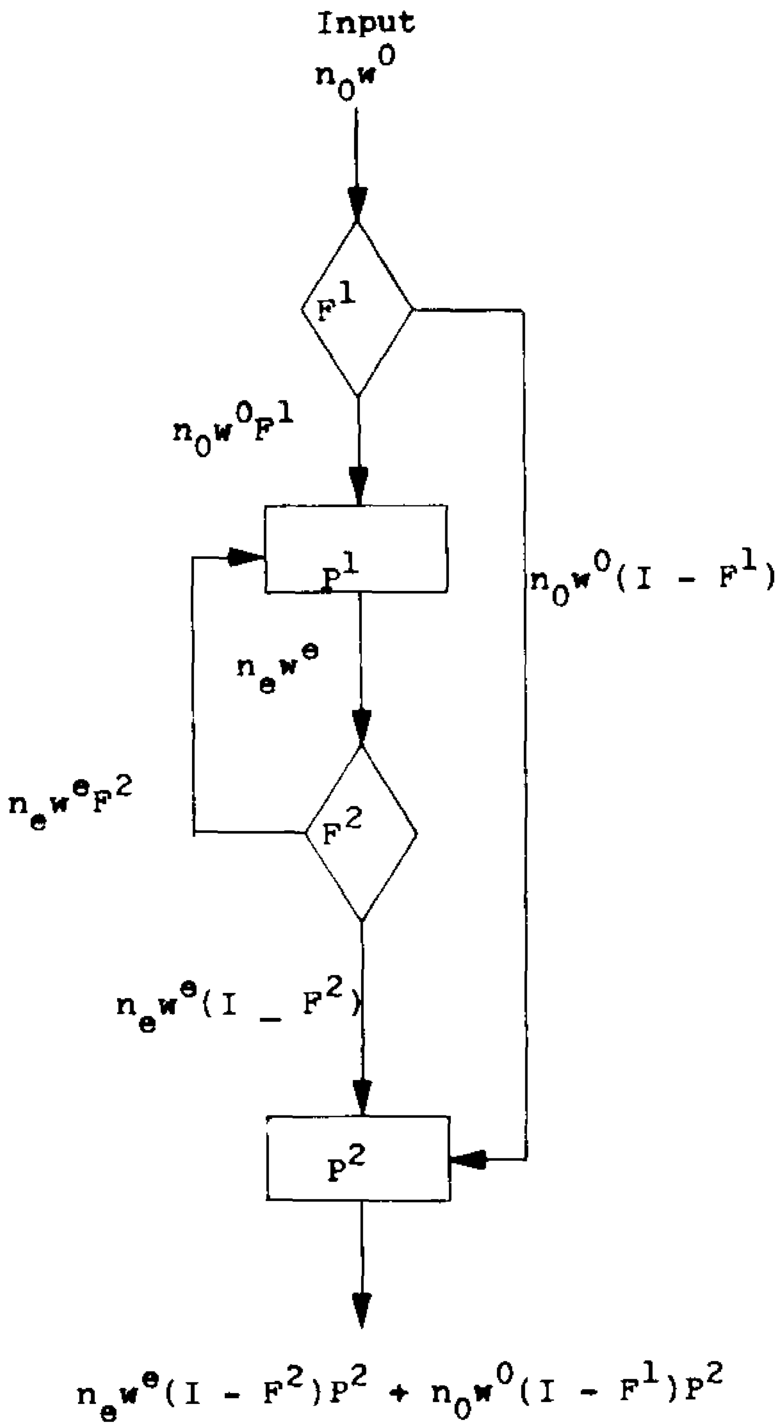


Figure 22. A feedback loop nested within a feedforward loop.

Additionally, the analysis of the feedback loop is identical to the analysis presented in Chapter II, section F, except the input to the loop is $n_0w^0F^1$. The result of the analysis of the feedback loop leads to the following equilibrium equation:

$$n_ew^e = n_0w^0F^1P^1(I - F^2P^1)^{-1}.$$

When the conservation law is applied to this model, the following relationship results:

$$n_0w^0c = n_0w^0(I - F^1)P^2c + n_ew^e(I - F^2)P^2c,$$ and on substitution:

$$n_0w^0c = n_0w^0((I - F^1)P^2 + F^1P^1(I - F^2P^1)^{-1}(I - F^2)P^2)c,$$

where c is a column vector with all elements equal to 1.

Figure 23 also describes a system in which there are two nested loops; however, this time there is a feedforward loop enclosed within a feedback loop. The first step in the analysis of this model is to apply the feedback rule to node P^1 because it has a feedback input. Then the treatment of the feedforward loop is identical to the treatment in Chapter II, section E, except the input is n_ew^e instead of n_0w^0.

Analysis of node P^1 leads to the following relationship:

$$n_ew^e = n_0w^0P^1 + n_ew^e(F^2P^2F^1P^1 + (I-F^2)S^2P^2F^1P^1)$$

and solving for the equilibrium conditions yields:

$$n_ew^e = n_0w^0P^1(I - (F^2P^2F^1P^1 + (I-F^2)S^2P^2F^1P^1))^{-1}.$$

Note that in this case the profile matrix for the whole system is

$$F^2P^2F^1P^1 + (I - F^2)S^2P^2F^1P^1 .$$

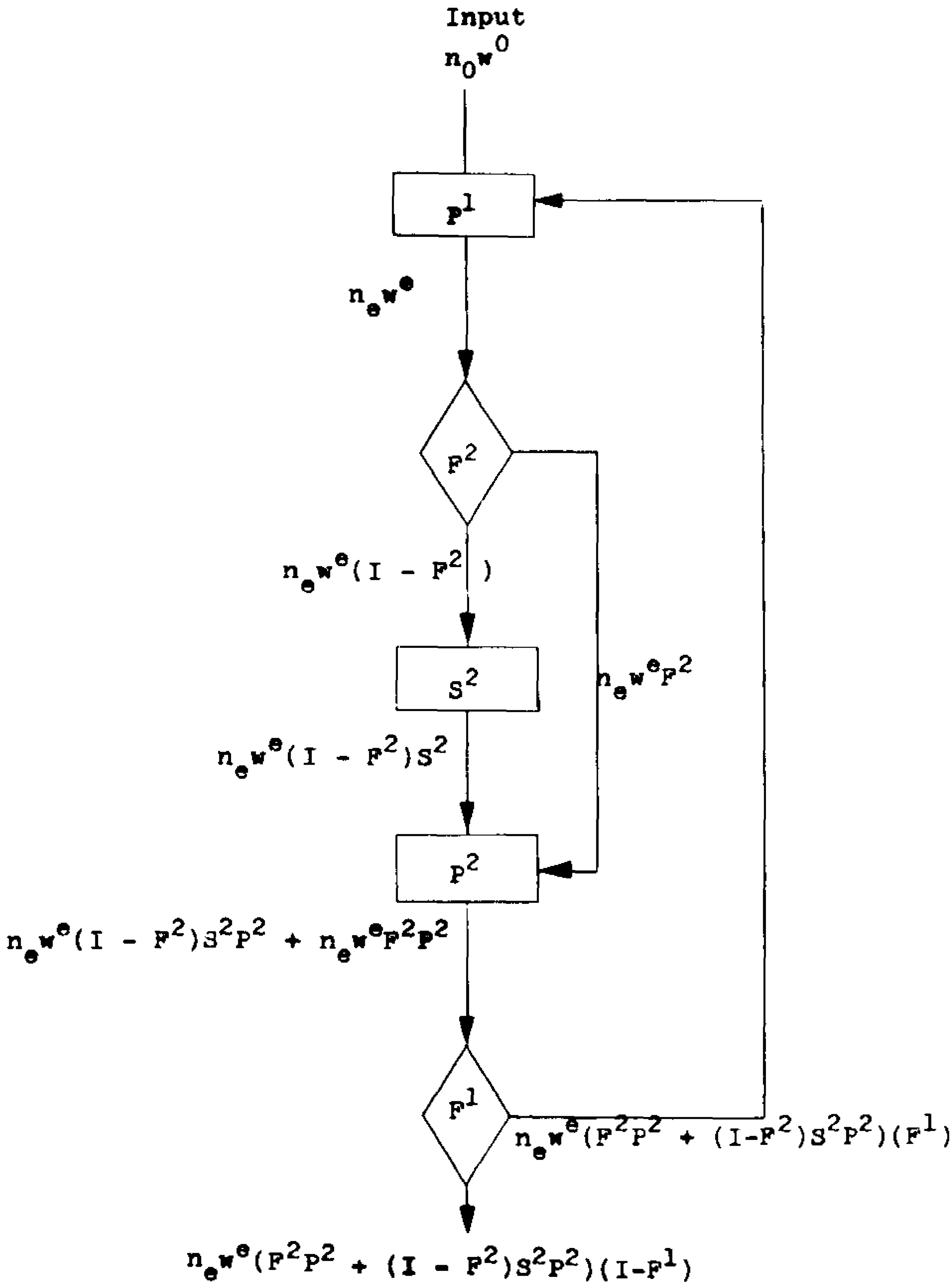


Figure 23. A feedforward loop nested within a feedback loop.

Application of the conservation rule to the entire system yields:

$$n_0 w^0 c = n_e w^e (F^2P^2 + (I - F^2)S^2P^2)(I - F^1)c$$

and on substitution for $n_e w^e$:

$$n_0 w^0 c = n_0 w^0 P^1 (I-(F^2P^2F^1P^1+(I-F^2)S^2P^2F^1P^1)^{-1} \cdot$$
$$(F^2P^2 + (I - F^2)S^2P^2)(I - F^1)c$$

where c is a column vector with all elements equal to 1.

C. INTERSECTING LOOPS

Figure 24 illustrates a system in which a feedback loop is locked into a feedforward loop. Here it is necessary to apply the feedback rule to node P^1 since it is the recipient of a feedback input. Thereafter, it is possible to evaluate $n_e w^e$ from the following relationship:

$$n_e w^e = n_0 w^0 (I-F^1)P^1 + n_0 w^0 F^1P^2F^2P^1 + n_e w^e P^2F^2P^1.$$

Then, on rearrangement:

$$n_e w^e = n_0 w^0 ((I-F^1)P^1 + F^1P^2F^2P^1)(I-P^2F^2P^1)^{-1}.$$

Again, to analyze the transient states use the matrix profile, $P^2F^2P^1$. Application of the conservation law to this model yields:

$$n_0 w^0 c = n_0 w^0 (F^1P^2(I - F^2) + (I-F^1)P^1 + F^1P^2F^2P^1)(I-P^2F^2P^1)^{-1}c,$$

where c is a column vector with all elements equal to 1.

Figure 25 differs from the previous figure in that the model it depicts has a feedforward loop locked into a feedback loop. Here, again, it is necessary to apply the feedback rule to node P^1 because it has a feedback input. It is then possible to evaluate the equilibrium conditions from the following equation:

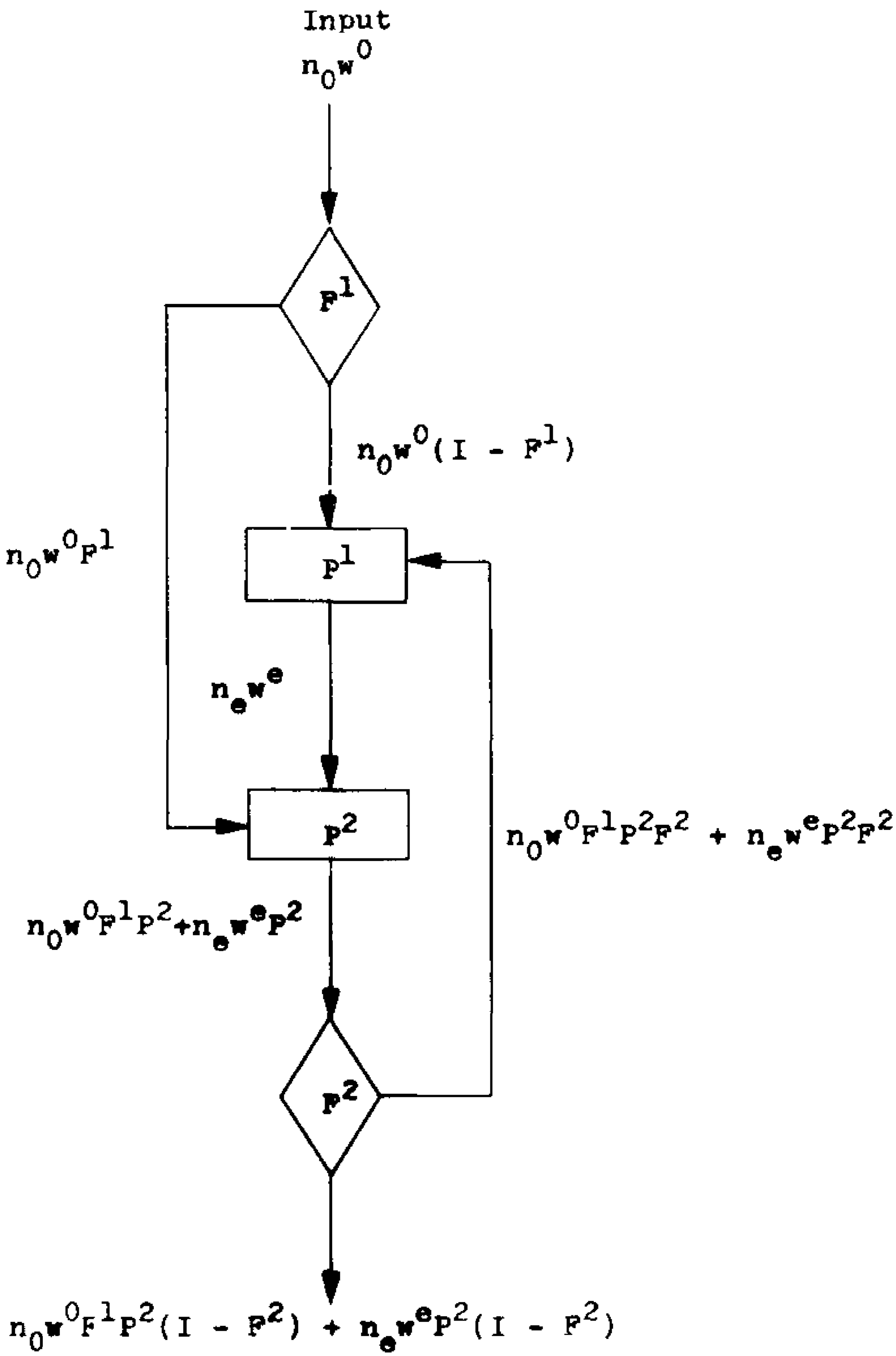


Figure 24. A feedforward loop interlocking with a feedback loop.

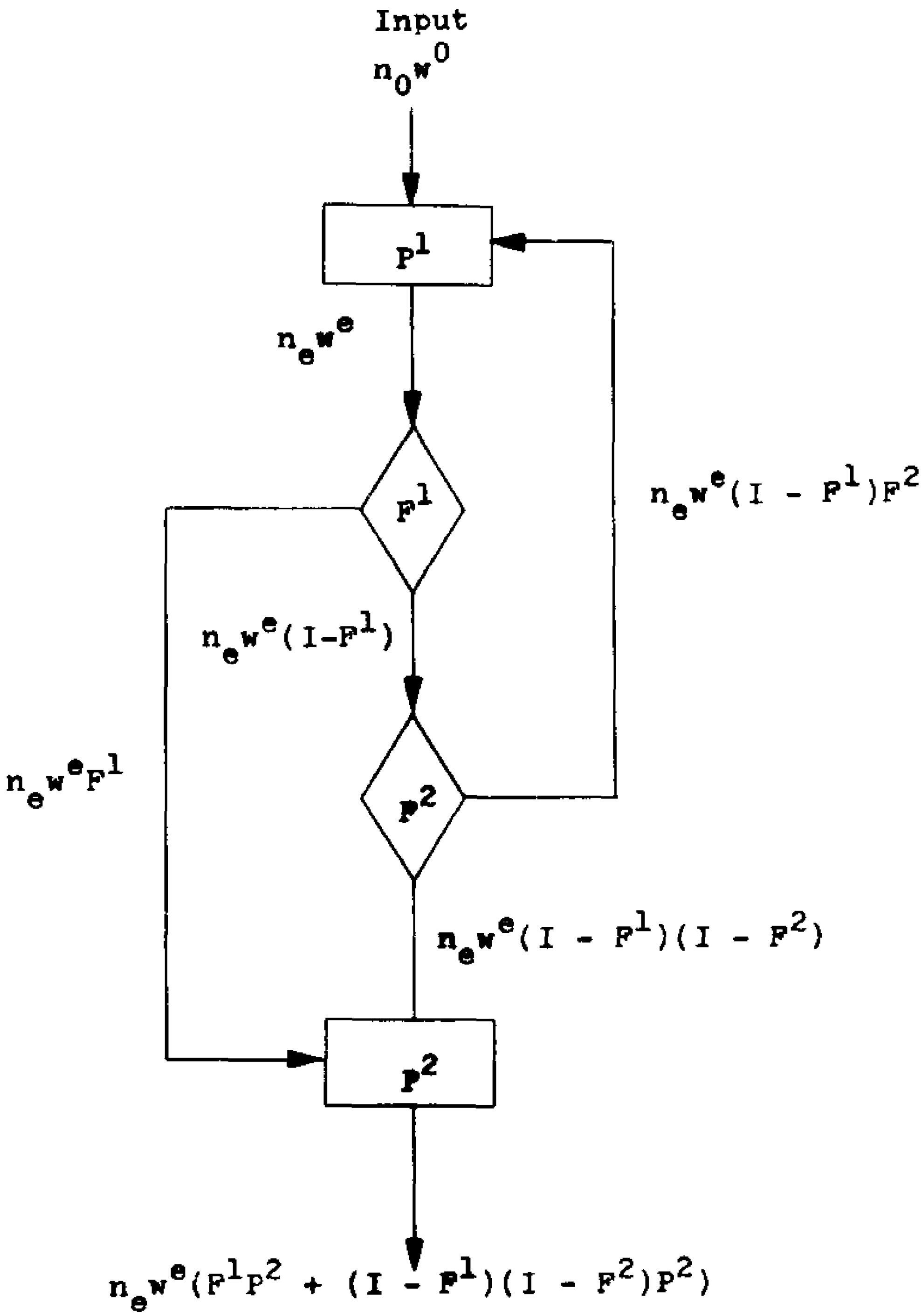


Figure 25 A feedback loop interlocking with a feedforward loop.

$$n_e w^e = n_0 w^0 P^1 + n_e w^e (I - F^1) F^2 P^1.$$

Then:

$$n_e w^e = n_0 w^0 P^1 (I - (I - F^1) F^2 P^1)^{-1}.$$

Note that the transient solution to the model can be analyzed using the profile matrix: $(I - F^1)F^2P^1$. Application of the conservation law to this model yields:

$$n_0 w^0 c = n_e w^e (F^1 P^2 + (I - F^1)(I - F^2) P^2) c$$

and on substitution:

$$n_0 w^0 c = n_0 w^0 P^1 (I-(I-F^1)F^2P^1)^{-1}(F^1P^2+(I-F^1)(I-F^2)P^2)c,$$

where c is a column vector with all elements equal to 1.

This completes an exhaustive analysis of all systems containing two feedback loops, two feedforward loops, or a combination of a feedforward and feedback loop. The objective of these examples is to demonstrate the ease with which the basic model can be applied to more complex models and the value of the profile matrix in extending the results of the basic model to more complex models.

CHAPTER VI

FURTHER EXTENSIONS OF THE MODEL AND CONCLUSIONS

A. EXTENSIONS OF THE MODEL

Possible extensions of the model described in this dissertation result from relaxing the conditions assumed in Chapter I, section A. In some cases, the extension is trivial; in others, new, more involved models are required.

There are several ways in which the first condition related to uniformity of flow can be relaxed. Consider first systems in which the rate of flow of items is dependent upon their quality level. Here, the present model would be valid in the equilibrium state but the transient solutions would differ. The result occurs because the quality associated with the items that travel slowly would take a long time to reach their equilibrium level, but once they reached that level, new items of the same quality would replace the old ones as soon as they flowed to the next operation. The equilibrium error-level distribution would not be changed because each item is counted the same number of times whether it travels rapidly or slowly. This type of model describes situations where some items are more difficult to work or more likely to contain errors. Algebraically, the solution is obtained through the application of the superposition principle.

A trivial extension of the present model would be to adjust the n-coefficient that describes the number of items in a channel to reflect the separation of, say, carbon copies or the merging of duplicate items. For example, one typographical error on the master of a set of carbon copies is multiplied by the number of copies and compounded when the copies are separated and travel separate paths. So long as the copies travel together, it is reasonable to consider the error as one-fold and the present model is adequate.

The second condition requires that the error level be a discrete variable. This can be relaxed by assuming that the degree of error can vary continuously. There are, of course, many cases in which the deviation from the standard is equivalent to error. In these cases, the measure of error could very well be a continuous quantity. It is, of course, possible to accommodate this condition within the framework of the present model by assuming that each of the discrete error levels refers to a continuous range of deviations.

An additional modification of the second condition would be a simplification of the error levels to a simple go-no go situation in which items would be classified as either "good" or "bad", as was done in Chapter II, section C.

If the second condition of Chapter I is relaxed so that the error-levels are allowed to be continuous, then the third condition must also be relaxed so that the error-level distribution is described by a probability density function rather than a probability distribution vector. Likewise, the fourth condition related to the existence of an error-level transition matrix must be relaxed for the continuous case so that the transition probabilities are described by a continuous Markov transition function. A mathematical development of the generalization of a discrete Markov model to one in which the state space is continuous is presented by Doob (1953, pp. 255-273).

Strictly speaking, the fifth condition guarantees a first-order Markov process in which the transition probability p_{ij} is dependent only on the state i. It is, however, possible to consider higher-order Markov processes in which the transition probabilities p_{ij} are dependent on the state i and a fixed number of states preceding state i. This modification of the present model would, of course, require a re-definition of the transition matrix to reflect the dependence on p_{ij} of the previous states. Again, Doob (1953, pp. 89-90) has carried out a mathematical development of the higher-order Markov processes.

An important extension of the model involves relaxing the sixth condition which requires that the transition probabilities p_{ij} be independent of time. This important condition requires that the machines and operators in the system not change their quality characteristics over time and hence the model does not describe situations in which quality learning or quality deterioration over time take place. Fortunately, mathematical treatments of Markov processes in continuous time are readily available in works by Bailey (1964, pp. 75-82) and Doob (1953, pp. 235-255).

The seventh, and final condition requires that the incoming error-level distribution, w^0, be known. It is of course possible to run a parametric analysis by studying the sensitivity of the models to various incoming distributions. One can also study the transient and equilibrium states of the system for a whole range of incoming distributions when the particular incoming error-level distribution is unknown.

B. COLLECTION OF REAL DATA

In the previous chapters, the models have been developed algebraically and examples given using hypothetical data. Though the absence of real data provides no difficulty in developing and demonstrating the model, it is clear that the application of the model is dependent upon real data. It is therefore worthwhile to investigate the data requirements of the model. These requirements involve the measurement of the error-level distribution in a stream of products, the transition probabilities associated with an operation, and the filter coefficients of a branching process.

Furthermore, each of these requirements is dependent upon a clear operational definition of error. In many cases, it may be possible to describe the system's quality in terms of only one type of error. On the other hand, it may be necessary to describe the system in terms of several classes of errors - each with its own state variables, transition matrices, and filter matrices.

Once error has been defined, the measurement of the error-level distribution at any point can be developed from an observation of the errors in a sample of the items from the stream of products passing a particular point. Similarly, an investigation of the streams of products leaving a branch operation should provide sufficient information to develop the appropriate filter matrix.

The problem of measuring transition probabilities for Markovian models has been discussed in the literature by Cain, Lee, and Judge (1969, pp. 374-397), Telser (1969, pp. 270-292), Theil and Guido (1966,pp. 714-721), and Lee, Judge, and Takayama (1965, pp. 742-762).

Often the particular model provides a clue to expected transition probabilities in various operations. For example, consider the number of burned-out light bulbs on a movie marquee. Here the use of the marquee can only introduce defective bulbs - the deterioration matrix which describes this operation must have all zero elements below its diagonal. Furthermore, the operation of replacing defective bulbs with good bulbs must be described by a repair matrix in which the first element of each row is 1 and all other elements are zero. The 1's, of course, indicate that the result of replacing all burned out bulbs is to make the probability of the system being in state zero equal to 1.

One model appropriate for many operations considers errors to be generated according to a Poisson distribution

$$\lambda = \frac{\sum_{i=1}^{n} \lambda_i r_i}{\sum_{i=1}^{n} r_i} \quad \text{and} \quad p = \frac{\sum_{i=1}^{n} p_i r_i}{\sum_{i=1}^{n} r_i} .$$

Again, the value of λ_i, p_i, and r_i should be obtainable from direct measurement of individual performance.

C. CONCLUSIONS

In addition to the uses demonstrated in this thesis, there appear to be several potential uses of expanded versions of the model. One extension would be the addition of an elapsed time parameter, i.e., the duration of time from the moment the item in the stream entered the system. This would permit the modeling of replacement problems in which items wait in queue to be repaired. Additionally, this parameter would be of value in selecting the optimal system configurations where timeliness is a factor.

A second extension of the model would be the addition of a cost parameter to accumulate the cost due to errors from the moment the item in the stream entered the system. This cost could reflect all expenses associated with the correction of errors, e.g., reworking, extra quality review, and delay in processing. Once either of these extensions were made it would be possible to use linear or dynamic programming to minimize the transit time through the system, the dollar cost of errors, or both by rearrangement of the system or addition of facilities at certain prescribed costs.

Let $m = j - k$, then:

$$P(s_1,s_2) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} \sum_{m=0}^{\infty} \binom{i}{i-k} p^{(i-k)}(1-p)^k e^{-\lambda}\frac{\lambda^m}{m!}(s_2)^m (s_1)^i (s_2)^k$$

Therefore:

$$= e^{-\lambda+\lambda s_2} \sum_{i=0}^{\infty} \sum_{k=0}^{i} \binom{i}{i-k} p^{(i-k)}(1-p)^k (s_1)^i (s_2)^k$$

$$= e^{-\lambda+\lambda s_2} \sum_{i=0}^{\infty} (p + (1-p)s_2)^i (s_1)^i$$

$$P(s_1,s_2) = \frac{e^{-\lambda+\lambda s_2}}{1 - ps_1 - (1-p)s_1 s_2} .$$

Finally, as a result of definition (6-1) the individual values of p_{ij} can be determined from the following formula:

$$p_{ij} = \frac{1}{i!j!} \left. \frac{\partial^{(i+j)} P(s_1,s_2)}{\partial^{(i)} s_1 \, \partial^{(j)} s_2} \right|_{s_1=0,\ s_2=0} \qquad \text{for all } i,\ j.$$

The advantage of using a model such as this is that if it is possible to obtain the mean rate of generating errors λ and the mean rate for correcting errors present on arrival at an operation p, it is then possible to generate all of the elements of the transition matrix describing the operation.

When an operation is carried out by one operator or machine for the entire system, the problem of computing the mean rates is simplified. However, in general, it may be necessary to weigh the means for a group of operators or machines carrying out a single operation in order to correct for their respective rates of production. Therefore, when there are n operators or machines with production rates $r_1, r_2, \ldots, r_n$ and individual mean rates (λ_1,p_1), (λ_2,p_2), $\ldots$, (λ_n,p_n), then:

and detected according to a binomial distribution. This model is appropriate where the generation of errors is random with a constant rate, say λ, and the correction of each error is independent of the other errors with the probability of correction of any one error present at the arrival of the item fixed at a constant value, say p. Then the probability of correcting enough errors to go from error level i to error level k is given by the binomial distribution

$$p_{ik} = \binom{i}{i-k} p^{(i-k)} (1-p)^k \quad , \text{ where } k \leq i.$$

Further, the probability of generating enough errors to go from error level k to error level j is given by the Poisson distribution:

$$p_{kj} = \frac{e^{-\lambda} \lambda^{(j-k)}}{(j-k)!} \quad , \text{ where } k \leq j.$$

Then, the overall probability of passing from state i to state j is given by p_{ij} according to the following rule:

$$p_{ij} = \sum_{k=0}^{i} p_{ik} p_{kj} \quad .$$

It is now possible to compute the bivariate moment generating function for p_{ij} using the following definition (Feller, 1950, p. 261):

$$P(s_1, s_2) = \sum_i \sum_j p_{ij} \, (s_1)^i \, (s_2)^j \qquad (6\text{-}1)$$

Then, on substitution:[5]

$$P(s_1, s_2) = \sum_{i=0}^{\infty} \sum_{j=k}^{\infty} \sum_{k=0}^{i} \binom{i}{i-k} p^{(i-k)} (1-p)^k e^{-\lambda} \frac{\lambda^{(j-k)}}{(j-k)!} (s_1)^i (s_2)^j$$

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[5]The author would like to express his appreciation to Dr. Ronald S. Dick for his assistance in solving this death-immigration generating function.

The lexicon of models presented in this thesis are intended to demonstrate the usefulness and generalizability of the basic model developed in Chapter II. Using these models, it appears that one should be able to represent the error-level distributions of a large class of systems in terms of their inputs, configurations, characteristic error-level transition matrices, and branching properties. In the final analysis, this effort has attempted to produce a structured analysis of designing systems for quality as well as economy and speed.

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