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THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY; THE PARAMETRIZED POST-NEWTONIAN FORMALISM

Thesis by

Clifford Martin Will

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

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THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY;

THE PARAMETRIZED POST-NEWTONIAN FORMALISM

by Clifford Martin Will

ABSTRACT

Increasing sophistication and precision of experimental tests of relativistic gravitation theories has led to the need for a detailed theoretical framework for analysing and interpreting these experiments. Such a framework is the Parametrized Post-Newtonian (PPN) formalism, which treats the post-Newtonian limit of arbitrary metric theories of gravity in terms of nine metric parameters, whose values vary from theory to theory. The theoretical and experimental foundations of the PPN formalism are laid out and discussed, and the detailed definitions and equations for the formalism are given. It is shown that some metric theories of gravity predict that a massive, self-gravitating body's passive gravitational mass should not be equal to its inertial mass, but should be an anisotropic tensor which depends on the body's self-gravitational energy (violation of the "principle of equivalence"). Two theorems are presented which probe the theoretical structure of the PPN formalism. They state that (i) a metric theory of gravity possesses post-Newtonian integral conservation laws if and only if its nine PPN parameters have values which satisfy a set of seven constraint equations, and (ii) a metric theory of gravity is invariant under asymptotic Lorentz transformations if

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and only if its PPN parameters satisfy a set of three constraint equations. Some theories of gravity (including Whitehead's theory and theories which violate one of the "Lorentz-invariance" parameter constraints) are shown to predict an anisotropy in the Newtonian gravitational constant. Gravimeter data on the tides of the solid Earth are used to put an upper limit on the magnitude of the predicted anisotropy, and thence to rule out such theories.

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TABLE OF CONTENTS

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A. INTRODUCTION

1. Brief Summary of the Thesis

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Recent progress in space and laboratory technology has made high-precision testing of relativistic theories of gravitation an active and exciting branch of experimental physics. Radar ranging to spacecraft and planets, laser ranging to the Moon, long-baseline interferometry, application of low temperature techniques to spacecraft, Mössbauer techniques, high-precision optical astrometry, accurate gravimeter devices --- all these technological advances have made relativists hopeful and confident that experiment will soon tell us which theory of gravity is the correct theory.

As these experimental techniques become more accurate and more sophisticated and probe deeper into the effects of relativistic gravity in the solar system, we theorists have two important duties to perform: (i) to examine and evaluate the theoretical significance of each proposed experimental test and (ii) to suggest new possible experimental tests of gravitation theories.

This has motivated us to devise a theoretical framework which is ideally suited to the performance of these two theoretical duties. This framework, called the Parametrized Post-Newtonian (PPN) Formalism, is not new; it dates back to Eddington (1922), Robertson (1962), and Schiff (1967), and has recently been generalized by Nordtvedt (1968). Our version of the PPN formalism is a further generalization (and, we feel, an improvement) of the Nordtvedt version. It treats the post-Newtonian limit of arbitrary metric theories of gravity in terms of a series of nine metric parameters, whose values vary from theory to theory.

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Such a formalism allows one to study the solar-system effects of relativistic gravity in an elegant and useful form: each effect --- perihelion shift, radar time delay, massive-body equivalence-principle violations, etc. --- finds a simple expression in terms of linear combinations of PPN parameters. This gives a convenient method for interpreting the results of experiment: any experimental measurement of a relativistic effect in the solar system is viewed as a measurement of the values of the corresponding PPN parameters. In order to decide which theory of gravity is "correct", we compare these measured parameter values with the values predicted by various theories.

The PPN formalism is also useful as a purely theoretical tool. The theorist can use the PPN parameters as "tracers" to determine exactly how the various "pieces" of the metric of spacetime contribute to each observable effect. The formalism also permits the theorist to catch a glimpse of the structure of arbitrary metric theories (at the post-Newtonian level, at least) by analysing theoretical concepts such as invariances and conservation laws within the PPN framework. Such theoretical analyses will contribute to our understanding of relativistic gravity, no matter which theory turns out to be the "correct" theory of gravity.

The remainder of this thesis is a detailed exposition of the PPN formalism. In Part A, following this Brief Summary of the Thesis, we analyse and discuss the theoretical and experimental foundations on

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which we have built the PPN formalism. A study of elementary particle experiments and of the gravitational redshift experiment leads us to conclude that (i) there exists a metric which governs the ticking rates of atomic clocks and the measurements made by physical rods; and (ii) freely falling test bodies move along geodesics of the metric, and stressed matter responds to the metric according to the standard curved-spacetime equation of motion "divergence of the stress-energy tensor vanishes". These two postulates, plus a few elementary physical considerations, are enough to create the PPN formalism.

The detailed structure of the PPN formalism is explored in Part B. Section 3 is concerned with setting up the formalism and giving key definitions and formulas to be used in any PPN analysis. We then apply the formalism to a particular problem in relativistic gravity: the breakdown in the principle of equivalence for massive self-gravitating bodies ("Nordtvedt effect"). We show that some theories of gravity (not including general relativity) predict that a massive self-gravitating body's passive gravitational mass should not be equal to its inertial mass, but should differ from it by terms which depend on the body's self-gravitational energy and the PPN parameters (and which may be anisotropic). In Section 4 we show that an arbitrary set of values for the nine PPN parameters does not necessarily correspond to a theoretically "well-behaved" theory. In particular, we prove that a metric theory of gravity possesses post-

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Newtonian integral conservation laws for momentum, angular momentum, and center-of-mass motion if and only if its PPN parameter values satisfy a set of seven constraint equations. Such a "conservative" theory has only two freely specifiable PPN parameters. We also show that the metric of a given theory is invariant under a "Post-Galilean transformation" (a transformation which reduces asymptotically to a Lorentz transformation far from the matter) if and only if its PPN parameter values satisfy a set of three constraint equations. A theory whose parameters did not satisfy these three constraints would be an "ether theory" of gravity, i.e. it would be a theory which would demand that all calculations involving gravity be done in a particular uniquely defined reference frame (the rest frame of some cosmological ether, for example). Asymptotic Lorentz transformations to other reference frames would give different physical results for any calculation involving gravity.

Such an "ether" theory would have observable consequences. One of them --- an anisotropy in the locally-measured Newtonian gravitational constant --- is examined in Part C. The predicted anisotropy in G would cause variations (as the Earth rotates) in the acceleration of gravity as measured by a gravimeter at rest on the Earth ("Earth-tides"). By examining Earth-tide gravimeter data, we put a limit of one part in 10^9 on a possible G-anisotropy, and we show that this represents a three per cent confirmation of one of the three "Lorentz-invariance" parameter constraints derived in Section 4. In a sep-

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arate computation we show that Whitehead's (1922) theory cannot be the correct theory of gravity since it predicts a G-anisotropy (of a different type than the PPN anisotropy) and hence an Earth tide, 200 times larger than the experimental limit of $1/10^9$.

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REFERENCES

Eddington, A. S. 1922, The Mathematical Theory of Relativity (London: Cambridge University Press).

Nordtvedt, K., Jr. 1968, Phys. Rev., 169, 1017.

Robertson, H. P. 1962, in Space Age Astronomy, ed. A. J. Deutsch and W. B. Klemperer (New York: Academic Press).

Schiff, L. I. 1967, in Relativity Theory and Astrophysics. I. Relativity and Cosmology, ed. J. Ehlers (Providence, R. I.: American Mathematical Society).

Whitehead, A. N. 1922, The Principle of Relativity (Cambridge: Cambridge University Press).

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2. Theoretical Frameworks for Testing Relativistic Gravity

I. Foundations

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THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY. I. FOUNDATIONS*

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ABSTRACT

This is the first in a series of theoretical papers which will discuss the experimental foundations of general relativity. This paper reviews, modifies, and compares two very different theoretical frameworks, within which one devises and analyzes tests of gravity. The *Dicke framework* assumes almost nothing about the nature of gravity; and it uses a variety of experiments to delineate the gross features of the gravitational interaction. Two of its tentative conclusions (the presence of a metric, and the "gravitational response equation," $\nabla \cdot T = 0$, for stressed matter) become the postulates of the *Parametrized Post-Newtonian framework*. The PPN framework encompasses most, if not all, of the theories of gravity that are currently compatible with experiment. Future papers in this series will develop the PPN framework in detail, and will use it to analyze a variety of relativistic gravitational effects that should be detectable in the solar system during the coming decade.

I. INTRODUCTION AND SUMMARY

Since 1963 a number of astronomical discoveries and observations have forced astrophysicists to make general relativity a working tool in their theoretical model building: The cosmic microwave radiation, QSOs, pulsars, gravitational waves—models for all these are constrained by or involve relativistic gravity in a fundamental way.[1]

We theorists, who wish to build models for these phenomena, are hamstrung: Experiment has not yet told us which relativistic theory of gravity is correct—general relativity, the Brans-Dicke theory, one of Bergmann's (1968) multitudinous scalar-tensor theories, a theory which nobody has yet constructed, ... The answer is of the utmost importance to astronomy today!

It would be naïve to expect that the very astronomical phenomena in which relativistic gravity is crucial will provide the answer. In cosmology, in QSOs and pulsars, and in the sources of gravity waves, gravitational effects are inextricably interwoven with the local behavior of matter and magnetic fields. There is little hope of separating them sufficiently to get *clean* tests of the nature of gravity. The astrophysical enterprise must be largely one of using the laws of gravity as an input, and trying to get out information about what the matter and fields are doing " 'way out there."

The greatest hopes for clean tests of relativistic gravity lie in today's rapidly advancing space and laboratory technology. Atomic clocks, very-long-baseline interferometry,

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This is the first in a series of papers on "Theoretical Frameworks for Testing Relativistic Gravity." With the exception of the present one, the papers in this series will tend to be rather theoretical and mathematical. A companion series on "Relativistic Gravity in the Solar System" will be more observationally oriented. It will concentrate on the nature of various relativistic effects and on prospects for their detection in the next few decades.

With great pleasure my students and I dedicate this series of papers to my close friend and colleague, Professor S. Chandrasekhar. This dedication is in honor of Professor Chandrasekhar's beautiful and systematic development of post-Newtonian hydrodynamics, which is an indispensable foundation for the "PPN framework" developed and used in this series of papers.

† Gulf Oil Graduate Fellow.

[1] For a very readable treatise on relativistic astrophysics see Zel'dovich and Novikov (1967, 1971, 1972).

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interplanetary radar, laser ranging, spacecraft-transponder ranging, and superconducting gyroscopes should make possible, in the next decade, a number of independent solar-system tests of non-Newtonian gravitational effects, with precisions as great as 3×10^{-4} (see Thorne and Will 1970 for a review). In a very real sense, the 1970's will be "*the decade for testing general relativity.*"

Although there are many new experimental possibilities, the cost of carrying each one out in terms of manpower and money is very high. (We can expect the megabuck to be a useful unit of measure for some of the tests.) For this reason, it is crucial that we have as good a theoretical framework as possible for comparing the relative values of the various experiments—and for proposing new ones which might have been overlooked.

The most simple-minded theoretical framework would be a direct comparison of general relativity with Newtonian theory. Indeed, it was just such a comparison that motivated Einstein's original three tests: the gravitational redshift, the deflection of light, and the perihelion shift of Mercury. One might think that we should merely continue to measure these and other non-Newtonian, general-relativistic effects to higher and higher accuracy; and only if a discrepancy between experiment and theory is found should we begin to consider other theories.

This would be a reasonable approach if we had enormous confidence in general relativity; but we do not—at least, some of us don't. So we would prefer to design the experiments to be as unbiased as possible; we would like to see them force us, with very few *a priori* assumptions about the nature of gravity, toward general relativity or some other theory. And, of course, this can happen only if we first open our minds to a wide variety of theoretical possibilities.

A leading exponent of this viewpoint is Robert H. Dicke.[2] It has led him and others to perform several high-precision null experiments (Dicke-Eötvös experiment; Hughes-Drever experiment; ether-drift experiments) which greatly strengthen our faith in the foundations of general relativity (see Dicke 1964; also § II below). Without this viewpoint, some of the null experiments might not have been performed, and we would certainly not understand their significance so well.

Dicke himself has suggested one type of theoretical framework for comparing various theories of gravity and analyzing the significance of various experiments. His framework (see § II below) is particularly powerful for discussing the null experiments, for delineating the qualitative nature of gravity, and for devising new covariant theories of gravity. However, in our opinion it is not so well suited to the analysis of the high-precision solar-system tests which may dominate the coming decade.

A second theoretical framework, one better suited to the solar-system tests, is the Parametrized Post-Newtonian (PPN) formalism of Eddington (1922), Robertson (1962), and Schiff (1967) (§ III below). Although it has been very useful in the past, the PPN formalism is too narrow and unsophisticated in its original form to serve the needs of the 1970's. Improved versions due to Baierlein (1967) and to Nordtvedt (1968) look much more promising, though they are still not broad enough.

The purpose of this paper is to review and compare these theoretical frameworks, which have been used in the past, and to make several modifications—or, we would prefer to say, *improvements*—in them. This paper contains little new material. Its chief *raison d'être* is to lay out a particular way of thinking about old material—a way that will become a guide for future papers in this series.

In § II we will discuss and modify the Dicke framework. In § III and in Paper II of this series we will deal with the PPN framework.

Fundamental to our viewpoint is the following assessment of the relationship between our versions of the Dicke and PPN frameworks:

The Dicke framework assumes almost nothing about the nature of gravity. It helps

[2] See also pp. 100–101 of Schild (1962) for a very convincing discussion of it.

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one to design and discuss experiments which test, at a very fundamental level, the nature of spacetime and gravity. Within it one asks such questions as: Do all bodies respond to gravity with the same acceleration? Is space locally isotropic in its intrinsic properties? What types of fields, if any, are associated with gravity—scalar fields, vector fields, tensor fields, affine fields, . . . ?

The PPN framework starts where the Dicke framework leaves off: By analyzing a number of experiments within the Dicke framework one arrives at (among others) two "fair-confidence" conclusions about the nature of gravity. These are (i) that gravity is associated, at least in part, with a symmetric tensor field, the "metric"; and (ii) that the response of matter and fields to gravity is described by "$\nabla \cdot T = 0$," where $\nabla\cdot$ is the divergence with respect to the metric, and T is the stress-energy tensor for all matter and nongravitational fields. These two conclusions in the Dicke framework become the postulates upon which the PPN framework is built.

We call theories of gravity that satisfy these two postulates "*metric theories*." The PPN framework takes the slow-motion, post-Newtonian limit of all conceivable metric theories and characterizes that limit by a set of nine real-valued parameters (see Will 1971 [Paper II] for details). Each metric theory of gravity is characterized by a set of particular values for these PPN parameters. The task of solar-system gravity experiments in the coming decade can be regarded as one of measuring the values of these PPN parameters and thereby delineating, hopefully, which theory of gravity is correct.

It is important for the future that experimenters concentrate not only on measuring the PPN parameters. They should also perform new experiments within the Dicke framework to strengthen—or destroy—the foundation which it lays for the PPN framework.

II. THE DICKE FRAMEWORK

a) *Statement of the Framework*

The Dicke framework for analyzing experimental tests of gravity was expounded in Appendix 4 of Dicke's (1964) Les Houches lectures. Here we shall present a slightly generalized version of Dicke's framework, and we shall couch it in slightly different language.

Dicke begins with two statements about the type of mathematical formalism to be used in discussing gravity. These statements have little physical content;[3] they serve primarily to delineate the vantage point from which gravity will be viewed. They say:[4]

Statement (i).—Spacetime will be regarded as a four-dimensional manifold, with each point of the manifold corresponding to a physical event. The manifold need not *a priori* have either a metric or an affine connection.

Statement (ii).—The theory of gravity will be expressed in a form that is independent of the particular coordinates used; i.e., the equations of gravity and the mathematical entities in them will be put into covariant form.

Notice that even if there is some physically preferred coordinate system in spacetime, the theory can still be put into covariant form. For example, one can introduce four scalar fields, whose numerical values are equal to the values of the preferred coordinates:

$$\alpha(q) = x(q)\,, \quad \beta(q) = y(q)\,, \quad \gamma(q) = z(q)\,, \quad \delta(q) = t(q)\,, \qquad (1)$$
$$q \text{ a point in spacetime}, \quad (x, y, z, t) \text{ preferred coordinates;}$$

and one can then regard these fields as associated with gravity.

The Newtonian theory of gravity is an example of a theory that is not normally

[3] See, however, Trautman's (1965, p. 101) remarks about the physical significance of assuming spacetime to be a differentiable manifold.

[4] These statements are equivalent to items 1, 2, and 3 on p. 50 of Dicke (1964).

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expressed in covariant language; the Newtonian equations, $\nabla^2 U = -4\pi G\rho$, $F = m\nabla U$, are valid only in a particular class of coordinate frames. However, as Cartan has shown (see Trautman 1965 for a review), Newtonian theory can be expressed in an alternative covariant form involving a nonmetrical affine connection.

Having laid down his mathematical viewpoint [statements (i) and (ii) above], Dicke then imposes two constraints, which he requires of all acceptable theories of gravity. They are:[5]

Constraint 1.—Gravity must be associated with one or more fields of tensorial character (i.e., scalars, vectors, and tensors of various ranks).

Constraint 2.—The dynamical equations which govern gravity must be derivable from an invariant action principle.

These constraints have deep significance; they strongly confine the theory. For this reason, we should be willing to accept them only if they are fundamental to our subsequent arguments. For most applications of the Dicke framework they are not needed at all. Therefore, *we shall usually not assume them.* If we ever need and use them, we shall state so explicitly.

There is one final item in the Dicke framework—an item of great significance:

Guiding principle.—Ockham's (1495) razor: Nature likes things as simple as possible.[6]

This guiding principle is used, of course, to tell us what kinds of theories of gravity are the most likely to be correct—and, therefore, what kinds of experiments are the most important ones to perform.

Notice that by telling us to apply Ockham's razor within a covariant mathematical framework, Dicke builds a very particular bias into his formalism. Only those theories which look simple when expressed in covariant form are deemed promising. By this criterion, general relativity is very promising—perhaps the most promising theory of all! However, Newtonian theory is not. In its covariant form (Trautman 1965; Misner 1969*b*), in contrast to its conventional form, Newtonian theory is exceedingly complicated. A physicist working in the Dicke framework would never be so pathological as to dream up a theory like that of Newton!

Keeping this bias in mind, but not referring to it again, we shall proceed to discuss experiments within the Dicke framework. Note that our present discussion does not attempt to be rigorous. As we have said, this paper is intended only to develop a particular viewpoint, which will function as a guide for later, more rigorous papers.

b) The Fields Associated with Gravity

The Dicke framework is particularly useful for designing and interpreting experiments which ask what types of fields are associated with gravity. When Dicke himself uses it for this purpose, he imposes constraint 1 (above)—i.e., he considers only scalar, vector, and tensor fields. We think this is a dangerous policy. Since Newtonian theory, in its covariant form, attributes gravity to a non-Riemannian affine connection, we should at least admit non-Riemannian affinities as well as scalars, vectors, and tensors. To be on the safe side, we shall go all the way and admit any field that takes on a covariant form; i.e., we shall abandon constraint 1.

i) *Second-Rank Tensor Field (Metric)*

First let us consider tensor fields of rank $\binom{0}{2}$. There is very strong experimental evidence that at least one such field exists in the Universe: a symmetric "metric" field

[5] These constraints correspond to items 5 and 4 on p. 50 of Dicke (1964).

[6] This corresponds to item 6 on p. 50 of Dicke (1964). Actually, this "principle of economy" (*pluralitas non est ponenda sine necessitate*) did not originate with Ockham (ca. 1300–1349), but can be traced back to Aristotle. Ockham's use of it was new because of his empiricism.

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g, whose orthonormal tetrads are related by Lorentz transformations, and which determines the ticking rates of atomic and nuclear clocks and the lengths of laboratory rods.

The evidence for a metric field comes largely from elementary-particle physics. It is of two types: *first*, experiments which measure space and time intervals directly, e.g., measurements of the time dilation of the decay rates of unstable particles;[7] *second*, experiments which reveal the fundamental role played by the Lorentz group in particle physics,[8] including everyday, high-precision verifications of four-momentum conservation and of the relativistic laws of kinematics. To cast out the metric tensor entirely would destroy the theoretical backing of such experiments.

Let us notice what particle-physics experiments *do* and *do not* tell us about the metric tensor, g: *First*, they *do not* guarantee that there exist global Lorentz frames—i.e., coordinate systems extending throughout all of spacetime, in which[9]

$$g_{ij} = \text{Minkowski metric } \eta_{ij} \equiv \text{diag}\,(1, -1, -1, -1)\,. \tag{2a}$$

However, they *do* demand that at each event q there exist local frames, related by Lorentz transformations, in which $g_{ij}(q) = \eta_{ij}$. Moreover, given such a frame, elementary differential geometry guarantees that we can construct coordinates in which

$$g_{ij} = \eta_{ij} + 0(\Sigma_k |x^k - x^k(q)|^2)\,; \quad \partial g_{ij}/\partial x^k = 0 \quad \text{at} \quad q\,. \tag{2b}$$

Such a coordinate system we shall call a "*local Lorentz frame at q.*"

Second, particle experiments *do not* guarantee that freely falling particles move along geodesics of the metric field, i.e., along straight lines in the local Lorentz frames. In particular, we do not know from elementary-particle experiments whether the local Lorentz frames in an Earth-bound laboratory are freely falling (so they fly up from the center of the Earth and then fall back with Newtonian acceleration $g = 980$ cm sec^{-2}); whether they are forever at rest relative to the laboratory walls; or whether they undergo some other type of motion. The strong equivalence principle (Einstein elevator argument) predicts that the local Lorentz frames should fall freely, so that a free particle initially at rest in one frame would always remain at rest in it. Contrast this with flat-spacetime theories of gravity, in which rods and atomic clocks are governed by the global Minkowski metric (2a), and gravity, like electromagnetism, is described by a field (scalar, vector, tensor, or combination) which resides in flat spacetime. In such theories a Lorentz frame initially at rest in an Earth-bound laboratory would remain always at rest (except for accelerations $\ll 980$ cm sec^{-2} due to the Earth's rotation and orbital motion). These possibilities and others are permitted by all elementary-particle experiments to date (except the Mössbauer redshift experiments discussed in § II*c* below).

Third, elementary-particle experiments *do* tell us that the times measured by atomic clocks depend only on velocity, not upon acceleration. The measured squared interval is $ds^2 = g_{ab}dx^a dx^b$, independently of acceleration. Equivalently but more physically, the time interval measured by a clock moving with velocity v^α relative to a local Lorentz frame is

$$ds = (\eta_{ab}dx^a dx^b)^{1/2} = [1 - (v^x)^2 - (v^y)^2 - (v^z)^2]^{1/2} dt\,, \tag{3}$$

[7] For a 2 percent test of time dilation with muons of $(1 - v^2)^{-1/2} \sim 12$ in a storage ring, see Farley *et al.* (1966). For earlier time-dilation experiments see Frisch and Smith (1963); Durbin, Loar, and Havens (1952); Rössi and Hall (1941); Ives and Stilwell (1938, 1941). For an experiment which verifies, to one part in 10^4, that the speed of light (γ-rays) is independent of the velocity of its source (decaying π^0) for source velocities $v > 0.99975c$, see Alväger *et al.* (1964).

[8] See Lichtenberg (1965) for a discussion of Lorentz invariance, spin and statistics, the *TCP* theorem, and relevant experiments.

[9] Here and throughout most of this paper we use units in which the speed of light is unity.

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independently of the clock's acceleration d^2x^α/dt^2. If this were not so, then particles moving in circular orbits in strong magnetic fields would exhibit different decay rates from those of freely moving particles, which they do not (Farley *et al.* 1966);[10] and ^{57}Fe nuclei would show acceleration dependence in the frequency of their Mössbauer transitions, which they do not (Sherwin 1960).

We shall henceforth assume the existence of the symmetric metric tensor; and we shall use it to raise and lower indices on all vectors and tensors.

We shall discuss the relationship between the metric and gravity in § II*c*, below.

ii) *More than One Second-Rank Tensor Field*

The Hughes-Drever experiments rule out, with very high precision, the existence of more than one second-rank tensor field (see pp. 14–22 of Dicke 1964 for discussion).

iii) *Vector Field*

Various ether-drift experiments make it unlikely that a vector field is present (see pp. 22–25 of Dicke 1964; also Turner and Hill 1964; Champeney, Isaak, and Khan 1963).

iv) *Scalar Field*

No experiment performed thus far has been able to rule out or to reveal the presence of a scalar field. However, future studies of the polarization properties of cosmic gravitational waves might reveal the scalar field, if it is present. The deformations produced in a disk placed perpendicular to the incoming waves are area-preserving (quadrupolar) if the waves are purely tensor in nature; but they can be area-changing (monopolar) if the waves have a scalar component. Other ways of experimentally delineating a scalar field are discussed by Dicke (1964).

v) *Scalar, Vector, and Tensor Densities*

When Dicke (1964) writes down his constraint 1 (cf. § II*a* above), he explicitly states that he will not consider theories in which boson fields, such as gravity, transform as tensor densities; he admits only tensorial transformation laws. However, once we have concluded that a metric field is present, such a constraint becomes superfluous. Any scalar, vector, or tensor density can be expressed in terms of the determinant of the metric and a corresponding pure scalar, vector, or tensor. Hence, without loss of generality we can ignore the densities.

vi) *Affine-Connection Field*

The metric endows spacetime with one affine-connection field—the "Riemannian affinity"

$$\left\{ {a \atop bc} \right\} = \tfrac{1}{2} g^{ae}(g_{eb,c} + g_{ec,b} - g_{bc,e}) \,. \tag{4}$$

However, there might be some other affine field $\Gamma^a{}_{bc}$ present. If so, the difference between it and the Riemannian affinity is guaranteed to be a third-rank tensor:[11]

[10] The experiment of Farley *et al.* is a 2 percent check of acceleration independence of the rate of muon decay for energies $E/m = (1 - v^2)^{-1/2} \sim 12$ and for accelerations, as measured in the muon rest frame, of $a = 5 \times 10^{20}$ cm sec^{-2} = 0.6 cm^{-1}. Note that, at accelerations a factor 10^{13} larger than this ($a \sim 10^{33}$ cm sec^{-2} $\sim 10^{12}$ cm^{-1}), in 1 light travel time across the muon it accelerates up to near the speed of light, if it was initially at rest. Such large accelerations will probably affect the decay rates—not because of any breakdown in relativity theory, but because the decay cannot be analyzed within a single comoving local Lorentz frame. The muon ceases to be a valid special relativistic clock. See Ageno and Amaldi (1966) and Bailey and Picasso (1970).

[11] Here and elsewhere in this paper we use well-known results from differential geometry without proof or comment. The reader can consult such texts as Hicks (1965) or Trautman (1965) for the necessary mathematical background.

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$$S^a{}_{bc} \equiv \Gamma^a{}_{bc} - \left\{ {a \atop bc} \right\} = \text{tensor}. \tag{5}$$

Thus, searching experimentally for another affinity $\Gamma^a{}_{bc}$ is equivalent to searching for a third-rank tensor $S^a{}_{bc}$. Since affinities, from a certain viewpoint, are very simple and fundamental entities, we should, in applying Ockham's razor, give higher priority to third-rank tensors than we might at first wish to.

The most attractive way to incorporate a non-Riemannian affinity $\Gamma^a{}_{bc}$ into the laws of gravity would be through the trajectories of freely falling bodies: those trajectories might be geodesics of $\Gamma^a{}_{bc}$. However, as we shall see in the next section, this possibility is made unlikely by gravitational-redshift experiments, which suggest that the free-fall trajectories are probably geodesics of the Riemannian affinity $\left\{ {a \atop bc} \right\}$ rather than of some other affinity $\Gamma^a{}_{bc}$. Unfortunately, those experiments do not have very high precision.

Free-fall trajectories can be geodesics simultaneously of $\Gamma^a{}_{bc}$ and of $\left\{ {a \atop bc} \right\}$ without the two affinities being identical. But then one must find ways other than free-fall motion to incorporate $\Gamma^a{}_{bc}$ into the laws of gravity. The most obvious other ways are explored in the Appendix and are shown to be fruitless. Thus, it seems to us that, to within the accuracy of the redshift experiments, a non-Riemannian affinity is probably absent from the laws of gravity.

c) *Test-Body Trajectories and the Gravitational Redshift*

According to the Dicke-Eötvös experiment (see, e.g., Dicke 1964), the trajectory of a freely falling, neutral, laboratory-sized object ("test body") is independent of its structure and composition—at least to a high degree of accuracy. We shall assume complete independence (Dicke's "weak equivalence principle").

This means that spacetime is filled with a family of preferred curves, the *test-body trajectories* (called "free-fall" trajectories in the preceding section). Any initial event in spacetime and initial velocity through that event determine a test-body trajectory which is unique except for parametrization. If we knew all the test-body trajectories, we would know a great deal—perhaps everything—about gravity.

There is a second family of preferred curves filling all of spacetime: the *geodesics* of the metric g. It is tempting to identify these geodesics with the test-body trajectories (Einstein's "strong equivalence principle"). However, we should not do so without rather convincing experimental proof.

In order to see what kinds of experiments are relevant, let us elucidate the physical significance of the geodesics.

A geodesic of g is most readily identified locally by the fact that it is a straight line in the local Lorentz frames. Put differently, a body's motion is unaccelerated as measured in a local Lorentz frame if and only if the body moves along a geodesic of g. Hence, to determine whether test-body trajectories are geodesics, we must compare experimentally the motion of a local Lorentz frame with the motion of a test body.

It is easy to study experimentally the motions of test bodies; relative to an Earth-bound laboratory they accelerate downward with $g = 980$ cm sec^{-2}; and this acceleration can be measured at a given location on the Earth to a precision of one part in 10^6 (Cook 1965).

Unfortunately, it is much more difficult to measure the motion of a local Lorentz

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frame. It seems to the authors that the only experimental handle we have on this today is gravitational-redshift experiments.[12]

The redshift experiment of highest precision is that of Pound and Rebka (1960), as improved by Pound and Snider (1965). It reveals a redshift of $z = \Delta\lambda/\lambda = (gh/c^2)(1 \pm 0.01)$ for photons climbing up through a height h in the Earth's locally homogeneous gravitational field—if the emitter and receiver are at rest relative to the Earth's surface. This tells us that the local Lorentz frames are not at rest relative to the Earth's surface as predicted by flat-spacetime theories of gravity; rather, as predicted by the strong equivalence principle, they accelerate downward with the same acceleration g as that which acts on a free particle (to within 1 percent precision). To arrive at this conclusion from the experiment, we argue as follows.[13]

We wish our argument to be as independent of the special-relativistic laws of physics as possible. The only aspects of special relativity that we shall use are (i) the relationship between the Minkowski metric of the local Lorentz frames and the ticking rates of atomic clocks; and (ii) the conservation of wave fronts in electromagnetic waves. Let us *assume* (falsely) that the local Lorentz frames were unaccelerated relative to the walls of the tower used in the Pound-Rebka experiment. We can then perform a calculation in that particular Lorentz frame which was attached to the walls of the tower and was large enough to cover the entire tower. The static nature of the emitter, receiver, gravitational field, and Lorentz coordinate system guaranteed that, although the spacetime trajectories of the wave crests might have been bent by gravity, they were certainly the same from one crest to another, except for a translation Δt_L in the Lorentz time coordinate. Hence, the coordinate rates $1/\Delta t_L$ of emission and reception of wave crests were the same. But by assumption these Lorentz coordinate rates were also the proper rates measured by the atomic clocks (^{57}Fe nuclei) of the experiment. Hence, theory predicts zero redshift, in contradiction with experiment. Our assumption that the local Lorentz frames were unaccelerated must be wrong!

We must assume, then, that the local Lorentz frames were accelerated relative to the tower. Since gravity pointed vertically and all horizontal directions were equivalent in all respects, the acceleration of the Lorentz frames must have been vertical. Denote by a its value in the downward direction. As in our previous argument, in a static coordinate system (i.e., in coordinates at rest relative to emitter, receiver, and Earth's static gravitational field) the wave-crest trajectories must have been identical, except for a time translation Δt_s from one crest to the next. But in this case the static coordinates were not Lorentz coordinates. Rather, they were accelerated upward (in the $+z$ direction) relative to the Lorentz frames (here we show the speed of light explicitly):

$$ct_L = (z_s + c^2/a) \sinh (at_s/c)\,, \qquad z_L = (z_s + c^2/a) \cosh (at_s/c)\,,$$

$$x_L = x_s\,, \quad y_L = y_s\,.$$

[12] Thus, we regard the redshift experiments as a crucial link in the chain of reasoning which will point, eventually, to the correct theory of gravity. By contrast, Dicke (1964, pp. 5 and 6) believes that "the gravitational redshift is not a very strong test of general relativity" because it can be derived from the weak equivalence principle, plus energy conservation, plus equivalence of inertial mass and conserved energy. We do not find Dicke's argument fully compelling. The fact that general relativity has no satisfactory local energy-conservation law, except in static external gravitational fileds, makes us worry about the *a priori* assumption of energy conservation. More importantly, we see no convincing *a priori* arguments why the inertial mass must equal the conserved energy to the precision required by Dicke's argument. In fact, this is not true in some theories with two tensor fields (see Peebles and Dicke 1963; we thank Professor Dicke for pointing this out to us). Finally, there exists a variety of relativistic gravitation theories which have been considered viable and attractive at one time or another but which disagree with the gravitational-redshift experiments (see p. 100 of Schild 1962).

[13] For a variety of somewhat similar arguments see chapter 5 of Schild (1962). The argument in the original Orange-Aid-Preprint version of this paper contained a flaw, which Charles W. Misner kindly pointed out to us; the corrected argument presented here is due largely to him.

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(For an elementary derivation and discussion of this transformation law between Lorentz frames and accelerated frames, see, e.g., chapter 5 of Misner, Thorne, and Wheeler 1971.) Hence, proper time as measured by atomic clocks was given by

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt_L^2 - dx_L^2 - dy_L^2 - dz_L^2 \\ &= (1 + az_s/c^2)^2 c^2 dt_s^2 - dx_s^2 - dy_s^2 - dz_s^2 . \end{aligned}$$

Since, as before, the wave-crest emission and reception rates were the same $(1/\Delta t_s)$ when measured in static coordinate time, they were related by

$$\frac{\Delta\lambda}{\lambda} = \frac{\nu_{em}}{\nu_{rec}} - 1 = \frac{[1 + (az_s)_{rec}/c^2]\Delta t_s}{[1 + (az_s)_{em}/c^2]\Delta t_s} - 1 = a[(z_s)_{rec} - (z_s)_{em}]/c^2 = ah/c^2 ,$$

when measured in the proper time of the atomic clocks. But the experimentally measured redshift was gh/c^2 to a precision of 1 percent. *Hence, the downward acceleration of the inertial frames was the same as that of a free particle, $g = 980$ cm sec^{-2}, to precision of 1 percent.*

The Pound-Rebka-Snider experiment is the easiest redshift experiment to interpret theoretically because it was performed in a uniform gravitational field. Complementary to it is the experiment by Brault (1963), which measured the redshift of spectral lines emitted on the surface of the Sun and received at Earth. To a precision of 5 percent he found a redshift of $GM_\odot/R_\odot c^2$, where $M_\odot$ and $R_\odot$ are the mass and radius of the Sun. This is just the redshift to be expected if the local Lorentz frames, at each point along the photon trajectory, are unaccelerated relative to freely falling test bodies. It certainly could not result if there were a single global Lorentz frame, extending throughout the solar system and at rest relative to its center of mass![14]

In summary, the redshift experiments reveal that, to a precision of $\sim 0.01\ GM/R^2$, where M and R are the mass and radius of the Earth, the local Lorentz frames at the Earth's surface are unaccelerated relative to freely falling test bodies. Equivalently, test bodies move along straight lines in the local Lorentz frames. Equivalently, *the test-body trajectories are geodesics of the metric g.*

Because this conclusion is crucial to the foundations of the PPN framework (see below), as well as to general relativity, it is very important that the precision of the redshift experiments be improved as much as possible, both on Earth (homogeneous field) and elsewhere in the solar system (inhomogeneous fields). Of particular interest will be experiments in which atomic clocks are flown in spacecraft (see, e.g., Kleppner, Vessot, and Ramsey 1970; Havas 1970; Geisler and McVittie 1965).

d) The Response of Stressed Matter to Gravity

For discussing solar-system tests of gravity in the PPN formalism, we will need to assume something about the response of stressed matter (e.g., the matter inside planets) to gravity. Our assumption will be that, as in special relativity (gravity absent), so also in the real world where gravity is present,

$$\nabla \cdot T = 0 . \tag{6}$$

Here T is the total stress-energy tensor for all matter and *non-gravitational* fields; and $\nabla\cdot$ is the divergence with respect to the metric g and its affine connection $\left\{ {a \atop b\,c} \right\}$.

Unfortunately, we do not have any firm experimental basis for the validity of equation

[14] See chapter 5 of Schild (1962) for further discussion of this point.

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(6) in the presence of gravity. However, we can make it seem reasonable—perhaps even compelling—by the following argument.

Geodesic motion for test bodies and $\nabla \cdot T = 0$ for stressed matter go hand in hand. In particular, from the assumption $\nabla \cdot T = 0$ we can derive geodesic motion (see, e.g., Fock 1964). From geodesic motion, i.e., straight-line motion in local Lorentz frames, we can derive $\nabla \cdot T = 0$ for the smeared-out stress-energy tensor of a swarm of noninteracting test particles. For test particles that interact only by means of instantaneous collisions, each of which conserves energy and momentum in the local Lorentz frames, geodesic motion again guarantees $\nabla \cdot T = 0$.

Unfortunately, one cannot prove that geodesic motion implies $\nabla \cdot T = 0$ in all circumstances. The closest one can come is the following: Consider a laboratory-sized object made of stressed material. Geodesic motion and conservation of rest mass mean that the body's four-momentum is conserved as seen in any local Lorentz frame:

$$\int_{\Sigma} T^{ab} dS_b = P^a \text{ is independent of } \Sigma . \tag{7}$$

Here Σ is any spacelike three-surface, contained entirely within the local Lorentz frame, which passes all the way through the body.[15] Using Stokes's theorem in the local Lorentz frame, we can infer from equation (7) that

$$\int_{\mathcal{V}} T^{ab}{}_{;b} d\mathcal{V} = \int_{\mathcal{V}} (\nabla \cdot T)^a d\mathcal{V} = 0 . \tag{8}$$

Here $\mathcal{V}$ is *any* four-volume contained entirely within the local Lorentz frame, which is intersected by all parts of the body. Equation (8) is equivalent to geodesic motion. The most straightforward way to guarantee the validity of equation (8) is by imposing $\nabla \cdot T = 0$. But that is not the only way. For example, if n is some spacelike vector field whose variation through the body is completely negligible, and if $T = T^a{}_a$ is the trace of the stress-energy tensor, then

$$\nabla \cdot T + n \cdot \nabla T = 0$$

would imply equation (8) and thence geodesic motion. However, there is no obvious, satisfactory way to pick out the vector n.

It is tempting, as another alternative to $\nabla \cdot T = 0$, to demand that $D \cdot T = 0$, where $D\cdot$ is the covariant derivative with respect to some affine connection $\Gamma^a{}_{bc}$ different from $\left\{ {a \atop b\,c} \right\}$—for example, $\left\{ {a \atop b\,c} \right\}$ plus a torsion. We show explicitly in the Appendix that this is untenable.

It is very important to seek, in the future, direct experimental proof that $\nabla \cdot T = 0$. To the accuracy of all laboratory experiments performed thus far (i.e., measurements of the behavior of stressed bodies in the Earth's gravitational field), $\nabla \cdot T = 0$ is true. But these experiments are probably not of sufficiently high precision for the purposes of the PPN formalism.[16]

[15] Here and elsewhere in the argument we ignore small corrections due to the Christoffel symbols, $\left\{ {a \atop bc} \right\}$, which vanish only at the origin of the local Lorentz frame. Clearly those corrections go to zero linearly with L, the size of the spacetime region under consideration—i.e., the "size of the local Lorentz frame."

[16] Note the great difference in spirit between the above discussion and the usual viewpoint. One usually assumes $\nabla \cdot T = 0$; and when confronted by any apparent violation of it (e.g., the apparent breakdown in energy-momentum conservation in β-decay), one normally seeks a modification of the stress-energy tensor T which will then restore the validity of $\nabla \cdot T = 0$ (e.g., Pauli's 1930 postulate of the existence of neutrinos). By contrast, we are assuming (without much justification) that all the contributions to T are known, and that the metric and covariant derivative ∇ are known; and we are then asking whether $\nabla \cdot T = 0$.

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III. THE PPN FRAMEWORK

a) Postulates; Metric Theories of Gravity

In constructing the PPN framework we shall use as postulates two, and only two, of the fair-confidence conclusions gleaned from analyzing experiments in the Dicke framework:

POSTULATE I. *There exists a metric of signature -2, which governs proper length and proper time measurements in the usual manner:*

$$ds^2 = g_{ij}dx^i dx^j . \tag{9}$$

POSTULATE II. *Stressed matter, being acted upon by gravity, responds in accordance with the equation*

$$\nabla \cdot T = 0 , \tag{10}$$

where T is the total stress-energy tensor for all matter and nongravitational fields.

It is interesting to notice that these two postulates can be obtained directly from a single, attractive assumption: *the existence of local Lorentz frames everywhere, in which all the laws of special relativity take on their usual form.*[17] However, we prefer to put the PPN formalism on the narrower base of metric plus $\nabla \cdot T = 0$, so that its experimental justification can be discussed more clearly.

Those theories of gravity which can be given mathematical representations that satisfy postulates I and II will be called "*metric theories of gravity*" throughout this series of papers.

One should keep in mind that any metric theory of gravity can perfectly well be given a mathematical representation that violates postulates I and II. For example, the Brans-Dicke theory, in the mathematical representation of Dicke (1962), does *not* satisfy our postulates: Dicke's scalar field causes deviations from geodesic motion, and physical rods and clocks do not measure $ds^2 = g_{ij}dx^i dx^j$. However, in the original mathematical representation of Brans and Dicke (1961), the theory does satisfy our postulates.

Notice that, in that representation of a metric theory where postulates I and II are satisfied, the metric is the only gravitational field which enters into the response equation $\nabla \cdot T = 0$ and into the resultant geodesic equation for test-body trajectories. (The metric determines ∇; and T contains no gravitational fields.) This does not mean that the metric is the only gravitational field present. On the contrary, as in Brans-Dicke theory, there may be other fields. However, the role of the other fields can only be that of helping to generate the spacetime curvature associated with the metric. Matter may create them, and they plus matter may create the curvature, but they cannot act back directly on the matter. The matter responds only to the metric!

Throughout this series of papers, when dealing with a metric theory of gravity, we shall use the mathematical representation which satisfies postulates I and II unless we state otherwise.

[17] In applying this assumption, one must be careful to allow for coupling to the Riemann curvature tensor in certain of the usual special-relativity equations. For example, the usual laws of vacuum electrodynamics in terms of the physical observables E and B,

$$\nabla \cdot E = \nabla \cdot B = 0 , \quad \nabla \times B = -\partial E/\partial t , \quad \nabla \times E = \partial B/\partial t ,$$

in curved spacetime imply that the vector potential A in the Lorentz gauge ($\nabla \cdot A = 0$) satisfies

$$\Box A - R \cdot A = 0$$

rather than $\Box A = 0$. Here $\Box$ is the wave operator (d'Alembertian) and R is the Ricci tensor.

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b) The Post-Newtonian Limit

The comparison of metric theories of gravity with each other and with experiment becomes particularly simple when one takes the slow-motion, post-Newtonian limit. Fortunately, the post-Newtonian limit is sufficiently accurate to encompass all solar-system tests that can be performed in the foreseeable future.[18]

The most primitive type of post-Newtonian limit for metric theories is a limit in which one assumes geodesic motion for planets as well as for test bodies, and one idealizes the solar-system metric as that of a spherical, nonrotating Sun. In this limit one obtains the original Eddington (1922)-Robertson (1962)-Schiff (1967) version of the PPN formalism. The solar-system metric in this limit reads

$$ds^2 = [1 - 2M^*/r + 2\beta(M^*/r)^2]dt^2 - (1 + 2\gamma M^*/r)(dx^2 + dy^2 + dz^2)\,, \quad (11)$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ is a radial coordinate; $M^* = GM_\odot/c^2$ is the geometrized mass of the Sun; and β and γ are parameters that differ from one metric theory to another. For general-relativity theory, $\beta = \gamma = 1$. For Brans-Dicke theory, $\beta = 1$, $\gamma = (1 + \omega)/(2 + \omega)$, where ω is the Dicke coupling constant. For the last 45 years the goal of light-deflection and perihelion-shift measurements has been to measure the parameters β and γ.

Schiff (1960), Nordtvedt (1968), and Baierlein (1967) have shown that the Eddington-Robertson-Schiff version of the post-Newtonian limit is too idealized for the 1970's. It is too narrow to encompass (i) the precession of a gyroscope due to the dragging of inertial frames by the rotating Earth (Schiff 1960); (ii) periodic terms in the Earth-Moon separation due to the nonlinear superposition of the Earth's and the Sun's gravitational fields (Baierlein 1967; Krogh and Baierlein 1968); (iii) an anomalous time-varying eccentricity in the Earth-Moon system due to a breakdown in geodesic motion for the Earth (Nordtvedt 1968). These effects, and others like them, should all be measurable in the coming decade.

Each of the above researchers has modified the original Eddington-Robertson-Schiff post-Newtonian limit to encompass the types of effects that interested him. Schiff added a metric term associated with the dragging of inertial frames. Baierlein began over again by taking Chandrasekhar's post-Newtonian limit for general-relativistic fluids and putting arbitrary parameters in front of some of the terms. Nordtvedt began over again by taking the Einstein-Infeld-Hoffman post-Newtonian limit for general-relativistic point particles, by adding several terms which are absent in general relativity but could be present in other theories, and by putting an arbitrary parameter in front of each term.

None of these new versions of the post-Newtonian limit is fully adequate. The Schiff version is clearly too narrow. The Baierlein version leaves out some fairly simple terms that are absent from general relativity but could appear in other metric theories, and it lumps together five terms in the metric that should each carry its own arbitrary parameter. The Nordtvedt version treats the planet as a swarm of noninteracting point particles moving in the smoothed-out gravitational fields of each other (self-consistent-field approach), which is much too idealized for reality.

In Paper II of this series one of us (C. M. W.) will present a new—and we hope definitive—version of the post-Newtonian limit, valid for any metric theory of gravity. This version will combine the best of the Nordtvedt and Baierlein approaches. The essence of this post-Newtonian limit is that it contains a series of parameters, β, β_1, β_2,

[18] There is one exception: gravity-wave experiments. (Gravity waves do not exist in the post-Newtonian limit.) However, gravity-wave experiments can be discussed more fruitfully in the Dicke framework than in the restricted realm of metric theories.

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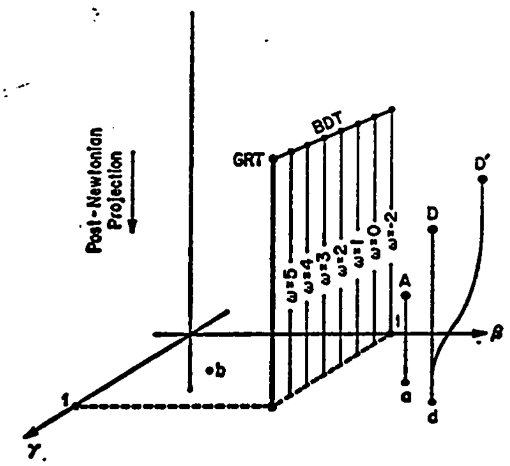
β_3, β_4, γ, Δ_1, Δ_2, ζ, with undetermined values. These parameters distinguish the various metric theories of gravity from each other: the post-Newtonian limit of general relativity has one set of values for these parameters; the post-Newtonian limit of Brans-Dicke theory, with fixed Dicke coupling constant ω, has another set of values, etc. In solar-system experiments, the task of the experimenter is to measure the values of one or more of these parameters. The implications of such measurements for relativistic gravitation theory are summarized schematically in Figure 1.

We shall refer to our post-Newtonian limit for metric theories of gravity as the Parametrized Post-Newtonian (PPN) framework. Subsequent papers in this series will use the PPN framework to analyze various relativistic effects in the solar system and experiments to measure them.



Parametrized Post-Newtonian Hyperplane

FIG. 1.—Schematic diagram of the PPN framework. A variety of exact, self-consistent metric theories of gravity exists or can be constructed, including among others general relativity theory (GRT), Brans-Dicke theory (BDT), and three hypothetical theories, A, D, and D′. By and large there is no simple relationship between the various exact metric theories. Some contain only a metric field and differ in their field equations; others contain, besides the metric, also a second tensor field, or a vector field, or a scalar field. The post-Newtonian limit is obtained (schematically in the diagram) by projecting each theory down into the "parametrized post-Newtonian hyperplane," which has nine dimensions corresponding to the nine parameters β, β_1, β_2, β_3, β_4, γ, Δ_1, Δ_2, ζ. (In the diagram seven of the dimensions are suppressed for ease of visualization.) In the post-Newtonian limit the theories are distinguished from each other completely by where they lie in the hyperplane—i.e., by the values of their nine PPN parameters. All reference to gravitational fields other than the metric has been lost at the PPN level (see Paper II for details); all the theories are on the same footing.

Experimental measurements of the PPN parameters could land us at one of three types of points: (i) At a point such as *a*, to which there corresponds precisely one self-consistent, exact theory, A; then we might be happy. (ii) At a point such as *b*, to which there correspond no self-consistent, exact theories; then we would have to go back to the theoretical drawing boards. (iii) At a point such as *d*, to which there corresponds more than one self-consistent, exact theory; then we would have to wait for sufficient technology to permit solar-system experiments at the post-post-Newtonian level, or we would have to devise some other means for experimentally distinguishing the theories.

Of course, this is a very highly idealized story; reality is never so simple. Nevertheless, this story is useful for organizing one's theoretical thoughts. In particular, it suggests that one should attempt to determine which points of the PPN hyperplane are of type *a*, which are *b*, and which are *d*. Hopefully a subsequent paper in this series will discuss this question.

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APPENDIX

GRAVITATION THEORIES WITH A METRIC AND A NON-RIEMANNIAN AFFINE CONNECTION

Since the covariant mathematical representation of Newtonian gravitation theory involves a non-Riemannian affine connection, it is tempting to speculate that the correct relativistic theory might similarly have a non-Riemannian affinity $\Gamma^a{}_{bc}$, in addition to its Riemannian affinity

$$\left\{ {a \atop bc} \right\} = \tfrac{1}{2}g^{ae}(g_{eb,c} + g_{ec,b} - g_{bc,e}) . \tag{A1}$$

The two affinities must then differ by a third-rank tensor

$$\Gamma^a{}_{bc} = \left\{ {a \atop bc} \right\} + S^a{}_{bc} . \tag{A2}$$

Let us attempt to build a viable theory of gravity in which this tensor comes into play.

In Newtonian theory the test-body trajectories are the geodesics of the nonmetrical affinity Γ. However, the gravitational-redshift experiments suggest that the relativistic test-body trajectories are the geodesics of $\left\{ {a \atop bc} \right\}$ and its metric. Both statements are possible—the geodesics of $\Gamma^a{}_{bc}$ and $\left\{ {a \atop bc} \right\}$ can coincide without $S^a{}_{bc}$ vanishing. If the geodesics are to be identical, including parametrization, then $S^a{}_{bc}$ must be a "torsion"; i.e., it must be antisymmetric. If the geodesics are the same except for parametrization, then for any vector u^a, $S^a{}_{bc}u^bu^c$ must point in the u^a direction. For example, $S^a{}_{bc}$ could have the form

$$S^a{}_{bc} = S^a{}_{[bc]} + \delta^a{}_{(b}f_{c)} , \tag{A3}$$

where brackets denote antisymmetrization, parentheses denote symmetrization, and f_c is an arbitrary vector.

Let us suppose that, except perhaps for parametrization, the test-body trajectories are geodesics both of $\Gamma^a{}_{bc}$ and $\left\{ {a \atop bc} \right\}$. Then $S^a{}_{bc}$ cannot be measured by means of test-body motion. How, then, can it be measured? We might hope that it would affect the manner in which a gyroscope transports its spin axis. For example, if a geodesically moving gyroscope were to parallel-transport its spin with respect to $\Gamma^a{}_{bc}$ rather than $\left\{ {a \atop bc} \right\}$, then $S^a{}_{bc}$ would have an effect on it (Misner 1969*a*).

We cannot simply postulate that gyroscope spins are parallel-transported with respect to $\Gamma^a{}_{bc}$. Rather, we must derive this fact from the response of stressed matter to gravity. The response equation which is usually assumed is $\nabla\cdot T = 0$, where ∇ is the Riemannian covariant derivative. But this necessarily leads to parallel transport with respect to $\left\{ {a \atop bc} \right\}$.

The most obvious way to incorporate the non-Riemannian affinity $\Gamma^a{}_{bc}$ into the response equation is to assume $D\cdot T = 0$ rather than $\nabla\cdot T = 0$, where D is the covariant derivative with respect to Γ. However, as we shall show below, the condition $D\cdot T = 0$, plus the condition of metrical-geodesic motion for test bodies, guarantees $\nabla\cdot T = 0$. This means that $S^a{}_{bc}$ can have no effect whatsoever on the behavior of stressed matter—in particular, it cannot affect the spin axis of a gyroscope!

We have been unable to find any other viable way to incorporate a non-Riemannian affinity into the response equation, and at the same time guarantee that test bodies move along geodesics of the metric. It seems likely that all viable theories with non-Riemannian affinities will violate metrical-geodesic motion, and will thus violate the gravitational-redshift experiments

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to a greater or lesser degree. Thus, redshift experiments might be viewed as tests for a non-Riemannian affinity.

We conclude with a proof that $D\cdot T = 0$ plus metrical-geodesic motion for all test bodies guarantees $\nabla\cdot T = 0$. Our proof assumes nothing, *a priori*, about the nature of the tensor $S^a{}_{bc} = \Gamma^a{}_{bc} - \left\{ {a \atop bc} \right\}$.

Consider a test body made of stressed material. Let the test body be so small that coupling to the curvature is negligible. Then we can work in a local Lorentz frame, where $\left\{ {a \atop bc} \right\} = 0$. The condition $D\cdot T = 0$ reads

$$0 = (D\cdot T)^a = T^{a0}{}_{,0} + T^{a\alpha}{}_{,\alpha} + S^a{}_{bc}T^{bc} + S^b{}_{bc}T^{ac} . \tag{A4}$$

Integrating this equation over the volume of the body, on a hypersurface of constant local-Lorentz time, we obtain

$$dP^a/dt \equiv (\textstyle\int T^{a0}d^3x)_{,0} = -(S^a{}_{bc}\mathfrak{T}^{bc} + S^b{}_{bc}\mathfrak{T}^{ac}) . \tag{A5}$$

Here

$$\mathfrak{T}^{ab} \equiv \textstyle\int T^{ab}d^3x \tag{A6}$$

is the integral of the stress-energy tensor over the body. The motion of the test body must be metrically geodesic, and its rest mass must be conserved. (Nonconservation of rest mass would make electrons with different past histories be distinguishable, in violation of experiment.) Consequently, dP^a/dt must vanish; i.e.,

$$S^a{}_{bc}\mathfrak{T}^{bc} + S^b{}_{bc}\mathfrak{T}^{ac} = 0 . \tag{A7}$$

Now $\mathfrak{T}^{ab}$ is completely arbitrary in its algebraic properties—except for the constraint that it be symmetric.[19] Hence, if all test bodies are to move along geodesics, conserving their rest masses, then $S^a{}_{bc}$ must satisfy the algebraic constraint

$$S^a{}_{bc}B^{bc} + S^b{}_{bc}B^{ac} = 0 \text{ for all symmetric } B . \tag{A8}$$

(The tensor (A3) with $S^b{}_{[ab]} + \frac{7}{2}f_a = 0$ is an example of such an S.) Condition (A8) implies directly that $D\cdot T = \nabla\cdot T$, so that if $D\cdot T = 0$ then $\nabla\cdot T = 0$. *Q.E.D.*

REFERENCES

Ageno, M., and Amaldi, E. 1966, *Accademia Nazionale dei Lincei*, ser. 8, **8**, 1.
Alväger, T., Farley, F. J. M., Kjellman, J., and Wallin, I. 1964, *Phys. Letters*, **12**, 260.
Baierlein, R. 1967, *Phys. Rev.*, **162**, 1275.
Bailey, J., and Picasso, E. 1970, *Progr. Nucl. Phys.*, **12**, 43.
Bergmann, P. G. 1968, *Int. J. Theoret. Phys.*, **1**, 25.
Brault, J. 1963, *Bull. Am. Phys. Soc.*, **8**, 28.
Brans, C., and Dicke, R. H. 1961, *Phys. Rev.*, **124**, 925.
Champeney, D. C., Isaak, G. R., and Khan, A. M. 1963, *Phys. Letters*, **7**, 241.
Cook, A. H. 1965, *Metrologia*, **1**, 84.
Dicke, R. H. 1962, *Phys. Rev.*, **125**, 2163.
———. 1964, *The Theoretical Significance of Experimental Relativity* (New York: Gordon & Breach).
Durbin, R. P., Loar, H. H., and Havens, W. W., Jr. 1952, *Phys. Rev.*, **88**, 179.
Eddington, A. S. 1922, *The Mathematical Theory of Relativity* (London: Cambridge University Press).
Farley, F. J. M., Bailey, J., Brown, R. C. A., Giesch, M., Jöstlein, H., van der Meer, S., Picasso, E., and Tannenbaum, M. 1966, *Nuovo Cimento*, **45**, 281.
Fock, V. 1964, in *The Theory of Space, Time and Gravitation* (New York: Pergamon Press), p. 238.
Frisch, D. H., and Smith, J. H. 1963, *Am. J. Phys.*, **31**, 342.
Geisler, P., and McVittie, G. C. 1965, *A.J.*, **70**, 14.

[19] One might think otherwise. If the body has static structure (no rotation; no time-changing deformation), and is viewed in a comoving Lorentz frame, then $\nabla\cdot T = 0$ guarantees that the volume integral of the *stress* tensor, $\mathfrak{T}^{\alpha\beta}$, vanishes. However: (i) We are assuming $D\cdot T = 0$, not $\nabla\cdot T = 0$; (ii) if the body is nonstatic, then even $\nabla\cdot T = 0$ does not guarantee $\mathfrak{T}^{\alpha\beta} = 0$.

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Havas, P. 1970, paper in press.
Hicks, N. J. 1965, *Notes on Differential Geometry* (Princeton: D. Van Nostrand Co., Inc.).
Ives, H. E., and Stilwell, G. R. 1938, *J. Opt. Soc. Am.*, **28**, 215.
———. 1941, *ibid.*, **31**, 369.
Kleppner, D., Vessot, R. F., and Ramsey, N. F. 1970, *Ap. and Space Sci.*, **6**, 13.
Krogh, C., and Baierlein, R. 1968, *Phys. Rev.*, **175**, 1576.
Lichtenberg, D. B. 1965, in *Meson and Baryon Spectroscopy* (New York: Springer-Verlag), p. 18.
Misner, C. W. 1969*a*, unpublished lecture notes on general relativity, University of Maryland.
———. 1969*b*, in *Astrophysics and General Relativity*, Vol. 1, ed. M. Chrétien, S. Deser, and J. Goldstein (New York: Gordon & Breach), p. 113.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1971, *Gravitation* (San Francisco: W. H. Freeman & Co.)
Nordtvedt, K., Jr. 1968, *Phys. Rev.*, **169**, 1017.
Ockham, William of. 1495, *Super quattuor libros sententiarum subtilissimae quaestiones* (Lyons).
Peebles, P. J., and Dicke, R. H. 1962, *Phys. Rev.*, **127**, 629.
Pound, R. V., and Rebka, G. A. 1960, *Phys. Rev. Letters*, **4**, 337.
Pound, R. V., and Snider, J. L. 1965, *Phys. Rev.*, **140**, B788.
Robertson, H. P. 1962, in *Space Age Astronomy*, ed. A. J. Deutsch and W. B. Klemperer (New York: Academic Press), p. 228.
Rössi, B. and Hall, D. B. 1941, *Phys. Rev.*, **59**, 223.
Schiff, L. I. 1960, *Proc. Nat. Acad. Sci.*, **46**, 871.
———. 1967, in *Relativity Theory and Astrophysics. I. Relativity and Cosmology*, ed. J. Ehlers (Providence, R.I.: American Mathematical Society).
Schild, A. 1962, in *Evidence for Gravitational Theories*, ed. C. Møller (New York: Academic Press).
Sherwin, C. W. 1960, *Phys. Rev.*, **120**, 17.
Thorne, K. S., and Will, C. M. 1970, *Comments Ap. and Space Phys.*, **2**, 35.
Trautman, A. 1965, in *Lectures on General Relativity*, by A. Trautman, F. A. E. Pirani, H. Bondi (Englewood Cliffs, N.J.: Prentice-Hall, Inc.).
Turner, K. C., and Hill, H. A. 1964, *Phys. Rev.*, **134**, B252.
Will, C. M. 1971, *Ap. J.*, **163**, 611 (Paper II).
Zel'dovich, Ya. B., and Novikov, I. D. 1967, *Relyativistkaya Astrofisika* (Moscow: Nauka).
———. 1971, *Relativistic Astrophysics*, Vol. 1, *Stars and Relativity* (Chicago: University of Chicago Press).
———. 1972, *Relativistic Astrophysics*, Vol. 2, *The Universe and Relativity* (Chicago: University of Chicago Press).

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B. THE PARAMETRIZED POST-NEWTONIAN FORMALISM: THEORY

3. Theoretical Frameworks for Testing Relativistic Gravity

II. Parametrized Post-Newtonian Hydrodynamics,

and the Nordtvedt Effect

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THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY. II. PARAMETRIZED POST-NEWTONIAN HYDRODYNAMICS, AND THE NORDTVEDT EFFECT*

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ABSTRACT

Chandrasekhar's post-Newtonian equations of hydrodynamics are generalized to encompass any metric theory of gravity by the use of arbitrary metric parameters. The resultant Parametrized Post-Newtonian (PPN) hydrodynamical equations are then used to calculate the Newtonian acceleration of the center of mass of a massive body of perfect fluid toward a very distant point mass. This acceleration can be written in terms of a gravitational-mass tensor and the gradient of the Newtonian potential: $ma^\alpha = m^{\alpha\beta} U_{,\beta}$. The gravitational-mass tensor $m^{\alpha\beta}$ is equal to the isotropic inertial mass ($m\delta^{\alpha\beta}$), plus a small correction ("Nordtvedt effect"), which depends on the gravitational internal energy of the body and on the metric parameters that characterize the particular theory being used. In general relativity, the gravitational mass is precisely equal to the inertial mass (correction terms vanish), in accordance with the equivalence principle. In Brans-Dicke theory the two masses differ by a small isotropic correction which varies from body to body, in violation of the equivalence principle. A simple explanation of these two results is discussed. This work generalizes and substantially agrees with previous calculations by Nordtvedt.

I. INTRODUCTION AND SUMMARY

In Paper I of this series (Thorne and Will 1971), the theoretical and experimental foundations for metric theories of gravity were described, and qualitative aspects of the Parametrized Post-Newtonian (PPN) formalism were discussed. Central to this formalism were the postulates that there exist a metric of signature -2 which governs proper length and proper time measurements, and that the response of stressed matter to gravity be described by the equation

$$\nabla \cdot T = 0 . \tag{1}$$

Here ∇ is the covariant derivative with respect to the metric, and T is the total stress-energy tensor for all matter and nongravitational fields. In this paper a series of metric parameters will be used to develop in detail from these postulates the PPN formalism for perfect fluids. The formalism will then be applied to a particular problem in relativistic gravity: the "Nordtvedt effect."

The Nordtvedt effect is a violation of the principle of equivalence in the motions of massive, self-gravitating bodies. Recent calculations by Nordtvedt (1968*b*, 1969) and by Dicke (1969) have shown that, according to a wide class of metric theories of gravity, such violations should occur; that is, in an external gravitational field, different massive bodies should fall with different accelerations. In Newtonian language, this means that the gravitational mass of such a body is no longer equal to its inertial mass. We may define a ***gravitational-mass tensor*** $m^{\alpha\beta}$ by

$$ma^\alpha = m^{\alpha\beta} U_{,\beta} , \tag{2}$$

where m is the inertial mass of the body, a^α is the acceleration of its center of mass, and U is the external (Newtonian) potential, which can be measured by means of test

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particles with negligible self-gravity. (In this paper, Greek indices will take the values 1, 2, and 3; Roman indices will take the values 0, 1, 2, 3; and summation over repeated indices will be employed. Commas will denote partial differentiation.) Experiment has shown that laboratory-size bodies obey the equivalence principle; i.e., within an accuracy of one part in 10^{11} (Roll, Krotkov, and Dicke 1964), they have

$$m^{\alpha\beta} = m\delta^{\alpha\beta} . \tag{3}$$

But for a massive-body configuration without radiation pressure, Nordtvedt (1968*b*) finds, to lowest order in a post-Newtonian expansion,

$$m^{\alpha\beta} = m\{\delta^{\alpha\beta}(1 - \eta|\Omega|/m) + \xi\Omega^{\alpha\beta}/m + O([\Omega/m]^2)\} , \tag{4}$$

where η and ξ are dimensionless constants depending on the theory of gravity, $\Omega^{\alpha\beta}$ is the body's Newtonian gravitational potential tensor, and $\Omega \equiv \Sigma_\alpha \Omega^{\alpha\alpha}$ is its gravitational potential energy. We use units for which the velocity of light is unity and the Newtonian gravitational constant in the outer regions of the solar system today is unity.[1] In general relativity theory, $\eta = \xi = 0$, while in Brans-Dicke theory, $\eta = 1/(2 + \omega)$ and $\xi = 0$, where ω is the Dicke coupling constant. We shall use the name "Nordtvedt effect" for this breakdown in the equivalence principle ($m^{\alpha\beta} \neq m\delta^{\alpha\beta}$) for massive bodies.

In his calculation, Nordtvedt makes use of a post-Newtonian metric which has been generalized through the use of arbitrary metric parameters to encompass a broad set of gravity theories, and with this metric he calculates the post-Newtonian equations of motion for a system of non-interacting point particles ("EIH problem"). He then computes the acceleration of the Newtonian center of mass of a gravitationally bound spherical cloud of such particles (model of a planet) toward a distant point mass, retaining only those terms which decrease as the inverse square of the distance from the center of mass of the body to the distant object. It is important to note that Nordtvedt's cloud of point particles which do not interact except through mutual gravitation is by definition an "ideal" gas. Nordtvedt (1968*b*, 1969) obtains equation (4) for a sphere in hydrostatic equilibrium, for a pulsating sphere, and for a rotating sphere. But when he includes the effects of radiation pressure, he finds a result slightly different from the above:

$$m^{\alpha\beta} = m\{\delta^{\alpha\beta}(1 - \eta|\Omega|/m + \tfrac{1}{3}(1 - \gamma)E_r/m) + \xi\Omega^{\alpha\beta}/m\} . \tag{5}$$

Here E_r is the total energy in radiation in the body, and γ is one of the metric parameters; in general relativity, $\gamma = 1$, while in Brans-Dicke theory $\gamma = (1 + \omega)/(2 + \omega)$.

In this paper we will show that Nordtvedt's original expression (4) for the gravitational-mass tensor, with the metric parameters reinterpreted in terms of the fluid picture, is actually valid for massive bodies of arbitrary shape, with arbitrary matter distributions and internal motions (such as convection), and obeying arbitrary equations of state. Since, from the fluid viewpoint, radiation in a star merely changes its equation of state, our result that equation (4) is valid in general disagrees with Nordtvedt's correction term for radiation (eq. [5]).

We will make only two assumptions regarding the nature of the massive body:

a) The body is composed of "perfect fluid." A fluid is said to be "perfect" if, in its rest frame, it cannot support shear stresses. Nordtvedt's ideal gas is a special case of a perfect fluid.

b) The (anisotropic) flux of radiation and heat through the body is negligible compared with the internal energy density.

[1] In some theories of gravity, the gravitational "constant" varies in space and time. Such variations in G must be small over the solar system or they would result in violations of Newtonian theory. Variations in G generated by the matter in the solar system (local matter) can be incorporated into the PPN metric parameters. Variations associated with cosmological models (such as a time-varying G) cannot be so easily incorporated and may require a modification of the formalism.

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These assumptions will be justified below for those massive bodies of interest, namely, the Sun and planets.

Our calculation will then proceed along lines similar to that of Nordtvedt, except that, instead of EIH point-mass equations of motion, the PPN generalization of the hydrodynamical formalism of Chandrasekhar (1965) will be used. This formalism has the advantage of including pressure (and hence the equation of state) in a straightforward and realistic way, and of taking care of large-scale internal fluid motions (such as convection and rotation).

Dicke (1969) has calculated $m^{\alpha\beta}$ for a massive body in Brans-Dicke theory, using a very different approach. From Dicke's point of view, the Nordtvedt effect arises from the fact that the gravitational "constant" G (related to the scalar field) is not constant, but depends on the gravitational potential at each point in space. Hence, the internal gravitational energy, E_g, of any body in an external, Newtonian gravitational field, U, depends on the body's position in space. Conservation of energy then demands that the body feel an acceleration a^α, given by (see Appendix)

$$ma_\alpha = (mU_{,\alpha} - \partial E_g/\partial x^\alpha) \text{ at body's center of mass}\,, \tag{6}$$

where m is the inertial mass, and the partial derivative is taken holding the body's structure fixed. In Brans-Dicke theory (Nutku 1969)

$$G = 1 - U/(2+\omega)\,. \qquad (7)^2$$

The internal gravitational energy E_g is directly proportional to G:

$$E_g = G\Omega = -\tfrac{1}{2}G\int \frac{\rho(x)\rho(x')}{|x-x'|}dxdx' = \Omega\{1 - U/(2+\omega)\}\,, \tag{8}$$

so

$$\partial E_g/\partial x^\alpha = -U_{,\alpha}\Omega/(2+\omega)\,, \tag{9}$$

and

$$ma_\alpha = U_{,\alpha}\{m + \Omega/(2+\omega)\}\,. \tag{10}$$

The gravitational-mass tensor is thus (cf. eq. [2])

$$m^{\alpha\beta} = m\delta^{\alpha\beta}\{1 - [1/(2+\omega)]|\Omega|/m\}\,. \tag{11}$$

The nature of this derivation shows that equation (11) holds for very general massive-body configurations in Brans-Dicke theory. In particular, it agrees with our result (and disagrees with Nordtvedt's) that radiation energy in the body has no effect on the gravitational mass.

Dicke's calculation also sheds some light on the problem of why massive bodies do obey the equivalence principle in general relativity. Since the physical constants are truly constant in general relativity, the internal structure of a massive body, and in particular its binding energy, are independent of location in any uniform external field. There is no anomalous acceleration due to a change in internal energy. Thus the gravitational mass is equal to the inertial mass.

A second way of understanding the absence of a Nordtvedt effect in general relativity has been suggested by Richard Price (private communication): Consider a massive body located in an external gravitational field which can be considered uniform over a region that is very large compared with the body's gravitational radius. Focus attention on a large volume of space V surrounding the massive body—a volume so large that in its

[2] The variation in the gravitational constant (due to the scalar field) used in this argument is an example of a variation generated by local matter (see n. 1). If we use the (1961) representation of Brans-Dicke theory (which is the representation used by Nutku 1969, and in this paper), the effect of the scalar field shows up in the metric parameters, and the gravitational constant is treated as a true constant, namely, unity.

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outer regions the spacetime curvature produced by the body itself is small to some desired accuracy (asymptotic flatness); but a volume still small enough that throughout it the external gravitational field is homogeneous to some other desired accuracy. In the outer regions of V one can introduce an inertial reference frame that falls freely in the homogeneous external field. Of course, that inertial frame cannot be extended into the massive body; but it does completely surround the body (asymptotic flatness). Conservation of the body's total four-momentum in that frame (valid for any massive body in asymptotically flat spacetime) guarantees that, if it is initially at rest in our inertial frame, it will always remain at rest there. Similarly, any *test particle* (far from the massive body) initially at rest will remain at rest. Thus both test particles and the massive body are tied to the inertial frame. This means that, as seen in the original accelerated frame where the external field is manifest, they fall with identical accelerations.

This classical, "Einstein elevator" type of argument fails in Brans-Dicke theory because there a uniform Newtonian field U carries with it a position-dependent gravitational "constant," so it is *not* completely equivalent to an accelerated frame.

The coupling between the massive body and the inhomogeneities of the external field, which is neglected in the above argument, should lead to multipole-type forces on the body which die out faster than 1/(distance to external body)^2. Such forces are not the subject of this paper.

Later in this paper the result that massive bodies obey the equivalence principle in general relativity will be seen to be due to the exact cancellation of various metric parameters.

The Nordtvedt effect can be put to observational test. In particular, it means that, according to Brans-Dicke theory, in a Newtonian gravitational field the Sun will fall with an acceleration which is less by about one part in 10^6 than that of a test body, Jupiter will fall more slowly by one part in 10^9, and the Earth by one in 10^{10}. Nordtvedt (1968*a*, *c*, 1970) has discussed this point in connection with the Trojan asteroids, the lunar laser reflection experiment, and interplanetary radar experiments; Thorne and Will (1970) have discussed it in connection with tests of gravity using spacecraft. Future systematic studies of the motions of the planets will have to include the Nordtvedt effect.

The rest of this paper details the hydrodynamical computation of the gravitational-mass tensor for an arbitrary massive body. Section II presents the generalization of the post-Newtonian fluid equations of motion using metric parameters (PPN hydrodynamics). In § III, we calculate the acceleration of a massive body toward a distant body (with both bodies assumed momentarily at rest in the post-Newtonian coordinate system), keeping only those terms which vary as the inverse square of the bodies' separation. This calculation yields a post-Newtonian expression for the gravitational-mass tensor in any geometric theory of gravity. In § IV, the gravitational-mass tensor is specialized to general relativity and to Brans-Dicke theory. Brief concluding remarks are presented in § V; and the details of Dicke's derivation of $m^{\alpha\beta}$ are given in an Appendix.

II. PARAMETRIZED POST-NEWTONIAN EQUATIONS OF HYDRODYNAMICS

Consider a perfect nonviscous fluid, which, in the Newtonian limit, obeys the usual Eulerian equations of hydrodynamics:

$$\partial\rho/\partial t + \partial(\rho v^\alpha)/\partial x^\alpha = 0\,, \quad \rho\frac{dv^\alpha}{dt} = \rho\frac{\partial U}{\partial x^\alpha} - \frac{\partial p}{\partial x^\alpha}, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v^\alpha\frac{\partial}{\partial x^\alpha}, \tag{12}$$

where v^α is the velocity of an element of fluid, ρ is the density of matter in the element, p is the total pressure (matter plus radiation) on the element, d/dt is the time derivative "following the fluid," and U is the Newtonian gravitational potential, defined by

$$\nabla^2 U = -4\pi\rho\,. \tag{13}$$

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The same fluid in a metric theory of gravity is described by an energy-momentum tensor of the form

$$T^{ij} = (\epsilon + p)u^i u^j - p g^{ij}\,, \tag{14}$$

where ϵ is the total mass-energy density, u^i is the fluid's four-velocity, and g^{ij} is the metric. We separate ϵ into a rest-mass density ρ, and a density $\rho\Pi$ consisting of all other types of energy density (radiation energy, compressional energy, thermal energy, etc.),

$$\epsilon = \rho(1 + \Pi)\,. \tag{15}$$

Before writing down the parametrized metric to be used in this paper, we must make a few comments about orders of magnitude involved in the post-Newtonian approximation. The fundamental quantity in the post-Newtonian approximation is the Newtonian gravitational potential U. In the solar system, U is everywhere less than $\sim 10^{-5}$. Other dimensionless quantities which are also small in the solar system are planetary and fluid velocities ($v^2 \lesssim U$), the ratio of pressure to matter density, p/ρ (10^{-5} in the Sun, 10^{-10} in the Earth), and the ratio of energy density to matter density, Π (10^{-5} in the Sun, 10^{-9} in the Earth). These three quantities will affect the motion of massive bodies only at the post-Newtonian level. This fact entitles us to neglect two quantities which are even smaller, namely, radiation transport and shear stresses. The assumption that there is negligible radiation transport in the massive bodies in the solar system allows us to include radiation as an additional energy density in $\rho\Pi$, completely "tied" to each element of fluid. This is a good approximation in the solar system, since the flux of radiation momentum through the Sun is less than 10^{-15} of the internal energy density, $\rho\Pi$. It is even less in the planets. The other quantity which we can neglect is shear stress. In the Earth, for example, the shear stresses are about 10^{-3} of the hydrostatic pressure (Jeffreys 1959); and in the Sun they are totally negligible. Of course, shear stresses are important in determining the shape of the Earth, even at the Newtonian level. There, a 10^{-3} deviation from isotropic pressure is important; but in the equation of motion for the planet in an external field, where ρ rather than p is the dominant factor, a 10^{-3} shear correction to p should be negligible.

By comparing these numbers with the precision to which solar-system tests of the Nordtvedt effect can be made in the next decade (see Thorne and Will 1970 for numbers), one finds that the perfect-fluid approximation is of adequate accuracy even for the planets.

Let us now turn to the metric parametrization. We use a parametrization similar to that of Nordtvedt (1968*b*), except that ours involves the fluid idealization, while his involved the less-satisfactory point-particle idealization. The fluid parametrization used by Baierlein (1967) is not general enough for our purposes.

We write the post-Newtonian metric as an expansion in terms of functionals of the small quantities U, v^2, Π, and p/ρ (which are all of the same order of magnitude), using ten parameters β, β_1, β_2, β_3, β_4, Σ, ζ, Δ_1, Δ_2, γ:

$$g_{00} = 1 - 2U + 2\beta U^2 - 4\Phi + \zeta\mathcal{A} + \Sigma\mathcal{B}\,,$$

$$g_{0\alpha} = \tfrac{7}{2}\Delta_1 V_\alpha + \tfrac{1}{2}\Delta_2 W_\alpha, \qquad g_{\alpha\beta} = -(1 + 2\gamma U)\delta_{\alpha\beta}\,, \tag{16}$$

where

$$U(x, t) = \int \frac{\rho(x', t)}{|x - x'|}\,dx'\,,$$

$$\Phi(x, t) = \int \frac{\rho(x', t)\phi(x', t)}{|x - x'|}\,dx'\,,$$

$$\phi = \beta_1 v^2 + \beta_2 U + \tfrac{1}{2}\beta_3\Pi + \tfrac{3}{2}\beta_4 p/\rho\,,$$

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$$\mathcal{A}(x,t) = \int \frac{\rho(x',t)[(x_\alpha - x'_\alpha)v_\alpha(x')]^2}{|x-x'|^3}dx' ,$$

$$\mathcal{B}(x,t) = \int \frac{\rho(x',t)\rho(x'',t)(x_\alpha - x'_\alpha)(x'_\alpha - x''_\alpha)}{|x-x'||x'-x''|^3}dx'dx'' ,$$

$$V_\alpha(x,t) = \int \frac{\rho(x',t)v_\alpha(x')}{|x-x'|}dx' ,$$

$$W_\alpha(x,t) = \int \frac{\rho(x',t)v_\beta(x')(x_\beta - x'_\beta)(x_\alpha - x'_\alpha)}{|x-x'|^3}dx' . \tag{17}$$

This is the most general post-Newtonian metric which can be written to satisfy the following conditions:

a) The deviations of the metric from flat space are all of Newtonian or post-Newtonian order; no post-post-Newtonian or higher-order deviations are included. (For a discussion of the distinction between Newtonian, post-Newtonian, and post-post-Newtonian terms, see, e.g., Chandrasekhar 1965.)

b) The metric becomes Minkowskian (flat space) as the distance $|x - x'|$ between the field point and the matter becomes large. This condition prevents the appearance of terms such as

$$\int v(x')^2\Pi(x')dx' \quad \text{or} \quad \int \Pi(x')[p(x')/\rho(x')]dx'$$

in g_{00}, for example.

c) The metric is generated only by the rest mass, energy, pressure, and velocity of the matter; not by their gradients. This is a reasonable condition to put on physically acceptable metric theories, and is a condition which can be relaxed quite easily if there is ever any reason to do so. Terms involving gradients, such as

$$\int v_\beta(x')(x_\beta - x'_\beta)[p(x')/\rho(x')]_{,\alpha}\, dx'$$

in $g_{0\alpha}$, for example, are prohibited by this condition.

d) The coordinates are chosen such that the metric coefficients are dimensionless. This rules out terms like

$$\int \frac{\rho(x')}{|x-x'|^2}dx'$$

in g_{00}, for example.

A further restriction on the form of the metric in equation (16) is the choice of gauge. One can make an infinitesimal coordinate transformation

$$(x^i)^\dagger = x^i + \xi^i(x^k) . \tag{18}$$

Then the metric changes to

$$g^\dagger_{ij} = g_{ij} - \xi_{i,j} - \xi_{j,i} . \tag{19}$$

We have chosen ξ^1, ξ^2, ξ^3 in such a manner as to make $g^\dagger_{12} = g^\dagger_{13} = g^\dagger_{23} = 0$ to post-Newtonian accuracy. In other words, we have chosen that coordinate system in which the spatial part of the metric is diagonal. We are still free to choose ξ_0, however. For example, the choice

$$\xi_0 = \alpha(\partial/\partial t)\int \rho(x',t)|x-x'|dx' , \tag{20}$$

where α is a constant, and the resultant change of gauge

$$g^\dagger_{00} = g_{00} - 2\xi_{0,0} , \quad g^\dagger_{0j} = g_{0j} - \xi_{0,j} , \tag{21}$$

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will only transform the various parameters into each other. The above transformation (eq. [20]) causes the following changes in the values of the parameters:

$$(\Delta_1)^\dagger = \Delta_1 + \tfrac{2}{7}\alpha\,, \qquad \zeta^\dagger = \zeta + 2\alpha\,,$$
$$(\Delta_2)^\dagger = \Delta_2 - 2\alpha\,, \qquad \Sigma^\dagger = \Sigma - 2\alpha\,,$$
$$(\beta_1)^\dagger = \beta_1 + \tfrac{1}{2}\alpha\,, \qquad (\beta_4)^\dagger = \beta_4 + \tfrac{2}{3}\alpha\,. \tag{22}$$

This means that the individual parameters do not have direct physical significance, since they are coordinate-system dependent. Only certain linear combinations of the parameters may be measured. In this series of papers we shall always work in a *standard gauge*—that in which $\Sigma = 0$. Thus our PPN formalism will be characterized by nine parameters: β, β_1, β_2, β_3, β_4, γ, Δ_1, Δ_2, ζ. Equation (16) in the standard gauge is thus written

$$g_{00} = 1 - 2U + 2\beta U^2 - 4\Phi + \zeta\mathcal{A}\,, \qquad g_{0\alpha} = \tfrac{7}{2}\Delta_1 V_\alpha + \tfrac{1}{2}\Delta_2 W_\alpha\,,$$
$$g_{\alpha\beta} = -(1 + 2\gamma U)\delta_{\alpha\beta}\,. \tag{23}$$

The post-Newtonian limit of any metric theory of gravity can be expressed in a representation and gauge where the metric has the form (23). Only the values of the parameters will vary from theory to theory. The field equations and all gravitational fields except the metric, which go along with a particular theory, are automatically incorporated into the formalism by writing the metric in terms of volume integrals over the matter. No further reference to the field equations or other fields is needed. They now disappear from the PPN formalism.

Before deriving the equations of motion, we must first relate this perfect-fluid metric parametrization to Nordtvedt's point-particle formalism, and to Baierlein's formalism. In the limiting case of a fluid composed of point masses (for details, see e.g., Estabrook 1969), our PPN metric has the same form as Nordtvedt's (1969) point-mass metric. The relation between the PPN parameters and Nordtvedt's parameters can be seen by comparing the parameters in front of each term (Table 1). Similarly, when the PPN metric is written in terms of the "conserved density" (see below, eq. [28]), it has the

TABLE 1

RELATION BETWEEN NORDTVEDT'S PARAMETERS, BAIERLEIN'S PARAMETERS, AND THE PPN PARAMETERS*

PPN	Nordtvedt	Baierlein
β..........	β	β'
β_1..........	$\frac{1}{4}(4\alpha''+1)$	$\frac{1}{4}(3\beta'+1)$
β_2..........	$\frac{1}{2}(3\gamma-\alpha')$	$\frac{1}{2}(3\gamma-\beta')$
β_3..........	Absent	β'
β_4..........	Absent	β'
γ..........	γ	γ
Δ_1..........	$\frac{8}{7}\Delta$	$\frac{1}{7}(8\eta-\eta')$
Δ_2..........	$8\Delta'$	η'
Σ..........	$-\chi$	Absent
ζ..........	α'''	Absent

* The relationship between the PPN and the Nordtvedt parameters is obtained by taking the point-particle limit of the PPN formalism, in the manner of Estabrook (1969). The PPN-Baierlein relationship is obtained by expressing the PPN fluid metric in terms of the conserved density ρ^* (cf. eq. [28]).

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same form as Baierlein's (1967) perfect-fluid metric—except that Baierlein's lacks some terms that PPN contains. The correspondence between parameters is given in Table 1. Notice that the parameters β_3 and β_4 do not appear in Nordtvedt's point-particle approximation, since pressure p and energy density Π are zero for point masses which do not interact except through gravity.

We now calculate the components of T^{ij} using definition (14) and the metric (eq. [23]), and we also calculate the Christoffel symbols. To the order required in the post-Newtonian equations of motion,

$$T^{00} = \rho(1 + v^2 + 2U + \Pi)\,,$$

$$T^{0\alpha} = \rho v_\alpha(1 + v^2 + 2U + \Pi + p/\rho)\,,$$

$$T^{\alpha\beta} = \rho v_\alpha v_\beta(1+v^2 + 2U + \Pi + p/\rho) + p\delta_{\alpha\beta}(1 - 2\gamma U)\,,$$

$$\Gamma^0{}_{00} = -\,\partial U/\partial t\,, \qquad \Gamma^0{}_{0\alpha} = -\,\partial U/\partial x^\alpha\,,$$

$$\Gamma^0{}_{\alpha\beta} = \gamma\delta_{\alpha\beta}\partial U/\partial t + \tfrac{7}{4}\Delta_1(V_{\alpha,\beta} + V_{\beta,\alpha}) + \tfrac{1}{4}\Delta_2(W_{\alpha,\beta} + W_{\beta,\alpha})\,,$$

$$\Gamma^\alpha{}_{00} = -\,\partial U/\partial x_\alpha + (\partial/\partial x^\alpha)\{(\beta + \gamma)U^2 - 2\Phi + \tfrac{1}{2}\zeta\mathfrak{A}\}$$

$$-\,\tfrac{7}{2}\Delta_1\partial V_\alpha/\partial t - \tfrac{1}{2}\Delta_2\partial W_\alpha/\partial t\,,$$

$$\Gamma^\alpha{}_{0\beta} = \gamma\delta_{\alpha\beta}\partial U/\partial t - \tfrac{1}{4}(7\Delta_1 + \Delta_2)(V_{\alpha,\beta} - V_{\beta,\alpha})\,,$$

$$\Gamma^\alpha{}_{\beta\zeta} = \gamma(\delta_{\alpha\beta}\partial U/\partial x^\zeta + \delta_{\alpha\zeta}\partial U/\partial x^\beta - \delta_{\beta\zeta}\partial U/\partial x^\alpha)\,. \tag{24}$$

In these calculations we have used the fact, readily verifiable from equations (17), that $W_{\alpha,\beta} - W_{\beta,\alpha} = V_{\alpha,\beta} - V_{\beta,\alpha}$.

The equation of motion for the fluid (called the "response equation" in Paper I) is (cf. eq. [1])

$$T^{ij}{}_{,j} + \Gamma^i{}_{jk}T^{jk} + \Gamma^j{}_{kj}T^{ik} = 0\,. \tag{25}$$

The $i = 0$ equation reduces to

$$(\partial/\partial t)\{\rho(1 + v^2 + 2U + \Pi)\} + (\partial/\partial x^\alpha)\{\rho v^\alpha(1 + v^2 + 2U + \Pi + p/\rho)\}$$

$$+ (3\gamma - 2)\rho\partial U/\partial t + (3\gamma - 3)\rho v^\alpha\partial U/\partial x^\alpha = 0\,. \tag{26}$$

Following Chandrasekhar (1965), we simplify all post-Newtonian terms in the above using the Newtonian hydrodynamical equations (12). Doing this, and making use of the first law of thermodynamics,

$$\rho d\Pi/dt = (p/\rho)(d\rho/dt)\,, \tag{27}$$ [3]

we can rewrite equation (26) in the form of an equation of continuity for the so-called conserved density,

$$\rho^* = \rho(1 + \tfrac{1}{2}v^2 + 3\gamma U)\,, \tag{28}$$

namely,

$$\partial\rho^*/\partial t + \partial(\rho^* v^\alpha)/\partial x^\alpha = 0\,. \tag{29}$$

This conserved density is useful (as opposed to physically significant), because for any function f defined in a volume V whose boundary is outside the fluid,

$$(d/dt)\int_V \rho^* f dx = \int_V \rho^*(df/dt)dx\,. \tag{30}$$

[3] The derivative d/dt is the "convective derivative," or the rate of change of the quantity for an observer following the fluid as it moves along; $\partial/\partial t$ is the rate of change for an observer who is at rest relative to some external coordinate system. The relation between the two is given in equation (12).

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The $i = \alpha$ ($\alpha = 1, 2, 3$) equation of motion yields, after similar manipulation,

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\sigma v^\alpha) + \frac{\partial}{\partial x^\beta}(\sigma v^\alpha v^\beta) - \rho\frac{\partial U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}\{p[1 + (3\gamma - 1)U]\} \\
&\quad + \rho\frac{d}{dt}\{(5\gamma - 1)Uv^\alpha - \tfrac{1}{2}(7\Delta_1 + \Delta_2)V_\alpha\} - \tfrac{1}{2}\Delta_2\rho\frac{\partial}{\partial t}(W_\alpha - V_\alpha) \\
&\quad + \tfrac{1}{2}(7\Delta_1 + \Delta_2)\rho v^\beta\frac{\partial V_\beta}{\partial x^\alpha} + \tfrac{1}{2}\rho\zeta\frac{\partial \mathfrak{A}}{\partial x^\alpha} - 2\rho\frac{\partial\Phi}{\partial x^\alpha} \\
&\quad - \left[(\gamma + 1)v^2 + (3\gamma - 2\beta + 1)U + \Pi + 3\gamma\frac{p}{\rho}\right]\rho\frac{\partial U}{\partial x^\alpha} = 0\,, \qquad (31)
\end{aligned}
$$

where

$$\sigma = \rho(1 + v^2 + 2U + \Pi + p/\rho)\,. \qquad (32)$$

It will be convenient in this equation of motion to express σ in terms of ρ^*. Using Newtonian equations where appropriate, we get, to the desired post-Newtonian accuracy,

$$
\begin{aligned}
&\rho^*\frac{dv^\alpha}{dt} - \rho^*\frac{\partial U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}[p(1 + 3\gamma U)] - \frac{\partial p}{\partial x^\alpha}\left(\tfrac{1}{2}v^2 + \Pi + \frac{p}{\rho^*}\right) \\
&\quad + \rho^*\frac{d}{dt}[(2\gamma + 2)Uv^\alpha - \tfrac{1}{2}(7\Delta_1 + \Delta_2)V_\alpha] - v^\alpha\left(\rho^*\frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}\right) \\
&\quad - \tfrac{1}{2}\Delta_2\rho^*\frac{\partial}{\partial t}(W_\alpha - V_\alpha) + \tfrac{1}{2}(7\Delta_1 + \Delta_2)\rho^* v^\beta\frac{\partial V_\beta}{\partial x^\alpha} + \tfrac{1}{2}\rho^*\zeta\frac{\partial\mathfrak{A}}{\partial x^\alpha} \\
&\quad - 2\rho^*\frac{\partial\Phi}{\partial x^\alpha} - \rho^*\left(\gamma v^2 - (2\beta - 2)U + 3\gamma\frac{p}{\rho^*}\right)\frac{\partial U}{\partial x^\alpha} = 0\,. \qquad (33)
\end{aligned}
$$

Equations (29) and (33) are the required generalized perfect-fluid equations of motion. Taken together with equations (23) and (17) for the metric, and the law of geodesic motion for test bodies, they completely characterize each geometric theory of gravity at the post-Newtonian level.

III. CALCULATION OF THE GRAVITATIONAL-MASS TENSOR

In Newtonian physics, we write the equation of motion of a massive body being attracted by a distant point source of strength m_0 in the form (cf. eq. [2])

$$ma^\alpha = -m_0 m^{\alpha\beta}R^\beta/R^3\,, \qquad (34)$$

where R is the vector from the distant source to the Newtonian center of mass of the massive body, and

$$R^3 = (\Sigma_\beta R^\beta R^\beta)^{3/2}\,. \qquad (35)$$

To calculate $m^{\alpha\beta}$, we consider a density function $\rho^*(x)$ made up of a density distribution $\rho^v(x)$, localized in a volume V, and a distant point source of strength m_0 located at x_0:

$$\rho^*(x) = m_0\delta(x - x_0) + \rho^v(x, t)\,. \qquad (36)$$

The center of mass of the body will be defined by

$$Mx_c^\alpha = \int\rho^v(x, t)x^\alpha dx\,, \qquad (37)$$

$$M = \int\rho^v(x, t)dx\,. \qquad (38)$$

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This definition is somewhat arbitrary, since we need not have used the "conserved" density ρ^*. But it has the advantage that the velocity and acceleration of the center of mass are given by (cf. eq. [30])

$$Mv_c^\alpha = \int \rho^*(x, t)[dx^\alpha/dt]dx\,, \tag{39}$$

$$Ma^\alpha = \int \rho^*(x, t)[dv^\alpha/dt]dx\,. \tag{40}$$

Note that M is the total rest mass of the particles making up the massive body. From equations (38), (28), and (23), we get, neglecting post-Newtonian corrections,

$$M = \int \rho(1 + \tfrac{1}{2}v^2 + 3\gamma U)dx = \int [\rho u^0\sqrt{(-g)}]dx$$

$$= \int \rho d\ (\text{proper volume}) = \text{total rest mass of particles}\,. \tag{41}$$

This M is not equal to the inertial mass m (which in special relativity is the total mass-energy of the body), but can differ from it at most by terms of post-Newtonian order (such as internal energy $\rho\Pi$, internal kinetic energy, and internal self-gravitational energy). However, we have been careful to express both the acceleration $\rho^* dv^\alpha/dt$ and the Newtonian gravitational force $\rho^*\partial U/\partial x^\alpha$ in equation (33) in terms of the *same* conserved density ρ^* (or ρ^* when integrated over the volume containing the body). The fact that we will express the final Newtonian equation of motion (eqs. [65] and [80] below) in terms of inertial mass m instead of rest mass M will not change the final answer at all. Of course, in all post-Newtonian terms, m and M can be used interchangeably, since their difference will contribute terms of a higher order than those we are interested in.

We now consider the center-of-mass coordinate system for the massive-body-point-mass system. This is the frame in which the center-of-mass velocity of the entire system vanishes (cf. eq. [39]),

$$Mv^\alpha{}_{\text{total}} = \int_{\text{system}} \rho^*(x, t)v^\alpha dx = 0\,. \tag{42}$$

This frame can be obtained from our initial coordinate frame by a "post-Galilean" transformation of the form (Chandrasekhar and Contopoulos 1967)

$$x^{\alpha\prime} = x^\alpha - (1 + \tfrac{1}{2}u^2)u^\alpha t + \tfrac{1}{2}x\cdot u u^\alpha + q(x \times u)^\alpha$$

$$t' = t(1 + \tfrac{1}{2}u^2) - x\cdot u + (\text{other post-Newtonian terms}). \tag{43}$$

Here u and u^α denote the relative velocity between the two frames and u^α is equal to $-v^\alpha{}_{\text{total}}$ (cf. eq. [42]) of the system in the initial frame; and q is a measure of the rotational motion between the frames. The metric parameters appear in the "other post-Newtonian terms." Under this transformation, the metric and the equations of motion (eqs. [23], [29], and [33]) are invariant. (Notice that since this transformation leaves the metric unchanged, it does not affect our choice of gauge.)

We assume that the massive body and the point source are both momentarily at rest with respect to the center-of-mass coordinate system. By this we mean

$$v_c = 0\,, \quad \text{and} \quad v_0 = dx_0/dt = 0\,. \tag{44}$$

We then re-express the dv^α/dt of equation (40) in terms of the metric functions and thermodynamic variables by means of the equation of motion (33); and we expand our result in powers of $1/R$, keeping only the dominant R^{-2} terms. The sum of the coefficients appearing in front of such terms will give $m^{\alpha\beta}$.

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First, we evaluate explicitly the functions appearing in the metric. To the required order of approximation (cf. eqs. [17], [28], and [36],

$$U = \int \frac{\rho^*(x', t)}{|x - x'|}[1 - \tfrac{1}{2}v(x')^2 - 3\gamma U(x')]dx' = \frac{m_0}{|x - x_0|} + \int \frac{\rho^v(x', t)}{|x - x'|}dx'$$

$$- \tfrac{1}{2}\int \frac{\rho^v(x', t)v(x')^2}{|x - x'|}dx' - \frac{3\gamma m_0}{|x - x_0|}\int \frac{\rho^v(x', t)}{|x_0 - x'|}dx'$$

$$- 3\gamma m_0 \int \frac{\rho^v(x', t)dx'}{|x - x'||x' - x_0|} - 3\gamma \int\int \frac{\rho^v(x', t)\rho^v(x'', t)}{|x - x'||x' - x''|}dx'dx'' , \tag{45}$$

$$\Phi = \beta_1 \int \frac{\rho^v(x', t)v(x')^2}{|x - x'|}dx' + \frac{\beta_2 m_0}{|x - x_0|}\int \frac{\rho^v(x', t)}{|x_0 - x'|}dx'$$

$$+ \beta_2 m_0 \int \frac{\rho^v(x', t)dx'}{|x - x'||x' - x_0|} + \beta_2 \int\int \frac{\rho^v(x', t)\rho^v(x'', t)}{|x - x'||x' - x''|}dx'dx''$$

$$+ \tfrac{1}{2}\beta_3 \int \frac{\rho^v(x', t)\Pi(x')}{|x - x'|}dx' + \tfrac{3}{2}\beta_4 \int \frac{p(x')dx'}{|x - x'|} , \tag{46}$$

$$\mathfrak{A} = \int \frac{\rho^v(x', t)[v_\alpha(x')(x_\alpha - x'_\alpha)]^2}{|x - x'|^3}dx' , \tag{47}$$

$$V_\alpha = \int \frac{\rho^v(x', t)v_\alpha(x')}{|x - x'|}dx' , \tag{48}$$

$$W_\alpha = \int \frac{\rho^v(x', t)[v_\beta(x')(x_\beta - x'_\beta)](x_\alpha - x'_\alpha)}{|x - x'|^3}dx' . \tag{49}$$

We now evaluate the acceleration (eq. [40]) by integrating the equation of motion (33) over the volume of the massive body, replacing ρ^* everywhere by ρ^v since the external point mass is outside the region of integration. In the following computations, the arrow is used to signify that we have dropped all terms which decrease faster than R^{-2}, where

$$R_\alpha = x_{c\alpha} - x_{0\alpha} , \quad R = |x_c - x_0| , \tag{50}$$

and we have also dropped all terms which are constant, i.e., totally internal terms. To clarify this point, the second term in equation (33) will be discussed in detail. Using expression (45) for U, we find

$$\int \rho^v(x, t)\frac{\partial U}{\partial x^\alpha}dx \rightarrow -m_0 \int \frac{\rho^v(x, t)(x_\alpha - x_{0\alpha})}{|x - x_0|^3}dx$$

$$+ 3\gamma m_0 \int\int \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x'_\alpha)}{|x - x'|^3|x' - x_0|}dxdx' . \tag{51}$$

To first order, the first term in equation (51) varies as R^{-2}, and the second term as R^{-1}. Thus these terms are retained. We have neglected terms which vary as R^{-3}, such as

$$3\gamma m_0 \int\int \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x_{0\alpha})}{|x - x_0|^3|x' - x_0|}dxdx' ,$$

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and terms which do not depend on R at all ("internal terms"), such as

$$\tfrac{1}{2}\int \frac{\rho^v(x,t)\rho^v(x',t)v(x')^2(x_\alpha - x'_\alpha)}{|x-x'|^3}\,dxdx'$$

or

$$3\gamma\int\int\int \frac{\rho^v(x,t)\rho^v(x',t)\rho^v(x'',t)(x_\alpha - x'_\alpha)}{|x-x'|^3|x'-x''|}\,dxdx'dx''\,.$$

For the special case of spherically symmetric bodies, all the "internal" terms which appear in the acceleration (eq. [40]) are identically zero. For arbitrary bodies, however, they represent a constant post-Newtonian acceleration in some preferred direction—the spin axis of a rotating body, for instance. These terms arise because of the manner in which we have defined "center of mass" (cf. eqs. [37] and [38]). It can be shown that the momentum of the center of mass (as defined by eq. [39]) of an isolated system is not conserved, in contrast to what is expected in Newtonian physics. This violation involves just those "internal" terms discussed above. In general relativity and Brans-Dicke theory, it is possible to define a "conserved momentum" (cf. Chandrasekhar 1965; Nutku 1969), which involves $\rho^* v^\alpha$ (as in our definition) plus post-Newtonian corrections involving V_α, W_α, U, Π, v^2, and p. Then the momentum of the center of mass defined in this sense remains constant for an isolated system, and "internal terms" do not appear. However, this formulation is not used here because the generalization of the "conserved momentum" to arbitrary metric gravity theories is difficult, if not impossible. Also, since the "conserved momentum" is no longer directly proportional to v^α, the interpretation of the "inertial mass" becomes unclear.

Before the motions of the planets in the solar system can be calculated with confidence using the PPN formalism (with arbitrary values of the parameters), the "self-accelerations" due to the "internal terms" must be understood more clearly and explicitly. However, the self-accelerations will be small (of post-Newtonian order, diminished further by their essentially nonspherical nature). Therefore, they will just add linearly to the externally produced accelerations ($\propto R^{-2}$) calculated in this paper.

Henceforth, we will use definitions (39) and (40), and ignore constant "internal terms." Then, by definition, the first term in the integral of equation (33) over the volume V is Ma^α, where a^α is the acceleration of the center of mass. The remaining terms are

$$\int_V (\partial/\partial x^\alpha)\{p[1+3\gamma U]\}\,dx = 0 \tag{52}$$

since $p = 0$ at the surface of the body,

$$\int_V (\partial p/\partial x^\alpha)(\tfrac{1}{2}v^2 + \Pi + p/\rho^v)dx \to 0\,, \tag{53}$$

$$\begin{aligned}\int \rho^v \frac{d}{dt}(Uv^\alpha)dx \to &-m_0\int \frac{\rho^v(x,t)v^\alpha(x)[v_\beta(x)(x_\beta - x_{\beta 0})]}{|x-x_0|^3}\,dx \\ &- m_0\int\int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x'_\alpha)}{|x-x_0|\,|x-x'|^3}\,dxdx' - m_0\int_V \frac{\partial p(x)/\partial x^\alpha}{|x-x_0|}\,dx \\ &- m_0\int\int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x_{0\alpha})}{|x-x'|\,|x-x_0|^3}\,dxdx'\,,\end{aligned} \tag{54}$$

$$\int \rho^v \frac{dV_\alpha}{dt}\,dx \to -m_0\int\int \frac{\rho^v(x,t)\rho^v(x',t)(x'_\alpha - x_{0\alpha})}{|x-x'|\,|x'-x_0|^3}\,dxdx'\,, \tag{55}$$

$$\int \rho^v v^\alpha(\partial U/\partial t)dx \to 0\,, \tag{56}$$

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$$\int v^\alpha(\partial p/\partial t)dx \to 0\,, \tag{57}$$

$$\begin{aligned}\int \rho^v \frac{\partial}{\partial t}(W_\alpha - V_\alpha)dx \\ \to -m_0 \int\!\!\int \frac{\rho^v(x,t)\rho^v(x',t)(x'_\beta - x_{0\beta})(x_\beta - x'_\beta)(x_\alpha - x'_\alpha)}{|x-x'|^3|x'-x_0|^3}\,dxdx' \\ + m_0 \int\!\!\int \frac{\rho^v(x,t)\rho^v(x',t)(x'_\alpha - x_{0\alpha})}{|x-x'|\,|x'-x_0|^3}\,dxdx'\,,\end{aligned} \tag{58}$$

$$\int \rho^v v^\beta(\partial V_\beta/\partial x^\alpha)dx \to 0\,, \tag{59}$$

$$\int \rho^v \frac{\partial \mathfrak{A}}{\partial x^\alpha}dx \to 0\,, \tag{60}$$

$$\int \rho^v \frac{\partial \Phi}{\partial x^\alpha}dx \to -\beta_2 m_0 \int\!\!\int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x'_\alpha)}{|x-x'|^3|x'-x_0|}\,dxdx'\,, \tag{61}$$

$$\int \rho^v v^2 \frac{\partial U}{\partial x^\alpha}dx \to -m_0 \int \frac{\rho^v(x,t)v(x)^2(x_\alpha - x_{0\alpha})}{|x-x_0|^3}\,dx\,, \tag{62}$$

$$\begin{aligned}\int \rho^v U \frac{\partial U}{\partial x^\alpha}dx \to -m_0 \int\!\!\int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x'_\alpha)}{|x-x_0|\,|x-x'|^3}\,dxdx' \\ - m_0 \int\!\!\int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x_{0\alpha})}{|x-x_0|^3|x-x'|}\,dxdx'\,,\end{aligned} \tag{63}$$

$$\int p(x)\frac{\partial U}{\partial x^\alpha}dx \to -m_0 \int_V \frac{p(x)(x_\alpha - x_{0\alpha})}{|x-x_0|^3}\,dx\,. \tag{64}$$

Bringing these terms together, we get

$$\begin{aligned}Ma^\alpha + m_0 \int \frac{\rho^v(x,t)(x_\alpha - x_{0\alpha})}{|x-x_0|^3}\,dx \Big\{ 1 + (\tfrac{7}{2}\Delta_1 - 2\gamma - 2\beta)\int \frac{\rho^v(x',t)}{|x-x'|}\,dx' \\ + \gamma v(x)^2 + (\gamma - 2)p(x)/\rho^v(x,t) \Big\} \\ + m_0 \int \frac{\rho^v(x,t)\rho^v(x',t)(x_\alpha - x'_\alpha)}{|x-x'|^3|x'-x_0|}\,dxdx' \Big\{ (2\beta + 2\beta_2 - \gamma) \\ + \tfrac{1}{2}\Delta_2 \frac{(x_\beta - x'_\beta)(x'_\beta - x_{0\beta})}{|x'-x_0|^2} \Big\} \\ - (2\gamma + 2)m_0 \int_V \frac{\rho^v(x,t)v^\alpha(x)v_\beta(x)(x_\beta - x_{0\beta})}{|x-x_0|^3}\,dx = 0\,.\end{aligned} \tag{65}$$

Now,

$$\frac{1}{|x-x_0|} = \frac{1}{R} - \frac{R^\beta(x_\beta - x_{c\beta})}{R^3} + O\left(\frac{1}{R^3}\right), \tag{66}$$

$$\frac{1}{|x'-x_0|} = \frac{1}{R} - \frac{R^\beta(x'_\beta - x_{c\beta})}{R^3} + O\left(\frac{1}{R^3}\right), \tag{67}$$

$$\frac{x_\alpha - x_{0\alpha}}{|x-x_0|^3} = \frac{R^\alpha}{R^3} + O\left(\frac{1}{R^3}\right). \tag{68}$$

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Thus, retaining only terms which decrease as R^{-2} (all the R^{-1} terms are identically zero), and using the fact that

$$\iint \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x'_\alpha)(x'_\beta - x_{c\beta})}{|x - x'|^3} dxdx'$$

$$= \tfrac{1}{2}\iint \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x'_\alpha)(x'_\beta - x_\beta)}{|x - x'|^3} dxdx' , \tag{69}$$

we can transform equation (65) into the form

$$Ma^\alpha = -\frac{m_0 R^\alpha}{R^3}\Big\{M + (\tfrac{7}{2}\Delta_1 - 2\gamma - 2\beta)\iint \frac{\rho^v(x, t)\rho^v(x', t)}{|x - x'|} dxdx'$$

$$+ \gamma\int \rho^v(x, t)v(x)^2 dx + (\gamma - 2)\int_V p(x)dx\Big\}$$

$$- \tfrac{1}{2}(2\beta + 2\beta_2 - \gamma + \Delta_2)\frac{m_0 R^\beta}{R^3}\iint \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x - x'|^3} dxdx'$$

$$+ (2\gamma + 2)\frac{m_0 R^\beta}{R^3}\int \rho^v(x, t)v_\alpha(x)v_\beta(x')dx . \tag{70}$$

In order to simplify the post-Newtonian corrections in equation (70), we will use the Newtonian tensor virial theorem. By manipulating the Euler equations (cf. eq. [12]), with the "conserved" density ρ^v in the center-of-mass system of the body, we find

$$\rho^v \frac{d^2}{dt^2}(x_\alpha x_\beta) - 2\rho^v v_\alpha v_\beta = \rho^v\left(x^\alpha \frac{\partial U^v}{\partial x^\beta} + x^\beta \frac{\partial U^v}{\partial x^\alpha}\right) - \left(x_\alpha \frac{\partial p}{\partial x^\beta} + x_\beta \frac{\partial p}{\partial x^\alpha}\right), \tag{71}$$

where U^v is now the internal Newtonian gravitational potential. Integrating equation (71) over the volume of the body and making use of equation (30), we get the Newtonian tensor virial theorem (cf. Chandrasekhar 1964)

$$\tfrac{1}{2}d^2 I_{\alpha\beta}/dt^2 = 2\mathfrak{T}_{\alpha\beta} + \Omega_{\alpha\beta} + \delta_{\alpha\beta}P , \tag{72}$$

where the moment-of-inertia tensor, kinetic-energy tensor, and gravitational-potential tensor are defined, respectively, as

$$I_{\alpha\beta} = \int \rho^v(x, t)x_\alpha x_\beta dx , \tag{73}$$

$$\mathfrak{T}_{\alpha\beta} = \tfrac{1}{2}\int \rho^v(x, t)v_\alpha(x)v_\beta(x)dx , \tag{74}$$

$$\Omega_{\alpha\beta} = -\tfrac{1}{2}\iint \frac{\rho^v(x, t)\rho^v(x', t)(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x - x'|^3} dxdx' , \tag{75}$$

and where

$$P = \int_V p(x)dx . \tag{76}$$

For a body in static equilibrium, $I_{\alpha\beta}$ is constant, so the virial theorem becomes

$$0 = 2\mathfrak{T}_{\alpha\beta} + \Omega_{\alpha\beta} + \delta_{\alpha\beta}P . \tag{77}$$

For a body in which $I_{\alpha\beta}$ is periodic with a period small compared with the time required for the body to acquire a significant velocity toward the external point mass, the virial

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theorem can be averaged over a period of oscillation, with the same result. (The period is always sufficiently small for this if the body and the source are sufficiently well separated.)

Contracting equation (77) on α and β, we get the usual scalar virial theorem,

$$0 = 2\mathfrak{T} + \Omega + 3P\,, \tag{78}$$

where

$$\Omega = \Sigma_\alpha \Omega_{\alpha\alpha} = -\tfrac{1}{2}\int \frac{\rho^*(x,t)\rho^*(x',t)}{|x-x'|}\,dxdx'\,. \tag{79}$$

By using the virial theorems (77) and (78) in the equation of motion (70), and by making use of the definition (34) of the gravitational-mass tensor $m^{\alpha\beta}$, we finally obtain for $m^{\alpha\beta}$:

$$m^{\alpha\beta}/m = \delta^{\alpha\beta}\{1-(7\Delta_1-3\gamma-4\beta)(\Omega/m)\} - \{2\beta+2\beta_2-3\gamma+\Delta_2-2\}(\Omega^{\alpha\beta}/m)\,, \tag{80}$$

where we have replaced M by m in the post-Newtonian correction terms (cf. eq. [41] and the discussion following it).

This result is a very general one. It applies to bodies which obey a broad class of equations of state, and which have arbitrary macroscopic interior fluid motions, including convection and rotation. The only restrictions are that, to the desired degree of accuracy, the body be made up of perfect fluid and have negligible radiation transport, and that the moment of inertia be, at most, periodic in time. This derivation and result have the advantage of including the effects of radiation and equation of state in a realistic and straightforward way. The breakdown of the equivalence principle is seen to depend only on the gravitational-potential tensor, for that class of metric theories described by the metric of equation (23).

IV. THE MASS TENSOR IN GENERAL RELATIVITY AND IN BRANS-DICKE THEORY

We will now specialize to general relativity and Brans-Dicke theory. Comparing the metric of equation (23) with the post-Newtonian fluid metrics obtained by Chandrasekhar (1965) for general relativity and by Nutku (1969) for Brans-Dicke theory, we find the values for the ten parameters given in Table 2. These particular values are not unique; they depend on the specific gauge chosen by Chandrasekhar and Nutku—which

TABLE 2

VALUES OF THE METRIC PARAMETERS

Parameter	General Relativity	Brans-Dicke
β	1	1
β_1	1	$(3+2\omega)/(4+2\omega)$
β_2	1	$(1+2\omega)/(4+2\omega)$
β_3	1	1
β_4	1	$(1+\omega)/(2+\omega)$
γ	1	$(1+\omega)/(2+\omega)$
Δ_1	1	$(10+7\omega)/(14+7\omega)$
Δ_2	1	1
ζ	0	0
Σ*	0	0

* In the standard gauge of this series of papers, Σ vanishes identically, independently of the theory.

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happens to agree, in both cases, with our "standard gauge." In another gauge (for example, that used by Estabrook 1969) the parameters take on different values, but the linear combinations of parameters which would then appear in expression (80) would be numerically the same.

In general relativity and Brans-Dicke theory, we have

$$2\beta + 2\beta_2 - 3\gamma + \Delta_2 - 2 = 0 . \tag{81}$$

This result means that anisotropic terms proportional to

$$R^\beta \Omega^{\alpha\beta}/R^3$$

will not appear in the Newtonian acceleration of the massive body. Such terms would cause Newtonian accelerations transverse to the line from the center of mass of the body to the external source. In general relativity,

$$7\Delta_1 - 3\gamma - 4\beta = 0 ; \tag{82}$$

so

$$m^{\alpha\beta} = m\delta^{\alpha\beta} , \tag{83}$$

as expected. In Brans-Dicke theory,

$$7\Delta_1 - 3\gamma - 4\beta = -1/(2 + \omega) ; \tag{84}$$

so

$$m^{\alpha\beta} = m\delta^{\alpha\beta}\{1 - [1/(2 + \omega)](|\Omega|/m)\} . \tag{85}$$

This agrees with the general result found by using the method of Dicke (1969) (see Appendix), and with Nordtvedt's result for spherical bodies without radiation (Nordtvedt 1968*b*, 1969). By including radiation pressure in massive bodies, Nordtvedt finds an extra term, $\Delta m^{\alpha\beta}$, in $m^{\alpha\beta}$:

$$\Delta m^{\alpha\beta} = m\delta^{\alpha\beta}[\tfrac{1}{3}(1 - \gamma)E_r/m] , \tag{86}$$

where E_r is the total radiation energy in the body. In general relativity,

$$1 - \gamma = 0 ; \tag{87}$$

and in Brans-Dicke theory

$$1 - \gamma = 1/(2 + \omega) . \tag{88}$$

The source of this erroneous term is, presumably, the artificial way by which Nordtvedt tries to introduce the effects of radiation pressure into his pressureless gas of noninteracting point particles. Our hydrodynamical formalism treats this probem in a more straightforward and rigorous way.

V. CONCLUSIONS

In the post-Newtonian approximation, massive, self-gravitating bodies were found to violate the equivalence principle in arbitrary metric theories of gravity, with the exception of general relativity. In Brans-Dicke theory, the breakdown in the equivalence principle (Nordtvedt effect) was seen to depend only on the body's internal gravitational-potential energy.

The calculations in this paper were restricted to massive bodies which, like the source of the external gravitational field, are momentarily at rest with respect to the coordinate system. Including the motion of such bodies, as in the solar system, might introduce further violations of the equivalence principle.

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APPENDIX

HEURISTIC CALCULATION OF THE GRAVITATIONAL-MASS TENSOR IN BRANS-DICKE THEORY

From the point of view of Dicke (1969), massive bodies violate the equivalence principle because the gravitational "constant," G, is a function of position in space, which causes the internal energy of a body to depend on its location.

Consider the following *gedankenexperiment*. Take n particles, each with rest mass μ, and create from them a bound massive body centered on a point at height h in an external field $U(h)$. There is a release of binding energy $E_B(h)$ in the process of formation. Raise the massive body a distance δh. The force on the body is

$$F = [n\mu - E_B(h)]a \, , \tag{A1}$$

where $n\mu - E_B(h)$ is the inertial mass (total energy) of the body and a is the acceleration it feels. The work done on the body is

$$E(\mathrm{up}) = -F\delta h = -[n\mu - E_B(h)]a\delta h + O(\delta h^2) \, . \tag{A2}$$

At the top of the cycle, the body is pulled apart. The energy $E_B(h + \delta h)$ required to do this is obtained by converting some of the particles into energy. The particles are then lowered one by one back down to the starting point. Since each particle is a "test body," it experiences an acceleration g which is the gradient of the external field U. So the energy retrieved from the body is

$$E(\mathrm{down}) = [n\mu - E_B(h)]g\delta h + O(\delta h^2) \, . \tag{A3}$$

The cycle is closed since we are now left with $n - E_B(h)/\mu$ particles plus the energy $E_B(h)$ released when we first created the massive body. Energy balance thus demands

$$[n\mu - E_B(h)](a - g) = -dE_B/dh \, . \tag{A4}$$

Denoting the inertial mass by m, we find that the force on the massive body is

$$ma = mg - dE_B/dh \, . \tag{A5}$$

Consider raising the body a distance δh, keeping its internal state fixed. Then the only change in binding energy is that due to the change in gravitational internal energy E_g, which is produced by the Brans-Dicke change in G. After the body has been raised, it is out of equilibrium, so it begins to pulsate; damping of the pulsations generates enough heat to keep the total non-gravitational internal energy constant, despite the readjustment of density and pressure due to the change in G. Hence, the only change in binding energy is the change in gravitational internal energy:

$$dE_B/dh = dE_g/dh \, . \tag{A6}$$

The final result for the force on the body (eq. [A5]) is, thus,

$$ma_\alpha = mU_{,\alpha} - \partial E_g/\partial x^\alpha \, . \tag{A7}$$

Since (cf. eq. [8])

$$E_g = \Omega\{1 - U/(2 + \omega)\} \, , \tag{A8}$$

then

$$ma_\alpha = U_{,\alpha}\{m + \Omega/(2 + \omega)\} \tag{A9}$$

and (cf. definition [2])

$$m^{\alpha\beta} = m\delta^{\alpha\beta}\{1 - [1/(2 + \omega)]|\Omega|/m\} \, . \tag{A10}$$

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The *gedankenexperiment* which leads to this result was first introduced by Dicke (1969); but the reasoning leading to the crucial equation (A6), and thence to the validity of equation (A10) for *any* massive body, is due to Kip S. Thorne (private communication).

REFERENCES

Baierlein, R. 1967, *Phys. Rev.*, **162**, 1275.
Brans, C., and Dicke, R. H. 1961, *Phys. Rev.*, **124**, 925.
Chandrasekhar, S. 1964, *Lectures in Theoretical Physics* (Boulder: University of Colorado Press) **6**, 1.
———. 1965, *Ap. J.*, **142**, 1488.
Chandrasekhar, S., and Contopoulos, G. 1967, *Proc. Roy. Soc.*, **298A**, 123.
Dicke, R. H. 1962, *Phys. Rev.*, **125**, 2163.
———. 1969, *Gravitation and the Universe*, Jayne Lectures for 1969 (Philadelphia: American Philosophical Society).
Estabrook, F. E. 1969, *Ap. J.*, **158**, 81.
Jeffreys, Sir H. 1959, *The Earth* (4th ed.; Cambridge: Cambridge University Press), p. 210.
Nordtvedt, K., Jr. 1968*a*, *Phys. Rev.*, **169**, 1014.
———. 1968*b*, *ibid.*, **169**, 1017.
———. 1968*c*, *ibid.*, **170**, 1186.
———. 1969, *ibid.*, **180**, 1293.
———. 1970, *Icarus*, **12**, 1.
Nutku, Y. 1969, *Ap. J.*, **155**, 999.
Roll, P. G., Krotkov, R., and Dicke, R. H. 1964, *Ann. Phys.* (*N.Y.*), **26**, 442.
Thorne, K. S., and Will, C. M. 1970, *Comments on Astrophysics and Space Physics*, **2**, 35.
———. 1971, *Ap. J.*, **163**, 595 (Paper I).

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4. Theoretical Frameworks for Testing Relativistic Gravity
III. Conservation Laws, Lorentz Invariance, and Values of the PPN Parameters

(Submitted in May, 1971, for publication in The Astrophysical Journal)

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I. INTRODUCTION

In Papers I and II of this series [Thorne and Will (1971) and Will (1971a), hereafter referred to as WII] the foundations for the Parametrized Post-Newtonian (PPN) framework were laid out and discussed. This framework treats the post-Newtonian limit of arbitrary metric theories of gravity in terms of nine metric parameters -- γ, β, β_1, β_2, β_3, β_4, Δ_1, Δ_2, ζ -- whose values vary from theory to theory. Hand in hand with the parametrized metric goes the equation of motion of stressed matter

$$T^{ij}{}_{;j} = 0 \quad , \tag{1}$$

where T^{ij} is the stress-energy tensor for matter and non-gravitational fields.

The PPN framework gives a complete description of gravity and of the response of stressed matter to gravity at the post-Newtonian level, once one chooses a set of values for the PPN parameters (see WII for the key formulas, eqs. [17], [23], [24], [25]). In this paper we will show that only a restricted set of PPN parameter values will lead to a description of gravity which has integral conservation laws at the post-Newtonian level. In particular, we will demand that the PPN perfect fluid equations of motion lead to conservation laws for energy, momentum, and angular momentum, as well as uniform motion of the center of mass; and we will show that the PPN parameters must then satisfy the following constraints:

$$\begin{aligned} \beta_1 &= \tfrac{1}{2}(\gamma + 1) , & \Delta_1 &= \tfrac{1}{7}(4\gamma + 3) , \\ \beta_2 &= \tfrac{1}{2}(3\gamma - 2\beta + 1) , & \Delta_2 &= 1 , \\ \beta_3 &= 1 , & \zeta &= 0 , \\ \beta_4 &= \gamma . \end{aligned} \tag{2}$$

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Thus the demand that conservation laws exist reduces the number of arbitrary parameters in the PPN formalism from nine to two -- for convenience we have chosen β and γ.

The constraints of equation (2) are satisfied by general relativity and Brans-Dicke Theory (WII); by generalized scalar-tensor theories (Nordtvedt 1970) and by Nordström's first and second theories (Ni 1971). They are not satisfied by scalar-metric theories with a "Universal time coordinate," including Einstein's, Whitrow and Morduch's, Yilmaz's, and Ni's [see Ni (1971) for discussion and references for these theories].

A second set of weaker PPN parameter constraints related to three of those in equations (2) can be derived by imposing "asymptotic Lorentz invariance" on metric theories of gravity. By demanding that the PPN metric for a moving point mass be the Lorentz transformation of the metric for a static point mass, and by requiring that the gravitational fields in the PPN metric be properly retarded (i.e., that gravity travel with the speed of light in flat, empty spacetime), Nordtvedt (1969), using his version of the PPN formalism, has obtained the constraints (when translated from his parameters into ours):

$$\begin{aligned} \Delta_2 + \zeta - 1 &= 0 \quad , \\ 7\Delta_1 + \Delta_2 &= 4\gamma + 4 \quad , \\ 4\beta_1 &= 2\gamma + 2 + \zeta \; . \end{aligned} \tag{3}$$

In this paper we will confirm and generalize Nordtvedt's result using the full PPN fluid metric rather than specializing to a static point mass, and using the formalism of <u>Post-Galilean Transformations</u> introduced by Chandrasekhar and Contopoulos (1967). We will prove that <u>the nine-parameter</u>

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PPN metric is invariant under a post-Galilean transformation (a transformation which reduces to a Lorentz transformation far from the matter) if and only if equations (3) are satisfied. We also discuss the physical significance of this invariance.

Any theory which is "conservative", i.e., which satisfies the conservation law constraints, is also asymptotically Lorentz invariant (compare eqs. [2] and [3]). We will show explicitly that the 10 conserved quantities of such a "conservative" theory behave as a four-vector and a second-rank antisymmetric tensor under Lorentz transformations.

We will also show that "conservative" theories of gravity predict equality of the active and passive gravitational masses of massive self-gravitating bodies. In Newtonian physics, equality of active and passive gravitational mass ("action equals reaction") goes hand in hand with conservation of momentum and uniform center-of-mass motion. We will thus show that this connection between conservation laws and "action equals reaction" holds at the post-Newtonian level as well.

The constraints on the PPN parameters which we will derive in this paper are theoretical constraints -- they do not tell us which theory is the correct theory of gravity. Only experiment can do that. Indeed, the constraints in equations (2) and (3) should be subjected to experimental test in order to confirm the existence of conservation laws and the validity of Lorentz invariance on astronomical scales where gravity is present. Several such experimental tests already exist: the author has recently used gravimeter data on the tides of the solid Earth to put an upper limit on a possible anisotropy in the Newtonian gravitational constant (Will 1971b), and has shown that the parameter combination $(\Delta_2 + \zeta - 1)$ should be zero to

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within about 3 percent. Thorne, Will, and Ni (1971) have discussed ways to put limits on β_1, β_3, and β_4 by using laboratory measurements of the equality of active and passive gravitational mass (Kreuzer 1968). Other possible tests of the constraints derived in this paper are currently under investigation.

In Table 1 we present a list of experimental tests of relativistic gravity and their PPN parameter dependence. We show the values predicted for these parameter combinations by theories of gravity with conservation laws and by theories of gravity which are asymptotically Lorentz invariant (but which might not have conservation laws).

In §II we derive the conservation-law constraints (eqs. [2]) using the equations of motion (eq. [1]) for matter in the PPN formalism. In §III the constraint equations (3) are obtained using post-Galilean transformations, and in §IV the transformation properties of the 10 conserved quantities are determined. We make concluding remarks in §V. Appendix A presents a discussion of the reasoning which leads us to five of the conservation-law constraints; Appendix B gives a derivation of the active gravitational mass of a massive self-gravitating body.

II. CONSERVATION LAWS AND PARAMETER CONSTRAINTS DERIVED FROM PPN EQUATIONS OF MOTION

It is well known [see, for example, Landau and Lifshitz (1962)] that integral conservation laws cannot be obtained directly from the equation of

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motion for stressed matter,

$$T^{ij}{}_{;j} = 0, \qquad (4)^{1}$$

[1] In this paper Greek indices will take the values 1, 2, and 3; Roman indices will take the values 0, 1, 2, 3; and summation over repeated indices will be employed. Commas will denote partial differentiation, and semi-colons will denote covariant differentiation. We use units for which the velocity of light is unity and the Newtonian gravitational constant in the outer regions of the solar system, today is unity. Square brackets enclosing indices will denote antisymmetrization and round brackets will denote symmetrization.

where T^{ij} is the stress-energy tensor for matter and non-gravitational fields, because of the presence of the Christoffel symbols in the covariant derivative. Rather, one searches for a quantity Θ^{ij} which reduces to T^{ij} in flat spacetime, and whose ordinary divergence is zero, i.e.,

$$\Theta^{ij}{}_{,j} = 0. \qquad (5)$$

Then, providing Θ^{ij} is symmetric, one finds that the quantities

$$P^{i} = \int_{\Sigma} \Theta^{ij}\, d\Sigma_{j} \quad , \qquad J^{ij} = 2 \int_{\Sigma} x^{[i}\Theta^{j]k}\, d\Sigma_{k} \, , \qquad (6)$$

are conserved, i.e., the integrals in equation (6) vanish when taken over a closed 3-dimensional hypersurface Σ. If one chooses a coordinate system in which Σ is a constant-time hypersurface and extends infinitely far in all directions, then P^{i} and J^{ij} are independent of time, and are given by

$$P^{i} = \int \Theta^{io}\, d\underset{\sim}{x} \quad , \qquad J^{ij} = 2 \int x^{[i}\Theta^{j]o}\, d\underset{\sim}{x} \, , \qquad (7)$$

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where $d\underline{x}$ is a volume element of ordinary three-dimensional space. An appropriate choice of Θ^{ij} allows one to interpret the components of P^i and J^{ij} in the usual way: as measured in the asymptotically flat spacetime far from the matter, P^0 is the total energy, P^α is the total momentum, $J^{\alpha\beta}$ is the total angular momentum and $J^{0\alpha}$ determines the motion of the center of mass of the matter.

The quantity Θ^{ij}, normally called the stress-energy complex, has been found for the exact versions of general relativity (Landau and Lifshitz 1962) and of Brans-Dicke theory (Dykla 1971). It has also been explicitly calculated in the post-Newtonian and post-post-Newtonian approximations of general relativity (Chandrasekhar 1969; Chandrasekhar and Nutku 1969). (A wide variety of non-symmetric stress-energy complexes have been found for general relativity, but only the symmetric version guarantees conservation of angular momentum.) In this section we will show that such a symmetric stress-energy complex can be found for the PPN formalism (and hence that conserved quantities exist) only if the parameter constraints of equations (2) are satisfied.

We now proceed to determine Θ^{ij}.

The PPN metric is given by

$$
\begin{aligned}
g_{00} &= 1 - 2U + 2\beta U^2 - 4\Phi + \zeta\mathcal{A} , \\
g_{0\alpha} &= \tfrac{7}{2}\Delta_1 V_\alpha + \tfrac{1}{2}\Delta_2 W_\alpha , \\
g_{\alpha\beta} &= -(1 + 2\gamma U)\,\delta_{\alpha\beta} ,
\end{aligned}
\tag{8}
$$

where

$$\Phi = \beta_1\Phi_1 + \beta_2\Phi_2 + \tfrac{1}{2}\beta_3\Phi_3 + \tfrac{3}{2}\beta_4\Phi_4 . \tag{9}$$

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Here U is the Newtonian gravitational potential. For explicit definitions of the functions appearing in the metric the reader is referred to WII, equation (17). We also state the following definitions: ρ is the rest-mass density of fluid; $\rho\Pi$ is the density of radiation energy, compressional energy, thermal energy, etc.; p is the pressure and $\underset{\sim}{v}$ is the fluid velocity.

The most general possible form for Θ^{ij} which reduces to T^{ij} in flat spacetime (negligible gravitational fields), and which is accurate to post-Newtonian order, is

$$\Theta^{ij} = (1 - \alpha U)(T^{ij} + t^{ij}), \tag{10}$$

where α is a constant (to be determined), and t^{ij} is a quantity (which may be interpreted as gravitational stress-energy) which vanishes in flat spacetime, and which is a function of the fields U, Φ, $\mathcal{A}$, V_α, and W_α and their derivatives (and may also contain the matter variables ρ, Π, p, and $\underset{\sim}{v}$). We reject terms in Θ^{ij} of the form

$$v^2T^{ij} \quad , \quad \Pi T^{ij} \quad , \quad (p/\rho)T^{ij} \; ,$$

since such terms do not vanish for arbitrary distributions of stressed matter in regions of negligible gravitational field.

By combining equations (4), (5), and (10) we find that t^{ij} must satisfy (to post-Newtonian order)

$$t^{ij}{}_{,j} - \alpha U_{,j}t^{ij} = \Gamma^i{}_{jk}T^{jk} + \Gamma^j{}_{kj}T^{ik} + \alpha U_{,j}T^{ij} \; . \tag{11}$$

In order to solve equation (11) for t^{ij} we will use the following equations (which are equivalent to the definitions for the metric functions given in WII) to express matter variables in terms of field quantities:

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$$\nabla^2 U = -4\pi\rho \quad , \quad \nabla^2 V_\alpha = -4\pi\rho v_\alpha \;,$$
$$\nabla^2 \Phi_1 = -4\pi\rho v^2 \quad , \quad \nabla^2 \Phi_2 = -4\pi\rho U \;, \tag{12}$$
$$\nabla^2 \Phi_3 = -4\pi\rho\Pi \quad , \quad \nabla^2 \Phi_4 = -4\pi p \;,$$
$$V_{\alpha,\alpha} = -U_{,0} \;;$$

and we will use the following identity, which is valid for any function f:

$$4\pi\rho f_{,\alpha} = -2(\partial/\partial x_\beta)(U_{,(\alpha}f_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}f_{,\gamma}) + U_{,\alpha}\nabla^2 f. \tag{13}$$

We substitute into equation (11) the formulas for T^{ij} and for the PPN Christoffel symbols given in WII (eq. [24]), and use equations (12) and (13) to obtain (to post-Newtonian order) for $i = 0$,

$$\begin{aligned} 4\pi t^{oj}{}_{,j} &= (\partial/\partial t)\,[\tfrac{1}{2}\,(6\gamma + 2\alpha - 5)\;|\nabla U|^2] \\ &\quad - (\partial/\partial x^\beta)[(3\gamma+\alpha-2)U_{,\beta}U_{,0} + (3\gamma+\alpha-3)U_{,\gamma}(V_{\gamma,\beta} - V_{\beta,\gamma})]. \end{aligned} \tag{14}$$

Equation (14) can be integrated directly (making use of the condition that t^{ij} vanish in flat spacetime) to yield

$$t^{00} = (8\pi)^{-1}(6\gamma + 2\alpha - 5)\;|\nabla U|^2 \;, \tag{15}$$

$$t^{0\alpha} = -(4\pi)^{-1}[(3\gamma+\alpha-2)U_{,\alpha}U_{,0} + (3\gamma+\alpha-3)U_{,\gamma}(V_{\gamma,\alpha} - V_{\alpha,\gamma})]. \tag{16}$$

An expression for the conserved energy can be obtained using equations (7), (10), and (15). The result is (after an integration by parts):

$$P^0 = \int \rho^*(1 + \tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi)\,d\mathbf{x} \;, \tag{17}$$

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where ρ^* is the so-called "conserved" density, given by

$$\partial\rho^*/\partial t + \nabla\cdot(\rho^* \mathbf{v}) = 0 \quad ,$$
$$\rho^* = \rho(1 + \tfrac{1}{2}v^2 + 3\gamma U) \ , \tag{18}$$

(WII, eqs. [28] and [29]). The first term in equation (17) is the total conserved rest-mass of particles in the fluid (WII, eq. [41]). The other terms in equation (17) are the total kinetic, gravitational, and internal energies in the fluid, whose sum is conserved according to Newtonian theory (which can be used in any post-Newtonian terms). Thus P^0 is simply the total mass-energy of the fluid. So far we have found nothing new. Equation (17) for P^0 can be found directly using the PPN equation of continuity (eq. [18]) and Newtonian theory.

For $i = \alpha$, we must first compute $t^{\alpha\beta}$ to Newtonian order. Equation (11) yields

$$4\pi t^{\alpha\beta}{}_{,\beta} = (\partial/\partial x^\beta)\ (U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\,\delta_{\alpha\beta}\, U_{,\gamma}U_{,\gamma}) \ , \tag{19}$$

from which we obtain the standard Newtonian result (Chandrasekhar 1969)

$$t^{\alpha\beta}{}_{\text{Newtonian}} = (4\pi)^{-1}\ (U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\,\delta_{\alpha\beta}\, U_{,\gamma}U_{,\gamma}) . \tag{20}$$

This Newtonian approximation for $t^{\alpha\beta}$ can now be used to simplify all post-Newtonian terms in equation (11). We obtain after a lengthy calculation,

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the post-Newtonian equation for $t^{\alpha j}$:

$$
\begin{aligned}
4\pi t^{\alpha j}{}_{,j} = (\partial/\partial t) \Big[& \tfrac{1}{2}(7\Delta_1 - \Delta_2) U_{,\alpha} U_{,0} + \tfrac{1}{2}(7\Delta_1 + \Delta_2) U_{,\gamma}(V_{\gamma,\alpha} - V_{\alpha,\gamma}) \\
& + (5\gamma + \alpha - 1) U_{,\gamma} V_{\alpha,\gamma} - (5\gamma + \alpha - 1)(\partial/\partial x^{\gamma})(U V_{\alpha,\gamma}) \Big] \\
& + (\partial/\partial x^{\beta}) \Big\{ [1 - \tfrac{1}{2}(4\beta + 4\beta_2 - \gamma - \alpha - 3) U](U_{,\alpha} U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta} U_{,\gamma} U_{,\gamma}) \\
& + \Big[U_{,(\alpha}(4\Phi - \zeta\mathcal{A})_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta} U_{,\gamma}(4\Phi - \zeta\mathcal{A})_{,\gamma} \Big] \\
& + \Big[U_{,(\alpha}(7\Delta_1 V_{\beta)} + \Delta_2 W_{\beta)})_{,0} - \tfrac{1}{2}\delta_{\alpha\beta} U_{,\gamma}(7\Delta_1 V_{\gamma} + \Delta_2 W_{\gamma})_{,0} \Big] \\
& - 2(7\Delta_1 + \Delta_2) \Big[V_{[\alpha,\gamma]} V_{[\beta,\gamma]} - \tfrac{1}{4}\delta_{\alpha\beta} V_{[\gamma,\delta]} V_{[\gamma,\delta]} \Big] \\
& - \tfrac{1}{4}(7\Delta_1 - \Delta_2)\delta_{\alpha\beta}(U_{,0})^2 + (5\gamma + \alpha - 1) U(\rho v^{\alpha} v^{\beta} + \delta^{\alpha\beta} p) \Big\} \\
& + 4\pi Q^{\alpha},
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
Q^{\alpha} = (\partial U/\partial x^{\alpha}) \Big[& (2\beta_1 - \gamma - 1)\rho v^2 + (8\pi)^{-1}(2\beta + 2\beta_2 - 3\gamma - 1)|\underset{\sim}{\nabla} U|^2 \\
& + (\beta_3 - 1)\rho\Pi + 3(\beta_4 - \gamma) p + (8\pi)^{-1}\zeta v^2 \mathcal{A} \Big] .
\end{aligned}
\tag{22}
$$

The term Q^{α} can <u>not</u> in general be written as a combination of gradients and time derivatives of fluid quantities and gravitational fields -- or so

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we believe. (We have been unable to develop a completely rigorous proof; but strong arguments that this is so are given in Appendix A.) Therefore, in order for $t^{\alpha j}$ to have a form which involves only matter and gravitational field variables and their derivatives, each parameter combination in Q^{α} (eq. [22]) must vanish separately, i.e., the parameters must satisfy

$$\begin{aligned} \beta_1 &= \tfrac{1}{2}(\gamma + 1) , \\ \beta_2 &= \tfrac{1}{2}(3\gamma - 2\beta + 1) , \\ \beta_3 &= 1 , \\ \beta_4 &= \gamma , \\ \zeta &= 0 . \end{aligned} \tag{23}$$

We have thus obtained five of the seven conditions of equations (2). These conditions can also be obtained using Chandrasekhar's (1965) technique which consists of integrating the hydrodynamic equations of motion over all space and calculating a conserved momentum P^{α}. Using the PPN formalism, the corresponding result is

$$(d/dt) \int \rho^* \left\{ v^{\alpha} [1 + \tfrac{1}{2} v^2 + (2\gamma + 1)U + \Pi + p/\rho] - \tfrac{1}{2}(7\Delta_1 V_{\alpha} + \Delta_2 W_{\alpha}) \right\} d\underset{\sim}{x} + \int Q^{\alpha} \, d\underset{\sim}{x} = 0. \tag{24}$$

The second term in equation (24) can be written as a total time derivative of an integral over all space, only if Q^{α} can be written as a combination of time derivatives, and spatial divergences (which lead to surface integrals at infinity that vanish). But according to the reasoning given in Appendix A, this can only be true if the five parameter constraints of equation (23) are

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satisfied. Then $Q^{\alpha} \equiv 0$, and we have a conserved momentum.

We now demand that t^{ij} be symmetric. The $t^{\alpha\beta}$ part of equation (21) is manifestly symmetric. Thus we learn nothing new -- the five conditions of equation (23) are necessary and sufficient to guarantee conservation of spatial angular momentum $J^{\alpha\beta}$. This can also be shown using the method of Chandrasekhar. It is the symmetry of $t^{0\alpha}$, i.e., uniform motion of the center of mass, which leads to the final two constraints. Comparing $t^{0\alpha}$ of equation (16) with $t^{\alpha 0}$ of equation (21) we find the conditions

$$\begin{aligned} \alpha &= 1 - 5\gamma \, , \\ 7\Delta_1 &= 4\gamma + 3 \, , \\ \Delta_2 &= 1 \, , \end{aligned} \tag{25}$$

which complete the list of parameter constraints of equations (2). Using these constraints along with equations (10), (15), (16), and (21), we obtain for t^{ij} and θ^{ij}:

$$t^{00} = -(8\pi)^{-1}(4\gamma+3)\,|\nabla U|^2 \, , \tag{26}$$

$$t^{0\alpha} = t^{\alpha 0} = (4\pi)^{-1}\Big[(2\gamma+1)\,U_{,\alpha}U_{,0} + (2\gamma+2)\,U_{,\beta}(V_{\beta,\alpha} - V_{\alpha,\beta})\Big] \, , \tag{27}$$

$$\begin{aligned} t^{\alpha\beta} = (4\pi)^{-1}\Big\{&\Big[1 - (5\gamma-1)U\Big]\left(U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}\right) \\ &+ 4\Big[U_{,(\alpha}\Phi_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}\Phi_{,\gamma}\Big] \\ &+ (4\gamma+3)\Big[U_{,(\alpha}V_{\beta),0} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}V_{\gamma,0}\Big] \\ &+ \Big[U_{,(\alpha}W_{\beta),0} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}W_{\gamma,0}\Big] \\ &- 8(\gamma+1)\Big[V_{[\alpha,\gamma]}V_{[\beta,\gamma]} - \tfrac{1}{4}\delta_{\alpha\beta}V_{[\gamma,\delta]}V_{[\gamma,\delta]}\Big] \\ &- \tfrac{1}{2}(2\gamma+1)\,\delta_{\alpha\beta}\,(U_{,0})^2\Big\} \, , \end{aligned} \tag{28}$$

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$$\theta^{ij} = [1 + (5\gamma - 1)\, U]\, (T^{ij} + t^{ij}). \tag{29}$$

Chandrasekhar (1969) and Chandrasekhar and Nutku (1969) have computed t^{ij} in the post-Newtonian limit of general relativity using the Landau-Lifshitz (1962) symmetric energy-momentum pseudo tensor. Their results are in complete agreement with equations (26), (27), (28), and (29), if we substitute the general-relativity parameter values $\gamma = \beta = 1$. Chandrasekhar and Nutku found that a careful derivation of the post-Newtonian version of $t^{\alpha\beta}$ from the exact Landau-Lifshitz version required knowledge of the post-post-Newtonian Christoffel symbols. They then found that the contributions of these higher-order terms to the post-Newtonian $t^{\alpha\beta}$ exactly cancelled (as they must). Our derivation of $t^{\alpha\beta}$ using the equations of motion <u>did not require any knowledge whatsoever</u> of higher corrections to the PPN Christoffel symbols, and thus confirms this cancellation.

In Brans-Dicke theory, Dykla (1971) has found an exact, symmetric energy-momentum pseudo tensor analogous to that of Landau and Lifshitz. Applying the post-Newtonian Christoffel symbols for Brans-Dicke theory (WII eq. [24], plus the Brans-Dicke parameter values) to Dykla's formula for t^{ij}, and assuming that the contributions of post-post-Newtonian Christoffel symbols cancel, one can show that the resulting expressions agree with equations (26), (27), (28), and (29).

Finally, we use equations (26), (27), (28), and (29) along with equation (7), to obtain expressions for the conserved quantities:

$$P^0 = \int \rho^* \left(1 + \tfrac{1}{2} v^2 - \tfrac{1}{2} U + \Pi\right) d\underset{\sim}{x}\,, \tag{30}$$

$$P^\alpha = \int \rho^* \left\{ v^\alpha \left[1 + \tfrac{1}{2} v^2 + (2\gamma + 1)\, U + \Pi + p/\rho\right] - \tfrac{1}{2}(4\gamma + 3) V^\alpha - \tfrac{1}{2} W^\alpha \right\} d\underset{\sim}{x}\,, \tag{31}$$

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$$J^{\alpha\beta} = 2\int \rho^* x^{[\alpha}\left\{v^{\beta]}\left[1+\tfrac{1}{2}v^2+(2\gamma+1)U+\Pi+p/\rho\right]-\tfrac{1}{2}(4\gamma+3)V^{\beta]}-\tfrac{1}{2}W^{\beta]}\right\}d\underset{\sim}{x}, \tag{32}$$

$$J^{\alpha 0} = \int \rho^* x^{\alpha}(1+\tfrac{1}{2}v^2-\tfrac{1}{2}U+\Pi)\,d\underset{\sim}{x} - P^{\alpha}t. \tag{33}$$

In general relativity and Brans-Dicke theory, these expressions agree with results obtained by Chandrasekhar (1965), Fock (1964), and Nutku (1969). By defining a center of mass X^{α}, given by

$$X^{\alpha} = \frac{\int \rho^* x^{\alpha}(1+\tfrac{1}{2}v^2-\tfrac{1}{2}U+\Pi)\,d\underset{\sim}{x}}{\int \rho^*(1+\tfrac{1}{2}v^2-\tfrac{1}{2}U+\Pi)\,d\underset{\sim}{x}}, \tag{34}$$

we find from equations (30) and (33) and the constancy of $J^{\alpha 0}$, that

$$dX^{\alpha}/dt = P^{\alpha}/P^0 , \tag{35}$$

i.e., the center of mass moves uniformly with velocity P^{α}/P^0.

In Newtonian gravitational theory, this uniform center-of-mass motion is a result of the law "action equals reaction", i.e., of the law "active gravitational mass equals passive gravitational mass." In the PPN formalism, one can still use such Newtonian language to describe the post-Newtonian motions of massive self-gravitating bodies in their mutual $1/(\text{separation})^2$ gravitational fields. The passive gravitational mass is a tensor given by (see Nordtvedt 1968, Will 1971a)

$$m_P^{\alpha\beta} = m\delta^{\alpha\beta}\left\{1-(7\Delta_1-3\gamma-4\beta)(\Omega/m)\right\} - (2\beta+2\beta_2-3\gamma+\Delta_2-2)\,\Omega^{\alpha\beta} , \tag{36}$$

where m is the body's total mass energy, given by (cf. eq. [17])

$$\begin{aligned} m &= \int \rho^*(1+\tfrac{1}{2}v^2-\tfrac{1}{2}U+\Pi) \\ &= M + E_{kin} + \Omega + E_{int} \end{aligned} \tag{37}$$

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Here, M is the total rest mass of particles, E_{kin} is the total kinetic energy, Ω and $\Omega^{\alpha\beta}$ are the body's self-gravitational energy and energy tensor respectively, and E_{int} is the internal energy. Similarly one has, for the active gravitational mass [Nordtvedt (1969); see also Appendix B]

$$m_a = m + 2(2\beta_1 - \beta_4 - 1)E_{kin} + (6\gamma - 4\beta_2 - \beta_4 - 1)\Omega + (\beta_3 - 1)E_{int} - \zeta\, E_{kin}^{\alpha\beta}\, e_\alpha e_\beta \,, \tag{38}$$

where $E_{kin}^{\alpha\beta}$ is the body's kinetic energy tensor, and $\underset{\sim}{e}$ is a unit vector joining the massive body to the field point at which its field is being measured.

Substituting the conservation-law parameter constraints (eq. [2]) into equations (36) and (38), we find that for conservative theories the two masses are isotropic and equal, and are given by

$$m_a = m_p = m\,[1 + (4\beta - \gamma - 3)(\Omega/m)] \,. \tag{39}$$

III. POST-GALILEAN TRANSFORMATIONS, LORENTZ INVARIANCE, AND PPN PARAMETER CONSTRAINTS

In this section we will prove that <u>the post-Newtonian metric of any theory of gravity is invariant under a post-Galilean transformation if and only if the PPN parameters for that theory satisfy three constraints</u>:

$$\begin{aligned} \Delta_2 + \zeta - 1 &= 0 \,, \\ 7\Delta_1 + \Delta_2 &= 4\gamma + 4 \,, \\ 4\beta_1 &= 2\gamma + 2 + \zeta \,. \end{aligned} \tag{40}$$

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A post-Galilean transformation (see Chandrasekhar and Contopoulos 1967) is a coordinate transformation which (i) reduces to a Lorentz transformation of velocity $|\underline{u}| \ll 1$ in the asymptotically flat region of spacetime far from the matter generating the metric, and (ii) preserves the post-Newtonian "gauge" being used -- i.e., preserves the post-Newtonian equations in their standard form.

Before proving the theorem in detail, we must first discuss its physical meaning. Consider two observers who set out to calculate the metric due to the same given distribution of perfect fluid, using the same particular theory of gravity. Each sets up a global coordinate system which satisfies the standard post-Newtonian gauge conditions, and which becomes inertial asymptotically at very large distances from the fluid; thus the two coordinate systems are related by what we have called a post-Galilean transformation. Each observer then uses the variables of the perfect fluid (density, velocity, etc.) as determined in his own coordinate system, along with the machinery of the metric theory of gravity, to compute a metric. The observers then compare their results. This theorem states that their results will be physically equivalent if and only if they used a theory whose PPN parameters satisfy equations (40). Such a theory is "asymptotically Lorentz invariant" in the sense that physics, including the generation of the metric by the matter is independent of the velocity of the (asymptotically Lorentz) frame in which it is calculated. Examples of such theories are general relativity, Brans-Dicke theory, and Nordstrom's theories (which in fact are more than just "asymptotically Lorentz invariant" -- they are "generally" covariant).

A theory which did not satisfy the conditions of the theorem

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(i.e., whose PPN parameters violated some of eqs. [40]), would therefore predict different physical results for computations of the metric in each asymptotic Lorentz frame. Such a theory would have to pick out some preferred reference frame (the rest frame of some cosmological "ether" for instance) in which the "correct" metric was to be calculated. Examples of such theories are a class of scalar-metric theories which contain a universal time coordinate, devised by Einstein, Yilmaz, Ni, and Whitrow and Morduch [see Ni (1971) for discussion and references]; another example is a theory which predicts different flat-space speeds for gravity and for light, given in Will (1971b). These theories -- like any theory (cf. §III of Thorne and Will 1971) -- can be written in a generally covariant form; but in the preferred reference frame their equations are particularly simple; they thus assign strong physical significance to this simplicity.

We obtain the form of the post-Galilean transformation by expanding the usual Lorentz transformation in powers of velocity u, assuming u^2 is of Newtonian order,[2] i.e., $\underset{\sim}{u}$ is O(1), u^2 is O(2), and so on. We then generalize

---

[2] For a discussion of the process of assigning Newtonian and post-Newtonian "orders" to various terms in the expansion see Chandrasekhar (1965). In our notation the Newtonian potential $U(\underset{\sim}{x})$ is O(2), velocity $\underset{\sim}{v}$ is O(1), time derivatives d/dt are O(1) and so on.

---

the resulting transformation (to take into account gravitation-induced deviations from perfect Lorentz invariance) using arbitrary functions. For a transformation from coordinates $(\underset{\sim}{x},t)$ to coordinates $(\underset{\sim}{\xi},\tau)$, the post-Galilean

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transformation has the form (Chandrasekhar and Contopoulos 1967):

$$\begin{aligned} \underset{\sim}{x} &= \underset{\sim}{\xi} - (1 + \tfrac{1}{2}u^2)\,\underset{\sim}{u}\tau + \tfrac{1}{2}(\underset{\sim}{\xi}\cdot\underset{\sim}{u})\underset{\sim}{u} + \underset{\sim}{Y}(\underset{\sim}{\xi},\tau), \\ t &= \tau\,(1 + \tfrac{1}{2}u^2 + \tfrac{3}{8}u^4) - (1 + \tfrac{1}{2}u^2)\,\underset{\sim}{\xi}\cdot\underset{\sim}{u} + Z(\underset{\sim}{\xi},\tau) + f(\underset{\sim}{\xi},\tau) \end{aligned} \qquad (41)$$

where $\underset{\sim}{Y}$ is $O(2)$, Z is $O(1)$, and f is $O(3)$ [we have assumed that $\underset{\sim}{u}\tau$ is $O(0)$]. This transformation must reduce asymptotically to a Lorentz transformation far from the matter, i.e., $\underset{\sim}{Y}$ Z and f must be bounded functions of $\underset{\sim}{\xi}$, or must satisfy

$$\frac{|\underset{\sim}{Y}|}{|\underset{\sim}{\xi}|} \to 0 \quad , \quad \frac{Z}{|\underset{\sim}{\xi}|} \to 0 \quad , \quad \frac{f}{|\underset{\sim}{\xi}|} \to 0 \quad , \quad \text{as } |\underset{\sim}{\xi}| \to \infty . \qquad (42)$$

Since equations (41) represent an asymptotic Lorentz transformation, with velocity $\underset{\sim}{u}$, the outer region of the $(\underset{\sim}{x},t)$ frame moves with velocity $\underset{\sim}{u}$ with respect to the outer region of the $(\underset{\sim}{\xi},\tau)$ frame, and conversely the $(\underset{\sim}{\xi},\tau)$ outer region moves with velocity $-\underset{\sim}{u}$ relative to the $(\underset{\sim}{x},t)$ outer region. This leads to the conditions

$$\begin{aligned} (d/d\tau)\,\underset{\sim}{Y}(\underset{\sim}{a} + \underset{\sim}{u}\tau,\tau) &= 0 \ , \\ (d/d\tau)\,\underset{\sim}{Y}(\underset{\sim}{a},\tau) &= 0 \ , \end{aligned} \qquad (43)$$

where $\underset{\sim}{a}$ is any constant vector in the outer region of the $(\underset{\sim}{\xi},\tau)$ frame.

We now apply this transformation to the PPN metric, equation (8), and demand that, in the new coordinates $(\underset{\sim}{\xi},\tau)$ the metric have exactly the same functional form as it had in the old coordinates $(\underset{\sim}{x},t)$. We use the standard transformation law ($x^0 = t$, $\xi^0 = \tau$):

$$g_{ij}(\underset{\sim}{\xi},\tau) = \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^\ell}{\partial \xi^j}\, g_{k\ell}(\underset{\sim}{x},t) \ . \qquad (44)$$

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We must also express the functions (fields) which appear in $g_{kl}(\underset{\sim}{x},t)$ in terms of the new coordinates. We note that since the density ρ^* is conserved (cf. eqs. [18]), then for any element of fluid,

$$\rho^*(\underset{\sim}{x},t)d\underset{\sim}{x} = \rho^*(\underset{\sim}{\xi},t)d\underset{\sim}{\xi} \tag{45}$$

= amount of rest mass in the corresponding volume elements $d\underset{\sim}{x}$ and $d\underset{\sim}{\xi}$.

If $\underset{\sim}{v}(\underset{\sim}{x},t)$ and $\underset{\sim}{\nu}(\underset{\sim}{\xi},\tau)$ are the fluid velocities in the two coordinate systems, they are related by

$$\underset{\sim}{v} = \underset{\sim}{\nu} - \underset{\sim}{u} - \tfrac{1}{2}u^2\underset{\sim}{\nu} + \underset{\sim}{\nu}\cdot\underset{\sim}{u}\,(\underset{\sim}{\nu} - \tfrac{1}{2}\underset{\sim}{u}) + (\underset{\sim}{\xi}\cdot\underset{\sim}{u})\,d\underset{\sim}{\nu}/d\tau. \tag{46}$$

Also, because they appear only in post-Newtonian terms, we can write

$$\begin{aligned} p(\underset{\sim}{x},t) &= p(\underset{\sim}{\xi},\tau) + O(4)\ , \\ \Pi(\underset{\sim}{x},t) &= \Pi(\underset{\sim}{\xi},\tau) + O(4)\ . \end{aligned} \tag{47}$$

We make use of a formula given by Chandrasekhar and Contopoulos (1967), namely

$$\begin{aligned} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{1}{|\underset{\sim}{\xi}-\underset{\sim}{\xi}'|}\Big\{1 &+ \tfrac{1}{2}(\underset{\sim}{n}'\cdot\underset{\sim}{u})^2 - (\underset{\sim}{n}'\cdot\underset{\sim}{u})(\underset{\sim}{n}'\cdot\underset{\sim}{\nu}') \\ &- \frac{\underset{\sim}{n}'}{|\underset{\sim}{\xi}-\underset{\sim}{\xi}'|}\cdot[(\underset{\sim}{Y}-\underset{\sim}{Y}') - (\underset{\sim}{\nu}'-\underset{\sim}{u})(Z-Z')]\Big\} + O(4), \end{aligned} \tag{48}$$

where

$$\underset{\sim}{n}' = (\underset{\sim}{\xi}-\underset{\sim}{\xi}')/|\underset{\sim}{\xi}-\underset{\sim}{\xi}'|\ . \tag{49}$$

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We define the potential $\chi(\xi,\tau)$ to be

$$\chi(\xi,\tau) = -\int \rho^*(\xi',\tau)\,|\xi-\xi'|\,d\xi' \,. \tag{50}$$

Then χ has the property that

$$\chi_{,\alpha\beta} = -\delta_{\alpha\beta}\,U(\xi,\tau) + \int \frac{\rho^*(\xi',\tau)n'^{\alpha}n'^{\beta}}{|\xi-\xi'|}\,d\xi' \,. \tag{51}$$

We then find using equations (45), (46), (47), (48), (51), along with the definitions of the metric functions (WII, eq. [17]), that

$$\begin{aligned} U(x,t) = {} & U(\xi,\tau) + u^{\alpha}V_{\alpha}(\xi,\tau) - u^{\alpha}W_{\alpha}(\xi,\tau) + \tfrac{1}{2}u^{\alpha}u^{\beta}\chi_{,\alpha\beta}(\xi,\tau) \\ & - \int \frac{\rho^*(\xi',\tau)d\xi'}{|\xi-\xi'|^2}\,n'\cdot[(Y-Y') - (v'-u)(Z-Z')] + O(6), \end{aligned} \tag{52}$$

$$\Phi(x,t) = \Phi(\xi,\tau) - 2\beta_1 u^{\alpha}V_{\alpha}(\xi,\tau) + \beta_1 u^2 U(\xi,\tau) + O(6), \tag{53}$$

$$\mathfrak{a}(x,t) = \mathfrak{a}(\xi,\tau) - 2u^{\alpha}W_{\alpha}(\xi,\tau) + u^2U(\xi,\tau) + u^{\alpha}u^{\beta}\chi_{,\alpha\beta}(\xi,\tau) + O(6), \tag{54}$$

$$V_{\alpha}(x,t) = V_{\alpha}(\xi,\tau) - u_{\alpha}U(\xi,\tau) + O(5), \tag{55}$$

$$W_{\alpha}(x,t) = W_{\alpha}(\xi,\tau) - u_{\alpha}U(\xi,\tau) - u^{\beta}\chi_{,\alpha\beta}(\xi,\tau) + O(5). \tag{56}$$

Applying the transformation equation (41) to the PPN metric equation (8), and making use of equations (44), (52), (53), (54), (55), and (56), we

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obtain for the metric in the $(\underset{\sim}{\xi},\tau)$ system, to post-Newtonian order,

$$
\begin{aligned}
g_{00}(\underset{\sim}{\xi},\tau) = {} & 1 - 2U(\underset{\sim}{\xi},\tau) + 2\beta U(\underset{\sim}{\xi},\tau)^2 - 4\Phi(\underset{\sim}{\xi},\tau) + \zeta\,\mathcal{A}(\underset{\sim}{\xi},\tau) \\
& + 2\,Z_{,0} \\
& + \{2\,f_{,0} + Z_{,0}(u^2 - 4U + Z_{,0}) + (7\Delta_1 + \Delta_2 - 2\gamma - 4\beta_1 - 2 + \zeta)u^2 U \\
& + (8\beta_1 - 7\Delta_1 - 2)u^{\alpha}V_{\alpha} - (\Delta_2 + 2\zeta - 2)u^{\alpha}W_{\alpha} + (\Delta_2 + \zeta - 1)u^{\alpha}u^{\beta}\,\chi_{,\alpha\beta} \\
& + 2\int \frac{\rho^*(\underset{\sim}{\xi}',\tau)d\underset{\sim}{\xi}'}{|\underset{\sim}{\xi} - \underset{\sim}{\xi}'|^2}\,\underset{\sim}{n}'\cdot[(\underset{\sim}{Y} - \underset{\sim}{Y}') - (\underset{\sim}{v}' - \underset{\sim}{u})(Z - Z')] + 2\underset{\sim}{u}\cdot\underset{\sim}{Y}_{,0}\},
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
g_{0\alpha}(\underset{\sim}{\xi},\tau) = {} & \tfrac{7}{2}\Delta_1 V_{\alpha}(\underset{\sim}{\xi},\tau) + \tfrac{1}{2}\Delta_2 W_{\alpha}(\underset{\sim}{\xi},\tau) \\
& + Z_{,\alpha} \\
& + \{f_{,\alpha} + Z_{,\alpha}(\tfrac{1}{2}u^2 - 2U) - u^{\alpha}Z_{,0} - Y^{\alpha}{}_{,0} + u^{\beta}Y_{\beta,\alpha} \\
& + \tfrac{1}{2}(4\gamma + 4 - 7\Delta_1 - \Delta_2)u^{\alpha}U - \tfrac{1}{2}\Delta_2 u^{\beta}\chi_{,\alpha\beta}\},
\end{aligned}
\tag{58}
$$

$$
g_{\alpha\beta}(\underset{\sim}{\xi},\tau) = -\,\delta_{\alpha\beta}\,[1 + 2\gamma U(\underset{\sim}{\xi},\tau)] - Y_{\alpha,\beta} - Y_{\beta,\alpha}\,. \tag{59}
$$

We demand that the transformed metric be of the same functional form as the untransformed metric. Then we must have [from eqs. (57) and (58), since $Z_{,0}$ is $O(2)$ and $Z_{,\alpha}$ is $O(1)$],

$$
Z_{,0} = 0, \text{ and } Z_{,\alpha} = 0. \tag{60}
$$

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Thus Z must be a constant, and since we have not specified our origin of time, it can be set equal to zero; without loss of generality

$$Z(\underset{\sim}{\xi},\tau) = 0. \tag{61}$$

In equation (59) we require

$$Y_{\alpha,\beta} + Y_{\beta,\alpha} = 0. \tag{62}$$

Thus $\underset{\sim}{Y}$ must be of the form

$$\underset{\sim}{Y} = \underset{\sim}{A}(\tau) + \underset{\sim}{B}(\tau) \times \underset{\sim}{\xi} \tag{63}$$

But conditions (42) and (43) imply $\underset{\sim}{A}(\tau) = \underset{\sim}{B}(\tau) = 0$, whence

$$\underset{\sim}{Y}(\underset{\sim}{\xi},\tau) = 0. \tag{64}$$

(Note the condition $\underset{\sim}{B} = 0$ means that our Lorentz transformations are pure boosts -- they contain no rotations.)

We obtain the remaining conditions from equations (57) and (58), using equations (61) and (64):

$$\begin{aligned} 0 = 2f_{,0} &+ (7\Delta_1 + \Delta_2 - 2\gamma - 4\beta_1 - 2 + \zeta)u^2U + (8\beta_1 - 7\Delta_1 - 2)u^\alpha V_\alpha \\ &- (\Delta_2 + 2\zeta - 2)u^\alpha W_\alpha + (\Delta_2 + \zeta - 1)u^\alpha u^\beta \chi_{,\alpha\beta} , \end{aligned} \tag{65}$$

$$0 = f_{,\alpha} + \tfrac{1}{2}(4\gamma + 4 - 7\Delta_1 - \Delta_2)u^\alpha U - \tfrac{1}{2}\Delta_2 u^\beta \chi_{,\alpha\beta} . \tag{66}$$

Equations (65) and (66) can be solved for the function $f(\underset{\sim}{\xi},\tau)$ (and hence a post-Galilean transformation exists for the metric), <u>if and only if f satisfies the integrability conditions</u>

$$f_{,[ij]} = 0. \tag{67}$$

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From equation (66) we have

$$f_{,[\alpha\beta]} = -\tfrac{1}{2}(4\gamma+4-7\Delta_1-\Delta_2)\, u_{[\alpha}U_{,\beta]} = 0 \; . \tag{68}$$

For this equation to hold for arbitrary physical systems and for arbitrary velocities $\underset{\sim}{u}$, we must have

$$7\Delta_1 + \Delta_2 = 4\gamma + 4. \tag{69}$$

From equations (65) and (66), the integrability condition leads to

$$\begin{aligned} f_{,[0\alpha]} = -\tfrac{1}{4}\Big[&(2\gamma + 2 + \zeta - 4\beta_1)(u^2U_{,\alpha} - 2u^{\beta}V_{\beta,\alpha}) \\ &+ (\Delta_2 + \zeta - 1)(u^{\beta}u^{\gamma}\,\chi_{,\alpha\beta\gamma} + 2u^{\beta}\,\chi_{,0\alpha\beta})\Big] = 0. \end{aligned} \tag{70}$$

Where we have made use of equation (69) where possible, and have used the relation (easily obtained from eq. [50])

$$\chi_{,0\alpha} = V_{\alpha} - W_{\alpha} \; . \tag{71}$$

For equation (70) to hold for arbitrary physical systems and for arbitrary velocities $\underset{\sim}{u}$, we must have

$$4\beta_1 = 2\gamma + 2 + \zeta \; , \tag{72}$$

$$\Delta_2 + \zeta - 1 = 0 \; . \tag{73}$$

We now solve equations (65) and (66) for f, using the parameter constraints, equations (69), (72), and (73), with the result

$$f(\underset{\sim}{\xi},\tau) = \tfrac{1}{2}\Delta_2 u^{\beta}\,\chi_{,\beta} + C, \tag{74}$$

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where C is a constant. Far from the matter, $\chi_{,\beta}$ takes the form

$$\chi_{,\beta} = -\int \frac{\rho^*(\underset{\sim}{\xi}',\tau)(\xi-\xi')_\beta}{|\underset{\sim}{\xi}-\underset{\sim}{\xi}'|}\, d\underset{\sim}{\xi}' \simeq -\frac{\xi_\beta}{|\underset{\sim}{\xi}|}\int \rho^*(\underset{\sim}{\xi}',\tau)\,d\underset{\sim}{\xi}' = -\frac{M\xi_\beta}{|\underset{\sim}{\xi}|}\,. \tag{75}$$

Thus f satisfies condition (42):

$$f/|\underset{\sim}{\xi}| \to -\tfrac{1}{2}\Delta_2 M(\underset{\sim}{u}\cdot\underset{\sim}{\xi})/|\underset{\sim}{\xi}|^2 + C/|\underset{\sim}{\xi}| \to 0, \quad \text{as } |\underset{\sim}{\xi}| \to \infty. \tag{76}$$

The constant C is arbitrary, and we can set it equal to zero by appropriately redefining our zero of time. Then the post-Galilean transformation takes the form

$$\begin{aligned} \underset{\sim}{x} &= \underset{\sim}{\xi} - (1+\tfrac{1}{2}u^2)\,\underset{\sim}{u}\tau + \tfrac{1}{2}(\underset{\sim}{\xi}\cdot\underset{\sim}{u})\,\underset{\sim}{u}\;, \\ t &= \tau\,(1+\tfrac{1}{2}u^2+\tfrac{3}{8}u^4) - (1+\tfrac{1}{2}u^2)\,\underset{\sim}{\xi}\cdot\underset{\sim}{u} + \tfrac{1}{2}\Delta_2 u^\beta\,\chi_{,\beta}(\underset{\sim}{\xi},\tau). \end{aligned} \tag{77}$$

If we had used a gauge for the PPN metric in which $\Delta_2 = 0$, the term in equation (77) involving $\chi(\underset{\sim}{\xi},\tau)$ would not appear, and our post-Galilean transformation would be identical to a Lorentz transformation.[a] The infinitesimal

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[a] I thank Yavuz Nutku for pointing this out to me.

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gauge transformation which leads to $\Delta_2 = 0$ has no physical significance, and changes only the form of the final post-Galilean transformation; the existence of a post-Galilean transformation for a given theory is not affected at all by the choice of gauge. This can be easily seen by repeating the calculations of this section using an arbitrary PPN gauge instead of the "standard" gauge (WII, eq. [23]). In an arbitrary gauge, the PPN metric differs from the

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metric in the standard gauge only in the g_{00} metric term (WII, eq. [16]):

$$g_{00}\left[\begin{smallmatrix}\text{arbitrary}\\ \text{gauge}\end{smallmatrix}\right] = g_{00}\left[\begin{smallmatrix}\text{standard}\\ \text{gauge}\end{smallmatrix}\right] + \Sigma\int \frac{\rho(\underset{\sim}{x}',t)\rho(\underset{\sim}{x}'',t)(\underset{\sim}{x}-\underset{\sim}{x}')\cdot(\underset{\sim}{x}'-\underset{\sim}{x}'')}{|\underset{\sim}{x}-\underset{\sim}{x}'||\underset{\sim}{x}'-\underset{\sim}{x}''|^3}\, d\underset{\sim}{x}'\, d\underset{\sim}{x}'', \tag{78}$$

where Σ is a "tenth" PPN parameter. But this additional term is not affected at all, to post-Newtonian order, by the post-Galilean transformation equation (41). Thus the calculation goes through as if this term were absent, and we obtain the same parameter constraints as before, independent of gauge.

IV. TRANSFORMATION PROPERTIES OF THE CONSERVED QUANTITIES

Any metric theory of gravity which possesses conservation laws, i.e., whose PPN parameters satisfy equations (2), automatically satisfies the conditions for post-Galilean invariance, equations (3). We can therefore use the post-Galilean transformation derived in the preceding section to determine the transformation properties of the conserved quantities of such "conservative" theories. We apply equations (77) to the integral formulas in equations (30), (31), (32), and (33), and take into account equations (45), (46), and (47). Denoting the conserved integrals in the $(\underset{\sim}{\xi},\tau)$ coordinate system by primes, we obtain to post-Newtonian order,

$$P^0 = P^{0'}(1 + \tfrac{1}{2}u^2) - \underset{\sim}{u}\cdot P' \ , \tag{79}$$

$$\underset{\sim}{P} = \underset{\sim}{P}' - (1 + \tfrac{1}{2}\underset{\sim}{u}^2)\underset{\sim}{u}P^{0'} + \tfrac{1}{2}\underset{\sim}{u}(\underset{\sim}{u}\cdot\underset{\sim}{P}') \ , \tag{80}$$

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$$J^{\alpha\beta} = J^{\alpha\beta'} - J^{\gamma[\alpha'}u^{\beta]}u^{\gamma} + 2(1 + \tfrac{1}{2}u^2)\, J^{0[\alpha'}u^{\beta]} \;, \tag{81}$$

$$J^{0\alpha} = J^{0\alpha'}\,(1 + \tfrac{1}{2}u^2) - u^{\gamma}J^{\alpha\gamma'} - \tfrac{1}{2}\, u^{\alpha}u^{\beta}J^{\beta 0'} \;. \tag{82}$$

Thus P^i and J^{ij} transform under a Lorentz transformation as a vector and an antisymmetric tensor, just as they do in special relativity (note that the gauge-dependent term involving χ in eq. [77] did not appear anywhere in this analysis -- its order was too high).

A Lorentz transformation of velocity $\underline{u} = \underline{P}'/P^{0'}$, leads to a frame in which

$$\begin{aligned} \underline{P} &= 0 \;, \\ P^0 &= P^{0'} - \tfrac{1}{2}\,\underline{P}\cdot\underline{P}/P^0 \approx [(P^{0'})^2 - \underline{P}'\cdot\underline{P}']^{\frac{1}{2}} \\ &= (\eta_{ij}\,P^i P^j)^{\frac{1}{2}} = m \;. \end{aligned} \tag{83}$$

In this frame -- the "rest" frame of the matter -- we also have

$$J^{0\alpha} = \int \rho^* x^{\alpha}(1 + \tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi)\, d\underline{x} \;. \tag{84}$$

By an appropriate redefinition of the origin: $y^{\alpha} = x^{\alpha} - a^{\alpha}$ where

$$a^{\alpha} = \frac{\int \rho^* x^{\alpha}(1 + \frac{1}{2}v^2 - \frac{1}{2}U + \Pi)\, d\underline{x}}{\int \rho^*(1 + \frac{1}{2}v^2 - \frac{1}{2}U + \Pi)\, d\underline{x}} \;, \tag{85}$$

we can set $J^{0\alpha} = 0$. In this frame we also denote $J^{\alpha\beta}$ by $S^{\alpha\beta}$, the "intrinsic" angular momentum of the matter. This frame is normally called the center-of-mass frame:

$$\underline{P} = 0, \quad P^0 = m, \quad J^{0\alpha} = 0, \quad J^{\alpha\beta} = S^{\alpha\beta}. \tag{86}$$

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In any other (asymptotically Lorentz) frame whose outer region moves with velocity-$\underset{\sim}{u}$ with respect to the center-of-mass frame, one gets the standard special relativistic result:

$$\begin{aligned} P^0 &= \gamma m , \qquad \underset{\sim}{P} = \gamma m \underset{\sim}{u} , \\ J^{\alpha\beta} &= (\delta^{\gamma[\beta} + u^\gamma u^{[\beta})S^{\alpha]\gamma} , \\ J^{\alpha 0} &= u^\gamma S^{\alpha\gamma} , \end{aligned} \tag{87}$$

where

$$\gamma = (1-u^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}u^2 + O(4) . \tag{88}$$

Notice that in this "moving" frame, the center of mass is obtained from the Lorentz invariant equation

$$P_j J^{ij} = 0 , \tag{89}$$

and is given by (eqs. [33], [34], [87], and [89])

$$X^\alpha = (P^0)^{-1} u^\gamma S^{\alpha\gamma} + u^\alpha \tau . \tag{90}$$

The asymptotic form of the metric of a "conservative" theory can be written, to lowest order, in terms of conserved quantities. Far from the matter, we can approximate:

$$\begin{aligned} 1/|\underset{\sim}{x} - \underset{\sim}{x}'| &\approx 1/R + \underset{\sim}{x}' \cdot \underset{\sim}{R}/R^3 , \\ \frac{(x-x')^\alpha (x-x')^\beta}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} &\approx \frac{R^\alpha R^\beta}{R^3} + \frac{3(\underset{\sim}{x}' \cdot \underset{\sim}{R}) R^\alpha R^\beta}{R^5} - \frac{x'^\alpha R^\beta + x'^\beta R^\alpha}{R^3} . \end{aligned} \tag{91}$$

Then the metric (eq. [8]) takes the form, for a "conservative" theory

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(cf. eq. [2])

$$g_{00} = 1 - (2/R)\int \rho(\underset{\sim}{x}',t)\,d\underset{\sim}{x}' - (2R^{\beta}/R^{3})\int \rho(\underset{\sim}{x}',t)\,\underset{\sim}{x}'^{\beta}\,d\underset{\sim}{x}' + O(4)$$

$$\begin{aligned} g_{0\alpha} = {} & \tfrac{1}{2}(4\gamma+3)(1/R)\int \rho' v'^{\alpha} d\underset{\sim}{x}' + \tfrac{1}{2}(R^{\alpha}R^{\beta}/R^{3})\int \rho' v'^{\beta} d\underset{\sim}{x}' \\ & + \tfrac{1}{2}(4\gamma+3)(R^{\beta}/R^{3})\int \rho' v'^{\alpha} x'^{\beta} d\underset{\sim}{x}' + \tfrac{3}{2}(R^{\alpha}R^{\beta}R^{\gamma}/R^{5})\int \rho' v'^{\beta} x'^{\gamma} d\underset{\sim}{x}' \\ & - \tfrac{1}{2}(R^{\beta}/R^{3})\int \rho' v'^{\beta} x'^{\alpha} d\underset{\sim}{x}' - \tfrac{1}{2}(R^{\alpha}/R^{3})\int \rho' \underset{\sim}{v}' \cdot \underset{\sim}{x}'\, d\underset{\sim}{x}' , \end{aligned} \tag{92}$$

$$g_{\alpha\beta} = -\delta_{\alpha\beta}\left[1 + 2\gamma(1/R)\int \rho' d\underset{\sim}{x}' + 2\gamma(R^{\beta}/R^{3})\int \rho' x'^{\beta} d\underset{\sim}{x}'\right] .$$

To lowest order, we have

$$\begin{aligned} \int \rho(\underset{\sim}{x}',t)d\underset{\sim}{x}' &= P^{0}\,[1 + O(2)] \\ \int \rho(\underset{\sim}{x}',t)v'^{\alpha}d\underset{\sim}{x}' &= P^{\alpha}\,[1 + O(2)] \\ \int \rho\,(\underset{\sim}{x}',t)x'^{\alpha}d\underset{\sim}{x}' &= (J^{\alpha 0} + P^{\alpha}t)[1 + O(2)] \end{aligned} \tag{93}$$

For a body with static structure, i.e., in which

$$(d/dt)\, I_{\alpha\beta} = (d/dt)\int \rho\,(\underset{\sim}{x}',t)\,x'^{\alpha}x'^{\beta}d\underset{\sim}{x}' = 0 \;, \tag{94}$$

we have

$$\begin{aligned} \int \rho\,(\underset{\sim}{x}',t)x'^{\alpha}v'^{\beta}d\underset{\sim}{x}' &= \tfrac{1}{2}\int \rho\,(\underset{\sim}{x}',t)(x'^{\alpha}v'^{\beta} - x'^{\beta}v'^{\alpha})\,d\underset{\sim}{x}' \\ &= \tfrac{1}{2}\,J^{\alpha\beta}\,[1 + O(2)]. \end{aligned} \tag{95}$$

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Then the metric takes the form

$$
\begin{aligned}
g_{00} &= 1 - 2P^0/R - 2J^{\beta 0}R^\beta/R^3 - 2(\underset{\sim}{P}\cdot\underset{\sim}{R}/R^3)t + O(4), \\
g_{0\alpha} &= \tfrac{1}{2}(4\gamma+3)\ P^\alpha/R + \tfrac{1}{2}(\underset{\sim}{P}\cdot\underset{\sim}{R})R^\alpha/R^3 \\
&\qquad + \tfrac{1}{2}(2\gamma+1)J^{\alpha\beta}R^\beta/R^3 + O(5)\ , \\
g_{\alpha\beta} &= -\delta_{\alpha\beta}\,[1+2\gamma P^0/R + 2\gamma J^{\beta 0}R^\beta/R^3 + 2\gamma(\underset{\sim}{P}\cdot\underset{\sim}{R}/R^3)t] + O(4).
\end{aligned}
\tag{96}
$$

Thus the conserved quantities can be given a physically measurable meaning to lowest order. One can measure P^j and $J^{\alpha 0}$ by means of Keplerian orbits and $J^{\alpha\beta}$ by means of gyroscope precession (dragging of inertial frames) far from any distribution of matter. The results of equation (96) agree (for $\gamma = 1$) with those obtained in general relativity [see for example Misner, Thorne, and Wheeler (1972)] and [for $\gamma = (1+\omega)/(2+\omega)$] with those obtained in Brans-Dicke theory (Dykla 1971).

V. CONCLUSIONS

In this paper, we have tried to show how a theorist would go about deciding what the values of the PPN parameters should be in a reasonable universe. By demanding that the correct theory of gravity be Lorentz (or post-Galilean) invariant, he would restrict some of the parameter values. He would also demand that the correct theory possess conservation laws (since theorists like conservation laws), and would further restrict the PPN parameters, until his final restricted PPN formalism would have the

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metric

$$
\begin{aligned}
g_{00} &= 1 - 2U + 2\beta U^2 - 4\Phi , \\
g_{0\alpha} &= \tfrac{1}{2}(4\gamma+3)V_\alpha - \tfrac{1}{2} W_\alpha , \\
g_{\alpha\beta} &= -\delta_{\alpha\beta}(1+2\gamma U) ,
\end{aligned} \tag{97}
$$

where

$$
\Phi = \tfrac{1}{2}(\gamma+1)\Phi_1 + \tfrac{1}{2}(3\gamma-2\beta+1)\Phi_2 + \tfrac{1}{2}\Phi_3 + \tfrac{3}{2}\gamma\Phi_4 . \tag{98}
$$

But these are theorists' constraints. Since the ultimate test of relativistic gravity theory is experiment, these constraints themselves should be subjected to experimental tests. A number of possible tests have already been discussed (Will 1971b; Thorne, Will, and Ni 1971; Nordtvedt 1971) and future papers in the companion series to this one ("Relativistic Gravity in the Solar System") may deal with others.

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APPENDIX A

DETAILS OF REASONING WHICH LEADS TO FIRST FIVE CONSERVATION-LAW CONSTRAINTS

Here we discuss in more detail the reasoning which leads us to conclude that the term Q^{α} in equation (21) cannot be written as a combination of divergences and time derivatives, unless the parameters obey equations (23).

We first argue that Q^{α} cannot be written as a spatial divergence alone. In fact, in obtaining equation (21), we have "extracted" all the divergences from Q^{α}, using equations (12) and (13). Any further use of these equations to transform Q^{α} leads nowhere -- we are always left with a residue which is not in the form of a divergence. This is shown explicitly in the following equation which follows from equations (12) and (13):

$$\begin{aligned} Q^{\alpha} = &- (2\pi)^{-1} (\partial/\partial x^{\beta}) [U_{,(\alpha^{B},\beta)}] \\ &- \rho [(2\beta_1 - \gamma - 1)\phi_{1,\alpha} - \tfrac{1}{2} (2\beta + 2\beta_2 - 3\gamma - 1)(U^2)_{,\alpha} \\ &+ (\beta_3 - 1)\phi_{3,\alpha} + 3(\beta_4 - \gamma)\phi_{4,\alpha} - \tfrac{1}{2}\zeta \mathcal{A}_{,\alpha}] , \end{aligned} \tag{A1}$$

where

$$\begin{aligned} B = &(2\beta_1 - \gamma - 1)\phi_1 - \tfrac{1}{4} (2\beta + 2\beta_2 - 3\gamma - 1)U^2 + (\beta_3 - 1)\phi_3 \\ &+ 3(\beta_4 - \gamma)\phi_4 - \tfrac{1}{2}\zeta \mathcal{A} . \end{aligned} \tag{A2}$$

In order to attempt to write Q^{α} as a combination of divergences and time derivatives, we make use of the (Newtonian) equations of motion (WII, eq. [12])

$$\begin{aligned} \rho dv^{\alpha}/dt &= \rho U_{,\alpha} - p_{,\alpha} , \\ \partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) &= 0 , \end{aligned} \tag{A3}$$

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where d/dt is the derivative "following the fluid" given by

$$d/dt = \partial/\partial t + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \; . \tag{A4}$$

As an example of this, consider the term $pU_{,\alpha}$ in equation (22). Using equations (A3) and (A4), we get

$$\begin{aligned} pU_{,\alpha} = {} & (\partial/\partial t)(\rho v^{\alpha} U) \\ & + (\partial/\partial x^{\beta}) \, [(4\pi)^{-1} U(U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}) \\ & \qquad + (\rho v^{\alpha} v^{\beta} + \delta^{\alpha\beta} p) U] \\ & - (8\pi)^{-1} U_{,\alpha} |\underset{\sim}{\nabla} U|^2 - \rho v^{\alpha} dU/dt \; . \end{aligned} \tag{A5}$$

Thus, although we have extracted a time derivative and a divergence from $pU_{,\alpha}$, plus a term $(U_{,\alpha}|\underset{\sim}{\nabla} U|^2)$ which can be combined with one of the other terms in Q^{α}, we are left with a residue, $\rho v^{\alpha} dU/dt$ which cannot be combined with any other terms in Q^{α}. All manipulations of this kind (which we have performed) have led to the same result: Each term in equation (22) can be split in a variety of ways into a time derivative, a divergence, and a "residue"; but the residue is always independent of any of the other terms in Q^{α}. Thus the only way to make these "residues" disappear for arbitrary fluid configurations is to set each parameter combination in equation (22) equal to zero, i.e., to demand that the parameters satisfy the constraint equations (23).

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APPENDIX B

ACTIVE GRAVITATIONAL MASS IN THE PPN FORMALISM

The active gravitational mass of a configuration of perfect fluid can be given a physically measurable meaning: it is the mass which determines the periods of Keplerian orbits far from the matter, i.e., it is one-half the coefficient of the 1/(distance from matter) part of the g_{00} metric term. Far from the matter, we can approximate

$$1/|\underset{\sim}{x}-\underset{\sim}{x}'| \approx 1/|\underset{\sim}{x}-\underset{\sim}{x}_{(\text{center of mass})}| \equiv 1/R \, . \tag{B1}$$

Then the g_{00} part of the metric (eq. [8]) can be written

$$\begin{aligned} g_{00} = 1 - (2/R)\Big\{ &\int \rho\, d\underset{\sim}{x} + 2\beta_1 \int \rho v^2 d\underset{\sim}{x} + 2\beta_2 \int \rho U\, d\underset{\sim}{x} \\ &+ \beta_3 \int \rho \Pi\, d\underset{\sim}{x} + 3\beta_4 \int p\, d\underset{\sim}{x} - \tfrac{1}{2}\zeta \int \rho\, [\underset{\sim}{v}\cdot \underset{\sim}{R}/R]^2\, d\underset{\sim}{x} \Big\} + O(R^{-2}) \, . \end{aligned} \tag{B2}$$

From equations (18) we have

$$\begin{aligned} \int \rho\, d\underset{\sim}{x} &= \int \rho^* (1 - \tfrac{1}{2}v^2 - 3\gamma U)\, d\underset{\sim}{x} \\ &= M - E_{kin} + 6\gamma\Omega \end{aligned} \tag{B3}$$

where E_{kin} and Ω are the internal kinetic and gravitational energies, respectively, and M is the total rest mass of particles in the fluid. We define

$$E_{int} = \int \rho \Pi\, d\underset{\sim}{x} \, , \tag{B4}$$

$$\begin{aligned} m &= M + E_{kin} + \Omega + E_{int} \\ &= \text{total mass energy.} \end{aligned} \tag{B5}$$

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We also make use of the Newtonian virial theorem for a static configuration of fluid (WII, eq. [78])

$$0 = 2\,E_{kin} + \Omega + 3\int p\, d\underset{\sim}{x}\ , \tag{B6}$$

to obtain from equation (B2),

$$g_{00} = 1 - (2/R)\Big[m + 2(2\beta_1 - \beta_4 - 1)\,E_{kin} + (6\gamma - 4\beta_2 - \beta_4 - 1)\,\Omega + (\beta_3 - 1)\,E_{int} - \zeta\,E_{kin}^{\alpha\beta}\,e_\alpha e_\beta\Big] + O(R^{-2}), \tag{B7}$$

where $E_{kin}^{\alpha\beta}$ is the kinetic energy tensor, and $\underset{\sim}{e}$ is a unit vector directed from the fluid to the field point, given by

$$\underset{\sim}{e} = \underset{\sim}{R}/R. \tag{B8}$$

The active gravitational mass is thus the quantity in square brackets in equation (B7). For a massive body made up of point particles, without pressure or internal energy, equation (B7) agrees with the results of Nordtvedt (1969).

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TABLE 1

PREDICTIONS OF "CONSERVATIVE" AND ASYMPTOTICALLY LORENTZ-INVARIANT THEORIES FOR VARIOUS EXPERIMENTAL TESTS*

Experimental Test or Observable Effect	PPN Parameter Dependence	Parameter Dependence as predicted by "Conservative" Theories	Parameter Dependence as predicted by asymptotically Lorentz Invariant Theories
Light Deflection, Radar Time Delay	$\frac{1}{2}(1+\gamma)$	$\frac{1}{2}(1+\gamma)$	$\frac{1}{2}(1+\gamma)$
Perihelion Shift	$\frac{1}{3}(2+2\gamma-\beta)$	$\frac{1}{3}(2+2\gamma-\beta)$	$\frac{1}{3}(2+2\gamma-\beta)$
Geodetic Precession of a Gyroscope	$\frac{1}{3}(1+2\gamma)$	$\frac{1}{3}(1+2\gamma)$	$\frac{1}{3}(1+2\gamma)$
Dragging of Inertial Frames	$\frac{1}{8}(7\Delta_1+\Delta_2)$	$\frac{1}{2}(1+\gamma)$	$\frac{1}{2}(1+\gamma)$
Nordtvedt Effect ($m_{passive} \neq m_{inertial}$)			
i) isotropic	$7\Delta_1-3\gamma-4\beta$	$-(4\beta-\gamma-3)$	$7\Delta_1-3\gamma-4\beta$
ii) anisotropic	$2\beta+2\beta_2-3\gamma+\Delta_2-2$	0	$2\beta+2\beta_2-3\gamma+\Delta_2-2$
Perturbations in Earthbound Gravimeter Measurements			
i) Variation in G			
a) due to field of sun and planets	$2\beta+2\gamma-2\beta_2-2$	$(4\beta-\gamma-3)$	$2\beta+2\gamma-2\beta_2-2$
b) due to motion through "ether"	$4\beta_1+2\gamma+1-7\Delta_1$	0	0
c) anisotropy due to motion through ether	$\Delta_2+\zeta-1$	0	0
ii) Other Effects			
a) due to external field gradients	$7\Delta_1+\Delta_2-4\gamma-4$	0	0
b) due to internal structure of the earth	$5\gamma-4\beta_2-\Delta_2$	$(4\beta-\gamma-3)$	$5\gamma-4\beta_2-\Delta_2$

* For a review of the relevant formulae see Thorne, Will, and Ni (1971). For detailed calculations, see Nordtvedt (1968, 1971), Will (1971a,b).

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REFERENCES

Chandrasekhar, S. 1965, Ap. J., 142, 1488.

______________ 1969, ibid., 158, 45.

Chandrasekhar, S., and Contopoulos, G. 1967, Proc. Roy. Soc., 298A, 123.

Chandrasekhar, S., and Nutku, Y. 1969, Ap. J., 158, 55.

Dykla, J. J. 1971, paper in preparation.

Fock, V. 1964, The Theory of Space, Time and Gravitation (New York: The Macmillan Co.).

Kreuzer, L. B. 1968, Phys. Rev., 169, 1007.

Landau, L. D., and Lifshitz, E. M. 1962, The Classical Theory of Fields (2nd ed.; Reading, Massachusetts: Addison-Wesley Press).

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972, Gravitation (San Francisco: W. H. Freeman & Co.).

Ni, W.-T. 1971, paper in preparation.

Nordtvedt, K., Jr. 1968, Phys. Rev., 169, 1017.

______________ 1969, ibid., 180, 1293.

______________ 1970, Ap. J., 161, 1059.

______________ 1971, Phys. Rev. D., 000, 000.

Nutku, Y. 1969, Ap. J., 155, 999.

Thorne, K. S., and Will, C. M. 1971, Ap. J., 163, 595 (Paper I).

Thorne, K. S., Will, C. M., and Ni, W.-T. 1971, in Proceedings of the Conference on Experimental Tests of Gravitation Theories, held at the California Institute of Technology, November 11-13, 1970, in press.

Will, C. M. 1971a, Ap. J., 163, 611 (Paper II, referred to as WII).

__________ 1971b, paper in preparation.

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C. THE PARAMETRIZED POST-NEWTONIAN FORMALISM: A NEW EXPERIMENTAL TEST OF RELATIVISTIC GRAVITY

5. Relativistic Gravity in the Solar System

II. Anisotropy in the Newtonian Gravitational Constant

(Submitted in April 1971 for publication in The Astrophysical Journal)

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I. INTRODUCTION AND SUMMARY

Since the formulation of the Brans-Dicke scalar-tensor theory, considerable interest has focused on the constancy of the Newtonian gravitational constant G. One line of investigation examines the effect of the evolution of the universe on the value of G as measured far from any local distribution of matter (solar system and galaxy). This value of G we will denote G_∞. Brans-Dicke theory predicts a secular rate of change of G_∞, and this change has recently come under experimental scrutiny (Shapiro, Smith, Ash, Ingalls, and Pettengill 1971). A second line of investigation examines the effect of nearby matter (planets and stars) on the value of G measured in laboratory Cavendish experiments (Brans 1962a). This value is normally called the "locally-measured" value of the gravitational constant. In an idealized version of such a Cavendish experiment, one measures the relative acceleration of two bodies as a function of their masses and of the distance between them. (Later in this paper the two bodies will be the Earth and a gravimeter at rest on the Earth.) Distances and times are measured by means of physical rods and atomic clocks at rest in the laboratory. The gravitational constant G is then identified as that number with dimensions $cm^3\ g^{-1}\ sec^{-2}$ which appears in Newton's law of gravitation for the two bodies.

In this paper we will show that some theories of gravity predict an anisotropy in the "locally-measured" value of the gravitational constant. We will then make use of gravimeter data on the tides of the solid Earth to put an experimental upper limit on this anisotropy.

We focus attention on the locally-measured gravitational constant because this can be analyzed within the <u>Parametrized Post-Newtonian</u> (PPN)

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framework (Nordtvedt 1968; Will 1971a; see also Appendix A), whereas the cosmological variation in the gravitational constant cannot be analyzed within the PPN framework.[1] Our result, in the nine-parameter PPN formalism

[1] In scalar-tensor theories this variation is due to the changing value of the scalar field in an evolving cosmological model. But in the PPN formalism, all reference to scalar fields and other fields has disappeared (see Will 1971a for discussion). Thus to compute the cosmological variation in G, one must go back to the original full theory of gravity.

of Will (1971a), for the gravitational constant as measured in a laboratory moving with velocity $\underset{\sim}{v}$ relative to the PPN coordinate system is[2]

[2] Here and throughout we use "geometrized" units; units in which the velocity of light is unity, and the gravitational constant as measured today, at rest, far from the solar system and galaxy (i.e., G_∞ today) is unity. For the purposes of our computation of the locally measured G, we will ignore cosmological variations in G_∞, and set $G_\infty = 1$.

$$G = 1 - (2\beta + 2\gamma - 2\beta_2 - 2)U + \tfrac{1}{2}(4\beta_1 + 2\gamma + 1 - 7\Delta_1)v^2 - \tfrac{1}{2}(\Delta_2 + \zeta - 1)(\underset{\sim}{v}\cdot\underset{\sim}{e}_r)^2 \; . \qquad (1)$$

Here $\underline{U} > 0$ is the Newtonian gravitational potential due to all the matter in the solar system and galaxy outside the laboratory, and $\underset{\sim}{e}_r$ is a unit vector in the direction separating the two bodies of the experiment. The parameters β, β_1, β_2, γ, Δ_1, Δ_2, ζ are PPN parameters. Nordtvedt (1970)

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has performed the same calculation using his version of the PPN formalism (Nordtvedt 1968), but neglecting the effects of velocity. His results are in agreement with equation (1) assuming $\underset{\sim}{v} = 0$.

In both Brans-Dicke theory and general relativity (Will 1971a),

$$\begin{aligned} 4\beta_1 + 2\gamma + 1 - 7\Delta_1 &= 0, \\ \Delta_2 + \zeta - 1 &= 0, \end{aligned} \tag{2}$$

so the velocity effects are absent in these two theories. The parameter $(2\beta + 2\gamma - 2\beta_2 - 2)$ is zero in general relativity and is equal to $1/(2+\omega)$ in Brans-Dicke theory; thus our results agree with those obtained by Brans (1962a,b).

We will concentrate our attention on the velocity-dependent terms in equation (1), and especially on the possible anisotropy in G predicted by the $(\underset{\sim}{v} \cdot \underset{\sim}{e}_r)^2$ term.

In every theory for which the PPN parameters have been calculated, the parameter combination $(\Delta_2 + \zeta - 1)$ is zero, i.e., the anisotropy in G is absent (Will 1971a, Ni 1971, Nordtvedt 1970). In fact it can be shown (although we will not do so in this paper; see Will 1971b) that any theory whose post-Newtonian metric admits a "post-Galilean" transformation (Chandrasekhar and Contopoulos 1967) should have PPN parameters whose values satisfy equations (2). [These "post-Galilean" transformations are transformations which reduce to simple Lorentz transformations far from the sources of the metric, and which leave the form of the metric unchanged. See also Nordtvedt (1969).] Put differently, the combinations $(4\beta_1 + 2\gamma + 1 - 7\Delta_1)$ and $(\Delta_2 + \zeta - 1)$ should be zero for any theory whose linearized equations for the metric (i.e., linearized in small deviations

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from flat spacetime) are Lorentz invariant. Instead of giving a proof of this using post-Galilean transformations, we will simply illustrate it (§IV) by giving an example of a theory of gravity which does not satisfy Lorentz invariance in its linearized field equations, and which thus predicts that, in regions of negligible gravitational fields (flat spacetime), gravity and light should have different propagation speeds. Such a theory picks out a preferred reference frame (the rest frame of some cosmological "ether", for instance) in which all gravitational fields are to be calculated. Lorentz transformations to other frames of reference will then not give the same physical results for any computation involving gravity -- though all the rest of physics will remain Lorentz invariant. If $v \neq 1$ is the speed of propagation of gravity in the rest frame of the "ether", we will show that, for this theory of gravity,

$$\begin{aligned} 4\beta_1 + 2\gamma + 1 - 7\Delta_1 &= (1/v^2) - 1, \\ \Delta_2 + \zeta - 1 &= (1/v^2) - 1. \end{aligned} \tag{3}$$

Note that, in this theory the calculation which leads to equation (1) is valid only in the rest frame of the "ether", and hence the velocity $\underset{\sim}{v}$ which appears in equation (1) must be the velocity of the laboratory relative to this ether.

From this point of view, one can consider an experimental measurement of the anisotropy in G to be a test of Lorentz invariance in linearized gravity or of the equality of the speeds of gravity and light.

There is another type of anisotropy in G which does not involve velocities, but which can arise in two-tensor theories of gravity. Peebles and Dicke (1962), [see also Peebles (1962)], have argued that theories of gravity

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which contain two tensor fields should exhibit anisotropies in the masses of bodies. They use the Hughes-Drever experiment to rule out, with precision one part in 10^{23}, anisotropies in the inertial masses of atoms. From this they conclude that a second tensor field probably does not occur in nature. However, their analysis did not include theories which couple matter to the second tensor field purely gravitationally. An example of such a theory is Whitehead's (1922) theory,[3] which is too complicated to fit

[3] Recall: Whitehead's theory agrees with general relativity in its predictions for the classical experimental tests -- redshift, perihelion shift, light deflection, time delay.

into the 9-parameter PPN formalism. Whitehead's second tensor field is a background global Lorentz metric. This Lorentz metric appears only in the equations used to compute the physical metric; otherwise it has no effect on the motions of material particles or other fields. Therefore it predicts for atoms (with self-gravitational energy $\sim 10^{-39}$ of rest-mass energy) highly isotropic inertial masses, in agreement with the Hughes-Drever experiment. But it predicts, according to our calculations (§III), that active gravitational masses are anisotropic, at the level 1 part in 10^{6}. This anisotropy is produced by a corresponding anisotropy in the locally measured G of Whitehead's theory:

$$G_{\text{Whitehead}} = 1 + 2U + \sum_k U_k \, (\underset{\sim}{e}_k \cdot \underset{\sim}{e}_r)^2. \qquad (4)$$

Here U_k is the Newtonian gravitational potential due to the $\underline{k}^{\text{th}}$ external body (including sun, moon, planets, and stars) and $\underset{\sim}{e}_k$ is a unit vector from

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the laboratory to the center of mass of the $\underline{k}^{th}$ body. The anisotropy will be dominated by the influence of the central regions of the galaxy:

$$G_{Whitehead} = 1 + 2U + (M_{gal}/R_{gal})(\underset{\sim}{e}_{gal} \cdot \underset{\sim}{e}_r)^2 . \tag{5}$$

Since the equations used to calculate the metric in Whitehead's theory are Lorentz invariant, there is no $(\hat{\underset{\sim}{v}} \cdot \underset{\sim}{e}_r)^2$ anisotropy in equation (4).

From the above viewpoint, an experiment to seek anisotropies in G (or equivalently in active gravitational masses) is complementary to the Hughes-Drever experiment.

We will now discuss the observational consequences of the possible anisotropy in G. As the orientation of a Cavendish experiment is rotated (by the rotating Earth, for example), the unit vector $\underset{\sim}{e}_r$ rotates relative to the $\underset{\sim}{v}$ of the PPN formalism and the $\underset{\sim}{e}_k$ of Whitehead's theory. Hence the anisotropy terms in equations (1) and (4) vary. The amplitude of the anisotropy and hence of these time variations is

$$\Delta G/G \sim \tfrac{1}{2}\,(\Delta_2 + \zeta - 1)[v_{(solar\ system)}/c]^2 \sim 2\times 10^{-7}\,(\Delta_2 + \zeta - 1) \qquad [\text{PPN}],$$

$$\Delta G/G \sim U_{(galaxy)}/c^2 \sim 5\times 10^{-7} \qquad [\text{Whitehead}],$$

where we have used the sun's velocity around the galaxy ($\sim$ 215 km/sec) as a rough estimate for the relative Earth-"ether" velocity. Of course, one can obtain any desired periodicity in the effect of the anisotropy by rotating the laboratory relative to the Earth (or by doing the experiment in a rotating space probe).

Recent progress in the design of Cavendish experiments has opened up the possibility of measuring the absolute magnitude of the gravitational

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constant to accuracies of at least one part in 10^4 and perhaps even one part in 10^6 (Rose, Parker, Lowry, Kuhlthau, and Beams 1969). Such Cavendish experiments would have to be made sensitive to periodic changes in G of a part in 10^8 in order to test for the anisotropy with ten per cent precision.

However, there already exists a body of experimental data -- gravimeter measurements of the tides of the solid Earth -- which gives an accurate test of the anisotropy in G.[4] Such measurements can be regarded as Cavendish

[4] I am indebted to P. J. E. Peebles for pointing this out to me.

experiments with the Earth as one attracting body and a gravimeter at rest on the surface of the Earth as the other body. As the earth rotates, the anisotropy in G produces "Earth tides", i.e., variations in the acceleration $\underline{g}$ measured by the gravimeter, which are completely analogous to the tides produced by the Moon and Sun. These gravimeter measurements are affected not only by the variation in G, but also by the displacement of the Earth's surface relative to the center of the Earth and by the deformation of the Earth [see Melchior (1966) for a discussion]. By analogy with solar and lunar tides, one can show that the variation in gravimeter readings is related to the variation in G by

$$(\Delta g/g) = \alpha\, (\Delta G/G), \tag{6}$$

where α is a dimensionless number (a combination of so-called Love's Numbers) which depends only on the structure of the Earth, and has a value of about 1.18 (Melchior 1966). These anisotropy-induced variations have periods of 12 hours sidereal time, since the vectors $\underline{v}$ and $\underline{e}_k$ (cf. eqs. [1] and [4]) are fixed relative to inertial frames (the galaxy).

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Assume for concreteness that the velocity $\underset{\sim}{v}$ of the Earth relative to the cosmological "ether" is approximately the same as its velocity through the galaxy. Then $\underset{\sim}{v}$ is oriented at an angle of $\sim 62^{\circ}$ away from the Earth's equatorial plane, and the G-induced perturbations in gravimeter measurements on the equator have amplitude

$$\begin{aligned}(\Delta g/g)_{PPN} &\approx \tfrac{1}{2}\alpha(\Delta_2 + \zeta - 1)\cos^2(62^{\circ})(\tfrac{1}{2}v^2) \\ &\approx (\Delta_2 + \zeta - 1)(3 \times 10^{-8}),\end{aligned} \tag{7}$$

according to the 9-parameter PPN formalism. Similarly, since the galactic center is about 29° south of the equator, Whitehead's theory predicts the amplitude

$$\begin{aligned}(\Delta g/g)_{Whitehead} &\approx \alpha\cos^2(29^{\circ})(\tfrac{1}{2}U_{gal}) \\ &\approx 2 \times 10^{-7}\,.\end{aligned} \tag{8}$$

The semi-diurnal tides predicted by Newtonian theory have three principal frequency components, a 12-hour lunar-time component (denoted M_2), a 12-hour solar-time component (S_2), and a 12-hour sidereal-time component (K_2). (The sidereal component depends on the declination (tilt) of the lunar and solar orbits relative to the Earth's equatorial plane.) These components of the tides have amplitudes at the equator of

$$(\Delta g/g)_{M_2} \sim 9 \times 10^{-8},$$

$$(\Delta g/g)_{S_2} \sim 4 \times 10^{-8},$$

$$(\Delta g/g)_{K_2} \sim 1 \times 10^{-8}.$$

The M_2 tide is easily separated from the other two semi-diurnal tides by

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means of Fourier analysis of one month's gravimeter data. However, separation of the S_2 and K_2 tides requires at least one year of continuous gravimeter data. Because of gravimeter drift and long-period tides, such a separation is not easy to obtain [see Barsenkov (1967) for a partial separation of S_2 and K_2 using 19 months of data taken at Talgar, U.S.S.R.].

Experimental measurements of the combined S_2 and K_2 tides (amplitude $\sim 5 \times 10^{-8}$ g) are found to agree with the predictions of Newtonian gravitation (coupled with reasonable models for the structure of the Earth) to a precision of 2 per cent (Harrison, Ness, Longman, Forbes, Kraut, and Slichter 1963; Pariiskii, Barsenkov, Volkov, Gridnev, and Kramer 1967). Thus any discrepancy between Newtonian theory and experiment for this component of the tides must be less than one part in 10^9, and hence the amplitude of the tides caused by the G-anisotropy must satisfy

$$(\Delta g/g)_{\text{anisotropy}} < 10^{-9} . \tag{9}$$

Comparing equation (9) with equation (8), we see that Whitehead's theory cannot be the correct theory of gravity, because it predicts an effect 200 times larger than the experimentally measured value. Since Whitehead's theory agrees with general relativity in its predictions for the standard experimental tests (redshift, light deflection, time delay, and perihelion shift), this is the first accurate experimental evidence ruling out this theory.

Equations (9) and (7) also show that the parameter combination $(\Delta_2 + \zeta - 1)$ must satisfy

$$(\Delta_2 + \zeta - 1) < 3 \times 10^{-2} . \tag{10}$$

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Thus, to a precision of about three per cent, the parameter combination $(\Delta_2 + \zeta - 1)$ must be zero. According to our discussion of Lorentz invariance, this means that, for the theory devised in §IV of this paper, the flat-space velocity of gravity must be the same as that of light to a precision of $\sim$ 1.5 per cent.

These are the central conclusions of this paper. The remaining sections are devoted to detailed calculations. In §II and §III, respectively, we calculate the locally-measured value of G using the PPN formalism, and using Whitehead's theory. In §IV we present and discuss a theory of gravity which predicts $(\Delta_2 + \zeta - 1) \neq 0$. Concluding remarks are made in §V. Appendix A gives the PPN n-body point-mass metric and Christoffel symbols; Appendix B gives the n-body point-mass metric and Christoffel symbols for Whitehead's theory.

II. CALCULATION OF THE LOCALLY-MEASURED GRAVITATIONAL CONSTANT

Since gravimeter measurements are the most sensitive of all experiments for seeking anisotropies in G, we shall use an Earth-gravimeter "Cavendish experiment" to calculate those anisotropies. A calculation for Cavendish experiments with both bodies of laboratory sizes would proceed similarly and would produce the same final answer (eq. [1]).

We idealize our Earth-gravimeter Cavendish experiment as follows: a body of mass m_1 (Earth) is freely falling through spacetime. A test body with negligible mass (gravimeter) is moving through spacetime, maintained at

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a constant proper distance r_p from the Earth by a four-acceleration $\mathbf{F}$.[5] An

[5] We will denote four-vectors by bold-face roman symbols and three-vectors by bold-face italic symbols. Greek indices will take the values 1, 2, and 3; Roman indices will take the values 0, 1, 2, 3; and summation over repeated indices will be employed. Exceptions to this rule are indices j and k, which will be used to label the masses in the problem.

invariant "radial" unit four-vector $\mathbf{E}_r$, carried by the gravimeter points directly away from the center of mass of the Earth. Then, according to Newton's law of gravitation, the radial component of the acceleration as measured by the gravimeter is given by

$$\mathbf{F} \cdot \mathbf{E}_r = Gm_1/r_p{}^2 + r_p\,(D\mathbf{E}_r/D\tau) \cdot (D\mathbf{E}_r/D\tau), \tag{11}$$

where $D/D\tau$ is the covariant derivative with respect to the gravimeter's proper time τ along the gravimeter's world line. The last term in equation (11) is simply the centrifugal acceleration, defined in an invariant way (except for corrections of order $10^{-9}\, G_\infty m_1/r_p{}^2$ which we ignore; see below). Since $\mathbf{F} \cdot \mathbf{E}_r$ is an invariant quantity, we can calculate it in the PPN coordinate system and then use equation (11) to identify the locally-measured gravitational constant G.

Before proceeding with the computation, we must say a few words about the approximation scheme we will be using. First, we work in the post-Newtonian approximation throughout.[6] Second we will neglect any terms which

[6] This means that we retain only Newtonian terms and terms which are O(2) or less beyond the Newtonian result. For a discussion of the process of assign-

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ing Newtonian and post-Newtonian "orders" to various terms in the expansion see Chandrasekhar (1965). In our notation the Newtonian potential $U(\underset{\sim}{x})$ is $O(2)$, velocity v^{α} is $O(1)$, time derivatives d/dt are $O(1)$, and so on.

---

produce accelerations of 10^{-9} g or smaller, as measured by the gravimeter. This amounts to neglecting Earth-generated post-Newtonian accelerations of the gravimeter, post-Newtonian corrections to the centrifugal acceleration and to tidal accelerations, and other, more complicated accelerations. We do this for two reasons: first, 10^{-9} g seems to be the limit at present of reliable gravimeter data on the tides of the Earth, and second, a number of these relativistic effects which produce accelerations smaller than 10^{-9} g have already been dealt with by Nordtvedt (1971). We will discuss these neglected accelerations in more detail later in this section, but for the purpose of ease of presentation, we will ignore them in most of the explicit computations to follow. Of course all these neglected accelerations would be even more negligible ($\ll 10^{-9}$ g) in a laboratory-type Cavendish experiment.

We will do our calculations in the PPN coordinate system, which is a quasi-cartesian coordinate system whose metric is given by equation (A1). At any given moment of PPN coordinate time t, each "particle", denoted by a subscript j, has a three-vector position denoted $\underset{\sim}{x}_j(t)$ and a velocity denoted $\underset{\sim}{v}_j(t)$. We will treat all "particles" of the problem -- the gravimeter ($j = 0$), the Earth ($j = 1$), and the sun, planets, and stars ($j = 2, 3, \ldots$) -- as point masses. [See Nordtvedt (1971) for the result of treating the Earth as a massive body, with resultant accelerations of magnitude $\lesssim 10^{-9}$ g due to the equivalence-principle breakdown, and accelerations dependent on the structure of the Earth.] We will separate the Newtonian gravitational potential U due to the Earth from that due to the other

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planets and the sun

$$U(\underset{\sim}{x}) = m_1/r_1 + \sum_{k \neq 1} m_k/r_k , \tag{12}$$

where

$$\underset{\sim}{r}_k = \underset{\sim}{x} - \underset{\sim}{x}_k , \qquad r_k = |\underset{\sim}{x} - \underset{\sim}{x}_k| = \left[\sum_\alpha (x - x_k)^\alpha (x - x_k)^\alpha\right]^{\frac{1}{2}} . \tag{13}$$

We also define, for later use,

$$\underset{\sim}{r}_{jk} = \underset{\sim}{x}_j - \underset{\sim}{x}_k , \qquad r_{jk} = |\underset{\sim}{x}_j - \underset{\sim}{x}_k| . \tag{14}$$

We first calculate the proper distance r_p from the gravimeter to the center of the Earth. We use a physically reasonable definition for r_p -- namely one half the proper time (as measured by the gravimeter) required for a photon to travel from the gravimeter to the center of the Earth and back:

$$r_p = \frac{1}{2} \int_{t_e}^{t_r} \left\{1 - U[\underset{\sim}{x}_0(t)] - \frac{1}{2} v_0^{\,2}(t)\right\} dt , \tag{15}$$

where t_e and t_r are the PPN coordinate times corresponding to emission and reception of the light signal, and the integral is taken along the gravimeter's world line. The round trip time $(t_r - t_e)$ is obtained by integrating the geodesic equation for the light signal along its path from its emission at $\underset{\sim}{x}_0(t_e)$ to its deflection at the center of the Earth $\underset{\sim}{x}_1(t_d)$ and back to its reception at $\underset{\sim}{x}_0(t_r)$, and is given by

$$\begin{aligned} t_r - t_e = {} & |\underset{\sim}{x}_0(t_r) - \underset{\sim}{x}_1(t_d)| + |\underset{\sim}{x}_1(t_d) - \underset{\sim}{x}_0(t_e)| \\ & + (1+\gamma) \int_{\sigma_e}^{\sigma_r} U[\underset{\sim}{x}(\sigma)]\, d\sigma + O(3) , \end{aligned} \tag{16}$$

where σ is PPN coordinate time t along the path of the light signal. We

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take into account the motion of the gravimeter and the Earth during the time of transit of the signal according to

$$
\begin{aligned}
x_0^{\alpha}(t_r) &\approx x_0^{\alpha}(t_e) + (t_r - t_e)v_0^{\alpha} + \tfrac{1}{2}(t_r - t_e)^2 dv_0^{\alpha}/dt, \\
x_1^{\alpha}(t_d) &\approx x_1^{\alpha}(t_e) + (t_d - t_e)v_1^{\alpha} + \tfrac{1}{2}(t_d - t_e)^2 dv_1^{\alpha}/dt.
\end{aligned}
\tag{17}
$$

The velocities v_0^{α} and v_1^{α} and accelerations dv_0^{α}/dt and dv_1^{α}/dt are all to be evaluated at $t = t_e$. Equations (15), (16), and (17) lead to the final result

$$
\begin{aligned}
r_p = r_{10} &\left\{ 1 + \frac{(\underset{\sim}{v}_1 - \underset{\sim}{v}_0)\cdot \underset{\sim}{r}_{10}}{r_{10}} + \tfrac{1}{2}\left(\frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{r}_{10}}{r_{10}}\right)^2 + \gamma \sum_{k\neq 1,0} \frac{m_k}{r_{1k}} \right\} \\
&+ \delta r_p ,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\delta r_p = r_{10} &\left\{ (d/dt)\left[(\underset{\sim}{v}_1 - \underset{\sim}{v}_0)\cdot \underset{\sim}{r}_{10}\right] - \tfrac{1}{2}(v_1 - v_0)^2 - m_1/r_{10} \right. \\
&\left. + \tfrac{1}{2}\gamma \sum_{k\neq 1,0} \frac{m_k}{r_{1k}^{\,3}} (\underset{\sim}{r}_{10}\cdot \underset{\sim}{r}_{1k}) + \tfrac{1}{2}\frac{(1+\gamma)}{r_{10}} \int_{\sigma_e}^{\sigma_r} \frac{m_1}{r_1(\sigma)}\, d\sigma + O(3) \right\}.
\end{aligned}
\tag{19}
$$

The proper distance r_p is to be kept constant (by the force which holds the gravimeter at rest on the surface of the Earth).[7] Thus

---

[7] Because of the response of the Earth to the time-varying g ("Earth tides"), the distance from the gravimeter to the center of the Earth actually varies by about one part in 10^7 (see discussion of Earth tides in §I above). This is too small to have any significant effect on the anisotropy in G; hence we will assume r_p is constant in our calculation. Of course, the actual

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gravimeter measurements will be affected by the Earth-tide variations in r_p caused by the lunar and solar tidal forces as well as by the G-anisotropy. Because this variation is so small, it can be accounted for by means of Newtonian theory [see Melchior (1966) for a discussion of this effect].

$$dr_p/d\tau = dr_p/dt = 0,$$
$$d^2r_p/d\tau^2 = d^2r_p/dt^2 = 0. \tag{20}$$

Equations (18) and (19) along with equations (20) then give the following results:

$$(\underset{\sim}{v}_1 - \underset{\sim}{v}_0)\cdot \underset{\sim}{r}_{10}/r_{10} = 0(3), \tag{21}$$

$$\left(\frac{d\underset{\sim}{v}_1}{dt} - \frac{d\underset{\sim}{v}_0}{dt}\right)\cdot\left[\frac{\underset{\sim}{r}_{10}}{r_{10}} + \underset{\sim}{v}_1\left(\frac{\underset{\sim}{v}_1\cdot\underset{\sim}{r}_{10}}{r_{10}}\right)\right] + \frac{(v_1 - v_0)^2}{r_{10}} + 0(3) = 0. \tag{22}$$

We also find that the term δr_p in equation (18) leads only to gravimeter-measured accelerations of less than 10^{-9} g, and can thus be ignored. It is equation (22) which we will use to determine the acceleration measured by the gravimeter.

Assume that the Earth follows a geodesic of spacetime (neglect equivalence-principle violations), but that the four-acceleration of the gravimeter is $\mathbf{F}$:

$$D\,u_{Earth}/D\tau_{Earth} = 0, \tag{23}$$

$$D\,u_{gravim}/D\tau_{gravim} = \mathbf{F}, \tag{24}$$

$$\mathbf{F}\cdot u_{gravim} = 0. \tag{25}$$

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In PPN coordinates, equations (23), (24), and (25) may be written

$$\frac{dv_1^{\alpha}}{dt} + \Gamma^{\alpha}{}_{bc}(\underset{\sim}{x}_1)v_1^b v_1^c - \Gamma^0{}_{bc}(\underset{\sim}{x}_1)v_1^b v_1^c v_1^{\alpha} = 0, \tag{26}$$

$$\frac{dv_0^{\alpha}}{dt} + \Gamma^{\alpha}{}_{bc}(\underset{\sim}{x}_0)v_0^b v_0^c - \Gamma^0{}_{bc}(\underset{\sim}{x}_0)v_0^b v_0^c v_0^{\alpha} = (d\tau_{gravim}/dt)^2\,(F^{\alpha} - v_0^{\alpha}F^0), \tag{27}$$

$$F^0 = \underset{\sim}{F}\cdot\underset{\sim}{v}_0 + O(4), \tag{28}$$

where $v_1^0 = v_0^0 = dt/dt = 1$, and

$$(d\tau_{gravim}/dt)^2 = 1 - 2\sum_{k\neq 1,0}(m_k/r_{1k}) - v_0^2 + (10^{-9}\text{ g terms}). \tag{29}$$

By making use of the PPN n-body point-mass Christoffel symbols (eqs. [A3]) along with equations (26), (27), and (28), and using the Newtonian equations of motion to simplify any post-Newtonian terms, and as usual ignoring small force terms, we get from equation (22):

$$\begin{aligned}\frac{\underset{\sim}{F}\cdot\underset{\sim}{r}_{10}}{r_{10}} = {} & \sum_{k\neq 1,0}\frac{m_k r_{10}^{\alpha} r_{10}^{\beta}(3r_{1k}^{\alpha} r_{1k}^{\beta} - r_{1k}^2\delta^{\alpha\beta})}{r_{1k}^5 r_{10}} + \frac{(v_0 - v_1)^2}{r_{10}} \\ & - \frac{m_1}{r_{10}^2}\Bigg[1 - (5\gamma + 2\beta - 2\beta_2 - 2)\sum_{k\neq 1,0}\frac{m_k}{r_{1k}} - \tfrac{1}{2}(4\beta_1 + 2\gamma + 1 - 7\Delta_1)v_1^2 \\ & - \tfrac{1}{2}(\Delta_2 + \zeta)(\underset{\sim}{v}_1\cdot\underset{\sim}{r}_{10}/r_{10})^2\Bigg].\end{aligned} \tag{30}$$

We must now compute the invariant radial unit vector E_r.

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The tangent four-vector to the photon path λ^α at the moment of emission by the gravimeter is given according to the photon's geodesic equation by (ignoring terms leading to small forces)

$$\lambda^0 = 1,$$

$$\lambda^\alpha = (r_{10}{}^\alpha/r_{10})\cdot\Big[1 - \underset{\sim}{v}_1\cdot\underset{\sim}{r}_{10}/r_{10} - \tfrac{1}{2}v_1{}^2 - (1+\gamma)\sum_{k\neq 1,0} m_k/r_{1k} + \tfrac{1}{2}(\underset{\sim}{v}_1\cdot\underset{\sim}{r}_{10}/r_{10})^2\Big] + v_1{}^\alpha + O(3). \tag{31}$$

The radial unit four-vector E_r is the direction of the emitted photon, as measured by the gravimeter. This is simply the projection of λ^i onto the hypersurface orthogonal to the gravimeter's four-velocity u^i, suitably normalized:

$$\begin{aligned}(E_r)^a &= (\delta^a{}_b - u^a u_b)\lambda^b/|(\delta^c{}_d - u^c u_d)\lambda^d| \\ &= [\lambda^a/(\lambda^b u_b)] - u^a\ .\end{aligned} \tag{32}$$

Then the invariant radial component of the gravimeter's four-acceleration is

$$F_r = (F^a\lambda_a)/(\lambda^b u_b) - F^a u_a\ . \tag{33}$$

From equations (18), (21), (28), (30), (31), and (33), we get for the radial acceleration measured by the gravimeter:

$$\begin{aligned}F_r = &-\sum_{k\neq 1,0}\frac{m_k r_{10}{}^\alpha r_{10}{}^\beta(3r_{1k}{}^\alpha r_{1k}{}^\beta - r_{1k}{}^2\delta^{\alpha\beta})}{r_{1k}{}^5 r_{10}} - \frac{(v_1 - v_0)^2}{r_p} \\ &+ (m_1/r_p{}^2)\Big[1 - (2\beta+2\gamma-2\beta_2-2)\sum_{k\neq 1,0} m_k/r_{1k} \\ &- \tfrac{1}{2}(4\beta_1+2\gamma+1-7\Delta_1)v_1{}^2 - \tfrac{1}{2}(\Delta_2+\zeta-1)(\underset{\sim}{v}_1\cdot\underset{\sim}{r}_{10}/r_{10})^2\Big]\ .\end{aligned} \tag{34}$$

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The first term in equation (34) is simply the Newtonian tidal acceleration, which is of the order of 10^{-7}g. The second term is the Newtonian centrifugal acceleration ($\sim 10^{-3}$ g), which is equivalent (to the necessary accuracy) to the invariant expression $r_p(D\,E_r/D\tau)\cdot(D\,E_r/D\tau)$ in equation (11). From the third term we get the locally-measured gravitational constant G, as in equation (1).

In Table 1 we list the accelerations which we have ignored as being too small ($\lesssim 10^{-9}$ g). The first two, obtained by Nordtvedt (1971), result from the fact that the Earth is not a point mass, but is a massive self-gravitating body. Our treatment of the Earth as a point mass has neglected these accelerations. The third acceleration in Table 1, also derived by Nordtvedt, depends on the fact that the externally-produced gravitational field is not uniform. This acceleration can also be obtained with the point-mass analysis of this paper by retaining the appropriate terms. The fourth acceleration is a velocity-dependent acceleration analogous to the two we have retained (cf. eq. [1]). However, this acceleration is smaller than 10^{-9} g. The other accelerations in Table 1 have more complicated forms and are not particularly enlightening.

III. LOCALLY-MEASURED GRAVITATIONAL CONSTANT ACCORDING TO WHITEHEAD'S THEORY

For Whitehead's theory, the calculation of the locally-measured G can be repeated using the n-body point-mass metric given in Appendix B.[8]

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[8] Whitehead's theory is too complex to be compatible with our nine-parameter version of the PPN formalism. More complicated versions of the formalism with more parameters would be required to handle Whitehead's theory.

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According to equation (B4), the Whitehead metric is the same as a PPN metric with parameter values

$$\begin{aligned} &\beta = 1, && \Delta_1 = \tfrac{1}{7}, && \gamma = 1, \\ &\beta_1 = -\tfrac{1}{2}, && \Delta_2 = 7, && \zeta = -6, \\ &\beta_2 = 1, \end{aligned} \tag{35}$$

except for the following additional term in g_{00}:

$$\delta g_{00}(\underset{\sim}{x}) = 2 \sum_j \sum_{k \neq j} \frac{m_j m_k (\underset{\sim}{x} - \underset{\sim}{x}_j)}{|\underset{\sim}{x} - \underset{\sim}{x}_j|^3} \cdot \left[\frac{(\underset{\sim}{x}_j - \underset{\sim}{x}_k)}{|\underset{\sim}{x} - \underset{\sim}{x}_k|} - \frac{(\underset{\sim}{x} - \underset{\sim}{x}_k)}{|\underset{\sim}{x}_j - \underset{\sim}{x}_k|} \right] . \tag{36}$$

This extra metric term changes only the equation of motion for the gravimeter via the Christoffel symbol Γ^{α}_{00} -- other effects of δg_{00} are either of post-post-Newtonian order or produce forces smaller than 10^{-9}g. The resulting change in the force $\underset{\sim}{F}$ is

$$\delta \underset{\sim}{F} \cdot \underset{\sim}{r}_{10}/r_{10} = -(m_1/r_{10}^2) \left[2 \sum_{k \neq 1,0} m_k/r_{1k} + \sum_{k \neq 1,0} m_k (\underset{\sim}{r}_{10} \cdot \underset{\sim}{r}_{1k})^2 / r_{1k}^3 \right] . \tag{37}$$

Combining equation (37) with equations (33) and (34), and substituting the parameter values of equation (35), we obtain

$$\begin{aligned} F_r = &- \sum_{k \neq 1,0} \frac{m_k r_{10}^{\alpha} r_{10}^{\beta} (3 r_{1k}^{\alpha} r_{1k}^{\beta} - r_{1k}^2 \delta^{\alpha\beta})}{r_{1k}^5 r_{10}} - \frac{(v_1 - v_0)^2}{r_p} \\ &+ (m_1/r_p^2) \left[1 + 2 \sum_{k \neq 1,0} m_k/r_{1k} + \sum_{k \neq 1,0} m_k (\underset{\sim}{r}_{10} \cdot \underset{\sim}{r}_{1k})^2 / r_{1k}^3 \right] . \end{aligned} \tag{38}$$

The value of G identified from equation (38) agrees with equation (4).

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Since the Whitehead metric must be calculated in a global Lorentz coordinate system of η_{ab} [or at best in a spacetime of constant curvature (Temple 1924)], the field due to the galaxy cannot be removed by transformation to a local inertial frame surrounding the solar system (as one would do in general relativity). Hence the galactic gravitational field must appear in equation (38) (see §I).

IV. EXAMPLE OF A THEORY WHICH PREDICTS A VELOCITY-DEPENDENT ANISOTROPY IN G

We take, as our example of a theory which violates Lorentz invariance in its linearized equations for the metric, linearized general relativity modified in a suitable way. In linearized general relativity, the field equations are [see, for example, Misner, Thorne, and Wheeler (1972)]

$$\Box \bar{h}_{ij} = -16\pi T_{ij} , \tag{39}$$

where $\Box$ is the ordinary flat space d'Alembertian, defined by

$$\Box = (\partial/\partial t)^2 - \nabla^2,$$

T_{ij} is the matter stress-energy tensor, and $\bar{h}_{ij}$ is related to the metric by

$$\begin{aligned} g_{ij} &= (1 - \tfrac{1}{2} h)\,\eta_{ij} + \bar{h}_{ij} , \\ h &= \eta^{ij}\bar{h}_{ij} . \end{aligned} \tag{40}$$

We modify equation (39) to allow the speed of propagation of gravity to differ from that of light, by replacing the Lorentz invariant d'Alembertian $\Box$ by

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the operator

$$(1/v^2)(\partial/\partial t)^2 - \nabla^2$$

where $v \neq 1$ is the speed of propagation of gravity in flat spacetime <u>in a particular reference frame</u> (the rest frame of the "ether"). All calculations involving gravity must now be performed in this frame. To the appropriate (post-Newtonian, but linearized) order, the modified equation (39) may be written {see Will (1971a) eq. [24] for expressions for T_{ij}}

$$\begin{aligned}(1/v^2)\,\bar{h}_{00,00} - \nabla^2\,\bar{h}_{00} &= -16\pi\rho(1+v^2+\Pi)\ ,\\ \nabla^2\,\bar{h}_{0\alpha} &= -16\pi\rho\, v_\alpha\ ,\\ \nabla^2\,\bar{h}_{\alpha\beta} &= 16\pi(\rho v_\alpha v_\beta + p\delta_{\alpha\beta})\ ,\end{aligned} \tag{41}$$

where $\rho\Pi$ and p are the matter's internal energy and pressure, respectively. These equations have the solutions

$$\begin{aligned}\bar{h}_{00} &= -4U - 4\Phi_1 - 4\Phi_3 + (2/v^2)\,\chi_{,00}\ ,\\ \bar{h}_{0\alpha} &= 4V_\alpha\ ,\quad \bar{h}_{\alpha\beta} = -4\Phi_{\alpha\beta} - 4\delta_{\alpha\beta}\,\Phi_4,\\ h &= -4U - 4\Phi_3 + 12\Phi_4 + (2/v^2)\,\chi_{,00},\end{aligned} \tag{42}$$

where Φ_1, Φ_3, Φ_4 are given in Will (1971a), and where $\Phi_{\alpha\beta}$ and χ are given by

$$\begin{aligned}\nabla^2\chi &= -2U\ ,\quad \chi = -\int\rho(\underset{\sim}{x}')|\underset{\sim}{x}-\underset{\sim}{x}'|d\underset{\sim}{x}',\\ \Phi_{\alpha\beta} &= \int\frac{\rho(\underset{\sim}{x}')v'_\alpha v'_\beta}{|\underset{\sim}{x}-\underset{\sim}{x}'|}d\underset{\sim}{x}'\ ,\quad \sum_\alpha \Phi_{\alpha\alpha} = \Phi_1\ .\end{aligned} \tag{43}$$

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Thus the metric is given by (eqs. [40] and [42])

$$g_{00} = 1 - 2U - 4\Phi_1 - 2\Phi_3 - 6\Phi_4 + (1/\nu^2)\, \chi_{,00} \quad ,$$
$$g_{0\alpha} = 4V_\alpha \, , \tag{44}$$
$$g_{\alpha\beta} = -\,(1 + 2U)\, \delta_{\alpha\beta} \, .$$

Making a gauge transformation {see Will (1971a) eqs. [18] and [20]} to the standard gauge in which the PPN metric (eq. [A1]) is written, we obtain

$$g_{00} = 1 - 2U - 4(\Phi_1 + \tfrac{1}{2}\Phi_3 + \tfrac{3}{2}\Phi_4) \quad ,$$
$$g_{0\alpha} = \tfrac{1}{2}\,[8 - (1/\nu^2)]\, V_\alpha + \tfrac{1}{2}\,(1/\nu^2)\, W_\alpha \, , \tag{45}$$
$$g_{\alpha\beta} = -\,(1 + 2U)\, \delta_{\alpha\beta} \, .$$

We can thus read off the PPN parameters by comparing equation (45) with equation (A1):

$$\begin{array}{lll} \beta = 0 \, , & \gamma_1 = 1 & , \\ \beta_1 = 1 \, , & \Delta_1 = \frac{1}{7}\,[8 - (1/\nu^2)] & , \\ \beta_2 = 0 \, , & \Delta_2 = (1/\nu^2) & , \\ \beta_3 = 1 \, , & \zeta = 0 & . \\ \beta_4 = 1 \, , & & \end{array} \tag{46}$$

(Note: β and β_2 are zero, because our theory is linearized in the gravitational fields.) Thus we have

$$\Delta_2 + \zeta - 1 = (1/\nu^2) - 1 \, ,$$
$$4\beta_1 + 2\gamma + 1 - 7\Delta_1 = (1/\nu^2) - 1 \, , \tag{47}$$

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and this theory predicts an anisotropy in the locally measured gravitational constant.

Nordtvedt (1969) has discussed the relation between Lorentz invariance and the values of the PPN parameters and has found that the parameter combination $(\Delta_2 + \zeta)$ should be unity in order that the g_{00} metric term be properly retarded. He also found that the parameters should satisfy

$$\begin{aligned} 4\beta_1 &= 2\gamma + 2 + \zeta \ , \\ 7\Delta_1 + \Delta_2 &= 4\gamma + 4 \ , \end{aligned} \tag{48}$$

in order that the PPN metric of a moving mass be obtained from that of a static mass by a Lorentz transformation. These results are in agreement with the results of this section and with the discussion in §I. [See also Will (1971b) for detailed discussion of this Lorentz invariance.]

V. CONCLUSIONS

The locally-measured Newtonian gravitational constant was calculated and was found to be anisotropic in Whitehead's theory and in a theory of gravity (devised in this paper) which predicted different flat-space propagation speeds for gravity and for light.

Earth-tide data were found to put an upper limit of $1/10^9$ on the magnitude of the anisotropy. This ruled out Whitehead's theory and showed that the speed of gravity and of light in the theory devised in this paper must be the same to within about 2 per cent.

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I would like to thank P. J. E. Peebles for suggesting Earth-tide data as experimental evidence limiting the anisotropy in G. I thank Wei-Tou Ni for helping me comprehend Whitehead's theory. Finally I wish to thank Kip S. Thorne, whose moral support made this paper possible and whose critical comments about presentation made it readable.

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APPENDIX A

THE PPN n-BODY POINT-MASS METRIC AND CHRISTOFFEL SYMBOLS

The metric for the Parametrized Post-Newtonian formalism has the form (Will 1971a)

$$
\begin{aligned}
g_{00} &= 1 - 2U + \beta U^2 - 4(\beta_1\Phi_1 + \beta_2\Phi_2 + \tfrac{1}{2}\beta_3\Phi_3 + \tfrac{3}{2}\beta_4\Phi_4) + \zeta\mathcal{A}, \\
g_{0\alpha} &= \tfrac{7}{2}\Delta_1 V_\alpha + \tfrac{1}{2}\Delta_2 W_\alpha, \qquad \text{(A1)} \\
g_{\alpha\beta} &= -(1 + 2\gamma U)\,\delta_{\alpha\beta},
\end{aligned}
$$

where β, β_1, β_2, β_3, β_4, γ, Δ_1, Δ_2, ζ are PPN parameters and U, Φ_1, Φ_2, Φ_3, Φ_4, $\mathcal{A}$, V_α, W_α are fields defined as integrals over the matter {for explicit definitions see Will (1971a) eq. [17]}. The point-mass metric can be obtained from equation (A1) by using a "conserved density" distribution (Will 1971a, eq. [28]) made up of point masses, or by taking Nordtvedt's (1969) point-mass metric and translating from his parameters to PPN parameters using Table 1 of Will (1971a). The result is

$$
\begin{aligned}
g_{00}(\underset{\sim}{x}) &= 1 - 2\sum_k \frac{m_k}{r_k} + 2\beta\left(\sum_k \frac{m_k}{r_k}\right)^2 - (4\beta_1 - 1)\sum_k \frac{m_k v_k^2}{r_k} \\
&\quad - (4\beta_2 - 6\gamma)\sum_k \frac{m_k}{r_k}\sum_{j\neq k}\frac{m_j}{r_{kj}} + \zeta\sum_k \frac{m_k}{r_k^2}(\underset{\sim}{v}_k\cdot\underset{\sim}{r}_k)^2, \qquad \text{(A2)} \\
g_{0\alpha}(\underset{\sim}{x}) &= \tfrac{7}{2}\Delta_1\sum_k \frac{m_k v_k^\alpha}{r_k} + \tfrac{1}{2}\Delta_2\sum_k \frac{m_k}{r_k^3}(\underset{\sim}{v}_k\cdot\underset{\sim}{r}_k)r_k^\alpha, \\
g_{\alpha\beta}(\underset{\sim}{x}) &= -(1 + 2\gamma\sum_k m_k/r_k)\,\delta_{\alpha\beta}.
\end{aligned}
$$

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The resulting point mass Christoffel symbols are

$$
\begin{aligned}
\Gamma^0{}_{00} &= -\sum_k (m_k/r_k{}^3)\, \underset{\sim}{v}_k \cdot \underset{\sim}{r}_k \quad , \qquad \Gamma^0{}_{0\alpha} = \sum_k m_k r_k{}^\alpha / r_k{}^3 \; , \\
\Gamma^0{}_{\alpha\beta} &= (\gamma + \tfrac{1}{2}\Delta_2)\, \delta_{\alpha\beta} \sum_k (m_k/r_k{}^3)\, \underset{\sim}{v}_k \cdot \underset{\sim}{r}_k - \tfrac{3}{2}\Delta_2 \sum_k (m_k/r_k{}^5)(\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k)\, r_k{}^\alpha r_k{}^\beta \\
&\quad - \tfrac{1}{4}(7\Delta_1 - \Delta_2) \sum_k (m_k/r_k{}^3)(v_k{}^\alpha r_k{}^\beta + v_k{}^\beta r_k{}^\alpha) \quad , \\
\Gamma^\alpha{}_{00} &= \sum_k \frac{m_k r_k{}^\alpha}{r_k{}^3} \Big[1 + (2\beta_2 - 3\gamma) \sum_{j \neq k} \frac{m_j}{r_{kj}} - (2\beta + 2\gamma) \sum_j \frac{m_j}{r_j} \\
&\quad + \tfrac{1}{2}(4\beta_1 + \Delta_2 - 1) v_k{}^2 - \tfrac{3}{2}(\Delta_2 + \zeta)(\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k / r_k)^2 + \tfrac{1}{2}\Delta_2 \sum_{j \neq k} (m_j / r_{kj}{}^3)\, \underset{\sim}{r}_k \cdot \underset{\sim}{r}_{kj} \Big] \qquad \text{(A3)} \\
&\quad + \tfrac{7}{2}\Delta_1 \sum_k \frac{m_k}{r_k} \sum_{j \neq k} \frac{m_j r_{kj}{}^\alpha}{r_{kj}{}^3} + \tfrac{1}{2}(2\zeta - 7\Delta_1 + \Delta_2) \sum_k \frac{m_k}{r_k{}^3} (\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k)\, v_k{}^\alpha , \\
\Gamma^\alpha{}_{0\beta} &= \gamma \delta_{\alpha\beta} \sum_k (m_k/r_k{}^3)\, \underset{\sim}{v}_k \cdot \underset{\sim}{r}_k + \tfrac{1}{4}(7\Delta_1 + \Delta_2) \sum_k (m_k/r_k{}^3)(v_k{}^\alpha r_k{}^\beta - v_k{}^\beta r_k{}^\alpha) \; , \\
\Gamma^\alpha{}_{\beta\eta} &= -\gamma \sum_k (m_k/r_k{}^3)(\delta_{\alpha\beta} r_k{}^\eta + \delta_{\alpha\eta} r_k{}^\beta - \delta_{\beta\eta} r_k{}^\alpha) .
\end{aligned}
$$

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APPENDIX B

THE n-BODY POINT-MASS METRIC IN WHITEHEAD'S THEORY[9]

---

9 I would like to acknowledge the contribution of Wei-Tou Ni to my understanding of Whitehead's theory. As a result of our discussions of his research on the post-Newtonian perfect-fluid metric for Whitehead's theory, I was able to learn enough to handle the point-mass version of the metric.

Whitehead's (1922) theory of gravity is a Lorentz-invariant action-at-a-distance metric theory. The metric g_{ij} determines the ticking rates of atomic clocks and the measurements made by physical rods, and determines the geodesics along which freely-falling test bodies move.[10] However, the

---

10 In its original form, Whitehead's theory could not describe measurements made by rods and atomic clocks, and said nothing about the trajectories of photons. The interpretation we use here was first introduced by Synge (1952) to make Whitehead's theory complete. For further discussion of Whitehead's theory, see Rayner (1954, 1955), Schild (1962), and Whitrow and Morduch (1965).

---

theory also contains a global Lorentz metric η_{ab} which is physically unobservable, but which appears in the equations used to calculate g_{ab}: for a field point with four-vector position X, g_{ab} is given by the following

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equations

$$g_{ab}(X) = \eta_{ab} - 2\sum_k m_k (y_k)_a (y_k)_b / w_k^3 ,$$

$$Y_k = X - X_k , \quad Y_k \cdot Y_k = 0 , \tag{B1}$$

$$w_k = Y_k \cdot (dX_k/d\sigma) ,$$

$$d\sigma^2 = \eta_{ab}\, dx^a\, dx^b ,$$

where "$\cdot$" means contraction with respect to η_{ab}. Thus the metric g_{ab} is determined at a point in spacetime by the effect of all other masses along the past η_{ab}-"light cone" of the point. By taking the usual low-velocity, weak-field approximation one can determine the post-Newtonian limit of the Whitehead metric. In the post-Newtonian metric, all field quantities are evaluated on a constant-time hypersurface rather than along the past η_{ab}-"light cone" at each point. The crucial formulas used in the derivation are

$$y_k^{\alpha} = r_k^{\alpha} + r_k v_k^{\alpha} + (\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k) v_k^{\alpha} - \tfrac{1}{2} a_k^{\alpha} r_k^2 ,$$

$$y_k^0 = r_k \left[1 + \underset{\sim}{v}_k \cdot \underset{\sim}{r}_k / r_k + \tfrac{1}{2} v_k^2 + \tfrac{1}{2} (\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k / r_k)^2 - \tfrac{1}{2} \underset{\sim}{a}_k \cdot \underset{\sim}{r}_k \right] , \tag{B2}$$

$$w_k = r_k \left[1 + \tfrac{1}{2} (\underset{\sim}{v}_k \cdot \underset{\sim}{r}_k / r_k)^2 + \tfrac{1}{2} \underset{\sim}{a}_k \cdot \underset{\sim}{r}_k \right] .$$

We make a gauge transformation to put the metric in the standard PPN form:

$$x^{\alpha} \rightarrow x^{\alpha} + \Sigma\, m_k r_k^{\alpha} / r_k ,$$

$$x^0 \rightarrow x^0 - 2\sum_k m_k \ln r_k + \tfrac{5}{2} (\partial/\partial t) \sum_k m_k r_k , \tag{B3}$$

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and obtain

$$g_{00} = 1 - 2\sum_k \frac{m_k}{r_k} + 2\left(\sum_k \frac{m_k}{r_k}\right)^2 + 3\sum_k \frac{m_k v_k^2}{r_k} + 2\sum_k \frac{m_k}{r_k}\sum_{j\neq k}\frac{m_j}{r_{kj}}$$

$$- 6\sum_k \frac{m_k}{r_k^3}(\underline{v}_k \cdot \underline{r}_k)^2 + 2\sum_k \frac{m_k \underline{r}_k}{r_k^3} \cdot \sum_{j\neq k} m_j\left(\frac{\underline{r}_{kj}}{r_j} - \frac{\underline{r}_j}{r_{jk}}\right),$$

(B4)

$$g_{0\alpha} = \tfrac{1}{2}\sum_k m_k v_k^{\alpha}/r_k + \tfrac{7}{2}\sum_k (m_k/r_k^3)\,(v_k \cdot r_k)\,r_k^2\,,$$

$$g_{\alpha\beta} = -\,(1 + 2\sum_k m_k/r_k)\,\delta_{\alpha\beta}\,.$$

Except for the final term in g_{00}, this is the same as a PPN metric with parameter values

$$\beta = 1\,, \qquad \Delta_1 = \frac{1}{7}\,, \qquad \zeta = -6\,,$$

$$\beta_1 = -\tfrac{1}{2}\,, \qquad \Delta_2 = 7\,, \qquad \gamma = 1\,, \qquad \text{(B5)}$$

$$\beta_2 = 1\,.$$

The Christoffel symbols for Whitehead's metric can be obtained by substituting the parameter values (eq. [B5]) into the PPN Christoffel symbols (eq. [A3]), and adding to Γ^{α}_{00} the contribution of the additional g_{00} term:

$$\delta\Gamma^{\alpha}_{00} = \sum_k \frac{m_k}{r_k^3}\sum_{j\neq k} m_j\left(\frac{r_{kj}^{\alpha}}{r_j} - \frac{r_j^{\alpha}}{r_{jk}}\right) - \sum_k \frac{m_k r_k^{\beta}}{r_k^3}\sum_{j\neq k} m_j\left(\frac{r_{kj}^{\beta} r_j^{\alpha}}{r_j^3} + \frac{\delta^{\alpha\beta}}{r_{jk}}\right)$$

(B6)

$$- 3\sum_k \frac{m_k r_k^{\alpha}}{r_k^5}\,\underline{r}_k \cdot \sum_{j\neq k} m_j\left(\frac{\underline{r}_{kj}}{r_j} - \frac{\underline{r}_j}{r_{jk}}\right).$$

In the case of two bodies, these results agree with those obtained by Clark (1954).

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TABLE 1

A CATALOGUE OF NEGLECTED ACCELERATIONS

Acceleration Neglected and Algebraic Form	PPN Parameter Dependence*	Magnitude	Typical Period of Variation
1) Equivalence Principle Breakdown†: $\sum_{k\neq 0,1} \frac{m_k}{r_{1k}^3} \frac{(\underset{\sim}{r}_{10}\cdot \underset{\sim}{r}_{1k})}{r_{10}} \left[\frac{E_1(\text{self-gravity})}{m_1}\right]$	$7\Delta_1 - 3\gamma - 4\beta$	$\lesssim 10^{-12}$ g	24 hours
2) Structure of the Earth†: $\sum_{k\neq 0,1} \frac{m_k}{r_{1k}^3} \frac{(\underset{\sim}{r}_{10}\cdot \underset{\sim}{r}_{1k})}{r_{10}} \left[\frac{\text{Mom. of Inertia}}{r_{10}^3}\right]$	$5\gamma - 4\beta_2 - \Delta_2$	$\lesssim 10^{-12}$ g	24 hours
3) Gradients of External Field†: $\frac{m_1}{r_p^2} \sum_{k\neq 0,1} \frac{m_k}{r_{1k}^3} (\underset{\sim}{r}_{10}\cdot \underset{\sim}{r}_{1k})$	$7\Delta_1 + \Delta_2 - 4\gamma - 4$	$\lesssim 10^{-12}$ g	24 hours
4) Ether Drift-Rotation Coupling: $\frac{m_1}{r_p^2} \left[\underset{\sim}{v}_1\cdot(\underset{\sim}{v}_1 - \underset{\sim}{v}_0)\right]$	$7\Delta_1 + \Delta_2 - 4\gamma - 4$	10^{-9} g	24 hours
5) Effects of Earth's Field: $\frac{m_1}{r_p^2} \times \left\{\frac{m_1}{r_p}, \int_{\text{photon path}} \frac{m_1}{r_1}\,dt\right\}$	—	10^{-9} g	constant

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TABLE 1 (continued)

Acceleration Neglected and Algebraic Form	PPN Parameter Dependence*	Magnitude	Typical Period of Variation
6) Post-Newtonian-Centrifugal: $\frac{(v_1 - v_0)^2}{r_p} \times \left\{ \frac{m_k}{r_{1k}}, \frac{m_1}{r_{10}}, v_k^2, \text{etc.} \right\}$	—	$< 10^{-9}$ g	various periods
7) Post-Newtonian-Tidal: $\sim \left(\frac{m_j r_{10}}{r_{1j}^3} \right) \times \left\{ \frac{m_k}{r_{1k}}, \frac{m_1}{r_{10}}, v_k^2, \text{etc.} \right\}$	—	$< 10^{-13}$ g	various periods
8) Gradient-Velocity Coupling: $\frac{m_j}{r_{1j}^2} v_k^2$	—	$< 10^{-10}$ g	various periods
9) Ether-Centrifugal Force Coupling: $\frac{[v_1 \cdot (v_1 - v_0)]^2}{r_p}$	—	10^{-9} g	12 hours

* A blank entry in this column means that the parameter dependence for that effect has never been precisely determined.

† Obtained by Nordtvedt (1971).

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REFERENCES

Barsenkov, S. N. 1967, Bull. (Izv.) Acad. Sci. U.S.S.R., Earth Phys., 3, 170.

Brans, C. 1962a, Phys. Rev., 125, 388.

_______ 1962b, ibid., 125, 2194.

Chandrasekhar, S. 1965, Ap. J., 142, 1488.

Chandrasekhar, S., and Contopoulos, G. 1967, Proc. Roy. Soc., 298A, 123.

Clark, G. L. 1954, Proc. Roy. Soc. Edinburgh, 64A, 49.

Harrison, J. C., Ness, N. F., Longman, I. M., Forbes, R. F. S., Kraut, E. A., and Slichter, L. B. 1963, J. Geophys. Res., 68, 1497.

Melchior, P. 1966, The Earth Tides (Oxford: Pergamon Press).

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972, Gravitation (San Francisco: W. H. Freeman and Co.) in press.

Ni, W.-T. 1971, paper in preparation.

Nordtvedt, K., Jr. 1968, Phys. Rev., 169, 1017.

_______ 1969, ibid., 180, 1293.

_______ 1970, Ap. J., 161, 1059.

_______ 1971, Phys. Rev. D., in press

Pariiskii, N. N., Barsenkov, S. N., Volkov., V. A., Gridnev, D. G., and Kramer, M. V. 1967, Bull. (Izv.) Acad. Sci. U.S.S.R., Earth Phys., 2, 107.

Peebles, P. J. E. 1962, Ann. Phys. (N.Y.), 20, 240.

Peebles, P. J. E., and Dicke, R. H. 1962, Phys. Rev., 127, 629.

Rayner, C. B. 1954, Proc. Roy. Soc., 222A, 509.

_______ 1955, ibid., 232A, 135.

Rose, R. D., Parker, H. M., Lowry, R. A., Kuhlthau, A. R., and Beams, J. W. 1969, Phys. Rev. Letters, 23, 655.

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Schild, A. 1962, in Evidence for Gravitational Theories, ed. C. Møller (New York: Academic Press).

Shapiro, I. I., Smith, W. B., Ash, M. B., Ingalls, R. P., and Pettengill, G. H. 1971, Phys. Rev. Letters, **26**, 27.

Synge, J. L. 1952, Proc. Roy. Soc., **211A**, 303.

Temple, G. 1924, Proc. Phys. Soc., **36**, 176.

Whitehead, A. N. 1922, The Principle of Relativity (Cambridge: Cambridge University Press).

Whitrow, G. J., and Morduch, G. E. 1965, in Vistas in Astronomy, Vol. 6, ed. A. Beer (Oxford: Pergamon Press).

Will, C. M. 1971a, Ap. J., **163**, 611.

__________ 1971b, paper in preparation.

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