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SOME PROBLEMS IN NONLINEAR ELASTICITY

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ABSTRACT

Two separate problems are discussed: axisymmetric equilibrium configurations of a circular membrane under pressure and subject to thrust along its edge, and the buckling of a circular cylindrical shell.

An ordinary differential equation governing the circular membrane is imbedded in a family of n-dimensional nonlinear equations. Phase plane methods are used to examine the number of solutions corresponding to a parameter which generalizes the thrust, as well as other parameters determining the shape of the nonlinearity and the undeformed shape of the membrane. It is found that in any number of dimensions there exists a value of the generalized thrust for which a countable infinity of solutions exist if some of the remaining parameters are made sufficiently large. Criteria describing the number of solutions in other cases are also given.

Donnell-type equations are used to model a circular cylindrical shell. The static problem of bifurcation of buckled modes from Poisson expansion is analyzed using an iteration scheme and pertubation methods. Analysis shows that although buckling loads are usually simple eigenvalues, they may have arbitrarily large but finite multiplicity when the ratio of the shell's length and circumference is rational. A numerical study of the critical buckling load for simple eigenvalues indicates that the number of waves along the axis of the deformed shell is roughly proportional to the length of the shell, suggesting the possibility of a "characteristic length." Further numerical work indicates that initial post-buckling curves are typically steep, although the load

may increase or decrease. It is shown that either a sheet of solutions or two distinct branches bifurcate from a double eigenvalue. Furthermore, a shell may be subject to a uniform torque, even though one is not prescribed at the ends of the shell, through the interaction of two modes with the same number of circumferential waves. Finally, multiple time scale techniques are used to study the dynamic buckling of a rectangular plate as well as a circular cylindrical shell; transition to a new steady state amplitude determined by the nonlinearity is shown. The importance of damping in determining equilibrium configurations independent of initial conditions is illustrated.

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INTRODUCTION

This study is concerned with two separate problems. The first is motivated by equations which model the behavior of a circular membrane. The resulting equation is imbedded in a class of equations, and the existence of multiple solutions is analyzed for this class. The second problem is to study the static and dynamic buckling of a circular cylindrical shell under axial loading.

For the first problem, we are concerned with studying the possibility of multiple equilibrium configurations of a circular shallow elastic membrane whose surface is subject to an axisymmetric pressure. A radial thrust is specified along the membrane's edge, and the edge is restrained from deforming normal to its midplane. Only axisymmetric deformations of the membrane are considered.

In chapter 1 we study the case of an initially flat membrane under a variable pressure. The situation of a flat membrane under constant pressure has been studied by A. Callegari, E. Reiss, and H. Keller [2]. In chapter 2 we consider a membrane which is not initially flat and is subjected to a variable pressure.

The reader is referred to the references [1, 2] for a derivation of the membrane theory. Notes on the final formulation of the problem are given in Appendix C. The resulting equation is

$$\frac{d}{dr}\left(r^3 \frac{du}{dr}\right) + \lambda^3 \frac{G}{(1-u)^2} = \lambda B r \phi^2 \tag{C.6}$$

The boundary conditions are

$$\frac{du}{dr} = 0 \quad \text{at } r = 0 \tag{C.7a}$$

$$u(1) = 0 \tag{C.7b}$$

When G(r) and $\phi(r)$ are of the form to be prescribed in chapters 1 and 2, it is found that equation (C.6) can be transformed into a second order autonomous system which is amenable to phase plane analysis. This method was first used by Gel′fand [4] to study solution multiplicity in certain problems arising in the theory of chemical reactors, viz.

$$\left.\begin{aligned} &\frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \frac{du}{dr}\right) + \lambda e^{u} = 0 \quad , \quad n = 1, 2, 3 \\ &\frac{du}{dr} = 0 \quad \text{at } r = 0 \\ &u(1) = 0 \end{aligned}\right\} \tag{0.1}$$

He found that there exists a value $\lambda_* > 0$ such that there are

(a) no solutions when $\lambda > \lambda_*$ $(n = 1, 2, 3)$

(b) one solution when $\lambda = \lambda_*$ $(n = 1, 2, 3)$

(c) two solutions when $0 < \lambda < \lambda_*$ $(n = 1, 2)$

(d) a countable infinity of solutions when $\lambda = \lambda_\infty = 2$ and $n = 3$

(e) a finite but large number of solutions when $n = 3$ and $|\lambda - \lambda_\infty|$ is small.

A. Callegari, E. Reiss, and H. Keller [2] applied this method to study an initially flat circular membrane under constant pressure, modeled by

$$\frac{1}{r^3} \frac{d}{dr}\left(r^3 \frac{du}{dr}\right) + \lambda (1-u)^{-2} = 0$$

Here the differential operator is a Laplacian in four dimensions $(n = 4)$. They found the behavior can be described by (a), (b), (d), and (e) with $\lambda_\infty = \sqrt[3]{16/9}$.

Joseph and Lundgren [8] studied (0.1) and

$$\frac{1}{r^{n-1}} \frac{d}{dr} (r^{n-1} \frac{du}{dr}) + \lambda(1-\alpha u)^{1-\beta} = 0 \qquad (0.2)$$

for arbitrary positive integers n and for $\beta > 1$, $\alpha > 0$. For (0.1) they found that (a) and (b) hold for $n \geq 1$, that (d) and (e) hold for $2 < n < 10$ with $\lambda_\infty = n(n-2)$, and that for $n \geq 10$ there exists one solution for $\lambda < 2(n-2)$. For (0.2) they found (a) and (b) hold for $n \geq 1$, (d) and (e) hold when $n-2 < f(\beta)$, and for $n-2 \geq f(\beta)$ there is only one solution in $0 < \lambda < \lambda_*$. Here

$$f(\beta) = \frac{4(\beta-1)}{\beta} + 4\sqrt{\frac{\beta-1}{\beta}}$$

(Note: [8] also includes a similar study for $\alpha < 0$ and $\beta < 1$).

In chapters 1 and 2 of this study we find that equation (C.6), for appropriate functions G and ϕ, is of the type

$$\frac{1}{r^{n-1}} \frac{d}{dr} (r^{n-1} \frac{du}{dr}) + \lambda^{\beta} r^{\mu} (1-\alpha u)^{1-\beta} = \lambda A\, r^{2+(\mu+2)/\beta} \qquad (0.3)$$

with $\beta > 1$ and $\alpha \neq 0$. We investigate solutions of (0.3) for all real λ, thereby generalizing the results of [8]. Of particular interest is the result that for $\alpha > 0$ there exist values μ^* (for $n \geq 1$) and A_0 (for $n \geq 3$) such that if $\mu > \mu^*$ or $A > A_0$, then the situation may be described by (a), (b), (d), and (e) with appropriate λ_∞. (The cases of A large and n = 1, 2 are not investigated here.) From this we see that the possibility of an infinity of solutions persists in all dimensions. We summarize our results below.

<u>For A = 0</u> there exist λ_* and μ^* such that, for $n \geq 1$, there are

(a) no solutions for $\alpha\lambda^{\beta} > \alpha\lambda_*^{\beta} > 0$

(b) one solution for $\alpha\lambda^{\beta} < 0$

(c) finitely many solutions for $0 < \alpha\lambda^{\beta} < \alpha\lambda_*^{\beta}$ if $\mu \le \mu^*$

(d) a countable infinity of solutions when $\lambda = \lambda_\infty$ if $\mu > \mu^*$

(e) a finite but large number of solutions if $|\lambda - \lambda_\infty| \ne 0$ is small and $\mu > \mu^*$.

Here μ^* is such that $\Phi(\mu^* + 2) = 0$, where

$$\Phi(\nu) = -4(\beta-1)[\nu + \tfrac{1}{2}\beta(n-2)]^2 + \beta^3(n-2)^2 \tag{1.23}$$

<u>For $A \ne 0$</u> we restrict β to integral values and take $n > 2$. When $|A| \ll 1$ the situation is the same as for $A = 0$, except that μ^* depends on A. For $|A|$ sufficiently large we find

(a) β odd, $A > 0$, $\alpha > 0$: For $\lambda > 0$ there exist λ_∞ and λ_* as above. For $\lambda < 0$ either there exists one solution for all λ or λ_*^- exists such that no solutions exist for $\lambda < \lambda_*^-$ and finitely many exist for $\lambda_*^- < \lambda < 0$.

(b) β odd, $A < 0$, $\alpha > 0$: For $\lambda > 0$, λ_* exists but there is no λ_∞ and hence there are only finitely many solutions. For $\lambda < 0$, there exists one solution for all λ.

(c) β even, $A > 0$, $\alpha > 0$: λ_*, λ_∞, and λ_*^- as above all exist.

(d) β even, $A > 0$, $\alpha < 0$: For $\lambda > 0$ either there exists one solution for all λ or λ_* exists but λ_∞ does not. For $\lambda < 0$ there exists one solution for all λ.

The cases omitted may be found by transforming $\alpha \to -\alpha$, $\lambda \to -\lambda$ for β odd and $A \to -A$, $\lambda \to -\lambda$ for β even.

For the second problem, we are concerned with studying the buckling of a circular cylindrical shell under axial loading. The fundamental equations for a Donnell-type model are developed in Appendix A.

In chapter 3 we consider the static problem. The classical solution known as Poisson expansion is introduced, and the problem of the bifurcation of equilibrium states from this solution is formulated. We analyze the multiplicity (i.e., the number of independent eigenfunctions) of eigenvalues or buckling loads and find that although they are typically simple, an arbitrarily large albeit finite multiplicity is possible when the ratio of the shell's circumference and length is rational. A numerical study is made of the mode corresponding to the critical buckling load, and it is found that the number of waves along the axis of the shell is roughly proportional to the length of the shell, suggesting the possibility of a "characteristic length" over which buckling occurs. An iteration scheme developed by H. Keller and W. Langford [13] is utilized to calculate the initial post-buckling curve for simple eigenvalues. We find that the load may increase or decrease, but regardless, the load-deflection curve is usually very steep. A pertubation scheme is used to study the number of bifurcating branches when the buckling load is a double eigenvalue. Several possibilities occur: there may exist a one or two-parameter "sheet" of solutions, or else there exist precisely two branches of solutions. A final calculation shows that, through the interaction of two modes with the same number of circumferential waves, the shell may be subjected to non-vanishing uniform torque even though no torque is prescribed at the ends of the shell.

Chapter 4 treats the dynamic problem when the load is such that Poisson expansion is unstable. The load is taken to be a "small distance" into the unstable regime, and a pertubation scheme employing multiple time scales is utilized. This method was first used by B. Matkowsky[14]. We first apply it to the dynamic buckling of a rectangular plate, and the results are compared to those of a study by Reiss and Matkowsky[15] of the buckling of rods. The equation governing the amplitude of the unstable mode is found to be a second order autonomous equation in the absence of damping. However, the equilibrium points depend on the initial conditions, which contradicts the fact that equilibrium configurations satisfy the time-independent steady state equations. When damping is present, the terms depending on the initial conditions vanish exponentially, and bounded solutions are shown to be asymptotic to the critical points of a reduced autonomous system. A similar discussion applies to the problem of a circular cylindrical shell. Qualitatively the two problems differ in that the reduced system for a plate is two dimensional, but that of a cylindrical shell is four dimensional. Also the plate has two physically distinct stable equilibrium configurations, but the cylindrical shell has only one. (For both problems we assume that the critical eigenvalue is simple.)

CHAPTER 1: INITIALLY FLAT MEMBRANES

In this chapter we will analyze the number of equilibrium configurations of an initially flat membrane subject to a pressure distribution of the form

$$p(r^*) = p_{max}(r^*/R)^c$$

for $c \geq 0$. Substituting this into the formulae given in equations (C.1) results in

$$G(r) = r^{\mu+3}$$

$$P = \frac{2}{(\mu+4)^2}\left(\frac{p_{max}}{E}\right)^2\left(\frac{R}{h}\right)^2$$

where we have set $\mu = 2c$. A flat membrane is described by $\phi(r) \equiv 0$; hence equation (C.6) becomes

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} + \lambda^3 \frac{r^\mu}{(1-u)^2} = 0$$

subject to boundary conditions (C.7).

Recognizing $\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr}$ as a spherically symmetric Laplacian in four dimensions motivates the following simple generalization of the membrane problem:

$$\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} + \lambda^\beta r^\mu (1-\alpha u)^{1-\beta} = 0 \quad , \quad 0 < r < 1 \tag{1.1}$$

$$\lim_{r\to 0} \left|\frac{1}{r}\frac{du}{dr}\right| < \infty \tag{1.2}$$

$$u(1) = 0 \tag{1.3}$$

The regularity condition (1.2) only mildly strengthens the previous boundary condition (C.7a). We assume $\alpha \neq 0$ so that the problem is truly nonlinear; furthermore, we retain the assumption that $\mu \geq 0$. Finally, we restrict β such that $\beta > 1$; this restriction on the form of the nonlinearity will play a strategic role in certain of the arguments to follow.

We seek a solution $u \in C^2(0, 1)$, so equation (1.1) implies that $1-\alpha u(r) \neq 0$ for $0 < r < 1$. $u(1) = 0$ and continuity then imply that $1-\alpha u > 0$ in $(0, 1]$. We extend this and require

$$1-\alpha u(r) > 0 \quad , \quad 0 \leq r \leq 1 \tag{1.4}$$

Elementary considerations using the theory of Lie lead to the following change of variables:

$$x = \log r \tag{1.5}$$

$$v(x) = (1-\alpha u)r^{\gamma} \tag{1.6}$$

where

$$\gamma \equiv -(\mu+2)/\beta \tag{1.7}$$

Note that $\gamma < 0$. These new variables transform equation (1.1) into the equivalent autonomous equation

$$\frac{d^2v}{dx^2} - (2\gamma+2-n)\frac{dv}{dx} + \gamma(\gamma+2-n)v - \alpha\lambda^{\beta}v^{1-\beta} = 0$$

or

$$\frac{d^2v}{dx^2} - (\gamma+\theta)\frac{dv}{dx} + \gamma\theta v - \alpha\lambda^{\beta}v^{1-\beta} = 0 \tag{1.8}$$

where we have defined $\quad \theta \equiv \gamma - (n-2) \quad$ (1.9)

Boundary conditions (1.2) and (1.3) become respectively

$$\lim_{x \to -\infty} e^{-(\gamma+2)x} \left| \frac{dv}{dx} - \gamma v \right| < \infty \tag{1.10}$$

and

$$v = 1 \quad \text{at } x = 0 \tag{1.11}$$

Remark also that condition (1.4) implies

$$v \to +\infty \quad \text{as } x \to -\infty \tag{1.12}$$

since $\gamma < 0$.

Although it is possible to study equation (1.8) in the phase plane directly, one last change of variables proves to greatly simplify the analysis. Set

$$y(x) = \alpha\lambda^{\beta} v^{-\beta} \tag{1.13}$$

$$z(x) = \frac{1}{v}\frac{dv}{dx} \tag{1.14}$$

We find that equation (1.8) is equivalent to the system

$$\dot{y} = -\beta yz \equiv f(y, z) \tag{1.15a}$$

$$\dot{z} = y - (z-\gamma)(z-\theta) \equiv g(y, z) \tag{1.15b}$$

where differentiation with respect to x is indicated by a dot. The boundary conditions become

$$\lim_{x\to-\infty} e^{-(\gamma+2)x}(z-\gamma)|y|^{-1/\beta} < \infty \tag{1.16}$$

$$y(0) = \alpha\lambda^{\beta} . \tag{1.17}$$

Furthermore, (1.12), (1.13) and the hypothesis $\beta > 1$ imply

$$y \to 0 \quad \text{as} \quad x \to -\infty \tag{1.18}$$

In making the transformation (1.13) we have tacitly assumed that $\lambda \neq 0$. When $\lambda = 0$ it is easily verified that equations (1.1), (1.2) and (1.3) have the unique solution

$$u(r) = 0 \quad , \quad 0 \le r \le 1$$

for $n > 0$. In the remainder of the chapter $\alpha\lambda^{\beta} \neq 0$ will be presumed.

Depending on the dimension, three cases arise in the phase plane analysis, namely: $n > 2$, $n = 2$ and $n = 1$.

<u>The case $n > 2$</u>

From (1.9) we see that $\theta < \gamma < 0$. System (1.15) has three critical points in the finite plane:

$$P_1: \; y = 0, \; z = \gamma$$

$$P_2:\ y = 0,\ z = \theta$$

$$P_3:\ y = \gamma\theta,\ z = 0$$

In figure 2 we indicate the field of tangent vectors corresponding to system (1.15), including the locus Γ where $\dot{z} = g(y, z) = 0$. Note that $y \equiv 0$, $\dot{z} = -(z-\gamma)(z-\theta)$ provides three exact solutions whose trajectories completely cover the z axis; consequently no trajectory can cross the z axis.

Next consider the local behavior about each critical point. We readily compute $f_y = -\beta z$, $f_z = -\beta y$, $g_y = 1$, $g_z = -2z+\gamma+\theta$ and so the equation for the characteristic exponent ℓ at the critical point (y_0, z_0) is

$$\begin{vmatrix} -\beta z_0 - \ell & -\beta y_0 \\ 1 & -2z_0+\gamma+\theta-\ell \end{vmatrix} = 0 \qquad (1.19)$$

For P_1, $y_0 = 0$, $z_0 = \gamma$ and (1.19) becomes

$$\begin{vmatrix} -\beta\gamma-\ell & 0 \\ 1 & \theta-\gamma-\ell \end{vmatrix} = 0$$

which has roots $\ell = \ell_1 = -\beta\gamma$ and $\ell = \ell_2 = \theta-\gamma$. Now $-\beta\gamma = \mu+2 > 0$ and $\theta-\gamma = -(n-2) < 0$ for $n > 2$. Consequently P_1 is always a saddle point.

For P_2, $y_0 = 0$, $z_0 = \theta$ and the roots are $\ell_1 = -\beta\theta$ and $\ell_2 = \gamma-\theta$. $\beta > 1$ and $\theta < 0$ mean $\ell_1 > 0$. $\gamma-\theta = n-2 > 0$ for $n > 2$ mean $\ell_2 > 0$. Consequently, P_2 is an unstable node. Furthermore, $\beta > 1$ and $n > 2$ imply that P_2 is an improper node, for recalling the definitions of γ and θ we find

$$\begin{aligned} \ell_1 &= -\beta\theta = -\beta\gamma + \beta(n-2) = (\mu+2) + \beta(n-2) \\ &> n-2 = \gamma-\theta = \ell_2 \end{aligned}$$

To describe the behavior near P_2 in more detail, set

$$y = \xi \quad , \quad z = \theta + \zeta$$

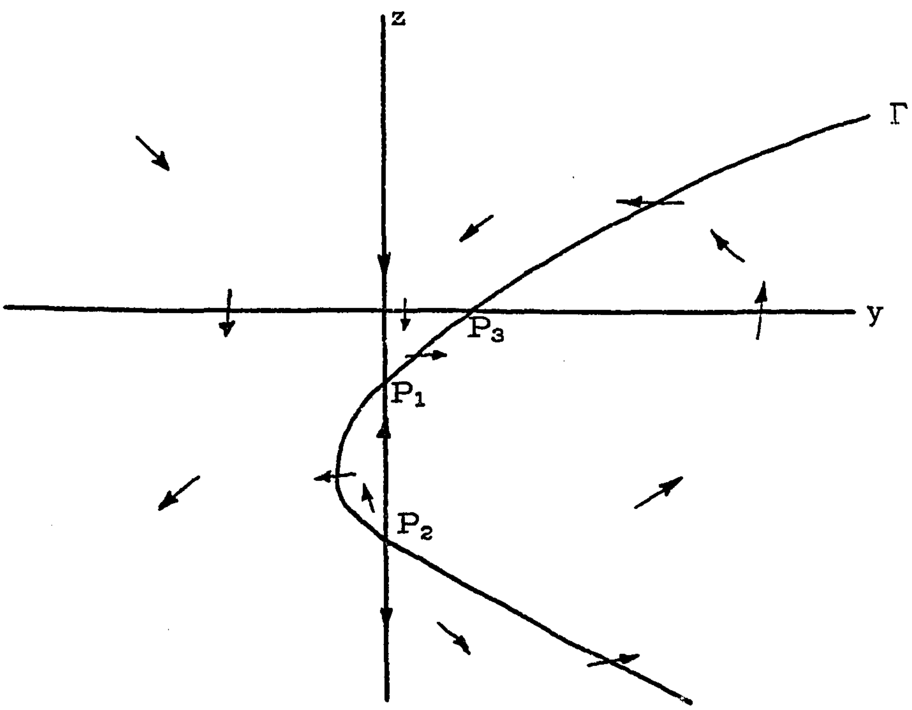


Figure 2 The field of tangent vectors for $n > 2$

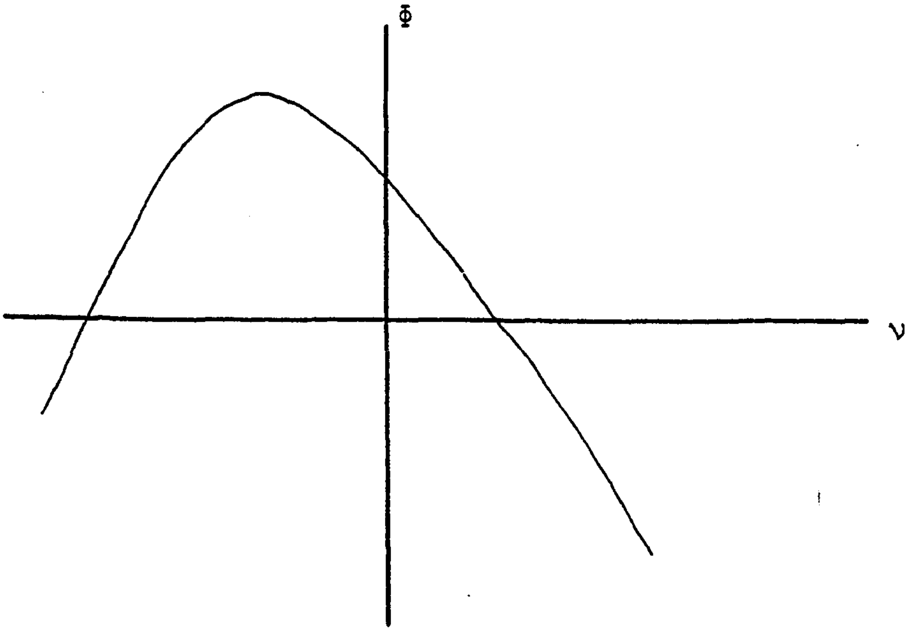


Figure 3 $\Phi(\nu)$ for $\beta > 1$, $n > 2$

We obtain the linearized form of equations (1.15)

$$\dot{\xi} = -\beta\theta\,\xi$$
$$\dot{\zeta} = \xi + (n-2)\zeta$$

which has solutions

$$\left.\begin{aligned} \xi &= a\,e^{-\beta\theta x} \\ \zeta &= b\,e^{(n-2)x} - \frac{a}{\beta\theta + n-2}\,e^{-\beta\theta x} \end{aligned}\right\} \qquad (1.20)$$

Since P_2 is an unstable node, trajectories approach P_2 for $x \to -\infty$. This, and the relation $-\beta\theta > n-2 > 0$ imply that trajectories are tangent to the ζ (or z) axis unless $b = 0$, in which case there exist two trajectories tangent to the line $\xi + (\beta\theta + n-2)\zeta = 0$.

For P_3, $y_0 = \gamma\theta$, $z_0 = 0$ and the characteristic equation is

$$\ell^2 - (\gamma + \theta)\ell + \beta\gamma\theta = 0$$

which has roots

$$\left.\begin{aligned} \ell_1 &= \tfrac{1}{2}(\gamma+\theta) + \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \\ \ell_2 &= \tfrac{1}{2}(\gamma+\theta) - \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \end{aligned}\right\} \qquad (1.21)$$

Now $4\beta\gamma\theta = -4(\mu+2)\theta > 0$ so that the real parts of ℓ_1 and ℓ_2 always have the same sign; consequently P_3 is always an attractor (i.e., a spiral point or a node). Furthermore, $\gamma+\theta < 0$ so that P_3 is a stable attractor.

Consider first when P_3 is a spiral point, i.e.,

$$(\gamma+\theta)^2 - 4\beta\gamma\theta < 0 \quad .$$

This relation is equivalent to the following inequality in terms of the original parameters β, μ and n:

$$\Phi(\mu+2) < 0 \qquad (1.22)$$

where

$$\Phi(\nu) \equiv -4(\beta-1)[\nu + \tfrac{1}{2}\beta(n-2)]^2 + \beta^3(n-2)^2 \qquad (1.23)$$

Figure 3 shows the qualitative behavior of $\Phi(\nu)$ for $\beta > 1$, $n > 2$. In the region $\nu > 0$, Φ decreases monotonically, so for given β and n, we have a spiral for all $\mu \geq \mu^*$ if and only if $\Phi(\mu^*+2) < 0$. In particular, P_3 is a spiral point for all $\mu \geq 0$ if $\Phi(2) < 0$, and this can be shown to be equivalent to

$$\beta^2(n-2)(n-10) + 8\beta(n-4) + 16 < 0 \tag{1.24}$$

For the membrane problem originally proposed, $n = 4$ and $\beta = 3$; it is a simple matter to verify that (1.24) is indeed satisfied for these values of the parameters. More generally, for given values of $n > 2$ and $\beta > 1$ there exists a value μ^* such that for all $\mu > \mu^*$, P_3 is a spiral point.

Next we consider the structure of the phase plane near P_3 when it is a node. (It will turn out that the local structure about P_3 critically determines the number of equilibrium solutions of the membrane problem.) Set $y = \gamma\theta + \xi$, $z = \zeta$; the linearized form of equations (1.15) is now

$$\begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma\theta \\ 1 & \gamma+\theta \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} .$$

Eigenfunctions

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = e^{\ell x} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

satisfy

$$\begin{pmatrix} -\ell & -\beta\gamma\theta \\ 1 & \gamma+\theta-\ell \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $c_1 = [\ell-(\gamma+\theta)]c_2$. Note from equations (1.21) that

$$\ell_1 - (\gamma+\theta) = -\ell_2$$

$$\ell_2 - (\gamma+\theta) = -\ell_1$$

The linearized theory in the neighborhood of P_3 thus provides the approximate relations

$$\begin{pmatrix}\xi\\ \zeta\end{pmatrix} \doteq a e^{\ell_1 x}\begin{pmatrix}-\ell_2\\ 1\end{pmatrix} + b e^{\ell_2 x}\begin{pmatrix}-\ell_1\\ 1\end{pmatrix}$$

for some constants a, b for a given trajectory. Since $\ell_2 < \ell_1 < 0$, we can conclude from this that in a neighborhood of P_3 all trajectories are tangent to the line $y - \gamma\theta + \ell_2 z = 0$, except for one pair of trajectories which is tangent to the line $y - \gamma\theta + \ell_1 z = 0$.

In the special case that $\ell_2 = \ell_1 < 0$, there is only one eigenvector, and the general theory for critical points shows that P_3 is again an improper node with all trajectories tangent to the same line.

In figure 4 we summarize the above results about the behavior of trajectories in a neighborhood of each of the finite critical points. Some reference to figure 2 may also be helpful. We note in passing that the tangent line to Γ at P_3 is $y - \gamma\theta + (\gamma+\theta)z = 0$. A justification that the linearized theory does in fact accurately describe the behavior of the solutions of the full nonlinear equations in neighborhoods of each of the critical points can be found in a standard reference, such as Coddington and Levinson [3].

By combining the results of figures 2, 4a, and 4b we can ascertain that the phase plane for $y \le 0$ has the qualitative form of figure 5, regardless of the behavior at P_3. We now concentrate on completing the phase plane for $y > 0$. First we show that no limit cycles exist. Introduce $\eta = \log y$ and compute

$$\left.\begin{aligned} \dot\eta &= \dot y/y = -\beta z \equiv F(\eta, z)\\ \dot z &= y - (z-\gamma)(z-\theta) = e^{\eta} - (z-\gamma)(z-\theta) \equiv G(\eta\ z) \end{aligned}\right\} \quad (1.25)$$

Suppose a limit cycle exists (necessarily it must lie entirely within the region $y > 0$). Then its image in the η, z plane must also be a simple

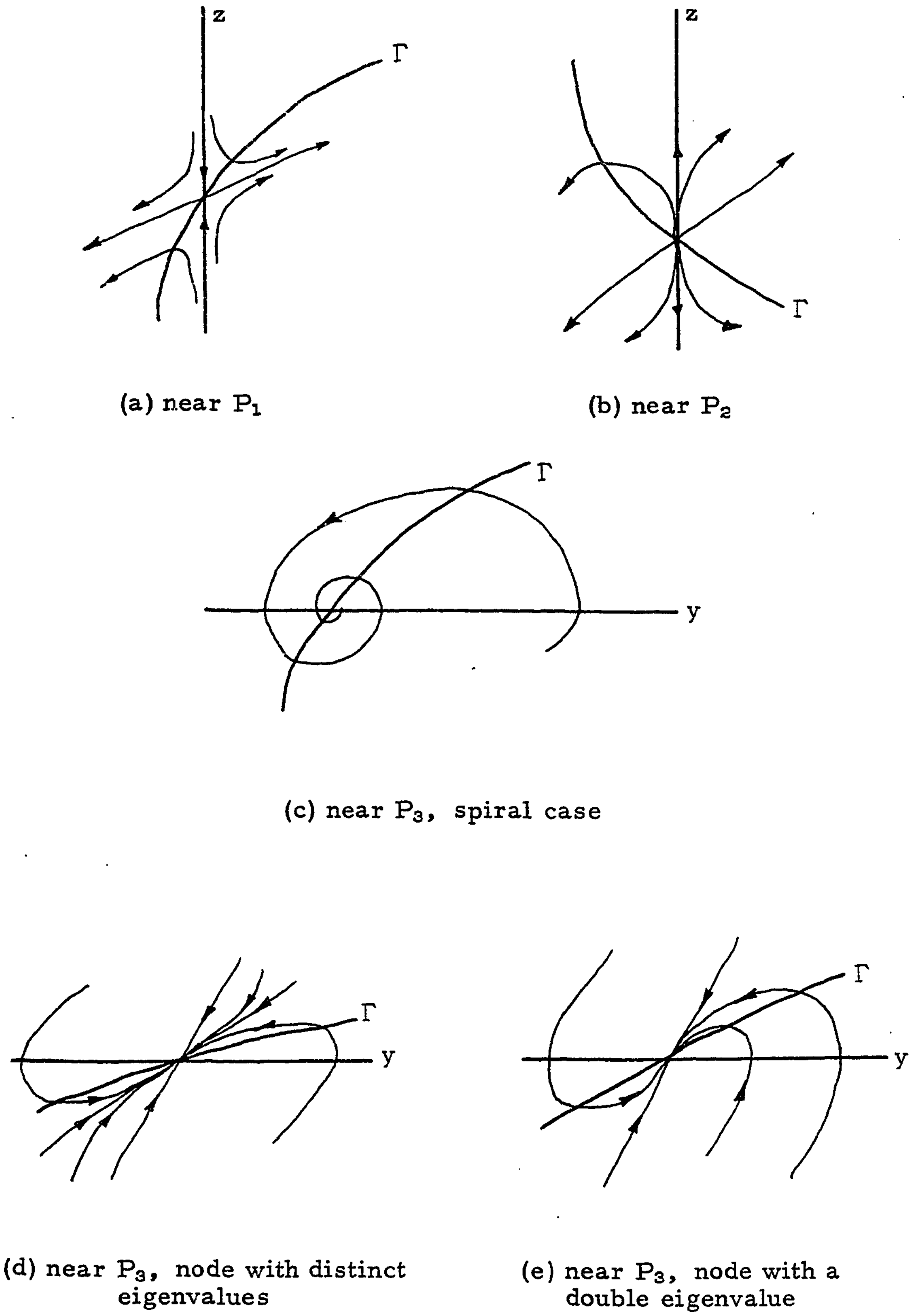


Figure 4 Phase plane structure near the critical points

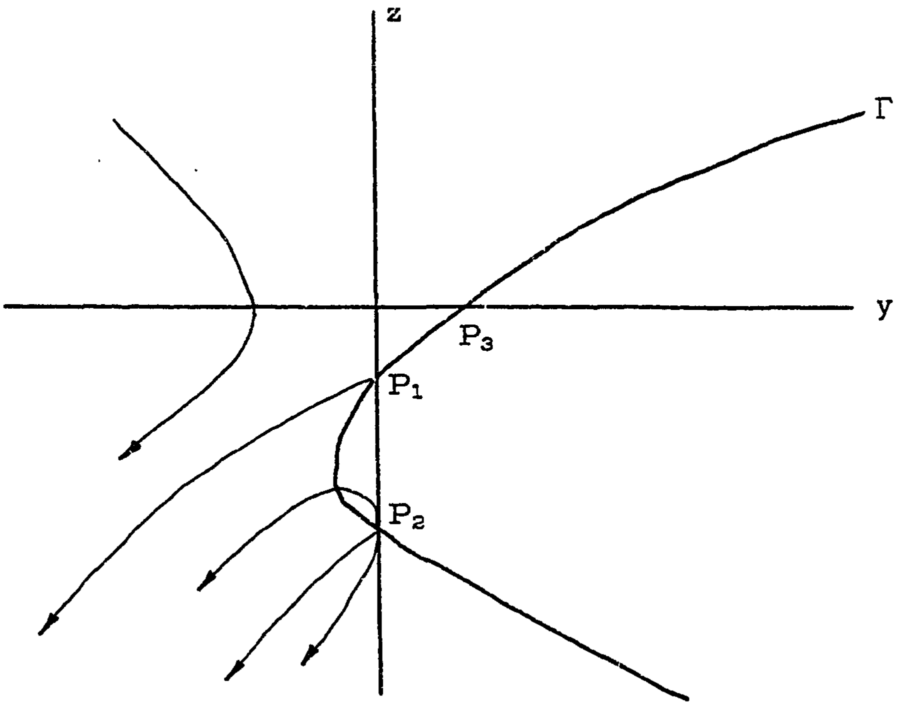


Figure 5 Phase plane for $y \leq 0$

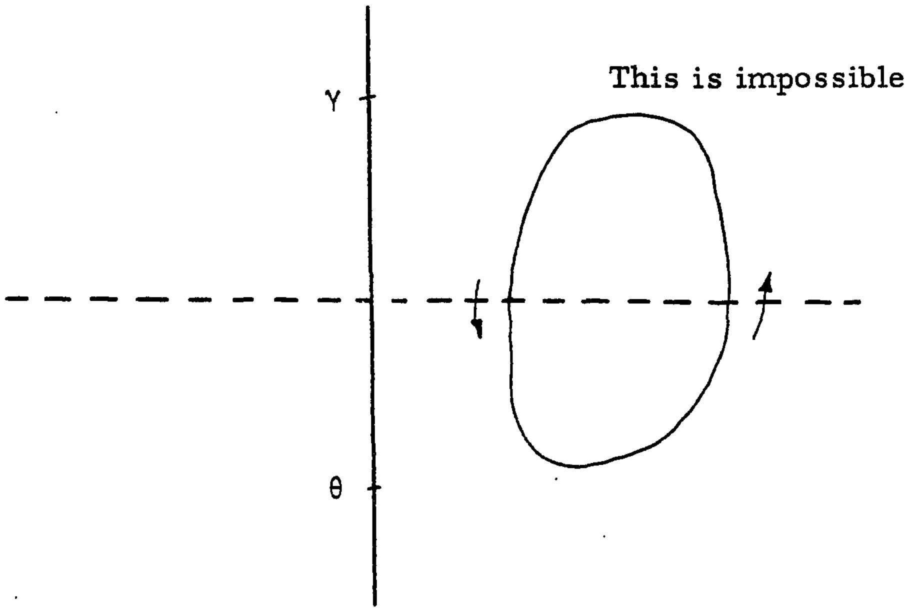


Figure 6 Non-existence of limit cycles

closed curve, say C, inclosing an area S. We calculate, integrating over one period in x:

$$0 = \int_C (\dot{\eta}\ddot{z} - \dot{z}\ddot{\eta})dx = \int_C (\dot{\eta}dz - \dot{z}d\eta) = \int_C (F dz - G d\eta)$$

$$= \int_S \nabla(\eta, z) \cdot (F, G) d\eta dz \qquad \text{(using Green's theorem)}$$

$$= \int_S (-2z + \gamma + \theta) d\eta dz$$

Consequently the limit cycle cannot lie wholly above nor wholly below the line $z = \frac{1}{2}(\gamma+\theta)$. (This is true in either phase plane, since η is merely a rescaling of the y axis.) However from either equation (1.15) or figure 2 we conclude that $\dot{z} > 0$ along the half-line $y > 0$, $z = \frac{1}{2}(\gamma+\theta)$. This means that a trajectory can only cross this line in one sense (Cf. figure 6). It follows that no limit cycles exist.

Now consider the (unique) trajectory eminating from the saddle point P_1 into the region $y > 0$. From the vector field we see that $\dot{y} > 0$, $\dot{z} > 0$ initially. Now either this trajectory intersects the y axis in some finite x, or else $z < z_0 \leq 0$ for all x, in which case $z \to z_0$ as $x \to +\infty$. But then we must also have $\dot{z} \to 0$ and $y \to +\infty$. This is inconsistent with $\dot{z} = y - (z-\gamma)(z-\theta)$, so in fact this trajectory must intersect the y axis at a finite point (necessarily to the right of the spiral point P_3). From the vector field (cf. figure 2) we can see that the trajectory then moves upward to the left, intersects Γ in the region $0 < z$, continues downward to the left, intersects the y axis in $0 < y < \gamma\theta$, and then continues downward to the right, intersecting Γ in the region $\gamma < z < 0$. Finally, the trajectory again moves upward to the right. We can now argue that the trajectory is bounded and, in the absence of limit cycles, necessarily spirals into P_3.

If we take any point on Γ in the region $y > 0$, $z < \theta$ and follow the trajectory through such a point as $x \to +\infty$, exactly analogous arguments apply. If we follow such a trajectory for $x \to -\infty$, the vector field forces it directly into the unstable node P_2. This discussion and the results illustrated in figures 2 and 4 permit us to construct the phase plane illustrated in figure 7 when P_3 is the spiral point. When P_3 is a node, the plane remains qualitatively the same except in the neighborhood of P_3, where trajectories either tend directly into the critical point or spiral about it at most a finite number of times before tending into it.

Two possible families of trajectories may exist which have not yet been discussed, their locations are indicated in figure 7 by region I and region II. First we consider a point in region I. Necessarily as $x \to +\infty$ the trajectory through such a point tends to P_3. We have not argued, however, that such a trajectory intersects the curve Γ (in the region $y > 0$, $z > 0$) as $x \to -\infty$. If this does not occur, then $y \to +\infty$ and $z \to +\infty$ as $x \to -\infty$. Similarly, the trajectory through a point in region II must tend to P_2 as $x \to -\infty$. We have not argued that such a trajectory intersects Γ (in the region $y > 0$, $z < \theta$) as $x \to +\infty$. Should this not occur, then $y \to +\infty$ and $z \to -\infty$ as $x \to +\infty$. Fortunately, it will turn out that these two possibilities are irrelevant to the boundary value problem posed, and so further investigation is unnecessary.

We now search for trajectories in the full plane satisfying boundary conditions (1.16) and (1.17), as well as the derived condition (1.18). Consider first (1.18), viz. $y(-\infty) = 0$. This eliminates all trajectories except the two eminating from the saddle point P_1 and those emanating from the (unstable) node P_2. In particular, this

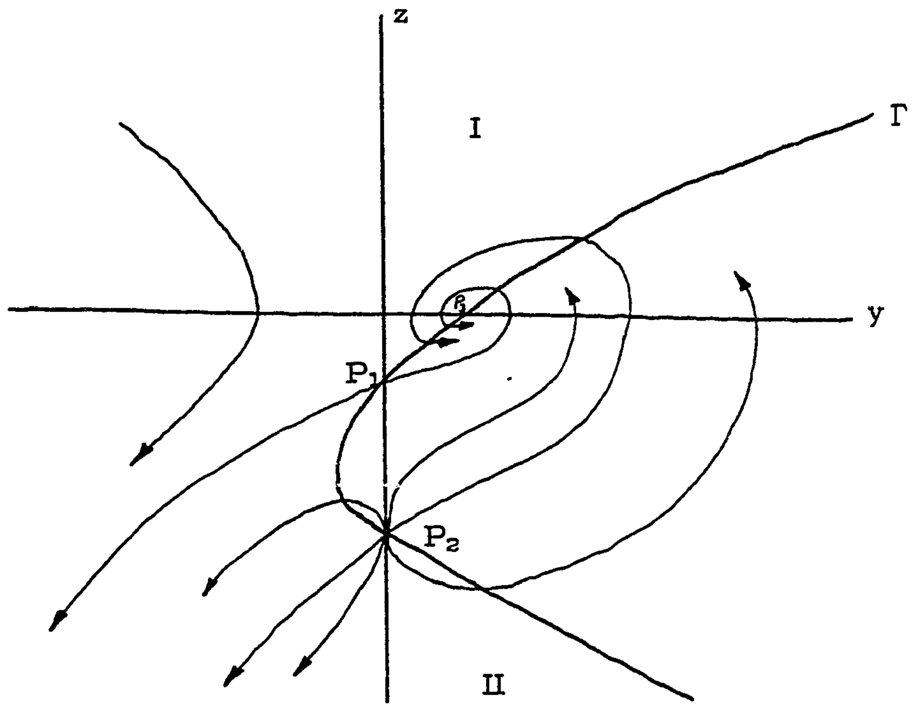


Figure 7 The phase plane

eliminates the possible trajectories in region I mentioned above.

Consider trajectories tending to P_2 as $x \to -\infty$. Along them $z - \gamma \to \theta - \gamma \neq 0$, so meeting boundary condition (1.16) is equivalent to satisfying $e^{-x(\gamma+2)}|y|^{-1/\beta} < \infty$ as $x \to -\infty$. We derived in equation (1.20) the asymptotic relation $y = \xi \sim a\, e^{-\beta\theta x}$ near P_2. The trajectories satisfying $a = 0$ locally are, in fact, segments of the z axis with $y \equiv 0$; hence they cannot satisfy (1.17). Near P_2 boundary condition (1.16) thus reduces to

$$e^{-(\gamma+2)x}\left(e^{-\beta\theta x}\right)^{-1/\beta} = e^{-(\gamma+2)x+\theta x}$$

$$= e^{-(\gamma+2)x + (\gamma-n+2)x} = e^{-nx} < \infty \text{ as } x \to -\infty$$

But this is not possible for $n > 2$, so that no trajectory tending to P_2 as $x \to -\infty$ can satisfy (1.16). In particular, this also eliminates the possible trajectories in region II described above.

Our only hope for a solution, then, lies with the two trajectories emanating from the saddle point P_1. With $y = \xi$, $z = \gamma + \zeta$ the linearized equations for (ξ, ζ) are

$$\dot{\xi} = -\beta\gamma\xi = (\mu+2)\xi$$

$$\dot{\zeta} = \xi + (\theta-\gamma)\zeta = \xi - (n-2)\zeta$$

with solutions

$$\xi = a\, e^{(\mu+2)x} = a\, e^{-\beta\gamma x}$$

$$\zeta = b\, e^{-(n-2)x} + \frac{a}{\mu+n}\, e^{(\mu+2)x}$$

In the parameter domain $n > 2$ trajectories tending to $\xi = \zeta = 0$ as $x \to -\infty$ must have $b = 0$. Thus as $x \to -\infty$

$$y \sim a\, e^{(\mu+2)x} = a\, e^{-\beta\gamma x}$$

$$z \sim \gamma + \frac{a}{n+\mu}\, e^{(\mu+2)x}$$

and

$$e^{-(\gamma+2)x}|y|^{-1/\beta}|z-\gamma| \sim e^{-(\gamma+2)x}e^{\gamma x}e^{(\mu+2)x} = e^{\mu x} < \infty,$$

as $x \to -\infty$ for $\mu \geq 0$. Thus these two trajectories do indeed satisfy the boundary condition at $x = -\infty$ (corresponding to $r = 0$).

In figure 8 we graph these two trajectories when P_3 is a spiral point. In figures 9 a, b, and c, we graph some typical examples when P_3 is a node. In these latter cases the trajectory in the right half-plane may tend directly into P_3 or may spiral about (at most) a finite number of times before tending into P_3. The analysis presented here is insufficient to determine precisely how many times P_3 is encircled in the case of a node.

The remaining boundary condition, (1.17), is trivial to satisfy; it merely requires that the solution trajectory intersect the line $y = \alpha\lambda^\beta$ when $x = 0$. If a trajectory intersects the line $y = \alpha\lambda^\beta$, then the translation invariance of the autonomous system (1.15) implies that a solution exists for which $x = 0$ at the point of intersection.

The question of the number of solutions to equation (1.1) with boundary conditions (1.2) and (1.3) is thus reduced to counting the number of intersections of the trajectories emanating from P_1 with the line $y = \alpha\lambda^\beta$ (each distinct point of intersection corresponds to a different transformation back to the independent variable $0 \leq r \leq 1$ and hence a distinct solution).

Regardless of whether P_3 is a spiral point or a node, precisely one solution exists for every value of $\alpha\lambda^\beta < 0$. We noted earlier in the chapter that when $\lambda = 0$ the unique solution $u(r) \equiv 0$ exists. It is equally simple to treat the linear problem resulting when $\alpha = 0$ to get

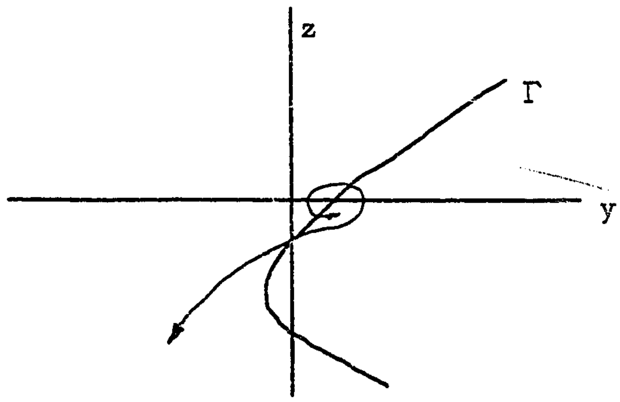


Figure 8 Trajectories satisfying (1.16) and (1.18), P_3 is a spiral point

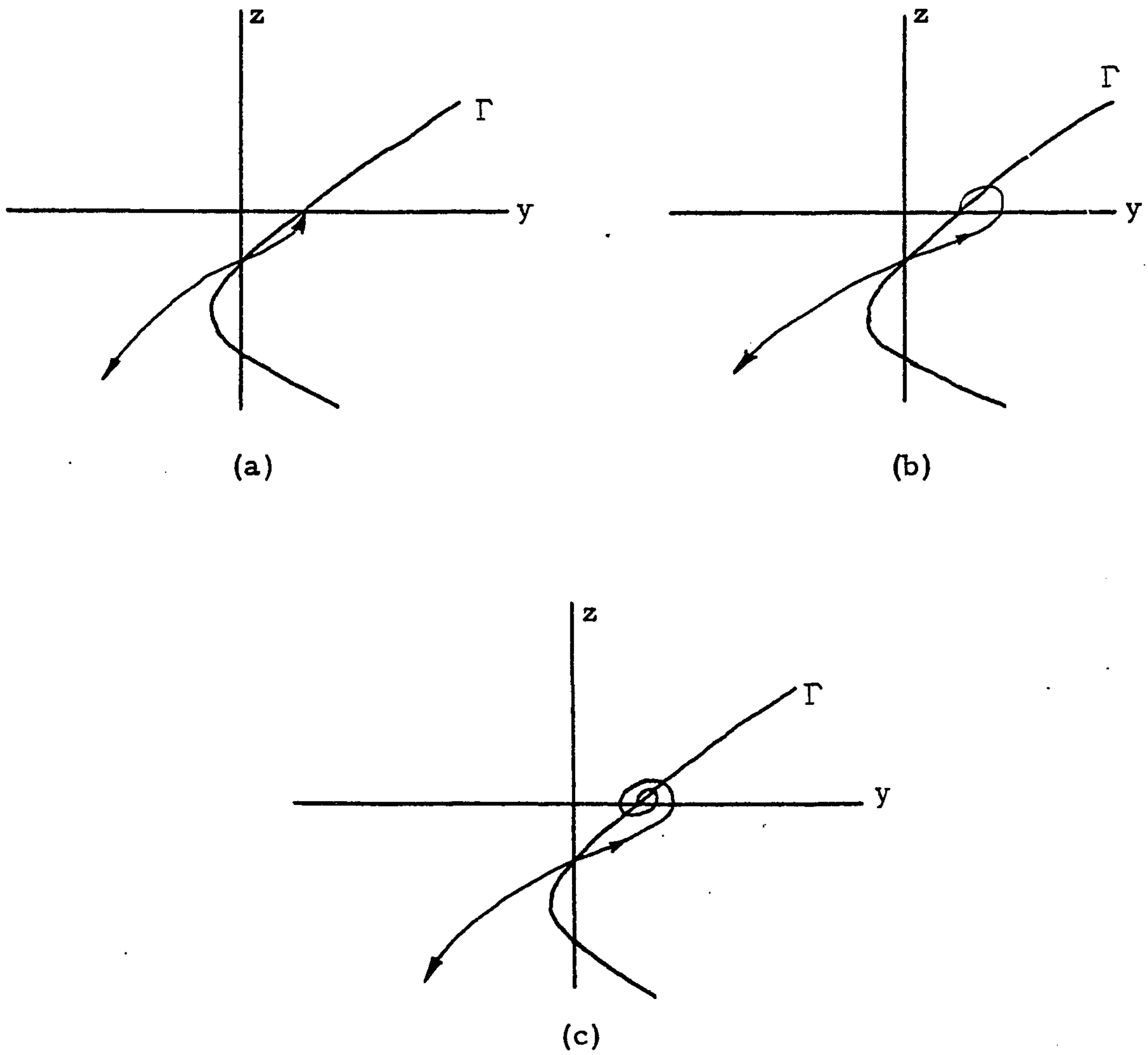


Figure 9 Trajectories satisfying (1.16) and (1.18), P_3 is a node

the unique solution

$$u(r) = \frac{\lambda}{(\mu+2)(\mu+n)} (r^{\mu+2} - 1) \quad .$$

Hence we see that even when $\alpha\lambda^{\beta} = 0$ a unique solution always exists.

The number of solutions for $\alpha\lambda^{\beta} > 0$ is of considerably more interest--particularly when P_3 is a spiral point (i.e. equation (1.22) is satisfied). Inspection of figures 8 and 11 shows that there exists a sequence of numbers $\{m_j\}$

$$m_0 = -\infty < 0 < m_2 < m_4 < \ldots < m_\infty < \ldots < m_3 < m_1 < m_{-1} = +\infty$$

such that

for $\alpha\lambda^{\beta} = m_j$	j solutions exist $j = 1, 2, \ldots \infty$
for $m_{2k} < \alpha\lambda^{\beta} \le m_{2k+2}$	$2k+1$ solutions exist $k = 0, 1, 2, \ldots$
for $m_{2k+1} < \alpha\lambda^{\beta} < m_{2k-1}$	$2k$ solutions exist $k = 0, 1, 2, \ldots$

In figure 10 these results are summarized graphically. It is evident that m_∞ is the abscissa of P_3, i.e. for $\alpha\lambda^{\beta} = m_\infty = \gamma\theta$ a countable infinity of solutions exists.

We remark in passing that it was found earlier that for the flat membrane P_3 is a spiral point for all values of μ. Consequently, figure 10 depicts the distribution of equilibrium configurations for various edge thrusts and pressure distributions.

When P_3 is a node the situation for $\alpha\lambda^{\beta} > 0$ is somewhat less dramatic and, unfortunately, more vague. As can be seen from figures 9a, b, and c, the separatrix connecting P_1 and P_3 can tend directly into P_3 without any spiral behavior, or it can spiral a finite number of times before tending into P_3. Consequently, a similar sequence of m_j can be constructed, with the difference that only a finite number of m_j exist and the number of solutions for arbitrary

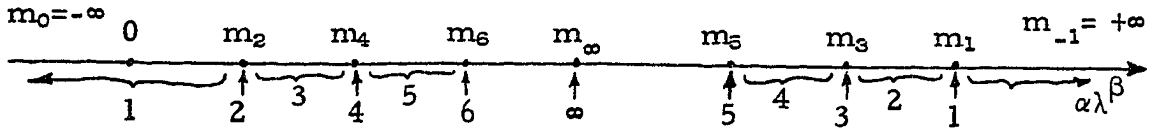


Figure 10 Number of solutions for each value of $\alpha\lambda^\beta$, P_3 is a spiral

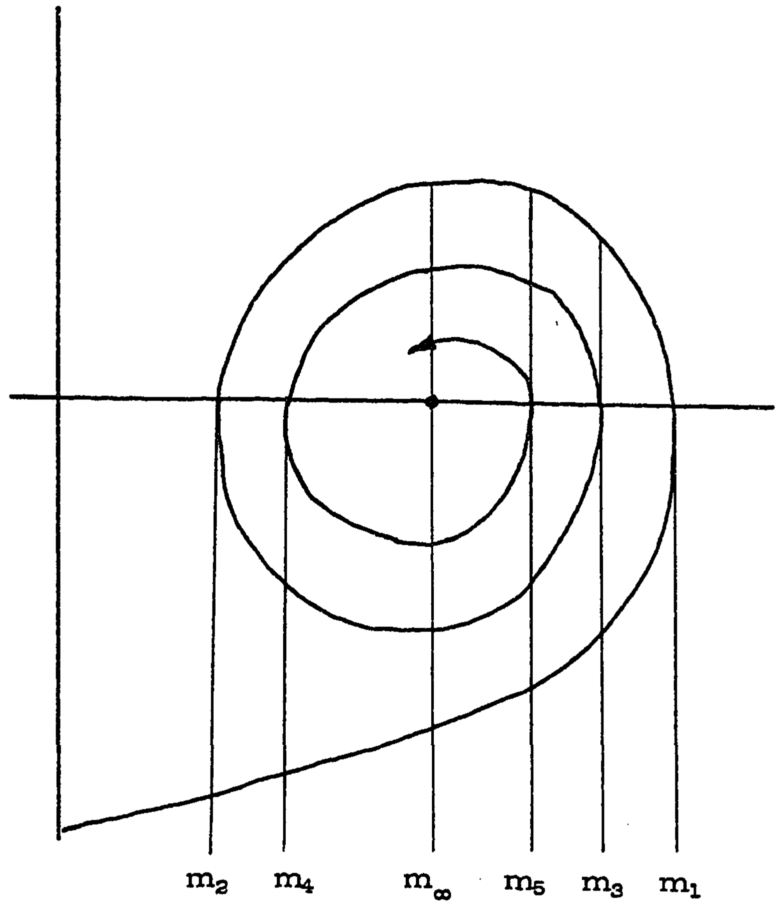


Figure 11 Location of the m_j

values of $\alpha\lambda^{\beta}$ is bounded for given values of β, n, and μ.

The case n = 2

In this case $\theta = \gamma < 0$ (cf. equation (1.9)) and the critical points P_1 and P_2 coalesce into a single point, say P^*, with coordinates $y = 0$, $z = \gamma$. The characteristic exponents at P^* are $\ell_1 = \mu + 2 > 0$ and $\ell_2 = 0$; since one of the exponents vanishes P^* is not an elementary critical point and a special analysis will be necessary.

The characteristic exponents at P_3 are given by equations (1.21) with $\theta = \gamma$; this yields

$$\ell_{1,2} = \gamma \mp \gamma\sqrt{-\beta+1}$$

Since $-\beta+1 < 0$, P_3 is always a spiral point, unlike the case $n > 2$. The previous argument that no limit cycles exist is still valid.

The tangent field is illustrated in figure 12. To discuss the behavior near P^* set $y = \xi$, $z = \gamma + \zeta$. Then

$$\left.\begin{aligned} \dot{\xi} &= -\beta\gamma\xi - \beta\xi\zeta \\ \dot{\zeta} &= \xi - \zeta^2 \end{aligned}\right\} \quad (1.26)$$

It is convenient to introduce local polar coordinates

$$\xi = r\cos\phi \quad , \quad \zeta = r\sin\phi$$

in terms of which (1.26) becomes

$$\begin{aligned} \dot{r} &= r(-\beta\gamma\cos^2\phi + \cos\phi\sin\phi) + r^2(-\beta\cos^2\phi\sin\phi - \sin^3\phi) \\ &\equiv r\,R(\phi) + r^2\rho(\phi) \\ \dot{\phi} &= (\cos^2\phi + \beta\gamma\cos\phi\sin\phi) + r(\beta-1)(\sin^2\phi\cos\phi) \\ &\equiv S(\phi) + r\,\sigma(\phi) \end{aligned} \quad (1.27)$$

The ζ axis is covered by two trajectories satisfying $\xi \equiv 0$, $\dot{\zeta} = -\zeta^2$, so that all other trajectories lie entirely within either the half-plane $\xi > 0$ or the half-plane $\xi < 0$. Consider trajectories tending to P^* as

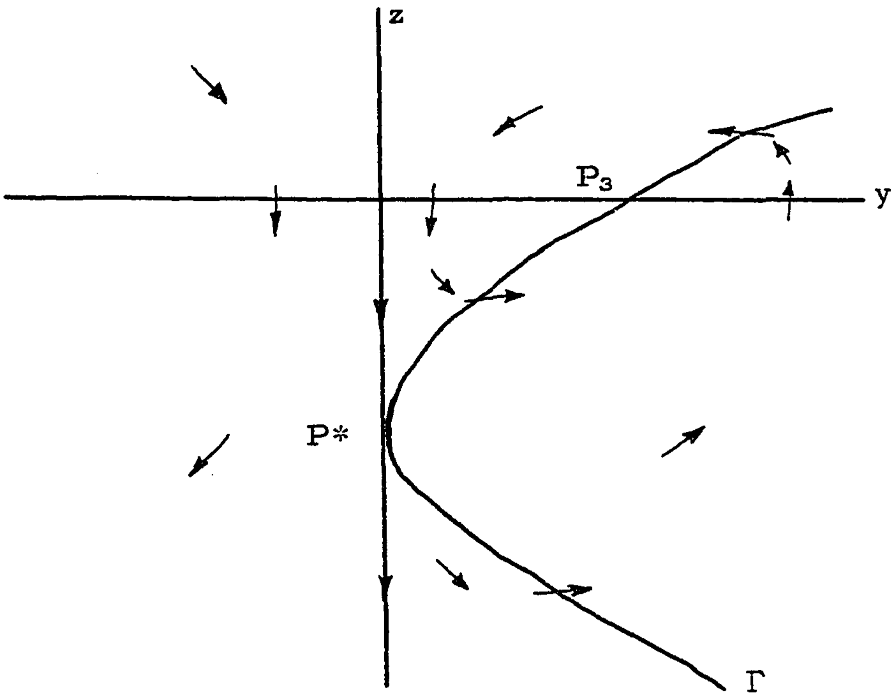


Figure 12 Tangent field, n = 2

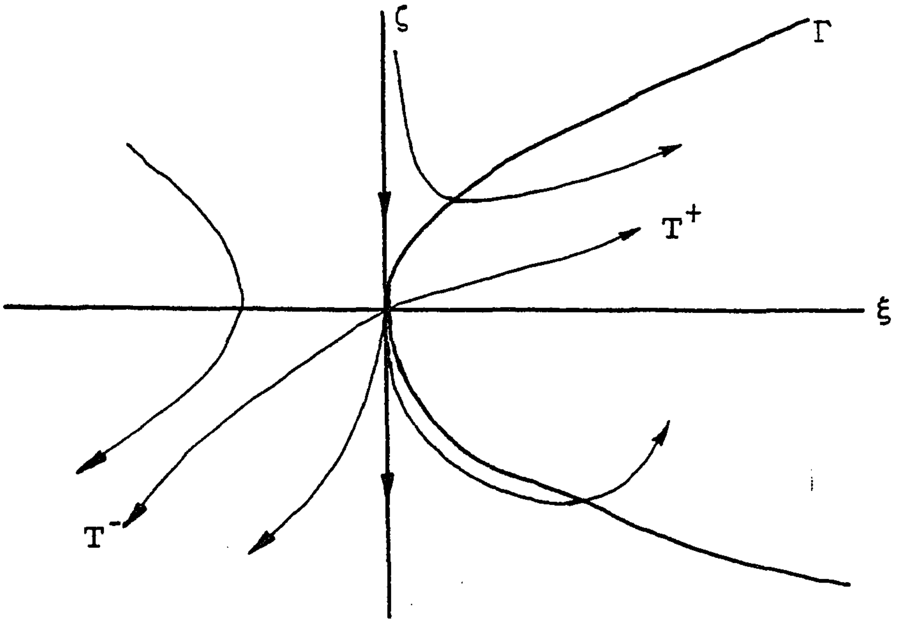


Figure 13 Phase plane in a neighborhood of P*

$x \to \pm\infty$. It is a standard result (e.g. see Hartman [5]) that as $r(x) \to 0$ either $|\phi(x)| \to \infty$ or else $\phi(x) \to \phi_0$, where ϕ_0 is a solution of $S(\phi_0) = 0$. The first alternative, that a trajectory spirals into P*, is impossible since no trajectory can cross the ζ axis. To find two possible angles of approach we solve

$$0 = S(\phi_0) = \cos\phi_0 (\cos\phi_0 + \beta\gamma \sin\phi_0)$$

to conclude either $\cos\phi_0 = 0$ or $\tan\phi_0 = -1/\beta\gamma = 1/(\mu+2)$. The first possibility yields $\phi_0 = \pm\pi/2$. We have already observed that the positive ζ axis is covered by a trajectory tending to P* as $x \to +\infty$ along $\phi_0 = \pi/2$, and the negative ζ axis is covered by a trajectory tending to P* as $x \to -\infty$ along $\phi_0 = -\pi/2$.

It is also a standard result that at least one trajectory exists which approaches P* along $\phi = \phi_0$ if, in addition to $S(\phi_0) = 0$, it is true that $S'(\phi_0) \neq 0$. Suppose that $\tan\phi_0 = -1/\beta\gamma$ and in addition

$$0 = S'(\phi_0) = -2\cos\phi_0 \sin\phi_0 + \beta\gamma \cos 2\phi_0$$

or $\tan 2\phi_0 = \beta\gamma$. The trigonometric identity

$$\tan 2\alpha = 2\tan\alpha \,/ (1 - \tan^2\alpha)$$

yields

$$\beta\gamma = (-2/\beta\gamma)/(1 - 1/\beta^2\gamma^2)$$

which is impossible for $0 < -1/\beta\gamma = 1/(\mu+2) < 1$. Consequently, taking into account the directions of the tangent field, we can conclude that there exists at least one trajectory T^+ in the half-plane $\xi > 0$ and at least one trajectory T^- in $\xi < 0$ such that $T^\pm$ tends to P* along the line $\tan\phi_0 = 1/(\mu+2)$ as $x \to -\infty$.

We can show that the trajectories $T^\pm$ are, in fact, unique by evoking a theorem due to K. A. Keil (see Sansone and Conti [7], page

257). Consider the system for $x(t)$, $y(t)$

$$\dot{x} = kx + f(x, y)$$
$$\dot{y} = g(x, y)$$

where $k \neq 0$. If the origin is an isolated singular point of this system, if $f, g \in C^1$ in a neighborhood of the origin, and if in addition $f = g = f_x = f_y = g_x = g_y = 0$ at the origin, then there exist two and only two trajectories with equations $y = y(x)$ defined to the right and to the left of $x = 0$, respectively, tangent to the x axis at the origin. It is simple to show that equations (1.26) satisfy the hypotheses of this theorem under the transformation

$$x = \xi \quad , \; y = \xi + \beta\gamma\zeta$$

where the x axis corresponds to the line $\xi + \beta\gamma\zeta = 0$.

We are now in a position to construct the local phase portrait about P* shown in figure 13. A solution trajectory must satisfy (1.16), (1.17), and (1.18). Equation (1.18) implies that the trajectory emanates from P*. Consider first trajectories tangent to $\phi = -\pi/2$ as $x \to -\infty$. Set $\phi = -\pi/2 + \phi^*$. As $x \to -\infty$, $\phi^* \to 0$; hence $\cos\phi \sim \phi^*$ and $\sin\phi \sim -1$. From equation (1.27) we get asymptotically

$$\dot{r} \sim r\phi^* + r^2 \tag{1.28a}$$
$$\dot{\phi}^* \sim -\beta\gamma\phi^* \tag{1.28b}$$

Equation (1.28b) implies

$$\phi^* \sim e^{-\beta\gamma x} \quad \text{as } x \to -\infty \;. \tag{1.29}$$

Substituting this into (1.28a) we get the further asymptotic relation

$$\dot{r} \sim r^2 \tag{1.30}$$

which implies

$$r \sim (x_0 - x)^{-1} \sim |x|^{-1} \quad \text{as } x \to -\infty \tag{1.31}$$

The result that r decays algebraically is consistent with the assumption that

$$r\phi^* = o(r^2)$$

which was made in deriving (1.30) from (1.28a). Also

$$\xi = r\cos\phi \sim r\phi^*$$

$$\zeta = r\sin\phi \sim -r \quad .$$

We can now use equations (1.29) and (1.31) to check whether or not trajectories tangent to $\phi = -\pi/2$ as $x \to -\infty$ satisfy (1.16).

$$e^{-(\gamma+2)x}(z-\gamma)\,|y|^{-1/\beta} = e^{-(\gamma+2)x}\zeta\,|\xi|^{-1/\beta}$$

$$\sim e^{-(\gamma+2)x}\, r\,(r\phi^*)^{-1/\beta}$$

$$\sim e^{-(\gamma+2)x}\, r^{(\beta-1)/\beta}\, e^{\gamma x} \quad .$$

$$\sim e^{-2x}\,|x|^{-(\beta-1)/\beta} \to \infty \quad \text{as } x \to -\infty \quad .$$

Thus such trajectories do not satisfy boundary condition (1.16).

Next consider trajectories $T^{\pm}$ which are tangent to the line $\xi = -\beta\gamma\zeta$ as $r \to 0$. Equations (1.26) yield

$$\dot{\xi} \sim -\beta\gamma\xi + \xi^2/\gamma$$

$$\sim -\beta\gamma\xi$$

$$\dot{\zeta} \sim -\beta\gamma\zeta - \zeta^2$$

$$\sim -\beta\gamma\zeta$$

which imply

$$\xi \sim e^{-\beta\gamma x}\ , \quad \zeta \sim e^{-\beta\gamma x} = e^{(\mu+2)x}$$

Checking equation (1.16), we calculate

$$e^{-(\gamma+2)x}\zeta\,|\xi|^{-1/\beta} \sim$$

$$e^{-(\gamma+2)x}e^{(\mu+2)x}e^{\gamma x} =$$

$$e^{\mu x} < \infty \quad \text{as } x \to -\infty$$

for $\mu \geq 0$. Thus trajectories $T^{\pm}$ meet this boundary condition.

We can now see that the case $n = 2$, for all values of $\beta > 1$ and $\mu \geq 0$, is identical with the case $n \geq 2$ when P_3 is a spiral point and can be summarized by figure 10.

The case $n = 1$

In this case $\theta = \gamma + 1$ and there are three subcases to consider, depending on whether $\theta < 0$ $(\gamma < -1)$, $\theta > 0$ $(-1 < \gamma < 0)$, or $\theta = 0$ $(\gamma = -1)$. In the first two cases there exist three distinct critical points (the same notation as before will be used); in the third case points P_2 and P_3 coalesce into P with coordinates $y = z = 0$. Only the salient features of the discussion will be mentioned since most of the details are similar to arguments used for $n \geq 2$.

For all three subcases, P_1 $(y = 0,\ z = \gamma)$ is an unstable improper node with characteristic exponents $\ell_1 = -\beta\gamma = \mu + 2 > 0$ and $\ell_2 = \theta - \gamma = 1$. All trajectories except one pair are tangent to the z axis as they approach P_1. Only the pair of trajectories tangent to the line $y = (\mu + 1)(z - \gamma)$ satisfy boundary condition (1.16). Now consider the subcase $\theta < 0$. Note that this is equivalent to $0 < -\beta\theta = -\beta(\gamma + 1)$ or

$$\beta < \mu + 2$$

Critical point P_2 $(y = 0,\ z = \theta)$ has characteristic exponents $\ell_1 = -\beta\theta > 0$ and $\ell_2 = \gamma - \theta = -1$ and is a saddle point. The separatrices are tangent to the lines $y = (1 - \beta\theta)(z - \theta)$ and $y = 0$ respectively. P_3 $(y = \gamma\theta,\ z = 0)$ is either a stable node or a stable spiral point, depending on the sign of the function $\Phi(\mu + 2)$ (evaluated with $n = 1$). Just as for the case $n > 2$, there exists a value of $\mu = \mu^*$ such that for $\mu > \mu^*$ P_3 is always a spiral point. The reader is referred back to figure 7 for the phase plane portrait, only with the labels for the points P_1 and P_2 exchanged. Existence and multiplicity are completely analogous to the case $n > 2$.

Next consider the subcase $\theta > 0$ ($\beta > \mu + 2$). Then the characteristic exponents of P_2 are $\ell_1 = -\beta\theta < 0$ and $\ell_2 = -1$, so it is a stable node. The characteristic exponents of P_3 are

$$\ell_+ = \tfrac{1}{2}(\gamma+\theta) + \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4\theta(\mu+2)} > 0$$
$$\ell_- = \tfrac{1}{2}(\gamma+\theta) - \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4\theta(\mu+2)} < 0$$

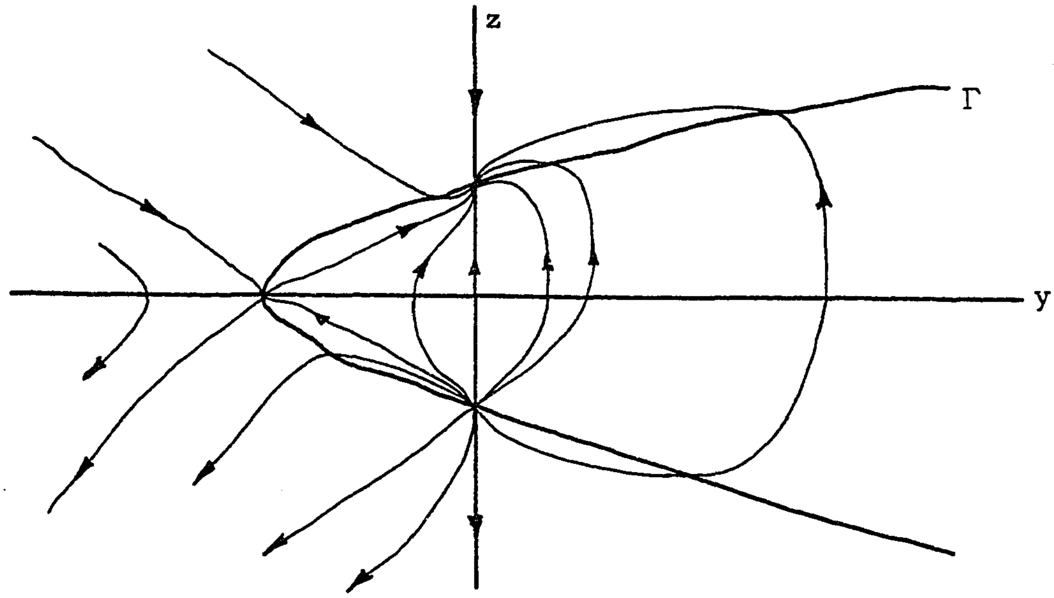
since $\theta(\mu+2) > 0$. Hence P_3 is a saddle point. From an earlier discussion we know that the separatrix corresponding to ℓ_+ is tangent to the line $(y - \gamma\theta) + \ell_- z = 0$ and the separatrix corresponding to ℓ_- is tangent to the line $(y - \gamma\theta) + \ell_+ z = 0$. Taking into account the tangent field, we are in a position to construct the phase plane portrait in figure 14a. In figure 14b we have isolated the two trajectories that satisfy conditions (1.16) and (1.18). Note that for $\alpha\lambda^\beta \le 0$ precisely one solution exists, and that there exists a number $m_1 > 0$ such that for $0 < \alpha\lambda^\beta < m_1$ two solutions exist, for $\alpha\lambda^\beta = m_1$, one solution exists, and for $\alpha\lambda^\beta > m_1$ no solutions exist.

Finally, consider the limiting case $\theta = 0$ ($\mu + 2 = \beta$). Critical points P_2 and P_3 coalesce into $\hat{P}$ with $y = z = 0$. Since this is not an elementary critical point, we introduce polar coordinates

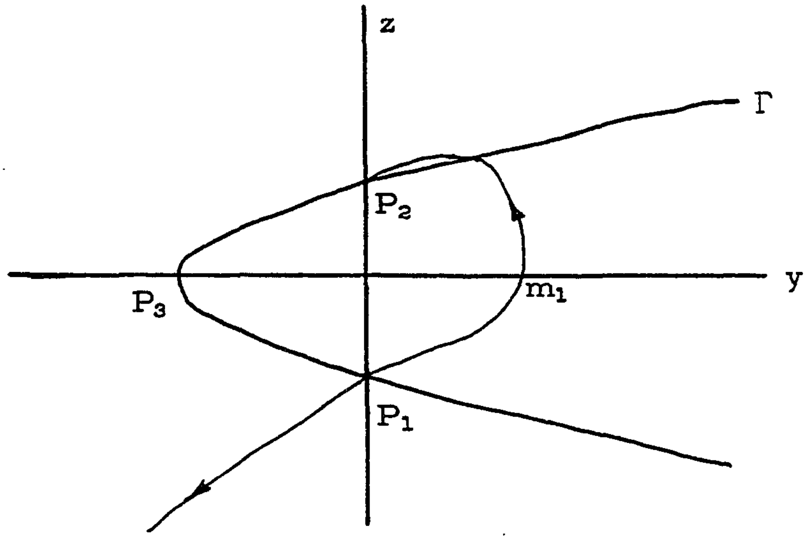
$$y = r\cos\phi \quad , \quad z = r\sin\phi$$

and find that equations (1.15) become

$$\dot{r} = r(\cos\phi\sin\phi + \gamma\sin^2\phi) - r^2(\beta\cos^2\phi\sin\phi + \sin^3\phi)$$
$$\equiv rR(\phi) + r^2\rho(\phi)$$
$$\dot{\phi} = (\cos^2\phi + \gamma\cos\phi\sin\phi) + r(\beta-1)\sin^2\phi\cos\phi$$
$$\equiv S(\phi) + r\sigma(\phi)$$



(a) The phase plane



(b) Solution separatrices only

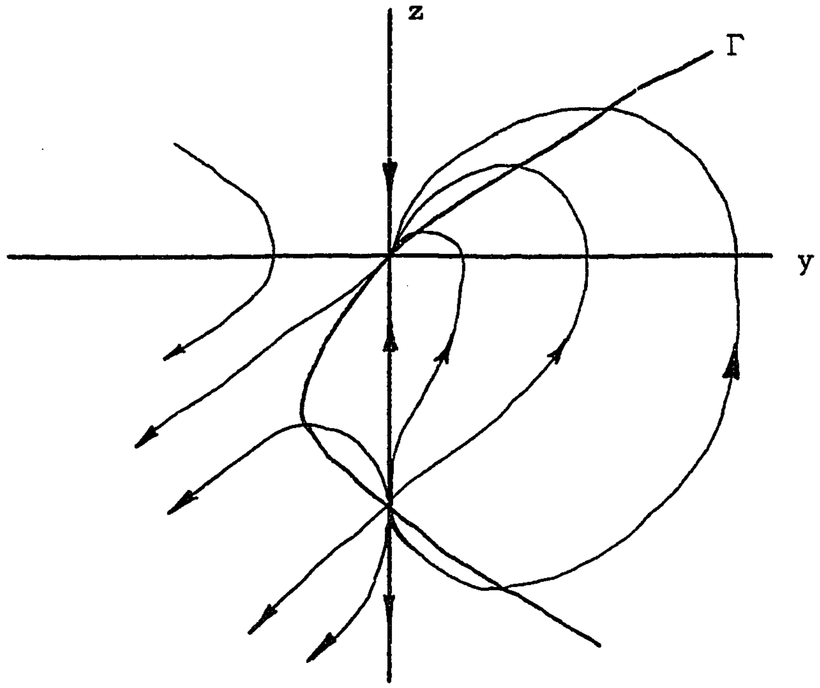
Figure 14 Phase plane for $n = 1$, $\theta > 0$

The z axis is covered by three trajectories, so $\hat{P}$ cannot be a spiral point. The possible angles of approach satisfying $S(\phi_0) = 0$ are

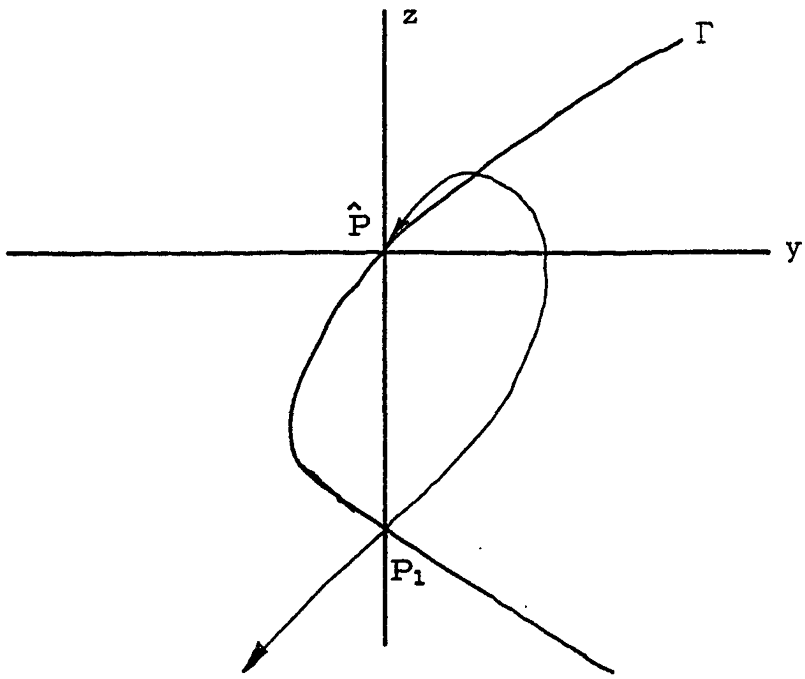
$$\phi_0 = \pm\pi/2 \qquad \text{and} \qquad \tan\phi_0 = -1/\gamma = 1$$

Using arguments similar to those for n = 2, it is possible to show that trajectories exist which tend to $\hat{P}$ for all four angles of approach, and that any trajectory which tends to $\hat{P}$ as $x \to -\infty$ does not satisfy boundary condition (1.16). In figure 15a we construct the phase plane portrait, taking the tangent field into consideration; in figure 15b we isolate the two trajectories satisfying (1.16) and (1.18). Note that the multiplicity of solutions is qualitatively the same as for the subcase $\theta > 0$.

Remark that for n = 1 and fixed β, there always exists a value μ^* such that for all $\mu > \mu^*$ a countable infinity of solutions exists for some value of $\alpha\lambda^\beta$ depending on μ.



(a) The phase plane



(b) Solution separatrices only

Figure 15 Phase plane for n = 1, θ = 0

CHAPTER 2: INITIALLY CURVED MEMBRANES

Recall from the introduction that the symmetric deformation of a circular membrane can be described by

$$\frac{d}{dr} \; r^3 \frac{du}{dr} \; + \lambda^3 \frac{G}{(1-\mu)^2} = \lambda \, Br\phi^2 \tag{C.6}$$

In chapter 1 we considered the situation when the membrane is initially flat, $\phi \equiv 0$, and is subjected to a pressure of the form $p = p_{max} r^{\mu/2}$. Recognizing $r^{-3}\frac{d}{dr}(r^3 \frac{du}{dr})$ as a spherically symmetric Laplacian, we generalized the problem to

$$\frac{1}{r^{n-1}} \frac{d}{dr} \; r^{n-1} \frac{du}{dr} \; + \lambda^{\beta} r^{\mu} (1-\alpha u)^{1-\beta} = 0 \tag{1.1}$$

for $\mu \geq 0$, $\beta > 1$, $\alpha \neq 0$.

We next consider the situation in which the pressure distribution remains the same, but the initial configuration of the membrane is given by $\phi = a\,r^b$, $b \geq 0$. The natural generalization of equation (C.6) is

$$\frac{1}{r^{n-1}} \frac{d}{dr} \; r^{n-1} \frac{du}{dr} + \lambda^{\beta} r^{\mu} (1-\alpha u)^{1-\beta} = \frac{\lambda A r^{2b+1}}{r^{n-1}} = \lambda A r^{2(b+1)-n} \tag{2.1}$$

with $A = Ba^2$.

The analysis of chapter 1 was possible because the transformations

$$x = \log r \tag{1.5}$$

$$v(x) = (1-\alpha u) r^{\gamma} \tag{1.6}$$

with
$$\gamma \equiv -(\mu+2)/\beta \tag{1.7}$$

yield a second order automonous equation. This is still true when the membrane is initially curved if the exponent b satisfies

$$2b = n - 4 - \gamma \quad .$$

Note that for the membrane problem $n = 4$, $\beta = 3$, and so

$$b = (\mu + 2)/6 > 0 \quad .$$

Consequently $\phi(0) = 0$ and the membrane is indeed flat at its apex. The response of the curved membrane is governed by

$$\frac{d^2 v}{dx^2} - (\gamma+\theta)\frac{dv}{dx} + \gamma\theta v - \alpha\lambda^{\beta} v^{1-\beta} + \alpha\lambda A = 0 \tag{2.2}$$

To facilitate analysis in the phase plane we introduce

$$w = \lambda/v \tag{2.3}$$

$$z = \frac{1}{v}\frac{dv}{dz} \tag{2.4}$$

Then equation (2.2) is equivalent to the system

$$\dot{w} = -wz \qquad \equiv f(w, z) \tag{2.5a}$$

$$\dot{z} = \alpha w^{\beta} - \alpha A w - (z-\gamma)(z-\theta) \equiv g(w, z) \tag{2.5b}$$

where differentiation with respect to $\dot{x}$ is indicated by a dot. The boundary and regularity conditions (1.10), (1.11), and (1.12) become

$$e^{-(\gamma+2)x} w^{-1} |z-\gamma| < \infty \quad \text{as } x \to -\infty \tag{2.6}$$

$$w = \lambda \quad \text{at} \quad x = 0 \tag{2.7}$$

$$w \to 0 \quad \text{as } x \to -\infty \tag{2.8}$$

In this chapter $A \neq 0$ so that the membrane is indeed curved, and $\alpha \neq 0$ so that the problem is truly nonlinear. Equation (2.1) can be solved exactly for a unique solution satisfying the appropriate boundary conditions when $\lambda = 0$, so we will also assume $\lambda \neq 0$.

In chapter 1 we found that the cases $n = 1$ and $n = 2$ required special analysis, although the results were not startlingly different than for $n > 2$. In this chapter we will limit ourselves to the case $n > 2$ insofar as the introduction of the new parameter A provides a multitude of possibilities to consider. In particular

$$\theta = \gamma - (n-2) < \gamma < 0$$

will always hold. Finally, to simplify the discussion, we will restrict β to integer values.

Consider the critical points of system (2.5). As before there exist points P_1 and P_2 given by $w = 0$, $z = \gamma$ and $w = 0$, $z = \theta$ respectively. Any remaining critical point is of the form $w = W$, $z = 0$ where W is a root of

$$p(w) \equiv \alpha w^{\beta} - \alpha A w - \gamma\theta = 0 \tag{2.9}$$

Although it is not possible, in general, to give explicit formulae for such roots, we can derive much qualitative information graphically. First note that $p(0) = -\gamma\theta < 0$ always. Suppose $0 = p'(\hat{w}) = \alpha\beta\hat{w}^{\beta-1} - \alpha A$ or

$$\hat{w}^{\beta-1} = A/\beta \tag{2.10}$$

If β is even there always exists precisely one point where $p' = 0$. If β is odd, there exist two values of $\hat{w}$ (equal in magnitude but opposite in sign) where $p' = 0$ when $A > 0$ and no such $\hat{w}$ when $A < 0$. Finally, $p'' = 0$ only at $w = 0$, but $p'(0) \neq 0$ so there exist no inflection points.

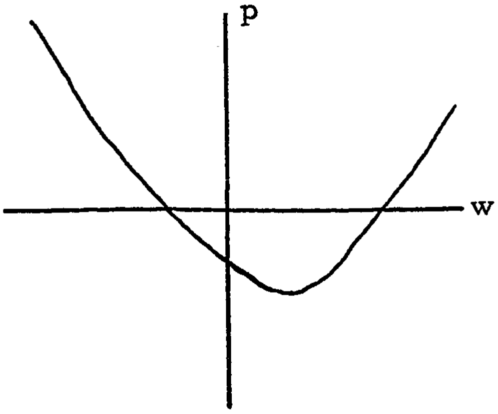
Using these simple facts we readily construct the various possible graphs of p(w) in figures 16. Note that when β is even we consider only $A > 0$, and when β is odd we consider only $\alpha > 0$. This is sufficient to give the qualitative behavior of the phase plane in all possible cases, because system (2.5) is invarient under

$$(w, z, \alpha, A) \rightarrow (-w, z, \alpha, -A)$$

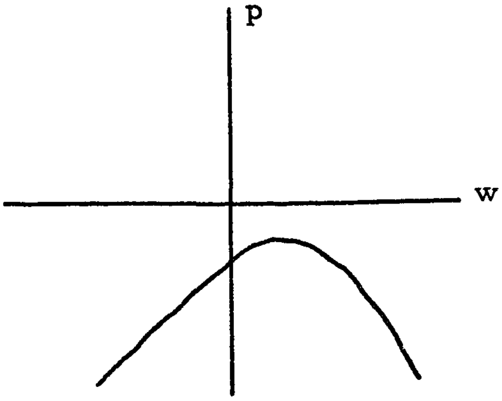
when β is even and

$$(w, z, \alpha, A) \rightarrow (-w, z, -\alpha, A)$$

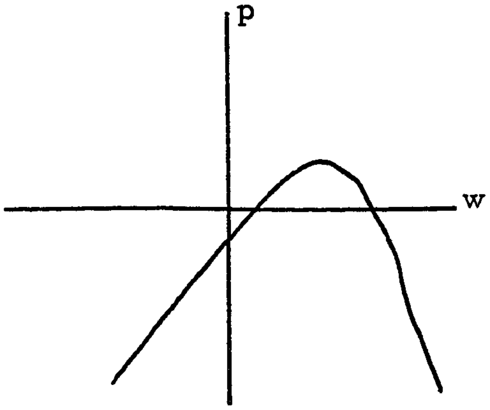
when β is odd. Consequently, the phase portraits not discussed can be obtained by simple reflection about the z axis.



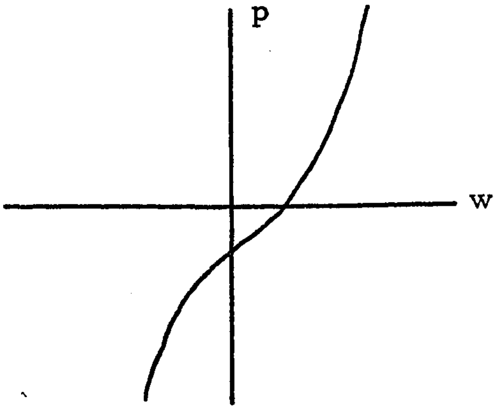
(a) β even, A > 0, $\alpha > 0$



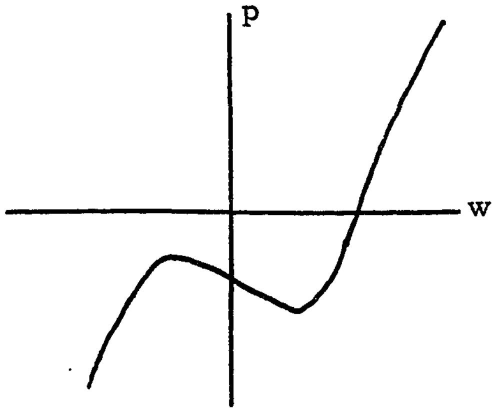
(b) β even, 1>> A > 0, $\alpha < 0$



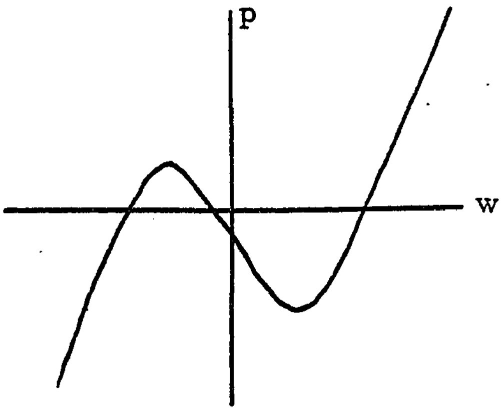
(c) β even, A >> 1, $\alpha < 0$



(d) β odd, A < 0, $\alpha > 0$



(e) β odd, 1 >> A > 0, $\alpha > 0$



(f) β odd, A >> 1, $\alpha > 0$

Figure 16 p(w) versus w

Note that for β even, $A > 0$ and $\alpha < 0$ there may exist no root, one, or two roots, depending on whether the maximum of p is negative, zero, or positive, respectively. Combining (2.9) and (2.10), the condition for two roots is

$$\alpha (A/\beta)^{\beta/(\beta-1)} - \alpha A(A/\beta)^{1/(\beta-1)} - \gamma\theta > 0 \quad .$$

or simplifying

$$A > A^* \equiv \beta[\gamma\theta/\alpha(1-\beta)]^{(\beta-1)/\beta} \tag{2.11}$$

Likewise, for $A = A^*$ one root exists, and for $0 < A < A^*$ no roots exist.

Similarly, for β odd, $A > 0$ and $\alpha > 0$ there may exist one, two, or three roots, depending on the sign of the local maximum of p. The local maximum occurs at $\hat{w} = -(A/\beta)^{1/(\beta-1)}$ and an analogous calculation shows that for

$A > A^*$	three roots exist
$A = A^*$	two roots exist
$0 < A < A^*$	one root exists

where A^* is again defined by (2.11).

Next consider the behavior locally about P_1 and P_2. Near P_1 we may set $w = \xi$, $z = \gamma+\zeta$ to obtain the linearized equations

$$\dot{\xi} = -\gamma\xi$$
$$\dot{\zeta} = -\alpha A\xi - (\gamma-\theta)\zeta = -\alpha A\xi - (n-2)\zeta$$

with solutions

$$\xi = a e^{-\gamma x} \ , \ \zeta = b e^{-(n-2)x} + a_1 e^{-\gamma x}$$

where $a_1 = -a\alpha A/(n-2-\gamma) = a\alpha A/\theta$. Since $-\gamma > 0$ and $-(n-2) < 0$, this is a saddle point.

$$\zeta/\xi = (b/a)e^{\theta x} + \alpha A/\theta \quad .$$

We designate the trajectory emanating from P_1 into $w > 0$ by T^+ and by T^- for $w < 0$. If, near P_2, we set $w = \xi$, $z = \theta + \zeta$ and linearize, we obtain

$$\dot{\xi} = -\theta\xi$$

$$\dot{\zeta} = -\alpha A\xi - \zeta(\theta-\gamma) = -\alpha A\xi + (n-2)\zeta$$

with solutions

$$\xi = a e^{-\theta x}, \quad \zeta = b e^{(n-2)x} + a_1 e^{-\theta x}$$

where

$$a_1 = a\alpha A/(n-2+\theta) = a\alpha A/\gamma$$

Since $-\theta > 0$ and $n-2 > 0$, P_2 is an unstable node.

$$\zeta/\xi = (b/a)e^{\theta x} + \alpha A/\gamma$$

Graphs of the phase plane in a neighborhood of P_1 and P_2 are summarized in figure 17.

To study the behavior in the neighborhood of a critical point $w = W$, $z = 0$ (should one exist) we examine the characteristic exponents ℓ which satisfy

$$0 = \begin{vmatrix} f_w - \ell & f_z \\ g_w & y_z - \ell \end{vmatrix} = \begin{vmatrix} -\ell & -W \\ \alpha\beta W^{\beta-1} - \alpha A & \gamma + \theta - \ell \end{vmatrix}$$

or

$$\ell^2 - (\gamma+\theta)\ell + \alpha\beta W^{\beta} - \alpha A W = 0 \quad .$$

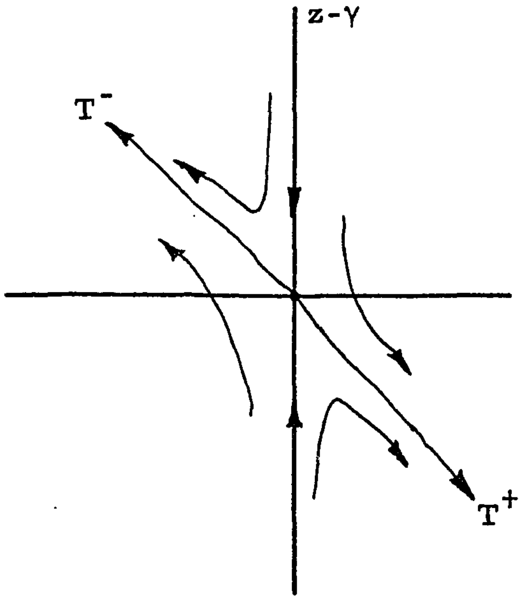
Using $p(W) = 0$ we simplify this to

$$\ell^2 - (\gamma+\theta)\ell + (\beta-1)\alpha A W + \beta\gamma\theta = 0$$

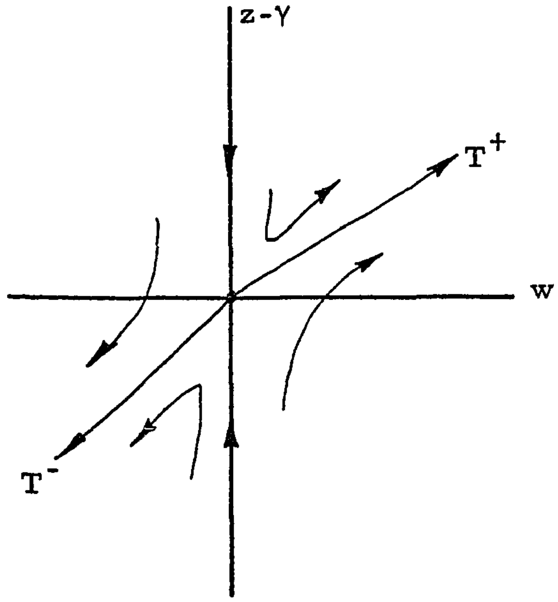
which yields the characteristic exponents

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4[(\beta-1)\alpha A W + \beta\gamma\theta]} \qquad (2.12)$$

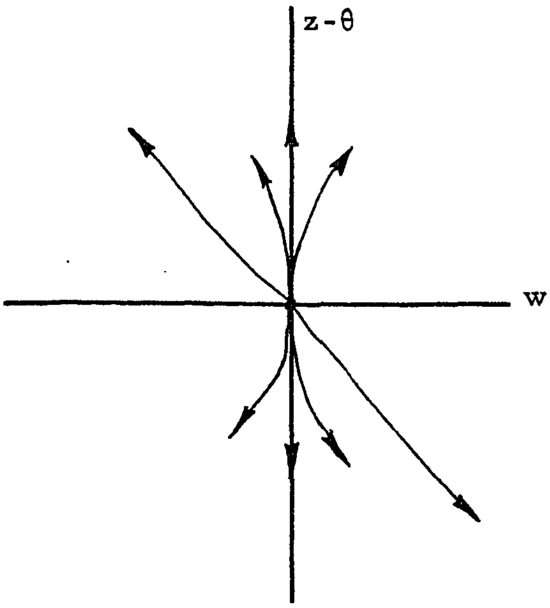
To analyze this in more detail, we shall consider the cases of small amplitude and large amplitude initial configurations. First, however, there are several more relevant observations.



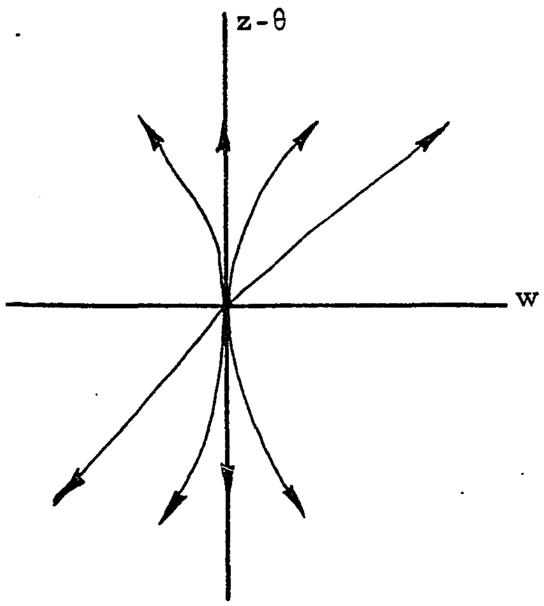
(a) Near P_1, $\alpha A > 0$



(b) Near P_1, $\alpha A < 0$



(c) Near P_2, $\alpha A > 0$



(d) Near P_2, $\alpha A < 0$

Figure 17 Local phase plane behavior near P_1 and P_2

Just as in the case of the initially flat membrane, the z axis is completely covered by trajectories satisfying

$$\dot{w} \equiv 0 \qquad \dot{z} = -(z-\gamma)(z-\theta) \quad .$$

Consequently, no trajectory can cross the z axis. In particular, if any limit cycles exist, they must lie entirely within the right half-plane or the left. By making the transformation $\eta = \log w$, an argument completely analogous to that of chapter 1 shows that any limit cycle must necessarily intersect the line $z = \frac{1}{2}(\gamma + \theta)$.

Arguments regarding the boundary conditions are also similar to those of chapter 1 and will not be repeated. Condition (2.8) requires that a solution trajectory emanate from P_1 or P_2. Condition (2.6) further requires that the solution trajectory be a separatrix emanating from the saddle point P_1, viz. T^+ or T^-. Condition (2.7) requires that we choose the separatrix in the right half-plane for $\lambda > 0$ and in the left half-plane for $\lambda < 0$.

<u>Small amplitude pertubations, $0 < |A| \ll 1$</u>

Case 1: β even, $A > 0$, $\alpha > 0$

From figure 16a it is clear that there exist two roots to p(w) and hence two critical points on the w axis. If W is one such root, expand

$$W = W_0 + AW_1 + A^2W_2 + \ldots \qquad (2.13)$$

and substitute into (2.9) to get

$$\alpha W_0{}^{\beta} - \gamma\theta = 0 \ , \quad \alpha\beta W_0{}^{\beta-1} W_1 - \alpha W_0 = 0$$

or

$$W_0 = \pm(\gamma\theta/\alpha)^{1/\beta} \qquad (2.14a)$$

$$W_1 = \frac{1}{\beta}(\alpha/\gamma\theta)^{(\beta-2)/\beta} > 0 \qquad (2.14b)$$

Thus to leading order in A

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4\beta\gamma\theta} \tag{2.15a}$$

i.e., the result is the same as for the unperturbed case: either a stable spiral or a stable node. In the special case that $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, we have

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{-(\beta-1)\alpha A W_0} \tag{2.15b}$$

so that the unperturbed critical point, a node, becomes a spiral in the right half-plane and remains a node in the left half-plane.

Each intersection of a separatrix from P_1 with the line $w = \lambda$ corresponds to a distinct solution of the boundary value problem. In the unperturbed problem there was no difference between λ and $-\lambda$. In the perturbed problem, when the critical points are spirals, for example, there still exist values λ_∞^+ and λ_∞^- such that there exist a countable infinity of solutions when λ attains either of these values, but it is no longer true that $\lambda_\infty^+ = -\lambda_\infty^-$. However, for $|\lambda|$ small there still exists a unique solution to the boundary value problem, and for $|\lambda|$ large no solution exists. In the special case that $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, an arbitrarily high multiplicity of solutions is possible only for $\lambda > 0$; for $\lambda < 0$ the multiplicity is bounded for given values of β, n, α, μ, and A. We note in passing that there cannot be a limit cycle about either of the critical points because, as is clear from the tangent field, the separatrices from P_1 bound the critical point s away from the line $z = \frac{1}{2}(\gamma+\theta)$. This will be true in all cases to be considered, for $A \gg 1$ and for $0 < |A| \ll 1$, so no further comment regarding the nonexistence of limit cycles will be made. In figure 18 the phase plane portrait is given. The locus of points where $\dot{w} = f(w, z) = 0$ is merely the w and z axes. The locus of points where $\dot{z} = g(w, z) = 0$ is denoted by Γ and

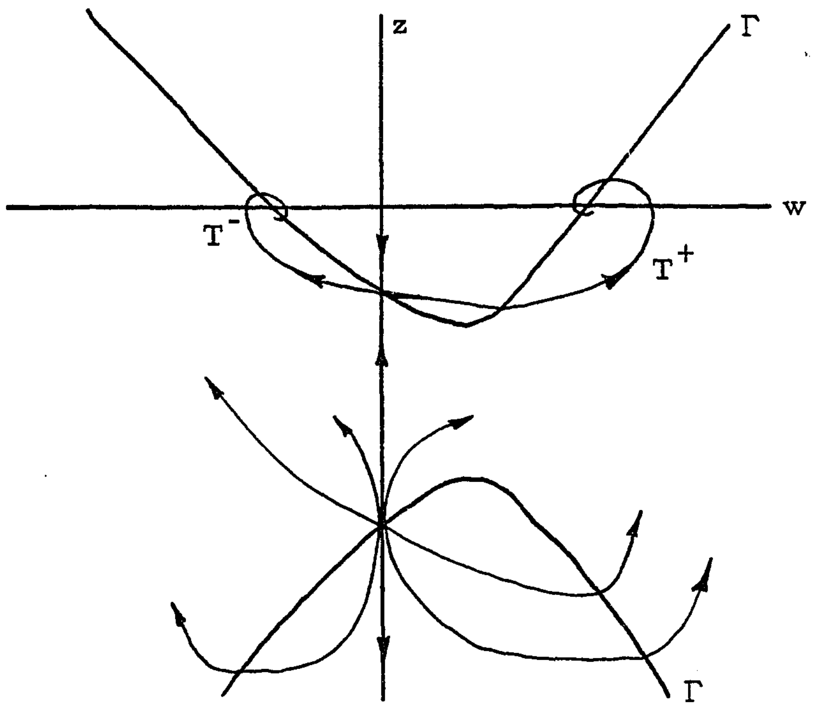


Figure 18 Phase plane for β even, $1 \gg A > 0$, $\alpha > 0$

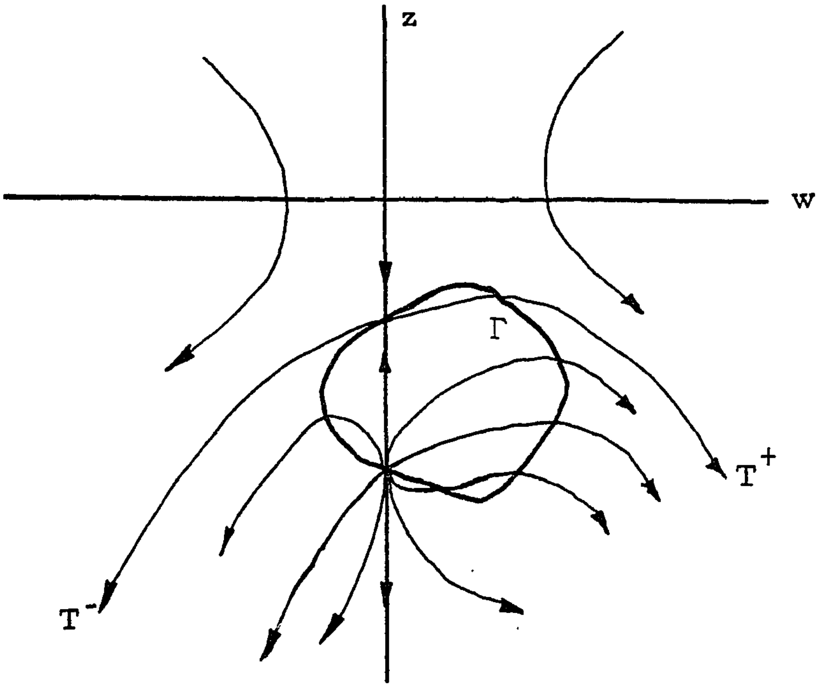


Figure 19 Phase plane for β even, $1 \gg A > 0$, $\alpha < 0$

has been included for clarity. We can also invert $g(w, z) = 0$ to get

$$z = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 + 4p(w)} \tag{2.16}$$

Reference to figure 16 and the fact that $z = \gamma$ or $z = \theta$ when $w = 0$ make it easy to sketch Γ. Recall that $z^+ = 0$ when $p = 0$.

Case 2: β even, $A > 0$, $\alpha < 0$

Refer to figure 16b. $p(w)$ has no roots, Γ is a simple closed curve enclosing the region of the phase plane where $g > 0$ and we can construct the phase plane portrait of figure 19. It is clear that precisely one solution exists for all values of λ.

Case 3: β odd, $A < 0$, $\alpha > 0$

From figure 16d we see that p has one root. This root may be given to leading order by (2.14a), except that it must be positive.

$$W_0 = (\gamma\theta/\alpha)^{1/\beta} > 0 \tag{2.17}$$

As in case 1, the nature of the corresponding critical point is unchanged from that for the unperturbed system, except when $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$. However, even in this instance we find that the point remains a stable node (Cf. equation (2.15b)). The phase plane is given in figure 20. For all $\lambda \le 0$, a unique solution exists. In fact, there exist numbers $m_1 > m_2 > 0$ such that for $\lambda < m_2$ a unique solution exists, and for $\lambda > m_1$ no solutions exist. The situation for $m_2 \le \lambda \le m_1$ depends on whether $(w, 0)$ is a spiral or a node. Depending on this, the appropriate discussion of chapter 1 applies (e.g. Cf. figure 10).

Case 4: β odd, $A > 0$, $\alpha > 0$

This is essentially the same as case 3. From figure 16e we see that p has one root which, to leading order, is given by equation

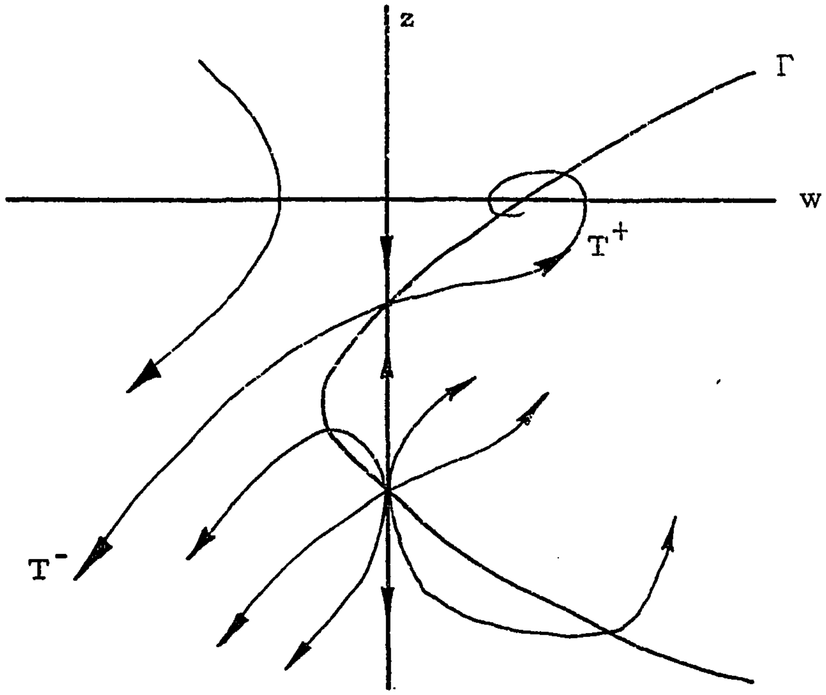


Figure 20 Phase plane for β odd, $A < 0$, $\alpha > 0$

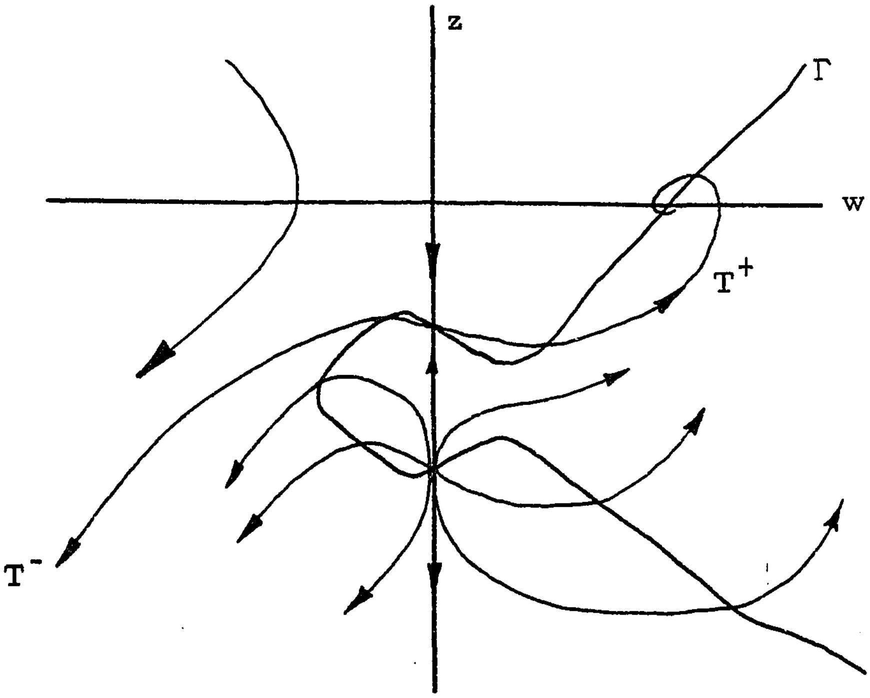


Figure 21 Phase plane for β odd, $1 >> A > 0$, $\alpha > 0$

(2.17). Although the phase plane, shown in figure 21, is slightly altered, the trajectories T^+ and T^-, the nature of the critical point (w, 0), and the possible number of solutions for a given λ are qualitatively the same as for case 3. The only exception is that when $(\gamma+\theta)^2 - 4\beta\gamma\theta = 0$, the unperturbed critical point (W, 0) is a node, but for $A > 0$ it becomes a spiral point (and hence there exist λ with arbitrarily many solutions).

Large amplitude pertubations, $|A| \gg 1$

Case 1: β odd, $A > 0$, $\alpha > 0$

From figure 16f we see that p(w) has three roots, say $W*$, W^-, and W^+ where

$$W^- < W* < 0 < W^+ \quad .$$

Because $p'(0) = -\alpha A \to -\infty$ as $A \to +\infty$ we expect $W* \to 0$. The local extrema of p occur at $w = \pm(A/\beta)^{1/(\beta-1)}$ (Cf. equation (2.10)) which tend to $\pm\infty$ as $A \to +\infty$; consequently $W^+ \to +\infty$ and $W^- \to -\infty$.

To find the roots W we substitute

$$W = U A^{1/(\beta-1)} \tag{2.18}$$

into equation (2.9) to get

$$U^\beta - U = (\gamma\theta/\alpha)\,\epsilon \tag{2.19}$$

where

$$\epsilon = A^{-\beta/(\beta-1)} \tag{2.20}$$

and $0 < \epsilon \ll 1$ for $A \gg 1$. We now expand

$$U = U_0 + \epsilon U_1 + \ldots \tag{2.21}$$

and substitute to get

$$U_0^\beta - U_0 = 0 \tag{2.22a}$$

$$\beta U_0^{\beta-1} U_1 - U_1 = \gamma\theta/\alpha \tag{2.22b}$$

We are only interested in the roots to leading order. Equation

(2.22a) yields solutions $U_0 = +1$, -1, and 0. The first two provide us with

$$W^+ \sim A^{1/(\beta-1)} \tag{2.23}$$

$$W^- \sim -A^{1/(\beta-1)} \tag{2.24}$$

If we take $U_0 = 0$ and substitute into (2.22b) we get

$$U_1 = -\gamma\theta/\alpha$$

and hence

$$W* \sim -(\gamma\theta/\alpha)\varepsilon A^{1/(\beta-1)} = -\gamma\theta/\alpha A \tag{2.25}$$

The characteristic exponents for each critical point are given by equation (2.12). If we substitute the values from (2.23), (2.24), and (2.25), we get, to leading order:

$$\text{at } (W^+, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm i\sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \tag{2.26}$$

$$\text{at } (W^-, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \tag{2.27}$$

$$\text{at } (W*, 0) \qquad \ell_\pm = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}(\gamma-\theta) = \gamma \text{ or } \theta \tag{2.28}$$

Consequently, $(W^+, 0)$ is a stable spiral, $(W^-, 0)$ is a saddle point, and $(W*, 0)$ is a stable improper node. In figures 22abc the three compatible phase planes are illustrated. They differ in that the separatrix T^- may tend into $(W^-, 0)$, it may tend into $(W*, 0)$, or it may be unbounded. For clarity, only T^+ is shown in the right half-plane. Note that the locus Γ where $g(w, z) = 0$ consists of two branches: a closed curve to the left and a parabolic-like curve to the right. If we only consider the vector field, it is conceivable that T^+ tends to infinity in the fourth quadrant without ever intersecting Γ. Were this not to occur, T^+ could not spiral into $(W^+, 0)$ as shown.

However, by examining the nature of the phase plane at infinity, we can verify that T^+ indeed intersects Γ. The method to be used was

(a)

(b)

(c)

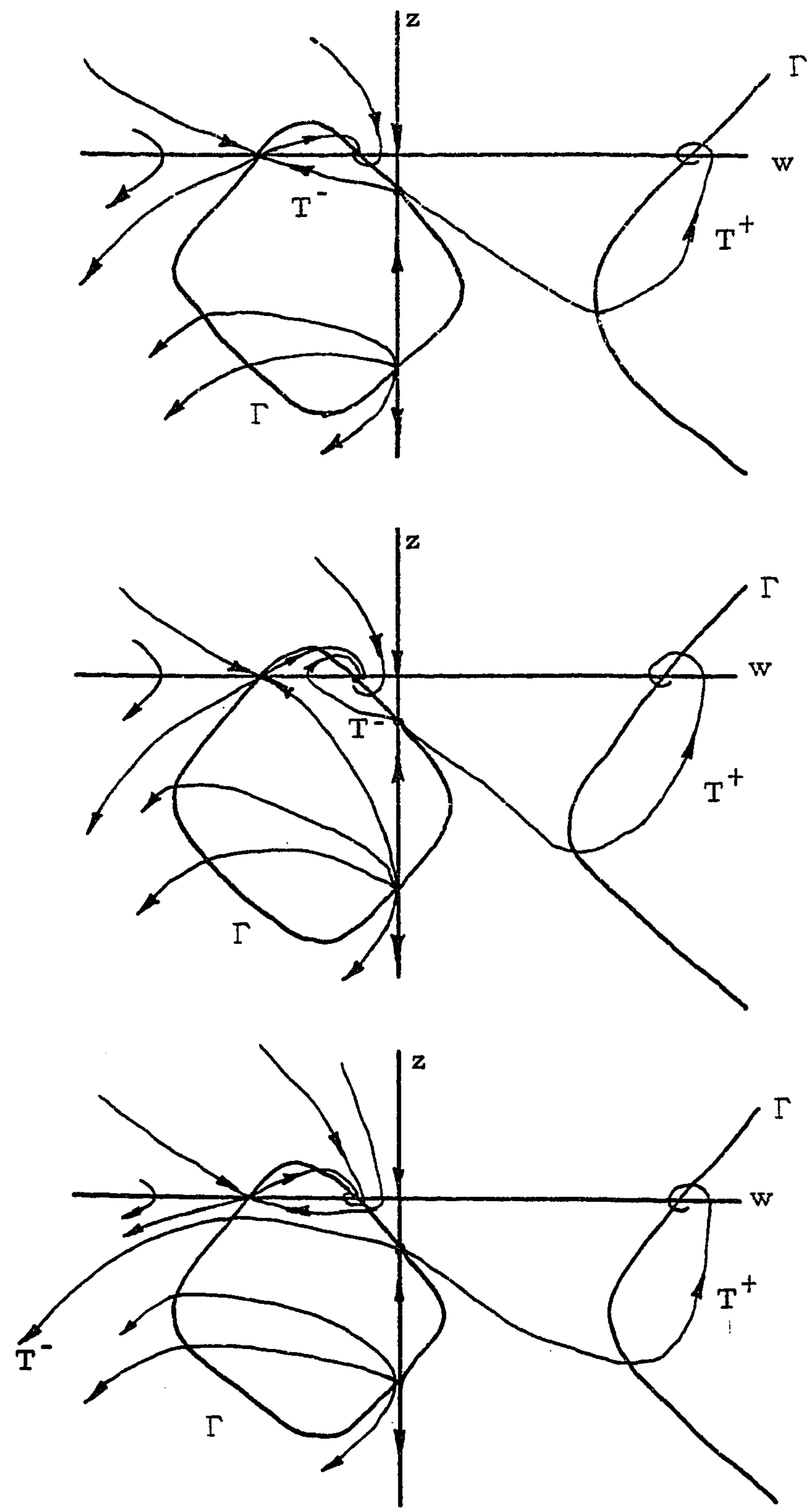


Figure 22 Phase plane for β odd, $A >> 1$, $\alpha > 0$

first introduced by Poincaré and may be found in the references [6], [7]. A point (w, z) in the plane can be represented in projective coordinates by (w, z, 1), or more generally, by $(\tilde{w}, \tilde{z}, \tilde{v})$ where

$$\tilde{w}/\tilde{v} = w \qquad \tilde{z}/\tilde{v} = z \qquad \tilde{v} \neq 0.$$

Points with $\tilde{v} = 0$ lie on the circle at infinity. If we take $\tilde{v} \equiv 1$, equations (2.5) can be rewritten as

$$\left(\frac{\dot{\tilde{w}}}{\tilde{v}}\right) = f\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \tag{2.29a}$$

$$\left(\frac{\dot{\tilde{z}}}{\tilde{v}}\right) = g\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \tag{2.29b}$$

Poincaré introduces

$$\begin{aligned} f^* &= \tilde{v}^2 f\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \\ &= -\tilde{w}\tilde{z} \end{aligned} \tag{2.30a}$$

$$\begin{aligned} g^* &= \tilde{v}^\beta g\left(\frac{\tilde{w}}{\tilde{v}}, \frac{\tilde{z}}{\tilde{v}}\right) \\ &= \alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1} \end{aligned} \tag{2.30b}$$

where f is a polynomial of degree 2 and g is a polynomial of degree β. Equations (2.5) can be imbedded in the equation

$$0 = \begin{vmatrix} d\tilde{w} & d\tilde{z} & d\tilde{v} \\ \tilde{w} & \tilde{z} & \tilde{v} \\ \tilde{v}^{\beta-2} f^* & g^* & 0 \end{vmatrix} \tag{2.31}$$

which, when expanded, is

$$\begin{aligned} &-\tilde{v}[\alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1}]\,d\tilde{w} - \tilde{v}^{\beta-1}\tilde{w}\tilde{z}\,d\tilde{z} \\ &\quad + \{\tilde{w}[\alpha\tilde{w}^\beta - \alpha A\tilde{w}\tilde{v}^{\beta-1} - \tilde{z}^2\tilde{v}^{\beta-2} - \gamma\theta\tilde{v}^\beta + (\gamma+\theta)\tilde{z}\tilde{v}^{\beta-1}] + \tilde{v}^{\beta-2}\tilde{w}\tilde{z}^2\}\,d\tilde{v} \\ &= 0 \end{aligned} \tag{2.31'}$$

Critical points are characterized by the simultaneous vanishing of the coefficients of $d\tilde{w}$, $d\tilde{z}$, and $d\tilde{v}$. In addition to the finite points determined above with $\tilde{v} = 1$, we find points on the circle at infinity by setting $\tilde{v} = 0$. By examining the coefficient of $d\tilde{v}$, we conclude there is only one such point, viz.

$$(\tilde{w}, \tilde{z}, \tilde{v}) = (0, 1, 0)$$

We introduce coordinates

$$W = \tilde{w}/\tilde{z}, \quad V = \tilde{v}/\tilde{z}, \quad \tilde{z} \equiv 1$$

Note that points (w, z, 1) with $w > 0$, $z < 0$ corresponds to points (W, V) with $W < 0$, $V < 0$, and that $(W, V) = (0, 0)$ corresponds to infinity in the (w, z) plane.

With $\tilde{z} \equiv 1$, equation (2.31') is equivalent to the system

$$\dot{W} = \alpha W^{\beta-1} - \alpha A W^2 V^{\beta-1} - \gamma\theta W V^{\beta} + (\gamma+\theta) W V^{\beta-1} \tag{2.32a}$$

$$\dot{V} = \alpha W^{\beta+1} - \alpha A V^{\beta} W - V^{\beta-1} - \gamma\theta V^{\beta+1} + (\gamma+\theta) V^{\beta} \tag{2.32b}$$

We wish to study behavior in the neighborhood of the origin. Note that $W \equiv 0$, $\dot{V} = -V^{\beta-1} - \gamma\theta V^{\beta+1} + (\gamma+\theta)V^{\beta}$ and $V \equiv 0$, $\dot{W} = \alpha W^{\beta+1}$ provide exact solutions which completely cover the V and W axes, respectively. In particular, this implies that trajectories in the quadrant $W < 0$, $V < 0$ can only leave this quadrant by tending to the origin (infinity in the w-z plane), by tending to infinity (which means crossing the line $z = 0$ in the w-z plane), or by tending to one of the other finite critical points (which all correspond to finite critical points in the w-z plane).

Since $\beta > 1$, the origin is not an elementary critical point of equations (2.32). To study the behavior of trajectories, we introduce polar coordinates

$$W = r\cos\phi, \qquad V = r\sin\phi$$

which result in the system

$$\dot{r} = -r^{\beta-1}\sin^{\beta}\phi + (\gamma+\theta)r^{\beta}\sin^{\beta-1}\phi$$
$$+ r^{\beta+1}(\alpha\cos^{\beta}\phi - \alpha A\cos\phi\sin^{\beta-1}\phi - \gamma\theta\sin^{\beta}\phi) \qquad (2.33a)$$
$$\dot{\phi} = -r^{\beta-2}\cos\phi\sin^{\beta-1}\phi \qquad (2.33b)$$

Since trajectories cannot cross the W axis and the V axis, and hence cannot spiral into the origin, they must approach the origin along angles ϕ_0 satisfying

$$\cos\phi_0\sin^{\beta-1}\phi_0 = 0$$

This leaves only $\phi_0 = 0, \pi/2, \pi, 3\pi/2$, so that a trajectory can only approach the origin tangent to one of the axes.

Now consider a trajectory through a point in the quadrant $W < 0$, $V < 0$. There $\pi < \phi < 3\pi/2$ so $\cos\phi < 0$. Since β is odd, $\sin^{\beta-1}\phi > 0$. Consequently, $\dot{\phi} > 0$ and the trajectory must approach the -V axis. Near the origin

$$r \sim -r^{\beta-1}\sin^{\beta}\phi > 0$$

since $\sin\phi < 0$ and β is odd. This implies that as the trajectory approaches the -V axis, it must move away from the origin. In short, as regards the quadrant $W < 0$, $V < 0$, the origin is a saddle point! We conclude that a trajectory in this quadrant must tend to a finite critical point other than the origin or must tend to infinity. Correspondingly, a trajectory in the quadrant $w > 0$, $z < 0$ must tend to a finite critical point or must cross the line $z = 0$. Since the only possible limit point as $x \to +\infty$ is a spiral point on the +w axis, the trajectory necessarily crosses the line $z = 0$. This in turn implies that the trajectory crosses the curve Γ and the argument is completed.

Having confirmed the behavior shown in figure 22, we summarize: For $\lambda > 0$ the behavior can be described by figure 10, including the existence of numbers m_1, m_2, and m_∞ such that

for $0 < \lambda < m_2$ one solution exists

for $\lambda > m_1$ no solutions exist

for $\lambda = m_\infty$ a countable infinity of solutions exist.

For $\lambda < 0$ one of three alternatives occurs, viz.

either there exists a number $m < 0$ such that

for $m < \lambda < 0$ one solution exists

for $\lambda \leq m$ no solutions exist (Cf. figure 22a)

or there exist numbers m_1, m_2 such that

for $m_1 < \lambda < 0$ one solution exists

for $m_2 < \lambda < m_1$ finitely many solutions exist, but more than one

for $\lambda = m_2$ one solution exists

for $\lambda < m_2$ no solutions exist (Cf. figure 22b)

or for all $\lambda < 0$ precisely one solution exists (Cf. figure 22c).

Case 2: β odd, $A < 0$, $\alpha > 0$

From figure 16d we see that p has one root W which tends to zero as $A \to -\infty$. Referring to the discussion of case 1, we conclude

$$W = W^* \sim -\gamma\theta/\alpha A > 0 \tag{2.25}$$

and the characteristic exponents are

$$\ell_+ \doteq \gamma \text{ and } \ell_- \doteq \theta \quad . \tag{2.28}$$

The corresponding point is a stable node. Figure 20 accurately describes the phase plane, and figure 10 describes the multiplicities, with the restriction that there exist only finitely many numbers m_k.

Case 3: β even, $A > 0$, $\alpha > 0$

From figure 16a we see that p has two roots, viz.

$$W^{+} \sim A^{1/(\beta-1)} > 0 \text{ and } W^{*} \sim -\gamma\theta/\alpha A < 0.$$

The characteristic exponents of the corresponding critical points are, respectively,

$$\ell_{\pm} \doteq \tfrac{1}{2}(\gamma+\theta) \pm i\sqrt{(\beta-1)\alpha A^{\beta/(\beta-1)}} \tag{2.26}$$

and

$$\ell_{+} \doteq \gamma, \quad \ell_{-} \doteq \theta \quad .$$

Hence $(W^{+}, 0)$ is a stable spiral point and $(W^{*}, 0)$ is a stable improper node. The phase plane is described in figure 23. An argument similar to that used in case 1 confirms that T^{+} indeed intersects Γ. For $|\lambda|$ large no solutions exist, and for $|\lambda|$ sufficiently small a unique solution exists. For $\lambda < 0$ the multiplicity of any λ is bounded, whereas for $\lambda > 0$ the multiplicity is unbounded; in particular, there exists a unique value $\lambda = m_{\infty} = W^{+}$ for which a countable infinity of solutions exists.

Case 4: β even, $A > 0$, $\alpha < 0$

From figure 16e we see that p has two positive roots, which must be

$$W^{+} \sim A^{1/(\beta-1)} > 0 \text{ and } W^{*} \sim -\gamma\theta/\alpha A > 0$$

The characteristic exponents of the corresponding critical points are, respectively,

$$\ell_{\pm} \doteq \tfrac{1}{2}(\gamma+\theta) \pm \sqrt{-(\beta-1)\alpha A^{\beta/(\beta-1)}}$$

and

$$\ell_{+} \doteq \gamma \ , \quad \ell_{-} \doteq \theta$$

so that $(W^{+}, 0)$ is a saddle point and $(W^{*}, 0)$ is a stable node. There exist three alternate phase planes, differing in the region $w > 0$ just as the phase planes for case 1 differ for $w < 0$. In fact, the possible multiplicities of $\lambda > 0$ for case 4 are analogous to those for $\lambda < 0$ in

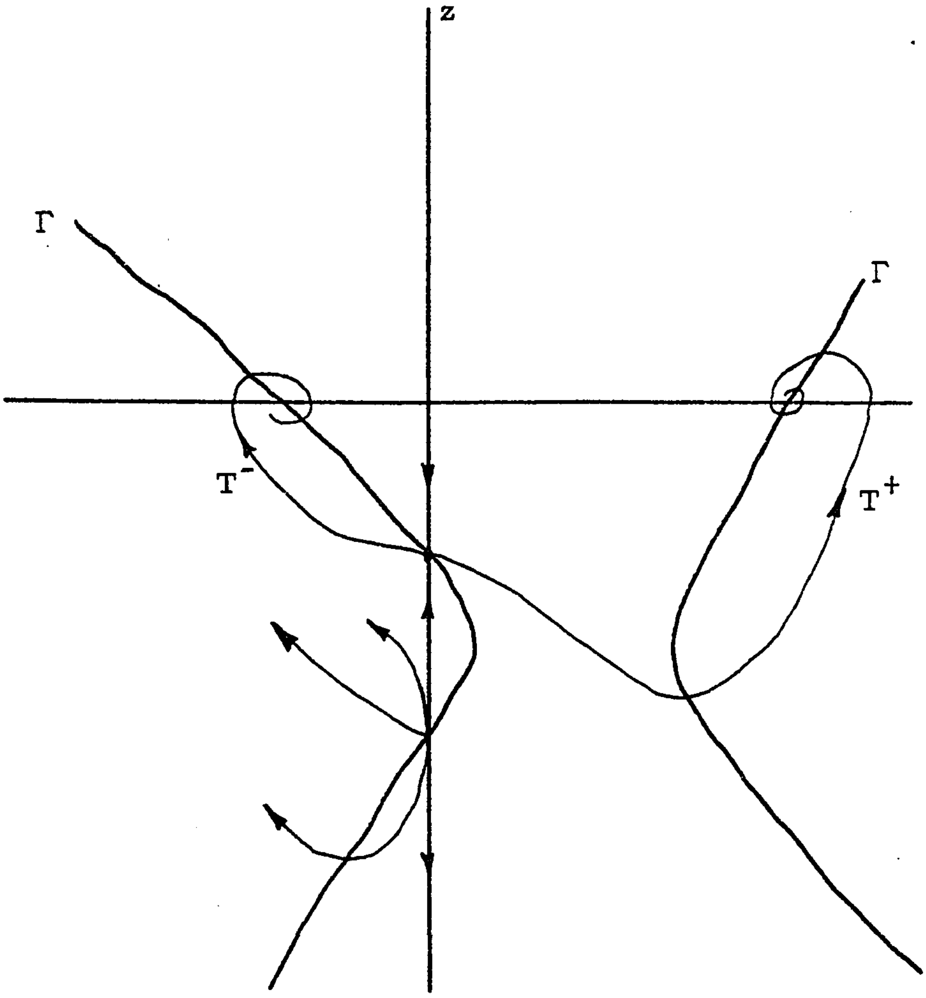


Figure 23 Phase plane for β even, A >> 1, $\alpha > 0$

case 1. For $\lambda < 0$, case 4 always has a unique solution. The phase planes are illustrated in figure 24.

Infinite multiplicities

When $A > 0$ and $\alpha > 0$ there always exists an attractor in the region $w > 0$. When this attractor is a spiral point, there exists a value λ_∞ with a countable infinity of solutions. When $A \ll 1$ this attractor may be a node, but by making μ sufficiently large it can be made a spiral point. When $A \gg 1$ it is always a spiral point. The question arises: if the attractor is a spiral point for $A = A_0$, is it a spiral point for $A > A_0$? We answer this in the affirmative.

Denote the coordinates of the attractor by $w = W$, $z = 0$. From figures 16a, e, f it is clear that

$$W > \hat{w} > 0 \tag{2.27}$$

where $\hat{w}$ is the (positive) root of $\frac{dp}{dw}$. Recall that $\hat{w}$ satisfies

$$\hat{w}^{\beta-1} = A/\beta \tag{2.10}$$

By differentiating the relation $p(W) = 0$ we obtain

$$\frac{dW}{dA} = W/(\beta W^{\beta-1} - A) \tag{2.28}$$

From (2.27) it is clear that W is an increasing function of A. Now the characteristic exponents at (W, 0) are

$$\ell_{\pm} = \tfrac{1}{2}(\gamma+\theta) \pm \tfrac{1}{2}\sqrt{(\gamma+\theta)^2 - 4[(\beta-1)\alpha AW + \beta\gamma\theta]} \tag{2.12}$$

For $\beta > 1$ and $\alpha > 0$

$$(\gamma+\theta)^2 - 4[(\beta-1)\alpha AW + \beta\gamma\theta]$$

is a decreasing function of A; if it is negative for $A = A_0$, it remains negative for $A > A_0$. Hence the desired result is shown.

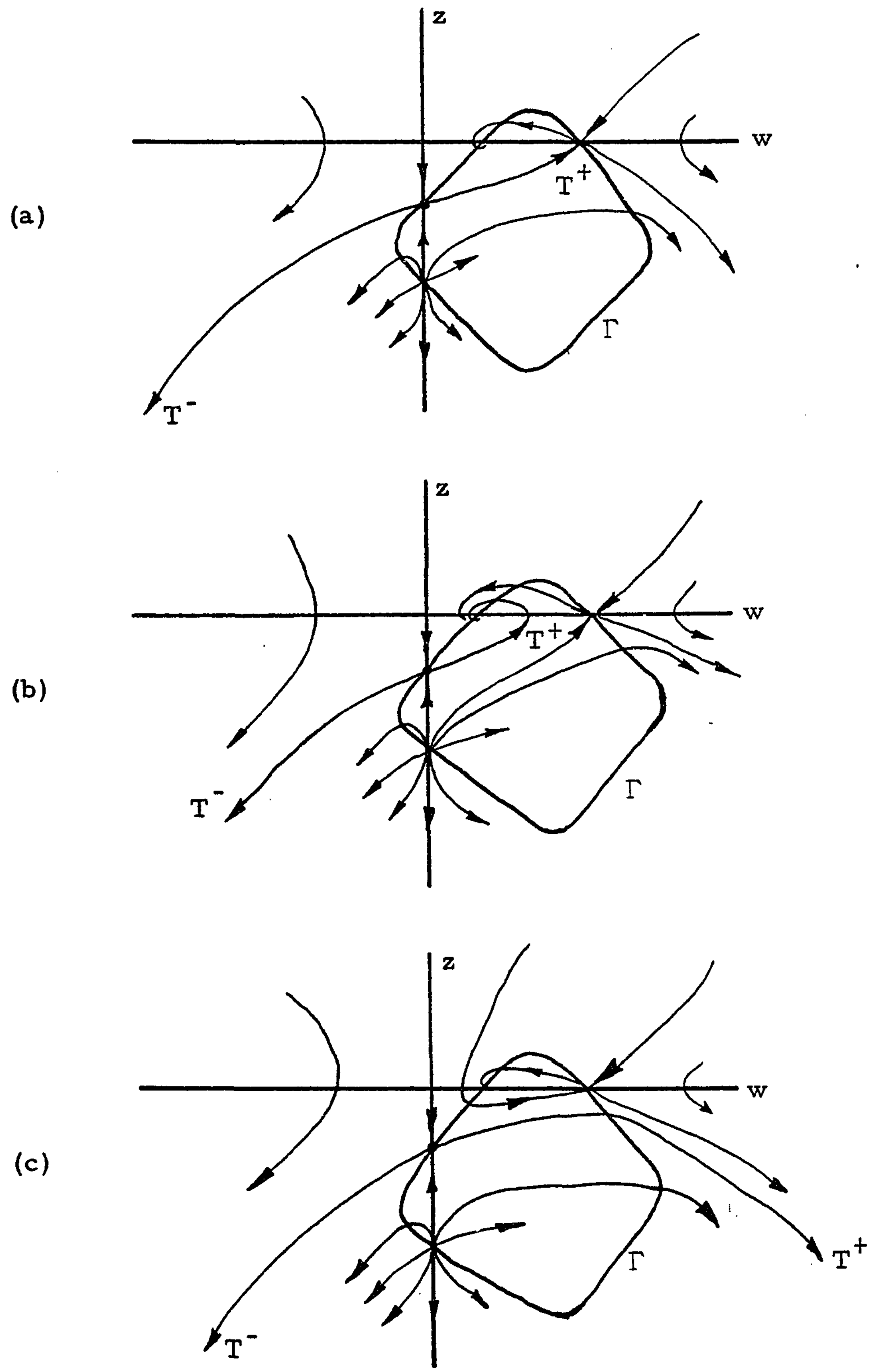


Figure 24 Phase plane for β even, $A >> 1$, $\alpha < 0$

CHAPTER 3

CIRCULAR CYLINDRICAL SHELLS: THE STATIC PROBLEM

The following two chapters are concerned with the problem of the buckling of a circular cylindrical shell under axial loading. In this chapter we treat the static problem as an example of bifurcation phenomena, using primarily methods from pertubation theory. In the next chapter we use multi-time scaling methods to analyze the dynamic buckling behavior of such a shell.

Donnell-type equations are used to model the cylinder. A derivation of these equations is in Appendix A. We assume that the shell is made from a homogeneous isotropic medium, and that locally the body may be assumed to be in a state of plane stress. The resulting equations are given below.

The shell is described by axial, circumferential, and radial coordinates x, y, and r, respectively; the corresponding displacements are denoted by u, v, and w. The components σ_r, σ_{xr}, σ_{yr} of the stress tenor vanish by the assumption of plane stress. In terms of the remaining components we define the axial, circumferential, and shear forces per unit width

$$N_x = h\sigma_x \ , \quad N_y = h\sigma_y \ , \quad N_{xy} = h\sigma_{xy}$$

where h is the thickness of the shell. For a body in a state of plane stress, Hooke's law becomes

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \end{aligned}\right\} \qquad (3.1)$$

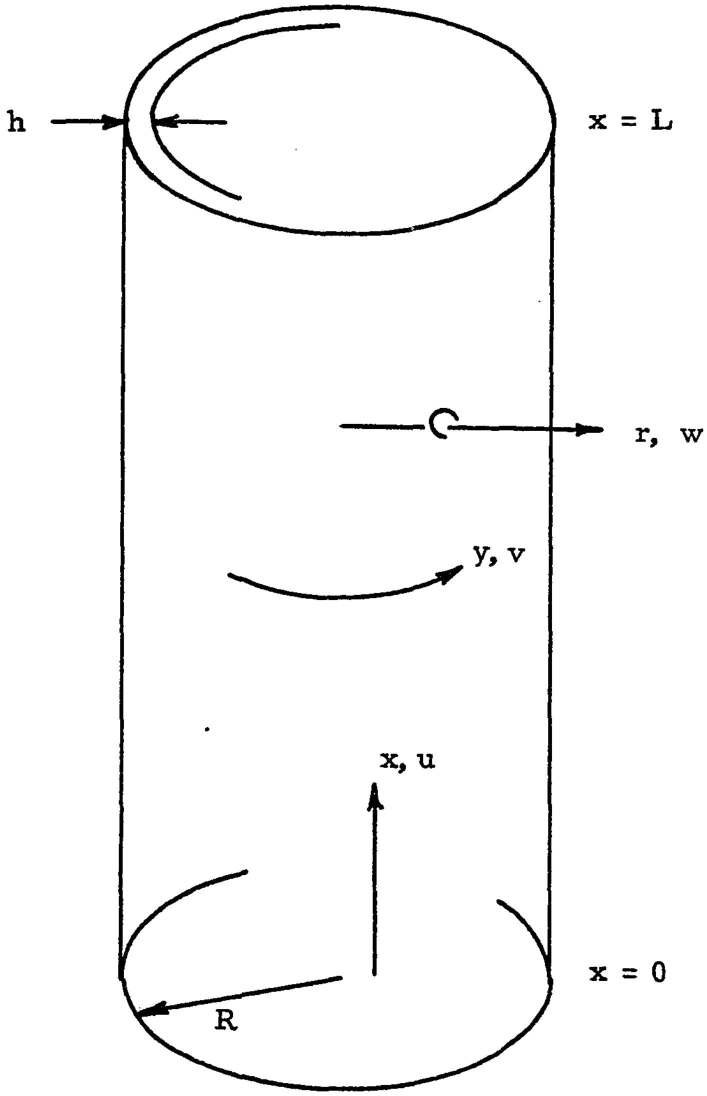


Figure 25 Geometry of the shell

$$\sigma_{xy} = \frac{E}{1-\nu}\,\epsilon_{xy}$$

where E is Young's modulus, ν is Poisson's ratio, and the strains are approximated (Cf. Appendix A) by

$$\left.\begin{aligned} \epsilon_x &= u_x + \tfrac{1}{2}w_x^2 \\ \epsilon_y &= v_y + \frac{1}{R}w + \tfrac{1}{2}w_y^2 \\ 2\epsilon_{xy} &= v_x + u_y + w_x w_y \end{aligned}\right\} \quad (3.2)$$

With the deflectional rigidity defined by

$$D = h^3 E/12(1-\nu^2)$$

the equations of equilibrium are

$$\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{xy} = 0 \qquad (3.3a)$$

$$\frac{\partial}{\partial y}N_y + \frac{\partial}{\partial x}N_{xy} = 0 \qquad (3.3b)$$

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} + N_y\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R}N_y = 0 \qquad (3.3c)$$

Equations (3.3a) and (3.3b) are satisfied if we introduce the Airy stress function F such that

$$N_x = \partial^2 F/\partial y^2 \qquad N_y = \partial^2 F/\partial x^2 \qquad N_{xy} = -\partial^2 F/\partial x\partial y$$

Then equation (3.3c) becomes

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R}\frac{\partial^2 F}{\partial x^2} = 0 \qquad (3.4)$$

Equations (3.1) and (3.2) may be viewed as a system of three equations in the four unknowns F, u, v, and w. If we eliminate u and v from these, we get a second equation in w and F, commonly known as the compatability relation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 F = (hE)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right] \tag{3.5}$$

The problem may be made dimensionless by introducing

$$x = L\tilde{x}, \quad y = L\tilde{y} \text{ for } 0 \le x \le L, \quad 0 \le y \le 2\pi R$$

$$u = L\tilde{u}, \quad v = L\tilde{v}, \quad w = R\tilde{w}$$

$$\sigma_x = E\tilde{\sigma}_x, \quad \sigma_y = E\tilde{\sigma}_y, \quad \sigma_{xy} = E\tilde{\sigma}_{xy}$$

$$F = L^2 hE\tilde{F}$$

$$\tilde{h}^2 \equiv h^2/12(1-\nu^2)L^2$$

$$\omega \equiv L/R, \quad \Omega \equiv 2\pi/\omega$$

If we now suppress the ~ notation, we get the following non-dimensional statement of the static problem on $0 \le x \le 1$, $0 \le y \le \Omega$:

$$\left.\begin{aligned} \sigma_x &= \frac{1}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= \frac{1}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\ \sigma_{xy} &= \frac{1}{1+\nu}\epsilon_{xy} \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} \epsilon_x &= u_x + \frac{1}{2\omega^2} w_x^2 \\ \epsilon_y &= v_y + w + \frac{1}{2\omega^2} w_y^2 \\ 2\epsilon_{xy} &= v_x + u_y + \frac{1}{\omega^2} w_x w_y \end{aligned}\right\} \tag{3.7}$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{3.8}$$

Equilibrium:

$$h^2 \Delta^2 w + \omega^2 \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{3.9}$$

Compatability:

$$\omega^2(\Delta^2 F - \frac{\partial^2 w}{\partial x^2}) = (\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \tag{3.10}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian.

Various boundary conditions are possible. We assume that the cylinder is constrained against radial expansion or contraction at the ends and is simply supported there so

$$w = 0 \quad \text{at } x = 0, 1 \tag{3.11a}$$

$$w_{xx} = 0 \quad \text{at } x = 0, 1 \tag{3.11b}$$

Also the edges are restrained from twisting, which implies

$$v = 0 \quad \text{at } x = 0, 1 \tag{3.11c}$$

A prescribed uniform axial load is applied at the ends, say

$$\sigma_x = -\sigma \equiv \text{constant} \quad \text{at } x = 0, 1 \tag{3.11d}$$

Finally, all physical quantities (and their derivatives) must be periodic in y:

$$u, v, w, \sigma_x, \sigma_y, \sigma_{xy} \text{ have period } \Omega \text{ in } y \tag{3.11e}$$

Decomposition of the Airy stress function

The requirement that σ_x, σ_y, and σ_{xy} be periodic in y places restrictions on the structure of F. Define

$$k(x, y) \equiv F(x, y+\Omega) - F(x, y)$$

Then

$$k_{yy}(x, y) = F_{yy}(x, y+\Omega) - F_{yy}(x, y) = \sigma_x(x, y+\Omega) - \sigma_x(x, y) = 0$$

and we can write

$$k(x, y) = k_0(x) + yk_1(x) \quad .$$

Now define f(x, y) by

$$F(x, y) = f(x, y) + y[\frac{1}{\Omega} k_0(x) - \tfrac{1}{2}k_1(x)] + y^2 \frac{1}{2\Omega} k_1(x) \quad .$$

We compute

$$\begin{aligned} k_0(x) + yk_1(x) &= k(x, y) \\ &= F(x, y+\Omega) - F(x, y) \\ &= \{f(x, y+\Omega) + (y+\Omega)[\frac{1}{\Omega} k_0(x) - \tfrac{1}{2}k_1(x)] + (y+\Omega)^2 \frac{1}{2\Omega} k_1(x)\} \\ &\quad -\{f(x, y) + y[\frac{1}{\Omega} k_0(x) - \tfrac{1}{2}k_1(x)] + y^2 \frac{1}{2\Omega} k_1(x)\} \\ &= f(x, y+\Omega) - f(x, y) + [k_0(x) - \frac{\Omega}{2} k_1(x)] + (y + \frac{\Omega}{2}) k_1(x) \\ &= f(x, y+\Omega) - f(x, y) + k_0(x) + yk_1(x) \quad . \end{aligned}$$

This implies

$$f(x, y+\Omega) = f(x, y)$$

so that f is necessarily periodic in y. If we set $m_0(x) = \frac{1}{\Omega} k_0(x) - \frac{1}{2} k_1(x)$ and $m_1(x) = \frac{1}{\Omega} k_1(x)$, then

$$F(x, y) = f(x, y) + ym_0(x) + \tfrac{1}{2} y^2 m_1(x) \quad .$$

Thus far we have only used the periodicity of $F_{yy} = \sigma_x$. Now $-\sigma_{xy} = F_{xy} = f_{xy} + m_0'(x) + ym_1'(x)$ must be periodic, so $m_1'(x) = 0$. Furthermore, $\sigma_y = F_{xx} = f_{xx} + ym_0''(x)$, so $m_0''(x) = 0$. Hence there exist constants M_0, M_1, M_2 such that

$$m_1(x) = M_1 \quad m_0(x) = M_0 + xM_2 \quad .$$

In summary,

$$F(x, y) = f(x, y) + M_0 y + \tfrac{1}{2} M_1 y^2 + M_2 xy \quad .$$

Note that F is arbitrary within any function G such that $G_{xx} = G_{yy} = G_{xy} = 0$; i.e., if F_1 is an Airy stress function which solves a particular problem, then so is

$$F_2 = F_1 + G$$

where

$$G = ax + by + c$$

for constants a, b, and c. The freedom to choose a, b, and c permits us to specify that

$$M_0 = 0$$

and, as we will find convenient below,

$$\int_0^\Omega f(x, y)dy = 0 \quad \text{for } x = 0 \text{ and } x = 1 \quad .$$

Now at $x = 0$ and $x = 1$, boundary condition (3.11d) becomes

$$-\sigma = \sigma_x = F_{yy} = f_{yy} + M_1$$

Periodicity of f and hence f_y then imply

$$-\sigma\Omega = \int_0^\Omega (f_{yy} + M_1)dy = f_y\Big|_0^\Omega + M_1\Omega = M_1\Omega$$

or $M_1 = -\sigma$ and $f_{yy} = 0$ for $x = 0$ and $x = 1$. Setting $M_2 = \ell$, we have

$$F(x, y) = f(x, y) - \tfrac{1}{2}\sigma y^2 + \ell_{xy} \tag{3.12}$$

For $x = 0$ and $x = 1$ expand

$$f(x, y) = \tfrac{1}{2}a_0(x) + \sum_{n>0} a_n(x)\cos n\omega y + b_n(x)\sin n\omega y \quad .$$

Then $f_{yy} = 0$ implies

$$0 = \int_0^\Omega f_{yy}\cos n\omega y\, dy = -(n\omega)^2\int_0^\Omega f\cos n\omega y\, dy$$

where we have integrated by parts and used the periodicity of f. For $n > 0$ we conclude

$$a_n(x) = 0 \qquad \text{for } x = 0, 1$$

and similarly $b_n(x) = 0$. As noted above, without any loss of generality we can require that

$$a_0(x) = \frac{2}{\Omega}\int_0^{\Omega} f(x, y)\,dy = 0 \qquad \text{for } x = 0 \text{ and } x = 1 \;.$$

Thus we conclude

$$f(x, y) = 0 \quad \text{for } x = 0 \text{ and } x = 1 \;. \tag{3.13}$$

It remains to see what (3.11c) implies about f at $x = 0$ and $x = 1$. For the moment relax condition (3.11a) to the more general condition

$$w = \text{constant} \qquad \text{at } x = 0, 1 \tag{3.11a'}$$

Now $v = 0$ and w = constant imply $v_y = w_y = 0$. The second of the strain relations (3.7) simplifies to

$$\epsilon_y = w \quad \text{at } x = 0, 1 \;.$$

Invert the first two equations of Hooke's law (3.6) to obtain

$$\epsilon_y = \sigma_y - \nu\sigma_x = F_{xx} - \nu F_{yy} = f_{xx} - \nu f_{yy} + \nu\sigma = f_{xx} + \nu\sigma$$

or

$$f_{xx} = w - \nu\sigma \quad \text{at } x = 0 \text{ and } x = 1 \tag{3.14}$$

<u>Symmetric solutions</u>

We now seek solutions angularly symmetric about the shell axis, i.e., assume u, v, w, σ_x, σ_y, and σ_{xy} are independent of y. First note that the strain relations simplify to

$$\left.\begin{aligned} \epsilon_x &= u_x + \frac{1}{2\omega^2} w_x^2 \\ \epsilon_y &= w \\ 2\epsilon_{xy} &= v_x \end{aligned}\right\} \tag{3.7S}$$

We calculate

$$\frac{\partial}{\partial x}\sigma_x = \frac{\partial}{\partial x}F_{yy} = \frac{\partial}{\partial y}F_{xy} = -\frac{\partial}{\partial y}\sigma_{xy} = 0$$

so that σ_x = constant. From boundary condition (3.11d) we conclude

$$\sigma_x = -\sigma$$

Similarly

$$\frac{\partial}{\partial x}\sigma_{xy} = -\frac{\partial}{\partial x}F_{xy} = -\frac{\partial}{\partial y}F_{xx} = -\frac{\partial}{\partial y}\sigma_y = 0$$

so that σ_{xy} = constant. From Hooke's law (3.6) we see that ϵ_{xy} is constant, and hence (3.7S) implies that v is linear in x. Boundary condition (3.11c) in turn requires that v vanish identically

$$v = 0$$

from which we infer that $\epsilon_{xy} = 0$ and hence

$$\sigma_{xy} = 0$$

From (3.6) and (3.7S) calculate

$$w = \epsilon_y = \sigma_y - \nu\sigma_x = \sigma_y + \nu\sigma$$

or

$$\sigma_y = w - \nu\sigma$$

Also

$$\sigma_x - \nu\sigma_y = \epsilon_x = u_x + \frac{1}{2\omega^2}w_x^2 = -\sigma - \nu w + \nu^2\sigma$$

Integrating this yields

$$u(x) = u(0) - \int_o^x (1-\nu^2)\sigma + \nu w + \frac{1}{2\omega^2}w_x^2 \, dx$$

All that remains is to find w(x). The equilibrium equation (3.9) reduces to

$$Lw \equiv h^2 w_{xxxx} + \sigma w_{xx} + \omega^2 w = \omega^2 \nu\sigma \tag{3.15}$$

subject to boundary conditions (3.11a, b). The linear operator L is self-adjoint; consequently (3.15) has a unique solution unless there exists a non-trivial solution to

$$Lz = 0$$

satisfying (3.11a, b). In that case (3.15) has no solution or a continuum of solutions, depending on whether $\langle z, 1\rangle \neq 0$ or $\langle z, 1\rangle = 0$, respectively.

Suppose $Lz = 0$ and expand $z = \sum_{k=1}^{\infty} c_k \sin k\pi x$.

Integrating by parts, we find

$$0 = \int_0^1 (h^2 z_{xxxx} + \sigma z_{xx} + \omega^2 z)\sin m\pi x \, dx$$

$$= [h^2(m\pi)^4 - \sigma(m\pi)^2 + \omega^2]\, cm/2$$

Now $z \neq 0$ implies $c_m \neq 0$ for some c_m; hence

$$\sigma = \sigma_{m,1} \equiv h^2(m\pi)^2 + \omega^2/(m\pi)^2 \qquad (3.16)$$

If we treat $m > 0$ as a continuous parameter and graph $\sigma_{m,1}$ as a function of m, it is clear that for a given value of σ there exist at most two values of m such that (3.16) holds. If a (non-unique) solution to (3.15) exists when $\sigma = \sigma_{m,1}$, then necessarily

$$0 = \langle z, 1\rangle = \int_0^1 \sin m\pi x \, dx$$

i.e., m must be even.

When $\sigma \neq \sigma_{m,1}$ for any m, it is a simple exercise to find the (unique) solution to (3.15), viz.

$$\begin{aligned} w(x) = \nu\sigma \Bigg[1 &+ \frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\sin\mu_1} (\sin\mu_1 x + \sin\mu_1(1-x)) \\ &+ \frac{\mu_1^2}{(\mu_2^2 - \mu_1^2)\sin\mu_2} (\sin\mu_2 x + \sin\mu_2(1-x)) \Bigg] \end{aligned} \qquad (3.17)$$

where μ_1 and μ_2 are defined by

$$\left.\begin{aligned} \mu_1^2 &= (\sigma + \sqrt{\sigma^2 - 4h^2\omega^2})/2h^2 \\ \mu_2^2 &= (\sigma - \sqrt{\sigma^2 - 4h^2\omega^2})/2h^2 \end{aligned}\right\} \qquad (3.18)$$

Numerical evaluation of w suggests that away from the boundaries of the interval $0 < x < 1$, $w(x) \sim \nu\sigma$. We can obtain this result for small values of σ by setting $\sigma = sh$ in (3.18) and letting $h \to 0+$. For s such that $0 < s < 2\omega$, μ_1 and μ_2 are complex and $|\mu_1| = |\mu_2| = O(h^{-\frac{1}{2}})$. Then for $0 < x < 1$

$$\sin\mu_1 x/\sin\mu_1 \to 0 \quad \sin\mu_1(1-x)/\sin\mu_1 \to 0 \quad \text{as } h\to 0$$

and similarly for μ_2, but

$$\mu_2^2/(\mu_1^2-\mu_2^2) = O(1),\ \mu_1^2/(\mu_2^2-\mu_1^2) = O(1) \quad .$$

From equation (3.17) it is clear that $w(x) \sim \nu\sigma$.

Poisson expansion and bifurcation

This motivates seeking a solution of the problem compatible with w = constant. From the equilibrium equation (3.9) we conclude $\sigma_y = F_{xx} = 0$; then (3.3b) implies $\sigma_{xy,x} = 0$ and $\sigma_{xy} = \alpha(y)$ for some function α. From equation (3.3a) we obtain $\sigma_{x,x} = -\alpha'(y)$ or $\sigma_x = \beta(y)-\alpha'(y)x$ for some function $\beta(y)$. The boundary condition $\sigma_x = -\sigma$ at $x = 0, 1$ implies $\beta(y) = -\sigma$ and $\alpha'(y) = 0$, whence $\sigma_x = -\sigma$ and $\sigma_{xy} = \alpha$ = constant. Now

$$\epsilon_y = \sigma_y - \nu\sigma_x = \nu\sigma$$

$$= v_y + w + \frac{1}{2\omega^2} w_y^2 = v_y + w$$

Hence $v = (\nu\sigma - w)y + \gamma(x)$ for some function $\gamma(x)$. Periodicity of v in y forces $w = \nu\sigma$, and $v = 0$ at $x = 0, 1$ then gives $\gamma(0) = \gamma(1) = 0$.

$$\epsilon_x = \sigma_x - \nu\sigma_y = -\sigma$$

$$= u_x + \frac{1}{2\omega^2} w_x^2 = u_x$$

which implies $u = -\sigma x + \delta(y)$ for some function $\delta(y)$.

$$2\epsilon_{xy} = 2(1+\nu)\sigma_{xy} = 2(1+\nu)\alpha$$

$$= v_x + u_y + \frac{1}{\omega^2} w_x w_y = \gamma'(x) + \delta'(y)$$

Therefore $\gamma'(x)$ = constant and $\delta'(y)$ = constant. Since $\gamma(0) = \gamma(1) = 0$, we conclude $\gamma = 0$. Periodicity of u in y forces δ = constant which can be chosen arbitrarily by a solid body translation along the x axis. Finally δ = constant and $\gamma = 0$ yield $\alpha = 0$. In summary, the only solution compatible with w = constant is

$$w = \nu\sigma \quad u = -\sigma(x-\tfrac{1}{2}) \quad v = 0$$

$$\sigma_x = -\sigma \quad \sigma_y = 0 \quad \sigma_{xy} = 0$$

$$f = 0 \quad \ell = 0$$

The solution is known as Poisson expansion; it satisfies the problem with boundary condition (3.11a) replaced by

$$w = \nu\sigma \quad \text{at } x = 0, 1 \quad . \tag{3.11A}$$

Note that with (3.11A), condition (3.14) becomes

$$f_{xx} = 0 \quad \text{at } x = 0, 1 \tag{3.14'}$$

Equations (3.9) and (3.10) are two equations in w and F; they can be converted into equations in w and f if we find an appropriate expression for ℓ. From (3.6) and (3.7) we calculate

$$2\epsilon_{xy} = 2(1+\nu)\sigma_{xy} = -2(1+\nu)F_{xy} = -2(1+\nu)(f_{xy} + \ell)$$

and

$$2\epsilon_{xy} = v_x + u_y + \frac{1}{\omega^2} w_x w_y$$

Periodicity of u and f imply

$$\int_0^\Omega f_{xy} dy = \int_0^\Omega u_y dy = 0$$

while boundary condition (3.11c) yields

$$\int_0^1 v_x dx = 0$$

By integrating by parts and using the periodicity of w, we have

$$\int_0^\Omega w_x w_y dy = -\int_0^\Omega w w_{xy} dy$$

Putting all this together, we can conclude

$$\ell = \frac{1}{4\pi\omega(1+\nu)} \int_0^\Omega \int_0^1 w w_{xy} dx dy \tag{3.19}$$

For the remainder of this chapter we will be concerned with the bifurcation of solutions from Poisson expansion. We will have to solve a hierarchy of constant coefficient linear equations with inhomogeneous terms which are known explicitly at each step. If, instead of Poisson expansion, we took the axisymmetric solution (3.17) as the state from which bifurcation occurs, the relevant system would be

$$h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} - f_{yy} w_{s,xx} - f_{s,xx} w_{yy}$$
$$= f_{yy} w_{xx} - 2 f_{xy} w_{xy} + f_{xx} w_{yy} - 2\ell w_{xy}$$

$$\omega^2(\Delta^2 f - w_{xx}) + w_{s,xx} w_{yy} = w_{xy}^2 - w_{xx} w_{yy}$$

Here (w_s, f_s) denotes the solution of (3.17) and $(w_s+w,\ f_s+f)$ is the solution of the full problem. The linearized equations for the pertubations w and f have variable coefficients and require numerical solution insofar as an explicit analytic solution is not possible. Numerical studies by Almroth [9] have shown that the value of the critical load σ_0 at which buckling (or bifurcation) occurs is not changed sufficiently by the boundary conditions to explain the well-known discrepancy between theory and test data. Consequently we will use Poisson expansion as the pre-buckling state.

Let $w = \nu\sigma + \hat{w}$; then drop the ^ notation so that w represents the displacement from Poisson expansion. Equations (3.9), (3.10), and (3.12), as well as boundary conditions (3.11A), (3.11b), (3.13), (3.14'), and (3.11e), are transformed into the following final formulation of the problem (equation (3.19) is unchanged under this transformation):

$$h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} = f_{yy}w_{xx} - 2f_{xy}w_{xy} + f_{xx}w_{xy} - 2\ell w_{xy} \quad (3.20)$$

$$\omega^2(\Delta^2 f - w_{xx}) = w_{xy}^{\ 2} - w_{xx}w_{yy} \quad (3.21)$$

$$w = w_{xx} = 0 \quad \text{at } x = 0, 1 \quad (3.22a)$$

$$f = f_{xx} = 0 \quad \text{at } x = 0, 1 \quad (3.22b)$$

$$w \text{ and } f \text{ have period } \Omega \text{ in } y \quad (3.22c)$$

We propose to attack this problem by seeking solutions which bifurcate from Poisson expansion (which corresponds to $w = f = 0$ in this notation). We seek solutions of the form:

$$\left.\begin{aligned} w &= \epsilon w_1 + \epsilon^2 w_2 + \ldots \\ f &= \epsilon f_1 + \epsilon^2 f_2 + \ldots \\ \sigma &= \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots \\ \ell &= \epsilon^2 \ell_2 + \ldots \end{aligned}\right\} \quad (3.23)$$

where the small parameter ϵ is defined by

$$\epsilon^2 = \frac{4}{\Omega}\int_0^\Omega\int_0^1 w^2 + f^2\,dxdy \quad (3.24)$$

Substituting (3.23) into equations (3.19), (3.20), and (3.21) yields the following hierarchy:

$$O(\epsilon): \quad h^2\Delta^2 w_1 + \sigma_0 w_{1_{xx}} + \omega^2 f_{1_{xx}} = 0 \quad (3.25a)$$

$$\omega^2(\Delta^2 f_1 - w_{1_{xx}}) = 0 \quad (3.25b)$$

$$O(\epsilon^2) \qquad h^2\Delta^2 w_2 + \sigma_0 w_{2xx} + \omega^2 f_{2xx} = f_{1yy}w_{1xx} - 2f_{1xy}w_{1xy} + f_{1xx}w_{1yy} - \sigma_1 w_{1xx} \tag{3.26a}$$

$$\omega^2(\Delta^2 f_2 - w_{2xx}) = w_{1xy}^2 - w_{1xx}w_{1yy} \tag{3.26b}$$

$$\ell_2 = \frac{1}{4\pi\omega(1+\nu)} \int_0^\Omega \int_0^1 w_1 w_{1xy}\, dxdy \tag{3.26c}$$

$$O(\epsilon^3) \qquad h^2\Delta^2 w_3 + \sigma_0 w_{3xx} + \omega^2 f_{3xx} = f_{1yy}w_{2xx} + f_{2yy}w_{1xx} - 2f_{1xy}w_{2xy} - 2f_{2xy}w_{1xy} + f_{1xx}w_{2yy} + f_{2xx}w_{1yy} - \sigma_1 w_{2xx} - \sigma_2 w_{1xx} - 2\ell_2 w_{1xy} \tag{3.27a}$$

$$\omega^2(\Delta^2 f_3 - w_{3xx}) = 2w_{1xy}w_{2xy} - w_{1xx}w_{2yy} - w_{2xx}w_{1yy} \tag{3.27b}$$

$$\ell_3 = \frac{1}{4\pi\omega(1+\nu)} \int_0^\Omega \int_0^1 w_1 w_{2xy} + w_2 w_{1xy}\, dxdy \tag{3.27c}$$

The normalization condition (3.24) yields

$$\begin{aligned}
\Omega/4 &= \int_0^\Omega \int_0^1 w_1^2 + f_1^2\, dxdy \\
0 &= \int_0^\Omega \int_0^1 w_1 w_2 + f_1 f_2\, dxdy \\
0 &= \int_0^\Omega \int_0^1 w_2^2 + 2w_1 w_3 + f_2^2 + 2f_1 f_3\, dxdy
\end{aligned} \tag{3.28}$$

All the w_j and f_j inherit the linear homogeneous boundary conditions (3.22).

Recall $\omega = 2\pi/\Omega$ and $0 \le y \le \Omega$. The set of functions

$$\begin{aligned}
\psi_{mn}(x, y) &= \sin m\pi x \cos n\omega y \\
\hat{\psi}_{mn}(x, y) &= \sin m\pi x \sin n\omega y
\end{aligned} \qquad \left\{ \begin{aligned} m &= 1, 2, 3, \ldots \\ n &= 0, 1, 2, \ldots \end{aligned} \right.$$

is complete on $[0, 1] \times [0, \Omega]$. If we multiply equations (3.25) by one such function and integrate by parts, we obtain

$$\left.\begin{aligned} h^2 Q^2_{mn} a_{mn} - \sigma_0 (m\pi)^2 a_{mn} - \omega^2 (m\pi)^2 A_{mn} &= 0 \\ Q^2_{mn} A_{mn} + (m\pi)^2 a_{mn} &= 0 \end{aligned}\right\} \quad (3.29)$$

and similarly for b_{mn}, B_{mn}, where

$$a_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 w_1 \sin m\pi x \cos n\omega y \, dxdy$$

$$A_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 f_1 \sin m\pi x \cos n\omega y \, dxdy$$

$$b_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 w_1 \sin m\pi x \sin n\omega y \, dxdy$$

$$B_{mn} = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 f_1 \sin m\pi x \sin n\omega y \, dxdy$$

and we define

$$Q_{mn} \equiv (m\pi)^2 + (n\omega)^2 \quad (3.30)$$

Equations (3.29) have a non-trivial solution if and only if

$$\sigma_0 = \omega^2 (m\pi)^2 / Q^2_{mn} + h^2 Q^2_{mn} / (m\pi)^2 \quad (3.31)$$

If there exists a unique integer pair M, N such that $M > 0$, $N \geq 0$, and $\sigma_0 = \sigma_0(M, N)$ we say σ_0 is a simple eigenvalue. Note, however, that for $N > 0$ there exist two eigenfunctions corresponding to (M, N). However, if $\{w(x, y), f(x, y)\}$ is a solution of the problem, then the translation invariance in y of the equations and boundary conditions implies that $\{w(x, y+y_0), f(x, y+y_0)\}$ is also a solution for any constant y_0. These merely correspond to a solid body rotation of the cylinder. Consequently, even though the solution of (3.25)is, for some constants c_1 and c_2,

$$w_1(x, y) = c_1 Q^2_{MN} \sin M\pi x \cos N\omega y + c_2 Q^2_{MN} \sin M\pi x \sin N\omega y$$

$$f_1(x, y) = -c_1 (M\pi)^2 \sin M\pi x \cos N\omega y - c_2 (M\pi)^2 \sin M\pi x \sin N\omega y$$

we may take $c_2 = 0$ without any loss of generality. The normalization condition (3.28) then yields

$$c_1 = \pm 1/\sqrt{Q^4_{MN} + (M\pi)^4}$$

Multiplicity of the eigenvalues

If we write $\sigma_0(t) = \omega^2 t + h^2/t$, then the eigenvalues are $\sigma_0(t_{mn})$ where

$$t_{mn} = (m\pi/Q_{mn})^2 \quad .$$

From the convexity of the graph of $\sigma_0(t)$ for $t > 0$ it follows that there exist at most two values t_1 and t_2 such that $\sigma_0(t_1) = \sigma_0(t_2)$. For distinct t_1 and t_2, a short algebraic manipulation shows that $\sigma_0(t_1) = \sigma_0(t_2)$ is equivalent to

$$t_1 t_2 = h^2/\omega^2 \quad .$$

Since the values t_{mn} depend on ω but not on h, given t_{mn} and $t_{m'n'}$ distinct it is possible to choose h (in a unique fashion) so that $\sigma_0(t_{mn}) = \sigma_0(t_{m'n'})$. Thus we see that σ_0 may have "multiplicity" at most two in regard to the number of corresponding values of t_{mn}, and indeed "multiplicity" two does occur. We can guarantee that $\sigma_0(t_{mn})$ has "multiplicity" one by choosing $h = t_{mn}\omega$, for in that case t_{mn} occurs at the global minimum of $\sigma(t)$ in $t > 0$ ($\sigma = 2\omega h$ there). Consequently, the question of the actual multiplicity of σ_0 (i.e., the number of integer pairs $m > 0$, $n \geq 0$ such that $\sigma_0 = \sigma_0(m, n)$ can be reduced to studying the multiplicity of t_{mn}.

Suppose $t_{m_1 n_1} = t_{m_2 n_2}$; then $m_1 \pi/Q_{m_1 n_1} = m_2 \pi/Q_{m_2 n_2}$, or

$$\omega^2/\pi^2 = m_1 m_2 (m_1 - m_2)/(m_1 n_2{}^2 - m_2 n_1{}^2)$$

so that ω^2/π^2 must be rational. Thus the irrationality of ω^2/π^2 is sufficient to assure that all t_{mn} are simple, and consequently the

$\sigma_0(m, n)$ are simple (except when $\sigma_0(t_{mn}) = \sigma_0(t_{m'n'})$ for $t_{mn} \neq t_{m'n'}$).

To further investigate the possible multiplicity of σ_0, consider when $q = \omega/\pi$ is rational. Suppose $t^* = t_{MN}$; we wish to find all other (m, n) such that $t^* = t_{mn}$. Manipulating,

$$t^* = (m\pi/Q_{mn})^2 = (m\pi)^2/[(m\pi)^2+(n\omega)^2]^2$$

leads to

$$(m^2 + n^2q^2)/m = 1/\pi\sqrt{t^*} \equiv 2c$$

Note that c is prescribed by q (or ω) and an integer pair (M, N); also c is rational. Under these circumstances, determing the multiplicity of σ_0 is reduced to finding the number of integer pairs (m, n) with $m > 0$, $n \geq 0$, such that

$$(m-c)^2 + (nq)^2 = c^2 \quad .$$

We will show that there exist q and c (and hence $\sigma_0(M, N)$) with arbitrarily large (but nonetheless finite) multiplicity. This will be done by construction, using integral values of c and special values of q. When c is a positive integer it is possible, for any value of q, to find a pair (M, N) such that $c = c(q, M, N)$; simply let $M = 2c$ and $N = 0$. Designate $\mu = m-c$ so that $-c < \mu \leq c$ and $n \geq 0$. When $q = 1$ there exists a one-to-one correspondence between such pairs (μ, n) satisfying

$$\mu^2 + n^2 = c^2$$

and the eigenfunctions ψ_{mn} (disregarding the translational invariance in y) corresponding to $\sigma_0(t^*)$.

The key to solving this problem is a standard result from the theory of numbers (Cf. reference [12], sections 16.9 and 16.10).

We define R(C) as the number of representations of C in the form $C = A^2 + B^2$, where A and B are integers (not necessarily positive).

We count representations as distinct even when they differ only trivially, i.e., in the sign or order of A and B. For example,

$$0 = 0^2 + 0^2 \qquad R(0) = 1$$

$$1 = (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2 \qquad R(1) = 4$$

$$5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2 \qquad R(5) = 8$$

The main result is as follows:

write $C = 2^{\alpha} \prod p_k^{r_k} \prod q_j^{s_j}$ where the p_k are distinct prime numbers of the form $4m+1$ and the q_j are distinct prime numbers of the form $4m+3$. Then

$$R(C) = \begin{cases} 4 \prod (r_k+1) & \text{if all } s_j \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

To apply this to the problem $\mu^2 + n^2 = c^2$, write $c = 2^{\alpha'} \prod p_k^{r'_k} \prod q_j^{s'_j}$. Then $C = c^2 = 2^{2\alpha'} \prod p_k^{2r'_k} \prod q_j^{2s'_j}$ so that

$$\alpha = 2\alpha' \quad r_k = 2r'_k \quad s_j = 2s'_j \quad .$$

Clearly all the s_j are even, so

$$R(c^2) = 4 \prod (2r'_k + 1)$$

However, $R(c^2)$ gives the number of pairs A, B satisfying

$$A^2 + B^2 = c^2 \quad -c \le A, \ B \le c$$

Identify μ with A and n with B. Obviously $A = \pm c$, $B = 0$ are solutions. Therefore there are R-2 solutions with $B \neq 0$. Of these $\frac{1}{2}(R-2)$ are solutions with $B > 0$ by symmetry. Now $B > 0$ implies $-c < A < c$, but we must also include the case $\mu = A = c$, $n = B = 0$ (and exclude the case $A = -c$, $B = 0$). This results in $\frac{1}{2}(R-2) + 1$ cases of pairs μ, n within the given bounds. Simplifying, the multiplicity of t^* for $q = 1$ and c an integer is

$$2\prod(2r'_k + 1)$$

which can be made arbitrarily large by merely taking enough prime factors of the form $4m + 1$.

The above construction is limited to multiplicities which are twice an odd number. Next consider $q = 2$ and c an odd integer. Set $\nu = 2n$ so that

$$\mu^2 + \nu^2 = c^2 \quad .$$

When c is odd, precisely one of the numbers μ, ν must be even; this must always be ν. Since the function R treats A and B symmetrically, there are $\frac{1}{2}R$ cases with B (or ν) even. Except for the case $\nu = 0$, they occur in symmetric pairs with $\nu > 0$ and $\nu < 0$; this leaves $\frac{1}{2}(\frac{1}{2}R-2)$ cases with $\nu > 0$ and ν even. Finally, we add the case $\mu = A = c$, $\nu = B = 0$, resulting in $\frac{1}{2}(\frac{1}{2}R-2)+1$ pairs μ, ν. Simplifying, the multiplicity of t^* for $q = 2$ and c an odd integer is

$$\prod(2r'_k + 1)$$

which can be set equal to an arbitrary odd number.

We can construct σ_0 with an arbitrary finite multiplicity. To achieve an even multiplicity, decompose the even number into a sum of two odd numbers and find corresponding values t_1 and t_2 (with, say, $q = 2$); then choose h so as to satisfy $t_1 t_2 = h^2/\omega^2$.

There is a physical significance to q being rational. $q = \omega/\pi = L/R\pi$, where L is the length of the cylinder and R is its radius. When q is rational, L and $2\pi R$ are commensurable, i.e., there exists a unit of length which divides both the length and the circumference of the cylinder an integral number of times. In this situation alone is it possible to cover the cylinder's surface with squares.

The buckling mode

The smallest buckling load

$$\sigma_{min} = \min_{m>0,\ n\geq 0} \sigma_0(t_{mn})$$

is of the most physical interest. In table 1 the values of M and N corresponding to σ_{min} for various combinations of (dimensional) h, R, and L are given, assuming that $\nu = 0.3$ (in all cases σ_{min} is simple). The values of N show no particular pattern; however for a fixed thickness and radius, the values of M tend to increase somewhat linearly with the length. This suggests the existence of a characteristic length over which buckling occurs. Indeed, buckling (or bifurcating) from Poisson expansion is characterized by a rather large number of waves along the entire axis of the shell for common values of L/R. In experiments, however, one typically observes only a few tiers (~ 2) located roughly midway along the axis (see reference [18]). We noted earlier that symmetric solutions with the ends restrained (but simply supported) tend to undergo Poisson expansion away from the ends of the shells; however, as the load increases, so does the width of the boundary layers. Near buckling only a fraction of the length of the cylinder is undergoing Poisson expansion, and we might conjecture that it is this effective length which undergoes the deformations observed in buckling.

Bifurcation for simple eigenvalues

We return to the problem of computing $\sigma = \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots$ along the bifurcating solution branch when σ_0 is a simple eigenvalue. Although it is possible to continue with the scheme indicated by equations (3.23) through (3.28), an iterative scheme exists which has

R/h \ L/R	0.1	0.25	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
1,000	1, 29	4, 19	8, 19	16, 19	24, 19	32, 19	44, 11	41, 25	43, 27	67, 16	72, 19	88, 11
900	1, 27	1, 23	5, 27	17, 8	26, 2	15, 27	32, 24	52, 2	11, 21	67, 10	78, 2	64, 24
800	1, 25	4, 8	7, 18	14, 18	20, 20	28, 18	31, 22	40, 20	22, 25	21, 24	8, 16	81, 5
700	1, 23	2, 24	4, 24	8, 24	12, 24	16, 24	38, 4	37, 19	53, 5	61, 3	32, 24	76, 4
600	1, 20	3, 16	3, 22	6, 22	9, 22	12, 22	15, 22	18, 22	21, 22	24, 22	27, 22	51, 20
500	1, 17	3, 11	5, 17	10, 17	8, 20	20, 17	19, 20	16, 20	35, 17	35, 19	24, 20	38, 20
400	1, 12	2, 17	4, 17	5, 18	17, 5	23, 3	26, 11	15, 18	40, 4	45, 6	52, 1	33, 18
300	1, 1	2, 13	5, 1	10, 1	14, 8	19, 7	25, 1	28, 8	5, 11	38, 7	42, 8	43, 11

TABLE 1: VALUES OF M, N ($\nu = 0.3$) h, R, L dimensional corresponding to σ_{min}

rigorously been shown to be convergent for simple eigenvalues (see reference [13]). When N = 0, equations (3.19)-(3.22) reduce to

$$h^2 w_{xxxx} + \sigma w_{xx} + \omega^2 w = 0$$

which has a continuum of solutions of arbitrary amplitude for $\sigma = \sigma_0$; since we can solve for this bifurcating branch exactly, we will assume $N > 0$ without any loss of generality.

Introduce the vector $\underline{u} = (u_1, u_2, u_3)$, where $u_1 = w$, $u_2 = f$, and $u_3 = \ell$, and define an inner product by

$$<\underline{u}, \underline{v}> = \frac{4}{\Omega}\int_0^{\Omega}\int_0^1 u_1 v_1 + u_2 v_2 \, dxdy + u_3 v_3 \quad .$$

The problem may be formulated as $G(\underline{u}, \sigma) = 0$ where

$$G(\underline{u}, \sigma) \equiv \begin{pmatrix} h^2\Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} - f_{yy} w_{xx} + 2f_{xy} w_{xy} - f_{xx} w_{yy} + 2\ell w_{xy} \\ \omega^2 \Delta^2 f - \omega^2 w_{xx} + w_{xx} w_{yy} - w^2_{xy} \\ 4\pi\omega(1+\nu)\,\ell - \int_0^{\Omega}\int_0^1 w\, w_{xy} dxdy \end{pmatrix} \tag{3.32}$$

The linearized problem about $\underline{u} = \underline{0}$ is given, for $\underline{\tilde{z}} = (\tilde{w}, \tilde{f}, \tilde{\ell})$, by

$$G^o_{\underline{u}} \underline{\tilde{z}} = \begin{pmatrix} h^2\Delta^2\tilde{w} + \sigma_0 \tilde{w}_{xx} + \omega^2 \tilde{f}_{xx} \\ \omega^2\Delta^2\tilde{f} - \omega^2\tilde{w}_{xx} \\ 4\pi\omega(1+\nu)\tilde{\ell} \end{pmatrix} = 0 \tag{3.33}$$

This coincides with our pertubation analysis. A nontrivial solution exists if and only if $\sigma_0 = \sigma_0(t_{MN})$ for some (M, N). We assume this is the case, and we further assume σ_0 is simple. If we remove the translational arbitrariness of the solution, we have

$$\underline{\tilde{z}} = (\rho Q^2 \sin M\pi x \cos N\omega y, \ -\rho(M\pi)^2 \sin M\pi x \cos N\omega y, \ 0) \tag{3.34}$$

where

$$\rho = 1/\sqrt{Q^4 + (M\pi)^4} \quad , \quad Q = Q_{MN} \quad .$$

The factor ρ was chosen so that

$$\| \underline{\phi} \|^2 = \langle \underline{\phi}, \underline{\phi} \rangle = 1$$

The adjoint operator is

$$G_{\underline{u}}^{o\,\dagger} \underline{\phi}^* = \begin{pmatrix} h^2 \Delta^2 w^* + \sigma_0 w^*_{xx} - \omega^2 f^*_{xx} \\ \omega^2 \Delta^2 f^* + \omega^2 w^*_{xx} \\ 4\pi\omega(1+\nu)\, \ell^* \end{pmatrix} \tag{3.35}$$

where $\underline{\phi}^* = (w^*, f^*, \ell^*)$; the appropriate member of the null space is

$$\underline{\phi}^* = \left(Q^2 \sin M\pi x \cos N\omega y,\ (M\pi)^2 \sin M\pi x \cos N\omega y,\ 0\right) \tag{3.36}$$

We seek a functional $\Lambda(\underline{u})$ satisfying

$$\langle \underline{\phi}^*, G(\underline{u}, \Lambda(\underline{u})) - G^o_{\underline{u}} \underline{u} \rangle = 0 \quad .$$

Substituting and carrying out the required manipulations yields

$$\Lambda(\underline{u}) = \sigma_0 + N(\underline{u}) / D(\underline{u})$$

where

$$N(\underline{u}) = \int_0^\Omega \int_0^1 [Q^2 (2\ell w_{xy} - f_{yy} w_{xx} + 2 f_{xy} w_{xy} - f_{xx} w_{yy}) + (M\pi)^2 (w_{xx} w_{yy} - w^2_{xy})] \sin M\pi x \cos N\omega y \, dxdy$$

and

$$D(\underline{u}) = Q^2 (M\pi)^2 \int_0^\Omega \int_0^1 w \sin M\pi x \cos N\omega y \, dxdy$$

The iteration scheme yields the solution $(\underline{u}, \sigma)$ as the limit of $\{(\underline{u}^k, \sigma^k)\}$, where

$$\underline{u}^k = \epsilon(\underline{\phi} + \epsilon \underline{v}^k) \tag{3.38a}$$

$$\sigma^{k+1} = \Lambda(\underline{u}^k) \tag{3.38b}$$

and $\underline{v}^{k+1}$ solves

$$G^o_{\underline{u}} \underline{v}^{k+1} = \epsilon^{-2} \{ G^o_{\underline{u}} \underline{u}^k - G(\underline{u}^k, \sigma^{k+1}) \}, \quad \langle \underline{\phi}^*, \underline{v}^{k+1} \rangle = 0 \tag{3.38c}$$

with $\underline{v}^o = 0$. Furthermore, the error decreases as ϵ^{k+1}.

With this notation $\underline{u}^o = \tilde{\underline{z}}$, as defined in (3.34); substituting this into (3.38b) yields $\sigma^1 = \sigma_0$. For the next iterate set $\underline{v}^1 = (w_1, f_1, \ell_1)$. After a little simplification, equation (3.38c) can be written as

$$h^2\Delta^2 w_1 + \sigma_0 w_{1xx} + \omega^2 f_{1xx} = \tilde{w}_{xx}\tilde{f}_{yy} - 2\tilde{w}_{xy}\tilde{f}_{xy} + \tilde{w}_{yy}\tilde{f}_{xx} \tag{3.39a}$$

$$\omega^2(\Delta^2 f_1 - w_{1xx}) = \tilde{w}^2_{xy} - \tilde{w}_{xx}\tilde{w}_{yy} \tag{3.39b}$$

$$4\pi\omega(1+\nu)\ell_1 = \int_0^\Omega\int_0^1 \tilde{w}\tilde{w}_{xy}\, dxdy \tag{3.39c}$$

along with the normalization condition

$$\int_0^\Omega\int_0^1 w_1 w^* + f_1 f^*\, dxdy = 0 \quad . \tag{3.39d}$$

We can evaluate the right sides of equations (3.39a, b, c) using (3.34); the results are

$$h^2\Delta^2 w_1 + \sigma_0 w_{1_{xx}} + \omega^2 f_{1_{xx}} = \rho^2 (M\pi)^4 (N\omega)^2 Q^2 (\cos 2M\pi x - \cos 2N\omega y)$$

$$\omega^2(\Delta^2 f_1 - w_{1_{xx}}) = \tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4 (\cos 2M\pi x - \cos 2N\omega y)$$

$$\ell_1 = 0$$

A particular solution for w_1 and f_1 is

$$\left.\begin{aligned} w_1 &= a_0(x) + a_2(x)\cos 2N\omega y \\ f_1 &= A_0(x) + A_2(x)\cos 2N\omega y \end{aligned}\right\} \tag{3.40}$$

with

$$\left.\begin{aligned} h^2 a_0^{iv} + \sigma_0 a_0'' + \omega^2 A_0'' &= \rho^2 (M\pi)^4 (N\omega)^2 Q^2 \cos 2M\pi x \\ \omega^2(A_0^{iv} - a_0'') &= \tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4 \cos 2M\pi x \end{aligned}\right\} \tag{3.41}$$

and

$$\begin{aligned} h^2[a_2^{iv} - 2(2N\omega)^2 a_2'' + (2N\omega)^4 a_2] + \sigma_0 a_2'' + \omega^2 A_2'' &= -\rho^2 (M\pi)^4 (N\omega)^2 Q^2 \\ \omega^2[A_2^{iv} - 2(2N\omega)^2 A_2'' + (2N\omega)^4 A_2 - a_2''] &= -\tfrac{1}{2}\rho^2 (M\pi)^2 (N\omega)^2 Q^4 \end{aligned} \tag{3.42}$$

Note that a_0, a_2, A_0, A_2 satisfy the boundary conditions

$$a_0 = a_{0_{xx}} = 0 \quad \text{at} \quad x = 0, 1, \quad \text{etc.}$$

This particular solution is in fact the desired solution, for it satisfies (3.39d) with $\underline{\phi}^*$ given by (3.36), and also with $\underline{\phi}^*$ given by

$$\underline{\psi}^* = (Q^2 \sin M\pi x \sin N\omega y, \quad (M\pi)^2 \sin M\pi x \sin N\omega y \;, \; 0) \tag{3.36'}$$

($\underline{\phi}^*$ and $\underline{\psi}^*$ together span the nullspace of $G_{\underline{u}}^{o\dagger}$). The assumption that σ_0 is a simple eigenvalue assures us that there exist no non-trivial solutions to the homogeneous forms of equations (3.41) and (3.42) satisfying the boundary conditions; consequently the equations are invertible and we are guaranteed that a_0, a_2, A_0, and A_2 exist.

To solve (3.41) we first need the characteristic exponents of the corresponding homogeneous constant coefficient system. Setting $a_0 = c\,e^{i\mu x}$, $A_0 = C\,e^{i\mu x}$, we find for nontrivial c and C that

$$\mu^4 (h^2\mu^4 - \sigma_0\mu^2 + \omega^2) = 0$$

which has roots $\mu = 0$ and $\mu = \pm\mu_+$, $\mu = \pm\mu_-$ where we define $\mu_+ > 0$ and $\mu_- > 0$ by

$$\mu_+^2 = (\sigma_0 + \sqrt{\sigma_0^2 - 4\omega^2 h^2}\,)/\,2h^2$$

$$\mu_-^2 = (\sigma_0 - \sqrt{\sigma_0^2 - 4\omega^2 h^2}\,)/\,2h^2$$

(Remark that μ_+ and μ_- are real because $\sigma_0 = \sigma_0(t_{MN}) \geq 2\omega h$ insofar as $2\omega h$ is the global minimum of $\sigma(t)$ for $t > 0$.) This generates eight linearly independent solutions of the homogeneous system:

$$\begin{pmatrix} a_0 \\ A_0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} -\mu_+^2 \cos\mu_+ x \\ \cos\mu_+ x \end{pmatrix}, \begin{pmatrix} -\mu_+^2 \sin\mu_+ x \\ \sin\mu_+ x \end{pmatrix}, \begin{pmatrix} -\mu_-^2 \cos\mu_- x \\ \cos\mu_- x \end{pmatrix}, \begin{pmatrix} -\mu_-^2 \sin\mu_- x \\ \sin\mu_- x \end{pmatrix}$$

A particular solution can be found as a constant vector times $\cos 2M\pi x$. Now equations (3.41) and their associated boundary conditions admit a solution which is symmetric about $x = \frac{1}{2}$; consequently we can find constants $\beta_1, \ldots, \beta_6$ such that

$$\begin{pmatrix} a_0(x) \\ A_0(x) \end{pmatrix} = \beta_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_3 \cos\mu_+(x-\tfrac{1}{2})\begin{pmatrix} -\mu_+^2 \\ 1 \end{pmatrix} + \beta_4 \cos\mu_-(x-\tfrac{1}{2})\begin{pmatrix} -\mu_-^2 \\ 1 \end{pmatrix}$$

$$+ \begin{pmatrix} \beta_5 \\ \beta_6 \end{pmatrix} \cos 2M\pi x$$

β_5 and β_6 are determined by the inhomogeneous terms; then β_1, β_2, β_3, and β_4 are determined from the boundary conditions. After carrying out the indicated algebra, we find

$$\begin{aligned}
\beta_1 &= -\lambda_2/\omega^2(2M\pi)^2 \\
\beta_2 &= (\lambda_1\omega^2 + \lambda_2\sigma_0)/\omega^4(2M\pi)^2 \\
\beta_3 &= (2M\pi)^2(\beta_5 + \beta_6\mu_-^2)/(\mu_+^2 - \mu_-^2)\,\mu_+^2 \cos\tfrac{1}{2}\mu_+ \\
\beta_4 &= (2M\pi)^2(\beta_5 + \beta_6\mu_+^2)/(\mu_-^2 - \mu_+^2)\,\mu_-^2 \cos\tfrac{1}{2}\mu_- \\
\beta_5 &= (\lambda_1(2M\pi)^2 + \lambda_2)/D \\
\beta_6 &= (\lambda_2(2M\pi)^2 h^2 - \lambda_2\sigma_0 - \omega^2\lambda_1)/\omega^2 D
\end{aligned}$$

where

$$\lambda_1 = \rho^2(M\pi)^4(N\omega)^2Q^2 \; , \qquad \lambda_2 = \tfrac{1}{2}\,\rho^2(M\pi)^2(N\omega)^2Q^4$$

$$D = (2M\pi)^2[h^2(2M\pi)^4 - \sigma_0(2M\pi)^2 + \omega^2]$$

We solve equations (3.42) in a similar fashion. It is easy to see that a particular solution is $a_{2p} = -\lambda_1/h^2(2N\omega)^4$, $A_{2p} = -\lambda_2/\omega^2(2N\omega)^4$. Solutions of the homogeneous system exist of the form $a_2 = c\,e^{i\nu x}$, $A_2 = C\,e^{i\nu x}$, where ν must satisfy

$$h^2[\nu^2 + (2N\omega)^2]^4 - \sigma_0[\nu^2 + (2N\omega)^2]^2\nu^2 + \omega^2\nu^4 = 0 \; .$$

Again the solution is symmetric about $x = \frac{1}{2}$, and we write

$$\begin{pmatrix} a_2(x) \\ A_2(x) \end{pmatrix} = \begin{pmatrix} -\lambda_1/h^2(2N\omega)^4 \\ -\lambda_2/\omega^2(2N\omega)^4 \end{pmatrix} + \sum_{j=1}^{4} \gamma_j \cos \nu_j(x-\tfrac{1}{2}) \begin{pmatrix} Q_j^2 \\ -\nu_j^2 \end{pmatrix} .$$

where $Q_j = \nu_j^2 + (2N\omega)^2$ and γ_1, γ_2, γ_3, γ_4 are constants to be determined by the boundary conditions. In general, there exist four distinct complex values for ν^2. Although an exact solution exists for the roots of a quartic equation, the answer is unwieldly and impracticable. For the purposes of numerical computation, it was found to be easier to find roots ν^2 by Newtonian iteration. When $\sigma_0 = \sigma_{min} = 2\omega h + \delta^2$ with $\delta^2 << 2\omega h$, good first estimates for two of the roots are roots of

$$h[\nu^2 + (2N\omega)^2]^2 - \omega\nu^2 = 0$$

viz.,

$$\nu^2 = \frac{\omega}{2h} - (2N\omega)^2 \pm \sqrt{\frac{\omega^2}{4h^2} - \frac{\omega}{h}(2N\omega)^2} .$$

Once two values of ν^2 have been found by iteration, the quartic may be reduced to a managable quadratic in the remaining roots. Alternately, an analytic expression may be found for ν^2 by expanding as a power series in δ.

Finally, $\underline{v}^1$ is known, and thence $\underline{u}^1$; substituting into (3.38b) we have an expression for the second iterate, σ^2, which after considerable simplification can be written as

$$\sigma^2 = \sigma_0 + \epsilon^2(N\omega)^2 H/Q^2 = \sigma_0 + \epsilon^2\sigma_2$$

where

$$H = \int_0^1 -8(M\pi)^2 a_0(x)\cos 2M\pi x + 4Q^2 A_0(x)\cos 2M\pi x + 4(M\pi)^2 a_2(x) - 2Q^2 A_2(x)\, dx .$$

The integrations indicated can be carried out explicitly:

$$\int_0^1 a_0(x)\cos 2M\pi x\,dx = \frac{2\beta_3\mu_+^3}{(2M\pi)^2-\mu_+^2} + \frac{2\beta_4\mu_-^3}{(2M\pi)^2-\mu_-^2} + \frac{\beta_5}{2}$$

$$\int_0^1 A_0(x)\cos 2M\pi x\,dx = \frac{2\beta_3\mu_+}{\mu_+^2-(2M\pi)^2} + \frac{2\beta_4\mu_-}{\mu_-^2-(2M\pi)^2} + \frac{\beta_6}{2}$$

$$\int_0^1 a_2(x)\,dx = -\lambda_1/h^2(2N\omega)^4 + \sum_{j=1}^{4} 2\gamma_j Q_j^2 \sin(\nu_j/2)/\nu_j$$

$$\int_0^1 A_2(x)\,dx = -\lambda_2/\omega^2(2N\omega)^4 - \sum_{j=1}^{4} 2\gamma_j \nu_j \sin(\nu_j/2)$$

Table 2 lists values of σ_2 computed for the modes (M, N) of table 1. The data reflect few common features. They may be either negative or positive, although these occur in proportion four to one. Also, they vary in magnitude from 11 to 21, 116, 509. However, most values are rather large--on the order of 10^3 or more--indicating that the bifurcating branch is steep and that σ changes value rapidly.

Bifurcation for double eigenvalues

We next investigate the ramifications of σ_0 being a double eigenvalue. Assume that there exist distinct pairs (M_1, N_1) and (M_2, N_2) such that $\sigma_0 = \sigma_0(M_1, N_1) = \sigma_0(M_2, N_2)$ and that there exist no other pairs. We further assume that $N_1 > 0$, for if both $N_1 = 0$ and $N_2 = 0$, then the leading order solution which bifurcates from Poisson expansion is symmetric; however, for symmetric solutions the equations become linear and we get a two parameter family of solutions of arbitrary amplitude, all corresponding to $\sigma = \sigma_0$.

We refer back to the pertubation expansion used to derive

R/h \ L/R	0.1	0.25	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
1,000	93,085	-1,018	-1,822	-561	-457	-707	-2,048	13,659	-1,107	-929	-485	-3,042
900	16,883	-267	-32,005	-2,271	1,124	14,114	348	1,114	-324	-35,232	-119	3,234
800	4,023	-452	-542	-518	-669	-836	-18,182	-6,059	-7,865	-1,303	-50	-7,955
700	782	-335,820	-183,605	-29,864	21,116,509	135,476	-39,388	-376	-1,499	-292,157	-579,578	-1,072
600	-212	-412	146,988	-3,981	-19,430	-3,853	2,555	-108	-5,671	-12,126	-1,802	7,229
500	-505	-1,990	-70	-16,132	2,305	-296	-3,775	14,223	-951	-294	-1,601	-10,968
400	-555	13,302	-303	1,871	-1,707	-626	-1,740	6,273	-630	-829	-11	-2,447
300	-1,106	-37	-60	-16	-132	-1,141	-14	-1,148	268	-20,007	-1,045	-152

TABLE 2: VALUES OF σ_2 ($\nu = 0.3$) h, R, L dimensional

equations (3.25) and (3.26). Since $N_1 > 0$, we can remove the translational indeterminancy in y by suppressing the mode $\sin M_1 \pi x \sin N_1 \omega y$ from the leading order solution. Hence, for some constants c_1, c_2, d_2

$$\left.\begin{aligned} w_1 &= c_1 Q_1^2 \sin M_1 \pi x \cos N_1 \omega y + c_2 Q_2^2 \sin M_2 \pi x \cos N_2 \omega y \\ &\qquad + d_2 Q_2^2 \sin M_2 \pi x \sin N_2 \omega y \\ \phi_1 &= -c_1 (M_1\pi)^2 \sin M_1 \pi x \cos N_1 \omega y - c_2 (M_2\pi)^2 \sin M_2 \pi x \cos N_2 \omega y \\ &\qquad - d_2 (M_2\pi)^2 \sin M_2 \pi x \sin N_2 \omega y \end{aligned}\right\} \quad (3.43)$$

where $Q_i = (M_i\pi)^2 + (N_i\omega)^2$, $i = 1, 2$. The normalization condition (3.28) integrates to yield

$$1 = c_1^2 [Q_1^4 + (M_1\pi)^4] + c_2^2 [Q_2^4 + (M_2\pi)^4] + d_2^2 [Q_2^4 + (M_2\pi)^4] \qquad (3.44)$$

Using (3.43), equations (3.26a) and (3.26b) become, after some simplification:

$$h^2 \Delta^2 w_2 + \sigma_0 w_{2_{xx}} + \omega^2 f_{2_{xx}} = R_1(x, y) \qquad (3.45a)$$

where

$$\begin{aligned} R_1(x, y) \equiv{} & \sin^2 N_1\omega y \cos M_1\pi x [2c_1^2 Q_1^2 (N_1\omega)^2 (M_1\pi)^4] \\ & -\cos^2 N_1\omega y \sin^2 M_1 \pi x [2c_1^2 Q_1^2 (N_1\omega)^2 (M_1\pi)^4] \\ & +\sin^2 N_2\omega y \cos^2 M_2 \pi x [2c_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & -\cos^2 N_2\omega y \sin^2 M_2 \pi x [2c_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & +\cos^2 N_2\omega y \cos^2 M_2 \pi x [2d_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & -\sin^2 N_2\omega y \sin^2 M_2 \pi x [2d_2^2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4] \\ & -\cos N_1\omega y \cos N_2\omega y \sin M_1 \pi x \sin M_2 \pi x \; c_1 c_2 \\ & \quad [(N_1\omega)^2 (M_1\pi)^2 Q_2^2 (M_2\pi)^2 + (M_1\pi)^4 Q_2^4 (N_2\omega)^2 \\ & \quad + Q_1^2 (M_1\pi)^2 (N_2\omega)^2 (M_2\pi)^2 + Q_1^2 (N_1\omega)^2 (M_2\pi)^4] \\ & -\cos N_1\omega y \sin N_2\omega y \sin M_1 \pi x \sin M_2 \pi x \; c_1 d_2 \\ & \quad [Q_1^2 (M_1\pi)^2 (N_2\omega)^2 (M_2\pi)^2 + Q_1^2 (N_1\omega)^2 (M_2\pi)^4 \\ & \quad + (N_1\omega)^2 (M_1\pi)^2 Q_2^2 (M_2\pi)^2 + (M_1\pi)^4 Q_2{}^2 (N_2\omega)^2] \end{aligned}$$

$$-\cos N_2\omega y \sin N_2\omega y \sin^2 M_2\pi x[4c_2 d_2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4]$$

$$+\sin N_1\omega y \sin N_2\omega y \cos M_1\pi x \cos M_2\pi x \; 2c_1 c_2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)[Q_1^2 (M_2\pi)^2 + Q_2^2 (M_1\pi)^2]$$

$$-\sin N_1\omega y \cos N_2\omega y \cos M_1\pi x \cos M_2\pi x \; 2c_1 d_2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)[Q_1^2 (M_2\pi)^2 + Q_2^2 (M_1\pi)^2]$$

$$-\sin N_2\omega y \cos N_2\omega y \cos^2 M_2\pi x[4c_2 d_2 Q_2^2 (N_2\omega)^2 (M_2\pi)^4]$$

$$+\cos N_1\omega y \sin M_1\pi x[\sigma_1 c_1 Q_1^2 (M_1\pi)^2]$$

$$+\cos N_2\omega y \sin M_2\pi x[\sigma_1 c_2 Q_2^2 (M_2\pi)^2]$$

$$+\sin N_2\omega y \sin M_2\pi x[\sigma_1 d_2 Q_2^2 (M_2\pi)^2]$$

and

$$\omega^2(\Delta^2 f_2 - w_{2_{xx}}) = R_2(x, y) \tag{3.45b}$$

where

$$R_2(x, y) \equiv \sin^2 N_1\omega y \cos^2 M_1\pi x[c_1^2 Q_1^4 (N_1\omega)^2 (M_1\pi)^2]$$

$$-\cos^2 N_1\omega y \sin^2 M_1\pi x[c_1^2 Q_1^4 (N_1\omega)^2 (M_1\pi)^2]$$

$$+\sin^2 N_2\omega y \cos^2 M_2\pi x[c_2{}^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

$$-\cos^2 N_2\omega y \sin^2 M_2\pi x[c_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

$$+\cos^2 N_2\omega y \cos^2 M_2\pi x[d_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

$$-\sin^2 N_2\omega y \sin^2 M_2\pi x[d_2^2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

$$-\cos N_1\omega y \cos N_2\omega y \sin M_1\pi x \sin M_2\pi x \; c_1 c_2 Q_1^2 Q_2^2 [(N_1\omega)^2 (M_2\pi)^2 + (M_1\pi)^2 (N_2\omega)^2]$$

$$-\cos N_1\omega y \sin N_2\omega y \sin M_1\pi x \sin M_2\pi x \; c_1 d_2 Q_1^2 Q_2^2 [(N_1\omega)^2 (M_2\pi)^2 + (M_1\pi)^2 (N_2\omega)^2]$$

$$-\cos N_2\omega y \sin N_2\omega y \sin^2 M_2\pi x[2c_2 d_2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

$$+\sin N_1\omega y \sin N_2\omega y \cos M_1\pi x \cos M_2\pi x [2c_1 c_2 Q_1^2 Q_2^2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)]$$

$$-\sin N_1\omega y \cos N_2\omega y \cos M_1\pi x \cos M_2\pi x [2c_1 d_2 Q_1^2 Q_2^2 (N_1\omega)(N_2\omega)(M_1\pi)(M_2\pi)]$$

$$-\sin N_2\omega y \cos N_2\omega y \cos^2 M_2 \pi x[2c_2 d_2 Q_2^4 (N_2\omega)^2 (M_2\pi)^2]$$

The linear operator

$$L\begin{pmatrix} w \\ f \end{pmatrix} \equiv \begin{pmatrix} h^2\Delta^2 w + \sigma_0 w_{xx} + \omega^2 f_{xx} \\ \omega^2(\Delta^2 f - w_{xx}) \end{pmatrix}$$

is not invertible. The nullspace of its adjoint is spanned by the four vectors

$$\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = (Q_1^2 \cos N_1\omega y \sin M_1 \pi x,\ \ (M_1\pi)^2 \cos N_1\omega y \sin M_1\pi x)^t$$

$$\begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} = (Q_1^2 \sin N_1\omega y \sin M_1 \pi x,\ \ (M_1\pi)^2 \sin N_1\omega y \sin M_1\pi x)^t$$

$$\begin{pmatrix} \phi_3 \\ \psi_3 \end{pmatrix} = (Q_2^2 \cos N_2\omega y \sin M_2 \pi x,\ \ (M_2\pi)^2 \cos N_2\omega y \sin M_2\pi x)^t$$

$$\begin{pmatrix} \phi_4 \\ \psi_4 \end{pmatrix} = (Q_2^2 \sin N_2\omega y \sin M_2 \pi x,\ \ (M_2\pi)^2 \sin N_2\omega y \sin M_2\pi x)^t$$

We write equations (3.45) as

$$L\begin{pmatrix} w_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

the Fredholm Alternative [16] implies that (3.45) has a solution if and only if

$$\left\langle \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\rangle = 0 \ , \quad k = 1, 2, 3, 4 \qquad (3.46)$$

where the inner product is defined by

$$\left\langle \begin{pmatrix} v \\ g \end{pmatrix}, \begin{pmatrix} u \\ f \end{pmatrix} \right\rangle \equiv \frac{4}{\Omega} \int_0^{\Omega} \int_0^1 uv + fg \ dxdy \quad .$$

Equations (3.46) and (3.44) provide the additional information needed to calculate c_1, c_2, and d_2. A table of integrals (see Table 3) facilitates

F	$\int_0^1 F \sin M\pi x \, dx$
$\sin^2 M\pi x$	0 M even , $4/3\,M\pi$ M odd
$\cos^2 M\pi x$	0 M even , $2/3\,M\pi$ M odd
$\sin^2 m\pi x$	0 M even , $4m^2/\pi M(4m^2 - M^2)$ M odd
$\cos^2 m\pi x$	0 M even , $(4m^2 - 2M^2)/M\pi(4m^2 - M^2)$ M odd
$\sin M\pi x \sin m\pi x$	0 m even , $4M^2/\pi m(4M^2 - m^2)$ m odd
$\cos M\pi x \cos m\pi x$	0 m even , $2M/\pi(4M^2 - m^2)$ m odd

for $n > 0$, $N > 0$:

F	$\int_0^\Omega F \cos N\omega y \, dy$	$\int_0^\Omega F \sin N\omega y \, dy$	$\int_0^\Omega F \cos n\omega y \, dy$	$\int_0^\Omega F \sin n\omega y \, dy$
$\sin^2 N\omega y$	0	0	$-\Omega/4$, $n = 2N$	0
$\cos^2 N\omega y$	0	0	$\Omega/4$, $n = 2N$	0
$\cos N\omega y \cos n\omega y$	$\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$	0
$\cos N\omega y \sin n\omega y$	0	$\Omega/4$, $n = 2N$	0	$-\Omega/4$, $N = 2n$
$\cos n\omega y \sin n\omega y$	0	$\Omega/4$, $N = 2n$	0	0
$\sin N\omega y \sin n\omega y$	$\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$	0
$\sin N\omega y \cos n\omega y$	0	$-\Omega/4$, $n = 2N$	0	$\Omega/4$, $N = 2n$

TABLE 3: "HANDY INTEGRALS"
(Note: if $n \neq 2N$, etc., integrals vanish)

the computation in (3.46). Equations (3.46) fall into four cases.

Case 1: M_1 and M_2 even, or $N_1 \neq 2N_2$ and $N_2 \neq 2N_1$ (but $N_2 > 0$), or both

This is the simplest case; one of the equations in (3.46) is vacuous, and the remaining three give

$$\sigma_1 c_1 = \sigma_1 c_2 = \sigma_1 d_2 = 0$$

From the normalization condition (3.44) it follows that $c_1 = c_2 = d_2 = 0$ is impossible. Consequently $\sigma_1 = 0$ and these three equations are satisfied for arbitrary values of c_1, c_2, and d_2. The normalization relation puts one constraint on these parameters and, in addition, guarantees that they are bounded. The result is a two-parameter family of solutions, even after the translational indeterminancy in y has been removed.

Case 2: $N_2 = 0$ (Recall $N_1 > 0$ always)

Take $d_2 = 0$ insofar as the corresponding terms in w_1 and f_1 are absent. Two of equations (3.46) are vacuous.

Subcase 2a: M_2 even

The remaining equations reduce to

$$\sigma_1 c_1 = \sigma_1 c_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 = 1$$

where

$$B_1 = Q_1^4 + (M_1\pi)^4 \qquad B_2 = Q_2^4 + (M_2\pi)^4$$

Again $\sigma_1 = 0$; for this case we find a one-parameter family of solutions.

Subcase 2b: M_2 odd

Now the remaining equations reduce to

$$\sigma_1 c_1 = c_1 c_2 A_1 \qquad \sigma_1 c_2 = c_1^2 A_2$$

$$B_1 c_1^2 + B_2 c_2^2 = 1$$

where

$$A_1 = \pi[Q_1^2 + 2(M_1\pi)^2 (M_2\pi)^2](N_1\omega)^2 (2M_2)^3 / Q_1^2 (4M_1^2 - M_2^2)$$

$$A_2 = 2[Q_1^2 + 2(M_1\pi)^2 (M_2\pi)^2] Q_1^2 (N_1\omega)^2 M_1^2 / (M_2\pi)^7 (4M_1^2 - M_2^2)$$

The solutions of this system are:

(i) $c_1 = 0 \quad , \quad c_2 = 1/\sqrt{B_2} \quad , \quad \sigma_1 = 0$

(Note: $c_2 = -1/\sqrt{B_2}$ corresponds to the same branch with $\epsilon < 0$.)

(ii) $c_1 = \sqrt{A_1/(A_1 B_1 + A_2 B_2)} \qquad c_2 = \sqrt{A_2/(A_1 B_1 + A_2 B_2)}$

$$\sigma_1 = A_1 \sqrt{A_2/(A_1 B_1 + A_2 B_2)}$$

(iii) $c_1 = -\sqrt{A_1/(A_1 B_1 + A_2 B_2)} \qquad c_2 = \sqrt{A_2/(A_1 B_1 + A_2 B_2)}$

$$\sigma_1 = A_1 \sqrt{A_2/(A_1 B_1 + A_2 B_2)}$$

The solution represented by (ii) has the form

$$w_1 = c_1 Q_1^2 \cos N_1\omega y \sin M_1\pi x + c_2 Q_2^2 \sin M_2\pi x \quad .$$

If we translate $y \to y + \pi/N_1\omega$ this solution transforms into

$$w_1 = -c_1 Q_1^2 \cos N_1\omega y \sin M_1\pi x + c_2 Q_2^2 \sin M_2\pi x$$

which is the solution given in (iii). Consequently, we see that for this case there exist two physically distinct solution branches that bifurcate from Poisson expansion: one corresponding to $\sigma_1 = 0$ and another corresponding to $\sigma_1 = A_1 \sqrt{A_2/(A_1 B_1 + A_2 B_2)}$.

Case 3: $N_2 = 2N_1$

Subcase 3a: M_2 even

One of equations (3.46) is vacuous, the remaining equations and (3.44) yield:

$$\sigma_1 c_1 = \sigma_1 c_2 = \sigma_1 d_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 + B_2 d_2^2 = 1$$

which is the same as case 1.

Subcase 3b: M_2 odd

The relevant equations become

$$\sigma_1 c_1 = c_1 c_2 A_3 \qquad \sigma_1 c_2 = c_1{}^2 A_4$$

$$\sigma_1 d_2 = 0 \qquad c_1 d_2 = 0$$

$$B_1 c_1{}^2 + B_2 c_2{}^2 + B_2 d_2{}^2 = 1$$

where

$$A_3 = \frac{4\pi\omega^2 (Q_1^2 M_2^2 + 2Q_2^2 M_1^2)(N_2^2 M_1^2 - N_1 N_2 M_2^2 + N_1^2 M_2^2)}{(4M_1^2 - M_2^2) Q_1^2 M_2}$$

$$= 4\pi(N_1\omega)^2 (Q_1{}^2 M_2{}^2 + 2Q_2{}^3 M_1{}^2)/Q_1{}^2 M_2$$

$$A_4 = 2\pi Q_1{}^2 (N_1\omega)^2 M_1{}^2 (Q_1{}^2 M_2{}^2 + 2Q_2{}^2 M_1{}^2)/Q_2{}^4 M_2{}^3$$

The solutions fall into three sets:

(i) a one-parameter family of solutions described by

$$\sigma_1 = 0 \ , \quad c_1 = 0 \quad c_2{}^2 + d_2{}^2 = 1/B_2$$

(ii) a branch described by

$$c_1 = \sqrt{A_3/(A_3 B_1 + A_4 B_2)} \qquad c_2 = \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

$$d_2 = 0 \qquad \sigma_1 = A_3 \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

(iii) a branch described by

$$c_1 = -\sqrt{A_3/(A_3 B_1 + A_4 B_2)} \qquad c_2 = \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

$$d_2 = 0 \qquad \sigma_1 = A_3 \sqrt{A_4/(A_3 B_1 + A_4 B_2)}$$

It should be noted that solution (i) has reintroduced the translational indeterminancy, since setting $c_1 = 0$ removes one mode. Thus we are free to set $d_2 = 0$, for example, in which case (i) represents the non-degenerate bifurcation of a branch given by

$$c_1 = 0 \ , \quad c_2 = 1/\sqrt{B_2} \ , \quad d_2 = 0 \ , \quad \sigma_1 = 0 \ .$$

As in case 2b, solutions (ii) and (iii) represent the same physical solution. Consequently, there exist precisely two physically distinct bifurcation branches.

Case 4: $N_1 = 2N_2$

Subcase 4a: M_1 even

This case is identical with cases 1 and 3a.

Subcase 4b: M_1 odd

Equation (3.44) and simplified forms of equations (3.46) are

$$\sigma_1 c_2 = A_5 c_1 c_2 \qquad \sigma_1 d_2 = -A_5 c_1 d_2$$

$$\sigma_1 c_1 = A_6 (c_2^2 - d_2^2) \qquad c_2 d_2 = 0$$

$$B_1 c_1^2 + B_2 c_2^2 + B_2 d_2^2 = 1$$

where

$$A_5 = 4\pi(N_2\omega)^2 (Q_2^2 M_1^2 + 2Q_1^2 M_2^2)/Q_2^2 M_1$$

$$A_6 = 2\pi Q_2^2 (N_2\omega)^2 M_2^2 (Q_2^2 M_1^2 + 2Q_1^2 M_2^2)/Q_1^4 M_1^3$$

There are five solution sets.

(i) $c_1 = 1/\sqrt{B_1}$, $c_2 = 0$, $d_2 = 0$, $\sigma_1 = 0$

(ii) $c_1 = \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$, $c_2 = \sqrt{A_5/(A_6 B_1 + A_5 B_2)}$

$d_2 = 0$, $\sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(iii) $c_1 = \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$, $c_2 = -\sqrt{A_5/(A_6 B_1 + A_5 B_2)}$

$d_2 = 0$, $\sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(iv) $c_1 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}$, $c_2 = 0$

$d_2 = \sqrt{A_5/(A_6 B_1 + A_5 B_2)}$, $\sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

(v) $c_1 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}$, $c_2 = 0$

$d_2 = -\sqrt{A_6/(A_6 B_1 + A_5 B_2)}$, $\sigma_1 = A_5 \sqrt{A_6/(A_6 B_1 + A_5 B_2)}$

The translation $y \to y + \pi/N_2\omega$ shows that solutions (ii) and (iii) coincide, and that solutions (iv) and (v) coincide. Using the fact that $N_1 = 2N_2$, we find that the translation $y \to y + \pi/N_1\omega$ takes solution (ii)

$$w_1 = c_1 Q_1^2 \cos N_1\omega y \sin M_1 \pi x + c_2 Q_2^2 \cos N_2\omega y \sin M_2 \pi x$$

into

$$w_1 = -c_1 Q_1^2 \cos N_1\omega y \sin M_1 \pi x - c_2 Q_2^2 \sin N_2\omega y \sin M_2 \pi x$$

which is solution (v). Consequently, solutions (ii)-(v) all represent the same physical solution, and so two branches bifurcate from Poisson expansion. Actually this just serves as a verification of case 3, since the two are physically symmetric to one another.

Reference [13] also provides a rigorous justification for the branches found by the pertubation expansion when c_1, c_2, d_2, and σ_1 are isolated roots of the algebraic bifurcation equations. This is true for cases 2b, 3b, and 4b.

Multiple eigenvalues can exhibit a curious effect not possible for simple eigenvalues. Recall that for a simple eigenvalue, ℓ vanishes to $O(\varepsilon^2)$. Using equations (3.26c) and (3.43) we find that for a double eigenvalue, $\ell_2 = 0$ unless $N_1 = N_2 = N > 0$ and $M_1 + M_2$ is odd. Under these special circumstances we calculate

$$\omega(1+\nu)\ell_2 = c_1 d_2 Q_1^2 Q_2^2 N M_1 M_2/(M_1^2 - M_2^2) \quad .$$

From our pertubation analysis (case 1) there exists a two-parameter family of values c_1, c_2, d_2; consequently, we can find values with $c_1 d_2 \neq 0$. It follows that the cylinder can be subjected to a uniform torque, even though there is no tangential displacement ($v = 0$) at the edges.

CHAPTER 4

CIRCULAR CYLINDRICAL SHELLS: THE DYNAMIC PROBLEM

In this chapter we propose to study the dynamic buckling of a circular cylindrical shell using multi-time scale perturbation methods [10]. When studying the stability of a solution to a nonlinear problem, one commonly considers the linearized equations for small perturbations to the solution. If all such perturbations vanish (exponentially) for large time, the solution is said to be stable; but if even one perturbation grows (exponentially), it is said to be unstable. In the latter case the linearized equations become an invalid approximation as the solution grows in magnitude. Matkowsky [14] has found that it is sometimes possible to examine the effect that nonlinearities have on curbing such growth for parameters which are only "a small distance" into the unstable regime. Reiss and Matkowsky [15] have applied this method to study the buckling of rods.

We will first illustrate the method by applying it to study the buckling of a rectangular plate. Although the governing equations are closely related to those for a circular cylindrical shell, the computations are considerably simpler, thus rendering the exposition clearer. It will turn out that the equation describing the nonlinear growth is almost identical with that for rods; however, we will make several observations not found in [15]. Following that, we proceed to apply the method to the problem of the cylinder.

The rectangular plates

The static equations governing the buckling of plate of length L_x and width L_y may be obtained from the local equations (3.1)-(3.5) for a circular cylindrical shell by letting the radius of curvature R become infinite. We assume that the plate is subjected to an end thrust directed along its length, and that the edges are simply supported. We make the problem dimensionless by measuring in units of length L_x; this necessitates introducing a parameter

$$\rho \equiv L_x/L_y$$

The dynamic equations are obtained by adding to equilibrium equation (3.3c) a term representing the acceleration and one representing damping effects. One obtains the non-dimensional equations for $0 \le x \le 1,\ 0 \le y \le \rho$

$$w_{tt} + 2\Gamma w_t + h^2 \Delta^2 + \sigma w_{xx} =$$

$$f_{yy} w_{xx} - 2f_{xy} w_{xy} + f_{xx} w_{yy} - 2\ell w_{xy} \tag{4.1a}$$

$$\Delta^2 f = w_{xy}^2 - w_{xx} w_{yy} \tag{4.1b}$$

$$2(1+\nu)\rho\ell = \int_0^\rho \int_0^1 w\, w_{xy}\, dxdy \tag{4.1c}$$

subject to

$$\left.\begin{array}{llll} w = w_{xx} & = 0 & \text{at} & x = 0, 1 \\ w = w_{yy} & = 0 & \text{at} & y = 0, \rho \\ f = f_{xx} & = 0 & \text{at} & x = 0, 1 \\ f = f_{yy} & = 0 & \text{at} & y = 0, \rho \end{array}\right\} \tag{4.2}$$

We will be primarily concerned with behavior when damping is small, i.e. $\Gamma = \epsilon\gamma$ for some small parameter $\epsilon > 0$.

The equilibrium configuration whose stability we analyze is $w = 0,\ f = 0,\ \ell = 0$. (For small loads, a plate remains unaltered, whereas a cylinder undergoes Poisson expansion.) The linearized form of equations (4.1) about this state is

$$w_{tt} + h^2 \Delta^2 w + \sigma w_{xx} = 0$$
$$\Delta^2 f = 0 \tag{4.3}$$

where we have assumed $w = O(\epsilon)$ and $f = O(\epsilon)$. A complete set of functions satisfying (4.2) is $\{Y_{mn}\}$, where

$$Y_{mn}(x, y) \equiv \sin m\pi x \sin n\omega y$$

with $\omega = \pi/\rho$ now. Let

$$w = \Sigma\, w_{mn}(t)\, Y_{mn}$$
$$f = \Sigma\, f_{mn}(t)\, Y_{mn}$$

From (4.3) we conclude

$$w_{mn,tt} + (h^2 Q^2_{mn} - \sigma(m\pi)^2) w_{mn} = 0$$
$$Q^2_{mn}\, f_{mn} = 0$$

where we have retained the notation $Q_{mn} = (m\pi)^2 + (n\omega)^2$. It follows that $f = 0$ and $w_{mn} = a \cos\lambda_{mn} t + b \sin\lambda_{mn} t$, with

$$\lambda^2_{mn} = h^2 Q^2_{mn} - \sigma(m\pi)^2$$

λ^2_{mn} is a monotonically decreasing function of σ. If, for some (m, n) $\lambda^2_{mn} < 0$, then w_{mn} grows exponentially. Hence the solution $w = 0$, $f = 0$ is unstable for $\sigma > \sigma_o$, with

$$\sigma_o = \min_{m,n} \sigma_{mn} \tag{4.4a}$$

$$\sigma_{mn} = h^2 Q^2_{mn}/(m\pi)^2 = h^2/t_{mn} \tag{4.4b}$$

using the same definition for t_{mn} as in Chapter 3. We note in passing that the same arguments apply regarding the multiplicity of t_{mn} and consequently the multiplicity of σ_{mn} as an eigenvalue.

To find the behavior in the region $\sigma > \sigma_o$, we expand

$$\left.\begin{aligned} \sigma &= \sigma_0 + \sigma_1 \epsilon + \sigma_2 \epsilon^2 + \cdots \\ w &= w_1 \epsilon + w_2 \epsilon^2 + \cdots \\ f &= f_1 \epsilon + f_0 \epsilon^2 + \cdots \\ \ell &= \epsilon^2 \ell_2 + \cdots \end{aligned}\right\} \tag{4.5}$$

Thus we are perturbing away from the transition boundary between stability and instability. We introduce multiple time scales t_k defined by

$$t_k = t\,\epsilon^k \; ; \quad k = 0, 1, 2, \ldots \tag{4.6}$$

and treat them formally as independent variables. The differential operator $\partial/\partial t$ transforms according to

$$\partial_t = \partial_{t_0} + \epsilon\, \partial_{t_1} + \epsilon^2 \partial_{t_2} + \cdots \tag{4.7}$$

To simplify notation we shall write $t_0 = \tau$, $t_1 = s$, $t_2 = \eta$. Substituting (4.5) and (4.7) into equations (4.1) generates the following hierarchy:

$O(\epsilon)$
$$\partial^2_\tau w_1 + h^2 \Delta^2 w_1 + \sigma_0\, \partial^2_x w_1 = 0 \tag{4.8a}$$

$$\Delta^2 f_1 = 0 \tag{4.8b}$$

$O(\epsilon^2)$ $$\partial_T^2 w_2 + 2\partial_{TS} w_1 + 2\gamma\partial_T w_1 + h^2\Delta^2 w_2$$

$$+ \sigma_0 \partial_x^2 w_2 + \sigma_1 \partial_x^2 w_1 =$$

$$\partial_y^2 f_1 \partial_x^2 w_1 - 2\partial_{xy} f_1 \partial_{xy} w_1 + \partial_x^2 f_1 \partial_y^2 w_1 \tag{4.9a}$$

$$\Delta^2 f_2 = (\partial_{xy} w_1)^2 - \partial_x^2 w_1 \partial_y^2 w_1 \tag{4.9b}$$

$$2(1+\nu)\rho \ell_2 = \int_0^t \int_0^1 w_1 \partial_{xy} w_1 \, dxdy \tag{4.9c}$$

$O(\epsilon^3)$ $$\partial_T^2 w_3 + 2\partial_{TS} w_2 + \partial_S^2 w_1 + 2\partial_{T\eta} w_1 + 2\gamma\partial_T w_2$$

$$+ 2\gamma\partial_S w_1 + h^2\Delta^2 w_3 + \sigma_0 \partial_x^2 w_3 + \sigma_1 \partial_x^2 w_2 + \sigma_2 \partial_x^2 w_1$$

$$+ \ell_2 \partial_{xy} w_1 =$$

$$\partial_y^2 f_2 \partial_x^2 w_1 + \partial_y^2 f_1 \partial_x^2 w_2 - 2\partial_{xy} f_2 \partial_{xy} w_1 - 2\partial_{xy} f_1 \partial_{xy} w_2$$

$$+ \partial_x^2 f_2 \partial_y^2 w_1 + \partial_x^2 f_1 \partial_y^2 w_2 \tag{4.10a}$$

$$\Delta^2 f_3 = 2\partial_{xy} w_1 \partial_{xy} w_2 - \partial_x^2 w_1 \partial_y^2 w_2 - \partial_x^2 w_2 \partial_y^2 w_1 \tag{4.10b}$$

$$2(1+\nu)\rho \ell_3 = \int_0^\rho \int_0^1 w_1 \partial_{xy} w_2 + w_2 \partial_{xy} w_1 \, dxdy \tag{4.10c}$$

We assume that σ_o is a simple eigenvalue; i.e. there exists a unique integer pair (M, N) with $M \geq 1$, $N \geq 1$ such that $\sigma_o = \sigma_o(M, N)$. Now

$$\sigma_o = \min_{m,n} h^2 Q_{mn}^2 / (m\pi)^2$$

$$= \min_{m,n} h^2 [(m\pi)^2 + (n\omega)^2]^2 / (m\pi)^2$$

But Q_{mn} is an increasing function of n, so that necessarily $N = 1$.

For initial conditions we take

$$w = \epsilon\phi, \quad w_t = \epsilon^2\psi \quad \text{at} \quad t = 0 \tag{4.11}$$

The motivation for taking $w_t = O(\epsilon^3)$ will be explained in the course of the calculations. Expressed in terms of the perturbation expansion, (4.11) becomes

$$\left.\begin{array}{lll} w_1 = \phi & \partial_T w_1 = 0 & \\ w_2 = 0 & \partial_T w_2 + \partial_s w_1 = \psi & \\ w_3 = 0 & \partial_T w_3 + \partial_s w_2 + \partial_\eta w_1 = 0 & \end{array}\right\} \text{at } t = 0 \qquad (4.12)$$

Introduce notation for the Fourier coefficients of a function g_k by

$$g^k_{mn} \equiv \frac{4}{\rho} \int_0^\rho \int_0^1 g_k \, Y_{mn} \, dxdy$$

Then equations (4.8) imply

$$\partial^2_T \, w^1_{mn} + \lambda^2_{mn} \, w^1_{mn} = 0 \qquad (4.13a)$$

$$Q^2_{mn} \, f^1_{mn} = 0 \qquad (4.13b)$$

where

$$\lambda^2_{mn} = h^2 \, Q^2_{mn} - \sigma_o (m\pi)^2 \qquad (4.14)$$

The solutions to (4.8) are

$$\left.\begin{array}{l} w_1 = \sum w^1_{mn} \, Y_{mn} \\ f_1 = 0 \end{array}\right\} \qquad (4.15)$$

where

$$w^1_{mn} = a^1_{mn} \cos \lambda_{mn} \tau + b^1_{mn} \sin \lambda_{mn} \tau \qquad (4.16a)$$

for $(m, n) \neq (M, 1)$ (in which case $\lambda^2_{mn} > 0$)

and

$$w^1_{M1} = w^1_{M1} + b^1_{M1}\tau \tag{4.16b}$$

since $\lambda^2_{M1} = 0$ by construction. We seek a solution which is bounded for all time; consequently, we conclude that $b^1_{M1} = 0$. Note that this in turn implies that $\partial_\tau w^1_{M1} = 0$. This is the reason for choosing $\partial_t w = O(\epsilon^2)$; otherwise we would have to require that one particular mode is absent from the initial velocity. Rather than place such an awkward constraint on the initial data, we find that we can circumvent the difficulty using this simple device.

The solution satisfying initial conditions (4.12) is that of (4.16) where

$$a^1_{mn} = a^1_{mn}(s, \eta, \ldots)$$

$$b^1_{mn} = b^1_{mn}(s, \eta, \ldots)$$

and

$$a^1_{mn} = \phi_{mn} \qquad b^1_{mn} = 0 \qquad \text{at } t = 0 \ . \tag{4.17}$$

It is worth noting that choosing $b^1_{M1} \equiv 0$ is not the only possible resolution to the problem of keeping w^1_{M1} bounded--it is merely the simplest, and consequently, the most natural to try first. If we retain b^1_{M1} and later discover, for example, that it decays exponentially in s, this would also be sufficient.

Using (4.15), equation (4.9a) reduces to

$$\partial^2_\tau w_2 + 2\partial_{\tau s} w_1 + 2\gamma\partial_\tau w_1 + h_2 \Delta^2 w_2 + \sigma_0 \partial^2_x w_1 + \sigma_1 \partial^2_x w_1 = 0$$

which can be Fourier analyzed to give

$$\partial_\tau^2 w_{mn}^2 + h^2 Q_{mn}^2 w_{mn}^2 - \sigma_0 (m\pi)^2 w_{mn}^2 = \partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2 =$$

$$-2\,\partial_{\tau s} w_{mn}^1 - 2\gamma \partial_\tau w_{mn}^1 + \sigma_1 (m\pi)^3 w_{mn}^1 \qquad (4.18)$$

In order that w_{mn}^2 be bounded in τ, one applies a familiar argument to "suppress secular terms" [8]. Integrating by parts, we compute

$$\frac{1}{T}\int_0^T (\partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2)(\alpha \cos \lambda_{mn}\tau + \beta \sin \lambda_{mn}\tau)d\tau =$$

$$\frac{1}{T}\Big[\partial_\tau w_{mn}^2(\alpha \cos\lambda_{mn}\tau + \beta\sin\lambda_{mn}\tau)$$

$$+ \lambda_{mn} w_{mn}^2(\alpha \sin\lambda_{mn}\tau - \beta\cos\lambda_{mn}\tau)\Big]\Big|_0^T$$

$\to 0$ as $T \to \infty$ if w_{mn}^2 and $\partial_\tau w_{mn}^2$ are bounded. Hence it is necessary that

$$0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T (2\partial_{\tau s} w_{mn}^1 + 2\gamma\partial_\tau w_{mn}^1 - \sigma_1 (m\pi)^2 w_{mn}^1)(\alpha\cos\lambda_{mn}\tau$$

$$+ \beta\sin\lambda_{mn}\tau)d\tau \qquad (4.19)$$

Consider first the (M, 1) mode. $w_{M1}^1 = a_{M1}^1(s,\ldots)$, $\lambda_{M1} = 0$, and so with $\alpha = 1$, $\beta = 0$, equation (4.19) becomes

$$\lim_{T\to\infty} \sigma_1 (m\pi)^2 a_{M1}^1(s,\ldots) = 0$$

which implies $\sigma_1 = 0$. Then for $(m, n) \neq (M, 1)$ equation (4.16a) yields

$$0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\partial_{\tau s} w_{mn}^1 + \gamma\partial_\tau w_{mn}^1)(\alpha\cos\lambda_{mn}\tau + \beta\sin\lambda_{mn}\tau)d\tau$$

$$= -\tfrac{1}{2}\beta\lambda_{mn}(\partial_s a_{mn}^1 + \gamma a_{mn}^1) + \tfrac{1}{2}\alpha\,\lambda_{mn}(\partial_s b_{mn}^1 + \gamma b_{mn}^1)$$

or since α and β are independent,

$$\partial_s a^1_{mn} + \gamma a^1_{mn} = 0 \qquad \partial_s b^1_{mn} + \gamma b^1_{mn} = 0$$

Applying initial conditions (4.17), we find for $(m, n) \neq M1$

$$a^1_{mn} = \hat{a}_{mn}\, e^{-\gamma s} \qquad b^1_{mn} = \hat{b}_{mn} e^{-\gamma s} \tag{4.20}$$

with

$$\hat{a}_{mn} = \hat{a}_{mn}(\eta, \ldots) \qquad \hat{b}_{mn} = \hat{b}_{mn}(\eta, \ldots)$$

$$\hat{a}_{mn} = \phi_{mn}, \quad \hat{b}_{mn} = 0 \quad \text{at} \quad t = 0 \tag{4.21}$$

Using (4.16) and (4.20), equation (4.18) simplifies to

$$\partial^2_\tau w^2_{mn} + \lambda^2_{mn} w^2_{mn} = 0$$

with solutions

$$w^2_{mn} = a^2_{mn} \cos\lambda_{mn}\tau + b^2_{mn} \sin\lambda_{mn}\tau, (m, n) \neq (M, 1) \tag{4.22a}$$

$$w^2_{M1} = a^2_{M1}(s, \eta, \ldots) \tag{4.22b}$$

From equation (4.9b) we derive

$$Q^2_{mn} f^2_{mn} = \frac{4}{\rho} \int_0^\rho \int_0^1 (\partial_{xy} w_1\, \partial_{xy} w_1 - \partial^2_x w_1\, \partial^2_y w_1)\, Y_{mn} dxdy \tag{4.23}$$

We can characterize (to leading order) the large time behavior of w_1 by (4.15) and (4.16) once we find a^1_{M1}. To do this we must proceed to the $O(\varepsilon^3)$ equations. With $f_1 = 0$ and $\sigma_1 = 0$, equation (4.10a) is

$$\partial^2_\tau w_3 + h^2 \Delta^2 w_3 + \sigma_0\, \partial^2_x w_3 + 2\partial_{\tau s} w_2 + \partial^2_s w_1$$

$$+ 2\, \partial_{\tau\eta} w_1 + 2\gamma\partial_\tau w_2 + 2\gamma\partial_s w_1 + \sigma_2\, \partial^2_x w_1 =$$

$$2\, \partial^2_y f_2\, \partial^2_x w_1 - 2\partial_{xy} f_2\, \sigma_{xy} w_1 + \partial^2_x f_2\, \partial^2_y w_1 - \ell_2\, \partial_{xy} w_1$$

Using the facts that $\lambda_{M1} = 0$ and $\partial_\tau w^1_{M1} = \partial_\tau w^2_{M1} = 0$, one readily calculates

$$\partial^2_\tau w^3_{M1} + \partial^2_s w^1_{M1} + 2\gamma \partial_s w^1_{M1} - \sigma_2 (M\pi)^2 w^1_{M1} = K \qquad (4.24a)$$

where

$$K = \frac{4}{\rho} \int_0^\rho \int_0^1 (\partial^2_y f_2 \partial^2_x w_1 - 2\partial_{xy} f_2 \partial_{xy} w_1 + \partial^2_x f_2 \partial^2_y w_1 - \ell_2 \partial_{xy} w_1) Y_{M1} dxdy \qquad (4.24b)$$

Up to this point ϵ has remained undefined; we have only used $\epsilon > 0$. We effectively define ϵ by requiring that $\sigma_2 = 1$. Note then that

$$\sigma = \sigma_o + \epsilon^2 + O(\epsilon^3) > \sigma_o \quad \text{for} \quad 0 < \epsilon \ll 1$$

so that $\epsilon > 0$ puts σ into the unstable region.

Next we again "suppress secular terms," assuming that $\partial_\tau w^3_{M1}$ is bounded. Writing $w^1_{M1} = a(s, \eta, \dots)$, we have

$$\partial^2_s a + 2\gamma \partial_s a - (M\pi)^2 a = H \qquad (4.25a)$$

with

$$H = \lim_{T\to\infty} \frac{1}{T} \int_0^T K \, d\tau \qquad (4.25b)$$

The actual calculation of H is lengthy and is left to Appendix B. We cite the result here, derived under the further assumption that all the λ_{mn} are distinct. Introduce the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$
$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y$$

and define the operators M and J by

$$M[g] = \lim_{T\to\infty} \frac{1}{T}\int_o^T g(\tau)d\tau \tag{4.26}$$

$$J[G] = \frac{4}{\rho}\int_o^{\rho}\int_o^{1} G(x,y)dxdy \tag{4.27}$$

Then it can be shown that

$$M\{J[\,\partial_y^2 f_2\,\partial_x^2 w_1 - 2\partial_{xy} f_2\,\partial_{xy} w_1 + \partial_x^2 f_2\,\partial_y^2 w_1)Y_{M1}]\}$$

$$= k_1\ a + k_2\ a^3 \tag{4.28a}$$

with

$$k_1 = \tfrac{1}{2}\pi^4\omega^4 \sum_{\substack{mn\neq M1\\ \underline{mn}}} N^1_{mn}\, Q^{-2}_{\underline{mn}}\, m^2 n^2 J_1\,(\,(M^2\underline{n}^2+\underline{m}^2)J_2 - 2M\underline{m}\,\underline{n}\,J_3\,)$$

$$- N^1_{mn}Q^{-2}_{\underline{mn}}\,(\,(m^2\underline{n}^2+\underline{m}^2n^2)J_4 - 2mn\,\underline{mn}\,J_5\,)\ x$$

$$(\,(M^2n^2+m^2)J_6 - 2M_{mn}\,J_7\,)$$

$$k_2 = \pi^4\omega^4 \sum_{mn}(\,(M^2n^2+m^2)J_8 - 2M_{mn}J_9\,)Q^{-2}_{mn}M^2 J_{10} \tag{4.28c}$$

The notation in (4.28b, c) is

$$N^1_{mn} = (a^1_{mn})^2 + (b^1_{mn})^2\ ,\quad (m,n)\neq(M,1) \tag{4.28d}$$

and

$$J_1 = J[(c_m^2 C_n^2 - s_m^2 S_n^2)s_{\underline{m}} S_{\underline{n}}\,] \tag{4.28e}$$

$$J_2 = J[s_{\underline{m}}\, s_M^2\, S_{\underline{n}}\, S_1^2]\quad J_3 = J[c_{\underline{m}}c_M s_M C_{\underline{n}} C_1 S_1]$$

$$J_4 = J[s_{\underline{m}}\, s_m s_M S_{\underline{n}} S_n S_1]\quad J_5 = J[c_{\underline{m}} c_m s_M C_{\underline{n}}\, C_n S_1]$$

$$J_6 = J[s_{\underline{m}}\, s_m\, s_M\, S_{\underline{n}} S_n S_1]\quad J_7 = J[s_{\underline{m}} c_m c_M C_n C_1\, S_{\underline{n}}]$$

$$J_8 = J[s_m\, s_M^2\, S_n\, S_1^2]\qquad J_9 = J[c_m c_M s_M\, C_n C_1 S_1]$$

$$J_{10} = J[(c_M^2 C_1^2 - s_M^2 S_1^2)s_m\, S_n]$$

It can also be shown that

$$M\{J[\ell_2\ \partial_{xy}\, w_1\ Y_{M1}]\} = k_3\ a \tag{4.29a}$$

with

$$k_3 = \frac{\pi^2\omega^2}{16(1+\nu)}\ \Sigma_{mn\neq M1}\ m^2n^2 N^1_{mn} J^2_{11} + M_{mn}N^1_{mn}J_{12}J_{13} \tag{4.29b}$$

and

$$J_{11} = J[s_M c_m S_1 C_n] \quad J_{12} = J[s_m c_M S_n C_1] \quad J_{13} = J[s_M c_m S_1 C_n] \tag{4.29c}$$

Now from (4.20) and (4.28d)

$$N^1_{mn} = [(\hat{a}_{mn})^2 + (\hat{b}_{mn})^2]e^{-2\gamma s}\ ,\quad (m,n)\neq(M,1)$$

and so we can set

$$k_1 + k_3 = -\alpha\, e^{-2\gamma s} \qquad k_2 = -\beta$$

with α and β independent of τ and s. Equation (4.25) becomes

$$\partial^2_s\, a + 2\gamma\, \partial^2_s a - (M\pi)^2 a + \alpha\, e^{-2\gamma s}\, a + \beta\, a^3 = 0 \tag{4.30}$$

Recall $a^1_{M1} = a = \phi_{M1}$ at $t = 0$ (equation (4.17)). To get the second initial condition, note that (4.12) implies $\partial_\tau\, w^2_{mn} + \partial_s\, w^1_{mn} = \psi_{mn}$ at $t = 0$. Using (4.22b) we conclude $\partial_s a = \psi_{M1}$ initially.

Several comments are in order. First remark that equation (4.30) is essentially the same as the equation governing a rod derived in [15]. However, in [15] the constants α and β are clearly positive, whereas for the plate this no longer seems to be true. Consequently, the solutions of (4.30) need not be bounded, and the perturbation scheme may fail to show how the nonlinearity stops the exponential growth of small perturbations.

In the case of no damping ($\gamma = 0$) (4.30) is autonomous and may be analyzed by phase plane methods. However, the N^1_{mn}, and hence α, depend on the initial conditions. It follows that the location of the equilibrium points (critical points) varies with the initial data. This runs counter to general experience, since equilibrium configurations are typically properties of the differential equations. The corresponding term in the equation for a rod also depends on the initial data, but the authors of [15] do not comment on the significance of this. It appears that damping is necessary for this model to yield physically meaningful results.

We are interested in bounded solutions of (4.30) when $\gamma > 0$ (when solutions are unbounded this model is no longer an accurate model of large time behavior). An energy relation shows that solutions are bounded when $\alpha > 0$ and $\beta > 0$:

$$\frac{\partial}{\partial s}\left(\frac{1}{2}\left(\frac{\partial a}{\partial s}\right)^2 - \frac{(M\pi)^2}{2}\, a^2 + \frac{1}{2}\,\alpha\, e^{-2\gamma s}\, a^2 + \frac{1}{4}\,\beta\, a^4\right) = -2\gamma\left(\frac{\partial a}{\partial s}\right)^2 - \alpha\gamma e^{-2\gamma s}\, a^2 < 0 \tag{4.31}$$

For large s it is tempting to ignore the term $\alpha\, e^{-2\gamma s}$ in (4.30) asymptotically. A simple geometric argument shows that this approximation is indeed valid. Introduce $b = a'$ and $g = \alpha\, e^{-2\gamma s}$, where ' denotes ∂_s. Solutions of (4.30) are contained among trajectories of the autonomous system

$$\left.\begin{aligned} a' &= b \\ b' &= (M\pi)^2 a - 2\gamma b - ga - \beta a^3 \\ g' &= -2\gamma g \end{aligned}\right\} \tag{4.32}$$

In the region $g \neq 0$ the g-component of every tangent vector is directed towards the a-b plane. Consequently, the limit points of any trajectory lie within the a-b plane. The limit points of a bounded trajectory are a closed connected set consisting of either a limit point, a limit cycle, or a separatrix connecting limit points [5]. Now the reduced system

$$\left.\begin{aligned} a' &= b \\ b' &= (M\pi)^2 a - 2\gamma b - \beta a^3 \end{aligned}\right\} \qquad (4.33)$$

always has a saddle point at the origin (the characteristic exponents there are $\lambda = -\gamma \pm \sqrt{\gamma^2 + (M\pi)^2}$). If $\beta \leq 0$ there are no other critical points and trajectories are unbounded. Hence $\beta > 0$ for bounded solutions. In that case there exist two attractors (stable nodes or spirals) at $a = \pm M\pi/\beta^{\frac{1}{2}}$, $b = 0$ with characteristic exponents $\lambda = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$) at either one. There are no limit cycles and all trajectories tend to one or another of the attractors as $s \to \infty$. Substituting this result into our earlier calculations shows that for $\sigma \sim \sigma_o + \epsilon^2$, $\gamma > 0$, and $\beta > 0$, solutions of (4.1) with small initial displacement and velocity tend to one of the states

$$w \sim \pm (M\pi/\beta^{\frac{1}{2}}) Y_{M1}$$

to leading order as $t \to \infty$. Note that when damping is present the equilibrium points do not depend on the initial data.

The circular cylindrical shell

We proceed to study the dynamic buckling of a circular cylindrical shell for small initial displacements and velocities when the load is a "small distance" into the unstable regime. The appropriate equations, with small damping $\Gamma = \epsilon\gamma$, are

$$w_{tt} + 2\epsilon\gamma\, w_t + h^2 \Delta^2 w + \sigma w_{xx} + \omega^2 f_{xx} =$$

$$f_{yy}\, w_{xx} - 2f_{xy}\, w_{xy} + f_{xx}\, w_{yy} - 2\ell\, w_{xy} \tag{4.34a}$$

$$\omega^2\, \Delta^2 f - \omega^2 w_{xx} = w_{xy}^2 - w_{xx}\, w_{yy} \tag{4.34b}$$

$$4\pi\omega(1+\nu)\ell = \int_0^{\Omega}\int_0^1 w\, w_{xy}\, dxdy \tag{4.34c}$$

The initial conditions and boundary conditions are

$$w = \epsilon\phi, \qquad w_t = \epsilon^2\psi \qquad \text{at} \qquad t = 0 \tag{4.35}$$

$$w = w_{xx} = f = f_{xx} = 0 \qquad \text{at} \qquad x = 0, 1 \tag{4.36}$$

$$w, f \ \text{ have period } \Omega = 2\pi/\omega \ \text{ in } y \tag{4.37}$$

The small parameter ϵ will be determined in the course of the calculations. We seek solutions of the form

$$w = \epsilon w_1 + \epsilon^2 w_2 + \cdots$$

$$f = \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

$$\ell = \epsilon^2 \ell_2 + \cdots$$

$$\sigma = \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \cdots$$

where σ_0 is the smallest load for which the state $w = f = 0$ is unstable to infinitesimal perturbations. Introduce time scales $t_k = \epsilon^k t$ $(k = 0, 1, 2, \ldots)$, and set $t_0 = \tau$, $t_1 = s$, $t_2 = \eta$ for convenience. As for the rectangular plate, these expansions lead to a hierarchy of equations:

$O(\varepsilon)$ $$\partial_\tau^2 w_1 + h^2\Delta^2 w_1 + \sigma_0\,\partial_x^2 w_1 + \omega^2\partial_x^2 f_1 = 0 \tag{4.38a}$$

$$\omega^2\Delta^2 f_1 - \omega^2\,\partial_x^2 w_1 = 0 \tag{4.38b}$$

$O(\varepsilon^2)$ $$\partial_\tau^2\, w_2 + h^2\Delta^2 w_2 + \sigma_0\partial_x^2 w_2 + \omega^2\partial_x^2 f_2 + 2\partial_{\tau s} w_1 + 2\gamma\partial_\tau w_1 + \sigma_1\,\partial_x^2 w_1$$

$$= \partial_y^2 f_1\,\partial_x^2 w_1 - 2\,\partial_{xy} f_1\,\partial_{xy} w_1 + \partial_x^2 f_1\,\partial_y^2 w_1 \tag{4.39a}$$

$$\omega^2\Delta^2 f_2 - \omega^2\partial_x^2 w_2 = (\partial_{xy} w_1)^2 - \partial_x^2 w_1\,\partial_y^2\, w_1 \tag{4.39b}$$

$$4\pi\omega(1+\nu)\ell_2 = \int_0^\Omega\int_0^1 w_1\,\partial_{xy} w_1\,dxdy \tag{4.39c}$$

$O(\varepsilon^3)$ $$\partial_\tau^2 w_3 + h^2\Delta^2 w_3 + \sigma_0\,\partial_x^2 w_3 + \omega^2\partial_x^2 f_3 + 2\,\partial_{\tau s} w_2 + \partial_s^2 w_1 + 2\,\partial_{\tau\eta} w_1$$

$$+ 2\gamma\partial_\tau w_2 + 2\gamma\,\partial_s w_1 + \sigma_1\,\partial_x^2 w_2 + \sigma_2\,\partial_x^2 w_1 =$$

$$\partial_y^2 f_2\,\partial_x^2\, w_1 + \partial_y^2\, f_1\,\partial_x^2\, w_2 - 2\partial_{xy} f_2\,\partial_{xy} w_1 - 2\partial_{xy} f_1\,\partial_{xy} w_2$$

$$+ \partial_x^2 f_2\,\partial_y^2\, w_1 + \partial_x^2 f_1\,\partial_y^2\, w_2 - 2\,\ell_2\,\partial_{xy}\, w_1 \tag{4.40a}$$

$$\omega^2\Delta^2 f_3 - \omega^2\,\partial_x^2\, w_3 = 2\partial_{xy} w_1\,\partial_{xy} w_2 - \partial_x^2 w_1\,\partial_y^2 w_2 - \partial_x^2 w_2\,\partial_y^2 w_1 \tag{4.40b}$$

Equations (4.38) are identical with the equations for a linearized stability analysis of the state w = f = 0; consequently we determine σ_o so that the solution of (4.38) is conditionally stable.

If we expand

$$\left.\begin{aligned} w_1 &= \Sigma\,\xi_n w^1_{mn}(\tau, s, \eta, \ldots)Y_{mn} + \bar{w}^1_{mn}\bar{Y}_{mn} \\ f_1 &= \Sigma\xi_n f^1_{mn}(\tau, s, \eta, \ldots)Y_{mn} + \bar{f}^1_{mn}\,\bar{Y}_{mn} \end{aligned}\right\} \tag{4.41}$$

where $\xi_n = \frac{1}{2}$ for $n = 0$, $\xi_n = 1$ for $n > 0$

and $Y_{mn}(x, y) = \sin m\pi x \cos n\omega y$

$\bar{Y}_{mn}(x, y) = \sin m\pi x \cos n\omega y$

for $m = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$, then (4.38) implies

$$\partial_\tau^2 w^1_{mn} + (h^2 Q^2_{mn} - \sigma_o (m\pi)^2) w^1_{mn} - (m\pi)^2 \omega^2 f^1_{mn} = 0 \tag{4.42a}$$

$$\omega^2 Q^2_{mn} f^1_{mn} + \omega^2 (m\pi)^2 (m\pi)^2 w^1_{mn} = 0 \tag{4.42b}$$

Here

$$Q_{mn} = (m\pi)^2 + (n\omega)^2 \tag{3.30}$$

A pair of equations analogous to (4.42) holds for $\bar{w}^1_{mn}$ and $\bar{f}^1_{mn}$. From (4.42) we conclude

$$\partial_\tau^2 w^1_{mn} + \lambda^2_{mn} w^1_{mn} = 0 \tag{4.43a}$$

$$f^1_{mn} = -((m\pi)^2 / Q^2_{mn}) w^1_{mn} \tag{4.43b}$$

with

$$\lambda^2_{mn} = (m\pi)^2 (\sigma_{mn} - \sigma_o) \tag{4.44}$$

and

$$\sigma_{mn} = h^2 Q^2_{mn} / (m\pi)^2 + \omega^2 (m\pi)^2 / Q^2_{mn} \tag{4.45}$$

Solutions of (4.43) grow exponentially if $\lambda^2_{mn} < 0$, so conditional stability occurs for $\lambda^2_{mn} = 0$. The smallest value σ_o such that $\lambda^2_{mn} = 0$ for some pair (m, n) is

$$\sigma_o = \min_{m,n} \sigma_{mn} \tag{4.46}$$

We will assume that σ_o is simple, i.e. that there exists a unique pair (M, N) such that $\sigma_o = \sigma_{MN}$.

We continue to denote Fourier components by the notation

$$g^k_{mn} = \frac{4}{\Omega} \int_o^\Omega \int_o^1 g_k Y_{mn} \, dxdy$$

$$\bar{g}^k_{mn} = \frac{4}{\Omega} \int_o^\Omega \int_o^1 g_k \bar{Y}_{mn} \, dxdy .$$

In general, for each equation in a variable g^k_{mn} there exists a symmetric equation in $\bar{g}^k_{mn}$. We will suppress the second equation in most instances for brevity.

Initial conditions (4.35), when expanded as in (4.12), lead to

$$\left.\begin{array}{ll} w^1_{mn} = \phi_{mn} & \partial_\tau w^1_{mn} = 0 \\ w^2_{mn} = 0 & \partial_\tau w^2_{mn} + \partial_s w^1_{mn} = \psi_{mn} \\ w^3_{mn} = 0 & \partial_\tau w^3_{mn} + \partial_s w^2_{mn} + \partial_\eta w^1_{mn} = 0 \end{array}\right\} \text{ at } t = 0 \qquad (4.47)$$

Thus the solution of (4.43) is

$$\left.\begin{array}{l} w^1_{mn} = a^1_{mn} \cos \lambda_{mn}\tau + b^1_{mn} \sin \lambda_{mn}\tau \quad (m,n) \neq (M,N) \\ w^1_{MN} = a^1_{M1} + b^1_{M1}\tau \end{array}\right\} \qquad (4.48)$$

where

$$a^1_{mn} = a^1_{mn}(s, \eta, \ldots) \text{ etc.}$$

and

$$a^1_{mn} = \phi_{mn} \quad b^1_{mn} = 0 \quad \text{at } t = 0 \qquad (4.49)$$

Boundedness of w^1_{MN} implies that $b^1_{M1} \equiv 0$.

To determine the behavior of the a^1_{mn} and b^1_{mn} we proceed to the $O(\epsilon^3)$ equations. We introduce the operators

$$J[g] = \frac{4}{\Omega} \int_0^\Omega \int_0^1 g \, dxdy$$

$$J_{mn}[g] = J[g\, Y_{mn}] \quad \bar{J}_{mn}[g] = J[g\, \bar{Y}_{mn}].$$

Equations (4.39 a, b) yield

$$\partial_\tau^2 w_{mn}^2 + (h^2 Q_{mn}^2 - \sigma_o (m\pi)^2) w_{mn}^2 - (m\pi)^2 \omega^2 f_{mn}^2$$
$$+ 2\,\partial_{\tau s} w_{mn}^1 + 2\gamma\,\partial_\tau w_{mn}^1 - \sigma_1 (m\pi)^2 w_{mn}^1 = P_{mn}^1 \tag{4.50a}$$

$$\omega^2 Q_{mn}^2 f_{mn}^2 + \omega^2 (m\pi)^2 w_{mn}^2 = P_{mn}^2 \tag{4.50b}$$

with

$$P_{mn}^1 = J_{mn}[\partial_y^2 f_1\, \partial_x^2 w_1 - 2\,\partial_{xy} f_1\, \partial_{xy} w_1 + \partial_x^2 f_1\, \partial_y^2 w_1]$$

$$P_{mn}^2 = J_{mn}[(\partial_{xy} w_1)^2 - \partial^2 w_1\, \partial_y^2 w_1]$$

Setting $P_{mn} = P_{mn}^1 + (m\pi/Q_{mn})^2 P_{mn}^2$, we have

$$\partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2 = P_{mn} - 2\,\partial_{\tau s} w_{mn}^1 - 2\gamma\,\partial_\tau w_{mn}^1 + \sigma_1 (m\pi)^2 w_{mn}^1 \tag{4.51}$$

Using the operator M defined in (4.26), the "suppression of secular terms" follows from

$$M[(\partial_\tau^2 w_{mn}^2 + \lambda_{mn}^2 w_{mn}^2) z] = 0$$

where z is any (bounded) solution of the (adjoint) equation

$$\partial_\tau^2 z + \lambda_{mn}^2 z = 0\,. \tag{4.52}$$

Thus

$$M[(2\partial_{\tau s} w_{mn}^1 + 2\gamma\,\partial_\tau w_{mn}^1 - \sigma_1 (m\pi)^2 w_{mn}^1) z] = M[P_{mn} z] \tag{4.53}$$

Consider $(m,n) = (M,N)$; $\lambda_{MN} = 0$ and so $z = 1$. From (4.48) we know that $\partial_\tau w_{MN}^1 = 0$, so (4.53) simplifies to

$$-\sigma_1 (M\pi)^2 w_{MN}^1 = M[P_{MN}]$$

Now if $M[P_{MN}] \neq 0$, σ_1 will depend on the initial conditions.

Since we seek a solution which is valid for arbitrary initial conditions, this possibility must be excluded. Consequently, for this perturbation scheme to be valid, it is necessary that

$$M[P_{MN}] = 0 \tag{4.54}$$

from which we conclude

$$\sigma_1 = 0 \tag{4.55}$$

Note that nothing equivalent to (4.54) was required for the problem of a rectangular plate. It can be shown (cf. Appendix B) that condition (4.54) is

$$O = \sum_{mn \neq MN} \left(\frac{Q_{MN}^2 m^4 n^2}{Q_{mn}^2} + \frac{M^2 m^2 n^2}{2} \right) [A_{mn}(\alpha_m + \beta_n) + \bar{A}_{mn}(\alpha_m + \gamma_n)] \tag{4.56}$$

where

$$A_{mn} = (a^1_{mn})^2 + (b^1_{mn})^2 \quad \bar{A}_{mn} = (\bar{a}^1_{mn})^2 + (\bar{b}^1_{mn})^2$$

and

$$\alpha_m = J[s_m^2 s_M C_N] \quad \beta_n = J[s_M S_n^2 C_N] \quad \gamma_n = J[s_M C_n^2 C_N] .$$

Here we are again using the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$

$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y$$

In order that (4.54) or (4.56) hold, it is sufficient that α_m, β_n, and γ_n vanish for all m and n. This will be the case if either N is odd or M is even. In the general case, the contradiction inherent in (4.54) can be circumvented by taking $w_1 \equiv 0$, $f_1 \equiv 0$ and assuming $w_2 \neq 0$,

$f_2 \neq 0$. For the remainder of this investigation we will assume that N is odd.

With $\sigma_1 = 0$, it remains to calculate (4.53) in detail for $(m, n) \neq (M, N)$. Details of the computation are indicated in Appendix B. We note that one further assumption is needed for the calculation. A sum of the form

$$\lambda_{mn} \pm \lambda_{m_1 n_1} \pm \lambda_{m_2 n_2}$$

can vanish if $(m, n) = (M, N)$ and $(m_1, n_1) = (m_2, n_2)$ since $\lambda_{MN} = 0$. We assume that this is the only way in which such a sum can vanish. In particular, this implies that

$$\lambda_{m_1 n_1} = \lambda_{m_2 n_2}$$

if and only if $(m_1, n_1) = (m_2, n_2)$.

Using $z = \cos\lambda_{mn}\tau$ and $z = \sin\lambda_{mn}\tau$, we find that for $(m, n) \neq (M, N)$, equation (4.53) yields

$$\partial_s a^1_{mn} + \gamma\, a^1_{mn} = 0 \qquad \partial_s\, b^1_{mn} + \gamma\, b^1_{mn} = 0$$

when N is odd (or M is even). Hence

$$a^1_{mn} = \hat{a}_{mn}\, e^{-\gamma s} \qquad b^1_{mn} = \hat{b}_{mn}\, e^{-\gamma s} \qquad (m, n) \neq (M, N) \tag{4.57}$$

where

$$\hat{a}_{mn} = \hat{a}_{mn}(\eta, \ldots) \qquad \hat{b}_{mn} = \hat{b}_{mn}(\eta, \ldots)$$

and

$$\hat{a}_{mn} = \phi_{mn}, \quad \hat{b}_{mn} = 0 \qquad \text{at } t = 0 \;.$$

To determine the behavior of a^1_{MN} we proceed to the $O(\epsilon^3)$ equations. For this we need the solutions w_2 and f_2, or equivalently, we need w^2_{mn} and f^2_{mn}. With the results (4.55) and (4.57), equation (4.51) simplifies to

$$\partial^2_\tau w^2_{mn} + \lambda^2_{mn} w^2_{mn} = P_{mn} \tag{4.58}$$

From (4.47) and (4.57) we can determine the initial conditions

$$w^2_{mn} = 0, \quad \partial_\tau w^2_{mn} = \psi_{mn} + \gamma \phi_{mn} \quad \text{at } t = 0$$

when $(m, n) \neq (M, N)$. We also know $w^2_{MN} = 0$ initially. The initial value of $\partial_\tau w^2_{MN}$ requires more subtle consideration. Solutions of

$$\partial^2_\tau w^2_{MN} = P_{MN}$$

can become unbounded due to two sources: P_{MN} may contain a "resonant term," even though it is bounded itself, or w^2_{MN} may contain a term linear in τ which satisfies the homogeneous equation. The first possibility was eliminated by (4.54). Removing the homogeneous solution proportional to τ fixes the initial value of $\partial_\tau w^2_{MN}$.

Fortunately, it turns out that it is not necessary to explicitly find w^2_{mn} in order to determine the equation for a^1_{MN}. It will suffice to be able to evaluate

$$M[w^1_{\underline{mn}} \, w^2_{mn}]$$

when $(\underline{m}, \underline{n}) \neq (m, n)$ (and hence, by assumption, $\lambda_{\underline{mn}} \neq \lambda_{mn}$). From (4.43a)

$$\int_0^T \partial^2_\tau w^1_{\underline{mn}} \, w^2_{mn} \, d\tau = -\lambda^2_{\underline{mn}} \int_0^T w^1_{\underline{mn}} \, w^2_{mn} \, d\tau .$$

Using (4.58) and integrating by parts, we also calculate

$$\int_0^T \partial_\tau^2 w^1_{\underline{mn}} w^2_{mn}\, d\tau$$

$$= [\partial_\tau w^1_{\underline{mn}} w^2_{mn} - w^1_{\underline{mn}} \partial_\tau w^2_{mn}]\Big|_0^T + \int_0^T w^1_{\underline{mn}} \partial_\tau^2 w^2_{mn}\, d\tau$$

$$= [\ldots]\Big|_0^T + \int_0^T w^1_{\underline{mn}} (P_{mn} - \lambda^2_{mn} w^2_{mn}) d\tau .$$

Hence

$$(\lambda^2_{mn} - \lambda^2_{\underline{mn}}) \int_0^T w^1_{\underline{mn}} w^2_{mn} d\tau = [\ldots]\Big|_0^T + \int_0^T w^1_{\underline{mn}} P_{mn}\, d\tau .$$

Multiply by T^{-1} and take the limit as $T \to \infty$. The result is

$$M[w^1_{\underline{mn}} w^2_{mn}] = (\lambda^2_{mn} - \lambda^2_{\underline{mn}})^{-1} M[w^1_{\underline{mn}} P_{mn}] \tag{4.59}$$

when $\lambda_{mn} \neq \lambda_{\underline{mn}}$.

To determine an equation for a^1_{M1}, we apply "suppression of secular terms" to the $O(\epsilon^3)$ equations. From (4.40) we deduce

$$\partial_\tau^2 w^3_{MN} + (h^2 Q^2_{MN} - \sigma_o (M\pi)^2) w^3_{MN} - (M\pi)^2 \omega^2 f^3_{MN}$$

$$+ 2\, \partial_{\tau s} w^2_{MN} + 2\gamma \partial_\tau w^2_{MN} + \partial_s^2 w^1_{MN} + 2\, \partial_{\tau\eta} w^1_{MN}$$

$$+ 2\gamma \partial_s w^1_{MN} - \sigma_2 (M\pi)^2 w^1_{MN} = P^3_{MN} \tag{4.60a}$$

$$\omega^2 Q^2_{MN} f^3_{MN} + \omega^2 (M\pi)^2 f^3_{MN} = P^4_{MN} \tag{4.60b}$$

with

$$P^3_{MN} = J_{MN}[\partial_y^2 f_2\, \partial_x^2 w_1 + \partial_y^2 f_1\, \partial_x^2 w_2 - 2\partial_{xy} f_2\, \partial_{xy} w_1 - 2\partial_{xy} f_1\, \partial_{xy} w_2$$

$$+ \partial_x^2 f_2\, \partial_y^2 w_1 + \partial_x^2 f_1\, \partial_y^2 w_2 - 2\, \ell_2\, \partial_{xy} w_1]$$

$$P^4_{MN} = J_{MN}[2\partial_{xy} w_1\, \partial_{xy} w_2 - \partial_x^2 w_1\, \partial_y^2 w_2 - \partial_x^2 w_2\, \partial_y^2 w_1] .$$

(ℓ_2 is given by (4.39c).) Setting $R = P^3_{MN} + (M\pi/Q_{MN})^2 P^4_{MN}$ we have, since $\lambda_{MN} = 0$,

$$\partial^2_\tau w^3_{MN} + 2\partial_{\tau s} w^2_{MN} + 2\gamma\partial_\tau w^2_{MN} + 2\partial_{\tau\eta} w^1_{MN}$$
$$+ \partial^2_s w^1_{MN} + 2\gamma\partial_s w^1_{MN} - \sigma_2 (M\pi)^2 w^1_{MN} = R \tag{4.61}$$

We effectively define ϵ at this point by requiring that $\sigma_2 = 1$, and so $\sigma \sim \sigma_o + \epsilon^2$. Write $w^1_{MN} = a^1_{MN} = A$. Operating on equation (4.61) with M results in

$$\partial^2_s A + 2\gamma\partial_s A - (M\pi)^2 A = M[R] \tag{4.62}$$

The explicit calculation of M[R] is lengthy. Relevant details are indicated in Appendix B; we only cite the results here. Recall that, corresponding to the mode $\overline{Y}_{MN}$, there also exists a Fourier coefficient $\overline{w}^1_{MN} = \overline{a}^1_{MN} = B$. We find

$$\left.\begin{aligned} &\partial^2_s A + 2\gamma\partial_s A - (M\pi)^2 A + (c_1 A + c_2 B)e^{-2\gamma s} + k_1 A^3 + k_2 AB^2 = 0 \\ &\partial^2_s B + 2\gamma\partial_s B - (M\pi)^2 B + (c_2 A + c_1 B)e^{-2\gamma s} + k_2 A^2 B + k_1 B^3 = 0 \end{aligned}\right\} \tag{4.63}$$

where the c_j and k_j are constants with respect to τ and $s (j = 1, 2)$. Furthermore, the c_j depend on the initial conditions, but k_j do not. The expressions for the k_j are unwieldy, and it is not clear whether or not they are positive or negative or both.

We propose to analyze solutions of (4.63) in phase space. When $\gamma = 0$ the system is autonomous, but the equilibrium points (or critical points) depend on the initial conditions through the c_j. Consequently we will only be concerned with the more physical case $\gamma > 0$. Furthermore, we will only discuss bounded solutions insofar as unbounded solutions of this model do not depict large time behavior accurately. Denote ∂_s by a prime $'$ and introduce $a = \partial_s A$, $b = \partial_s B$, and $g = e^{-2\gamma s}$. Then solutions of (4.63) are contained among trajectories satisfying

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - g(c_1 A + c_2 B) - k_1 A^3 - k_2 A B^2 \\ B' &= b \\ b' &= (M\pi)^2 B - 2\gamma b - g(c_2 A + c_1 B) - k_2 A^2 B - k_1 B^3 \\ g' &= -2\gamma g \end{aligned}\right\} \quad (4.64)$$

This system is analogous to (4.32) which describes buckling of a rectangular plate. We argue that all tangent vectors are directed towards the AaBb hyperplane, and consequently the limit points of any trajectory lie within this hyperplane. As before, the limit points of a bounded trajectory constitute a closed connected set consisting of either a limit point, a limit cycle, or a separatrix connecting limit points [5]. Thus to describe large time behavior we are led to consider the reduced system

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - k_1 A^3 - k_2 AB^2 \\ B' &= b \\ b' &= (M\pi)^3 B - 2\gamma b - k_2 A^2 B - k_1 B^3 \end{aligned}\right\} \quad (4.65)$$

Remark that if $B = b = 0$ initially, a solution to (4.65) satisfies

$$A' = a$$
$$a' = (M\pi)^2 A - 2\gamma a - k_1 A^3$$
$$B' = 0$$
$$b' = 0$$

From this we conclude that $k_1 > 0$ if solutions are to be bounded. The qualitative behavior of (4.65) is unaltered under the transformation $A \to A/\sqrt{k_1}$, $a \to a/\sqrt{k_1}$, $B \to B/\sqrt{k_1}$, $b \to b/\sqrt{k_1}$. Setting $k_2/k_1 = \alpha$, one obtains the canonical form of (4.65)

$$\left.\begin{aligned} A' &= a \\ a' &= (M\pi)^2 A - 2\gamma a - A^3 - \alpha AB^2 \\ B' &= b \\ b' &= (M\pi)^2 B - 2\gamma b - \alpha A^2 B - B^3 \end{aligned}\right\} \quad (4.66)$$

To find equilibrium configurations of the shell, we investigate the critical points of (4.66). Necessarily $a = b = 0$, so the problem reduces to studying the characteristic exponents associated with roots of

$$A^3 + \alpha AB^2 = (M\pi)^2 A, \quad B^3 + \alpha A^2 B = (M\pi)^2 B$$

(i) <u>A=B=0</u>: There are two double exponents $\lambda_{1,2} = -\gamma + \sqrt{\gamma^2 + (M\pi)^2}$ and $\lambda_{3,4} = -\gamma - \sqrt{\gamma^2 + (M\pi)^2}$. This generalizes a saddle point; the origin is an unstable equilibrium point.

(ii) <u>A=0, B=±Mπ; A=±Mπ, B=0</u>: At each of these four points, the characteristic exponents are $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$ and $\lambda_{3,4} = -\gamma \pm \sqrt{\gamma^2 + (1-a)(M\pi)^2}$. $\mathrm{Re}\{\lambda_{1,2}\} < 0$. If $\alpha > 1$, $\mathrm{Re}\{\lambda_{3,4}\} < 0$ and these points are stable; if $a < 1$, $\lambda_3 > 0$ and $\lambda_4 < 0$ and these points are unstable.

(iii) $A \neq 0,\ B \neq 0$: Then we have

$$A^2 + \alpha B^2 = (M\pi)^2 \qquad B^2 + \alpha A^2 = (M\pi)^2 \ .$$

If $\alpha = 1$ there exists a one-dimensional locus of (non-isolated) critical points. This merely reflects the translational degeneracy of the solutions. These solutions contain case (ii).

If $\alpha \neq 1$, we solve to find $A^2 = B^2 = (M\pi)^2/(1+\alpha)$. For $\alpha \le -1$ no such solutions exist, but for $\alpha > -1$ four solutions exist. In the latter case, characteristic exponents satisfy

$$\lambda^2 + 2\gamma\lambda + 2(M\pi)^2 = 0$$

and

$$\lambda^2 + 2\gamma\lambda + 2(M\pi)^2(1-\alpha)/(1+\alpha) = 0$$

The first equation gives rise to roots $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2}$, so $\mathrm{Re}\{\lambda_{1,2}\} < 0$. The second equation has roots

$$\lambda_{3,4} = -\gamma \pm \sqrt{\gamma^2 - 2(M\pi)^2(1-\alpha)/(1+\alpha)}$$

For stability we must have $2(M\pi)^2(1-\alpha)/(1+\alpha) > 0$ or

$$1 > \alpha > -1$$

This is the opposite of case (ii). We classify the four solutions of (iii) as physically identical under translation.

Summarizing the behavior of (i)-(iii), we conclude that if solutions are bounded, there exists only one physical equilibrium configuration.

APPENDIX A: DERIVATION OF SHELL EQUATIONS

In this appendix we derive Donnell-type equations describing a circular cylindrical shell in equilibrium. In equilibrium the potential energy V has a stationary value

$$\delta V = 0 \tag{A.1}$$

The potential energy can be computed from the strain energy (or internal energy) S and the applied work W

$$V = S - W \tag{A.2}$$

We assume that the strain energy is the same as in linear theory, viz.

$$S = \frac{1}{2}\int \sigma_{ij}\,\varepsilon_{ij}\,d\underline{x} \tag{A.3}$$

Here σ_{ij} and ε_{ij} are the physical components of the stress and strain tensors, respectively; summation convention is used, and integration is carried out over the volume of the shell walls. Using tensor calculus one can derive [11] the exact relations

$$\left.\begin{aligned}
\varepsilon_r &= \frac{\partial w}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r}\right)^2\right] \\
\varepsilon_\theta &= \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2r^2}\left[\left(\frac{\partial w}{\partial \theta} - v\right)^2 + \left(\frac{\partial v}{\partial \theta} + w\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right] \\
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] \\
\varepsilon_{r\theta} &= \frac{1}{2r}\left[r\frac{\partial v}{\partial r} + \frac{\partial w}{\partial \theta} - v + \frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial \theta} - v\right) + \frac{\partial v}{\partial r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}\right] \\
\varepsilon_{rx} &= \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial u}{\partial x}\right] \\
\varepsilon_{\theta x} &= \frac{1}{2r}\left[r\frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial \theta} - v\right) + \frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial \theta}\right]
\end{aligned}\right\} \tag{A.4}$$

Here $\theta = y/r$ is the angle about the axis of the cylinder.

The stresses can be determined using Hooke's law, which becomes, for a homogeneous isotropic medium [17],

$$\sigma_{ij} = \lambda \Theta \delta_{ij} + 2\mu\varepsilon_{ij} \tag{A.5}$$

Here $\Theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ and the 1, 2, 3 directions refer to some cartesian coordinates. Also

$$\lambda = E\nu/(1+\nu)(1-2\nu) \qquad \mu = E/2(1+\nu)$$

where E is Young's modulus and ν is Poisson's ratio. A body is said to be in a state of plane stress parallel to the x_1, x_2 plane when $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. From (A.5) and $\sigma_{33} = 0$ one can deduce

$$\varepsilon_{33} = \frac{-\lambda}{\lambda+2\mu}(\varepsilon_{11} + \varepsilon_{22}) \tag{A.6}$$

From (A.6), it follows that

$$\left.\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \\ \sigma_{22} &= \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \\ \sigma_{12} &= \frac{E}{1+\nu}\varepsilon_{12} \end{aligned}\right\} \tag{A.7}$$

We assume that locally the cylinder is in a state of plane stress parallel to the tangent plane; hence (A.7) holds with

$$\sigma_{11} = \sigma_x \qquad \sigma_{22} = \sigma_\theta \qquad \sigma_{12} = \sigma_{x\theta} \qquad \sigma_{33} = \sigma_r$$

$$\varepsilon_{11} = \varepsilon_x \qquad \varepsilon_{22} = \varepsilon_\theta \qquad \varepsilon_{12} = \varepsilon_{x\theta}$$

Symmetry of the stress tensor yields

$$\sigma_{21} = \sigma_{12} \qquad \sigma_{31} = \sigma_{13} = 0 \qquad \sigma_{32} = \sigma_{23} = 0$$

We can now approximate (A.3) by

$$S = \frac{1}{2}\int (\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\, d\underline{x} \tag{A.8}$$

If we substitute (A.4) and (A.7) into (A.8), the result is still rather unwieldly. Assume further that displacements and their gradients are small compared to 1. If it is also assumed (on intuitive grounds) that most of the change occurs in the radial direction, one can argue that

$$\left|\frac{\partial w}{\partial y}\right| \gg \left|\frac{\partial u}{\partial y}\right| , \qquad \left|\frac{\partial w}{\partial x}\right| \gg \left|\frac{\partial v}{\partial x}\right|$$

Then (A.4) simplifies to

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$2\varepsilon_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial x} \tag{A.9}$$

Recall R designates the distance from the axis to the midsurface of the undeformed shell. Set r-R = ρ and expand, for small ρ,

$$\left.\begin{aligned} u(r,\theta,x) &\doteq u_0(\theta,x) + \rho\, u_1(\theta,x) \\ v(r,\theta,x) &\doteq v_0(\theta,x) + \rho\, v_1(\theta,x) \\ w(r,\theta,x) &\doteq w_0(\theta,x) \end{aligned}\right\} \qquad (A.10)$$

To find u_1 we use

$$0 = 2\varepsilon_{rx} \doteq \frac{\partial w}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial x} \doteq \frac{\partial w_0}{\partial x} + u_1 + v_1\frac{\partial v_0}{\partial x}$$

Since $v_1 \frac{\partial v_0}{\partial x}$ is small to second order, we have approximately

$$u_1 \doteq -\frac{\partial w_0}{\partial x} \qquad (A.11a)$$

Similarly, from $\varepsilon_{r\theta} = 0$ we conclude

$$v_1 \doteq -\frac{1}{r}\frac{\partial w_0}{\partial \theta} \qquad (A.11b)$$

(Note: $\varepsilon_{rx} = \varepsilon_{r\theta} = 0$ follows from $\sigma_{13} = \sigma_{23} = 0$ and Hooke's law.)

Approximate $1/r = 1/R$. Then (A.9, 10, 11) imply

$$\varepsilon_x = \varepsilon_{x0} + \rho\,\varepsilon_{x1} \qquad \varepsilon_\theta = \varepsilon_{\theta 0} + \rho\,\varepsilon_{\theta 1} \qquad \varepsilon_{\theta x} = \varepsilon_{\theta x 0} + \rho\,\varepsilon_{\theta x 1} \qquad (A.12a)$$

with

$$\begin{aligned} \varepsilon_{x0} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 & \varepsilon_{x1} &= -\frac{\partial^2 w_0}{\partial x^2} \\ \varepsilon_{\theta 0} &= \frac{w_0}{R} + \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{1}{2R^2}\left(\frac{\partial w_0}{\partial \theta}\right)^2 & \varepsilon_{\theta 1} &= -\frac{1}{R^2}\frac{\partial^2 w_0}{\partial \theta^2} \\ 2\varepsilon_{\theta x 0} &= \frac{\partial v_0}{\partial x} + \frac{1}{R}\frac{\partial u_0}{\partial \theta} + \frac{1}{R}\frac{\partial w_0}{\partial \theta}\frac{\partial w_0}{\partial x} & \varepsilon_{\theta x 1} &= -\frac{1}{R}\frac{\partial^2 w_0}{\partial \theta \partial x} \end{aligned} \qquad (A.12b)$$

For a shell of thickness h we have

$$S = \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{R-h/2}^{R+h/2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\, r\,dr\,d\theta\,dx$$

$$\doteq \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + 2\sigma_{x\theta}\varepsilon_{x\theta})\, R\,d\rho\,d\theta\,dx \qquad (A.8')$$

We substitute (A.7) and (A.12) into (A.8') and carry out the indicated integration in ρ; then we calculate δS and simplify by integrating by parts, using the boundary conditions

$$\left.\begin{aligned} w_0 &= w_{0,xx} = 0 \\ \sigma_{x0} &= -\sigma_0 = \text{constant} \\ v_0 &= 0 \end{aligned}\right\} \text{ at } x=0,L \tag{A.13}$$

The result is

$$\begin{aligned} \delta S = \frac{E}{1-\nu^2} Rh \int_0^L \int_0^{2\pi} d\theta\, dx \Big\{ \Big[& \frac{1}{R}\varepsilon_{\theta 0} - \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\varepsilon_{\theta 0}\frac{\partial w}{\partial\theta}\right) - \frac{1}{12}\frac{h^2}{R^2}\frac{\partial^2\varepsilon_{\theta 1}}{\partial\theta^2} \\ & + \frac{\nu}{R}\varepsilon_{x0} - \frac{\nu}{R^2}\frac{\partial}{\partial\theta}\left(\varepsilon_{x0}\frac{\partial w_0}{\partial\theta}\right) - \frac{\nu}{12}\frac{h^2}{R^2}\frac{\partial^2\varepsilon_{x1}}{\partial\theta^2} - \frac{\partial}{\partial x}\left(\varepsilon_{x0}\frac{\partial w_0}{\partial x}\right) \\ & - \nu\frac{\partial}{\partial x}\left(\varepsilon_{\theta 0}\frac{\partial w_0}{\partial x}\right) - \frac{1}{12}h^2\frac{\partial^2\varepsilon_{x1}}{\partial x^2} - \frac{\nu}{12}h^2\frac{\partial^2\varepsilon_{\theta 1}}{\partial x^2} \\ & - \frac{(1-\nu)}{R}\frac{\partial}{\partial\theta}\left(\varepsilon_{\theta x0}\frac{\partial w_0}{\partial x}\right) - \frac{(1-\nu)}{R}\frac{\partial}{\partial x}\left(\varepsilon_{\theta x0}\frac{\partial w_0}{\partial\theta}\right) \\ & - \frac{(1-\nu)}{12}\frac{h^2}{R}\frac{\partial^2\varepsilon_{\theta x1}}{\partial\theta\partial x}\Big]\delta w_0 - \Big[\frac{\partial}{\partial x}\varepsilon_{x0} + \nu\frac{\partial}{\partial x}\varepsilon_{\theta 0} + \frac{(1-\nu)}{R}\frac{\partial}{\partial\theta}\varepsilon_{\theta x0}\Big]\delta u_0 \\ & - \Big[\frac{1}{R}\frac{\partial}{\partial\theta}\varepsilon_{\theta 0} + \frac{\nu}{R}\frac{\partial}{\partial\theta}\varepsilon_{x0} + (1-\nu)\frac{\partial}{\partial x}\varepsilon_{\theta x0}\Big]\delta v_0 \Big\} \\ + \Big[\frac{E}{1-\nu^2} Rh \int_0^{2\pi} & (\varepsilon_{x0} + \nu\varepsilon_{\theta 0})\,\delta u_0\, d\theta\Big]\Big|_{x=0}^{x=L} \end{aligned} \tag{A.14}$$

But

$$\left[\frac{E}{1-\nu^2} Rh \int_0^{2\pi} (\varepsilon_{x0}+\nu\varepsilon_{\theta 0})\,\delta u_0\, d\theta\right]\Big|_{x=0}^{x=L} = \left[\int_0^{2\pi} \sigma_0\, \delta u_0\, h R\, d\theta\right]\Big|_{x=0}^{x=L} = \delta W$$

so that only the surface integral in (A.14) contributes to δV. The variations δu_0, δv_0, δw_0 are independent and give rise to three equations. Using Hooke's law (A.7) we can write these equations as

$$\frac{\partial}{\partial x}\sigma_{x0} + \frac{1}{R}\frac{\partial}{\partial\theta}\sigma_{\theta x0} = 0 \tag{A.15a}$$

$$\frac{1}{R}\frac{\partial}{\partial\theta}\sigma_{\theta 0} + \frac{\partial}{\partial x}\sigma_{\theta x0} = 0 \tag{A.15b}$$

$$\begin{aligned} & \frac{1}{R}\sigma_{\theta 0} - \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\sigma_{\theta 0}\frac{\partial w_0}{\partial\theta}\right) - \frac{\partial}{\partial x}\left(\sigma_{x0}\frac{\partial w_0}{\partial x}\right) - \frac{1}{R}\frac{\partial}{\partial\theta}\left(\sigma_{\theta x0}\frac{\partial w_0}{\partial x}\right) \\ & - \frac{1}{R}\frac{\partial}{\partial x}\left(\sigma_{\theta x0}\frac{\partial w_0}{\partial\theta}\right) - \frac{1}{12}\frac{E}{1-\nu^2}\Big[\frac{h^2}{R^2}\frac{\partial^2}{\partial\theta^2}(\varepsilon_{\theta 1}+\nu\varepsilon_{x1}) \\ & + h^2\frac{\partial^2}{\partial x^2}(\varepsilon_{x1}+\nu\varepsilon_{\theta 1})\Big] - \frac{1}{6}\frac{E}{1+\nu}\frac{h^2}{R}\frac{\partial^2}{\partial\theta\partial x}\varepsilon_{\theta x1} = 0 \end{aligned} \tag{A.15c}$$

We can use (A.12b) and (A.15a, b) to simplify (A.15c). Also let $y = R\theta$

and write $w = w_o$, $\sigma_x = \sigma_{xo}$, $\sigma_y = \sigma_{\theta o}$, $\sigma_{xy} = \sigma_{\theta xo}$. Then equations (A.15) become

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\sigma_{xy} = 0 \\
&\frac{\partial}{\partial y}\sigma_y + \frac{\partial}{\partial x}\sigma_{xy} = 0 \\
&\frac{h^2 E}{12(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w - \left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) \\
&\qquad + \frac{1}{R}\sigma_y = 0
\end{aligned}\right\} \quad (A.16)$$

APPENDIX B: SOME CALCULATIONS FOR CHAPTER 4

We indicate how to calculate

$$H = M[K] \tag{4.25b}$$

where

$$K = J[(\partial_y^2 f_2\, \partial_x^2 w_1 - 2\,\partial_{xy} f_2\, \partial_{xy} w_1 + \partial_x^2 f_2\, \partial_y^2 w_1 - \ell_2\, \partial_{xy} w_1)\, Y_{M1}] \quad . \tag{4.24b}$$

The operators M and K are defined in (4.26) and (4.27), respectively.

Recall the notation

$$s_k = \sin k\pi x \qquad c_k = \cos k\pi x$$

$$S_k = \sin k\omega y \qquad C_k = \cos k\omega y \quad .$$

Now

$$M \circ J[\partial_y^2 f_2\, \partial_x^2 w_1\, s_M\, S_1] =$$

$$M \circ J[(-\Sigma (\underset{\sim}{n}\omega)^2 f^2_{\underset{\sim}{m}\underset{\sim}{n}} s_{\underset{\sim}{m}} S_{\underset{\sim}{n}})(-\Sigma (\underset{\sim}{m}\pi)^2 w'_{\underset{\sim}{m}\underset{\sim}{n}} s_{\underset{\sim}{m}} S_{\underset{\sim}{n}})\, s_M S_1] =$$

$$\Sigma\, (\underset{\sim}{n}\omega)^2 (\underset{\sim}{m}\pi)^2\, M[f^2_{\underset{\sim}{m}\underset{\sim}{n}}\, w'_{\underset{\sim}{m}\underset{\sim}{n}}]\; J[s_{\underset{\sim}{m}} s_{\underset{\sim}{m}} s_M S_{\underset{\sim}{n}} S_{\underset{\sim}{n}} S_1] \tag{B.1}$$

Recall

$$Q^2_{\underset{\sim}{m}\underset{\sim}{n}} f^2_{\underset{\sim}{m}\underset{\sim}{n}} = J[(\partial_{xy} w_1\, \partial_{xy} w_1 - \partial_x^2 w_1\, \partial_y^2 w_1)\, s_{\underset{\sim}{m}} S_{\underset{\sim}{n}}] \tag{4.23}$$

Thus we are led to consider

$$M \circ J[(\partial_{xy} w_1\, \partial_{xy} w_1\, s_{\underset{\sim}{m}} S_{\underset{\sim}{n}})\, w'_{\underset{\sim}{m}\underset{\sim}{n}}] \tag{B.2}$$

When $(\underset{\sim}{m}, \underset{\sim}{n}) = (M, 1)$ we have

$$w'_{\underset{\sim}{m}\underset{\sim}{n}} = a'_{M1}(s, \eta, \ldots)$$

and (B.2) becomes

$$a'_{M1}\, M \circ J[\partial_{xy} w_1\, \partial_{xy} w_1\, s_{\underset{\sim}{m}} S_{\underset{\sim}{n}}] =$$

$$a'_{M1}\, M \circ J[(\Sigma\, w'_{m_1 n_1} (m_1\pi)(n_1\omega)\, c_{m_1} C_{n_1})(\Sigma\, w'_{m_2 n_2} (m_2\pi)(n_2\omega)\, c_{m_2} C_{n_2})\, s_{\underset{\sim}{m}} S_{\underset{\sim}{n}}] =$$

$$a'_{M1}\, \Sigma\, (m_1 m_2 \pi^2)(n_1 n_2 \omega^2)\, M[w'_{m_1 n_1} w'_{m_2 n_2}]\, J[c_{m_1} c_{m_2} s_{\underset{\sim}{m}} C_{n_1} C_{n_2} S_{\underset{\sim}{n}}] \tag{B.3}$$

Since

$$w'_{mn} = a'_{mn} \cos \lambda_{mn} \tau + b'_{mn} \sin \lambda_{mn} \tau \tag{4.16a}$$

it follows from the assumption

$$\lambda_{m_1 n_1} = \lambda_{m_2 n_2} \text{ if and only if } (m_1, n_1) = (m_2, n_2)$$

that

$$M[w'_{m_1 n_1} w'_{m_2 n_2}] = 0 \qquad (m_1, n_1) \neq (m_2, n_2)$$

$$M[w'_{mn} w'_{mn}] = \tfrac{1}{2}[(a'_{mn})^2 + (b'_{mn})^2] \qquad (m,n) \neq (M,1) \tag{B.4}$$

$$M[w'_{M1} w'_{M1}] = (a'_{M1})^2$$

Using (B.4) we simplify (B.3) to

$$a'_{M1} \sum (m\pi)^2 (n\omega)^2 M[(w'_{mn})^2] \; J[c^2_m s_{\underline{m}} C^2_n S_{\underline{n}}] \tag{B.3'}$$

From elementary trigonometric formulae one can show

$$\left.\begin{aligned}
4C_pC_qC_r &= C_{p+q+r} + C_{-p+q+r} + C_{p-q+r} + C_{p+q-r} \\
4S_pS_qS_r &= S_{-p+q+r} + S_{p-q+r} + S_{p+q-r} - S_{p+q+r} \\
4S_pS_qC_r &= C_{-p+q+r} + C_{p-q+r} - C_{p+q-r} - C_{p+q+r} \\
4C_pC_qS_r &= S_{p+q+r} + S_{-p+q+r} + S_{p-q+r} - S_{p+q-r}
\end{aligned}\right\} \tag{B.5}$$

This implies that

$$M[w'_{m_1 n_1} w'_{m_2 n_2} w'_{m_3 n_3}] \neq 0$$

only if a sum vanishes of the form

$$\pm\lambda_{m_1 n_1} \pm \lambda_{m_2 n_2} \pm \lambda_{m_3 n_3} = 0 \quad . \tag{B.6}$$

Clearly (B.6) holds if

$$\lambda_{m_1 n_1} = 0 \qquad \lambda_{m_2 n_2} = \lambda_{m_3 n_3} \quad .$$

We further assume that this is the only way (B.6) can be true. With this assumption we can evaluate (B.2) when $(\underline{m}, \underline{n}) \neq (M, 1)$. We are led to

$$\sum (m_1 m_2 \pi^2)(n_1 n_2 \omega^2) \, M[w'_{\underline{m}\underline{n}} w'_{m_1 n_1} w'_{m_2 n_2}] \, J[c_{m_1} c_{m_2} s_{\underline{m}} C_{n_1} C_{n_2} S_{\underline{n}}] \, . \tag{B.7}$$

Since $(m, n) \neq (M, 1)$, there are only two possible terms which can be non-vanishing, viz.

$$(m_1, n_1) = (\underline{m}, \underline{n}) \;, \quad (m_2, n_2) = (M, 1)$$

and $$(m_2, n_2) = (\underset{\sim}{m}, \underset{\sim}{n}) \;, \quad (m_1, n_1) = (M, 1) \quad .$$

Thus when (m, n) $\neq$ (M, 1) we find that (B.2) reduces to

$$a'_{M1}\left[(a'_{\underset{\sim}{m}\underset{\sim}{n}})^2 + (b'_{\underset{\sim}{m}\underset{\sim}{n}})^2\right](M\underset{\sim}{m}\pi^2)(\underset{\sim}{n}\omega^2)\, J\left[c_{\underset{\sim}{m}}c_M s_{\underset{\sim}{m}}\, C_{\underset{\sim}{n}} C_1 S_{\underset{\sim}{n}}\right] \quad . \qquad (B.7')$$

In a similar fashion we calculate

$$M \cdot J\left[\,(\partial_x^2 w_1\, \partial_y^2 w_1\, s_{\underset{\sim}{m}} S_{\underset{\sim}{n}})\, w'_{\underset{\sim}{m}\underset{\sim}{n}}\,\right] \quad .$$

This gives us

$$M\left[\, f^2_{\underset{\sim}{m}\underset{\sim}{n}}\, w'_{\underset{\sim}{m}\underset{\sim}{n}}\right]$$

which, as we see from (B.1), is the quantity necessary to calculate H.

The arguments needed to calculate equations (4.53), (4.54), (4.56), and (4.62) (describing the buckling of circular cylindrical shells) are completely analogous to the above arguments (for a rectangular plate) and can be carried out by the persevering reader.

APPENDIX C: NOTES ON THE MEMBRANE EQUATIONS

The reader is referred to the references [1, 2] for a derivation of the membrane theory; only the final formulation of the problem is given here. The midsurface of the undeformed membrane is generated by rotating a curve C about the axis of symmetry (see figure 1). This surface extends a distance R from the axis. The curve C can be described by prescribing the angle $\theta(r^*)$ between the normal to the surface (at distance r^* from the axis, $0 \le r^* \le R$) and the axis of rotation. We will assume that $\theta(0) = 0$ so that the membrane is not pointed at the apex. The surface is deformed by a pressure $p(r^*)$ which is normal to the midsurface; p is positive if it is directed toward the center of curvature. $\sigma_r(r^*)$ is the radial stress, h is the thickness, and E is Young's modulus. Then with the definitions

$$\left.\begin{aligned}
r &\equiv r^*/R\ , \qquad 0 \le r \le 1 \\
Q &\equiv \tfrac{1}{2}h^2R^2 \\
P &\equiv \max_{0 \le r \le 1}\ \frac{Q}{r}\left[\int_0^{Rr} \frac{p(\xi)}{E}\,\xi\ d\xi\right]^2 \\
\sigma(r) &\equiv \sigma_r(r^*)/EP^{1/3} \\
\phi(r) &\equiv \theta(r^*) \\
G(r) &\equiv \frac{Q}{Pr}\left[\int_0^{Rr} \frac{p(\xi)}{E}\,\xi\ d\xi\right]^2 \\
B &\equiv \tfrac{1}{2}P^{1/3}
\end{aligned}\right\} \qquad (C.1)$$

the problem of interest can be formulated in terms of dimensionless variables as follows:

$$\frac{d}{dr}\left(r^3\frac{d\sigma}{dr}\right) + \frac{G}{\sigma^2} = Br\phi^2 \qquad (C.2)$$

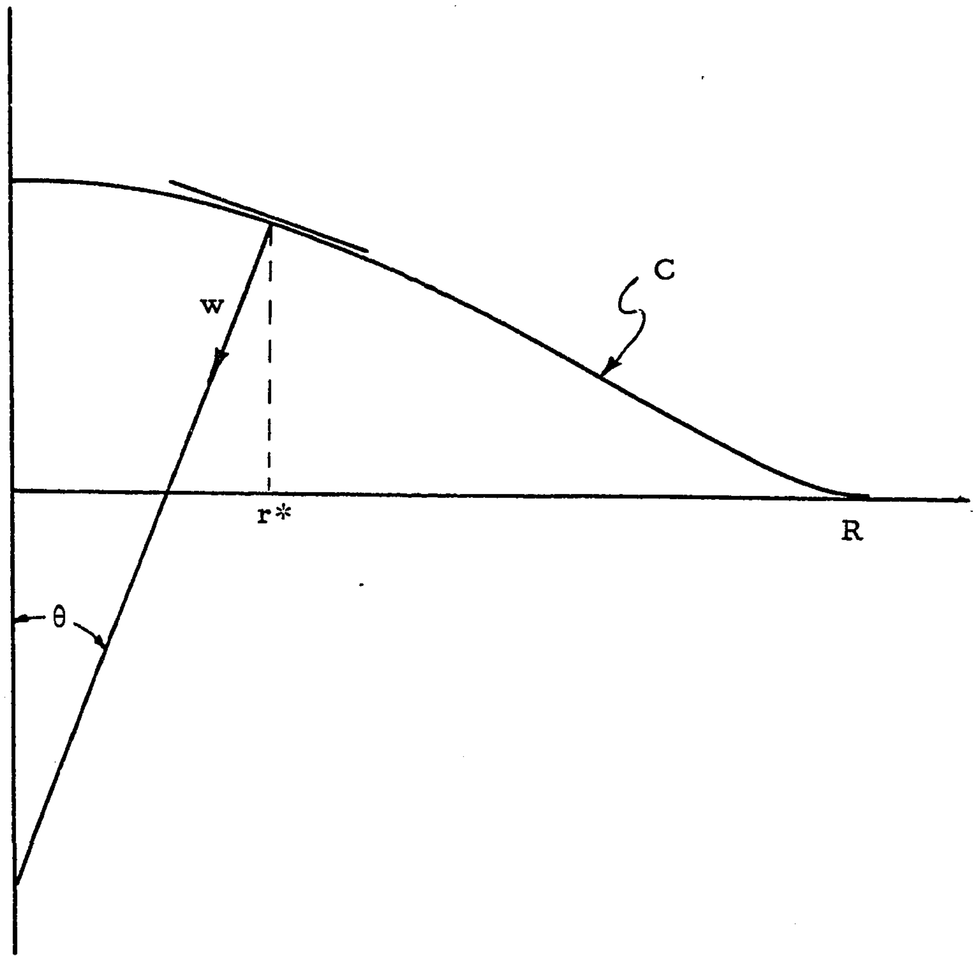


Figure 1 Geometry of the undeformed midsurface

Symmetry and boundedness of the stresses and displacements at $r = 0$ imply

$$|\sigma(0)| < \infty \tag{C.3a}$$

and

$$\frac{d\sigma}{dr} = 0 \quad \text{at} \quad r = 0 \tag{C.3b}$$

The prescribed radial stress at the edge yields the further boundary condition

$$\sigma(1) = -S \equiv T/EP^{1/3} \tag{C.4}$$

($T < 0$ when the stress is compressive.) The normal displacement of the midsurface $W(r^*) = Rw(r)$ can be regained from

$$\frac{dw}{dr} = \frac{-1}{\sigma}\left(2P^{1/3}\ \frac{G}{r}\right)^{\frac{1}{2}} - \phi \tag{C.5a}$$

$$w(1) = 0 \tag{C.5b}$$

Instead of studying σ directly we prefer to introduce

$$u \equiv S^{-1}\sigma + 1$$

and

$$\lambda \equiv S^{-1}$$

which results in the formulation

$$\frac{d}{dr}\left(r^3\ \frac{du}{dr}\right) + \lambda^3\ \frac{G}{(1-u)^2} = \lambda\, Br\phi^2 \tag{C.6}$$

The boundary conditions are then

$$\frac{du}{dr} = 0 \quad \text{at } r = 0 \tag{C.7a}$$

$$u(1) = 0 \tag{C.7b}$$

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Problem Solving Strategies in Learning Disabled and Normal Achieving Children

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by

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DEDICATION

To my

wife, Brooke, for her loving patience and to

my little daughter, Marianna, whose smile made it all worthwile.

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ABSTRACT OF THE DISSERTATION

Problem Solving Strategies
in Learning Disabled and Normal Achieving Children

by

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This investigation involved the ontogeny of problem solving strategies in learning disabled and normal achieving children. The study was focused on two processes, one having to do with aspects of categorical storage of information, the other with the use of categories in problem solving situations. Both may be relevant to academic problem solving and thus of importance to educational programming, especially for children with school achievement problems. An associative clustering test was used to measure the number of concepts stored and efficiency of their storage in categories, a twenty-questions test served as a measure of categorical retrieval and labeling in a problem solving situation, and transfer of categorical organization to new problem solving tasks was measured by a second associative clustering test and by immediate

and delayed classical sorting tasks.

Three suburban public school districts provided the subject pool for the study. Subjects were selected from upper elementary self-contained special education classrooms for learning disabled children. Normal achieving children selected as comparison subjects were enrolled in the same neighborhood schools as the learning disabled children with whom they were compared. Children composing the comparison group were matched individualy with learning disabled children on chronological age, ethnicity, sex, and where possible on IQ.

On the operational measures of number of concepts stored and efficiency of their categorical storage, learning disabled and normal achieving children performed equally. However, learning disabled children performed less adequately than normal achievers on the problem solving task designed to measure retrieval and labeling of conceptual categories. When this problem solving task was ordered categorically, the performance of learning disabled children approached that of normal achieving children, and equaled the performance of normal achievers when in-task categorical order was provided with verbal, visual and motoric cues. Operational measures of transfer of categorical organization showed no effects for categorical storage in either learning disabled or normal achieving children. On measures of immediate and delayed transfer of categorical organization to new problem solving tasks tapping retrieval of conceptual categories, learning disabled children showed significant effects as a function of in-task categorical ordering and cues. Normal achievers showed no such effects.

These results were inferred to suggest that learning disabled children acquire and categorically store information equally with normal achieving children, but differ from normal achievers by failing to retrieve and label conceptual categories as a strategy in problem solving under some circumstances. However, learning disabled children appear to approach normal achievers in the use and transfer of this strategy when the problem solving task is ordered categorically. Further, they seem to equal normal achievers in the use of categories as a strategy and in transfer effects when cues are added to the in-task categorical order. In essence, study findings suggest that learning disabled children have acquired an average amount of information and have organized it, but do not use it within the same context or in response to the same cues as the normal achiever does. Learning disabled children apparently require a highly structured context provided with cues if they are to use information as well as normal achieving children do.

In general, then, an important implication of the findings in this study is that contexts and cues presented by standard educational assessment and programming may not be optimal for tapping learning potential and for promoting learning in children identified as learning disabled. Consequently, educators may be identifying some children as deficient learners who are simply different learners compared with normal achievers. It also follows that children labeled as learning disabled may require modes of instruction different from those that are seemingly optimal for the majority of school children. If children classified as learning disabled are simply children who learn differently from the majority, then it may be that typical patterns of educational assessment and programming present these children with unwarranted barriers to academic learning.

Chapter I

REVIEW OF LITERATURE

Statement of the Problem

This investigation involved problem solving strategies in learning disabled (LD) and normal achieving children. The study was focused on two processes, one having to do with aspects of categorical storage of information, the other with the use of categories in problem solving situations. Both may be relevant to academic problem solving and thus of importance to educational programming, especially for children with school achievement problems.

Learning disabled (LD) children are, by definition, children with intelligence above the mentally retarded range as expressed by IQ, but who exhibit inadequate academic performance both in the classroom and on standardized academic achievement tests (National Advisory Committee, 1968; California Education Code, Section 6750, 1969). Typically, LD children score within limits defined as normal on IQ tests, suggesting adequate acquistion of information but inefficient use of information in academic settings.

This puzzling discrepancy between apparent intellectual ability and academic performance, the most general charateristic of LD children (Bateman, 1964; Keogh, 1970; Myers & Hammill, 1969), has traditionally been analyzed using a profile of psychometric indices (Hammill & Bartel, 1975). These indices may be viewed as operational measures of <u>how much</u> the child has or lacks of particular abilities believed to be critical to learning in school. However, such scores fail in themselves to describe <u>how</u> the child processes environmental information (Keogh, 1972). Researchers have recently stressed the importance of analyzing processes used by normal (Gibson, 1965, Junkala, 1973) and LD children (Torgesen, 1975) in academic learning situations. Torgesen's (1975) recent review of research on the development of memory, attention, perception, and learning supports the interpretation that LD children may perform inadequately on academic tasks as a result of failure to use efficient problem solving task strategies. Thus, it seems reasonable to assume that approaches to understanding processes used by LD children in solving problems may help to clarify the ability-performance discrepancy typical of these children.

One approach may be to analyze this ability-performance discrepancy in LD children in terms of categorical storage and usage (retrieval) of information in problem solving situations. LD children may acquire everyday concepts adequately as evidenced by their IQ scores within the normal range, yet not store them efficiently in categories (Haight, 1974). By consequence, they may perform poorly in problem solving that involves use of categories. Alternatively, LD children may be efficient in both acquisition and categorical storage of everyday concepts, but fail to retrieve these categories when solving problems. Problem solving performance of LD children on academic tasks involving conceptual categories may fall below expectancy for either of two reasons: (1) inefficient categorical storage of everyday concepts, or (2) inefficient use (categorical retrieval) of everyday concepts adequately stored in categories. Investigation of these two possibilities represents the focus of this study.

Background and Rationale

Definition of Terms. The terms "concept," "category," and "process" are central to the problem under investigation and therefore are defined here for the purposes of the study. Consistent with several theorists (Neisser, 1967; Vinacke, 1954; Vygotsky, 1962), a concept is considered to be a class of individual environmental stimuli having certain features or attributes in common, while a category is simply considered to be a superordinate class of individual concepts. Concepts and categories are also considered to be the units of intellectual activity employed during problem solving (Sigel, 1964). Such intellectual activity defines the term process as applied here. Vygotsky's (1962) emphasis on systematic hierarchical organization of concepts as a prerequisite to intellectual operations is in agreement with this definition of process, as is Blount's (1968) equating of process with strategies employed in the formation and use of concepts.

Theoretical Foundation. Vygotsky's (1962) analysis of mental organization is appropiate for understanding problem solving abilities of normal achieving and LD children in two important respects. First, it distinguishes between spontaneous concepts and scientific concepts; and second, it clarifies the functional significance of systematic relationships among concepts.

According to Vygotsky, spontaneous concepts are formed by the child from everyday life experiences, while scientific concepts are learned in school through instruction in systems of hierarchically organized abstract concepts. In Vygotsky's words, "... the absence of a system is the cardinal psychological difference distinguishing spontaneous from scientific concepts." (Vygotsky, 1962, p.116) Although spontaneous and scientific concepts develop from different sources, Vygotsky held that they interact during the course of the child's conceptual development, forming a single developmental pattern. Spontaneous concepts, which are "...saturated with experience," (Vygotsky, 1962, p.108) contribute the body and vitality of content, while scientific concepts supply systematic hierarchical organization. Scientific concepts act as powerful influences determining the school child's mental development.

> The formal discipline of scientific concepts gradually transforms the structure of the child's spontaneous concepts and helps organize them into a system; this furthers the child's ascent to higher developmental levels. (Vygotsky, 1962, p.116)

An example of systematically restructured spontaneous concepts is given by Vygotsky in the case of the spontaneous concepts "flower" and "rose," which originally have the same level of generality for the young child, the former not being understood by the child as superordinate to the latter. Only later in the child's development do the concepts "flower" and "rose" assume the hierarchical relationship of superordinate and subordinate, respectively. "Flower" therefore becomes the construct of greater generality, encompassing not only all roses but all species of flowers and individual flowers generally.

Systematization of spontaneous concepts accomplished through interaction with scientific concepts learned in school has, according to Vygotsky, the effect of making conscious intellectual operations (processes) available to the child:

> To us it seems obvious that a concept can become subject to conciousness and deliberate control only when it is part of a system. If consciousness means generalization, generalization in turn means the formation of a superordinate concept that includes the given concept as a particular case. A superordinate concept implies the existence of a series of subordinate concepts, and it also presupposes a hierarchy of concepts of different levels of generality. (Vygotsky, 1962, p.92)

> Concepts do not lie in the child's mind like peas in a bag, without any bonds between them. If that were the case, no intellectual operation requiring coordination of thought would be possible, nor any concepts as such could exist; their very nature presupposes a system. (Vygotsky, 1962, p. 110-111)

Based on the preceding consideration of spontaneous concepts, scientific concepts, and intellectual operation (processes) it is possible to restate the focus of this study in terms of Vygotsky's theory. Thus, the discrepancy in LD children between intellectual ability expressed on IQ tests and school performance evidenced by academic achievement tests may be a reflection of Vygotsky's distinction between spontaneous and scientific concepts (Haight, 1974). That is, IQ tests, on which LD children typically score within limits defined as normal, may tap the store of spontaneously formed concepts that a child brings to the school setting; while academic achievement tests, on which these children score below expectancy, may be measures of scientific concepts acquired at school.

If this assumption is admitted, two plausible but certainly not independent arguments regarding the ability-performance discrepancy in LD children may follow. The first is an inference by Haight (1974) that LD children have acquired the expected number of spontaneous concepts (as reflected by their IQ scores that typically fall in the normal range), but are inefficient in categorical storage of these concepts (as reflected in their poor academic achievement test scores). This argument suggests that the ability-performance discrepancy in LD children may be a consequence of difficulty with processes at the input organization stage of learning. Stated more fully in Vygotsky's terms LD children may fail on measures of academic achievement because their spontaneous concepts have not been efficiently stored in systematic, hierarchical categories through interaction with scientific concepts; thus the prerequisite mental organization for categorical intellectual operations (processes), such as academic problem solving, has not been attained. Alternatively, it may be argued that LD children have acquired and categorically stored spontaneous concepts at expected levels (as reflected by their IQ scores in the normal range), but are inefficient in the use (categorical retrieval) of these concepts (as reflected in their poor academic achievement test scores). This second argument clearly implicates difficulties with retrieval processes in the ability-performance discrepancy of LD children. Stating this argument more fully in Vygotsky's terms, LD children may fail on measures of academic achievement because they are inefficient in the use (categorical retrieval) of spontaneous concepts stored in systematic hierarchical categories through interaction with scientific concepts; thus the prerequisite mental organization for categorical intellectual operations (processes), such as academic problem solving, has been attained by LD children, but is not readily accessible to them for use.

Related Research

Categorical storage of information and use of categories have been studied extensively in normal children (Jablonski, 1974) and in mildly retarded children and adolescents (Blount, 1968). Little study has been devoted to storage and use of categories in LD children (Haight, 1974). Therefore, a review of related research in normal and mentally retarded children is appropriate to place the limited knowledge of the processes of categorical storage and usage in LD children in meaningful context.

Normal Children. Jablonski (1974) reviewed investigations of free recall in normal children in which individual words belonging to several categories were presented in random order for immediate recall. Results consistently showed a tendency in children to recall items from the same category in consecutive order at higher than chance frequencies, despite the random input order. This spontaneous inclination to recall randomly arranged items in categorical clusters was first reported by Bousfield (1953) in adults. He interpeted this tendency to be a reflection of mental organization. The term "associative clustering" has been used to describe this spontaneous disposition (Blount, 1968). Research findings have shown associative clustering to be a common phenomenon in children for diverse types of stimuli (Jablonski, 1974), and also general in adults regardless of number of presented categories, number of items per category, formula used for calculating associative clustering (Haight, 1974), or stimulus mode (Bousfield & Cohen, 1956). Associative clustering has also been identified as a developmental phenomenon, since number of items recalled and amount of clustering both increase in children with increasing chronological age (Jablonski, 1974). According to Jablonski 1974, results from a number of studies using experimenter introduced mediational techniques have demonstrated increased immediate recall and clustering in normal children. This was true when stimulus items were organized in blocks according to conceptual category at the input stage by the experimenter, and also when categorical retrieval cues were provided at the recall stage. Jablonski interpreted results from several of these studies (Cole, Frankel, & Sharp, 1971; Hagen, 1971; Nelson, 1969; Thurm & Glanzer, 1971) to suggest that developmental increases in recall and clustering may reflect underlying development of "schemes" or strategies for retrieval from long-term storage, such as subjective rehearsal strategies for chunking information. Consistent with this possibility is Flavell's (1970) contention that normal young children manifest a production deficiency in recall in which mediating strategies for enhancing recall are available to the child, but are not spontaneously used without prompting.

In short, based on present evidence, it seems reasonable to state that deficient (developmentally immature) recall and clustering patterns typically found in normal young children may implicate both input and retrieval processes. More specifically, these deficiencies may in part reflect an inability to use mediational strategies at the input stage of learning without the aid of short-term memory cues such as arranging stimuli in categorical blocks, but simultaneoulsy may reflect an inability to use mediational strategies for categorical retrieval of information from long-term storage without the aid of such prompting as instructions during recall to chunk (categorize) similar items. This conclusion is in agreement with Hagen's (1971) suggestion that both short-term and long-term memory develop as a function of changes in rehearsal strategies.

Whereas associative clustering has been used to study categorical mental organization (storage) of spontaneously formed concepts (Bousfield, 1953; Blount, 1968; Jablonski, 1974), the spontaneous tendency to ask categorical questions in a problem solving situation has been employed to study the use (retrieval and labeling) of categorically stored information (Ault, 1973; Mosher & Hornsby, 1966; Van Horn & Bartz, 1968). Mosher and Hornsby (1966) studied problem solving strategies in normal children using the parlor game of twenty-questions. This game is composed of an array of pictures of common objects belonging to various categories arranged in rows and columns. Rules of the game required the child to discover a picture in the array known only to the experimenter by asking questions that could be answered simply with a "yes" or "no." A categorical question including several stimulus pictures was considered an efficient question as it encompassed several possible solutions at once. These questions were called "constraint seeking" questions, were accepted as evidence of a constraint seeking strategy, and inferred to reflect categorical mental organization. By contrast, a question including only one picture at a time was considered less efficient as it included but one possible solution and gave no evidence of categorical mental organization. This type of question was called an "hypothesis seeking" question and taken as evidence of an hypothesis seeking strategy. A developmental continuum was found by Mosher and Hornsby in which six year old children asked only hypothesis seeking questions, eight year old children asked both hypothesis seeking and constraint seeking questions, while eleven year olds used constraint seeking questions almost exclusively. These investigators held that hierarchically arranged categorical mental organization of concepts is a prerequisite to the use of constraint seeking questions in problem solving. Van Horn and Bartz (1968) demonstrated that ordering the array of pictures in a twenty-questions game by clustering the stimuli into categorical groups promoted constraint seeking questions in normal six year old children. Similarly, Ault (1973) found that categorically ordering the array on a modified twenty-questions game aided the production of constraint seeking questions in first and third grade children, but not in fifth grade children.

Comparing results from constraint seeking problem solving studies with those from associative clustering investigations is instructive. Increasing the use of constraint seeking questions in young children by categorically ordering the problem solving task clearly parallels findings from associative clustering studies in which categorically cued recall increased clustering behavior in young children. Stated differently, categorical cuing at the recall stage has been found to promote spontaneous use of categories as a strategy in immediate free recall and categorical retrieval and labeling as a strategy in constraint seeking problem solving. In Flavell's (1970) terminology, these findings from associative clustering and problem solving research may be interpreted as evidence to support the notion that young children typically manifest a production defeciency; that is, young children may have mediators available for categorical retrieval and labeling of spontaneous concepts, but need prompting to use them as a strategy in free recall and problem solving situations.

Mentally Retarded Children. Blount (1968) reviewed the literature dealing with storage and use of conceptual categories in mildly retarded children and adolescents and found evidence to support several conclusions: 1) retardates have concepts available to them; 2) they may not be able to use concepts efficiently, especially when verbal labels are required; and 3) they can perform as well as or better than MA controls when their attention is focused on the relevant variables by some manipulation of the learning situation. Blount's conclusion that retardates may be unable to use concepts efficiently is consistent with an inference by Spitz (1966) that learning problems in retardates result from their failure to organize input information. This inference that retardates fail to organize, and by implication store, input information is strengthened by the finding that associative clustering in retardates is inefficient (Blount, 1968; Haight, 1974; Spitz, 1966), and thus by inference (Bousfield, 1953) that retardates have inefficient mental organization (storage) of concepts.

Findings from a number of the studies reviewed by Blount demonstrated that associative clustering is weak in retardates, but that mediational techniques introduced by the experimenter during either input or retrieval promote increased clustering in retarded children and adolescents during immediate recall. One of these studies (Gerjouy & Spitz, 1966) is particularly illustrative, as clustering and recall were significantly increased by mediation at both input and retrieval stages of learning in the same adolescent retardates using the same task. The specific mediations used were presenting items categorically blocked during input, and requesting items by category name during retrieval.

Summarizing the above findings and inferences, it may be stated that retardates have acquired concepts, but have not organized them efficiently in categorical storage for easy retrieval; and, that categorical mediators provided by the experimenter during either the input stage of learning or the retrieval stage significantly increase use of categories as a strategy during immediate free recall.

In an investigation of transfer of categorical clustering effects in free recall for adolescent retardates, Bilsky (1976) found that categorical blocking of stimulus items plus use of categorical labeling at the input stage produced significantly increased clustering and recall and a trend toward transfer of clustering effect to randomly organized lists composed of new categories. Blocking alone and blocking with instructions to cluster at the input stage did not result in comparable increases in clustering or transfer of clustering. In reviewing earlier clustering research, Bilsky (1976) pointed out that "... brief practice with a blocked list is not sufficient to establish a generalized set to employ a clustering strategy." (Bilsky, 1976, p.589) The increase of clustering and a transfer effect for clustering with addition of verbal labeling to categorical blocking is consistent with Blount's (1968) conclusion that retardates use concepts inefficiently particularly when verbal labels are required. That is, Bilsky's (1976) provision of categorical labels as a supplement to categorical blocking evidently compensated in part for the inefficient use of concepts in retardates especially when verbalization is involved as noted by Blount. Labeling and other forms of verbal madiation have also been used to increase retardates' performance in motor learning (Cantor & Hottel, 1957), discrimination learning (Smith & Means, 1961; Dickerson, Girardeau, & Spradlin, 1964), paired-associate learning (Cantor & Ryan, 1962; Vergason, 1964; Jensen & Rohwer, 1963; MacMillan, 1970a, 1972), and serial learning (MacMillan, 1970b, 1970c; Ryan 1969a, 1969b). One of MacMillan's (1970c) studies of serial learning is particularly important for clarifying the facilitating role of verbal mediation in categorical storage and retrieval by retardates. In this study MacMillan demonstrated that immediate recall for a series of visually presented digits in educable mentally retarded children was significantly increased when input was organized by grouping the digits as higher-decade numbers and by simultaneously requiring the subject to verbalize these higher-decade numbers prior to recall. Presenting digits ungrouped for verbalization as single digits did not increase recall, while presenting digits grouped as higher-decade numbers for verbalization as single digits facilitated recall at an intermediate level. Since higher-decade numbers are meaningful numerical categories, MacMillan's use of grouped digits and verbal labeling by subjects may be a mediational technique functionally analogous to Bilsky's (1976) use of categorically blocked word lists plus experimenter verbal labeling. Whereas both Bilsky and MacMillan employed categorical blocking and verbal categorical labels during input, Gerjouy and Spitz (1966) demonstrated increased immediate recall and clustering in retardates using verbal labels alone during the retrieval stage. The marked facilitating effect of verbal mediations on categorical use of concepts in retardates seems well established by the accumulation of evidence.

The foregoing studies of conceptual category usage in retardates dealt with immediate recall performance. Other studies (Blount, 1968) of conceptual category use in retardates have employed various sorting techniques tapping categorical retrieval of concepts from long-term memory. A number of these studies support the conclusions that retardates 1) have concepts available in long-term storage, 2) can use these concepts as well as equal MA controls in tasks requiring categorical retrieval, 3) have difficulty in verbalization of these conceptual categories, but 4) can use verbal cues such as category labels when they are provided by the experimenter.

In sum, it seems that retardates have concepts but are inefficient in categorical retrieval; and that categorical mediators (especially verbal mediators) provided by the experimenter during the input or retrieval stage of learning facilitate use of conceptual categories in retardates as a strategy in immediate free recall. Retardates apparently behave similarly to younger normal achieving children in their categorical storage and retrieval of concepts. Stated differently, these two groups of children appear equivalent in their failure to use categorical mediating strategies as an aid in conceptual learning at both input and retrieval stages, unless prompted to do so by some manipulation of the learning situation. Using Flavell's (1970) terminology, retardates and young children evidence a production deficiency at both input and retrieval stages.

Learning Disabled (LD) Children. According to Haight (1974), studies of concept usage in learning disabled (LD) children are practically nonexistent, but a few investigations have demonstrated atypical use of concepts in school children with characteristics similar to LD children. For example, Levy and Cuddy (1956) found that normal fourth grade students behind in school achievement performed less efficiently than normal achieving controls on a concept formation test using blocks; and Blank, Weider, and Bridger (1968) compared normal achieving first grade students with first grade poor readers on several coding tasks, finding a difference between the two groups only when verbal abstraction was involved. Haight (1974) inferred from her study of concept formation and concept usage in LD subjects, that these children acquire as many concepts from everyday life experiences as do normal achievers, but are inefficient in categorical storage of these concepts. Haight also inferred in the same investigation that LD children perform similarily to normal achievers on concept usage tasks involving categorical retrieval when the learning situation is provided with cues, context, and structure. In addition, she presented evidence that retarded and LD children verbalize fewer and less varied solutions than do normal achievers on a concept formation task. Based on these conclusions, LD children appear similar to normal achieving peers in number of spontaneous concepts acquired; but they apparently resemble retardates and younger normal achieving children in their inefficient categorical storage of spontaneous concepts, their tendency to use categorical retrieval only when cues, context, and structure are provided in the learning situation, and seem especially similar to retardates in their inefficient verbal ability as related to several aspects of conceptual thought.

Summary

Categorical storage of concepts and use of conceptual categories in learning disabled (LD), normal achieving, and retarded children were discussed in relation to strategies for recall and problem solving employed by these children. Difficulties with categorical storage of concepts and usage of conceptual categories were considered possible alternative explanations for inadequate problem solving strategies in LD children. Use of inadequate problem solving strategies with academic tasks was considered a plausible explanation for the characteristic discrepancy in LD children between ability as expressed by IQ and performance as evidenced in academic achievement. More specifically, it was argued that this ability-performance discrepancy in LD children may reflect Vygotsky's (1962) distinction between spontaneous concepts (learned from everyday life experiences) and scientific concepts (abstract, hierarchically organized conceptual categories learned in school). Two alternative arguments were stated: 1) that LD children acquire as many spontaneous concepts (reflected by IQ scores in the normal range) as do normal achieving children, but fail to store these concepts efficiently in hierarchical conceptual categories (reflected by depressed academic achievement test scores); or 2) that LD children acquire and store spontaneous concepts in abstract hierarchical categories (reflected by IQ scores in the normal range) as well as do normally achieving children, but fail to retrieve these categories efficiently (reflected by depressed academic achievement test scores). Either disturbance (inefficient categorical storage or inefficient categorical retrieval) was considered a failure to establish a prerequisite condition for use of categorical problem solving strategies in school and thus to be an important factor contributing to the ability-performance discrepancy in LD children.

A review of the literature was interpreted to suggest that retarded children and normal children of the same mental age acquire spontaneous concepts, but categorically store and retrieve these concepts inefficiently. Both groups of children were found to increase use of conceptual categories as a mediating strategy when the learning situation was presented with categorical structure and cues (especially verbal labels) during acquisition or retrieval. From findings in the only investigation of concepts usage in children identified as LD, it was inferred that these children acquire spontaneous concepts equally with normal achieving children of the same chronological age, but are inefficient in storing these concepts categorically. In this same study, LD children were shown to perform similarly to their normal achieving peers in categorical retrieval when a concept usage task was provided with cues, context and structure. Therefore, LD children would appear to be similar to normal achieving peers in amount of everyday conceptual information acquired, but similar to retardates and younger normal achieving children in categorical storage of information and the retrieval and labeling of categories as a strategy in a learning situation.

Major Questions

Two alternative but related questions were proposed for investigation. The first was whether LD children acquire as many spontaneous concepts as do normal achieving children, but fail to store these concepts systematically in hierarchical categories. This infers a lack of the prerequisite mental organization for use of problem solving strategies involving coordination of hierarchical conceptual categories. The second was whether LD children acquire and store spontaneous concepts systematically in hierarchical categories equally with normally achieving children, but fail to retrieve these categories efficiently, and, therefore, tend not to use problem solving strategies that involve coordination of hierarchical conceptual categories.

Chapter II

METHODS

Plan of Study

Problem solving strategies in learning disabled (LD) and normal achieving (NA) children was assessed using operational measures of categorical storage, retrieval, labeling, and transfer of spontaneously formed concepts. An associative clustering test was used to measure the number of spontaneous concepts stored and the efficiency of their storage in categories; retrieval and labeling of categorically stored spontaneous concepts were measured with a twenty-questions test; while transfer effects for categorical organization of spontaneous concepts were measured with an associative clustering test and with immediate and delayed sorting tasks.

Sample

Schools. Three geographically contiguous suburban public school districts in Los Angeles County provided the subject pool for this study. These three medium-sized districts, the Charter Oak Unified School District (total student population of 7,856), the Covina Valley Unified School District (total student population of 14,038), and the West Covina Unified School District (total student population of 10,850) were similar in socio-economic status and ethnic respresentation. The West Covina figures for ethnic populations (1975-76) were representative for all three districts: Indian (American or Alaskan native) 0.41%, Oriental (Asian or Pacific Islanders) 1.60%, Black (not of Hispanic origin) 5.66%, Hispanic (Mexican, Puerto Rican, Cuban, Central or South

American, or Spanish Culture) 16.01% and Caucasian (not of Hispanic origin) 76.31%. Total minority population for the district (1975-76 survey) was 23.68%.

Learning Disabled (LD) Subjects. LD children from this subject pool were drawn from upper elementary self-contained special education classrooms in neighborhood schools of the three districts. Based on practical administrative considerations within each participating district, six classrooms were made available for investigation, one from the Charter Oak Unified School District, two from the Covina Valley Unified School District, and three from the West Covina Unified School District. The districts' criteria for placement in these classrooms conformed with Section 6750 of the California Education Code requirements for pupil admission.

All children in these classrooms were scheduled for testing except for those children with known sensory impairment or with severe emotional disturbance. Mean chronological age of children composing the LD subsample was 11.31 years (N=30, SD=0.97, Range =9.67-13.17) and mean I.Q. was 90.68 (N=25, SD=11.19, and Range=70-117). I.Q. scores for five LD children were unavailable to the experimenter. Ethnic representation was 1 Black, 5 Hispanic, and 24 Caucasian. Three LD subjects were females and 27 were males.

Normal Achieving (NA) Subjects. NA children selected as comparison subjects were enrolled in the same upper elementary level neighborhood school as the LD children with whom they were compared. Children composing the comparison group (N=30) were matched individually with the LD children on chronological age, ethnicity, sex, and where possible on IQ. Matching was conducted by principals and/or teachers of LD and NA children participating in the study. Mean chronological age was 11.9 years (N=30, SD=0.73 and Range=9.83-12.75). Ethnic representation was 2 Oriental, 1 Black, 4 Hispanic, and 23 Caucasian. Five NA subjects were females and 25 were males. Matching patterns achieved between LD and NA samples for chronological age, ethnicity, and sex (Table 1) suggest similarity of samples on these characteristics.

Only 5 IQ scores for NA subjects were available to the experimenter; 3 were in the average range and 2 were in the below average range. Ability estimates of 20 NA children were summarized by school personnel in terms of classifications of above average, average, and below average. Grouping all available ability estimates for NA subjects, 7 fell in the above average range, 9 fell in the average range, 1 in the low average range, and 8 fell in the below average range. As all NA pupils were in regular classes, the "below average" estimates were not interpreted to suggest retardation.

Assignment to Treatments. LD and NA subjects were systematically assigned in equal numbers to three treatment groups. The treatment groups were for the purpose of administering three versions of a visual-verbal twenty-questions test in which 24 stimulus pictures of common items were presented in an array randomly arranged, in an array ordered by categorical clusters, and in an array ordered by identified categorical clusters. The three experimental groups were named respectively random, clustered, and identified clustered. Subjects were assigned to the three groups sequentially as they were released by teachers for testing in a repeated pattern of random, clustered, identified clus-

TABLE 1

Comparison of Learning Disabled (LD) and Normal Achieving (NA) Subjects on Independent Variables of Chronological Age, Ethnicity, and Sex

Independent Variables	Group LD	Group NA
	(N=30)	(N=30)
Chronological Age	$\bar{X}$=11.31 SD= 0.97 Range= 9.67-13.17	$\bar{X}$=11.19 SD= 0.73 Range= 9.83-12.75
Ethnicity		
Indian (American or Alaskan native)	0	0
Oriental (Asian or Pacific Islanders)	0	2
Black (not of Hispanic origin)	1	1
Hispanic (Mexican, Puerto Rican, Cuban, Central or South American, or Spanish Culture)	5	4
Caucasian (Not of Hispanic origin)	24	23
Sex		
Female	3	5
Male	27	25

tered; random, clustered, ... This assignment pattern was adopted to negate possible teacher introduced or other circumstantial biases reflected in the order of presentation of subjects for testing. When a subject listed for testing was absent, the next subject in line was moved up to take his place.

CA and IQ differences between subsamples assigned to treatment groups (Table 2) were tested with analyses of variance (Table 3). All differences for class (LD and NA) and for treatment (random, clustered, and identified clustered) were non-significant. There was no interaction between class and treatment for CA. Subgroups were, thus, considered comparable.

Procedure

Test Instruments. Three test instruments were used in this study. The first was an associative clustering test (a presumed measure of number of spontaneous concepts stored and efficiency of categorical storage of spontaneous concepts) described by Bousfield (1953), the second was a modified form of a twenty-questions test (a presumed measure of retrieval and labeling of categorically stored spontaneous concepts) described by Mosher and Hornsby (1966) and modified by Ault (1973), and the third was a classical sorting task (a presumed measure of transfer of treatment effects for categorical organization of spontaneous concepts).

The associative clustering test was an adapted form essentially similar to that employed by Haight (1974). This adaptation was composed of 24 words divided into two randomized lists of 12 words. Each list contained three implicit categories of four words (Figure 1), but words

TABLE 2

Means, Standard Deviations of Chronological Age (CA) and IQ for Learning Disabled and Normal Achieving Subjects in Three Experimental Treatments

Variable		Learning Disabled Random	Learning Disabled Clustered	Learning Disabled Identified Clustered	Normal Achieving Random	Normal Achieving Clustered	Normal Achieving Identified Clustered
CA	Mean	11.50	11.28	11.13	11.36	11.23	10.98
	SD	1.21	0.88	0.86	0.70	0.46	0.98
IQ	Mean	90.43	90.90	90.63	Not available for NA sample		
	SD	13.00	10.91	11.45			

TABLE 3

Analysis of Variance for Chronological Age (CA) and IQ for Learning Disabled and Normal Achieving Subjects in Experimental Treatment Group Subsamples

	CA			IQ		
Source	df	MS	F	df	MS	F
Classification (C)	1	30.813	0.277			
Treatment (T)	2	101.396	0.911	2	0.475	0.003
CXT	2	2.064	0.019			
Error	54	111.279		22	136.477	
Total	60			24		

Test 1

Birds	Mammals
Hawk	Tiger
Blackbird	Rabbit
Robin	Mouse
Goose	Bear
Fruit	Toys
Cherry	Doll
Strawberry	Jack-in-the-box
Plum	Sled
Tangerine	Marbles
Eating Equipment	Garden Tools
Teapot	Hoe
Salt Shaker	Wheel Barrow
Glass	Lawn Sprinkler
Dish	Trowel

Test 2

Birds	Mammals
Dove	Rat
Blue Jay	Deer
Crow	Chipmunk
Hummingbird	Lion
Fruit	Toys
Peach	Top
Lemon	Blocks
Apricot	Baseball Bat
Grapefruit	Jacks
Eating Equipment	Garden Tools
Napkin	Spade
Sugar Bowl	Hose
Pepper Shaker	Lawn Mower
Plate	Seed Spreader

Fig. 1. Associative Clustering Categories.

were randomly ordered. Lists were designed for oral presentation by the experimenter to individual subjects for immediate recall. In order to insure the subjects' attention to the words, each subject was asked to repeat each word immediately after it was presented. Each subject was then told to recall as many words as he could from each list. The second list was presented in identical fashion after the subject had recalled as many words as he was able from the first list. Presentation order of lists and of randomized words remained constant throughout the testing.

The twenty-questions test was composed of an array of 24 colored pictures of common objects belonging simultaneously to 6 perceptual categories and 6 semantic categories (Figure 2). The pictures were on 3" X $2\frac{1}{2}$" cards arranged in a 6 X 4 array of rows and columns in one of three conditions. In the first condition, pictures were randomly placed in the 24 locations of the array. In the second condition, the pictures were clustered by rows in 6 perceptual categories of 4 pictures each, and were simultaneously clustered in 6 semantic categories of 4 pictures each, arranged by quadrangles intersecting the rows. This intersecting pattern of clustered categories provided the task with structure and context that visually defined both perceptual and semantic categories. The third condition was the same as the second but with the addition of bright blue rectangles surrounding the semantic categories, pointing gestures by the experimenter tracing these rectangles, and accompanying statements by the experimenter labeling each category by name, "Notice that all of these are...." The rectangles, gestures, and statements were designed to provide visual,

Eagle	Sparrow	Dog	Monkey
Owl	Duck	Cat	Zebra
Apple	Orange	Ball	Balloon
Pear	Banana	Kite	Wagon
Spoon	Knife	Rake	Shovel
Coffee Pot	Cup	Pail	Sprinkling Can

<u>Perceptual categories, by rows:</u>

Color-brown; color-black and white; shape-round; color-yellow; distinctive feature-straight handles; distinctive feature-round handles.

<u>Abstract categories, by groups of four:</u>

Birds; Mammals: Fruit: Toys; Eating equipment; Garden tools.

Fig. 2. Ordered array for twenty-questions test.

motoric, and verbal cues clearly identifying the abstract categories. Individual subjects assigned to each treatment condition were instructed by the experimenter to ask questions in order to discover a "correct" picture in the array.

The third measure was a classical sorting task. This task was composed of the same 24 pictures used in the twenty-questions test. Rather than the 6 X 4 array used in the twenty-questions test, a randomly arranged array of these pictures was presented to the subject. Each subject was asked to put pictures he thought were alike together in groups.

Administration and Scoring: The associative clustering test was the first instrument administered and was given to all subjects individually. The procedure followed was essentially that used by Haight (1974). The two word lists were presented orally following instructions and a practice trial. Instructions read to each subject were:

> I am going to say some words and I want you to listen carefully and you say each word after I say it. After I have said all the words, I want you to tell me all the words you can remember. Do you understand? (If not, explain again). Let's practice to make sure you understand. You say each word after me. BOY--PUPPY--SHOE--BED. Now you tell me all the words you remember.

If two or more practice words were recalled correctly by the subject, the experimenter proceeded. If not, the practice list was repeated with the same instructions. All words were read at two or three second intervals, giving the subject time to repeat each word. When a subject recalled at least two words from the practice list, the following instructions were given:

> Now this time I will say more words. You say each one after me and remember as many as you can. You can forget those words now, remember just the new words I am going to say. Ready?

Immediately after presentation of the first list the experimenter said, "Tell me the words." Subjects who were hesitant to begin or continue saying the words or who asked if words were correct were passively encouraged with a non-discriminant nod or a "yes," whether words were correct or not. Although there was no time limit set for recall, subjects were asked after 30 seconds of silence, "Do you remember any more?" Following list 1, list 2 was presented; the experimenter said: "Good! Now let's do it again with new words. Remember only these new words. Ready?" All responses were recorded on tape.

In scoring results, the two lists were combined for all statistical analyses. The following scores were recorded:

(1) Total number of words recalled (correct words only, excluding perseverations and intrusions)

(2) Number of categorical intrusions (any category intrusion, even from a previous list)

(3) Number of irrelevant intrusions (any non-categorical word)

(4) Number of perseverations (words recalled more than once)

(5) Number of observed repetitions or number of times a stimulus word was followed by one or more stimulus words from the same category

(6) Number of words recalled from the various categories

If a word was correctly recalled more than once (perseveration), it was recorded as a correct response only when first stated. Scoring techniques used were proposed by Bousfield (1953), used by Dallett (1964) and adopted by Spitz (Gerjuoy & Spitz, 1966; Spitz, 1966). In this study an adjusted ratio of clustering (ARC) formula was used (Bousfield & Bousfield, 1966; Roenker, Thompson & Brown, 1971). The adjusted ratio of clustering (ARC) formula was:

$$ARC = \frac{O(R) - E(R)}{Max\ (R) - E(R)}$$

Where O(R) = Number of observed repetitions or number of times a stimulus word was followed by one or more stimulus words from the same category

(O(R) = 2 for 3 words recalled consecutively in the same category)

$$E(R) = \frac{M_1^2 + M_2^2 + M_3^2 + M_4^2 + M_5^2 + M_6^2}{n} - 1$$

$$Max(R) = (M_1-1) + (M_2-1) + (M_3-1) + (M_4-1) + (M_5-1) + (M_6-1)$$

M = Number of words recalled from the various categories

n = Total number of words recalled (Correct words only, excluding perseverations and intrusions)

E(R) = Number of times a stimulus word is expected (chance) to be followed by one or more stimulus words (expected number of category repetitions)

Max(R) = Maximum number of category repetitions

Immediately following the associative clustering test, one of the three previously described variants of the twenty-questions test was given, choice depending upon assignment of the individual subject to either the random, clustered or identified clustered treatment group. Stimulus pictures were presented in each case on a desk or table top at which the subject was seated facing the stimuli. As testing was conducted at one side of an on-going special education classroom, a visual screen was used to shield test materials from other students in the class who were to be tested. Instructions for all three groups were the same, except that in the identified clustered group visual, motoric and verbal cues were added as previously described. Instructions taken from Mosher and Hornsby (1966) were read to each subject.

> Now we're going to play some question-asking games. I'm thinking of one of these pictures, and your job is to find out which one it is that I have in mind. To do this you can ask any questions at all that I can answer by saying "yes" or "no," but I can't give any other answer but "yes" or "no." You can have as many questions as you need, but try to find out in as few questions as possible. (Mosher & Hornsby, 1966, p.90)

Before asking questions, each student was instructed to name every picture to assure attention and comprehension of the task stimuli.

Actually the experimenter had no correct picture in mind but allowed the subject to ask up to 7 questions. The experimenter gave a "no" response to the first 6 questions to prompt futher questions. The seventh question was answered "yes" and the subject was praised for his efforts and for finding the "correct" solution. Using a question asking strategy in which either semantic or perceptual categorical clusters are eliminated one by one, it is possible to eliminate all but one picture from the array with 7 questions. Using the most efficient question asking strategy possible in which half the array is eliminated by each question asked, all but one picture could be eliminated using 4 questions.

Statistical analysis of subject responses on the twenty-questions test was limited to the first four questions asked for two reasons. First, a number of subjects asked questions in such a way that the experimenter was forced to give "yes" responses before a total of six questions had been asked, but never before 4 questions had been asked. Second, by limiting analysis to the first 4 questions, it made possible future comparative analyses of present data relative to the logically most efficient question asking strategy.

Two general types of questions asked by subjects were recorded. The first general type of question included several stimulus pictures at once and represented the most efficient type of problem solving strategy as it simultaneously eliminated a number of possible solutions from further consideration. This efficient type of question was termed a constraint seeking question and was presumed to reflect the use of a constraint seeking strategy in the subject. Examples of constraint seeking questions were "Is it an animal?" or "Is it yellow?". Because constraint seeking questions included several similar stimuli under one heading as "bird" or "yellow," they were considered reflections of categorical mental organization. The total number of constraint seeking questions was recorded and also reported under three subdivisions: perceptual, semantic, and extrinsic. Perceptual constraint seeking questions employed perceptual characteristics as classifying principles and were of the sort "Is it yellow?". Semantic constraint seeking questions used abstractions as classifying principles and were of the sort "Is it a bird?". By contrast, extrinsic constraint seeking questions utilized the arrangement of the stimulus pictures as a classifying principle rather than their content and were of the sort "Is it in this box [blue rectangle] ?". Questions which potentially included several solutions but actually referred to only one stimulus picture were termed pseudoquestions. Pseudoquestions were of the sort "Is it red?" when only one red picture appeared in the array.

The second general type of question was one that included a single possible solution at a time and represented a less efficient strategy as it eliminated only one solution from further consideration. This inefficient type was termed an hypothesis seeking question and was presumed to reflect an hypothesis seeking strategy in the subject. Examples of hypothesis seeking questions were "Is it the owl?" or "Is it the pear?".

The following scores were recorded:

(1) Total number of constraint seeking questions (questions including more than one stimulus picture), subdivided as:

  (2) Number of perceptual constraint seeking questions

  (3) Number of semantic constraint seeking questions

  (4) Number of extrinsic constraint seeking questions

(5) Number of hypothesis seeking questions (questions including only one stimulus picture)

(6) Number of pseudoquestions (questions actually including only one stimulus picture, but potentially including several items; e.g., "Is it red?" when only one stimulus picture was red.

A second associative clustering test identical to the first was given immediately after the twenty-question test with the only difference that new stimulus words were used (Figure 1). The same categories of stimuli appeared in both associative clustering tests and in the twenty-questions test as well. The same measurements were recorded as for the first associative clustering test. The second associative clustering test was a presumed measure of the transfer of treatment effects for categorical organization of spontaneous concepts and number of spontaneous concepts stored resulting from imposed categorical task structure in the clustered and identified clustered variations of the twenty-questions test.

A sorting task using the twenty-questions pictures was administered immediately following the second associative clustering test. The stimulus pictures were presented in a scrambled pattern face up on a desk top at which the subject was seated facing the stimuli. The subject was instructed to "Put pictures together that you think belong together." A second, but identical, sorting task was given to each subject 24 hours later. These tests were presumed measures of immediate and delayed transfer effects for categorical organization of spontaneous concepts imposed by categorical task structure in the clustered and identified clustered variations of the twenty-questions test to a new problem solving task. All responses were recorded in writing. The following measurements were made:

(1) Number of categories constructed corresponding exactly to the given semantic and perceptual categories of the twenty-questions test.

(2) Number of categories constructed corresponding exactly to the given semantic categories of the twenty-questions test.

(3) Number of categories constructed corresponding exactly to the given perceptual categories of the twenty-questions test.

All testing was conducted in special education classrooms in the schools involved in order to simulate normal teaching conditions as closely as possible. An exception to this practice occurred at one school where a tutoring room familiar to the subjects was used.

Experimental Design

Data from the first associative clustering test were analyzed using a 2 X 3 analysis of variance design comparing differences between mean adjusted ratio of clustering (ARC) scores for LD and NA subjects collapsed across treatment groups. Similarly, a second analysis of variance design was used to compare differences between mean number of words recalled (n) on the first associative clustering test by LD and NA subjects collapsed across treatment groups. The twenty-questions test was analyzed using a 2 X 3 analysis of variance design comparing differences between mean number of constraint seeking questions asked for class (LD and NA), for treatment (random, clustered, identified clustered), and for interaction effects between class and treatment. The second associative clustering test was analyzed using two repeated measures analyses, the first with number of words recalled (n) as the repeated measures factor and the second with ratio of clustering (ARC) as the repeated measures factor. Data from the immediate and delayed sorting tasks were also analyzed using a repeated measures design with number of categories constructed exactly matching semantic or perceptual categories of the twenty-question test as the repeated measures factor.

As two experimenters (the author and a trained assistant) cooperated in testing subjects, a counter balanced testing schedule for LD and NA subjects was used to eliminate possible systematic experimenter bias for classification. Interrater reliability was not calculated as all measured subject responses were objective.

Operational Definitions of Major Questions

Categorical Storage. An associative clustering test was administered as a presumed measure of number of spontaneous concepts stored and efficiency of categorical storage of spontaneous concepts. LD children were expected to store the same number of spontaneous concepts as NA children, but fail to organize these stored concepts efficiently in conceptual categories when compared with NA children. Number of stored spontaneous concepts was estimated by number of words recalled, and efficiency of categorical storage of spontaneous concepts was calculated by formula using an adjusted ratio of clustering.

Categorical Usage (Retrieval and Labeling). A twenty-questions test was then used to measure retrieval and labeling of categorically stored spontaneous concepts. As evidenced by number of categorical (constraint seeking) questions asked on the random version of the twenty-questions test, LD children were expected to retrieve and label categories of spontaneous questions less frequently than NA children. However, with increasing degrees of categorical structure, context and cues provided by the clustered and identified clustered versions of the twenty-questions test, LD children were expected to ask categorical questions at a frequency approaching or equal to NA children.

Transfer Effects for Categorical Storage and Usage. A second associative clustering test was administered immediately following each version of the twenty-questions test; it was a measure of transfer of treatment effects for number of spontaneous concepts stored and efficiency of categorical storage of spontaneous concepts estimated respectively by the two dependent variables number of words recalled and adjusted ratio of clustering. LD and NA children were expected to be alike in showing no increase in number of spontaneous concepts stored. However, LD children were expected to show marked transfer effects for efficiency of categorical storage of spontaneous concepts as a function of increasing categorical structure, context, and cues imposed in the twenty-questions test; by contrast, NA children were expected to show no transfer effects on this categorical storage variable.

Transfer effects for categorical retrieval of spontaneous concepts were measured by immediate and delayed sorting tasks on the dependent variable number of categories constructed exactly matching categories of the twenty-questions test. LD children were expected to show marked transfer effects for categorical retrieval of spontaneous concepts as a function of increasing categorical structure, context, and cues imposed in the twenty-questions test; no transfer effects on this retrieval variable were expected for NA children.

Specific Hypotheses

Based on the operational measures chosen to assess categorical storage and usage of concepts, the following specific hypotheses were posed:

I. Categorical Storage

a. LD and NA children will recall correctly approximately the same number of words.

b. LD children will perform more poorly than NA children on the adjusted ratio of clustering.

II. Categorical Usage (Retrieval and Labeling)

a. LD children will ask fewer constraint seeking questions than NA children on the random version of the twenty-questions test, approach NA children on the clustered version, and equal NA children on the identified clustered version.

III. Transfer Effects (Categorical Storage and Usage)

a. LD and NA children will recall correctly approximately the same number of words across treatment versions and across class.

b. LD children will perform more poorly than NA children on the adjusted ratio of clustering with the random treatment version, approach NA children on the clustered version, and equal NA children on the identified clustered version.

c. On both immediate and delayed sorting tasks, LD children will construct fewer matching categories than NA children on the random treatment version, approach NA children on the clustered version, and equal NA children on the identified clustered version.

Chapter III

RESULTS

Categorical storage, retrieval, labeling, and transfer of spontaneously formed concepts were investigated in learning disabled (LD) and normal achieving (NA) children. An associative clustering test was used to measure the number of spontaneous concepts stored and efficiency of categorical storage of spontaneous concepts, a twenty-questions test presented in three versions served as a measure of categorical retrieval and labeling of spontaneous concepts in a problem solving situation, while transfer of categorical organization from a problem solving situation to new problem solving tasks was measured by a second associative clustering test and by immediate and delayed classical sorting tasks.

Categorical Storage

First Associative Clustering Test. Several scores obtained through use of the first associative clustering test were analyzed. One of these scores, a presumed measure of number of spontaneous concepts stored, was total number of words correctly recalled (n). Means, standard deviations, and ranges for n are given in Table 4 for total LD (N=30) and NA (N=30) samples, as well as for subsamples assigned differing versions of the twenty-questions test (random, clustered, and identified clustered). Although subjects were all given the same treatment on the first associative clustering test, descriptive statistics ($\bar{X}$, SD, R) for n were analyzed by subsample to allow identification of possible difference among subsample means.

TABLE 4

Means, Standard Deviations, and Ranges for Number of Words Correctly Recalled (n) on the First Associative Clustering Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Number of Words Correctly Recalled		
		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=7.70 SD=1.19 R=6-11	$\bar{X}$=7.60 SD=1.17 R=6-10	$\bar{X}$=7.00 SD=0.82 R=6-8	$\bar{X}$=8.50 SD=1.58 R=6-11
Normal Achieving	$\bar{X}$=8.47 SD=2.07 R=5-12	$\bar{X}$=9.10 SD=1.52 R=6-11	$\bar{X}$=8.20 SD=2.62 R=3-11	$\bar{X}$=8.10 SD=2.08 R=5-12

Difference among means for n according to subsample groups and difference between total NA and LD sample scores were tested using an analysis of variance; main effects for subsample groups and for class (total NA and LD samples) were found to be nonsignificant (Table 5).

Another score obtained by use of the first associative clustering test, presumed to be a measure of efficiency of categorical storage of spontaneous concepts, was calculated by formula as described in the previous chapter, and is called the adjusted ratio of clustering (ARC). Means, standard deviations, and ranges for ARC scores are given in Table 6 for total LD (N=30) and NA (NA=30) samples, as well as for subsamples reflecting the three versions of the twenty-questions test (random, clustered, and identified clustered). All subjects were given the same treatment on the first associative clustering test; however, descriptive statistics ($\bar{X}$, SD, R) for ARC are reported by subsample. An analysis of variance was used to test difference among means for ARC according to subsamples and difference between total LD and NA sample means; nonsignificant differences for subsample groups and for class (total LD and NA samples) were found, (Table 7).

Three additional scores were obtained with the first associative clustering test: number of categorical intrusions, number of irrelevant intrusions, and number of perseverations. Descriptive statistics for number of categorical intrusions were recorded for both the total LD sample (N=30, $\bar{X}$=0.90, SD=1.04, R=0-4) and the total NA sample (N=30, $\bar{X}$=1.00, SD=0.77, R=0-2). The same statistics were recorded for number of irrelevant intrusions for the total LD sample (N=30, $\bar{X}$=0.53, SD=0.77, R=0-3) and the total NA sample (N=30, $\bar{X}$=0.30, SD=0.59, R=0-2). Number

TABLE 5

Analysis of Variance for Number of Words Correctly Recalled (n) on the First Associative Clustering Test for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Classification (C)	1	8.817	2.933	0.089
Treatment (T)	2	3.517	1.770	0.318
C X T	2	5.217	1.736	0.184
Error	54	3.006		
Total	59	3.196		

TABLE 6

Means, Standard Deviations and Ranges for Adjusted Ratio of Clustering (ARC) on the First Associative Clustering Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

		Adjusted Ratio of Clustering (ARC)		
Group		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=29.00 SD=0.49 R=(-0.49)-(+1.00)	$\bar{X}$=0.35 SD=0.54 R=(-0.39)-(+1.00)	$\bar{X}$=0.12 SD=0.44 R=(-0.49)-(+1.00)	$\bar{X}$=0.40 SD=0.48 R=(-0.40)-(+1.00)
Normal Achieving	$\bar{X}$=33.00 SD=0.36 R=(-0.50)-(+1.00)	$\bar{X}$=0.57 SD=0.30 R=(+0.17)-(+1.00)	$\bar{X}$=0.32 SD=0.40 R=(-0.40)-(+1.00)	$\bar{X}$=0.10 SD=0.37 R=(-0.50)-(+0.68)

TABLE 7

Analysis of Variance for the Adjusted Ratio of Clustering (ARC) on the First Associated Clustering Test for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Classification (C)	1	0.002	0.122	0.728
Treatment (T)	2	0.336	1.832	0.170
C X T	2	0.438	2.387	0.102
Error	54	0.185		
Total	59			

of perseverations, the last of these scores, were similarly recorded for total LD sample (N=30, $\bar{X}$=0.33, SD+0.55, R=0-2) and the total NA sample (NA=30, $\bar{X}$=0.30, SD=0.57, R=0-2). Three separate analyses of variance testing mean differences between total LD and NA samples and among subsamples for each of these scores yielded nonsignificant values of F.

Categorical Retrieval and Labeling

Twenty-Questions Test. Several scores were obtained from the twenty-questions test. The first, total number of constraint seeking questions asked, was presumed to measure categorical retrieval and labeling of spontaneous concepts. Means, standard deviations, and ranges of total constraint seeking questions for the total LD (N=30) and NA (N=30) samples, as well as for the three LD subsamples (random, clustered, and identified clustered) and the three NA subsamples (random, clustered, and identified clustered) are found in Table 8. An analysis of variance was used to test difference among means for total number of constraint seeking questions according to treatment subsamples and difference between means for LD and NA samples (Table 9). The main effect for classification was significant, the main effect for treatment, while in the predicted direction, was nonsignificant, and the interaction reached significance. Mean total number of constraint seeking questions for LD and NA subsamples is plotted as a function of classification and treatment group in Figure 3 (Appendix).

The score for total number of constraint seeking questions asked was subdivided into: 1) number of perceptual constraint seeking questions asked, 2) number of semantic constraint seeking questions

TABLE 8

Means, Standard Deviations, and Ranges for Total Number of Constraint Seeking Questions Asked on the Twenty-Questions Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Number of Constraint Seeking Questions Asked		
		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=1.37 SD=1.53 R=0-4	$\bar{X}$=0.40 SD=0.97 R=0-3	$\bar{X}$=1.10 SD=1.72 R=0-4	$\bar{X}$=2.60 SD=1.90 R=0-4
Normal Achieving	$\bar{X}$=2.53 SD=1.68 R=0-4	$\bar{X}$=2.50 SD=1.43 R=0-4	$\bar{X}$=2.80 SD=1.62 R=0-4	$\bar{X}$=2.30 SD=2.00 R=0-4

TABLE 9

Analysis of Variance for Total Number of Constraint Seeking Questions Asked on the Twenty-Questions Test for Learning Disabled and Normal Achieving Children

Source	df	MS	F
Classification (C)	1	20.417	7.880*
Treatment (T)	2	4.999	1.930
C X T	2	8.267	3.191*
Error	54	2.591	

* $P < .05$

asked, and 3) number of extrinsic constraint seeking questions asked. Mean number of perceptual constraint seeking questions is plotted as a function of classification and treatment group in Figure 4, while mean number of semantic constraint seeking questions is similarly plotted in Figure 5. Only six individual extrinsic constraint seeking questions were asked, representing 2.5% of total questions for NA and LD children; all extrinsic questions were asked by LD children assigned to the identified clustered subsample.

An additional score obtained from the twenty-questions test was number of hypothesis seeking questions asked. Mean number of hypothesis seeking questions is plotted in Figure 6 as a function of classification and treatment group. Hypothesis seeking questions together with total number of constraint seeking questions equaled approximately 92% of all questions asked on the twenty-questions test. The approximately 8% remaining was composed of pseudoquestions, which were rather evenly distributed across treatment groups for both LD and NA children.

Transfer Effects

Second Associative Clustering Test. As on the first associative clustering test, the means, standard deviations, and ranges for total number of words correctly recalled (n) and for ARC were summarized, and the means analyzed for total LD and NA samples and treatment group subsamples. Descriptive statistics ($\bar{X}$, SD, R) are reported for n according to LD and NA samples and treatment subsamples in Table 10. An analysis of variance yielded a statistically significant ($p<.05$) main effect for classification (LD and NA) and an interaction ($p<.01$) (Table 11). These classification and interaction effects are plotted

TABLE 10

Means, Standard Deviations, and Ranges for Number of Words Correctly Recalled (n) on the Second Associative Clustering Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Number of Words Correctly Recalled: Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=7.90 SD=1.52 R=5-12	$\bar{X}$=7.90 SD=1.45 R=6-11	$\bar{X}$=7.20 SD=1.03 R=6-9	$\bar{X}$=8.60 SD=2.07 R=5-12
Normal Achieving	X=8.83 SD=1.29 R=6-12	X=9.80 SD=0.92 R=8-11	X=8.90 SD=1.73 R=6-12	X=7.80 SD=1.23 R=6-9

TABLE 11

Analysis of Variance for Number of Words Correctly Recalled (n) on the Second Associative Clustering Test for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Classification (C)	1	13.067	6.136	0.016*
Treatment (T)	2	3.617	1.698	0.191
C X T	2	11.317	5.314	0.008**
Error	54	2.130		
Total	59	2.677		

* P<.05
** P<.01

in Figure 7 and compared with results on the first associative clustering test. An analysis of repeated measures for n (Table 12) comparing the first and second associative clustering test for LD and NA subjects, yielded a significant ($p < .01$) main effect for classification (LD and NA) and a significant interaction ($p < .01$). Descriptive statistics ($\bar{X}$, SD, R) are reported for ARC according to LD and NA samples and treatment subsamples in Table 13. An analysis of variance testing for main and interaction effects for ARC yielded nonsignificant values of F for all effects (Table 14). Similarly nonsignificant was an analysis of repeated measures with ARC as the repeated measures factor (Table 15).

Sorting Tasks. Only one score (total number of categories constructed exactly matching perceptual and semantic categories in the twenty-questions test) was analyzed statistically for both immediate and delayed sorting tasks for LD and NA subjects. This score was presumed to measure transfer of categorical organization from a problem solving task to a new but similar problem solving task encountered immediately and again after a delay of twenty-four hours. For the immediate sorting task, means, standard deviations, and ranges were recorded by total LD and NA samples and by treatment subsamples (Table 16). Means for the immediate sorting task are plotted in Figure 8 as a function of classification and treatment group. Scores for exactly matching semantic categories considered independently of perceptual categories are similarly plotted in Figure 9, as are scores for exactly matching perceptual categories in Figure 10. For the delayed sorting task, means, standard deviations and ranges are reported by total LD and NA samples and by treatment subsamples in Table 17.

TABLE 12

Analysis of Repeated Measures of Words Correctly Recalled (n) Comparing First and Second Associative Clustering Tests for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Mean	1	8117.957	2742.3735	0.000
Classification (C)	1	21.675	7.322	0.009**
Treatment (T)	2	6.025	2.035	0.141
C X T	2	15.925	5.380	0.007**
Error	54	2.960		
W	1	2.408	1.107	0.297
W Classification (C)	1	0.208	0.096	0.758
W Treatment (T)	2	1.108	0.510	0.604
W C X T	2	0.608	0.280	0.757
Error	54	2.175		

** $P < .01$

TABLE 13

Means, Standard Deviations and Ranges for Adjusted Ratio of Clustering (ARC) on the Second Associative Clustering Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Adjusted Ratio of Clustering (ARC)		
		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=0.31 SD=0.32 R=(-0.50)-(+1.00)	$\bar{X}$=0.37 SD=0.43 R=(-0.04)-(+1.00)	$\bar{X}$=0.06 SD=0.32 R=(-0.05)-(+0.56)	$\bar{X}$=0.49 SD=0.21 R=(+0.20)-(+1.00)
Normal Achieving	$\bar{X}$=0.25 SD=0.41 R=(-0.74)-(+1.00)	$\bar{X}$=0.20 SD=0.41 R=(-0.67)-(+0.62)	$\bar{X}$=0.23 SD=0.41 R=(-0.74)-(+0.77)	$\bar{X}$=0.32 SD=0.42 R=(-0.40)-(+1.00)

TABLE 14

Analysis of Variance for the Adjusted Ratio of Clustering (ARC) on the Second Associated Clustering test for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Mean	1	4.587	32.750	0.000
Classification (C)	1	0.052	0.373	0.544
Treatment (T)	2	0.331	2.365	0.104
C X T	2	0.183	1.308	0.279
Error	54	0.140		
Total	60			

TABLE 15

Analysis of Repeated Measures of Adjusted Ratio of Clustering (ARC) Comparing the First and Second Associative Clustering Tests for Learning Disabled and Normal Achieving Children

Source	df	MS	F	Probability F Exceeded
Mean	1	10.331	66.266	0.000
Classification (C)	1	0.003	0.019	0.888
Treatment (T)	2	0.391	2.511	0.091
C X T	2	0.442	2.832	0.068
Error	54	0.156		
R	1	0.034	0.205	0.653
R Classification (C)	1	0.072	0.426	0.517
R Treatment (T)	2	0.276	1.645	0.202
R CXT	2	0.180	1.071	0.350
Error	54	0.168		

TABLE 16

Means, Standard Deviations, and Ranges for Total Number of Categories Constructed on Immediate Sorting Task Exactly Matching the Given Semantic and Perceptual Categories of the Twenty-Questions Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Number of Categories Corresponding Exactly		
		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=4.40 SD=1.67 R=0-6	$\bar{X}$=2.80 SD=2.82 R=0-6	$\bar{X}$=4.40 SD=2.22 R=0-6	$\bar{X}$=6.00 SD=0.0 R=6-6
Normal Achieving	$\bar{X}$=4.00 SD=2.46 R=0-6	$\bar{X}$=3.80 SD=2.30 R=0-6	$\bar{X}$=4.20 SD=2.15 R=1-6	$\bar{X}$=4.00 SD=2.94 R=0-6

TABLE 17

Means, Standard Deviations, and Ranges for Total Number of Categories Constructed on Delayed Sorting Task Exactly Matching the Given Semantic and Perceptual Categories of the Twenty-Questions Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Group		Number of Categories Corresponding Exactly		
		Random	Clustered	Identified Clustered
	(N=30)	(N=10)	(N=10)	(N=10)
Learning Disabled	$\bar{X}$=4.50 SD=1.44 R=0-6	$\bar{X}$=3.10 SD=2.51 R=0-6	$\bar{X}$=4.40 SD=1.90 R=0-6	$\bar{X}$=6.00 SD=0.0 R=6-6
Normal Achieving	$\bar{X}$=4.13 SD=2.22 R=0-6	$\bar{X}$=4.50 SD=1.90 R=0-6	$\bar{X}$=4.40 SD=2.01 R=1-6	$\bar{X}$=3.50 SD=2.76 R=0-6

Means for semantic and perceptual categories considered together on the delayed sorting task are plotted in Figure 11 as a function of classification and treatment group. Semantic and perceptual categories considered independently are similarly plotted in Figures 12 and 13. An analysis of repeated measures comparing immediate and delayed sorting performance of LD and NA subjects as assigned to treatment groups with total number of exactly matching semantic and perceptual categories constructed as the repeated measures factor is summarized in Table 18 showing a significant ($p < .05$) effect for the interaction between class (LD and NA) and treatment. No simple main effects for class or treatment were found.

Summary of Findings

On the first associative clustering test, scores were reported for number of words correctly recalled (n), an adjusted ratio of clustering (ARC), number of categorical intrusions, number of irrelevant intrusions, and number of perseverations. Separate analyses of variance were used to test differences among means for n and for ARC according to subsamples and difference between total LD and NA sample means. Main effects for treatment and for class were found to be nonsignificant both for n and for ARC. Similarly, separate analyses of variance for number of categorical intrusions, number of irrelevant intrusions, and number of perseverations all yielded nonsignificant F values.

Several scores were also reported for the twenty-questions test. For one of these, the total number of constraint seeking questions asked, an analysis of variance was used to test differences among subsample means and differences between means for LD and NA samples.

TABLE 18

Analysis of Repeated Measures of Sorting Comparing Sorting Tasks I and II on Total Number of Categories Exactly Matching the Given Semantic and Perceptual Categories of the Twenty-Questions Test for Learning Disabled and Normal Achieving Subjects Assigned to Treatment Groups

Source	df	MS	F	Probability F Exceeded
Mean	1	2175.974	258.962	0.000
Classification (C)	1	4.408	0.525	0.472
Treatment (T)	2	17.809	2.119	0.130
CXT	2	30.358	3.613	0.034*
Error	54	8.403		
S	1	0.408	0.646	0.425
SX Classification (C)	1	0.008	0.013	0.909
SX Treatment (T)	2	1.408	2.227	0.118
S X C X T	2	0.558	0.883	0.419
Error	54	0.632		

* $P < .05$

The main effect for class (LD and NA samples) was significant, the main effect for treatment was nonsignificant but in the predicted direction, and the interaction attained significance. Other scores from the twenty-questions test were graphed as a function of classification and treatment group, but not analyzed statistically. These other scores were the number of perceptual constraint seeking questions asked, the number of semantic constraint seeking questions asked, and the number of hypothesis seeking questions asked.

Scores for the second associative clustering test were reported and analyzed as on the first associative clustering test, but with the addition of analyses of repeated measures for n and ARC. An analysis of variance for n yielded a significant main effect for classification (LD and NA) and a significant interaction. An analysis of repeated measures for n also yielded a significant main effect for classification and a significant interaction. For ARC, an analysis of variance testing for main and interaction effects yielded nonsignificant F values for all effects; similarly, an analysis of repeated measures with ARC as the repeated measure factor yielded nonsignificance for all effects.

For both immediate and delayed sorting tasks, several scores were reported, but only one score was analyzed statistically. This score was the total number of categories constructed exactly matching perceptual and semantic categories in the twenty-question test. Using this score, an analysis of repeated measures comparing immediate and delayed sorting performance of LD and NA subjects showed a significant interaction, but no simple main effects. There were no differences on this measure between the two sorting tasks. Other scores reported for both sorting tasks were graphed as a function of classifications and treatment groups. These scores were number of categories constructed exactly matching semantic categories in the twenty-questions test, number of categories constructed exactly matching perceptual categories in the twenty-questions test, and number of hypothesis questions asked.

Hypothesis Testing

Findings were tested against the hypotheses proposed yielding the following results:

I. Categorical Storage

a. The hypothesis that LD and NA children will recall correctly approximately the same number of words was consistent with study findings.

b. The hypothesis that LD children will perform more poorly on the adjusted ratio of clustering than NA children was not confirmed.

II. Categorical Usage (Retrieval and Labeling)

a. The hypothesis that LD children will ask fewer constraint seeking questions than NA children on the random version of the twenty-questions test, approach NA children on the clustered version, and equal NA children on the identified clustered version was confirmed.

III. Transfer Effects (Categorical Storage and Usage)

a. The hypothesis that LD and NA children will recall correctly approximately the same number of words across treatment versions and across class was not consistent with study findings.

$$a_0(x) = \frac{2}{\Omega}\int_0^{\Omega} f(x,y)\,dy = 0 \qquad \text{for } x = 0 \text{ and } x = 1 \ .$$

Thus we conclude

$$f(x,y) = 0 \quad \text{for } x = 0 \text{ and } x = 1 \ . \tag{3.13}$$

It remains to see what (3.11c) implies about f at $x = 0$ and $x = 1$. For the moment relax condition (3.11a) to the more general condition

$$w = \text{constant} \qquad \text{at } x = 0, 1 \tag{3.11a'}$$

Now $v = 0$ and $w =$ constant imply $v_y = w_y = 0$. The second of the strain relations (3.7) simplifies to

$$\varepsilon_y = w \quad \text{at } x = 0, 1 \ .$$

Invert the first two equations of Hooke's law (3.6) to obtain

$$\varepsilon_y = \sigma_y - \nu\sigma_x = F_{xx} - \nu F_{yy} = f_{xx} - \nu f_{yy} + \nu\sigma = f_{xx} + \nu\sigma$$

or

$$f_{xx} = w - \nu\sigma \quad \text{at } x = 0 \text{ and } x = 1 \tag{3.14}$$

<u>Symmetric solutions</u>

We now seek solutions angularly symmetric about the shell axis, i.e., assume $u, v, w, \sigma_x, \sigma_y$, and σ_{xy} are independent of y. First note that the strain relations simplify to

$$\left.\begin{aligned} \varepsilon_x &= u_x + \frac{1}{2\omega^2} w_x^2 \\ \varepsilon_y &= w \\ 2\varepsilon_{xy} &= v_x \end{aligned}\right\} \tag{3.7S}$$

We calculate

$$\frac{\partial}{\partial x}\sigma_x = \frac{\partial}{\partial x}F_{yy} = \frac{\partial}{\partial y}F_{xy} = -\frac{\partial}{\partial y}\sigma_{xy} = 0$$

so that $\sigma_x =$ constant. From boundary condition (3.11d) we conclude

$$\sigma_x = -\sigma$$

Chapter IV

DISCUSSION

This study involved an analysis of problem solving strategies used by learning disabled (LD) and normal achieving (NA) children. More specifically, the study was designed to investigate aspects of information processing that may be relevant to understanding the characteristic discrepancy in LD children between ability and performance. Two information processing dimensions were considered: (1) categorical storage of information, and (2) retrieval of these categories as a strategy in problem solving. Based on Vygotsky's (1962) theory, it was reasoned that both processes may be essential to problem solving in school. Since LD children typically perform inadequately on school tasks and the processes of categorical storage and retrieval may be essential to academic problem solving, then LD children may be deficient in one or both of these processes. This possibility was investigated by comparing LD and NA children on operational measures of acquisition and categorical storage of concepts learned from everyday experiences, retrieval and labeling of these conceptual categories as a strategy in problem solving, and transfer of categorical organization to other problem solving situations.

<u>Summary of Results.</u> On operational measures of number of everyday concepts acquired and categorical storage of these concepts, LD and NA children performed equally. However, LD children performed less adequately than NA children on a problem solving task presumed to measure retrieval and labeling of conceptual categories. When this problem

solving task was ordered categorically, the performance of LD children approached that of NA children, and equaled the performance of NA children when in-task categorical order was cued. A test given immediately after the problem solving task was designed to measure transfer effects for both number of concepts acquired and categorical storage. On this test, only NA children showed significant effects for number of concepts acquired, and neither group showed effects for categorical storage. Given next were sorting tasks presumed to measure immediate and delayed transfer effects for retrieval of conceptual categories. LD children showed significant effects on these tasks as a function of the increased degrees of categorical order and cues in the first problem solving task.

Discussion of Results. Results of the study may be interpreted to suggest that LD children differ from NA children in problem solving by failing to retrieve and label conceptual categories as a strategy under some circumstances. However, LD children appear to approach NA children in the use and transfer of this strategy when the problem solving task is ordered categorically. Further, they seem to equal NA children in use of categories as a strategy and in transfer effects when cues are added to the in-task categorical order. This interpretation gains added meaning when compared with related empirical studies, when considered theoretically, and when discussed in the light of possible applications.

The study most closely related to this investigation is Haight's (1974) analyses of concept formation and concept usage in LD and NA children. Inferences drawn by Haight are consistent in part with those drawn in this study. The two investigations agree in concluding that

LD and NA children acquire the same number of concepts from everyday experience, but are in conflict in regard to categorical storage of these concepts by LD children. Whereas Haight found LD children inferior to NA children in categorical storage, LD and NA children were found equal on categorical storage in this study. Both studies used the same operational measure of categorical storage, the adjusted ratio of clustering. No explanation is apparent to account for this inconsistency of findings; additional research using the adjusted ratio of clustering or other measures of categorical storage in LD children is needed. However, using dissimilar measures, both investigations concluded that LD children are equivalent to NA children in use of categorical retrieval and labeling as a problem solving strategy when the task is presented with categorical structure and cues. The equivocal findings regarding categorical storage in LD children and the consistent findings relating to categorical retrieval and labeling in these children can be further clarified by reference to research in these areas with young NA children and retardates.

Regarding the issue of categorical storage of concepts in LD children, empirical findings may be cited in support of the interpretation that LD children are as efficient as NA children in this process. Associative clustering, a presumed measure of categorical mental organization (Bousfield, 1953), increases with increasing chronological age in normal children (Jablonski, 1974). In their investigation of categorical or "constraint seeking" questions asked in a problem solving situation, Mosher and Hornsby, (1966) held that categorical mental organization is a prerequisite to retrieving and labeling conceptual categories. It seems, then, that categorical mental organization increases over a period of years in young normal children and is a likely precondition to categorical retrieval and labeling. If this is so, then it may not be reasonable to contend as Haight (1974) has done that with one exposure to a concepts usage task provided with structure context and cues, these children can employ categorical retrieval and labeling equally with NA children. That is, it seems unreasonable to hold that a developing process such as categorical mental organization, which is presumed to be a prerequisite to categorical retrieval and labeling, could be generated in the LD child through one structured learning experience. The alternative interpretation inferred in this study seems more plausible; namely, that LD children have categorical mental organization equal to NA children but fail to retrieve and label those categories equally with NA children unless the task is categorically ordered and cued. This latter interpretation is also consistent with Flavell's (1970) statement that young normal children fail to use mediating strategies (such as categorical clustering) to facilitate recall unless prompted to do so. He termed this behavior a <u>production</u> deficiency in which a child fails to use organizational strategies he has, rather than a <u>mediational</u> deficiency in which a child does not have organizational strategies available. Findings in associative clustering research (Jablonski, 1974) substantiate Flavell's position. In that research, categorically structured stimulus items, cues, and suggestions to cluster items categorically were employed during input and recall to facilitate use of clustering strategies. The interpretation that LD children have categorical mental organization but use it only when the task is ordered and cued also agrees with Hagen's (1971) suggestion that improvement in both short-term and long-term memory is a function of changes in rehearsal strategies. Although not supported by the most closely related study (Haight, 1974), the finding in this investigation that LD children have categorical mental organization equal to NA children receives support from a variety of sources. In general it seems plausible to conclude that LD children are like NA children in the process of categorical information storage.

Turning specifically to the issue of categorical retrieval and labeling used as a strategy in problem solving, results in this study suggest that LD children are inferior to NA children in this process under some circumstances, but equal to NA children when the task is ordered categorically and cued. This inference was reached based on results from the twenty-questions test in which children were instructed to ask questions to discover one "correct" picture from an array of twenty-four pictures. In asking questions, NA children spontaneously tended to ask categorical or "constraint seeking" questions in which several pictures were included as a group or a class. As children were not directed to ask categorical questions, their spontaneous tendency to do so was taken as a clear indication not only of categorical mental organization, but also of retrieval and labeling of these mental categories. Although categorical mental organization (storage) seems clearly evidenced on the associative clustering test, the two processes of categorical mental organization (storage) and use of categories (retrieval and labeling) are not separable on the twenty-questions test and thus confounded. It is of importance to note that this confounding effect may be eliminated by the pattern of responses by LD children in this investigation. LD and NA children have been shown to perform equally in associative clustering, this finding suggesting efficient categorical mental organization (storage) in LD children. Further, in non-cued situations LD children have been shown inferior to NA children on the twenty-questions test, a presumed measure of both categorical mental organization (storage) and use (retrieval and labeling) of categories. Yet LD children were equal to NA children when the twenty-questions task was ordered and cued categorically. Thus, it may be reasonably stated that LD children perform more poorly than NA children on the twenty-questions test by consequence of inefficient use (retrieval and labeling) of categories, rather than as a result of inefficient categorical mental organization (storage). Stated simply, since LD children appear to be as efficient as NA children in storing information in categories, it seems likely that their failure to use these categories as a strategy in problem solving may result from inefficiency in the processes of retrieval and labeling categories.

It is possible that a motivational hypothesis might account for failure to use categories in problem solving. However, this is unlikely as LD children used categories as often as NA children on the categorically ordered and cued version of the twenty-questions test, but failed to use them on the random version. Ordering and cuing were the only conditions varied between these two test versions. This manipulation of cognitive in-task dimensions cannot be used reasonably to support a purely motivation hypothesis. No affective differences were built into the two situations and no affective differences were noted between samples on the two tests.

Comparing retardates with NA and LD children on aspects of categorical storage and usage sheds further light on findings reported in this study. Blount (1968) reviewed the literature on concept usage in retardates and concluded from evidence on tests of short-term memory that retardates are able to acquire concepts and to store them categorically as well as MA matches when the learning situation is appropriately mediated; that on sorting tasks, retardates are able to cluster stimuli categorically (categorical retrieval from long-term memory) as well as MA matches; and with concepts usage tasks requiring verbalization, retardes have particular difficulty. Several individual studies help to clarify categorical storage and retrieval in relation to verbal mediation in retardates. Although a number of investigations with retardates have demonstrated increased recall and categorical clustering in short-term recall for a list of words when the stimulus words were arranged in categorical blocks (Blount, 1968), "... brief practice with a blocked list is not sufficient to establish a generalized set to employ a clustering strategy" (Bilsky, 1976, p.589). Bilsky (1976), therefore, in an attempt to promote transfer effects for clustering, tested retardates with three conditions: blocking alone, blocking with instructions to cluster, and blocking with verbal categorical labels provided by the experimenter at the input stage. The second condition resulted in more recall and clustering than the first and the third still more recall and clustering than the second, with a tendency in the third condition toward transfer of clustering to lists of words belonging to new categories. Gerjouy and Spitz (1966) in a similar study demonstrated increased recall and clustering using verbal labels alone at the retrieval stage. In a serial recall task with numerals as stimulas items, MacMillan (1970c) demonstrated no increase in recall when single digits were presented ungrouped for verbalization and recall by subjects, some increase in recall when digits were grouped as higher-decade numerals but verbalized by subjects as single digits, and greater increase in recall when digits were grouped as higher-decade numerals and verbalized as such by subjects. Taken together, these findings for retardates appear to indicate that verbally labeled in-task categorical ordering facilitates use of categorical clustering as a strategy for both storage and retrieval of information. This conclusion for retardates is strikingly like that for LD children inferred from results on the twenty-questions test in which it was concluded that by first providing in-task categorical ordering and then adding categorical cues, including verbal labels, low levels of categorical retrieval and labeling by LD children can be raised to the levels achieved by NA matches. It seems that LD children behave like retardes in their inefficient retrieval and labeling of categories and also in their ability to equal their <u>mental age</u> peers in use of categorical organization as a strategy when in-task categorical ordering and verbal cuing are provided in the learning situation.

In essence then, when comparing problem solving performance of LD with empirical findings for NA children and retardates, it seems that LD children behave like NA children of the same chronological age in categorical storage of information, but under some circumstances like retarded children and younger NA children in their failure to use categorical retrieval and labeling as a strategy in a problem solving situation. However, when categorical ordering and cues, including verbal cues, are added to the problem solving task, LD children retrieve and label categories equally with NA children of the same chronological age.

Using Vygotsky's theoretical distinction between spontaneous concepts (concepts learned from everyday life) and scientific concepts (abstract hierarchical categories learned in school) as a framework for understanding the processes of categorical storage and retrieval, two alternatives were stated: (1) that IQ may represent the number of spontaneous concepts a child brings to the classroom, and academic achievement may reflect systematic organization and use (retrieval) of these concepts as hierarchically structured categories; or (2) that IQ may represent both number and systematic organization of spontaneous concepts in hierarchical categories, and academic achievement may reflect the use (retrieval) of these systematically structured hierarchical categories in the classroom. The latter alternative appears to be supported by the findings and interpretations in this study; namely, that LD children have acquired and stored information in systematic conceptual categories equally with NA children, but fail in some circumstances to retrieve and label these categories when attempting to solve academic problems such as those presented on academic achievement tests or classroom assignments. In further support of the second alternative, it seems plausible that IQ tests may be composed of spontaneous concepts (e.g., Information subtest on WISC) in addition to scientific concepts (e.g., Similarities subtest on the WISC). IQ is well known to correlate with academic achievement; however, such a correlation does not occur with LD children, and this puzzling condition has been termed the ability-performance discrepancy in LD children. This discrepancy is consistent with the latter theoretical alternative derived from Vygotsky.

In sum, it may be stated that, consistent with Vygotsky's theory, the ability-performance discrepancy in LD children may be described as an information processing difficulty in which spontaneous concepts have been acquired and categorically stored at the expected rate as measured by IQ, but that these conceptual categories are not efficiently retrieved and labeled as a strategy in problem solving situations such as those encountered on achievement tests and in many classroom assignments. This conclusion may be further clarified by considering the ontogeny of problem solving from the "cognitive-developmental" or "interactional" viewpoint as presented by Kohlberg (1968). According to Kohlberg the cognitive-developmental view represents a broad stream of educational thought espoused by such theorists as Baldwin, Dewey, Piaget and Vygotsky. This viewpoint suggests that: (1) cognition is a function or process, (2) cognition proceeds through stages of structural reorganization, (3) the source of cognitive structure and development is the interaction between structure and maturation of the organism and teaching structures of the environment, and (4) such structural reorganization requires an optimal balance of discrepancy and match between organism and environment. More concisely, this theoretical position views cognition as a process that passes through stages of structural reorganization impling an optimal interactional balance between behavioral structure of the organism and the teaching structure of the environment. According to Kohlberg, these "stages imply distinct or qualitative differences in children's modes of thinking or of solving the same problem at different ages." (Kohlberg, 1968, p. 81).

The major conclusion of this study may be understood in terms of the cognitive-developmental or interactional viewpoint. Thus, the ontogeny of problem solving in LD children may have proceeded to a stage like that of NA children of equal CA in respect to the processes of aquisition and storage of concepts in hierarichal categorical structures, but may have proceeded to a less developed stage than that of NA children in respect to the processes of categorical retrieval and labeling. It may be only when a more optimal match between the behavioral structure of the child and the teaching structure of the environment is attained that LD children proceed to a cognitive stage equal to NA children in respect to the processes of categorical retrieval and labeling as strategies in problem solving. Such an optimal match for LD children would seem, in the light of findings of this study, to require categorically organized and cued teaching structures.

Using this cognitive-developmental restatement of the major study finding, an argument for a difference, rather than a deficit, interpretation of LD children may be offered. LD children may be considered different from NA children in pattern of optimal match between in-child behavioral structures and teaching structures of the environment. Based on findings in this study, it seems that LD children require a more highly structured teaching environment than do NA children to optimize use of categorical retrieval and labeling as strategies in problem solving.
In general, it is logical to assume that LD children may fail academically because they learn problem solving strategies through interaction with

teaching structures of the environment different from structures which typify the regular school curriculum. On the other hand, the findings in this investigation that categorical retrieval and labeling in LD children is enhanced following one exposure to a highly structured teaching environment cannot reasonably be explained on the basis of the neurological deficit theory of LD children (Clements, 1966; Myklebust, 1968), which holds that extensive new learning is required for improved performance in LD children (Cruickshank, 1976).

Several additional points raised by study findings require discussion. Regarding the matter of the significant transfer effects suggested by results on the immediate and delayed sorting tasks, it is possible that these effects may reflect task memory rather than the establishment of a general set to retrieve information categorically. Memory for task seems implicated as the twenty-questions task and the sorting tasks were very similar in process and identical in content, as the same categories and stimulus items were used in both. Further experimentation is needed to clarify the relative contributions of task memory and strategy acquisition to the effects noted. For example, it would be valuable to know what effects would be shown if different stimulus items, but the same categories were used in the sorting tasks.

In respect to the absence of effects for categorical organization on the second associative clustering test, one possible interpretation is that the clear differences between the twenty-questions test and the associative clustering test may have represented a barrier to transfer effects for that process. However, NA, as opposed to LD children, recalled more words on the second associative clustering test. This increase in recall for NA children may have represented a mere practice effect. If this is so, the question remains as to why LD children did not show a comparable practice effect.

Other findings may be mentioned briefly. One is that on the random version of the twenty-questions test, NA children tended to ask perceptual rather than semantic constraint seeking questions. However, with the addition of semantic categorical order and cues, NA children continued to ask as many categorical questions as before, but shifted from perceptual to semantic categories. Also, of interest is the finding that NA children assigned the random version used far more semantic than perceptual categories on the transfer tasks, although these same children used more perceptual than semantic categories on the twenty-questions test. Another interesting finding is that LD children exceeded NA children in use of categories following the cued version of the twenty-questions test. Also to be noted is that extrinsic constraint seeking questions were asked only by LD children and only on the cued version of the twenty-questions test. No interpretation of these findings is attempted, but the tendencies noted may be worthy of further examination.

A final result, although not reported, may also be of interest for future study. Namely, both LD and NA children, but particularly LD children, tended on both associative clustering tests to recall more food related words by a noticeable margin than other categories of words. No interpretation of this unanalized finding is offered.

Interpretation of findings in which LD and NA samples were compared may be questionable as a confounding effect existed for IQ. The LD sample was considered for purposes of interpretation to be of approximately normal or average intellectual ability based on the sample mean IQ score of 90.68 for that group, but the range for these scores was 70 to 117 with a standard deviation of 11.19. This heterogeneity in IQ indicates that a number of individual children in the LD sample were well below the average range for IQ (often considered to be 90-110) and some were above. These children in the extremes of the sample IQ distribution cannot therefore be considered of "normal" intellectual ability. A further confounding effect exists for IQ since school personnel matched NA children individually on IQ with children in the LD sample, using estimated IQ's for the NA children. Assuming that these estimates were fairly accurate, the NA sample probably approximated the LD sample in heterogeneity of IQ. Thus, the NA sample clearly may have contained individual children who were potential candidates for assignment to LD classrooms. Despite these confounding effects for IQ, a different approach to conceptualizing LD children may make comparison berween LD and NA samples in this study less questionable. Such an approach might be to consider that, in addition to intellectual ability, individual behavioral style or temperament as manifested in the classroom may influence the child's learning (Chess, 1969). Chess identified and described nine categories of temperament: activity level, rhythmicity, approach or withdrawal, adaptability, intensity of reaction, threshold of responsiveness, quality of mood, distractibility, attention span and persistence. Several of these categories (e.g., activity level, distractibility, attention span and persistence) may relate directly to criteria typically used (Learner, 1971) to characterize LD children (e.g., hyperactivity, high distractibility, erratic behavior). Referral of children by the regular classroom teacher for evaluation by the school psychologist may be prompted as much by such temperamental behavioral characteristics as by concern over intellectual problems. Other categories of temperament (e.g., adaptibility, threshold of responsiveness) described by Chess (1969) are suggestive of the findings and interpretations in this study. For example, LD children were shown to adapt readily to demands of a problem solving task in response to altered structuring of the task (behavior similar to Chess' adaptibility category). It was also found that by raising the intensity level of stimulation with mediating cues presumably making important features of the problem solving task more salient, task appropriate responses were evoked (behavior similar to Chess' threshold of response category). Thus, the LD sample in this study appears more homogeneous when interpreted in Chess' terms than when characterized as being of normal or near-normal intellectual ability. This apparent homogeneity in regard to temperament as defined by Chess, suggests a commonality of behavioral style in the LD sample. This suggestion is consistent with the principle interpretation of the present investigation; namely, that LD children manifest a characteristic problem solving strategy or style that is different from that of NA children.

Educational Implications

The study findings suggest that the LD child has acquired an average amount of information and has organized it, but doesn't use it within the same context or in response to the same cues as the NA child does. LD children apparently require a highly structured context provided with cues if they are to use information as well as NA children do. These findings have implications for assessment and instruction of LD children.

Assessing a child's ability with a standardized IQ test involves imposing a rigidly structured set of conditions on the child along with restrictive scoring. An IQ test also supplies the child with limited and restrictive cues. The structure and cues provided may or may not correspond to an optimal context and set of cues for use of task-appropriate information by the LD child. A reasonably close match between test protocol and LD children seems to occur as these children generally score in the low average range, but this may not be the case. Perhaps the low average range and the characteristically uneven profile of subtest scores earned by LD children on IQ tests indicate that the LD child-test match is not at all optimal. Relatively low subtest scores in this typical profile are generally taken as evidence of specific deficits in the LD child. However, these low scores may not result from deficits in the child, but possibly from elements of the test incompatible with the child's most efficient mode of information processing.

The same features of rigidly structured test conditions, limited and restrictive cues, and restrictive scoring apply to assessment on standardized academic achievement tests. LD children, by definition, score poorly on these tests in contrast with their scores on IQ tests which typically fall within the normal range. This poor performance may occur because achievement tests present problems in a context and with cues that are incompatible with the LD child's preferred information processing pattern. One notable difference between achievement and IQ test is the abstract, symbolic content of the achievement test opposed to the more everyday content of the IQ test (pictures, puzzles, blocks, and common sense questions). Also, IQ tests typically involve far more verbal structure and a closer one-to-one relationship with an adult than do academic achievement tests. Such differences may in part account for the discrepancy in scores between these two measures charateristic of LD children. Achievement tests may be particularly void and IQ tests relatively rich in cues, especially verbal cues, that aid LD children in retrieving stored information.

A major implication for instruction is that LD children probably require in-task structure that suggests categories of information useful for completing the task efficiently, and that the addition of cues clearly identifying these categories may be especially important for enhanced learning in LD children. The optimal number and type of cues needed to facilitate learning may vary for individual LD children and for tasks. In this study, visual, motoric, and verbal cues were used together. It was not possible with this procedure to tell whether one cue was most important. Perhaps no one type of cue alone will produce expected levels of learning in LD children, but several different cues together will. Based on the well documented critical importance of verbal mediation for learning in retardates and a few suggestive findings relating to verbal learning patterns in LD children (Blank, Weider, & Bridges, 1968; Haight, 1974), a reasonable speculation might be that verbal cues are vital to information retrieval in LD children. The relative success of LD children on IQ tests which involve highly structured interaction with an adult on a one-to-one basis suggests that instruction of LD children may best be carried out in highly structured, verbally interactive one-to-one tutorial teaching situation. In such a teaching situation, task context and cues can be varied efficiently by the teacher to match idiosyncrasies of learning style in the LD child that originally set him apart from the general school population.

One instance illustrative of teacher introduction of increased categorical task structure and associated cues to match retrieval patterns inferred for LD children in this study may be offered. The illustration involves teaching the multiplication tables. LD children often fail or are erratic in memorization of the multiplication facts. Categorizing aspects of this task, teaching each categorized aspect separately according to one carefully verbalized rule, and using other specifically related cues often represent a basis for success in teaching LD children the times tables. One specific example of this approach involves the fact that LD children frequently fail to distinguish between the processes of multiplication by zero and multiplication by one. In such a case it is useful for the teacher to dwell on one of these processes as a category of responses (e.g., 0 X 1, 0 X 2, 0 X 3, ...) until the pattern becomes clear to the individual LD child, and only then to instruct the second process as a different category of responses. Choosing multiplication by zero to instruct first, the teacher might adopt a phrase such as "<u>nothing</u> times a number <u>must</u> be <u>nothing</u>!" as a verbal cue for the child to repeat and internalize for use whenever he encounters zero x some number. The underlined words might be heavily stressed by means of the teacher's intonation. Such defferential emphasis of words carring task-relevant semantic information is, of course, an important dimension of verbal mediation in teaching. Writing zero in color each time it appears in a multiplication problem presented to the child may be a very helpful supplementary cue to remind the child which process to employ. Motoric cuing may also be useful. Thus, the child might be presented with 3 wooden one inch cubes along with the written problem 0 X 3 and told to pick up the set of 3 blocks "zero times" or "no times." The child, if he understands the command, remains immobile and is usually amused or openly perplexed by the inherent contradiction presented by the notion zero x a number at the sensorimotor level of expression. This absence of a motoric counterpart for the concept zero x a number is usually very salient for a child who has been instructed using concrete experiences as he originally learned the basic arithmetic operations. Such an absence represents an excellent cue for learning this purely abstract concept.

The illustrated teaching technique has been used on many occasions with LD children, usually with satisfactory results after one or two exposures. It essentially involves teaching zero times a number as a principle or a category of similar processes. Meaningful verbal cues are included as mediators of the essential concepts to be taught. A concrete instance is associated with one of these verbal mediators as a parallel cue at a non-verbal level of experience. Also, a visual cue is suggested as a simple device to alert the child when he is confronted with an instance of the categorical response or principle he has been taught or is learning. These aspects of task manipulation clearly are specific examples of the categorical in-task structure and cuing inferred in this investigation to be of general importance for retrieval of categorically stored information by LD children when solving problems.

In general, then, an important implication of the findings in this study is that contexts and cues presented by standard educational assessment and programming may not be optimal for tapping learning potential and for promoting learning in children identified as learning disabled. Consequently, educators may be identifying some children as deficient learners who are simply <u>different</u> learners compared with normal achievers. It also follows that children labeled as learning disabled may require modes of instruction different from those that are seemingly optimal for the majority of school children. If children classified as learning disabled are simply children who learn differently from the majority, then it may be that typical patterns of educational assessment and programming present these children with unwarranted barriers to academic learning.

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Appendix

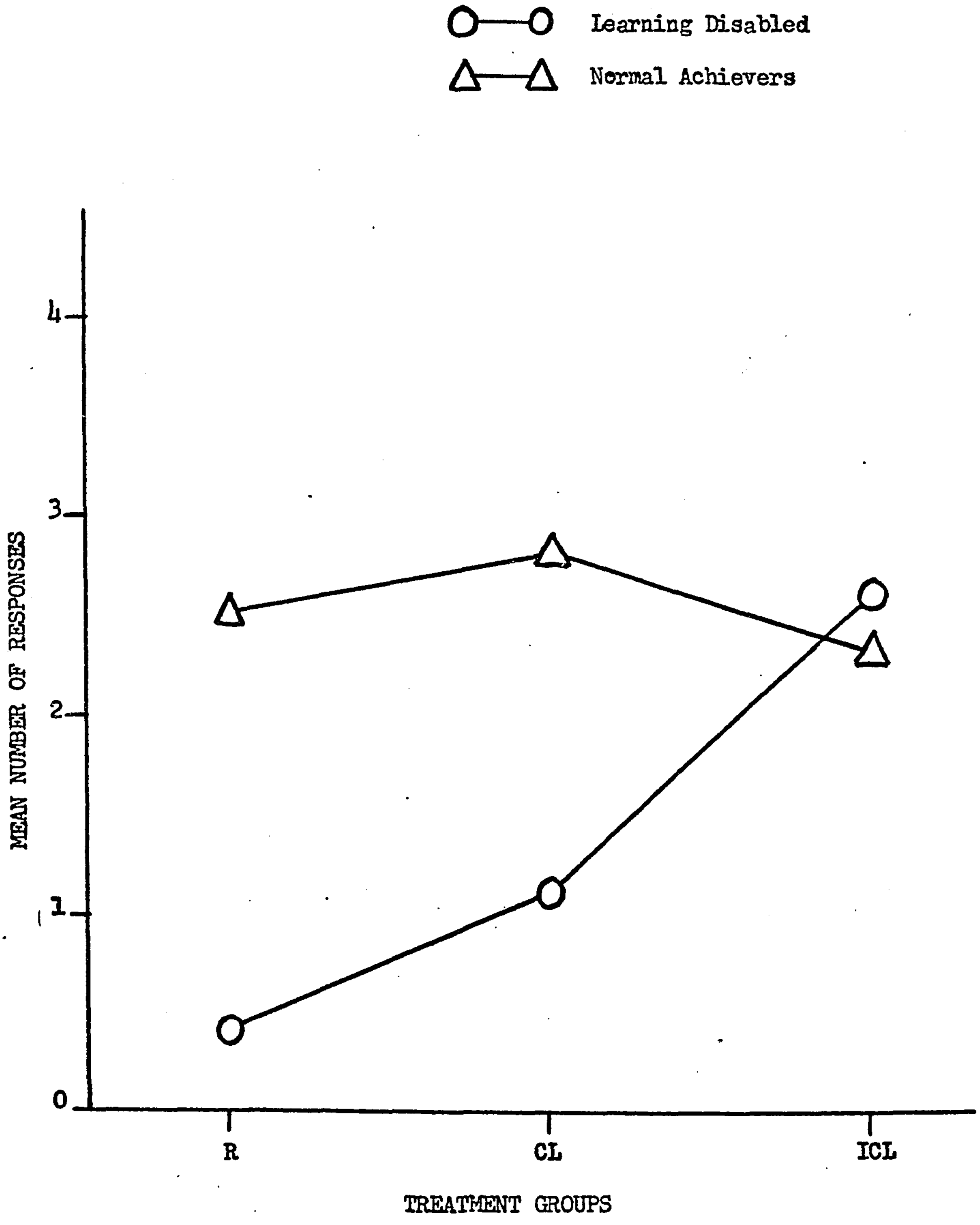

Fig. 3. Mean total number of constraint seeking questions on the twenty-questions test as a function of classification and treatment group.

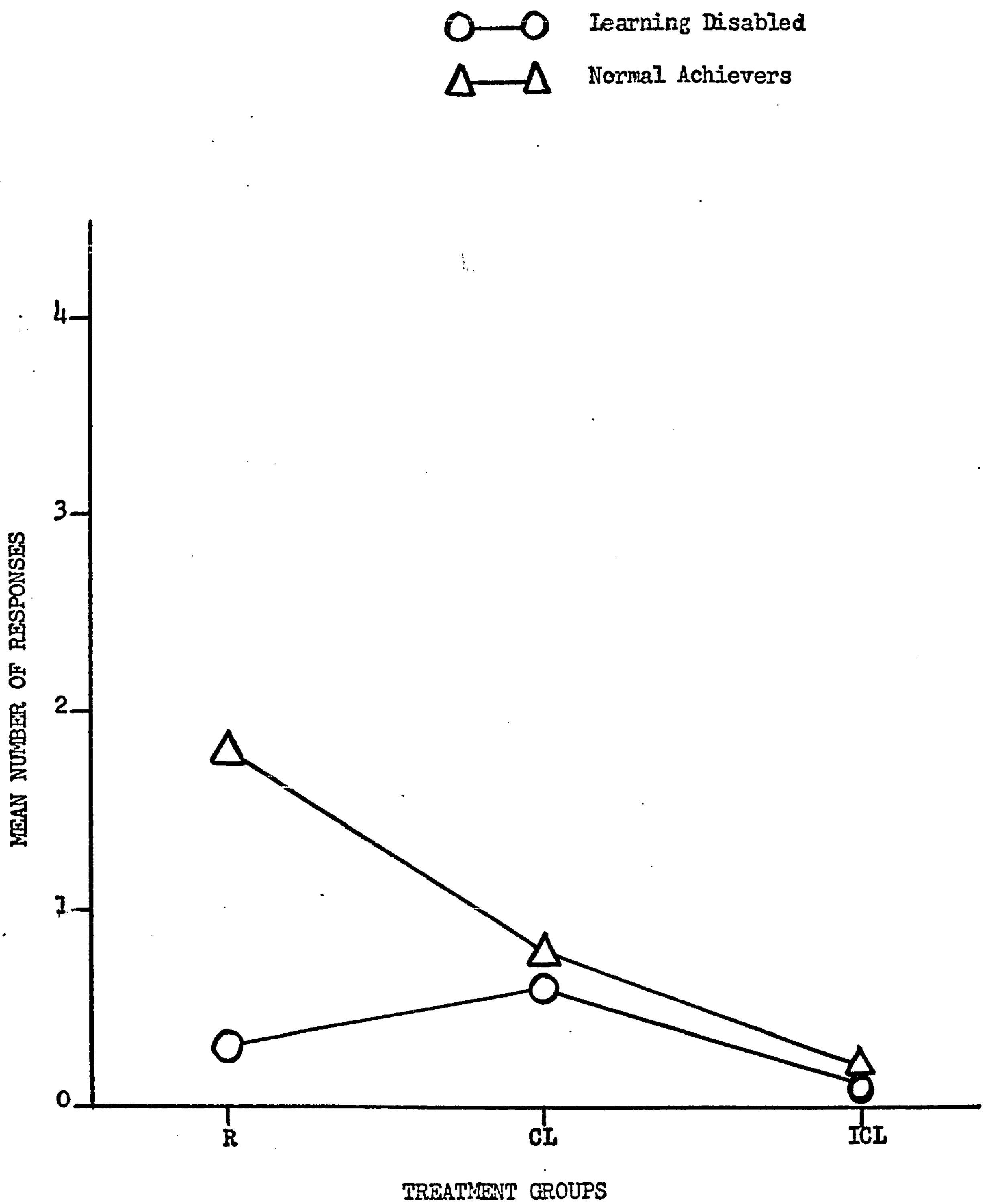

Fig. 4. Mean number of perceptual constraint seeking questions on the twenty-questions test as a function of classification and treatment group.

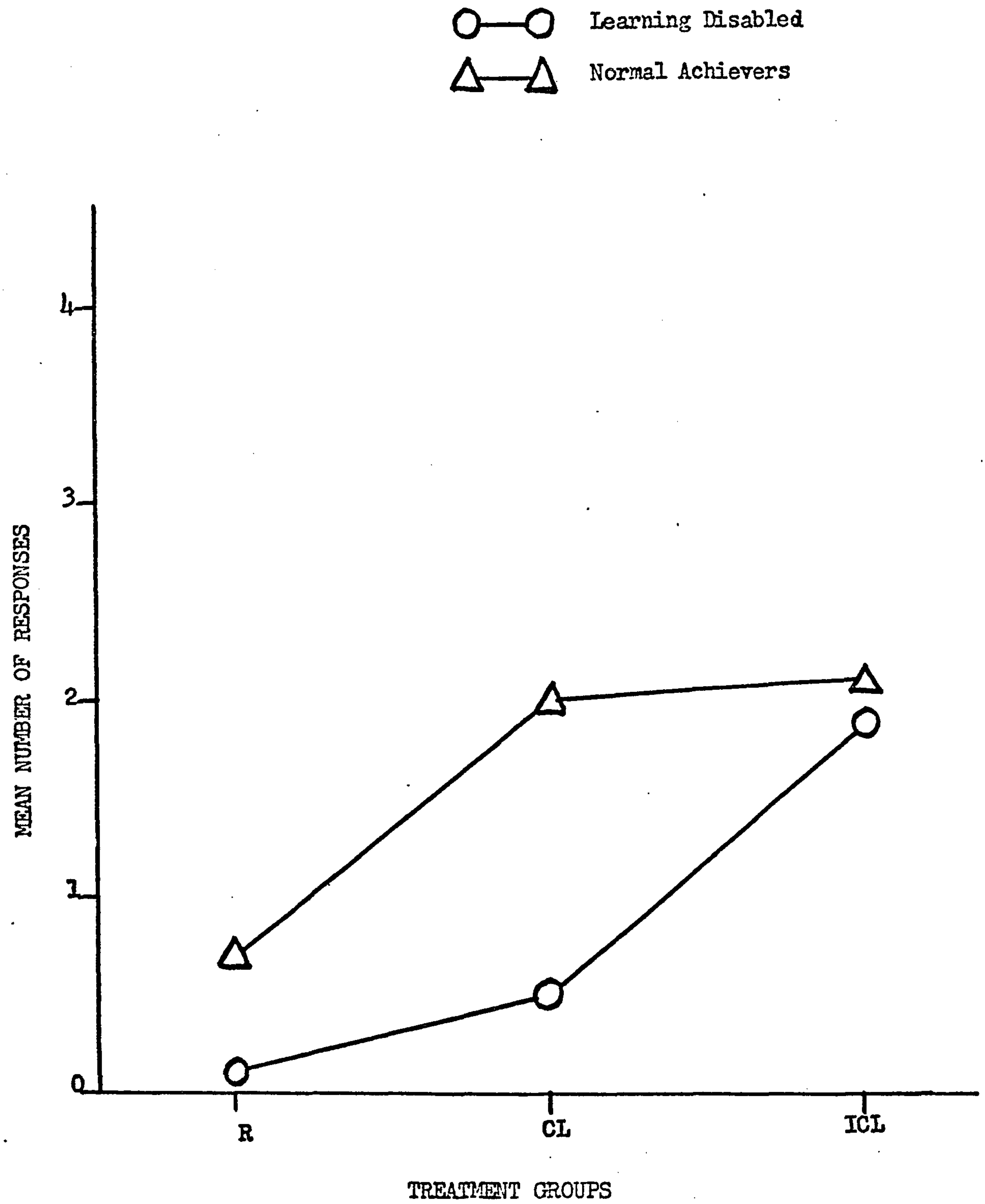

Fig. 5. Mean number of semantic constraint seeking questions on the twenty-questions test as a function of classification and treatment group.

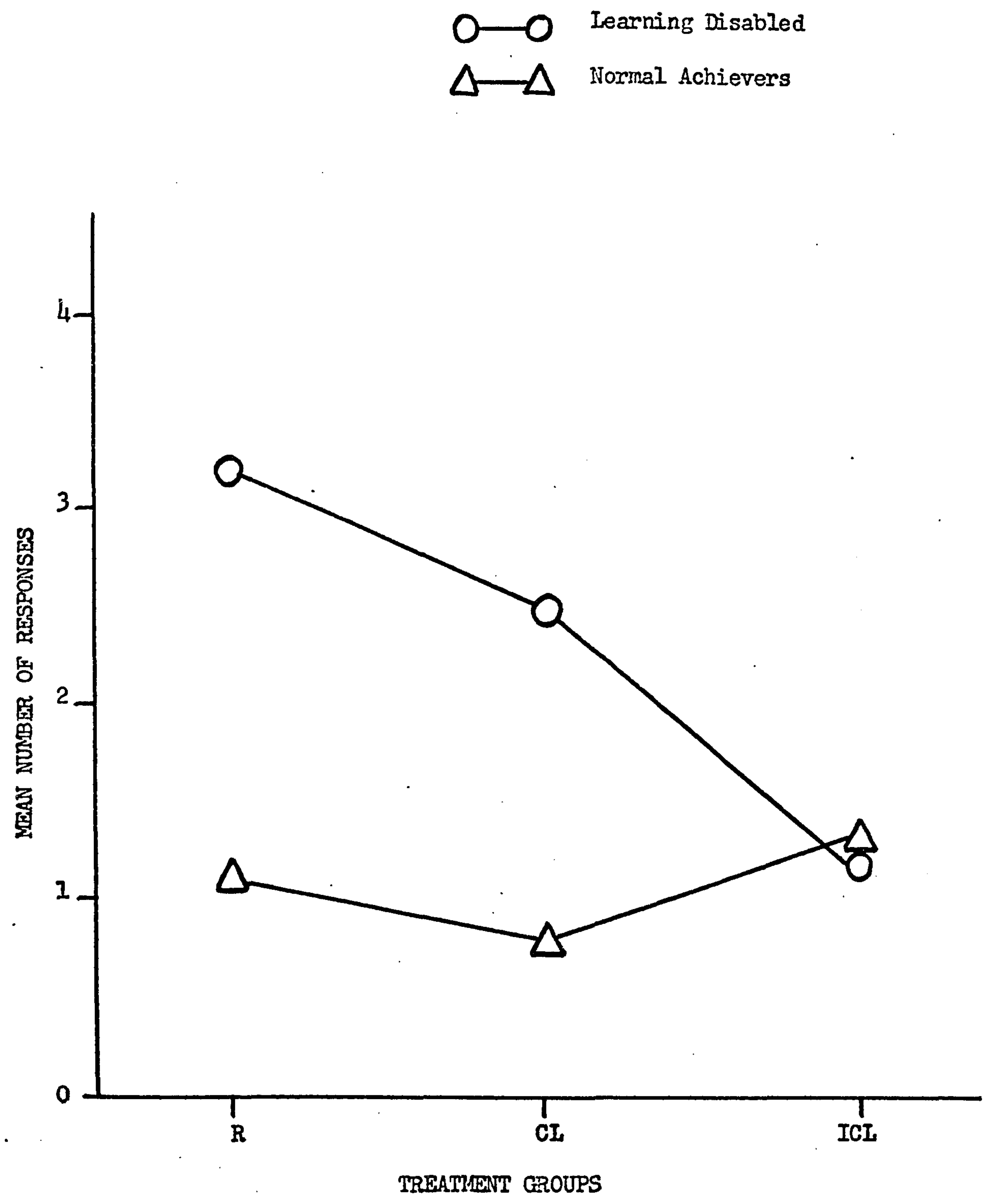

Fig. 6. Mean number of hypothesis seeking questions on the twenty-questions test as a function of classification and treatment group.

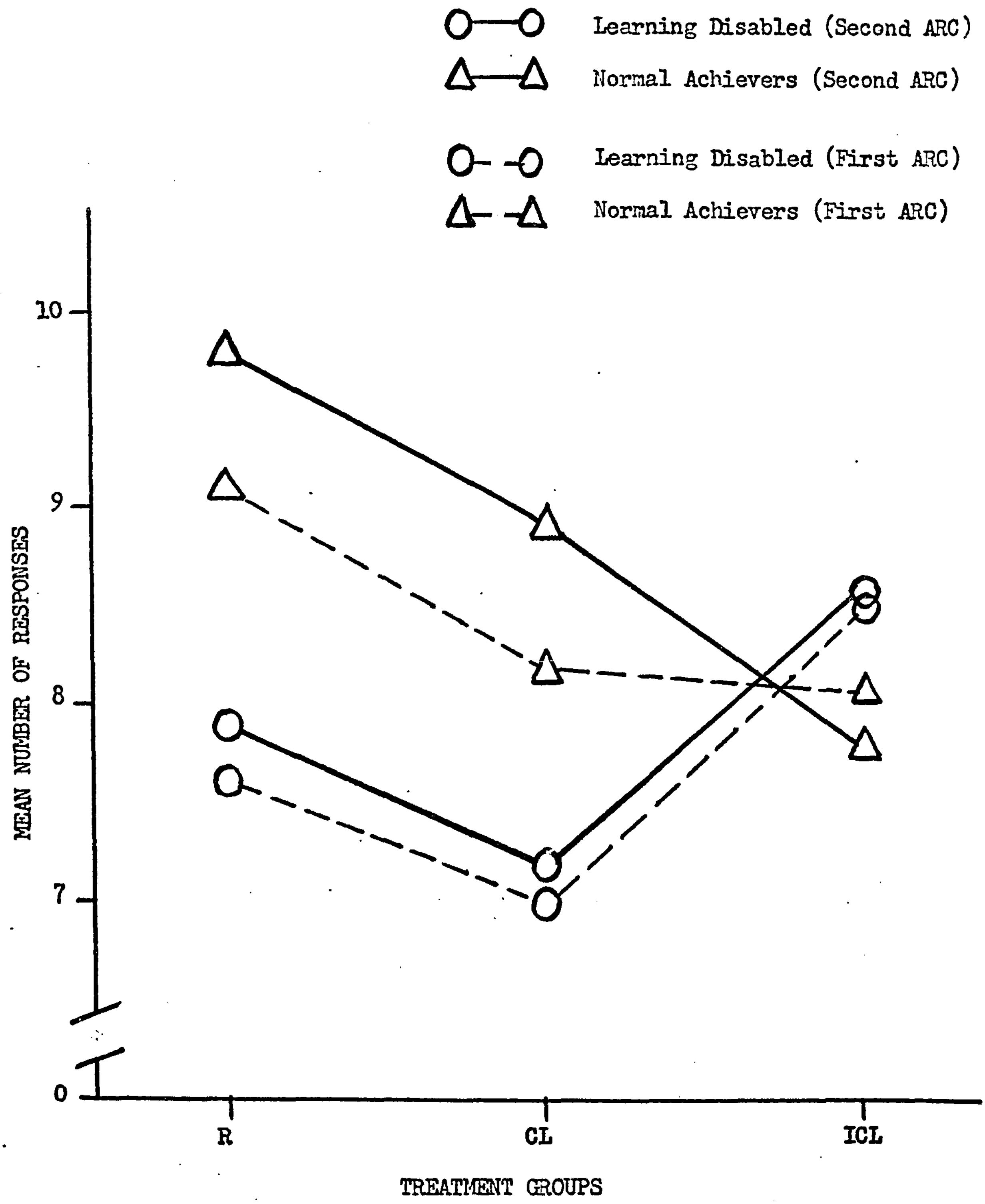

Fig. 7. Mean number of words correctly recalled on first and second associative clustering tests and as a function of classification and treatment group.

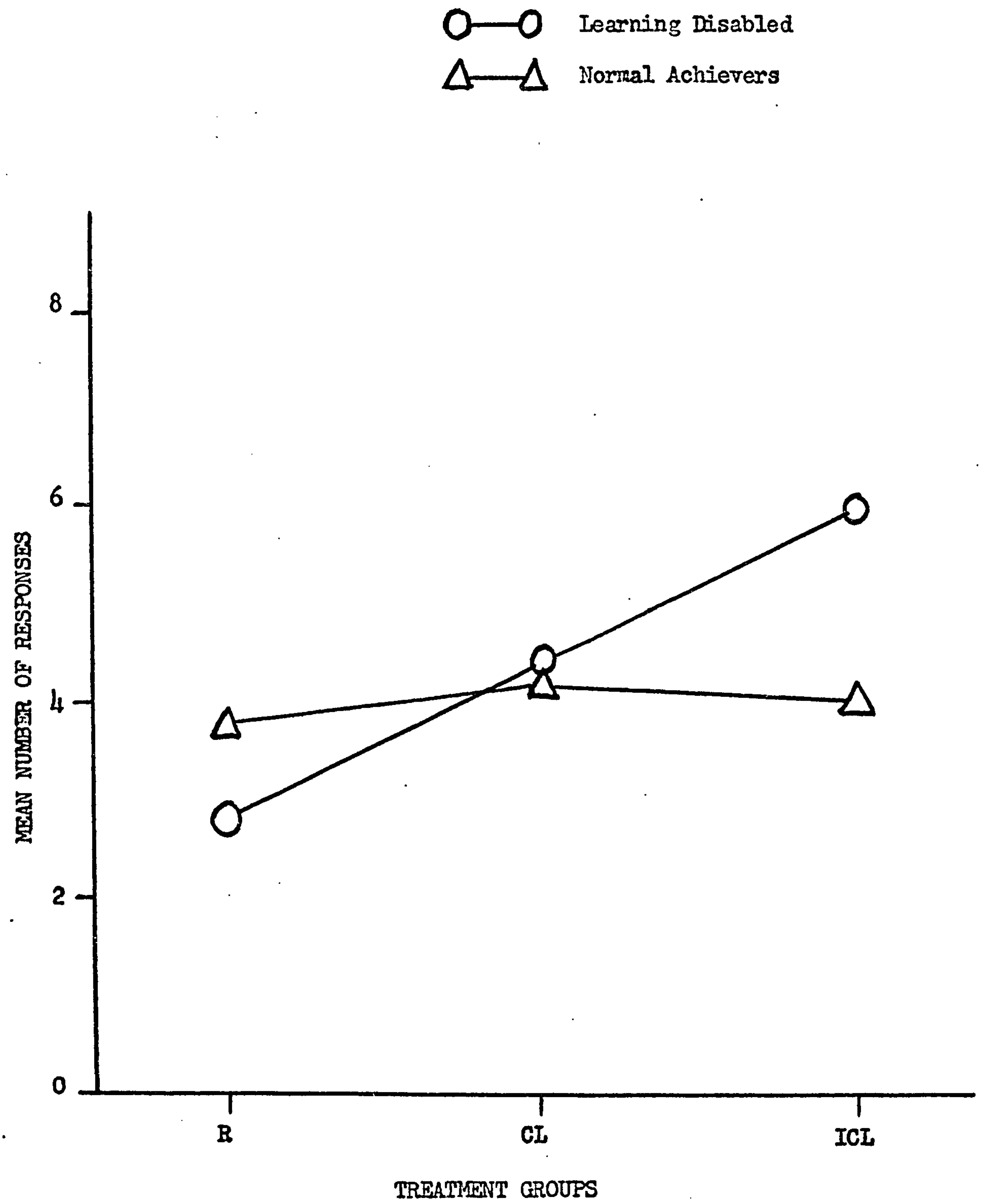

Fig. 8. Mean total number of categories constructed on the immediate sorting task exactly matching the given semantic and perceptual categories of the twenty-questions test as a function of classification and treatment group.

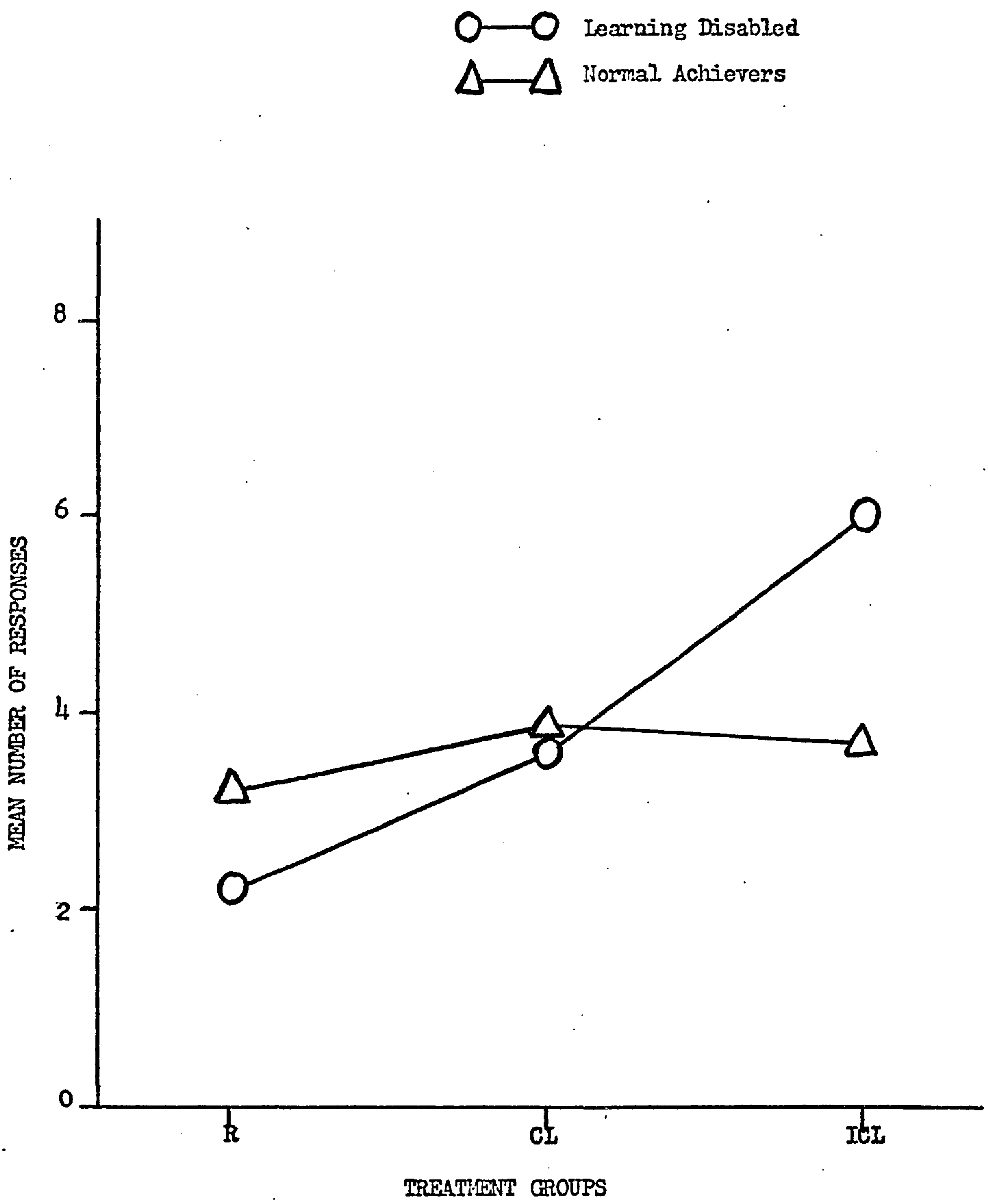

Fig. 9. Mean number of categories constructed on the immediate sorting task exactly matching the given semantic categories of the twenty-questions test as a function of classification and treatment group.

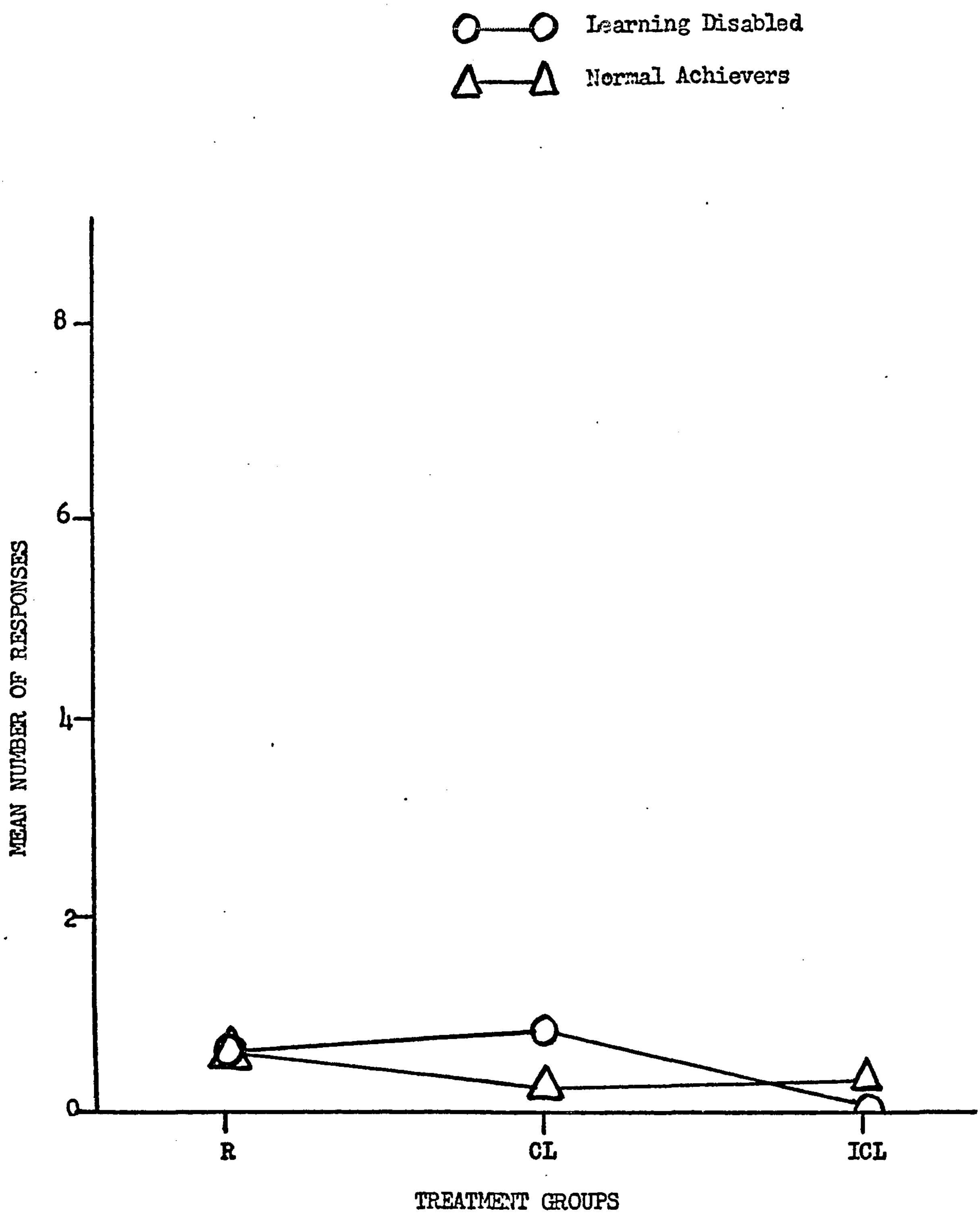

Fig. 10. Mean number of categories constructed on the immediate sorting task exactly matching the given perceptual categories of the twenty-questions test as a function of classification and treatment group.

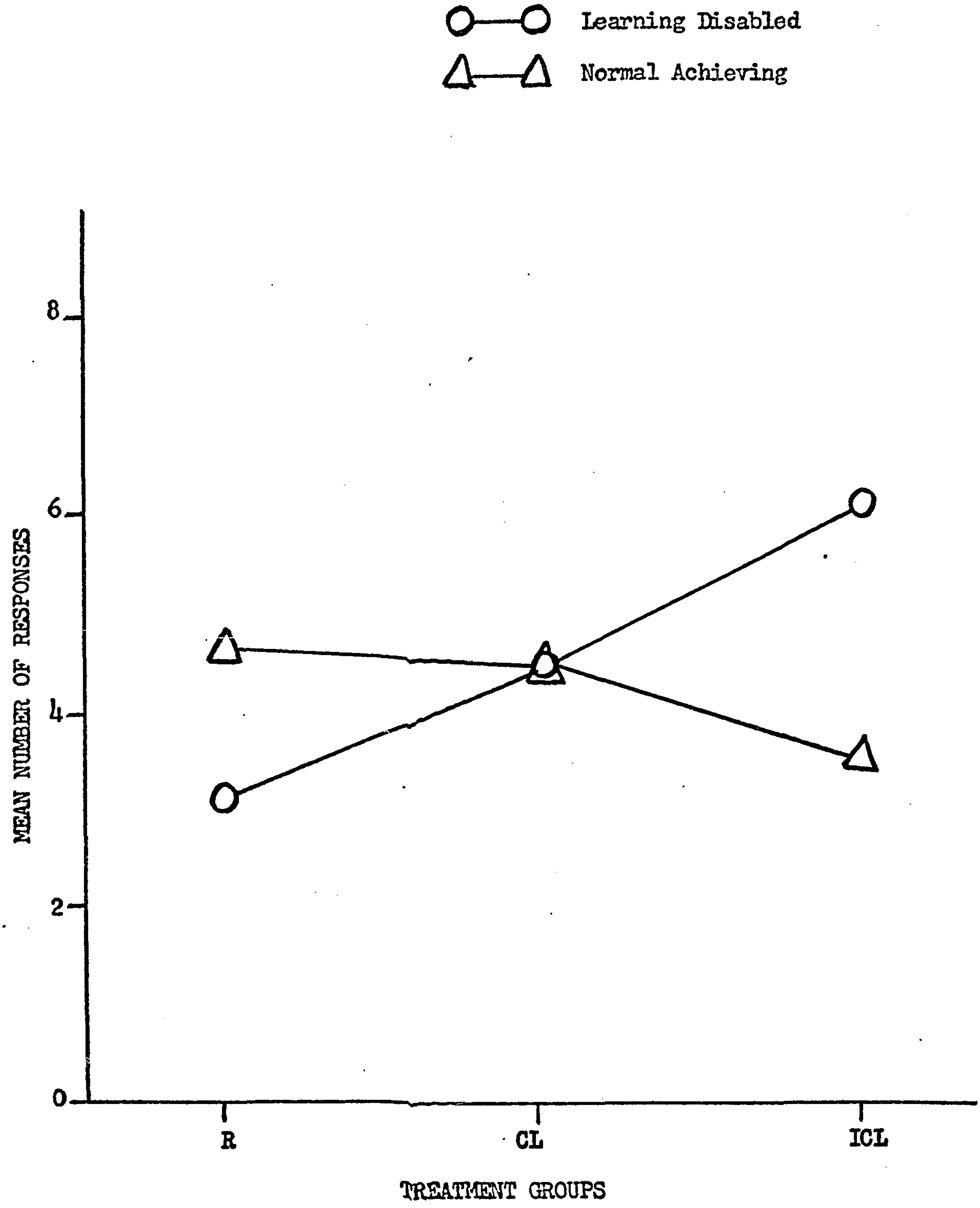

Fig. 11. Mean total number of categories constructed on the delayed sorting task exactly matching the given semantic and perceptual categories of the twenty-questions test as a function of classification and treatment group.

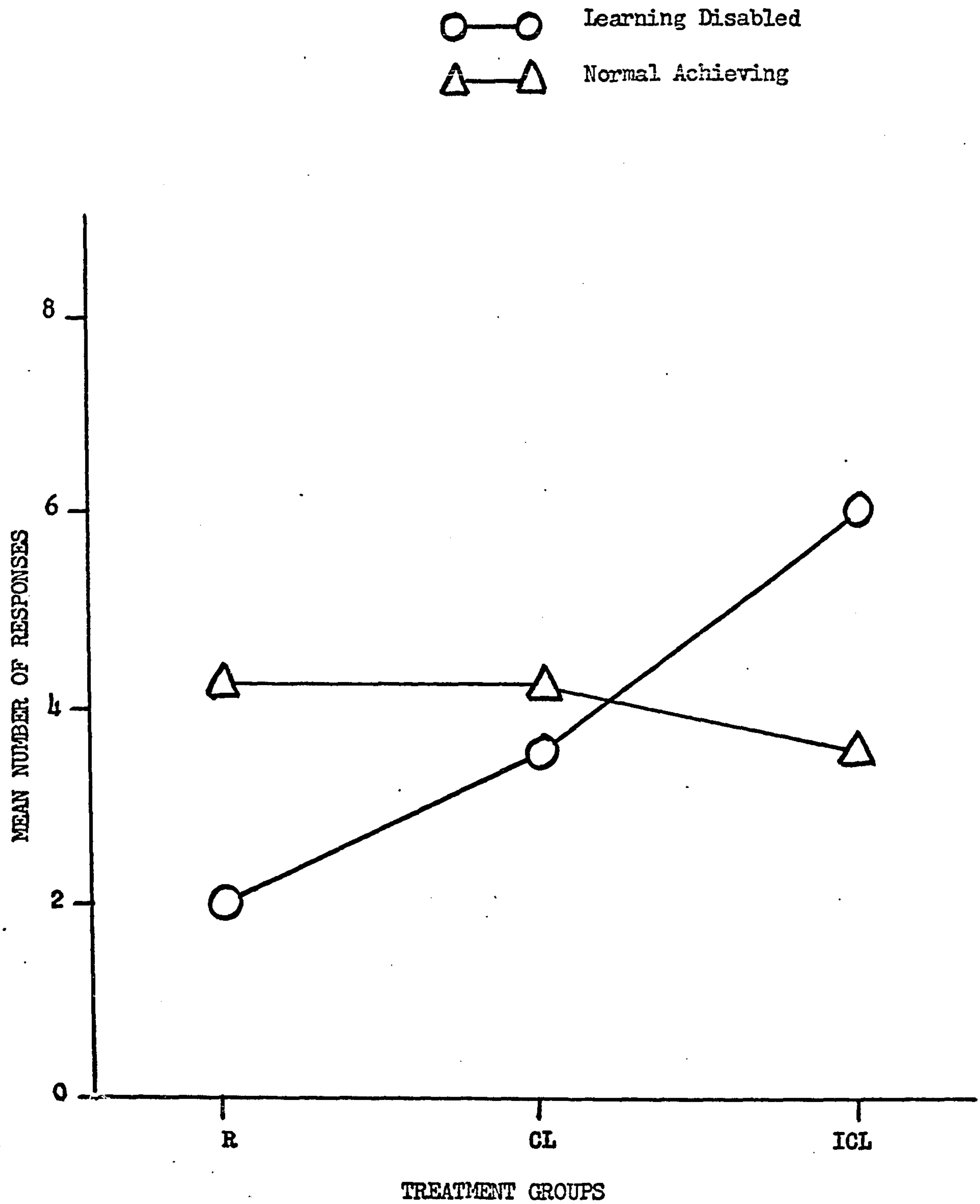

Fig. 12. Mean number of categories constructed on the delayed sorting task exactly matching the given semantic categories of the twenty-questions test as a function of classification and treatment group.

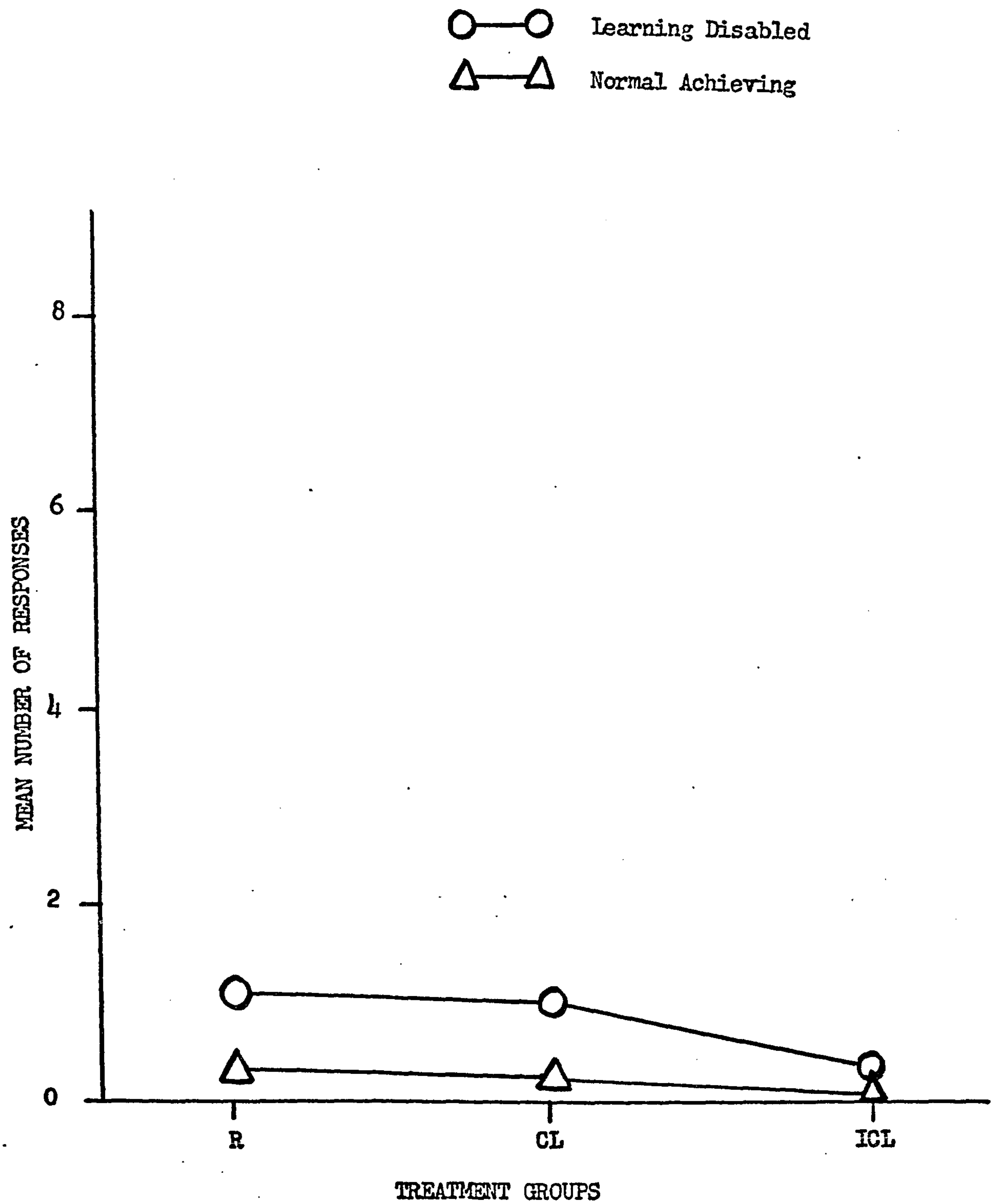

Fig. 13. Mean number of categories constructed on the delayed sorting task exactly matching the given perceptual categories of the twenty-questions test as a function of classification and treatment group.