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CS4800 Algorithms

Homework 1

1. The table, in order of increasingly faster growing functions, is below:

$\log_2 n$ $\log_{10} n^2$	$\sqrt{x}$	$\mathrm{n} \log_2 n$	$25 * n^2$ 1 + 2 + … + n	n^n $500 * 2^n$	$\frac{1}{312}n^6$	$\lvert 1 - 2 + 4 - 8 + \cdots + (-1)^n + 2^n \rvert$	$10 * n!$ $(n-1)!$

2.

A.) Simple function: $g(n) = n^{20}$

Proof: Check that $\lim_{\infty} \frac{f(n)}{g(n)} = c$, where c > 0. This will imply that $f(n) = \theta(g(n))$.

Substitute the two functions in $\lim_{\infty} \frac{(n^2+1)^{10}}{n^{20}}$

Simplify $\lim_{\infty} (\frac{n^2+1}{n^2})^{10}$

Separate the fractions $\lim_{\infty} (\frac{n^2}{n^2} + \frac{1}{n^2})^{10}$

Simplify $\lim_{\infty} (1 + \frac{1}{n^2})^{10}$

Analyze: As $\frac{1}{n^2}$ approaches infinity, it goes closer and closer to 0, so we can focus on the behavior of $(1)^{10}$ as it approaches infinity.

Simplify $\lim_{\infty} (1)^{10}$

Simplify $\lim_{\infty} (1) = 1$

Because the limit equals 1, and so c equals 1 which is greater than 0, we have proven that $f(n) = \theta(g(n))$.

B.) Simple function: $g(n) = n$

Proof: Check that $\lim_{\infty} \frac{f(n)}{g(n)} = c$, where c > 0. This will imply that $f(n) = \theta(g(n))$.

Substitute the two functions in $\lim_{\infty} \frac{\sqrt{10n^2+7n+3}}{n}$

Simplify $\lim_{\infty} (\frac{10n^2+7n+3}{n^2})^{\frac{1}{2}}$

Separate the fractions $\lim_{\infty} (\frac{10n^2}{n^2} + \frac{7n}{n^2} + \frac{3}{n^2})^{\frac{1}{2}}$

Simplify $\lim_{\infty} (10 + \frac{7}{n} + \frac{3}{n^2})^{\frac{1}{2}}$

Analyze: As $\frac{7}{n}$ and $\frac{3}{n^2}$ approach infinity, it goes closer and closer to 0, so we can focus on the behavior of $(10)^{\frac{1}{2}}$ as it approaches infinity.

Simplify $\lim_{\infty} (10)^{\frac{1}{2}}$

Simplify $\lim_{\infty} (\sqrt{10}) = \sqrt{10}$

Because the limit equals $\sqrt{10}$, and so c equals $\sqrt{10}$ which is greater than 0, we have proven that $f(n) = \theta(g(n))$.

C.) Simple function: $g(n) = n^2 \log_2 n$

There are 2 main parts to this equation:

$2n \log_2[(n+2)^2]$	and	$(n+2)^2 \log_2 \frac{n}{2}$
$= 4n \log_2(n+2)$		$= (n^2 + 4n + 4) \log_2(\frac{n}{2})$
$\approx n \log_2 n$		$\approx n^2 \log_2 n$

Since $n^2 \log_2 n$ is larger asymptotically than $n \log_2 n$, the simple function would be: $g(n) = n^2 \log_2 n$.

D.) Simple function: $g(n) = 3^n$

There are 2 main parts to this equation:

2^{n+1}	and	3^{n-1}
$\approx 2^n$		$\approx 3^n$

Since 3^n is larger asymptotically than 2^n, the simple function would be: $g(n) = 3^n$.

E.) Simple function: $g(n) = \log_2 n$

Explanation: Because the $\lfloor \log_2 n \rfloor$has $\log_2 n$ complexity, because the floor of any function is always less than or equal to the function itself.

3. I will try to prove that $max\{f(n), g(n)\} = \theta(f(n) + g(n))$.

In order to prove this, there must be two constants a and b, where $a \leq b$, and an integer n_p that is positive and all $n \geq n_p$, since $f(n)$ and $g(n)$ are non-negative functions.

Also, because $f(n)$ and $g(n)$ are non-negative functions, we know that $n \geq n_p \geq 0$.

This implies: $0 \leq a * \big(f(n) + g(n)\big) \leq max\{f(n), g(n)\} \leq b * (f(n) + g(n))$.

Now, we need to choose positive numbers for a and b.

- If we choose ½ for a, then:
  - If $f(n) > g(n)$:

$$\frac{1}{2}\big(f(n) + g(n)\big) < \frac{1}{2}(f(n) + f(n))$$

$$= \frac{1}{2}\big(f(n) + g(n)\big) < f(n)$$

$$= max\{f(n), g(n)\}$$

  - If $g(n) > f(n)$:

$$\frac{1}{2}\big(f(n) + g(n)\big) < \frac{1}{2}(g(n) + g(n))$$

$$= \frac{1}{2}\big(f(n) + g(n)\big) < g(n)$$

$$= max\{f(n), g(n)\}$$

- If we choose 2 for b, then:

$$max\{f(n), g(n)\} \leq 2(f(n) + g(n))$$

So, if we choose a = ½ and b = 2, the statement holds.

4. I can split the problem into two: the location and the distance traveled. As an example, if I start on a road with a number line and I start at 0, I will decide to go to the east at a distance of 1 (measure of distance does not matter, as I imagine I am on a number line). Then, I decide to go to the other direction to -1. Then I turn around and go to +2, and then -2, etc.
   The location that I am in is presented by the following formula:

   $T(k) = 1 - 2 + 4 - 8 + \cdots + 2^k$. But we can simplify this formula. The formula is $sum = \frac{next\ term - first\ term}{ratio - 1}$, where the next term is -2^{k+1}, first term is 1, and ratio is -2.

   This simplifies to $T(k) = \frac{(-2)^{k+1} - 1}{-3}$, which is the location that I am in as I move across the line.

   In terms of travel, it would be the absolute value of all the distances, so it will look like this: $1 + 2 + 4 + 8 + \cdots + 2^k$. This simplifies to: $2^{k+1} - 1$.

   Now, to calculate the rate-of-growth relationship between length traveled and distance of farthest points reached, we can split the problem into two parts.

   <u>If k is odd</u>: You have to calculate the absolute value of location, so it will look like this: $|location| = \frac{2^{k+1} - 1}{3}$ and for travel to equal location, you will have to multiply the absolute value of location by 3, so: $travel = 3|location|$.

   <u>If k is even</u>: You do not have to take the absolute value of location, so it will look like this: $location = \frac{-2^{k+1} - 1}{-3} = \frac{2^{k+1}}{3}$. So now $travel = 3 * location$.

   So now with this simplification, we can say $travel = 2^{k+1} + 1 = 3 * location$. So, the distance traveled to get to the well is *at most* 10 times the location.

5. A.) $T(1) = 1$

$T(2) = 3 + 2$

$T(3) = 3^2 + 2 * 3^1 + 2$

$T(4) = 3^3 + 2 * 3^2 + 2 * 3^1 + 2$

$T(5) = 3^4 + 2 * 3^3 + 2 * 3^2 + 2 * 3^1 + 2$

...

$T(n) = 3^{n-1} + 2\underbrace{(3^{n-2} + 3^{n-1} + \cdots + 1)}$

This is a geometric progression equal to $\frac{3^{n-1}-1}{2}$

$= 3^{n-1} + 2(\frac{3^{n-1}-1}{2})$

$= 2 * 3^{n-1} - 1$ **This is the guess to a solution for the recurrence**

Prove the base case:

$T(1) = 1$

$2 * 3^{1-1} - 1 => 2 * 3^0 - 1 => 2 * 1 - 1 => 2 - 1 => 1$

Prove T(n) is equal to guess using n-1:

$3T(n-1) + 2$

$= 3 * (2 + 3^{n-2} - 1) + 2$

$= 2(3^{n-1}) - 3 + 2$

$= 2(3^{n-1}) - 1$

$= T(n)$

So, we have proven that $2 * 3^{n-1} - 1$ is an appropriate expression for the recurrence.

B.) $T(1) = 1$

$T(3) = 3 + 1$

$T(9) = 3^2 + 3 * 1 + 1$

$T(27) = 3^3 + 3^2 * 1 + 3 + 1$

...

$T(n) = \underbrace{3^n + 3^{n-1} + 3^{n-2} + \cdots + 3^{n-n}}$

This is a geometric progression equal to $\frac{3n-1}{2}$

This is the guess to a solution for the recurrence

Prove the base case:

$$T(1) = 1$$

$$\frac{3(1)-1}{2} => \frac{3-1}{2} => \frac{2}{2} => 1$$

Prove T(n) is equal to guess using n-1:

$$3T\left(\frac{n-1}{3}\right) + 1$$

$$= 3 * [\frac{\frac{3(n-1)-1}{2}}{3}] + 1$$

$$= 3 * [\frac{\frac{3n-4}{2}}{3}] + 1$$

$$= (\frac{3n-4}{2}) + 1$$

$$= \frac{3n-1}{2}$$

$$= T(n)$$

So, we have proven that $\frac{3n-1}{2}$ is an appropriate expression for the recurrence.

6. The solution is $2n - 1$.
   To prove that, I will have to first prove the base case and then the recursive case.

   <u>Base Case</u>:
   $T(1) = 1$
   $2(1) - 1 => 2 - 1 => 1$

   <u>Induction Step</u>:
   We will use strong induction and assume $T(k) = 2k - 1$ is true for all $k < n$. Prove true for n. Also, we will make n > 1.
   $T(n) = T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1$
   Now, if we plug in $\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lceil\frac{n}{2}\right\rceil$ into my solution, $2n - 1$, we should get the same answer as if we had plugged them into T(n).

   The original recursion: $T\left(\frac{n}{2}\right) = T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1$
   Using the solution: $T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) = 2\left\lfloor\frac{n}{2}\right\rfloor - 1$ and $T\left(\left\lceil\frac{n}{2}\right\rceil\right) = 2\left\lceil\frac{n}{2}\right\rceil - 1$
   Substitute in those values for the ceiling and floor:
   $T\left(\frac{n}{2}\right) = 2\left\lfloor\frac{n}{2}\right\rfloor - 1 + 2\left\lceil\frac{n}{2}\right\rceil - 1 + 1$

Factor out the 2: $T\left(\frac{n}{2}\right) = 2(\left\lfloor\frac{n}{2}\right\rfloor + \left\lceil\frac{n}{2}\right\rceil) - 1 - 1 + 1$

There is a theorem that states that the floor and ceiling of any number $\frac{n}{2}$ equals n, so if we apply it here, the equation becomes: $T\left(\frac{n}{2}\right) = 2(n) - 1 - 1 + 1$

Simplify: $T\left(\frac{n}{2}\right) = 2(n) - 1$

$$T\left(\frac{n}{2}\right) = 2n - 1$$

The induction shows that $2n - 1$ is indeed a formula that is valid for all n.