# Chapter 12

Chi-Square and
Nonparametric Tests

# Learning Objectives

**In this chapter, you learn:**

- How and when to use the chi-square test for contingency tables
- How to use the Marascuilo procedure for determining pairwise differences when evaluating more than two proportions
- How and when to use nonparametric tests

# $\chi^2$ Test for the Difference Between Two Proportions

DCOVA

$H_0$: $\pi_1 = \pi_2$ (Proportion of females who are left handed is equal to the proportion of males who are left handed)

$H_1$: $\pi_1 \neq \pi_2$ (The two proportions are not the same)

- If $H_0$ is true, then the proportion of left-handed females should be the same as the proportion of left-handed males
- The two proportions above should be the same as the proportion of left-handed people overall

# The Chi-Square Test Statistic

DCOVA

The Chi-square test statistic is:

$$\chi^2_{STAT} = \sum_{all\ \text{cells}} \frac{(f_o - f_e)^2}{f_e}$$

- where:
  $f_o$ = observed frequency in a particular cell
  $f_e$ = expected frequency in a particular cell if $H_0$ is true

$\chi^2_{STAT}$ **for the 2 x 2 case has 1 degree of freedom**

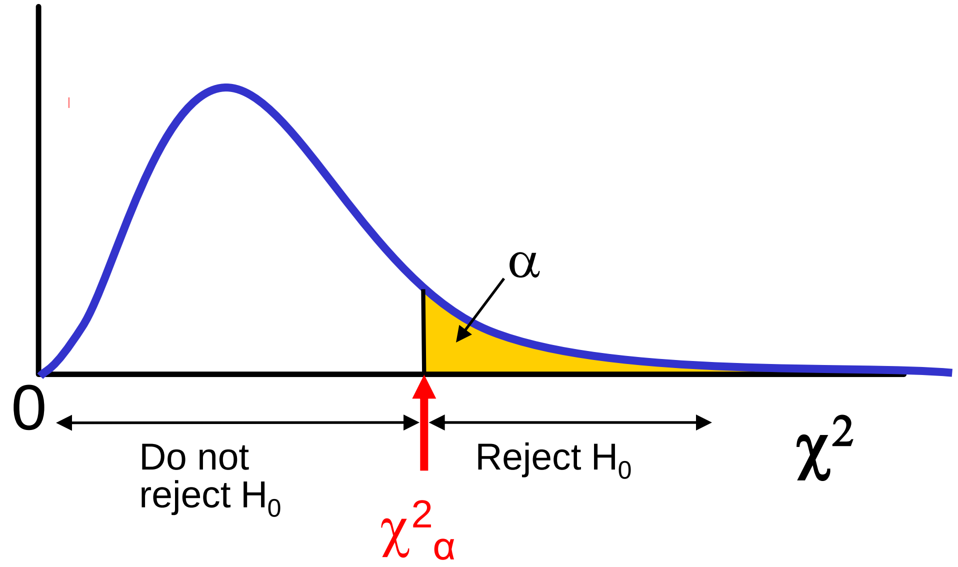
(Assumed: each cell in the contingency table has expected frequency of at least 5)

# Decision Rule

DCOVA

The $\chi^2_{STAT}$ test statistic approximately follows a chi-squared distribution with one degree of freedom

**Decision Rule:**
If $\chi^2_{STAT} > \chi^2_{\alpha}$, reject $H_0$, otherwise, do not reject $H_0$



# Observed vs. Expected Frequencies

DCOVA

| Gender | Hand Preference | | |
|---|---|---|---|
| | Left | Right | |
| Female | Observed = 12<br>Expected = 14.4 | Observed = 108<br>Expected = 105.6 | 120 |
| Male | Observed = 24<br>Expected = 21.6 | Observed = 156<br>Expected = 158.4 | 180 |
| | 36 | 264 | 300 |

$$f_e = \frac{\text{row total} \times \text{column total}}{\text{n}}$$

$$= \frac{120 \times 36}{300} = 14.4$$

# The Chi-Square Test Statistic

DCOVA

| Gender | Hand Preference | | |
|---|---|---|---|
| | Left | Right | |
| Female | Observed = 12<br>Expected = 14.4 | Observed = 108<br>Expected = 105.6 | 120 |
| Male | Observed = 24<br>Expected = 21.6 | Observed = 156<br>Expected = 158.4 | 180 |
| | 36 | 264 | 300 |

The test statistic is:

$$\chi^2_{STAT} = \sum_{\text{all cells}} \frac{(f_o - f_e)^2}{f_e}$$

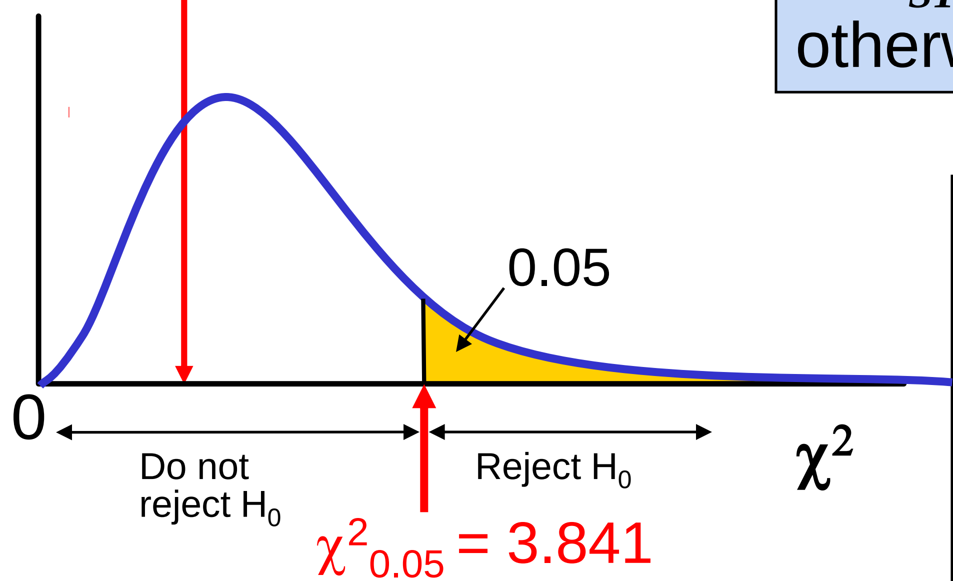
$$= \frac{(12-14.4)^2}{14.4} + \frac{(108-105.6)^2}{105.6} + \frac{(24-21.6)^2}{21.6} + \frac{(156-158.4)^2}{158.4} = 0.7576$$

# Decision Rule

DCOVA

The test statistic is $\chi^2_{STAT} = \boxed{0.7576}$; $\chi^2_{0.05}$ with 1 d.f. $= 3.841$

**Decision Rule:**
If $\chi^2_{STAT} > 3.841$, reject $H_0$, otherwise, do not reject $H_0$




Here,
$\chi^2_{STAT} = 0.7576 < \chi^2_{0.05} = 3.841$, so we do not reject $H_0$ and conclude that there is not sufficient evidence that the two proportions are different at $\alpha = 0.05$

# $\chi^2$ Test for Differences Among More Than Two Proportions

DCOVA

- Extend the $\chi^2$ test to the case with more than two independent populations:

$H_0$: $\pi_1 = \pi_2 = \cdots = \pi_c$

$H_1$: Not all of the $\pi_j$ are equal ($j = 1, 2, \cdots, c$)

# The Chi-Square Test Statistic

DCOVA

The Chi-square test statistic is:

$$\chi^2_{STAT} = \sum_{all\ \text{cells}} \frac{(f_o - f_e)^2}{f_e}$$

- Where:
  $f_o$ = observed frequency in a particular cell of the 2 x c table
  $f_e$ = expected frequency in a particular cell if $H_0$ is true

$$\chi^2_{STAT} \textbf{ for the 2 x c case has (2 - 1)(c - 1) = c - 1 degrees of freedom}$$

(Assumed: each cell in the contingency table has expected frequency of at least 1)

- Expected cell frequencies for the c categories are calculated as in the 2 x 2 case, and the decision rule is the same:

Decision Rule:
If $\chi^2_{STAT} > \chi^2_{\alpha}$, reject $H_0$,
otherwise, do not reject $H_0$

Where $\chi^2_{\alpha}$ is from the chi-squared distribution with c – 1 degrees of freedom

# The Marascuilo Procedure

DCOVA

- Used when the null hypothesis of equal proportions is rejected
- Enables you to make comparisons between all pairs
- Start with the observed differences, $p_j - p_{j'}$, for all pairs (for $j \neq j'$) then compare the absolute difference to a calculated critical range

# The Marascuilo Procedure

*(continued)*

DCOVA

- Critical Range for the Marascuilo Procedure:

$$\text{Critical range} = \sqrt{\chi^2_\alpha}\sqrt{\frac{p_j(1-p_j)}{n_j} + \frac{p_{j'}(1-p_{j'})}{n_{j'}}}$$

  - (Note: the critical range is different for each pairwise comparison)

- A particular pair of proportions is significantly different if

$$|p_j - p_{j'}| > \text{critical range for } j \text{ and } j'$$

# Marascuilo Procedure Example

DCOVA

A University is thinking of switching to a trimester academic calendar. A random sample of 100 administrators, 50 students, and 50 faculty members were surveyed

| Opinion | Administrators | Students | Faculty |
|---|---|---|---|
| Favor | 63 | 20 | 37 |
| Oppose | 37 | 30 | 13 |
| Totals | 100 | 50 | 50 |

Using a 1% level of significance, which groups have a different attitude?

# Chi-Square Test Results

DCOVA

$H_0: \pi_1 = \pi_2 = \pi_3$

$H_1$: Not all of the $\pi_j$ are equal ($j = 1, 2, 3$)

**Chi-Square Test: Administrators, Students, Faculty**

| | Admin | Students | Faculty | Total |
|---|---|---|---|---|
| **Favor** | 63 | 20 | 37 | 120 |
| | 60 | 30 | 30 | |
| **Oppose** | 37 | 30 | 13 | 80 |
| | 40 | 20 | 20 | |
| **Total** | 100 | 50 | 50 | 200 |

Expected

Observed

$$\chi^2_{STAT} = 12.792 > \chi^2_{0.01} = 9.2103 \text{ so reject } H_0$$

# Marascuilo Procedure: Solution

DCOVA

Excel Output:

compare

| Marascuilo Procedure | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | Sample | Sample | | Absolute | Std. Error | Critical | |
| Group | Proportion | Size | Comparison | Difference | of Difference | Range | Results |
| 1 | 0.63 | 100 | 1 to 2 | 0.23 | 0.084445249 | 0.2563 | Means are not different |
| 2 | 0.4 | 50 | 1 to 3 | 0.11 | 0.078606615 | 0.2386 | Means are not different |
| 3 | 0.74 | 50 | 2 to 3 | 0.34 | 0.092994624 | 0.2822 | Means are different |

At 1% level of significance, there is evidence of a difference in attitude between students and faculty

# $\chi^2$ Test of Independence

DCOVA

- Similar to the $\chi^2$ test for equality of more than two proportions, but extends the concept to contingency tables with r rows and c columns

$H_0$: The two categorical variables are independent
(i.e., there is no relationship between them)
$H_1$: The two categorical variables are dependent
(i.e., there is a relationship between them)

# $\chi^2$ Test of Independence

*(continued)*

The Chi-square test statistic is:

DCOVA

$$\chi^2_{STAT} = \sum_{all\ \text{cells}} \frac{(f_o - f_e)^2}{f_e}$$

- where:

  $f_o$ = observed frequency in a particular cell of the r x c table

  $f_e$ = expected frequency in a particular cell if $H_0$ is true

$\chi^2_{STAT}$ for the r x c case has (r - 1)(c - 1) degrees of freedom

(Assumed: each cell in the contingency table has expected frequency of at least 1)

# Expected Cell Frequencies

DCOVA

- Expected cell frequencies:

$$f_e = \frac{\text{row total} \times \text{column total}}{n}$$

Where:

row total = sum of all frequencies in the row

column total = sum of all frequencies in the column

n = overall sample size

# Decision Rule

DCOVA

- The decision rule is

If $\chi^2_{STAT} > \chi^2_{\alpha}$, reject $H_0$,

otherwise, do not reject $H_0$

Where $\chi^2_{\alpha}$ is from the chi-squared distribution with $(r – 1)(c – 1)$ degrees of freedom

# Example

DCOVA

- The meal plan selected by 200 students is shown below:

| Class Standing | Number of meals per week | | | Total |
|---|---|---|---|---|
| | 20/week | 10/week | none | |
| Fresh. | 24 | 32 | 14 | 70 |
| Soph. | 22 | 26 | 12 | 60 |
| Junior | 10 | 14 | 6 | 30 |
| Senior | 14 | 16 | 10 | 40 |
| Total | 70 | 88 | 42 | 200 |

# Example

DCOVA

*(continued)*

- The hypothesis to be tested is:

$H_0$: Meal plan and class standing are independent
(i.e., there is no relationship between them)
$H_1$: Meal plan and class standing are dependent
(i.e., there is a relationship between them)

# Example: Expected Cell Frequencies

*(continued)*

DCOVA

Observed:

| Class Standing | Number of meals per week | | | Total |
|---|---|---|---|---|
| | 20/wk | 10/wk | none | |
| Fresh. | 24 | 32 | 14 | 70 |
| Soph. | 22 | 26 | 12 | 60 |
| Junior | 10 | 14 | 6 | 30 |
| Senior | 14 | 16 | 10 | 40 |
| Total | 70 | 88 | 42 | 200 |

Expected cell frequencies if $H_0$ is true:

| Class Standing | Number of meals per week | | | Total |
|---|---|---|---|---|
| | 20/wk | 10/wk | none | |
| Fresh. | 24.5 | 30.8 | 14.7 | 70 |
| Soph. | 21.0 | 26.4 | 12.6 | 60 |
| Junior | 10.5 | 13.2 | 6.3 | 30 |
| Senior | 14.0 | 17.6 | 8.4 | 40 |
| Total | 70 | 88 | 42 | 200 |

Example for one cell:

$$f_e = \frac{\text{row total} \times \text{column total}}{n}$$

$$= \frac{30 \times 70}{200} = 10.5$$

# Example: The Test Statistic

*(continued)*

DCOVA

- The test statistic value is:

$$\chi^2_{STAT} = \sum_{all\ \text{cells}} \frac{(f_o - f_e)^2}{f_e}$$
$$= \frac{(24 - 24.5)^2}{24.5} + \frac{(32 - 30.8)^2}{30.8} + \cdots + \frac{(10 - 8.4)^2}{8.4} = 0.709$$

$\chi^2_{0.05}$ = 12.592 from the chi-squared distribution with (4 – 1)(3 – 1) = 6 degrees of freedom

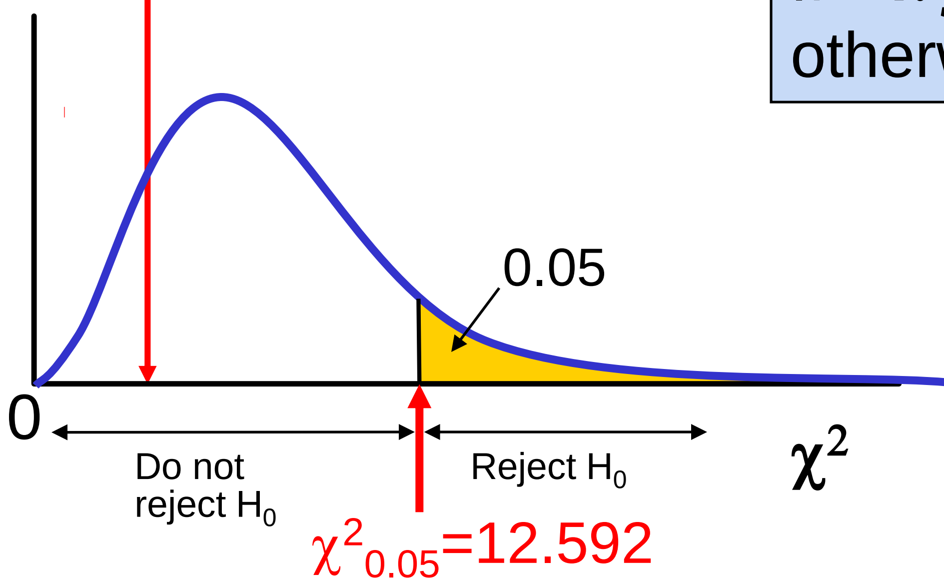
# Example: Decision and Interpretation

DCOVA

*(continued)*

**The test statistic is $\chi^2_{STAT} = \boxed{0.709}$; $\chi^2_{0.05}$ with 6 d.f. $= 12.592$**

Decision Rule:
If $\chi^2_{STAT} > 12.592$, reject $H_0$, otherwise, do not reject $H_0$




Here,
$\chi^2_{STAT} = 0.709 < \chi^2_{0.05} = 12.592$,
so do not reject $H_0$

Conclusion: there is not sufficient evidence that meal plan and class standing are related at $\alpha = 0.05$

# Chapter Summary

In this chapter we discussed

- Applying the $\chi^2$ test for the difference between two proportions
- Applying the $\chi^2$ test for differences in more than two proportions
- Applying the Marascuilo procedure for comparing all pairs of proportions after rejecting a $\chi^2$ test
- Applying the $\chi^2$ test for independence

# Chapter 12

On-Line Topic:
McNemar Test for the Difference Between Two Proportions (Related Samples)

# Learning Objectives

**In this topic, you learn:**

- How and when to use the McNemar test

# McNemar Test (Related Samples)

DCOVA

- Used to determine if there is a difference between proportions of two related samples

- Uses a test statistic the follows the normal distribution

# McNemar Test (Related Samples)

*(continued)*

DCOVA

- Consider a 2 X 2 contingency table:

| | Condition 2 | | |
|---|---|---|---|
| Condition 1 | Yes | No | Totals |
| Yes | A | B | A+B |
| No | C | D | C+D |
| Totals | A+C | B+D | n |

# McNemar Test (Related Samples)

*(continued)*

DCOVA

- The sample proportions of interest are

$$p_1 = \frac{A+B}{n} = \text{proportion of respondents who answer yes to condition 1}$$

$$p_2 = \frac{A+C}{n} = \text{proportion of respondents who answer yes to condition 2}$$

- Test $H_0: \pi_1 = \pi_2$

  (the two population proportions are equal)

  $H_1: \pi_1 \neq \pi_2$

  (the two population proportions are not equal)

# McNemar Test (Related Samples)

*(continued)*

DCOVA

- The test statistic for the McNemar test:

$$\boxed{Z_{STAT} = \frac{B - C}{\sqrt{B + C}}}$$

where the test statistic Z is approximately normally distributed

# McNemar Test Example

DCOVA

- Suppose you survey 300 homeowners and ask them if they are interested in refinancing their home. In an effort to generate business, a mortgage company improved their loan terms and reduced closing costs. The same homeowners were again surveyed. Determine if change in loan terms was effective in generating business for the mortgage company. The data are summarized as follows:

# McNemar Test Example

DCOVA

| Survey response before change | Survey response after change | | |
|---|---|---|---|
| | Yes | No | Totals |
| Yes | 118 | 2 | 120 |
| No | 22 | 158 | 180 |
| Totals | 140 | 160 | 300 |

Test the hypothesis (at the 0.05 level of significance):

$H_0$: $\pi_1 \geq \pi_2$: The change in loan terms was ineffective

$H_1$: $\pi_1 < \pi_2$: The change in loan terms increased business

# McNemar Test Example

DCOVA

| Survey response before change | Survey response after change | | |
|---|---|---|---|
| | Yes | No | Totals |
| Yes | 118 | 2 | 120 |
| No | 22 | 158 | 180 |
| Totals | 140 | 160 | 300 |

The critical value (0.05 significance) is $Z_{0.05} = -1.645$

The test statistic is:

$$Z_{STAT} = \frac{B - C}{\sqrt{B + C}} = \frac{2 - 22}{\sqrt{2 + 22}} = -4.08$$

Since $Z_{STAT} = -4.08 < -1.645$, you reject $H_0$ and conclude that the change in loan terms significantly increase business for the mortgage company.

# Topic Summary

In this topic we discussed

- How & when to use the McNemar test

# Chapter 12

On-Line Topic:
Chi-Square Test for the Variance or Standard Deviation

# Learning Objectives

**In this topic, you learn:**

- How to use the Chi-Square to test for a variance or standard deviation

# Chi-Square Test for a Variance or Standard Deviation

DCOVA

- A $\chi^2$ test statistic is used to test whether or not the population variance or standard deviation is equal to a specified value:

$H_0$: $\sigma^2 = \sigma_0^2$

$H_a$: $\sigma^2 \neq \sigma_0^2$

$$\chi^2_{STAT} = \frac{(n-1)S^2}{\sigma^2}$$

Where n = sample size

$S^2$ = sample variance

$\sigma^2$ = hypothesized population variance

$\chi^2_{STAT}$ follows a chi-square distribution with d.f. = n - 1

Reject $H_0$ if $\chi^2_{STAT} > \chi^2_{\alpha/2}$ or if $\chi^2_{STAT} < \chi^2_{1-\alpha/2}$

# Chi-Square Test For A Variance: An Example

DCOVA

Suppose you have gathered a random sample of size 25 and obtained a sample standard deviation of s = 7 and want to do the following hypothesis test:

$H_0$: $\sigma^2 = 81$

$H_a$: $\sigma^2 \neq 81$

$$\chi^2_{STAT} = \frac{(n-1)S^2}{\sigma^2} = \frac{24*49}{81} = 14.185$$

Since $\chi^2_{0.975} = 12.401 < 14.185 < \chi^2_{0.025} = 39.364$ you fail to reject $H_0$

# Topic Summary

In this topic we discussed

- How to use the Chi-Square to test for a variance or standard deviation