PHL345 Homework 2

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Question 1(i)

Show that if the Cut and Deduction Theorem hold, then $\vdash^* (\alpha \to \beta \to \gamma) \to ((\alpha \to \beta) \to \alpha \to \gamma)$

Consider the following lemma:

Lemma (1): $\emptyset \vdash \alpha \to \beta \implies \alpha \to \beta$.
Proof: Assume our claim. We know that $\alpha \vdash \alpha$. Using modus ponens on our two assumptions, we get that $\alpha \vdash \beta$ as needed. ■

We use this in combination with the Deduction Theorem to seamlessly move back and forth between $\vdash$
We know that

$$\alpha \to \beta \to \gamma \vdash^* \alpha \to \beta \to \gamma$$

Using Cut and Lemma (1), we get

$$\alpha \to \beta \to \gamma, \alpha, \beta \vdash^* \gamma$$

$$\alpha \to \beta \to \gamma, \beta \vdash^* \alpha \to \gamma$$

But we know that $\alpha, \alpha \to \beta \vdash^* \beta$. That means by Cut and Deduction, we get that

$$\alpha, \alpha \to \beta, \alpha \to \beta \to \gamma \vdash^* \alpha \to \gamma$$

$$\alpha, \alpha \to \beta \to \gamma \vdash^* (\alpha \to \beta) \to \alpha \to \gamma$$

$$\emptyset \vdash^* \alpha \to (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$$

But this is equivalent to

$$\emptyset \vdash^* (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$$

which completes the proof ■

Question 1(ii)

Show that if the Deduction and Inconsistency Theorem hold, then $\vdash^* \neg\alpha \to \alpha \to \beta$

Proof : We know that $\neg\alpha \vdash^* \neg\alpha$. That means by Lemma (1), $\neg\alpha \to \neg\alpha \vdash^* \emptyset$

By inconsistency theorem, that means $\neg\alpha \to \neg\alpha \vdash^* \alpha \to \beta$. By deduction, we get that $\emptyset \vdash^* \neg\alpha \to \neg\alpha \to \alpha \to \beta$

But this is equivalent to $\emptyset \vdash^* \neg\alpha \to \alpha \to \beta$, which completes the proof. ■

Question 2

Provide a sentence and prove that it is satisfied by all and only valuations based on structures with domains that have exactly two members.

Proof : Suppose $\mathbb{U}$ is a structure s.t its domain U has exactly two members. Let σ be a valuation based on $\mathbb{U}$. Our sentence will be

$$\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))$$

We want to show that $[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \top$
Suppose by reductio that $[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \bot$
We rewrite the above as,

$$[\neg \forall x \neg \neg \forall y (\neg (x \neq y) \lor \neg \forall z (z = x \lor z = y))]^\sigma = \bot$$

Which is equivalent to

$$[\forall x \neg \neg \forall y (\neg (x \neq y) \lor \neg \forall z (z = x \lor z = y))]^\sigma = \top$$

Using the definition of $\sigma(x/u)$, we further simplify

$$[\forall y (\neg (x \neq y) \lor \neg \forall z (z = x \lor z = y))]^{\sigma(x/u)} = \top$$

$$[\neg (x \neq y) \lor \neg \forall z (z = x \lor z = y))]^{\sigma(x/u)(y/v)} = \top$$

$$[x = y \lor z \neq x \land z \neq y)]^{\sigma(x/u)(y/v)(z/k)} = \top$$

for every $u, v, k \in U$. But we know that there are exactly two members of U. So let $z_1 \neq z_2 \in U$ s.t. for some $z_3 \in U$, $z_3 = z_1$ or $z_3 = z_2$. Using our above result, we get

$$[x = y \lor z \neq x \land z \neq y)]^{\sigma(x/z_1)(y/z_2)(z/z_3)} = \top$$

But this is equivalent to

$$\langle x^{\sigma(x/z_1)(y/z_2)(z/z_3)}, y^{\sigma(x/z_1)(y/z_2)(z/z_3)}, z^{\sigma(x/z_1)(y/z_2)(z/z_3)} \rangle \in =^{\sigma(x/z_1)(y/z_2)(z/z_3)}$$

But this is a contradiction, since that means

$$\langle z_1, z_2, z_3 \rangle \in id_U \iff z_1 = z_2 = z_3$$

meaning there is only member. Thus,

$$[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \top$$

as needed.

Next, suppose that the domain of $\mathbb{U}$ does NOT have at least two members. We want to show that $[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \bot$.
Suppose by reductio that $[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \top$. Based on our previous work, we arrive at

$$[x = y \lor z \neq x \land z \neq y)]^{\sigma(x/u)(y/v)(z/k)} = \bot$$

for every $u, v, k \in U$. which means

$$\langle x^{\sigma(x/z_1)(y/z_2)(z/z_3)}, y^{\sigma(x/z_1)(y/z_2)(z/z_3)}, z^{\sigma(x/z_1)(y/z_2)(z/z_3)} \rangle \notin =^{\sigma(x/z_1)(y/z_2)(z/z_3)}$$

That means $\langle u, v, z \rangle \notin id_U$. But by our assumption, we are allowed to have one element that we will call $z_0 \in U$ with the property that

$$\langle z_0, z_0, z_0 \rangle \notin id_U \iff z_0 \neq z_0$$

which is a contradiction. Thus,

$$[\exists x \exists y (x \neq y \land \forall z (z = x \lor z = y))]^\sigma = \bot$$

as needed. This shows that the provided sentence satisfies all and only valuations with structures that have domains containing exactly two members, which completes the proof. ■

Question 3(i)

Show that $\sigma \models \exists x (A(f(z), c) \to \forall y (A(y, x) \lor A(f(y), x)))$

<u>*Proof* :</u> We WTS that $[\exists x (A(f(z), c) \to \forall y (A(y, x) \lor A(f(y), x)))]^\sigma = \top$. Assume for reductio that $= \bot$

Rewriting, we get,

$$[A(f(z), c) \to (A(y, x) \lor A(f(y), x))]^{\sigma(x/c)(y/z)} = \bot$$

for some $c \in U$ and for each $z \in U$. This means that

$$[A(f(z), c)]^{\sigma(x/c)(y/z)} = \top$$

and

$$[(A(y,x) \vee A(f(y),x))]^{\sigma(x/c)(y/z)} = \bot$$

This means that $[A(y,x)]^{\sigma(x/c)(y/z)} = \bot$ and $A(f(y),x)^{\sigma(x/c)(y/z)} = \bot$, which means that

$$\langle x^{\sigma(x/c)(y/z)}, y^{\sigma(x/c)(y/z)} \rangle \notin A^{\sigma(x/c)(y/z)}$$

and

$$\langle f(y)^{\sigma(x/c)(y/z)}, x^{\sigma(x/c)(y/z)} \rangle \notin A^{\sigma(x/c)(y/z)}$$

But recall that we had $[A(f(z),c)]^{\sigma(x/c)(y/z)} = \top$, which means that $\langle f(z)^{\sigma(x/c)(y/z)}, c^{\sigma(x/c)(y/z)} \rangle \in A^{\sigma(x/c)(y/z)}$

But notice that $[f(y)]^{\sigma(x/c)(y/z)}$ and $[x]^{\sigma(x/c)(y/z)}$ are just $f(z)$ and c respectively.

Thus, we get that $\langle f(z)^{\sigma(x/c)(y/z)}, c^{\sigma(x/c)(y/z)} \rangle \in A^{\sigma(x/c)(y/z)}$ and $\langle f(z)^{\sigma(x/c)(y/z)}, c^{\sigma(x/c)(y/z)} \rangle \notin A^{\sigma(x/c)(y/z)}$, which is a contradiction. So $[\exists x(A(f(z),c) \to \forall y(A(y,x) \vee A(f(y),x)))]^{\sigma} = \top$ as needed.

Question 3(ii)

Give a different structure and variable assignment in which the formula is not satisfied.

Let $\mathcal{U}$ be the same structure described in the question, except $A^{\mathcal{U}} = \{\langle 1,2 \rangle\}$. That would mean

$$[\forall y(A(y,x) \vee A(f(y),x))]^{\sigma} = \bot$$

and

$$[\exists x(A(f(z),c))]^{\sigma} = \top$$

Question 4(i)

Prove that $\Gamma \vdash^* \phi \implies \Gamma \models^* \phi$

<u>*Proof*</u> : To prove this, we show by induction on the length of ϕ. Let σ be an arbitrary valuation, let $U = \{\top, \bot\}$

<u>Base Case</u>: We WTS that axioms 1-3 of PropCal are logically true in $\vdash^*$.

1. $\alpha \to \beta \to \alpha$
   WTS: $[\alpha \to \beta \to \alpha]^{\sigma} = \top$. Suppose by reductio that $[\alpha \to \beta \to \alpha]^{\sigma} = \bot$.

   That means $\alpha^{\sigma} = \top$ and $[\beta \to \alpha]^{\sigma} = \bot$. But that must mean $\beta^{\sigma} = \top$ and $\alpha^{\sigma} = \bot$, which is a contradiction. Thus, $[\alpha \to \beta \to \alpha]^{\sigma} = \top$ as needed.

2. $(\alpha \to \beta \to \gamma) \to ((\alpha \to \beta) \to \alpha \to \beta)$
   WTS: $[(\alpha \to \beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))]^\sigma = \top$. Suppose by reductio again.

   That means $[\alpha \to \gamma]^\sigma = \top$ and $[(\alpha \to \beta) \to (\alpha \to \gamma)]^\sigma = \bot$, meaning $[\alpha \to \beta]^\sigma = \top$ and $[\alpha \to \gamma]^\sigma = \bot$. However, that means $\alpha^\sigma = \top$ and $\gamma^\sigma = \bot$. But given that $\alpha^\sigma = \top$, it must be the case that $\beta^\sigma = \top$ as well, since $[\alpha \to \beta]^\sigma = \top$

   Thus, we get that $[\beta \to \gamma]^\sigma = \bot$, so that means $[\alpha \to \beta \to \gamma]^\sigma = \bot$, which is a contradiction. Thus, $[(\alpha \to \beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))]^\sigma = \top$ as needed.

3. $((\alpha \to \beta) \to \alpha) \to \alpha$
   WTS: $[((\alpha \to \beta) \to \alpha) \to \alpha]^\sigma = \top$. Suppose by reductio yet again.

   That means we get $[(\alpha \to \beta) \to \alpha]^\sigma = \top$ and $\alpha^\sigma = \bot$

   That means $[\alpha \to \beta]^\sigma = \bot$ as well, but that means $\alpha^\sigma = \top$, which is a contradiction. Thus, $[((\alpha \to \beta) \to \alpha) \to \alpha]^\sigma = \top$ as needed.

Inductive Step: Now we WTS that modus ponens preserves the validity of the proof.

Suppose that all previous lines of the proof α are valid, that is $\alpha^\sigma = \top$ and $[\alpha \to \beta]^\sigma = \top$. But that means it must be the case that $\beta^\sigma = \top$, thus modus ponens preserves validity.

This completes the inductive step, which completes the proof ■

Question 4(ii)

Prove that if $\Gamma \vdash^* \beta$, then $\beta \in \Gamma$

Proof : Let σ be an implicational valuation as defined in the question.

We know by 4(i) that if $\Gamma \vdash^* \beta$, then $\Gamma \models^* \beta$.

By definition, that means if $\Gamma \models^* \beta$, by every possible valuation of σ and for every $\gamma \in \Gamma$, $\gamma^\sigma = \top$, then $\beta^\sigma = \top$

Since Γ is maximally consistent s.t. $\Gamma \nvdash^* \alpha$ for some α, we know that

$$\Gamma = \{\gamma : \gamma^\sigma = \top \ \wedge \gamma^\sigma \neq \alpha^\sigma\}$$

That means $\beta^\sigma = \top$ so long as $\beta \neq \alpha$. But notice that if that were true, then we would get that $\Gamma \nvdash^* \beta$, which contradicts our assumption. Thus it must be the case that $\beta^\sigma = \top$ but this is equivalent to $\beta \in \Gamma$, which completes the proof. ■