PHL345 Homework 3

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Question 1

Find a formula in first-order logic that is satisfiable by a valuation only if the domain of the valuation is infinite

The following formula satisfies the above conditions:

$$\forall x \forall y \forall z (S(x, y) \wedge S(y, z) \rightarrow S(x, z) \wedge \neg S(x, x)) \wedge \forall x \exists y S(x, y)$$

where S is the successor relation, and the formula itself is the conjunction of the second and third axioms.

Question 2(i)

Prove that $\sigma \models \alpha \rightarrow \forall x \alpha$, where x does not occur free in α

Proof: We WTS that $[\alpha \rightarrow \forall x \alpha]^\sigma = 1$. Suppose otherwise. That means $[\alpha]^\sigma = 1$ and $[\forall x \alpha]^\sigma = 0$

If $[\forall x \alpha]^\sigma = 0$, that means $\alpha^{\sigma(x/u)} = 0$ for all $u \in U$, where U is the domain of the interpretation which σ is based on.

Since x is not free in $\forall x \alpha$, then x occurs in α, so we have that

$$\alpha^{\sigma(x/u)} = \alpha^\sigma = 0$$

which is a contradiction, since $[\alpha]^\sigma = 1$. This completes the proof.

Question 2(ii)

Prove that $\sigma \models s_1 = t_1 \rightarrow ... \rightarrow s_n = t_n \rightarrow fs_1...s_n = ft_1...t_n$

Proof: Show that $[s_1 = t_1 \rightarrow ... \rightarrow s_n = t_n \rightarrow fs_1...s_n = ft_1...t_n]^\sigma = 1$. Suppose for reductio that $= 0$

Since $[fs_1...s_n = ft_1...t_n]^\sigma = 0$, that means

$$\langle (fs_1...s_n)^\sigma, (ft_1...t_n)^\sigma \rangle \notin =^\sigma$$

which can be rewritten as

$$\langle f^\sigma(s_1^\sigma...s_n^\sigma), f^\sigma(t_1^\sigma...t_n^\sigma) \rangle \notin =^\sigma$$

We can also put $\langle s_1^\sigma...s_n^\sigma \rangle \in f^\sigma$ and $\langle t_1^\sigma...t_n^\sigma \rangle \in f^\sigma$ in place of $f^\sigma(s_1^\sigma...s_n^\sigma)$ and $f^\sigma(t_1^\sigma...t_n^\sigma)$ respectively. However, this is a contradiction, since that would mean that $\langle s_1, t_1 \rangle \notin =^\sigma, ..., \langle s_n, t_n \rangle \notin =^\sigma$, which contradicts $s_1 = t_1 \to ... \to s_n = t_n$, given that they are under the same n-ary function. This completes the proof.

Question 2(iii)

Prove that $\sigma \models \forall x \alpha \to \alpha(x/t)$

Proof: Suppose otherwise. That means $\alpha^{\sigma(x/u)} = 1$ for all $u \in U$, and $\alpha(x/t)^\sigma = \alpha^{\sigma(x/t^\sigma)} = 0$ for all $t^\sigma \in U$

This is clearly a contradiction, since u and t^σ are within the domain of the same universe.

Question 3

Show that $\sigma \models \exists x \forall y(\phi \iff x = y)$ in case $\sigma(y/u) \models \phi$ for exactly one $u \in U$. x must also not occur free in ϕ

Proof: We WTS that $[\exists x \forall y(\phi \iff x = y)]^\sigma = 1$

We know that $[\exists x \forall y(\phi \iff x = y)]^\sigma = 1$ iff

$$[\forall y(\phi \iff x = y)]^{\sigma(x/v)} = 1$$

for some $v \in U$ iff

$$[\phi \iff x = y]^{\sigma(x/v)(y/u)} = 1$$

for some $v \in U$ and for all $u \in U$ iff

$$\phi^{\sigma(x/v)(y/u)} = [x = y]^{\sigma(x/v)(y/u)}$$

for some $v \in U$ and for all $u \in U$ iff

$$\phi^{\sigma(x/v)(y/u)} = \langle x^{\sigma(x/v)(y/u)}, y^{\sigma(x/v)(y/u)} \rangle \in =^{\sigma(x/v)(y/u)}$$

for some $v \in U$ and for all $u \in U$ iff

$$\phi^{\sigma(x/v)(y/u)} = \langle v, u \rangle \in =^{\sigma(x/v)(y/u)}$$

for some $v \in U$ and for all $u \in U$.

Notice that $\langle v, u \rangle \in =^{\sigma(x/v)(y/u)}$ for some $v \in U$ and for all $u \in U$, is another way of saying that there is some v s.t. for all u, $u = v$. This itself is the formal way of saying that there is exactly one $u \in U$ s.t

$$\phi^{\sigma(x/v)(y/u)} = 1$$

But this is equivalent to $\sigma(y/u) \models \phi$, which completes the proof.

Question 4(i)

Show that the set of all formulas whose parity is 0 is a Hintikka set

Proof: Let Γ be the set of all first-order formulas α s.t. $pr(\alpha) = 0$. We WTS that Γ is Hintikka. We show this by proving that Γ has all the properties of a Hintikka set.

1. If α is atomic and $\alpha \in \Gamma$, then $\neg\alpha \notin \Gamma$
   Assume the antecedent. That means $pr(\alpha) = 0$. We know that
   $$pr(\neg\alpha) = 1 - pr(\alpha) = 1 - 0 = 1$$
   Thus, $\neg\alpha \notin \Gamma$ as needed,
2. If $\neg\neg\alpha \in \Gamma$, then $\alpha \in \Gamma$
   That means $pr(\neg\neg\alpha) = 0$. But since $\neg\neg\alpha$ is equivalent to α (by double negation property of PL), our result follows.
3. If $\alpha \to \beta \in \Gamma$, then either $\neg\alpha \in \Gamma$ or $\beta \in \Gamma$
   Assume the antecedent. That means
   $$pr(\alpha \to \beta) = (1 - pr(\alpha)) \times pr(\beta) = 0$$
   $$pr(\beta) - pr(\alpha) \times pr(\beta) = 0$$
   where $pr(\alpha)$ and $pr(\beta)$ are unknown variables. Solving the above equation, we get that $pr(\alpha) = 1$ or $pr(\beta) = 0$ iff $pr(\neg\alpha) = 0$ or $pr(\beta) = 0$, as needed.
4. If $\neg(\alpha \to \beta) \in \Gamma$, then $\alpha \in \Gamma$ and $\neg\beta \in \Gamma$
   That means $pr(\neg(\alpha \to \beta)) = 0$, which we can write as
   $$pr(\neg(\alpha \to \beta)) = 1 - pr(\alpha \to \beta) = 1 - ((1 - pr(\alpha)) \times pr(\beta)) = 0$$
   $$= 1 - (pr(\beta) - pr(\alpha) \times pr(\beta)) = 0$$
   $$pr(\beta) - pr(\alpha) \times pr(\beta) = 1$$
   Solving the above equation, we get that $pr(\alpha) = 0$ and $pr(\beta) = 1$, so $\alpha \in \Gamma$ and $\neg\beta \in \Gamma$ as needed.

5. If $\forall x\alpha \in \Gamma$, then $\alpha(x/t) \in \Gamma$ for every term t
   Assume antecedent. That means

$$pr(\forall x\alpha) = pr(\alpha) = 0$$

   for every term $t \in U$. Since $\alpha(x/t) = \alpha$ for every term t (by definition), that means $pr(\alpha(x/t)) = 0$ for every term $t \in \alpha$

6. If $\neg\forall x\alpha \in \Gamma$, then $\neg\alpha(x/t) \in \Gamma$ for some term t.
   That means

$$pr(\neg\forall x\alpha) = 1 - pr(\forall x\alpha) = 1 - pr(\alpha) = 0$$

   Since $pr(\alpha)$ is an unknown, solving for it gets us $pr(\alpha) = 1$. But this can be rewritten as $pr(\alpha(x/t)) = 1$ for some term t. Thus, $pr(\neg\alpha(x/t)) = 0$ for some term t.

So Γ satisfies all the conditions of a Hintikka set, thus, Γ is Hintikka.

Question 4(ii)

Define a valuation σ and prove that $\sigma \models \alpha$ iff α has parity 0. You can assume that the universe of the underlying interpretation is the singleton set.

Proof: Let Γ be the set in 4(i). We define a valuation σ s.t $\sigma \models \alpha$ for all $\alpha \in \Gamma$, as follows:

a) Let $x^\sigma = pr(x)$ for each variable x

b) Let f^σ be the operation on U s.t. $f(t_1^\sigma) = (ft_1)^\sigma = pr(ft_1)$

We prove this by induction on the length of α

Base Case: α is atomic

If α is atomic, and $\alpha^\sigma = pr(\alpha)$ by how we define σ, then $pr(\alpha) = 0$ by definition of parity.

In the reverse direction, if $pr(\alpha) = 0$ and α is atomic, then $pr(\alpha) = \alpha^\sigma \implies \sigma \models \alpha$

Inductive Step: Assume by inductive hypothesis, that $\sigma \models \alpha$ and $\sigma \models \beta \iff pr(\alpha) = pr(\beta) = 0$

1. WTS: $\sigma \not\models \neg\alpha \iff \neg\alpha$ does not have parity 0
   $\implies$: Suppose that $\sigma \not\models \alpha$. That means

$$\neg\alpha^\sigma = pr(\neg\alpha) = 1 - pr(\alpha) = 1 - 0 = 1 \neq 0$$

$\underline{\Longleftarrow:}$ Suppose that $pr(\neg\alpha) = 1$. That means

$$pr(\neg\alpha) = \neg\alpha^\sigma = 1 \implies \sigma \not\models \neg\alpha$$

2. WTS: $\sigma \models \alpha \to \beta \iff \alpha \to \beta$ has parity 0
$\underline{\Longrightarrow:}$ Suppose that $pr(\neg\alpha) = 1$ That means

$$(\alpha \to \beta)^\sigma = pr(\alpha \to \beta) = ((1 - pr(\alpha)) \times pr(\beta))$$

Since $pr(\alpha) = pr(\beta) = 0$ by our inductive hypothesis, we get that

$$((1 - 0) \times 0) = 0$$

as needed $\underline{\Longleftarrow:}$ Suppose that $pr(\alpha \to \beta) = 0$. Our result immediately follows

3. WTS: $\sigma \models \forall x\alpha \iff \forall x\alpha$ has parity 0
$\underline{\Longrightarrow:}$ Suppose that $\sigma \models \forall x\alpha$. That means

$$(\forall x\alpha)^\sigma = pr(\forall x\alpha) = pr(\alpha) = 0$$

by our Inductive Hypothesis, and by definition of parity.
$\underline{\Longleftarrow:}$ Suppose $pr(\forall x\alpha) = 0$. Since $pr(\alpha) = pr(\forall x\alpha) = 0$, our result follows

This completes the inductive step, thus $\sigma \models \alpha$ iff α has parity 0 as needed.