PHL345 Problem Set 4

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Question 1(i)

Show that $t^{f\sigma} = f(t^{\sigma})$ for every term t

Proof: We will show this by induction on the complexity of t

Base Case: t is a single variable
If t is a variable, it follows by definition that $t^{f\sigma} = f(t^{\sigma})$

Inductive Step: Suppose that $t^{f\sigma} = f(t^{\sigma})$ for every variable t. WTS that

1. $(st)^{f\sigma} = f((st)^{\sigma})$
   We know that $(st)^{f\sigma} = s^{f\sigma}(t^{f\sigma})$. Since $t^{f\sigma} = f(t^{\sigma})$,

   $$(st)^{f\sigma} = s^{f\sigma}(f(t^{\sigma})) = s^{^*R}(f(t^{\sigma})) = *^{s}(f(t^{\sigma}))$$

   But by definition, that means

   $$^*s(t^{\sigma}) = f(t^{\sigma} + 1)$$

   which is just $f((st)^{\sigma})$ as needed.

2. $(+rt)^{f\sigma} = f((+rt)^{\sigma})$ where r and t are any variables. We rewrite as

   $$(+rt)^{f\sigma} = +^{f\sigma}(r^{f\sigma}t^{f\sigma})$$

   $$= +^{f\sigma}(f(r^{\sigma})f(t^{\sigma})))$$

   From the definition of addition in R and pseudo-addition in $^*R$, we get

   $$= f(r^{\sigma})^* + f(t^{\sigma})$$

   $$f(r^{\sigma} + t^{\sigma}) = f((+rt)^{\sigma})$$

3. $(\times rt)^{f\sigma} = f((\times rt)^{\sigma})$ where r and t are any variables.
   Using similar reasoning to 2., we get the following string of equalities:

   $$(\times rt)^{f\sigma} = \times^{f\sigma}(r^{f\sigma}t^{f\sigma})$$

$$
\begin{aligned}
&= \times^{f\sigma}(f(r^{\sigma} f(t^{\sigma}))) \\
&= f(r^{\sigma})^{*} \times f(t^{\sigma}) \\
&= f(r^{\sigma} t^{\sigma}) = f((\times r t)^{\sigma})
\end{aligned}
$$

The final form our term could take would be of the single term 0, but our solution follows immediately by the expansion of $f(0) =^{*} 0$. This completes the inductive step, which completes our proof.

Question 1(ii)

Show that $f[\sigma(x/n)] = (f\sigma)(x/fn)$ for any variable x and number n

Proof: We know that $(f\sigma)(x/fn$ is the valuation that is based on $^{*}R$, where if y is a variable, then

$$y^{(f\sigma)(x/fn)} = fn$$

Similarly, $\sigma(x/n)$ is the valuation based on R where

$$y^{\sigma(x/n)} = n$$

But notice that $f[\sigma(x/n)]$ has the range fn, where $n \in^{*} R$. That means,

$$y^{f[\sigma(x/n)]} = fn$$

Thus,

$$
\begin{aligned}
y^{f[\sigma(x/n)]} &= y^{(f\sigma)(x/fn)} \\
f[\sigma(x/n)] &= (f\sigma)(x/fn)
\end{aligned}
$$

Question 1(iii)

Let f be an isomorphism between R to $^{*}R$. Show that $\alpha^{f\sigma} = \alpha^{\sigma}$

We proceed with an induction proof on the complexity of α

Base Case: α is a term of L

By i), that means $\alpha^{f\sigma} = f(\alpha^{\sigma})$. But since f is an isomorphism, we know that $^{*}R$ is an exact replica of R, so we can say that $f(\alpha^{\sigma}) = \alpha^{\sigma}$, so $\alpha^{f\sigma} = \alpha^{\sigma}$

Inductive Step: Assume that $\alpha^{f\sigma} = \alpha^{\sigma}$ for all terms α. We WTS that $\beta^{f\sigma} = \beta^{\sigma}$ for all formula β. We consider by cases:

1. $(\neg\alpha)^{f\sigma} = (\neg\alpha)^{\sigma}$
   Since σ is based on R and $f\sigma$ is based on $^{*}R$ and f is an isomorphism, we get that,
   $$(\neg\alpha)^{f\sigma} = f(\neg\alpha^{\sigma}) = \neg\alpha^{\sigma}$$

2. $(\alpha \to \gamma)^{f\sigma} = (\alpha \to \gamma)^{\sigma}$ for all terms γ.
   Same reasoning as above, we get that,

$$(\alpha \to \gamma)^{f\sigma} = f((\alpha \to \gamma)^{\sigma} = (\alpha \to \gamma)^{\sigma}$$

3. $(\forall x\alpha)^{f\sigma} = (\forall x\alpha)^{\sigma}$
   Now recall that

$$(\forall x\alpha)^{f\sigma} = (\alpha)^{f\sigma(x/u)}$$

   for all $u \in^* R$
   Notice that since f is an isomorphism, there is a bijective correspondence between every element in ${}^*R$ and R. Moreover, these elements behave exactly like there representations in either set. Thus, we can say that $u \in R$ as well, so we get that

$$(\alpha)^{\sigma(x/u)} = (\alpha)^{f\sigma(x/u)}$$

   which is just $(\forall x\alpha)^{\sigma}$

This completes the inductive step, which shows that the condition in question holds for all formulas α.

Question 2(i)

Show that f is injective

Proof: To show this, fix $nnm \in N$. Suppose that the following relation is in Ω

$$f(n) = s_n^{*R} = f(m) = s_m^{*R}$$

We WTS that $m = n$
Given that it's in Ω, we can say that $R \models s_n^{*R}$ and $R \models s_m^{*R}$. In other words,

$$(s_n^{*R})^{\sigma} = s_m^{*R} = 1$$

$$f(n) = f(m) = 1$$

where σ is the valuation based on R

Question 2(ii)

Show that f is an embedding

Proof: We know from 2(i) that f is injective for all n. To show that f is an embedding, we consider by cases, and WTS that f satisfies each case.

1. $f(0) =^* 0$
   Since $0 = s_0^R, f(0) = s_0^{*R} =^* 0$

2. $f(m+1) =^* s(f(m))$
   We know that $f(m+1) = s^{*R}_{m+1} = s^{*R}(s^{*R}_m)$ But this just means that

$$= s^{*R}(f(m)) =^* s(f(m))$$

3. $f(m+n) = f(m)^* + f(n)$
   We know that $f(m+n) = s^{*R}_{m+n}$

$$= \underbrace{s^{*R}(s^{*R}(s^{*R}(...}_{\text{n times}}(s^{*R}_m))))$$

   But this is equivalent to

$$= s^{*R}_m * +s^{*R}_n = f(m)^* + f(n)$$

4. $f(mn) = f(m)^* \times f(n)$
   This means $f(mn) = s^{*R}_{mn}$. Similar to the above argument, we get that

$$= s^{*R} * \times s^{*R}_n$$

$$= f(m)^* \times f(n)$$

Given that our function f is injective and satisfies all the conditions of an embedding from R to ${}^*R$, we get that f is an embedding as needed.