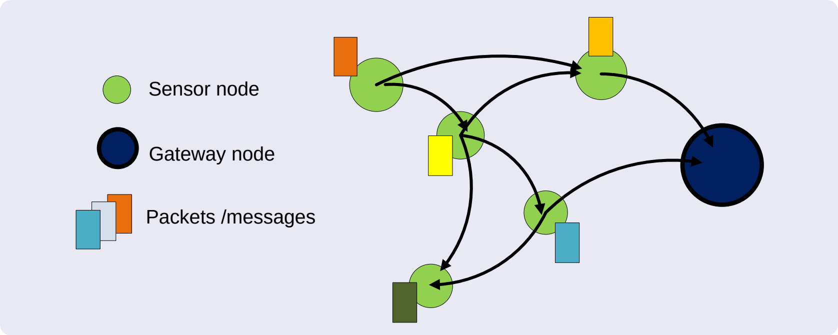
# Restricted Isometry Property in Quantized Network Coding of Sparse Messages

Mahdy Nabaee and Fabrice Labeau
ECE Dept, McGill University

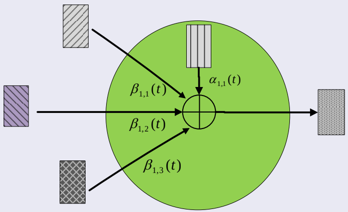
December 4, 2012

# Scenario: Data Gathering in Sensor Networks



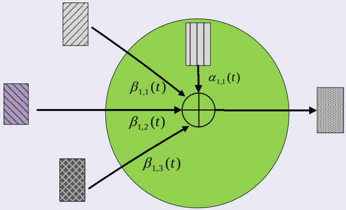

- messages are correlated,
- links are lossless without any interference.

# Linear Network Coding in lossless networks

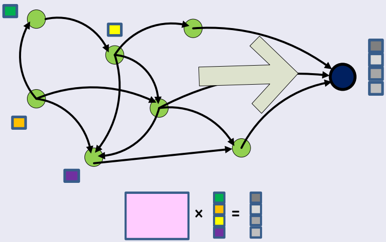



- Calculates linear combinations in a finite field, according to network coding coefficients.

# Linear Network Coding in lossless networks

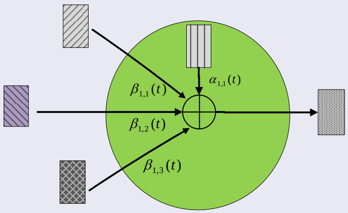



- Calculates linear combinations in a finite field, according to network coding coefficients.

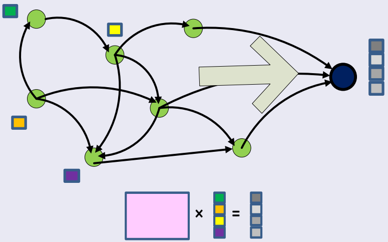


- Perfect decoding is possible, by using matrix inversion in the field, if measurement matrix is full rank.

# Linear Network Coding in lossless networks




- Calculates linear combinations in a finite field, according to network coding coefficients.

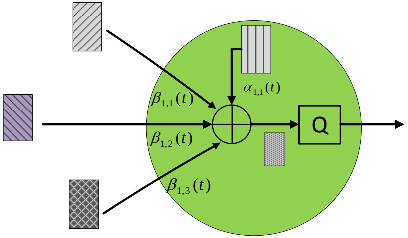



- Perfect decoding is possible, by using matrix inversion in the field, if measurement matrix is full rank.

**Using Quantized Network Coding (QNC),**

robust recovery is possible even if there are fewer measurements.

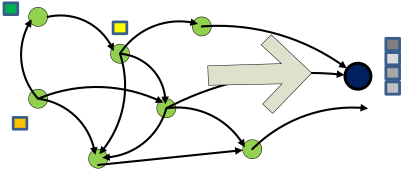
# Quantized Network Coding

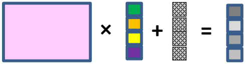



## Network Coding + Quantization $\rightarrow$ QNC

- Linear network coding in real field with semi-random coefficients,
- *Quantization* to cope with the finite capacity of links.

# QNC meets Compressed Sensing





$$\Psi_{tot} \cdot \underline{x} + \underline{n}_{eff,tot} = \underline{z}_{tot}$$

**Decoding for: $[\Psi_{tot}]_{m\times n} \cdot [\underline{x}]_{n\times 1} + [\underline{n}_{eff,tot}]_{m\times 1} = [\underline{z}_{tot}]_{m\times 1}$**

- If $\Psi_{tot}$ is full rank, a matrix inversion can recover $\underline{x}$, with respect to an error, caused by $\underline{n}_{eff,tot}$.
- If not, we have an under-determined set of equations, which for **Compressed Sensing** decoding ($\ell_1$-minimization) *may* help.

# $\ell_1$-min decoding for QNC

$$\text{meas. eq.}: \qquad [\Psi_{tot}]_{m\times n} \cdot [\underline{x}]_{n\times 1} + [\underline{n}_{eff,tot}]_{m\times 1} = [\underline{z}_{tot}]_{m\times 1}$$

**CS claims recovery is possible for $m < n$, if:**

- $\underline{x} = \phi \cdot \underline{s}$, where $\underline{s}$ is $k$-sparse,
- bounded measurement noise, $\left|\left|\underline{n}_{eff,tot}\right|\right|_{\ell_2} \leq \epsilon_{rec}$.

$$\Psi_{tot}\phi \cdot \underline{s} + \underline{n}_{eff,tot} = \underline{z}_{tot}$$

# $\ell_1$-min decoding for QNC

meas. eq. : $$[\Psi_{tot}]_{m\times n} \cdot [\underline{x}]_{n\times 1} + [\underline{n}_{eff,tot}]_{m\times 1} = [\underline{z}_{tot}]_{m\times 1}$$

**CS claims recovery is possible for $m < n$, if:**

- $\underline{x} = \phi \cdot \underline{s}$, where $\underline{s}$ is $k$-sparse,
- bounded measurement noise, $\left|\left|\underline{n}_{eff,tot}\right|\right|_{\ell_2} \leq \epsilon_{rec}$.

$$\Psi_{tot}\phi \cdot \underline{s} + \underline{n}_{eff,tot} = \underline{z}_{tot}$$

**$\ell_1$-min recovery**

$$\hat{\underline{x}} = \phi \cdot \arg\min_{\underline{s}'} \left|\left|\underline{s}'\right|\right|_{\ell_1}, \quad s.t. \left|\left|\underline{z}_{tot} - \Psi_{tot} \cdot \phi \cdot \underline{s}'\right|\right|_{\ell_2} \leq \epsilon_{rec}$$

# $\ell_1$-min decoding for QNC

$$[\Psi_{tot}]_{m\times n}\cdot[\phi]_{n\times n}\cdot[\underline{s}]_{n\times 1}+[\underline{n}_{eff,tot}]_{m\times 1}=[\underline{z}_{tot}]_{m\times 1}$$

**Advantage:**

If we have robust recovery, when $m < n$, we have a *saving* in the required number of channel uses $\longrightarrow$ **inter-node compression**.

# $\ell_1$-min decoding for QNC

$$[\Psi_{tot}]_{m\times n} \cdot [\phi]_{n\times n} \cdot [\underline{s}]_{n\times 1} + [\underline{n}_{eff,tot}]_{m\times 1} = [\underline{z}_{tot}]_{m\times 1}$$

## Advantage:

If we have robust recovery, when $m < n$, we have a *saving* in the required number of channel uses $\longrightarrow$ **inter-node compression**.

## How to ensure robust recovery?

**Theorem (Candes, 2008):** If $\Theta_{tot} = \Psi_{tot} \cdot \phi$ satisfies Restricted Isometry Property (RIP) of order $2k$ with constant $\delta_{2k} < \sqrt{2} - 1$, then:

$$||\underline{x} - \hat{\underline{x}}||_{\ell_2} = ||\underline{s} - \hat{\underline{s}}||_{\ell_2} \leq c_1(\delta_{2k}) \cdot \epsilon_{rec}.$$

# Satisfaction of RIP

## Definition

As a *norm conservation* property, $\Theta_{m\times n}$ is said to satisfy Restricted Isometry Property (RIP) of order $k$, with constant $\delta_k$, if for all $k$-sparse vector $\underline{s}$, we have:

$$1-\delta_k \leq \frac{||\Theta\cdot\underline{s}||_{\ell_2}^2}{||\underline{s}||_{\ell_2}^2} \leq 1+\delta_k.$$

# Satisfaction of RIP

## Definition

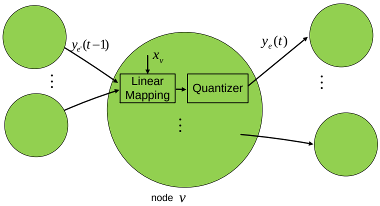
As a *norm conservation* property, $\Theta_{m\times n}$ is said to satisfy Restricted Isometry Property (RIP) of order $k$, with constant $\delta_k$, if for all $k$-sparse vector $\underline{s}$, we have:

$$1 - \delta_k \leq \frac{||\Theta \cdot \underline{s}||_{\ell_2}^2}{||\underline{s}||_{\ell_2}^2} \leq 1 + \delta_k.$$

## Example

Random matrices with i.i.d Gaussian entries satisfy RIP, with overwhelming probability.

# Appropriate NC Coefficients

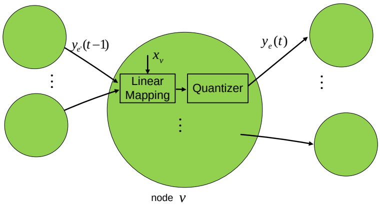



$$y_e(t) = \sum_{e' \in In(v)} \beta_{e,e'}(t) \cdot y_{e'}(t-1) + \alpha_{e,v}(t) \cdot x_v$$

## Semi-random coefficients...

We proposed design of local network coding coefficients, $\alpha_{e,v}(t)$ and $\beta_{e,e'}(t)$, which results in a Gaussian-like total measurement matrix, $\Psi_{tot}$.

# Appropriate NC Coefficients




$$y_e(t) = \sum_{e' \in In(v)} \beta_{e,e'}(t) \cdot y_{e'}(t-1) + \alpha_{e,v}(t) \cdot x_v$$

## Semi-random coefficients...

We proposed design of local network coding coefficients, $\alpha_{e,v}(t)$ and $\beta_{e,e'}(t)$, which results in a Gaussian-like total measurement matrix, $\Psi_{tot}$.

## In the following,

we show that the resulting $\Psi_{tot}$ has similar RIP behavior as i.i.d Gaussian.

# Tail Probability and RIP Satisfaction

## Definition

Tail Prob. $$\mathbf{p}_{tail}\left(\Psi_{tot}, \epsilon\right) = \max_{\underline{x}',\ ||\underline{x}'||_{\ell_2}=1} \mathbf{P}\left(\left| ||\Psi_{tot}\underline{x}'||_{\ell_2}^2 - 1 \right| > \epsilon\right)$$

# Tail Probability and RIP Satisfaction

## Definition

$$\text{Tail Prob.} \quad \mathbf{p}_{tail}\left(\Psi_{tot}, \epsilon\right) = \max_{\underline{x}',\ ||\underline{x}'||_{\ell_2}=1} \mathbf{P}\left(\left|\left|\left|\Psi_{tot}\underline{x}'\right|\right|_{\ell_2}^2 - 1\right| > \epsilon\right)$$

## Theorem

*For every orthonormal $\phi$, $\Theta_{tot} = \Psi_{tot}\phi$ satisfies RIP of order $k$ with constant $\delta_k$, with a probability exceeding:*

$$1 - \binom{n}{k} \left(\frac{42}{\delta_k}\right)^k \mathbf{p}_{tail}\left(\Phi, \epsilon = \frac{\delta_k}{\sqrt{2}}\right). \tag{1}$$

# Tail Probability and RIP Satisfaction

## Definition

Tail Prob. $$\mathbf{p}_{tail}\left(\Psi_{tot}, \epsilon\right) = \max_{\underline{x}',\ ||\underline{x}'||_{\ell_2}=1} \mathbf{P}\left(\left|\left|\left|\Psi_{tot}\underline{x}'\right|\right|_{\ell_2}^2 - 1\right| > \epsilon\right)$$

## Theorem

*For every orthonormal $\phi$, $\Theta_{tot} = \Psi_{tot}\phi$ satisfies RIP of order $k$ with constant $\delta_k$, with a probability exceeding:*

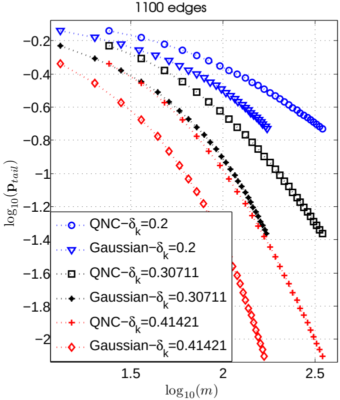
$$1 - \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{42}{\delta_k}\right)^k \mathbf{p}_{tail}\left(\Phi, \epsilon = \frac{\delta_k}{\sqrt{2}}\right). \qquad (1)$$

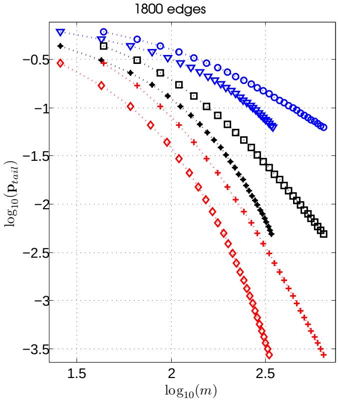
## Therefore,

we compare tail probabilities for i.i.d Gaussian and $\Psi_{tot}$ matrices.

# Numerical Evaluation: Tail Probabilities vs No of Measurements

Tail probabilities versus the number of received packets for different $\epsilon = \delta_k/\sqrt{2}$:



# Conclusion

- The measurement matrix, $\Psi_{tot}$, resulting from appropriate quantized network coding has almost the same behavior in terms of RIP satisfaction as a i.i.d Gaussian matrix.
- This enables to perform sparse recovery by using smaller number of measurements (received packets) than the size of data.
- Moreover, this can be done in a non-adaptive way, which provides the bases for joint distributed compression and network coding of sparse sources (messages).

# Effective Measurement Noise in QNC (Backup)

$$\Psi_{tot} \cdot \underline{x} + \underline{n}_{eff,tot} = \underline{z}_{tot}$$

## Quantization Noise at Nodes

- We use *uniform* quantizer in a constant range for all nodes and vary the step size, depending on the capacity of edges.

## Quantization Noise Propagation

- Quantization noises at each outgoing edge is considered as a random source which propagates noise in the network and has a transfer function to decoder ports.
- We have calculated an *upper bound* on the $\ell_2$-norm of effective noise, at the decoder ports, $\left|\left|\underline{n}_{eff,tot}\right|\right|_{\ell_2}$.

# Simulation Results (Backup)

- Random uniform deployment of 1400 edges, and 100 nodes,
- QNC with uniform quantizer,
- Packet forwarding via delay optimized routes,
- Different sparsity factors, $\frac{k}{n} = 0.1, 0.2, 0.3,$