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ON LINEAR STOCHASTIC EQUATIONS OF OPTIONAL SEMIMARTINGALES AND THEIR APPLICATIONS

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Abstract. Elements of the stochastic calculus of optional semimartingales are presented. A solution of the nonhomogeneous and general linear stochastic equations are given in this framework. Also, the Gronwall inequality is derived. Furthermore, a theory of martingale transforms and examples of applications to mathematical finance are presented.

Key words. Stochastic Equations; Martingale deflators; Optional Processes

1. Introduction.

Consider the probability space $(\Omega, \mathbf{F}, \mathbf{P})$ equipped with a non-decreasing family of sigma-algebras $\mathcal{F}_t \in \mathbf{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, for all $s \leq t$; In the theory of stochastic processes such a probability space is called the stochastic basis and satisfy to the so-called "usual conditions" ($\mathcal{F}_t$ is complete and right continuous for all t). Under these convenient conditions all adapted processes form a wide class of processes (semimartingales) with right continuous and left limits paths (RCLL). This theory has a number of excellent applications in different areas of modern probability theory and mathematical statistics. Moreover, many fundamental results of modern mathematical finance were proved with the help of this theory, and it is difficult to imagine how to get these results using other techniques and approaches.

Nevertheless the pervasiveness of the "usual conditions" in stochastic analysis, it is not difficult to give examples showing the existence of a stochastic basis without the "usual conditions", see for instance Fleming & Harrington (2011), p.24 [7]. Therefore, in the middle of 1970 famous experts in stochastic analysis Doob (1975) [5] and Dellacherie and Meyer (1975) [4] initiated studies of stochastic processes without this assumption. By the way, Dellacherie called this case the "un-usual conditions", and we will follow this terminology in the paper. Further developments were done by Mertens (1972) [19], Lepingle (1977) [16], Horowitz (1978) [12], Lenglart (1980) [15], and mostly by Gal'chouk in several papers published in the period 1975-1985 [8, 10, 11]. In these publications, a parallel theory of stochastic analysis was constructed for optional processes. The existence of such theory calls for a new initiative for further developments as well as for further applications.

The goal of the paper is to bring attention to the calculus of optional processes on unusual stochastic basis, develop new results and applications. The paper is organized as follows. The next section presents auxiliary materials on stochastic calculus of optional processes. The following section cover new results: extension of Gronwall lemma and solution of the nonhomogeneous linear and the general linear stochastic equations involving optional semimartingales. In section four we describe the theory of martingale transforms of optional semimartingales and give some useful examples. In section five we present some illustrative examples of possible applications of the calculus of optional processes to financial market modeling.

2. Auxiliary Material.

Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in [0,\infty)$, where $\mathcal{F}_t \in \mathbf{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s \leq t$, be a complete probability space – $\mathcal{F}$ contains all $\mathbf{P}$ null sets. However, the family $\mathbf{F}$ is not assumed to be complete, right or left continuous. We introduce $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ as the optional and predictable σ-algebras on $(\Omega, [0,\infty))$. A random process $X = (X_t)$, $t \in [0,\infty)$, is said to be *optional* if it is $\mathcal{O}(\mathbf{F})$-measurable. In general, optional processes have right and left limits but are not necessarily continuous in $\mathbf{F}$. For an optional process we can define the following properties: $X_- = (X_{t-})_{t\geq 0}$ and $X_+ = (X_{t+})_{t\geq 0}$, $\triangle X = (\triangle X_t)_{t\geq 0}$, $\triangle X_t = X_t - X_{t-}$ and $\triangle^+ X = (\triangle^+ X_t)_{t\geq 0}$, $\triangle^+ X_t = X_{t+} - X_t$. A random process (X_t), $t \in [0,\infty)$, is predictable if $X \in \mathcal{P}(\mathbf{F})$ and *strongly predictable*, in $\mathcal{P}_s(\mathbf{F})$, if $X \in \mathcal{P}(\mathbf{F})$ and $X_+ \in \mathcal{O}(\mathbf{F})$. An Optional semimartingale $X = (X_t)_{t\geq 0}$ can be decomposed to an optional local martingale and an optional finite variation; $X = X_0 + M + A$ where $M \in \mathcal{M}_{loc}$, $A \in \mathcal{V}$. A semimartingale X is called special if the above decomposition exists with a strongly predictable process $A \in \mathcal{A}_{loc}$. Let $\mathcal{S}(\mathbf{F},\mathbf{P})$ denote the set of optional semimartingales and $\mathcal{S}p(\mathbf{F},\mathbf{P})$ the set of special optional semimartingales. If $X \in \mathcal{S}p(\mathbf{F},\mathbf{P})$ then the semimartingale decomposition is unique. By optional martingale decomposition and decomposition of predictable processes [10, 11] we can decompose a semimartingale further to $X = X_0 + X^r + X^g$, $X^r = A^r + M^r$, $X^g = A^g + M^g$ and $M^r = M^c + M^d$ where A^r and A^g are finite variation processes right and left continuous, respectively. $M^r \in \mathcal{M}^r_{loc}$ right continuous local martingale, $M^d \in \mathcal{M}^d_{loc}$ discrete right continuous local martingale and $M^g \in \mathcal{M}^g_{loc}$ a left continuous local martingale. This decomposition is useful for defining integration with respect to optional semimartingales. A stochastic integral with respect to optional semimartingale was defined by Gal'chuk [11],

$$\varphi \circ X_t = \int_0^t \varphi_s dX_s = \int_{0+}^t \varphi_{s-} dX^r_s + \int_0^{t-} \varphi_s dX^g_{s+},$$

where $\int_{0+}^t \varphi_{s-} dX^r_s = \int_{0+}^t \varphi_{s-} dA^r_s + \int_{0+}^t \varphi_{s-} dM^r_s$, and $\int_0^{t-} \varphi_s dX^g_{s+} = \int_0^{t-} \phi_s dA^g_{s+} + \int_0^{t-} \phi_s dM^g_{s+}$.

The stochastic integral with respect to the finite variation processes or strongly predictable process A^r and A^g are interpreted as usual, in the Lebesgue sense. The integral $\int_{0+}^t \varphi_{s-} dM^r_s$ is our usual stochastic integral with respect to RCLL local martingale whereas $\int_0^{t-} \phi_s dM^g_{s+}$ is Gal'chuk stochastic integral [10, 11] with respect to left continuous local martingale. In general, the stochastic integral with respect to optional semimartingale X can be defined as a bilinear form $(\varphi, \phi) \circ X_t$ such that

$$Y_t = (\varphi, \phi) \circ X_t = \varphi \cdot X_t^r + \phi \odot X_t^g,$$
$$\varphi \cdot X^r = \int_{0+}^{t} \varphi_{s-} dX_s^r, \quad \phi \odot X^g = \int_{0}^{t-} \phi_s dX_{s+}^g$$

where Y is again an optional semimartingale $\varphi_- \in \mathcal{P}(\mathbf{F})$, and $\phi \in \mathcal{O}(\mathbf{F})$. Note that, the stochastic integral over optional semimartingale is defined on a much larger space of integrands, the product space of predictable and optional processes, $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$. From now on we are going to use the operator "$\circ$" to denote the stochastic optional integral "$\cdot$" the regular stochastic integral with respect to RCLL semimartingales and "$\odot$" for the Galtchouk stochastic integral $\phi \odot X^g$ with respect to left continuous semimartingales.

The stochastic exponential for RCLL processes has many applications in the theory of linear stochastic equations, in the statistics of stochastic processes of the exponential family and in mathematical finance. For optional semimartingales the stochastic exponential was defined by Gal'chuk [11]. If $X \in \mathcal{S}$ then there exists a unique semimartingale $Z \in \mathcal{S}$ such that $Z_t = Z_0 \mathcal{E}(X)_t = Z_0 + Z \circ X_t$ where $\mathcal{E}(X)_t$ is given by

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}\langle X^c, X^c\rangle\right) \prod_{0 \le s \le t} (1 + \Delta X_s) e^{-\Delta X_s} \prod_{0 \le s < t} (1 + \Delta^+ X_s) e^{-\Delta^+ X_s}.$$

For further study of the stochastic calculus of optional processes see [1, 11].

3. Linear Stochastic Equations.

The literature on linear stochastic equations is vast and their applications spans many areas of science and engineering. Furthermore, linear equations provide the first and most fundamental step in understanding nonlinear stochastic equations. Here, we will cover linear stochastic equations involving optional semimartingales, presenting new results: Gronwall lemma, solution of the nonhomogeneous and general linear equations.

3.1. Nonhomogeneous Linear Equation.

A generalization of the stochastic exponential equation is the nonhomogeneous linear stochastic integral equation [6], $X = G + X \circ H$. This equation has a natural application in finance: G is a stochastic cash flow, H is the interest rate of the money market account and X is the time value of the cash flow accumulated in the money market account. Here we will give the solution of the nonhomogeneous linear stochastic integral equation in the case of, G and H are optional semimartingales.

Theorem 3.1. *Consider the nonhomogeneous linear stochastic integral equation* $X_t = G_t + X \circ H_t$

where G is an optional semimartingale. The solution is

$$X_t = \mathcal{E}_t(H)\left[G_0 + \mathcal{E}(H)^{-1} \circ \tilde{G}_t\right], \tag{3.1}$$

$$\tilde{G}_t = G_t - \left[G, \tilde{H}\right]_t, \quad \tilde{H}_t = H_t^c + \sum_{0<s\leq t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\leq s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}.$$

Proof. Lets consider a solution of the following form $X_t = \mathcal{E}_t(H)Z_t$ where Z_t is related to G_t. The differential of X is

$$dX = \mathcal{E}(H)dZ = X dH + \mathcal{E}(H)d\left\{Z + [H, Z]\right\}. \tag{3.2}$$

Comparing the nonhomogeneuous equation to equation 3.2 we find that $dG = \mathcal{E}(H)d\left\{Z + [H, Z]\right\}$. We would like to solve this equation for Z. To do so we choose

$$\tilde{H}_t = H_t^c + \sum_{0<s\leq t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\leq s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}$$

and compute the quadratic variation of G with $\tilde{H}$. Note that, the quadratic variation for optional processes is defined as $[G, \tilde{H}] = \langle G^c, \tilde{H}^c\rangle + \left[\triangle G, \triangle\tilde{H}\right] + \left[\triangle^+ G, \triangle^+\tilde{H}\right]$. Hence,

$$\begin{aligned}
d\left[G, \tilde{H}\right]_t &= \mathcal{E}_t(H)\left\{d\left[Z, \tilde{H}\right]_t + d\left[[H, Z], \tilde{H}\right]_t\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \frac{\triangle H_t \triangle Z_t}{1+\triangle H_t} + \frac{\triangle^+ H_t \triangle^+ Z_t}{1+\triangle^+ H_t} + \frac{(\triangle H_t)^2}{1+\triangle H_t}\triangle Z_t + \frac{(\triangle^+ H_t)^2}{1+\triangle^+ H_t}\triangle^+ Z_t\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \triangle H_t \triangle Z_t\left(\frac{1+\triangle H_t}{1+\triangle H_t}\right) + \triangle^+ H_t \triangle^+ Z_t\left(\frac{1+\triangle^+ H_t}{1+\triangle^+ H_t}\right)\right\} \\
&= \mathcal{E}_t(H)\left\{d\left[Z^c, H^c\right]_t + \triangle H \triangle Z + \triangle^+ H_t \triangle^+ Z_t\right\} = \mathcal{E}_t(H)d\left[Z, H\right]_t.
\end{aligned}$$

Now, we calculate Z in the following way: $dG = \mathcal{E}(H)\left\{dZ + d[H, Z]\right\}$, $\mathcal{E}(H)^{-1}dG = dZ + d[H, Z] = dZ + \mathcal{E}(H)^{-1}d\left[G, \tilde{H}\right]$. Therefore, $dZ = \mathcal{E}(H)^{-1}\left[dG - d\left[G, \tilde{H}\right]\right]$. Note that,

$$\begin{aligned}
\left[[H, Z], \tilde{H}\right] &= \left[\langle H^c, Z^c\rangle + \sum_{s\leq\cdot} \triangle H \triangle Z + \sum_{s<\cdot} \triangle^+ H \triangle^+ Z, \tilde{H}\right] \\
&= \left[\langle H^c, Z^c\rangle, \tilde{H}\right] + \left[\left(\sum_{s\leq\cdot} \triangle H \triangle Z\right), \tilde{H}\right] + \left[\left(\sum_{s<\cdot} \triangle^+ H \triangle^+ Z\right), \tilde{H}\right]
\end{aligned}$$

$\left[\langle H^c, Z^c\rangle, \tilde{H}\right] = 0$ because $\langle H^c, Z^c\rangle$ is continuous locally bounded variation processes and $\tilde{H}$ a semimartingale. Then,

$$\begin{aligned}\left[[H,Z],\tilde{H}\right] &= \left[\left(\sum_{s\leq\cdot}\triangle H\triangle Z\right),\tilde{H}\right] + \left[\left(\sum_{s<\cdot}\triangle^+ H\triangle^+ Z\right),\tilde{H}\right]\\ &= \sum_{s\leq t}\left(\triangle H_s\right)^2\triangle Z_s + \sum_{s<t}\left(\triangle^+ H_s\right)^2\triangle^+ Z_s.\end{aligned}$$

$\square$

3.2. Gronwall Lemma.

The Gronwall lemma is a fundamental inequality in analysis and has far reaching consequences. For example, a fundamental problem in the study of differential or integral equations and their stochastic generalization is that of existence and uniqueness of solutions for which many variants of Gronwall's lemma were extensively used. Basically, Gronwall's lemmas allows us to put bounds on functions that satisfies an integral or differential inequality by a solution of the supposed equality. In stochastic analysis the lemma of Gronwall is essential and many extensions have been proposed see for example Metivier [21], Melnikov [17] and others [14, 22]. It is used to study the stability of solutions of stochastic equations of semimartingales. Here, we will extend Gronwell lemma to optional semimartingale under the unusual probability space.

Lemma 3.2 (Gronwall). *Assume an unusual probability space and let X be an optional process and H be optional increasing process and C an optional process such that $X_t \leq C_t + X\circ H_t$ for all $t\in[0,\infty)$. Then, $X_t \leq C_t\mathcal{E}_t(H)$.*

Proof. Let $N_t = C_t + X\circ H_t - X_t$ then $N_t \geq 0$ for all t. So, $X_t = C_t - N_t + X\circ H_t$ is a nonhomogeneous stochastic integral equation whose solution is given by

$$\begin{aligned}&X_t = \mathcal{E}_t(H)\left[G_0 + \mathcal{E}(H)^{-1}\circ\tilde{G}_t\right], \qquad (*)\\ &G_t = C_t - N_t, \quad \tilde{G}_t = G_t - \left[G,\tilde{H}\right]_t,\\ &\tilde{H}_t = H_t^c + \sum_{0<s\leq t}\frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\leq s<t}\frac{\triangle^+ H_s}{1+\triangle^+ H_s}.\end{aligned}$$

Since H is increasing then $\triangle H\geq 0$, $\triangle^+ H \geq 0$ and $\tilde{H}$ are increasing. Therefore, $\left[G,\tilde{H}\right] = 0$ and $\tilde{G}_t = G_t = C_t - N_t \leq C_t$ for all t since $N_t\geq 0$. Knowing all this, we can write (*) as

$$X_t = \mathcal{E}_t(H)\left[G_0 + \mathcal{E}(H)^{-1}\circ\tilde{G}_t\right] = \mathcal{E}_t(H)\left[G_0 + \mathcal{E}(H)^{-1}\circ G_t\right] \leq C_t\mathcal{E}_t(H),$$

where $\mathcal{E}(H)^{-1} \circ G_t \leq 0$ and $G_0 \leq C_0$. ☐

3.3. General Linear Equation.

Here, we give a solution to the general linear stochastic equation driven by optional local martingale and an increasing process.

THEOREM 3.3. *Consider the linear stochastic integral equation*

$$X_t = (AX + C) \circ H_t + B \circ M_t \tag{3.3}$$

where $X_0 = 0$, C is strongly predictable A and B are optional finite variation processes, H is increasing optional process and M is a local optional martingale. Then, the solution is

$$X_t = \mathcal{E}(-A \circ H)_t^{-1} \left[\mathcal{E}(-A \circ H) B \circ M_t + \mathcal{E}(-A \circ H) C \circ H_t \right].$$

Proof. We begin by rearranging terms of equation 3.3 as follows,

$$X = (AX + C) \circ H + B \circ M = AX \circ H + C \circ H + B \circ M = AX \circ H + G$$

where, $G = C \circ H + B \circ M$. Then, multiply X by $\mathcal{E}(-A \circ H)$, use the product rule and reduce,

$$\begin{aligned} \mathcal{E}(-A \circ H)X &= X \circ \mathcal{E}(-A \circ H) + \mathcal{E}(-A \circ H) \circ X \\ &= -AX\mathcal{E}(-A \circ H) \circ H + AX\mathcal{E}(-A \circ H) \circ H + \mathcal{E}(-A \circ H) \circ G = \mathcal{E}(-A \circ H) \circ G. \end{aligned}$$

Therefore,

$$\begin{aligned} X_t &= \mathcal{E}(-A \circ H)_t^{-1} \left(\mathcal{E}(-A \circ H) \circ G_t \right) \\ &= \mathcal{E}(-A \circ H)_t^{-1} \left(\mathcal{E}(-A \circ H) \circ (C \circ H + B \circ M)_t \right) \\ &= \mathcal{E}(-A \circ H)_t^{-1} \left[\mathcal{E}(-A \circ H) C \circ H + \mathcal{E}(-A \circ H) B \circ M_t \right] \end{aligned}$$

☐

The process, C, in equation 3.3 is known as the *control process* in *stochastic control* but in *mathematical finance* it is known as the *consumption plan*. Next we discuss a fundamental topic in the theory of stochastic processes with wide applications to finance known as martingale transforms (i.e. martingale deflators).

4. Martingale Transforms and Applications.

Martingale transforms are local optional martingale processes that transform an optional semimartingale to a local optional martingale given some appropriate conditions. So, if X is an optional semimartingale having the decomposition, $X = A + M$, then a local optional martingale Z is a martingale transform, if ZX is a local optional martingale. ZX is a local optional martingale if and only if $Z \circ A + [M, Z] = 0$. This result can be arrived to by the product rule,

$$\begin{aligned} ZX &= Z \circ X + X \circ Z + [X, Z] = Z \circ A + Z \circ M + X \circ Z + [A, Z] + [M, Z] \\ &= Z \circ A + [M, Z] + Z \circ M + X \circ Z, \end{aligned}$$

where $X \circ Z$ and $Z \circ M$ are local martingales because M and $Z \in \mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$. Suppose, we choose $Z = \alpha \circ M + N$ where N is any local optional martingale orthogonal to M (i.e. $[M, N] = 0$) and α is an optional process, to be defined. Then, it must be that $[Z, M] = [\alpha \circ M + N, M] = \alpha \circ [M, M]$. Hence, Z is a martingale transform of X if $[M, M]$ is absolutely continuous with respect to A, that is $\alpha = -Z dA/d[M, M]$. With the choice, $Z = \alpha \circ M + N$, one notices that: 1. It is a reasonable representation, that guarantees the existence of Z if $[M, M]$ is absolutely continuous with respect to A; 2. The local martingale space $\mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$ has a vector space flavor to it, in the sense that if $N = 0$ then every martingale in $\mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$ is an integral form of M, and we say, M span the space $\mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$. On the other hand, the space $\mathcal{M}_{loc}(\mathbf{F}, \mathbf{P})$ could be, such that there are many N with $[Z, M] = \alpha \circ [M, M]$. This means that, there are many local optional martingale transforms of the semimartingale X.

An alternative representation of the local martingale transform Z is $Z = \mathcal{E}(\alpha \circ M + N)$. With this form, the martingale condition is given by $Z\alpha \circ [M, M] = -Z \circ dA$, leading to $\alpha \circ [M, M] = -A$ if $Z \neq 0$. This particular choice of a martingale transform is useful in some problems arising in mathematical finance; Below, we will demonstrate such an application, that generalizes a corresponding result of the right continuous case by Melnikov *et al.* [17].

But, how are martingale transforms useful? Martingale transforms are of a broader class of transforms that change one semimartingale to another semimartingale or one stochastic equation to another stochastic equation that is easier to solve, or for which, the existence and uniqueness can be established under different conditions. Consider as an example, the *general stochastic linear equation* (3.3): the condition for finding the martingale transformation Z is $B \circ [Z, M] = -Z\,(AX + C) \circ H$ by which ZX is a local optional martingale.

In other words, ZX will have the following sum of integrals representation,

$$ZX = [Z(AX + C)] \circ H + ZB \circ M + X \circ Z + B \circ [M, Z] = ZB \circ M + X \circ Z.$$

Suppose, we choose $Z = \mathcal{E}(\alpha \circ M) \neq 0$ then $B\alpha \circ [M, M] = -(AX + C) \circ H$ from which we compute $\alpha = -(Bd[M, M])^{-1}(AX + C)\,dH$ if $B \neq 0$ and $[M, M] \ll H$. With this choice of Z, the general stochastic linear equation is transformed to $ZX = (\alpha ZX + ZB) \circ M$. Set $\tilde{X} = ZX$ and $\tilde{B} = ZB$, and solve $\tilde{X} = \left(\alpha\tilde{X} + \tilde{B}\right) \circ M$ for any α and $\tilde{B}$ (see [1] for a solution using stochastic logarithms). Then, with the previously specified choice of α and $Z = \mathcal{E}(\alpha \circ M)$, we are able to construct the solution X.

Let us consider another example, the special case of the *ratio process* $R = R_0\mathcal{E}(H)\mathcal{E}(h)^{-1}$ where h and H are optional semimartingale with the decomposition $h = a + m$ and $H = A + M$. This equation appears nonlinear, however, using the properties of stochastic exponentials it can be put in a linear form. Given that $\mathcal{E}^{-1}(h) = \mathcal{E}(-h^*)$ where

$$h_t^* = h_t - \langle h^c, h^c\rangle_t - \sum_{0<s\leq t} \frac{(\triangle h_s)^2}{1 + \triangle h_s} - \sum_{0\leq s<t} \frac{(\triangle^+ h_s)^2}{1 + \triangle^+ h_s}.$$

and using the product formula $\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U + V + [U, V])$ where $[U, V] = \langle U^c, V^c\rangle + [\triangle U, \triangle V] + [\triangle^+ U, \triangle^+ V]$. Therefore, $\mathcal{E}(H)\mathcal{E}^{-1}(h) = \mathcal{E}(H - h^* - [H, h^*])$ with

$$H_t - h_t^* - [H, h^*]_t = H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J_t^d + J_t^g$$
$$J_t^d = \sum_{0<s\leq t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1 + \triangle h_s}, \quad J_t^g = \sum_{0\leq s<t} \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s}.$$

Then the ratio R satisfies $R = R_0 + R \circ \Psi$. With the properties of stochastic integrals we get, $\Psi(h, H) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F}) \Rightarrow R \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. If $\Psi(h, H)$ is not a local martingale? Then, there exist a martingale transform, Z, such that $ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. We will do this with the following theorem.

Theorem 4.1. *Let $Z = \mathcal{E}(N)$, $N \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ then $ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ is a local optional martingale if and only if*

$$(A - a) + \langle m^c - N^c, m^c - M^c\rangle + \tilde{K}^d + \tilde{K}^g = 0,$$
$$K_t^d = \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1 + \triangle h_s}, \quad K_t^g = \sum_{0\leq s<t} \frac{(\triangle^+ h_s - \triangle^+ N_s)(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s}.$$

where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of the processes K^d and K^g.

Proof. Suppose $Z_t = \mathcal{E}(N)_t \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ then $ZR = \mathcal{E}(N + \Psi(h,H) + [N, \Psi(h,H)])_t = R_0\mathcal{E}(\Psi(h,H,N))$, where

$$
\begin{aligned}
\Psi(h,H,N) = {} & N_t + H_t - h_t + \langle h^c, h^c - H^c \rangle_t + J^d_t + J^g_t + \langle N^c, H^c \rangle_t - \langle N^c, h^c \rangle_t \\
& + \sum_{0<s\leq t} \triangle N_s \triangle H_s + \sum_{0\leq s<t} \triangle^+ N_s \triangle^+ H_s - \sum_{0<s\leq t} \triangle N_s \triangle h_s - \sum_{0\leq s<t} \triangle^+ N_s \triangle^+ h_s \\
& + \sum_{0<s\leq t} \triangle N_s \frac{\triangle h_s (\triangle h_s - \triangle H_s)}{1 + \triangle h_s} + \sum_{0\leq s<t} \triangle^+ N_s \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s} \\
& = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c \rangle_t \\
& + \sum_{0<s\leq t} \left[\frac{\triangle h_s (\triangle h_s - \triangle H_s)}{1 + \triangle h_s} + \triangle N_s\,(\triangle H_s - \triangle h_s) + \triangle N_s \frac{\triangle h_s (\triangle h_s - \triangle H_s)}{1 + \triangle h_s} \right] \\
& + \sum_{0\leq s<t} \left[\frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s} + \triangle^+ N_s \left(\triangle^+ H_s - \triangle^+ h_s \right) + \triangle^+ N_s \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s} \right].
\end{aligned}
$$

therefore,

$$
\Psi(h,H,N) = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c \rangle_t + K^d + K^g,
$$

$$
K^d = \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)\,(\triangle h_s - \triangle H_s)}{1 + \triangle h_s}, \quad K^g = \sum_{0<s\leq t} \frac{(\triangle^+ h_s - \triangle^+ N_s)\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s}.
$$

So, if $\Psi(h,H,N) \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$ then $ZR \in \mathcal{M}_{loc}(\mathbf{P}, \mathbf{F})$. Now, let us take into consideration the decomposition of H and h and write

$$
\Psi(h,H,N) = (A - a) + (M - m + N) + \langle (m - N)^c, (m - M)^c \rangle + K^d + K^g.
$$

So, $\Psi(h,H,N)$ is a local optional martingale under $\mathbf{P}$ if $(A-a) + \langle m^c - N^c, m^c - M^c \rangle + \tilde{K}^d + \tilde{K}^g = 0$ where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of K^d and K^g, respectively. □

In mathematical finance, R is a ratio of two securities where the principle asset $X = X_0\mathcal{E}(H) \geq 0$ and the numerarie asset $x = x_0\mathcal{E}(h) > 0$ (i.e. money market account). In this context Z is referred to as a local martingale deflator if $Z > 0$ a.s. $\mathbf{P}$.

5. Illustrative Examples.

Stochastic modeling of financial markets is a well developed area of research in usual probability spaces. However, there are no papers devoted to the applications of stochastic calculus of optional processes to financial market modeling. Therefore, the possibilities of its applications to finance ought to be explored. We can mention here few research problems that were *not* treated with the methods of optional semimartingale calculus but show some relevance to the calculus of optional processes on "unusual spaces", and should be studied in this framework. The first one, is pricing and hedging with *transaction costs* (see [3] for details) and the second is in models with stochastic dividends paid at random times [2]. Both of these problems we are going to consider in future work, after we lay the foundation to an optional calculus of financial markets.

So, here we are interested in market models of optional processes that are ladlag and their mathematical properties. To this end, we are going to present two examples: The first example is general and deals with a portfolio of ladlag assets, for example, a left continuous bond and a right continuous stock. The second example is specific, a ladlag jump diffusion model which can be a model of an asset that is right continuous but subjected to frictional forces, such as, transaction costs or sudden changes in market conditions.

5.1. The Portfolio Equation.

Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in [0, \infty)$, be the unusual stochastic basis and that the financial market stays on this space. Let a portfolio $\pi = (\eta, \xi)$, be of optional processes η and ξ, describing the volume of reference asset x and traded securities X, respectively. Suppose $x_t > 0$ and $X_t \geq 0$ for all $t \geq 0$ and write $R_t = X_t/x_t$ the *ratio process* or normalized security. Then, our value process (*portfolio equation*) is given by $Y_t = \eta_t + \xi_t R_t$. We restrict the portfolio π to be *self-financing*. Or that, the change in the value process, Y_t, can only be achieved as a result of a change in R. This fact can be described by $Y_t = Y_0 + \xi_t \circ R_t$. Therefore, it must be that $C_t = \eta_t + \xi \circ R_t + [\xi, R]_t = C_0$ where C_t is the consumption process with its initial value C_0. So, in a market that evolves on $[0, T]$ such that π is self-financing then the consumption profile, C, is constant. Given the integral $\xi_t \circ R_t$ must be well defined leads us to require additional assumptions to be put on the process ξ. That is, ξ must satisfy the following conditions: I. Since the ratio process R is an optional semimartingale evolving on an "unusual market" the volume of the traded asset, ξ, evolves in the space $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$ with the predictable part determines the volume of R^r and the optional part determines the traded volume of R^g. Also, η is not predictable; it belongs to the space $\mathcal{O}(\mathbf{F})$. So, (η, ξ) is not our usual predictable portfolio but contains predictable and optional parts. II. Also ξ must be R-integrable or in other words it must be that $\xi^2 \circ [R, R]_\infty \in \mathcal{A}_{loc}$.

In the previous section we have showed that we can find local martingale Z such that ZR is a local martingale. This fact is essential in proving the following lemma,

LEMMA 5.1. *If Z is a local martingale transform of R, that is ZR is a local optional martingale, and π is a self financing portfolio which is R-integrable then ZY_t^π is a local optional martingale.*

Proof. Z is a local martingale transform of R therefore $Z > 0$. $\pi = (\eta, \xi)$ is self financing and R-

integrable, then $Y_t^\pi = Y_0 + \xi \circ R_t$ and $Z_t Y_t^\pi$ can be written as,

$$
\begin{aligned}
d\left(Z_t Y_t^\pi\right) &= Z_t dY_t^\pi + dZ_t Y_t^\pi + d\left[Z_t, Y_t^\pi\right] = Z_t \xi_t dR_t + dZ_t \xi_t R_t + dZ_t \eta_t + \xi_t d\left[Z_t, R_t\right] \\
&= \xi_t \left[Z_t dR_t + dZ_t R_t + d\left[Z_t, R_t\right]\right] + dZ_t \eta_t = \xi_t d\left(Z_t R_t\right) + dZ_t \eta_t.
\end{aligned}
$$

This leads us to the following result $Z_t Y_t^\pi = \xi \circ Z_t R_t + \eta \circ Z_t . \eta \circ Z_t$ and $\xi \circ Z_t R_t$ are local optional martingales therefore their sum $Z_t Y_t^\pi$ is a well defined local optional martingale. Note, that we have implicitly used the fact that η is bounded, i.e. comes from the fact that π is a self financing and also that, $\xi^2 \circ [ZR]_\infty \in \mathcal{A}_{loc}$. □

On the other hand, if we know that there exist a Z such that ZY^π is a local optional martingale then what can we say about the portfolio π and the product ZR? It is reasonable to suppose that $Z = \mathcal{E}(N) > 0$, π-self-financing, ξ is R-integrable and η is bounded. In this case, $\xi \circ Z_t R_t = Z_t Y_t^\pi - \eta \circ Z_t$ is a sum of two local optional martingales and therefore a local optional martingale it self. Also, for any optional process ξ, in particular for $\xi = 1$. Therefore ZR is a local optional martingale.

5.2. LadLag Jump-Diffusion Model.

Consider the stochastic equations,

$$
x_t = x_0 + \int_{0+}^t r x_s ds, \quad X_t = X_0 + \int_{0+}^t X_{s-}\left(\mu ds + \sigma dW_s + a dL_s^r\right) + \int_0^{t-} b X_s dL_{s+}^g, \tag{5.1}
$$

where $L_t^r = L_t - \lambda t$, $L_t^g = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are independent Poisson with constant intensity λ and γ respectively. We can write X as $X_t = X_0 \mathcal{E}(H)$, where $H_t = \mu t + \sigma W_t + a\left(L_t - \lambda t\right) + b\left(\gamma t - \bar{L}_{t-}\right)$, with $H_0 = 0$, and $x_t = x_0 \exp(rt)$ so that $h_t = rt$. These equations resemble Merton jump diffusion model [20] widely used in finance. Here, we study the properties of the ratio process $R = X/x$;

$$
\begin{aligned}
R_t = X_0 \exp&\left\{\left(\mu - r - \frac{1}{2}\left(\sigma^2 - \lambda a^2 - \gamma b^2\right)\right) t + \sigma W_t\right\} \prod_{0<s\le t}\left[\left(1 + a\triangle L_t\right) e^{-a\triangle L_t}\right] \\
&\times \prod_{0\le s<t}\left[\left(1 - b\triangle^+ \bar{L}_{s-}\right) e^{-b\triangle^+ \bar{L}_{s-}}\right],
\end{aligned}
$$

which is not a local optional martingale. So, we want a martingale transform $Z = \mathcal{E}(N) > 0$ such that

$\Psi(h,H,N)$,

$$\Psi(h,H,N)=N_t+H_t-h_t+\langle h^c-N^c,h^c-H^c\rangle_t$$
$$+\sum_{0<s\le t}\frac{(\triangle h_s-\triangle N_s)(\triangle h_s-\triangle H_s)}{1+\triangle h_s}+\sum_{0<s\le t}\frac{(\triangle^+h_s-\triangle^+N_s)(\triangle^+h_s-\triangle^+H_s)}{1+\triangle^+h_s}.$$

It makes sense to guess $N_t=\varsigma W_t+c(L_t-\lambda t)+d\left(\gamma t-\bar{L}_{t-}\right)$ an optional local martingale that will render Z an optional scaling factor. If N is as we chose above then

$$\Psi(h,H,N)=\varsigma W_t+c(L_t-\lambda t)+d\left(\gamma t-\bar{L}_{t-}\right)+(\mu-r)t+\sigma W_t+a(L_t-\lambda t)+b\left(\gamma t-\bar{L}_{t-}\right)$$
$$+\left\langle\left[\varsigma W_t+c(L_t-\lambda t)+d\left(\gamma t-\bar{L}_{t-}\right)\right]^c,\left[\sigma W_t+a(L_t-\lambda t)+b\left(\gamma t-\bar{L}_{t-}\right)\right]^c\right\rangle$$
$$+\sum_{0<s\le t}ac\triangle L_s\triangle L_s+\sum_{0\le s\le t}bd\triangle_s^+\bar{L}_{s-}\triangle_s^+\bar{L}_{s-}$$
$$=(\varsigma+\sigma)W_t+(a+c+ac)(L_t-\lambda t)+(b+d-bd)\left(\gamma t-\bar{L}_{t-}\right)$$
$$+(\mu-r+\varsigma\sigma+2ac\lambda+2bd\gamma)t,$$

is local martingale if $\mu-r+\varsigma\sigma+2ac\lambda+2bd\gamma=0$ (*). So we have to find (ς,c,d) such that the last statement (*) is true. In other words, $[\sigma,2a\lambda,2b\gamma][\varsigma,c,d]^{\intercal}=r-\mu$. There are infinitely many solutions for this equation. One interesting solution is to let $d=0$ which leads to right continuous local martingale transform Z. Another solution is a one that will eliminate the effects of jumps on drift that is by letting $d=-1/b\gamma$ and $c=1/a\lambda$, in this case $\varsigma=(r-\mu)/\sigma$.

The process L^g in the ladlag jump diffusion model represents optional processes that are not uncommon in financial modeling; For example, L^g can be random dividend payments, changes in market conditions that impacts prices, transaction costs, the effects of external forces such as weather or, generally, any consumption plan.

In mathematical finance being able to transform portfolios to local martingales is essential in providing fare prices for contingent claims. From the above examples we showed that it is possible that given markets consisting of optional processes that are ladlag on unusual probability spaces and that it is possible to find local martingale deflators. This might, we hope, will pave the way for a consistent theory of financial market involving a much larger class of stochastic processes.

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