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A COMPARISON THEOREM FOR STOCHASTIC EQUATIONS OF OPTIONAL SEMIMARTINGALES

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Abstract. This paper is devoted to comparison of strong solutions of stochastic equations with respect to optional semimartingales. Optional semimartingales have right and left limits but are not necessarily continuous and therefore defined on "unusual" probability spaces. Integration theory with respect to optional semimartingales is well developed. However, not much attention is given to stochastic integral equations of optional semimartingales. A pathwise comparison result for strong solutions of a very general class of optional stochastic equations with nonlipshitz coefficients is given. Moreover, simple applications to mathematical finance is presented.

Key words. Stochastic Integral Equations, Stochastic Differential Equations, Non-Lipschitz Condition, Optional Semimartingales, LadLag Processes, Comparison Theorem, Existence and Uniqueness of Solutions

AMS subject classification. 1234.56

1. Introduction.

We study pathwise comparison of stochastic processes that are strong solutions of stochastic equations of optional semimartingales in "unusual" probability spaces. Stochastic equations have numerous applications in Engineering, Finance, and Physics and are a central part of stochastic analysis. Moreover, and perhaps a fundamental aspect of stochastic equations is that they provide a way of manufacturing complex processes from ones that are simpler. For example, a geometric Brownian motion is constructed from the simpler Weiner process. The impetus of progress in the study of stochastic equations came as a result of the development of semimartingale integration theory which gave the study of stochastic equations a strong theoretical basis. As research in stochastic integration theory progresses to realm beyond the usual probability spaces and RCLL processes, stochastic equations will also be advanced in those directions. This is the main goals of this paper, to bring attention to new stochastic integration theory, that is integrals with respect to optional semimartingales, and to study stochastic equations driven by optional semimartingales.

Briefly, optional semimartingales are RLL processes defined on "unusual" stochastic basis – a probability space that encompasses a much larger class of processes where the underlying filtration is not necessarily left or right continuous or complete. The study of optional semimartingales under unusual probability spaces have been advanced by many mathematicians; Dellacherie (1975) [3] began the study of optional stochastic processes without the usual conditions. Further developments of the theory of optional processes were done by Lepingle (1977) [16], Horowitz (1978) [11], Lenglart (1980) [15], and mostly by Gal'chuk [5, 7, 9]. And more recently, Abdelghani and Melnikov [1] provided a solution to the nonhomogeneous linear stochastic integral equation driven by optional semimartingales, constructed a version of Gronwall lemma for optional semimartingales and introduced the use of optional semimartingales to financial market modeling.

A central problem in the theory of stochastic equations is the study of existence and uniqueness of solutions under certain conditions placed on the semimartingale driver and on coefficients of this semimartingale. A plethora of stochastic equations, models and proofs were proposed (see [17, 20] for a review). Little was done in showing existence and uniqueness of solution of stochastic equations driven by optional semimartingales. This will be the subject of a follow up paper and will not be discussed here. Another equally important result in the study stochastic equations are comparison theorems. Comparison theorems allows us to compare solutions of related stochastic equations. With a comparison result one finds that knowing the structure of the stochastic equation and the set of possible initial conditions, a stochastic ordering of some sort can be established between processes that are solutions of these stochastic equations. Many have studied comparison of solutions of stochastic equations: Skorokhod [21] established a comparison theorem for diffusion equations with which he discovered that the solution of these equations is a nondecreasing function of their drift coefficient. The same result was demonstrated in [23] but with a weaker condition on the diffusion coefficient. In [17] and [22], the comparison theorem was proved for equations with respect to continuous martingales. Gal'chuk [8] considered stochastic equations with respect to continuous martingales and integer-valued random measures where the coefficients of the semimartingale are not Lipschitz but satisfy weaker conditions similar to those of Yamada [23]. Extensions of comparison theorem to multidimensional case required an additional condition on the drift coefficient, known as Kamke-Wazewski condition. It was first done in [18] for equations involving continuous semimartingale and in [10] for diffusion equations. Recently, an interesting application of path-wise comparison theorem to mathematical finance was demonstrated in the paper by Krasin and Melnikov [14].

Our goal for this paper is to extend stochastic equations to optional semimartingales on "unusual" stochastic basis and study comparison of solutions in this setting under more general conditions placed on the coefficient of the stochastic equation. The paper consists of the following sections: In section 2, we present some auxiliary materials about optional semimartingales. In section 3, we define stochastic equations with respect to components of optional semimartingales. Section 4 is devoted to comparison of stochastic processes. Finally, the paper ends with illustrative examples of applications of comparison to finance.

2. Auxiliary Facts.

We consider given a complete but "unusual" probability space.$(\Omega, \mathcal{F}, \mathbf{F}=(\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, $t\in\mathbb{R}_+$ on it a nondecreasing family $\mathcal{F}_s\subseteq\mathcal{F}_t$, $s\leq t$, family of σ-algebras. The probability space is complete because $\mathcal{F}$ contains all $\mathbf{P}$ null sets. It is "unusual" because the family $\mathbf{F}$ is not assumed to be complete, right or left continuous. We also introduce the following families of σ-algebras $\mathbf{F}_+=(\mathcal{F}_{t+})_{t\geq 0}$ which is right-continuous and $\mathbf{F}_-=(\mathcal{F}_{t-})_{t\geq 0}$ which is left-continuous and completion of $\mathbf{F}$ under $\mathbf{P}$ is $\mathbf{F}^{\mathbf{P}}$. Let us also introduce $\mathcal{G}(\mathbf{F})$ progressive, $\mathcal{O}(\mathbf{F})$ optional and $\mathcal{P}(\mathbf{F})$ predictable σ-algebras on $\Omega\times\mathbb{R}_+$. Recall that $\mathcal{G}$ is generated by all progressively measurable processes, $\mathcal{O}$ by all processes whose trajectories are right-continuous and have left limits in $\mathbf{F}$, $\mathcal{P}$ by all processes whose trajectories are left-continuous and have right limits in $\mathbf{F}$. A process $X=(X_t)_{t\geq 0}$ is progressive if $X\in\mathcal{G}(\mathbf{F})$ and optional if $X\in\mathcal{O}(\mathbf{F})$. A process $X=(X_t)_{t\geq 0}$ is predictable if it belongs to $\mathcal{P}(\mathbf{F})$ and *strongly predictable* if X_+ belongs to $\mathcal{O}(\mathbf{F})$. Let $\mathcal{P}_s(\mathbf{F})$ be set of strongly predictable processes.

We assume, unless otherwise specified, that all processes considered here are $\mathbf{F}$-consistent, and their trajectories have right and left limits but are not necessarily right or left continuous. Processes on this space have the following properties: $X_-=(X_{t-})_{t\geq 0}$ is $\mathcal{P}(\mathbf{F})$ where $X_{t-}=\lim_{s\uparrow t}X_s$; if $X\in\mathcal{O}(\mathbf{F})$ then $X_+=(X_{t+})_{t\geq 0}$ is $\mathcal{O}(\mathbf{F}_+)$ and right-continuous, where $X_{t+}=\lim_{s\downarrow t}X_s$; $\triangle X=(\triangle X_t)_{t\geq 0}$ where $\triangle X_t=X_t-X_{t-}$ and $\triangle^+X=(\triangle^+X_t)_{t\geq 0}$ where $\triangle^+X_t=X_{t+}-X_t$.

An optional process is an optional semimartingale $X=(X_t)_{t\geq 0}$ if it can be decomposed to an optional local martingale, $M\in\mathcal{M}_{loc}(\mathbf{F},\mathbf{P})$, and an optional finite variation, $A\in\mathcal{V}(\mathbf{F},\mathbf{P})$,

$$X=X_0+M+A.$$

A semimartingale X is called special if the above decomposition exists with a strongly predictable process $A\in\mathcal{A}_{loc}(\mathbf{F},\mathbf{P})$. Let $\mathcal{S}(\mathbf{F},\mathbf{P})$ denote the set of optional semimartingales and $\mathcal{S}p(\mathbf{F},\mathbf{P})$ the set of special optional semimartingales. If $X\in\mathcal{S}p(\mathbf{F},\mathbf{P})$ then the semimartingale decomposition is unique.

Optional local martingales can be decomposed to

$$M=M^r+M^g=M^c+M^d+M^g$$

where $M^r\in\mathcal{M}^r_{loc}$ right-continuous local martingales, $M^c\in\mathcal{M}^c_{loc}$ continuous local martingales, $M^d\in\mathcal{M}^d_{loc}$ discrete right-continuous local martingales and $M^g\in\mathcal{M}^g_{loc}$ left-continuous local martingales.

A finite variation process or an increasing process A can also be decomposed in the same way, $A=$

$A^r + A^g = A^c + A^d + A^g$ [7, 9]. Consequently, one can decompose a semimartingale further to

$$
\begin{aligned}
X &= X_0 + X^r + X^g \\
&= X_0 + A^r + M^r + A^g + M^g \\
&= X_0 + A^c + A^d + A^g + M^c + M^d + M^g.
\end{aligned}
$$

This decomposition is useful for defining stochastic integration with respect to optional semimartingales and for defining the *component* and *canonical* representation of optional semimartingales.

A stochastic integral with respect to optional semimartingale was defined by Gal'chuk as

$$
Y_t = h \circ X_t = \int_0^t h_s dX_s = h \cdot X_t + h \odot X_t,
$$

where

$$
\begin{aligned}
h \cdot X_t &= \int_{0+}^t h_{s-} dX_s^r = \int_{0+}^t h_{s-} dA_s^r + \int_{0+}^t h_{s-} dM_s^r, \\
h \odot X_t &= \int_0^{t-} h dX_{s+}^g = \int_0^{t-} h_s dA_{s+}^g + \int_0^{t-} h_s dM_{s+}^g.
\end{aligned}
$$

Note that the integral with respect to the finite variation processes or strongly predictable process A^r and A^g are interpreted as Lebesgue integrals. $\int_{0+}^t h_{s-} dM_s^r$ is our usual stochastic integral with respect to RCLL local martingale whereas $\int_0^{t-} h_s dM_{s+}^g$ is Gal'chuk stochastic integral [7, 9] with respect to left continuous local martingale. A direct extension of the above integral to a larger class of integrands is given by the bilinear form $(f, g) \circ X_t$,

$$
Y_t = (f, g) \circ X_t = f \cdot X_t^r + g \odot X_t^g,
$$

where Y_t is again an optional semimartingale $f_- \in \mathcal{P}(\mathbf{F})$, and $g \in \mathcal{O}(\mathbf{F})$ [1]. Hence, for the stochastic integral with respect to optional semimartingales, the space of integrands is the product space of predictable and optional processes, $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$.

Notation 2.1. *From now on we are going to use the operator "$\circ$" to denote the stochastic optional integral. The operator "$\cdot$" for the regular stochastic integral with respect to RCLL semimartingales and "$\odot$" for the Gal'chuk stochastic integral $\phi \odot X^g$ with respect to left continuous semimartingales. Sometimes we will use "$\circ$" to mean either the regular stochastic integral "$\cdot$" or the Gal'chuk integral "$\odot$" if it is clear from context which one we are working with. The superscript "^{r}" will denote RCLL processes, "^{d}" will denote* *discrete RCLL processes, "^{c}" denote continuous processes and "^{g}" will denote RLLC processes.*

The canonical and component representation of semimartingales is of fundamental importance in stochastic analysis. It is also essential to our development of stochastic integral equations driven by optional semimartingales. The canonical and component representation of optional semimartingale can be seen as a natural consequence of the decomposition

$$X = X_0 + X^c + X^d + X^g.$$

where X^c is a continuous optional semimartingale with decomposition, $X^c = a + m$, where a is continuous strongly predictable with locally integrable variation ($a \in \mathcal{P}_s \cap \mathcal{A}_{0,loc}$), and m a continuous local martingale ($m \in \mathcal{M}^c_{0,loc}$). The discrete optional semimartingale parts, $X^d = a^d + m^d$ and $X^g = a^g + m^g$, (a^d, $a^g \in \mathcal{A}_{loc}$, $m^d \in \mathcal{M}^d_{loc}$, $m^g \in \mathcal{M}^g_{loc}$) are representable in terms of an some underlying measures of right and left jumps, respectively. These measures' of jumps, are referred to as integer valued random measures. We describe the integer random measure representation of discrete martingales briefly and refer the reader to the paper by Gal'chuk [9] for details.

Consider the Lusin space $(\mathbb{E}, \mathscr{E})$ where $\mathbb{E} = (\mathbb{R}^d \backslash \{0\}) \cup \{\delta^d\} \cup \{\delta^g\}$; δ^d and δ^g are some supplementary points or is the set of processes with finite variation on any segment $[0, t]$, $\mathbf{P}$-a.s.; $\mathscr{E} = \mathcal{B}(\mathbb{E})$ is the Borel σ-algebra in $\mathbb{E}$. Also, define the spaces

$$\begin{gathered}\widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathbb{E}} = \mathbb{R}_+ \times \mathbb{E}, \quad \widetilde{\mathscr{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathscr{E}, \quad \widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{B}(\mathbb{E}), \\ \widetilde{\mathcal{O}}(\mathbf{F}) = \mathcal{O}(\mathbf{F}) \times \mathscr{E}, \quad \widetilde{\mathcal{O}}(\mathbf{F}_+) = \mathcal{O}(\mathbf{F}_+) \times \mathscr{E}, \text{ and } \widetilde{\mathcal{P}}(\mathbf{F}) = \mathcal{P}(\mathbf{F}) \times \mathscr{E}.\end{gathered} \tag{2.1}$$

It was shown in [9] that there exist sequences $\{S_n\}$, $\{T_n\}$, and $\{U_n\}$ for $n \in \mathbb{N}$ of predictable stopping time (s.t.), totally inaccessible stopping time and totally inaccessible stopping time in the broad sense (s.t.b.) respectively, absorbing all jumps of the process X such that the graphs of these stopping times do not intersect within each sequence. On $\widetilde{\Omega}$ let $\mu^i(\omega, \cdot, \cdot)$, $p^i(\omega, \cdot, \cdot)$ and $\eta^g(\omega, \cdot, \cdot)$ where $i \in (d, g)$ be integer valued measures defined on the σ-algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ that are associated with the sequences of stopping times that are associated with X. On the σ-algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ we define the random integer-valued measures by the relations,

$$p^d(B\times\Gamma)=\sum_n \mathbf{1}_{B\times\Gamma}(S_n,\beta^d_{S_n}),\quad p^g(B\times\Gamma)=\sum_n \mathbf{1}_{B\times\Gamma}(S_n,\beta^g_{S_n}),$$
$$\mu^d(B\times\Gamma)=\sum_n \mathbf{1}_{B\times\Gamma}(T_n,\beta^d_{T_n}),\quad \mu^g(B\times\Gamma)=\sum_n \mathbf{1}_{B\times\Gamma}(T_n,\beta^g_{U_n}),$$
$$\eta^g(B\times\Gamma)=\sum_n \mathbf{1}_{B\times\Gamma}(T_n,\beta^g_{T_n}),$$

where $B\in\mathcal{B}(\mathbb{R}_+)$, $\Gamma\in\mathcal{B}(\mathbb{E})$, $\beta^d_t=\Delta X_t$ if $\Delta X_t\neq 0$ and $\beta^d_t=\delta^d$ if $\Delta X_t=0$, $\beta^g_t=\Delta^+X_t$ if $\Delta^+X_t\neq 0$, $\beta^g_t=\delta^g$ if $\Delta^+X_t=0$, $t>0$, $\mathbf{1}_A(x)$ is the indicator of the set A. For the measures $\mu^d(\omega,\cdot)$ and $\mu^g(\omega,\cdot)$ there exists unique random measures $\nu^d(\omega,\cdot)$ and $\nu^g(\omega,\cdot)$, respectively, on $\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{E})$ such that, for any non-negative functions $\varphi_d\in\widetilde{\mathcal{P}}(\mathbf{F})$ and $\varphi_g\in\widetilde{\mathcal{O}}(\mathbf{F})$

(i) The process

$$\int_{0+}^{t}\int_{\mathbb{E}}\varphi_d(s,u)v^d(ds,du),\quad \int_{0}^{t-}\int_{\mathbb{E}}\varphi_g(s,u)v^g(ds,du)$$

is $\mathcal{P}(\mathbf{F})$-measurable and $\mathcal{O}(\mathbf{F})$-measurable respectively.

(ii) The equalities

$$\mathbf{E}\int_{0+}^{\infty}\int_{\mathbb{E}}\varphi_d(s,u)\mu^d(ds,du)=\mathbf{E}\int_{0+}^{\infty}\int_{\mathbb{E}}\varphi_d(s,u)\nu^d(ds,du)$$
$$\mathbf{E}\int_{0}^{\infty}\int_{\mathbb{E}}\varphi_g(s,u)\mu^g(ds,du)=\mathbf{E}\int_{0}^{\infty}\int_{\mathbb{E}}\varphi_g(s,u)\nu^g(ds,du)$$

are valid.

The measures ν^i, $i\in(d,g)$ possesses the property $0\le\nu^i(\omega,\{t\}\times\mathbb{E})\le 1$ for all ω and t except for some set of $\mathbf{P}$ measure zero. We denote by $\lambda^i(\omega,\cdot)$, $i\in(d,g)$ the analogous measures for $p^i(\omega,\cdot)$. and $\theta^g(\omega,\cdot)$ that of $\eta^g(\omega,\cdot)$. The measures ν^i, λ^i and θ^g are called the (dual) predictable projections (compensator) for the measures μ^i, p^i and η^g, respectively. Note, μ^g, p^g, and η^g are $\mathcal{O}(\mathbf{F}_+)$-optional with their compensators' being $\mathcal{O}(\mathbf{F})$-optional. On the other hand μ^d and p^d is $\mathcal{O}(\mathbf{F})$-optional with their compensators' being $\mathcal{P}(\mathbf{F})$-predictable.

Having defined integer valued measures and stochastic integrals with respect them one can write a representation of discrete optional semimartingales,

$$
\begin{aligned}
X^d_t &= \int_{0+}^{t}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^d-\nu^d)(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} u\mathbf{1}_{|u|>1}\mu^d(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} up^d(ds,du),\\
X^g_t &= \int_{0}^{t-}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^g-\nu^g)(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} u\mathbf{1}_{|u|>1}\mu^g(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} up^g(ds,du)\\
&\quad + \int_{0}^{t-}\int_{\mathbb{E}} u\eta^g(ds,du).
\end{aligned}
$$

With $X^c = a + m$, we can write the component decomposition of X as

$$
\begin{aligned}
X &= X_0 + a + m\\
&\quad + \int_{0+}^{t}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^d-\nu^d)(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} u\mathbf{1}_{|u|>1}\mu^d(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} up^d(ds,du),\\
&\quad + \int_{0}^{t-}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^g-\nu^g)(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} u\mathbf{1}_{|u|>1}\mu^g(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} up^g(ds,du)\\
&\quad + \int_{0}^{t-}\int_{\mathbb{E}} u\eta^g(ds,du).
\end{aligned}
$$

Note that the local martingales of the process X are,

$$
\begin{aligned}
m^d &= \int_{0+}^{t}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^d-\nu^d)(ds,du),\\
m^g &= \int_{0}^{t-}\int_{\mathbb{E}} u\mathbf{1}_{|u|\le 1}(\mu^g-\nu^g)(ds,du).
\end{aligned}
$$

and the characteristics of the process X is $\left(a, \langle m,m\rangle, \nu^d, \lambda^d, \nu^g, \lambda^g, \theta^g\right)$. For further details on the construction of the component decomposition of optional semimartingales, see [9].

Now lets consider integrals with respect to the components of X. The process, a, is continuous locally finite variation process that is strongly predictable, ($a \in \mathcal{P}_s \cap \mathcal{A}_{0,loc}$). An integral of a function, f, with respect, a, is well defined in the Lebesgue-Stieltjes sense, $f \cdot a_t \in \mathcal{A}_{loc}$, where the integral is over the interval $[0,t]$ and f is $\mathcal{P}(\mathbf{F})$-measurable. For the continuous local martingale $m \in \mathcal{M}^c_{0,1oc}$, If the function g is $\mathcal{P}(\mathbf{F})$-measurable and $\left[|g|^2 \cdot \langle m,m\rangle\right]^{j/2} \in \mathcal{A}_{1oc}$ then the stochastic integral, $g \cdot m \in \mathcal{M}^{j,c}_{1oc}$, $j = 1,2$, is well defined; again, the integral is over the interval $[0,t]$.

Integral with respect to random measures are

$$k_d * \mu^d_t = \int_{0+}^{t}\int_{\mathbb{E}} k_d(s,u)\mu^d(ds,du), \quad k_g * \mu^g_t = \int_{0}^{t-}\int_{\mathbb{E}} k_g(s,u)\mu^g(ds,du),$$
$$h_d * (\mu^d - \nu^d)_t = \int_{0+}^{t}\int_{\mathbb{E}} h_d(\omega,s,u)(\mu^d - \nu^d)(ds,du),$$
$$h_g * (\mu^g - \nu^g)_t = \int_{0}^{t-}\int_{\mathbb{E}} h_g(\omega,s,u)(\mu^g - \nu^g)(ds,du),$$
$$r_d * p^d = \int_{0+}^{t}\int_{\mathbb{E}} r_d(\omega,s,u)p^d(ds,du), \quad r_g * p^g = \int_{0}^{t-}\int_{\mathbb{E}} r_g(\omega,s,u)p^g(ds,du),$$
$$w_g * \eta^g = \int_{0}^{t-}\int_{\mathbb{E}} w_g(\omega,s,u)\eta^g(ds,du),$$

If the function h_d is $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable and $\left[|h_d|^2 * \nu^d\right]^{j/2} \in \mathcal{A}_{1oc}$, then the stochastic integral $h_d * (\mu^d - \nu^d) \in \mathcal{M}^{j,d}_{1oc}$, $j = 1,2$, is well defined. If the function h_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable and $\left[|h_g|^2 * \nu^g\right]^{j/2} \in \mathcal{A}_{1oc}$, then the stochastic integral $h_g * (\mu^g - \nu^g) \in \mathcal{M}^{j,g}_{1oc}$, $j = 1,2$, is also well defined. If k_d is $\widetilde{\mathcal{G}}(\mathbf{F})$-measurable and $|k_d| * \mu^d \in \mathcal{V}$, then the integral $f * \mu^d \in \mathcal{V}$ is defined (see [12]). And, If k_g is $\widetilde{\mathcal{G}}(\mathbf{F})$-measurable and $|k_g| * \mu^g \in \mathcal{V}$, then the integral $k_g * \mu^g \in \mathcal{V}$ is defined.

If r_d is $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable, $\left[|r_d|^2 * p^d\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any predictable stopping time S, we have that $\mathbf{E}\left[r_d(S, \beta^d_S)|\mathcal{F}_{S_-}\right] = 0$ a.s., then the stochastic integral $r_d * p^d \in \mathcal{M}^{j,d}_{1oc}$, $j = 1,2$, is defined. And, If $r_d \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|r_d| * p^d \in \mathcal{V}$ then the integral $r_d * p^d \in \mathcal{V}$ is defined (see [6]). Note that the facts used below in the theory of martingales can be found in [13, 19, 12]. If r_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable, $\left[|r_g|^2 * p^g\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any totally inaccessible stopping time T, $\mathbf{E}\left[r_g(T, \beta^g_T)|\mathcal{F}_T\right] = 0$ a.s., then the stochastic integral $r_g * p^g \in \mathcal{M}^{j,g}_{1oc}$, $j = 1,2$, is defined. And, If $r_g \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|r_g| * p^g \in \mathcal{V}$ then the integral $r_g * p^g \in \mathcal{V}$ is defined. If w_g is $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable, $\left[|w_g|^2 * \eta^g\right]^{j/2} \in \mathcal{A}_{1oc}$ and for any totally inaccessible stopping time in the broad sense U, $\mathbf{E}\left[w_g(U, \beta^g_U)|\mathcal{F}_U\right] = 0$ a.s., then the stochastic integral $w_g * \eta^g \in \mathcal{M}^{j,g}_{1oc}$, $j = 1,2$, is defined. And, If $w_g \in \widetilde{\mathcal{G}}(\mathbf{F})$ and $|w_g| * \eta^g \in \mathcal{V}$ then the integral $w_g * \eta^g \in \mathcal{V}$ is defined.

Notation 2.2. *We have used $i \in (d,g)$ to clearly identify the different types of optional semimartingales. However, from now on we are going to identify "d" by 1 the right-continuous discrete component of the semimartingale and "g" by 2 the left-continuous discrete part of the semimartingale to give a concise description (i.e. $i \in (1,2)$).*

With the new notation, the optional semimartingale X has the following components representation,

$$
\begin{aligned}
X &= X_0 + a + m \\
&+ \int_{0+}^{t}\int_{\mathbb{E}} U(\mu^1-\nu^1)(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} V\mu^1(ds,du) + \int_{0+}^{t}\int_{\mathbb{E}} up^1(ds,du) \\
&+ \int_{0}^{t-}\int_{\mathbb{E}} U(\mu^2-\nu^2)(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} V\mu^2(ds,du) + \int_{0}^{t-}\int_{\mathbb{E}} up^2(ds,du) \\
&+ \int_{0}^{t-}\int_{\mathbb{E}} u\eta(ds,du).
\end{aligned}
$$

where $U = u\mathbf{1}_{|u|\le 1}$ and $V = u\mathbf{1}_{|u|>1}$.

The component representation of optional semimartingale will be the representation form that we will use to construct the comparison lemma.

Before we get to the main theorem we need to extend the change of variables formula of the component representation of semimartingales in the usual conditions (cf. [6]) to optional semimartingales in the unusual case.

Lemma 2.3. *Suppose an optional semimartingale $Y = \left(Y^1, Y^2, ..., Y^k\right)$ is defined by the relation*

$$
\begin{aligned}
Y_t &= Y_0 + f\cdot a_t + g\cdot m_t + (r+w)*\eta_t, \\
&\quad + \sum_j UH_j*(\mu^j-\nu^j)_t + Vh_j*\mu^j_t + (k_j+l_j)*p^j_t,
\end{aligned}
$$

where all the integrals are well defined. Consider the function $F(y) = F(y^1, y^2, ..., y^k)$ to be twice continuously differentiable on $\mathbb{R}^k$.
Then the process $F(Y) = (F(Y_t))_{t\ge 0}$ is an optional semimartingale and has the representation

$$
\begin{aligned}
F(Y_t) &= F(Y_0) + F'(Y)f\cdot a_t + F'(Y)g\cdot m_t + \frac{1}{2}F''(Y)g^2\cdot\langle m,m\rangle_t \\
&\quad + \sum_j U\left[F\left(Y+H_j\right)-F(Y)\right]*(\mu^j-\nu^j)_t \\
&\quad + \sum_j V\left[F\left(Y+h_j\right)-F(Y)\right]*\mu^j_t \\
&\quad + \sum_j U\left[F\left(Y+H_j\right)-F(Y)+F'(Y)H_j\right]*\nu^j_t \\
&\quad + \sum_j \left[F\left(Y+(k_j+l_j)\right)-F(Y)\right]*p^j_t \\
&\quad + \left[F\left(Y+(r+w)\right)-F(Y)\right]*\eta_t.
\end{aligned}
$$

Proof. Gal'chuk [6] proved the change of variable formula for semimartingales under the usual conditions. Extending the proof to optional semimartingale is straight forward. □

3. Comparison of Stochastic Processes.

Let there be given an optional semimartingale Z with components: a continuous locally integrable process $a \in \mathcal{A}_{1oc}$ with $a_0 = 0$, a continuous martingale $m \in \mathcal{M}^c_{1oc}$ with $m_0 = 0$ and integer-valued measures μ^j, p^j for$j = 1,\ 2$ and η with predictable and optional projections ν^j, λ^j, and θ respectively.

We shall consider the equations

$$\begin{aligned} X^i_t = {} & X^i_0 + f^i(X^i) \cdot a_t + g(X^i) \cdot m_t \\ & + \sum_j U h_j(X^i) * (\mu^j - \nu^j)_t + V h^i_j(X^i) * \mu^j_t + \big(k^i_j(X^i) + l^i_j(X^i)\big) * p^j_t \\ & + \big(r^i(X^i) + w^i(X^i)\big) * \eta_t, \end{aligned} \tag{3.1}$$

where $U = \mathbf{1}_{|u| \leq 1}$ and $V = \mathbf{1}_{|u|>1}$ and the dependence on the arguments is as follows:

$$\begin{aligned} & f^i(X^i) = f^i(\omega, s, X^i_{s-}), && g(X^i) = g^i(\omega, s, X^i_{s-}) \\ & h_1(X^i) = h_1(\omega, s, u, X^i_{s-}), && h_2(X^i) = h_2(\omega, s, u, X^i_s) \\ & h^i_1(X^i) = h^i_1(\omega, s, u, X^i_{s-}), && h^i_2(X^i) = h^i_2(\omega, s, u, X^i_s) \\ & k^i_1(X^i) = k^i_1(\omega, s, u, X^i_{s-}), && k^i_2(X^i) = k^i_2(\omega, s, u, X^i_s) \\ & l^i_1(X^i) = l^i_1(\omega, s, u, X^i_{s-}), && l^i_2(X^i) = l^i_2(\omega, s, u, X^i_s) \\ & r^i(X^i) = r^i(\omega, s, u, X^i_s), && w^i(X^i) = w^i(\omega, s, u, X^i_s) \end{aligned}$$

for $i = 1, 2$; In another way to describe the processes X^i for $i = 1, 2$,

$$\begin{aligned} X^i_t &= X^i_0 + A^i_t(X^i) + M_t(X^i), \\ A^i_t(X^i) &= f^i(X^i) \cdot a_t + \sum_j V h^i_j(X^i) * \mu^j_t + \big(k^i_j(X^i) + l^i_j(X^i)\big) * p^j_t + \big(r^i(X^i) + w^i(X^i)\big) * \eta_t, \\ M_t(X^i) &= g(X^i) \cdot m_t + \sum_j U h_j(X^i) * (\mu^j - \nu^j)_t \end{aligned}$$

the martingale part $M(X^i) \in \mathcal{M}_{1oc}$ and finite variation process $A^i(X^i) \in \mathcal{V}$ form the process X^i.

It is also assumed that for, $i = 1, 2$, the functions above satisfy these conditions, (D1) $f^i(\omega, s, x)$ and $g(\omega, s, x)$ are defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}(\mathbb{R})$-measurable,

(D2) $Uh_1(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| \leq 1) \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| \leq 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D3) $Uh_2(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| \leq 1) \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| \leq 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D4) $Vh_1^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| > 1) \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| > 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D5) $Vh_2^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \cap (|u| > 1) \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E} \cap (|u| > 1)\right) \times \mathcal{B}\left(\mathbb{R}\right)$-measurable,

(D6) $k_1^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{P}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $k_1^i(X^i) * p^1 \in \mathcal{M}_{1oc}^{1,r}(\mathbf{F})$,

(D7) $k_2^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $k_2^i(X^i) * p^2 \in \mathcal{M}_{1oc}^{1,g}(\mathbf{F})$,

(D8) $l_1^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $l_1^i(X^i) * p^1 \in \mathcal{V}$,

(D9) $l_2^i(\omega, s, u, x)$ is defined on $\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $l_2^i(X^i) * p^2 \in \mathcal{V}$,

(D10) $r^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{O}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $r^i(X^i) * \eta \in \mathcal{M}_{1oc}^{1,g}(\mathbf{F})$,

(D11) $w^i(\omega, s, u, x)$ is defined on$\Omega \times \mathbb{R}_+ \times \mathbb{E} \times \mathbb{R}$

and $\mathcal{G}(\mathbf{F}) \times \mathcal{B}\left(\mathbb{E}\right) \times \mathcal{B}(\mathbb{R})$-measurable such that $w^i(X^i) * \eta \in \mathcal{V}$,

Now let us formulate the conditions under which the comparison theorem will be proved:

(A1) $X_0^2 \geq X_0^1$;

(A2) $f^2(s, x) > f^1(s, x)$ for any $s \in \mathbb{R}_+,\ x \in \mathbb{R},\ f^i(s, x)$ are continuous in (s, x), $i = 1, 2$;

(A3) There exists a non-negative nondecreasing function $\rho(x)$ on $\mathbb{R}_+$ and a $\mathcal{P}(\mathbf{F})$-measurable non-negative function G such that

$$|g(s, x) - g(s, y)| \leq \rho(|x - y|)G(s),$$

$$|G|^2 \cdot \langle m, m \rangle_s < \infty \ \ a.s., \quad \int_0^{\epsilon} \rho^{-2}(x)dx = \infty \ \ for\ any \ \ s \in \mathbb{R}_+,\ \epsilon > 0,\ x, y \in \mathbb{R};$$

(A4) There exists a non-negative $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable function H_1 and $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable function H_2 such that

$$|h_1(s,u,x)-h_1(s,u,y)| \leq \rho(|x-y|)H_1(s,u), \quad |H_1|^2 * \nu^1_s < \infty \;\; a.s. \;\; for\ any \;\; s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x,y\in\mathbb{R},$$
$$|h_2(s,u,x)-h_2(s,u,y)| \leq \rho(|x-y|)H_2(s,u), \quad |H_2|^2 * \nu^2_s < \infty \;\; a.s. \;\; for\ any \;\; s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x,y\in\mathbb{R};$$

(A5) For any $s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x,y\in\mathbb{R},\ y\geq x$,

$$h_1(s,u,y)\geq h_1(s,u,x), \quad h_2(s,u,y)\geq h_2(s,u,x)$$
$$y+h_1^2(s,u,y)\mathbf{1}_{|u|>1} \geq x+h_1^1(s,u,x)\mathbf{1}_{|u|>1},$$
$$y-h_2^2(s,u,y)\mathbf{1}_{|u|>1} \geq x-h_2^1(s,u,x)\mathbf{1}_{|u|>1},$$
$$y+h_1(s,u,y)\mathbf{1}_{|u|\leq 1}+(k_1^2+l_1^2)(s,u,y) \geq x+h_1(s,u,x)\mathbf{1}_{|u|\leq 1}+(k_1^1+l_1^1)(s,u,x),$$
$$y-h_2(s,u,y)\mathbf{1}_{|u|\leq 1}-(k_2^2+l_2^2)(s,u,y)-\left(r^2+w^2\right)(s,u,x) \geq$$
$$x-h_2(s,u,x)\mathbf{1}_{|u|\leq 1}-(k_2^1+l_2^1)(s,u,x)-\left(r^1+w^1\right)(s,u,x);$$

(A6) The functions $\left(r^i+w^i\right)(s,u,x)$ and $(k_j^i+l_j^i)(s,u,x)$ are continuous in (s,u,x), $i=1,2$ and $j=1,2$,

$$(k_j^2+l_j^2)(s,u,x) > (k_j^1+l_j^1)(s,u,x)$$
$$\left(r^2+w^2\right)(s,u,x) > \left(r^1+w^1\right)(s,u,x)$$

for any $s\in\mathbb{R}_+,\ u\in\mathbb{E},\ x\in\mathbb{R}$;

(A7) For $i=1,2$ and $j=1,2$,

$$\left|f^i(X^i)\right|\cdot a \in \mathcal{A}_{1oc},$$
$$\left|g(X^i)\right|^2\cdot\langle m,m\rangle \in \mathcal{A}_{1oc}, \quad \left|h_j(X^i)\right|^2 * \nu^j \in \mathcal{A}_{1oc},$$
$$\left|l_j^i(X^i)\right| * p^j \in \mathcal{A}_{1oc}, \quad \left[\left|k_j^i(X^i)\right|^2 * p^j\right]^{1/2} \in \mathcal{A}_{1oc},$$
$$\left|r^i(X^i)\right| * \eta \in \mathcal{A}_{1oc}, \quad \left[\left|w^i(X^i)\right|^2 * \eta\right]^{1/2} \in \mathcal{A}_{1oc},$$

and $\mathbf{E}[k_1^i(S,\beta^d_S,X^i_{S-})|\mathcal{F}_{S-}]=0$ a.s. for any predictable stopping time S and $\mathbf{E}[k_2^i(T,\beta^g_T,X^i_T)|\mathcal{F}_T]=0$ a.s., for any totally inaccessible stopping time T, and $\mathbf{E}\left[w(U,\beta^g_U)|\mathcal{F}_U\right]=0$ a.s., for any totally inaccessible stopping time in the broad sense $U,\ i=1,2$.

To formulate the next assumption we need to introduce the sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers

$a_0 = 1 > a_1 > \cdots$, $\lim_{n\to\infty} a_n = 0$, by the relations

$$\int_{a_{n+1}}^{a_n} \rho^{-2}(x)dx = n+1, \quad n = 0, 1, \cdots.$$

Now let us write the last assumption

(A8) We assume that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\varepsilon_n \le a_{n-1} - a_n$ for all $n \in \mathbb{N}$ and

$$\frac{1}{n}\left[\frac{\rho(a_{n-1})}{\rho(a_{n-1} - \varepsilon_n)}\right]^2 \to 0, \quad n \to \infty.$$

It is easy to verify that condition A8 is satisfied by Holder class functions ρ with index $\alpha = 1/2+\epsilon$, $\epsilon > 0$, and is not satisfied by functions of this class with index $\alpha = 1/2$.

Theorem 3.1. *Let there exist strong solutions X^i, $i = 1, 2$, of equations (1) and let conditions A1-A8 hold. Then off some set of $\mathbf{P}$-measure zero $X^2_t \ge X^1_t$ for any $t \in \mathbb{R}_+$.*

Before proving the theorem, let us perform a useful reduction of the problem.

Lemma 3.2. *If the comparison theorem is valid for the equations in 3.2,*

$$\begin{aligned} Y^i_t &= X^i_0 + f^i(Y^i)\cdot a_t + g(Y^i)\cdot m_t \\ &\quad + \sum_j Uh_j(Y^i) * (\mu^j - v^j)_t + \left(k^i_j(Y^i) + l^i_j(Y^i)\right) * p^j_t + \left(r^i(Y^i) + w^i(Y^i)\right) * \eta_t \end{aligned} \tag{3.2}$$

with functions X^i_0, f^i, g, h_j, k^i_j, l^i_j, r^i, and w^i satisfying conditions A1-A8, then, it is also valid for equations 3.1.

Proof. Let $\{\tau_n\}_{n\in\mathbb{N}}$, $\tau_0 = 0$, be a nondecreasing sequence of totally inaccessible stopping times and stopping times in the broad sense, absorbing the jumps of the processes $h^i_j(X^i) * \mu^j$, $i = 1, 2$ and $j = 1, 2$, from equations 3.1. On $]\tau_0, \tau_1[$ equations 3.1 and 3.2 coincide. Since $Y^2 \ge Y^1$, on this interval then $X^2 \ge X^1$ on this interval. On the boundary of this interval, from equations 3.1 we find that at time τ_1,

$$\begin{aligned} X^2_{\tau_1} &= X^2_{\tau_1 -} + \Delta X^2_{\tau_1} = X^2_{\tau_1 -} + h^2_1\left(\tau_1, \beta^1_{\tau_1}, X^2_{\tau_1 -}\right)\mathbf{1}_{|\beta^1_{\tau_1}|>1} \\ &\ge X^1_{\tau_1 -} + h^1_1(\tau_1, \beta^1_{\tau_1}, X^1_{\tau_1 -})\mathbf{1}_{|\beta^1_{\tau_1}|>1} = X^1_{\tau_1 -} + \Delta X^1_{\tau_1} = X^1_{\tau_1}, \end{aligned}$$

and

$$
\begin{aligned}
X^2_{\tau_0} &= X^2_{\tau_0+} - \Delta^+ X^2_{\tau_0} \\
&= X^2_{\tau_0+} - h^2_2\left(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0}\right)\mathbf{1}_{|\beta^2_{\tau_0}|>1} - \left(r^2(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0}) + w^2(\tau_0, \beta^2_{\tau_0}, X^2_{\tau_0})\right)\mathbf{1}_{|\beta^2_{\tau_0}|>1} \\
&\geq X^1_{\tau_0+} - h^1_2(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0})\mathbf{1}_{|\beta^2_{\tau_0}|>1} - \left(r^1(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0}) + w^1(\tau_0, \beta^2_{\tau_0}, X^1_{\tau_0})\right)\mathbf{1}_{|\beta^2_{\tau_0}|>1} \\
&= X^1_{\tau_0+} - \Delta^+ X^1_{\tau_0} = X^1_{\tau_0};
\end{aligned}
$$

Therefore, by condition A5 the comparison theorem holds for 3.1 on $[0, \tau_1]$.

Now let us suppose that the comparison theorem for 3.1 holds on $[0, \tau_n]$, $n \geq 1$ and prove it holds on $]\tau_n, \tau_{n+1}]$. On $]\tau_n, \infty[$ consider the equations ($i = 1, 2$)

$$
\begin{aligned}
Y^i_t = X^i_{\tau_n} &+ \int_{\tau_n+}^{t} f^i(Y^i_s)da_s + \int_{\tau_n+}^{t} g(Y^i_s)dm_s + \int_{\tau_n+}^{t}\int_{\mathbb{E}} Uh_1(Y^i)d(\mu^1 - \nu^1)_s \\
&+ \int_{\tau_n+}^{t-}\int_{\mathbb{E}} Uh_2(Y^i_s)d(\mu^2 - \nu^2)_s + \int_{\tau_n+}^{t}\int_{\mathbb{E}} (k^i_1(Y^i_s) + l^i_1(Y^i_s))dp^1 \\
&+ \int_{\tau_n+}^{t-}\int_{\mathbb{E}} (k^i_2(Y^i_s) + l^i_2(Y^i_s))dp^2_s + \int_{\tau_n+}^{t-}\int_{\mathbb{E}} \left(r^i(Y^i_s) + w^i(Y^i_s)\right)d\eta_s
\end{aligned}
\tag{3.3}
$$

Let us transform (3.3) to the form (3.2). For this we make the substitution $t - \tau_n = s$ and set

$$
\begin{gathered}
\mathcal{F}^{(n)}_s = \mathcal{F}_{s+\tau_n}, \quad s \in [0, \infty[, \quad a^{(n)}_s = a_{s+\tau_n}, \quad m^{(n)}_s = m_{s+\tau_n}, \\
\left(\mu^{1(n)} - \nu^{1(n)}\right)(]\rho, \varsigma], \Gamma) = \left(\mu^1 - \nu^1\right)(]\rho + \tau_n, \varsigma + \tau_n], \Gamma), \\
\left(\mu^{2(n)} - \nu^{2(n)}\right)([\rho, \varsigma[, \Gamma) = \left(\mu^2 - \nu^2\right)([\rho + \tau_n, \varsigma + \tau_n[, \Gamma), \\
p^{1(n)}(]\rho, \varsigma], \Gamma) = p^1(]\rho + \tau_n, \varsigma + \tau_n], \Gamma), \\
p^{2(n)}([\rho, \varsigma[, \Gamma) = p^2([\rho + \tau_n, \varsigma + \tau_n[, \Gamma), \\
\eta^{(n)}([\rho, \varsigma[, \Gamma) = \eta([\rho + \tau_n, \varsigma + \tau_n[, \Gamma), \\
Y^{i(n)}_s = Y^i_{s+\tau_n}, \quad X^{i(n)}_s = X^i_{s+\tau_n}.
\end{gathered}
$$

Introduce further the functions $f^{i(n)}$, $g^{(n)}$, $h^{(n)}_j$, $k^{i(n)}_j$, $l^{i(n)}_j$, $r^{i(n)}$, and $w^{i(n)}$ setting

$$
f^{i(n)}(s, x) = f^i(s + \tau_n, x), \quad h^{(n)}_j(s, u, x) = h_j(s + \tau_n, u, x),
$$

and proceed analogously for the remaining functions. Equations (3.3) take on the form

$$\begin{aligned}
Y_s^i &= X_0^{i(n)} + f^{i(n)}\left(Y^{i(n)}\right)\cdot a_s^{(n)} + g^{(n)}\left(Y^{i(n)}\right)\cdot m_s^{(n)} \\
&\quad + \sum_j Uh_j^{(n)}(Y^{i(n)}) * (\mu^j - \nu^j)_s^{(n)} + (k_j^{i(n)} + l_j^{i(n)})(Y^{i(n)}) * p_s^{j(n)} \\
&\quad + \left(r^{i(n)}(Y^{i(n)}) + w^{i(n)}(Y^{i(n)})\right) * \eta_t^{(n)}
\end{aligned}$$

for $(i = 1, 2)$ and $s \in [0, \infty[$.

These are equations of the form 3.2 with integrands satisfying conditions A1-A8. By the assumption of the lemma, the comparison theorem holds for these equations. Their solutions for $s \in]0, \tau_{n+1} - \tau_n[$ coincide with the solutions of 3.1 on $]\tau_n, \tau_{n+1}[$. Hence the comparison theorem for 3.1 can be extended to the interval $]0, \tau_{n+1}[$. Arguing just as in the case of $[0, \tau_1]$ we extend the comparison theorem for 3.1 to the points τ_n and τ_{n+1}. Then repeating these arguments, we prove the result for equation 3.1 for all $t \in [0, \infty[$. □

Proof. [Proof of Theorem 1] By lemma, it suffices to establish comparison result for 3.2. So it begins; By A7, all the integrals in (3.2) are defined. Not to resort to an additional localization arguments, we shall assume that for $(i = 1, 2)$,

$$\mathbf{E}\left[|G|^2 \cdot \langle m, m\rangle_\infty\right] < \infty, \quad \mathbf{E}\left[|H_j|^2 * \nu_\infty^j\right] < \infty,$$

$$\begin{aligned}
&\mathbf{E}\Big[|f^i(X^i)| \cdot a_\infty + |g(X^i)|^2 \cdot \langle m, m\rangle_\infty \\
&\quad + \textstyle\sum_j U|h_j(X^i)|^2 * \nu_\infty^j + |l_j^i(X^i)| * p_\infty^j + \left(|k_j^i(X^i)|^2 * p_\infty^j\right)^{1/2} \\
&\quad + \left|r^i(Y^i)\right| * \eta_\infty + w^i(Y^i) * \eta_\infty\Big] < \infty.
\end{aligned}$$

Let us introduce the sets

$$A = \left\{\omega : X_0^2(\omega) > X_0^1(\omega)\right\}, \quad B = \left\{\omega : X_0^2(\omega) = X_0^1(\omega)\right\}.$$

I. First we prove the theorem on the set B. Let $Y_0^i = X_0^i \mathbf{1}_B$, $\tilde{f}^i = f^i \mathbf{1}_B$, and similarly define $\tilde{g}$, $\tilde{h}_j$, $\tilde{k}_j^i$, $\tilde{l}_j^i$, $\tilde{r}^i$ and $\tilde{w}^i$. Consider the equations $(i = 1, 2)$,

$$\begin{aligned}
Y_t^i &= Y_0^i + \tilde{f}^i(Y^i) \cdot a_t + \tilde{g}(Y^i) \cdot m_t \\
&\quad + \sum_j U\tilde{h}_j(Y^i) * (\mu^j - \nu^j)_t + (\tilde{k}_j^i + \tilde{l}_j^i)(Y^i) * p_t^j \\
&\quad + \left(\tilde{r}^i(Y^i) + \tilde{w}^i(Y^i)\right) * \eta_t.
\end{aligned} \tag{3.4}$$

It is clear that $Y^i = X^i$ on B. Define the quantity T as follows:

$$\begin{aligned} T = \inf\Big\{ & t > 0 : \tilde f^1(t, Y^1_{t-}) > \tilde f^2(t, Y^2_{t-}) \\ & or \quad (\tilde k^1_1 + \tilde l^1_1)(t, \beta^1_t, Y^1_{t-}) > (\tilde k^2_1 + \tilde l^2_1)(t, \beta^1_t, Y^2_{t-}) \\ & or \quad (\tilde k^1_2 + \tilde l^1_2)(t, \beta^2_t, Y^1_t) > (\tilde k^2_2 + \tilde l^2_2)(t, \beta^2_t, Y^2_t) \\ & or \quad (\tilde r^1 + \tilde w^1)(t, \beta^2_t, Y^1_t) > (\tilde r^2 + \tilde w^2)(t, \beta^2_t, Y^2_t)\Big\} \end{aligned}$$

We have $Y^2_0 = Y^1_0$, and, by A2 and A6,

$$\begin{aligned} \tilde f^2(0, Y^2_0) &> \tilde f^1(0, Y^1_0), \\ (\tilde k^2_1 + \tilde l^2_1)(0, \beta^1_0, Y^2_0) &> (\tilde k^1_1 + \tilde l^1_1)(0, \beta^1_0, Y^1_0), \\ (\tilde k^2_2 + \tilde l^2_2)(0, \beta^2_0, Y^2_0) &> (\tilde k^1_2 + \tilde l^1_2)(0, \beta^2_0, Y^1_0), \\ (\tilde r^2 + \tilde w^2)(0, \beta^2_0, Y^2_0) &> (\tilde r^1 + \tilde w^1)(0, \beta^2_0, Y^1_0) \end{aligned}$$

Since $\beta^1_t \to \beta^1_0$ and $\beta^2_t \to \beta^2_0,\ Y^i_t \to Y^i_0,\ t \downarrow 0$, and the functions $\tilde f^i(t,x)$, $(\tilde k^i_j + \tilde l^i_j)(t,u,x)$ and $(\tilde r^i + \tilde w^i)(t,u,x)$ are continuous in (t,u,x), it follows from what has been said that $T > 0$ a.s. on the set B.

Let $v = t \wedge T$. Set $R = Y^2 - Y^1$,

$$\begin{aligned} R_v = Y^2_v - Y^1_v = R_0 &+ \left(\tilde f^2(Y^2) - \tilde f^1(Y^1)\right) \cdot a_t + \left(\tilde g(Y^2) - \tilde g(Y^1)\right) \cdot m_t \\ &+ \sum_j U\left(\tilde h_j(Y^2) - \tilde h_j(Y^1)\right) * (\mu^j - \nu^j)_t + \left[(\tilde k^2_j + \tilde l^2_j)(Y^2) - (\tilde k^1_j + \tilde l^1_j)(Y^1)\right] * p^j_t \\ &+ \left(\left(\tilde r^2 + \tilde w^2\right)(Y^2) - \left(\tilde r^1 + \tilde w^1\right)(Y^1)\right) * \eta_t. \end{aligned}$$

and considering the properties of stochastic integrals we have

$$\mathbf{E}R_v = \mathbf{E}\left[\left(\tilde f^2(Y^2) - \tilde f^1(Y^1)\right) \cdot a_v + \sum_j \left(\tilde l^2(Y^2) - \tilde l^1(Y^1)\right) * \lambda^j_v + \left(\tilde w^2(Y^2) - \tilde w^1(Y^1)\right) * \theta_v\right]. \tag{3.5}$$

Now let $\{\psi_n(x)\}_{n\in\mathbb{N}}$ be a sequence of non-negative continuous functions such that $\mathrm{supp}\psi_n \subseteq (a_n, a_{n-1})$,

$$\int_{a_n}^{a_{n-1}} \psi_n(x)dx = 1, \quad \psi_n(x) \le \frac{2}{n}\rho^{-2}(|x|), \quad x \in \mathbb{R},$$

and the maximum of ψ_n is attained at $a_{n-1} - \epsilon_n$, where the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ satisfies A8. Set

$$\varphi_n(x) = \int_0^{|x|} dy \int_0^y \psi_n(u)du, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Clearly,

$$\begin{gathered}\varphi_n \in C^2(R^1), \quad \varphi_n(x) \uparrow |x|, \quad n \to \infty, \quad |\varphi_n'| \le 1, \\ \varphi_n''(x) = \psi_n(x) \le \frac{2}{n}\rho^{-2}(|x|), \quad x \in \mathbb{R}. \end{gathered} \tag{3.6}$$

Then, by lemma on change of variable formula for the component representation of optional semimartignales,

$$\begin{aligned}\varphi_n(R_v) = &\varphi_n(R_0) + \varphi_n'(R)\left(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1)\right)\cdot a_v + \varphi_n'(R)\left(\tilde{g}(Y^2) - \tilde{g}(Y^1)\right)\cdot m_v \\ &+\frac{1}{2}\varphi_n''(R)\left(\left|\tilde{g}(Y^2) - \tilde{g}(Y^1)\right|^2\right)\cdot\langle m,m\rangle_v \\ &+\sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R)\right] * (\mu^j - \nu^j)_v \\ &+\sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R) + \varphi_n'(R)\left(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right)\right] * \nu^j_v \\ &+\sum_j \left[\varphi_n\left(R + (\tilde{k}^2_j + \tilde{l}^2_j)(Y^2) - (\tilde{k}^1_j + \tilde{l}^1_j)(Y^1)\right) - \varphi_n(R)\right] * p^j_v \\ &+\left[\varphi_n\left(R + \left(\tilde{r}^2 + \tilde{w}^2\right)(Y^2) - \left(\tilde{r}^1 + \tilde{w}^1\right)(Y^1)\right) - \varphi_n(R)\right] * \eta_v\end{aligned}$$

Let

$$\begin{aligned}I_1(v) &= \left[\varphi_n'(R)(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1))\right]\cdot a_v, \\ I_2(v) &= \frac{1}{2}\left[\varphi_n''(R)|\tilde{g}(Y^2) - \tilde{g}(Y^1)|^2\right]\cdot\langle m,m\rangle_v, \\ I_3(v) &= \sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R) - \varphi_n'(R)\left(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right)\right] * \nu^j_v \\ I_4(v) &= \sum_j \left[\varphi_n\left(R + (\tilde{k}^2_j + \tilde{l}^2_j)(Y^2) - (\tilde{k}^1_j + \tilde{l}^1_j)(Y^1)\right) - \varphi_n(R)\right] * p^j_v \\ I_5(v) &= \left[\varphi_n\left(R + \left(\tilde{r}^2 + \tilde{w}^2\right)(Y^2) - \left(\tilde{r}^1 + \tilde{w}^1\right)(Y^1)\right) - \varphi_n(R)\right] * \eta_v.\end{aligned}$$

and write

$$\begin{aligned}\varphi_n(R_v) = {}& [\varphi_n'(R)(\tilde{g}(Y^2) - \tilde{g}(Y^1))] \cdot m_v \\ &+ \sum_j U\left[\varphi_n\left(R + \tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right) - \varphi_n(R)\right] * (\mu^j - \nu^j)_v \\ &+ \sum_{\kappa=1}^{5} I_\kappa(v).\end{aligned}$$

Taking the expectation we get

$$\mathbf{E}\varphi_n(R_v) = \mathbf{E}\sum_{\kappa=1}^{5} I_\kappa(v). \tag{3.7}$$

Since $\tilde{f}^2(v, Y^2_{v-}) > \tilde{f}^1(v, Y^1_{v-})$ for $v < T$ and $|\varphi_n'| \leq 1$, we have

$$\mathbf{E}I_1(v) \leq \mathbf{E}\left[\left(f^2(Y^2) - \tilde{f}^1(Y^1)\right) \cdot a_v\right].$$

Further, by A3 and property (7), the relations

$$\begin{aligned}\mathbf{E}|I_2(v)| &\leq \frac{1}{2}\left(\max_{a_n \leq x \leq a_{n-1}}[\varphi_n''^2(|x|)\rho^2(|x|)]\right)\mathbf{E}\left[|G|^2 \cdot \langle m, m\rangle_v\right] \\ &\leq \frac{1}{n}\mathbf{E}|G|^2 \cdot \langle m, m\rangle_\infty \to 0,\ n \to \infty,\end{aligned}$$

are valid.

Applying Taylor's formula, noting A4 and A8 and property (7) we get

$$\begin{aligned}|I_3(v)| &\leq \sum_j \frac{1}{2}\left[\left|\varphi_n''\left(R + \alpha(\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1))\right)\right|\left|\tilde{h}_j(Y^2) - \tilde{h}_j(Y^1)\right|^2\right] * \nu^j_v \\ &\leq \sum_j \frac{1}{2}\left[|H_j|^2\rho^2(|R|)|\varphi_n''(R + \alpha(\tilde{h}(Y^2) - \tilde{h}(Y^1)))|\right] * \nu^j_v,\end{aligned}$$

where $0 \leq \alpha \leq 1$.

Consider here three cases:

(i) if $|R| \in [a_n, a_{n-1}]$, then by (7) and A8,

$$|I_3| \leq \sum_j \frac{1}{2}|H_j|^2 * \nu^j_\infty\left(\frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1} - \epsilon_n)\right)$$

(ii) if $|R| < a_n$, then, by A5 and A8,

$$\begin{aligned}|I_3| &\le \sum_j \frac{1}{2}\rho^2(a_n)\left[|H_j|^2\left|\varphi_n''\left(R+\alpha\left(\tilde h(Y^2)-\tilde h(Y^1)\right)\right)\right|\right] * \nu_\infty^j\\ &\le \sum_j \frac{1}{2}|H|^2 * v_\infty^j \frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1}-\epsilon_n);\end{aligned}$$

(iii) if $|R| > a_{n-1}$, then by A5 and property (7) we find that $\varphi_n''(R+\alpha(\tilde h(Y^2)-\tilde h(Y^1))) = 0$ and $I_3 = 0$.

From what has been said it follows that

$$\mathbf{E}|I_3(v)| \le \sum_j \frac{1}{2}\mathbf{E}|H_j|^2 * \nu_\infty^j\left(\frac{1}{n}\rho^2(a_{n-1})\rho^{-2}(a_{n-1}-\epsilon_n)\right) \to 0, \quad n\to\infty.$$

Again using Taylor's formula and noting that $(\tilde k_j^2+\tilde l_j^2)(v,\beta_v^1,Y_{v-}^2) > (\tilde k_j^1+\tilde l_j^1)(v,\beta_v^1,Y_{v-}^1)$ and $(\tilde k_2^2+\tilde l_2^2)(v,\beta_v^2,Y_v^2) > (\tilde k_2^1+\tilde l_2^1)(v,\beta_v^2,Y_v^1)$ for $v<T$, we have

$$\begin{aligned}\mathbf{E}I_4(v) &= \mathbf{E}\sum_j\left[\varphi_n'\left(R+\alpha\left(\left(\tilde k_j^2+\tilde l_j^2\right)(Y^2)-\left(\tilde k_j^1+\tilde l_j^1\right)(Y^1)\right)\right)\left(\left(\tilde k_j^2+\tilde l_j^2\right)(Y^2)-\left(\tilde k_j^1+\tilde l_j^1\right)(Y^1)\right)\right] * p_v^j\\ &\le \mathbf{E}\sum_j\left[\left(\tilde k_j^2+\tilde l_j^2\right)(Y^2)-\left(\tilde k_j^1+\tilde l_j^1\right)(Y^1)\right] * p_v^j.\end{aligned}$$

Now noting that $\mathbf{E}\left[\tilde k_j^2(Y^2)-\tilde k_j^1(Y^1)\right] * p_v^j = 0$, we obtain

$$\mathbf{E}I_4(v) \le \mathbf{E}\sum_j\left[\tilde l_j^2(Y^2)-\tilde l_j^1(Y^1)\right] * p_v^j = \mathbf{E}\sum_j\left[\tilde l_j^2(Y^2)-\tilde l_j^1(Y^1)\right] * \lambda_v^j.$$

Applying Taylor's formula one more time and noting that $\left(\tilde r^2+\tilde w^2\right)(v,\beta_v^2,Y_v^2) > \left(\tilde r^1+\tilde w^1\right)(v,\beta_v^2,Y_v^2)$ for $v<T$, we have

$$\begin{aligned}\mathbf{E}I_5(v) &= \mathbf{E}\left[\varphi_n'\left(R+\alpha\left(\left(\tilde r^2+\tilde w^2\right)(Y^2)-\left(\tilde r^1+\tilde w^1\right)(Y^1)\right)\right)\left(\left(\tilde r^2+\tilde w^2\right)(Y^2)-\left(\tilde r^1+\tilde w^1\right)(Y^1)\right)\right] * \eta_v\\ &\le \mathbf{E}\left[\left(\tilde r^2+\tilde w^2\right)(Y^2)-\left(\tilde r^1+\tilde w^1\right)(Y^1)\right] * \eta_v.\end{aligned}$$

Now noting that $\mathbf{E}\left[\tilde r^2(Y^2)-\tilde r^1(Y^1)\right] * \eta_v = 0$, we obtain

$$\mathbf{E}I_5(v) \le \mathbf{E}\left[\tilde w^2(Y^2)-\tilde w^1(Y^1)\right] * \eta_v = \mathbf{E}\left[\tilde w^2(Y^2)-\tilde w^1(Y^1)\right] * \theta_v.$$

by the estimates of $\mathbf{E}I_i,\ f = 1,...,5$, and the fact that

$$\mathbf{E}\varphi_n(R) = \mathbf{E}\varphi_n(Y^2 - Y^1) \uparrow \mathbf{E}\left|Y^2 - Y^1\right|$$

as $n \to \infty$, we have from (8)

$$\begin{aligned}\mathbf{E}|Y^2_v - Y^1_v| &\leq \mathbf{E}\left[(\tilde{f}^2(Y^2) - \tilde{f}^1(Y^1)) \cdot a_v + \sum_j (\tilde{l}^2_j(Y^2) - \tilde{l}^1_j(Y^1)) * \lambda^j_v + \left(\tilde{w}^2(Y^2) - \tilde{w}^1(Y^1)\right) * \theta_v\right] \\ &= \mathbf{E}\left(Y^2_v - Y^1_v\right),\end{aligned}$$

where the last equality follows as a result of equation 3.5. Since the processes $Y^i,\ i = 1, 2$, are RLL, it follows from the derived inequality that apart from some set of $\mathbf{P}$-measure zero, $Y^2_v \geq Y^1_v$ (hence, also $X^2_v \geq X^1_v$) for $v \leq T$ a.s. on the set B.

Now, consider the random time

$$\varrho = \inf\left(t > T : Y^2_t < Y^1_t\right).$$

Let us show that $\varrho = \infty$ a.s. Naturally for $t < \varrho$ or on $[0, \varrho[$ the inequality $Y^2_t \geq Y^1_t$ is valid a.s. whereas $]\varrho, \infty[\ Y^2_t < Y^1_t$ by definition of ϱ. Using (3.4) we obtain, that at time ϱ,

$$Y^i_\varrho = Y^i_{\varrho-} + \tilde{h}_1(\varrho, \beta^1_\varrho, Y^i_{\varrho-})\mathbf{1}_{|\beta^1_\varrho|\leq 1} + \left(\tilde{k}^i_1 + \tilde{l}^i_1\right)(\varrho, \beta^1_\varrho, Y^i_{\varrho-}).$$

And by A5 we obtain that $Y^2_\varrho \geq Y^1_\varrho$. Hence, $Y^2_\varrho \geq Y^1_\varrho$ is true a.s. on $[0, \varrho]$.

Now, let us introduce the sets

$$C = \left\{Y^2_\varrho > Y^1_\varrho, \varrho < \infty\right\}, \quad D = \{Y^2_\varrho = Y^1_\varrho, \varrho < \infty\}.$$

From the definition of ϱ and the processes $Y^i,\ i = 1, 2$, It follows that $\mathbf{P}(C) = 0$.

For the set, D, we carry out the same arguments as we have done for, B. Moreover, if $\varrho < \infty$, then there is a stopping time S, $\mathbf{P}(S > \varrho, \varrho < \infty) > 0$ such that $Y^2_t \geq Y^1_t$ for $\varrho < t \leq S$. The latter will contradict the definition of ϱ. Hence, $\varrho = \infty$ a.s.

Now let us prove the theorem on the set, A. Consider the quantity $\hat{\varrho} = \inf(t > 0 : X^2_t < X^1_t)$, where the $X^i,\ i = 1, 2$, are the solutions of equations (3.2). For $t < \hat{\varrho}$ we have: $X^2_t \geq X^1_t$ a.s. on A. As above let us compute the jumps of the X^i at time ϱ and using condition A5 we get: $X^2_{\hat{\varrho}} \geq X^1_{\hat{\varrho}}$ a.s. on A.

Construct the sets,

$$\hat{C} = \left\{X^2_{\hat{\varrho}} > X^1_{\hat{\varrho}}, \hat{\varrho} < \infty\right\}, \quad \hat{D} = \left\{X^2_{\hat{\varrho}} = X^1_{\hat{\varrho}}, \hat{\varrho} < \infty\right\}.$$

From the definition of the stopping time $\hat{\varrho}$ and from the right continuity of the processes X^i at stopping times, it follows that $\mathbf{P}(\hat{C}) = 0$. For the set $\hat{D}$ we repeat the arguments made in Section 1 for the set B. First by carrying out a time shift by the quantity $\hat{\varrho}$, as was done in Lemma 2.2. To finally obtain, $X^2 \geq X^1$ a.s. on A. □

The proof of the comparison theorem is completed. Next we give an example from finance.

4. Illustrative Example.

In this section we give an example showing how the stochastic domination (comparison) theorems can be used in mathematical finance. For the purpose of demonstration we shall restrict our attention to the simplest cases only.

Example 4.1. *The constant elasticity of variance (CEV) model was proposed by Cox and Ross [2]. It is often used in mathematical finance to capture leverage effects and stochasticity of volatility. It is also widely used by practitioners in the financial industry for modeling equities and commodities. Consider a modified version of the CEV model where the stock price is said to satisfy the following integral equation,*

$$\begin{aligned} S_t &= \rho S \cdot A_t + \sigma S^\alpha \cdot M_t, \quad S_0 = s, \\ A_t &= t + V * \mu^1 + V * \mu^2 \\ M_t &= W_t + U * \left(\mu^1 - \nu^1\right)_t + U * \left(\mu^2 - \nu^2\right)_t \end{aligned} \tag{4.1}$$

where ρ and σ are constants and the martingale M is a jump-diffusion process with left and right jumps. W_t is the Wiener process, $\mu^1 - \nu^1$ is the measure of right jumps and $\mu^2 - \nu^2$ is the measure of left jumps. For $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{B}(\mathbb{E})$ the jump measures are defined

$$\begin{aligned} \mu^1 (B \times \Gamma) &: = \#\left\{\left(t, \Delta L^1_t\right) \in B \times \Gamma | t > 0 \textit{ such that } \Delta L^1_t \neq 0\right\} \\ \mu^2 (B \times \Gamma) &: = \#\left\{\left(t, \Delta^+ L^2_t\right) \in B \times \Gamma | t > 0 \textit{ such that } \Delta^+ L^2_t \neq 0\right\} \end{aligned}$$

where L^1_t and L^2_t are independent Poisson with constant intensities γ^1 and γ^2 respectively and compensators $\nu^1 = \gamma^1 t$ and $\nu^2 = \gamma^2 t$.

Let

$$F(x) = \frac{1}{\sigma}\int_s^x u^{-\alpha}du = \frac{x^{1-\alpha} - s^{1-\alpha}}{\sigma(1-\alpha)},$$
$$F'(x) = \frac{x^{-\alpha}}{\sigma}, \quad F''(x) = \frac{-\alpha x^{-\alpha-1}}{\sigma}.$$

where $0 < \alpha < 1$. *Denote* $X_t = F(S_t)$ *and applying Itô's formula we get*

$$\begin{aligned} X_t &= \rho S F'(S) \circ A_t + \sigma S^\alpha F'(S) \circ M_t + \frac{\sigma^2}{2} F''(Y) S^{2\alpha} \circ [M,M]_t \\ &= \frac{\rho}{\sigma} S_t^{1-\alpha} \circ A_t - \frac{\alpha\sigma}{2} S^{\alpha-1} \circ [M,M]_t + M_t \\ &= \frac{\rho}{\sigma}\left(\sigma(1-\alpha)X + s^{1-\alpha}\right) \circ A_t - \frac{\alpha\sigma}{2}\left(\sigma(1-\alpha)X + s^{1-\alpha}\right)^{-1} \circ [M,M]_t + M_t \end{aligned}$$

where $[M,M]_t = \left(1 + \gamma^1 + \gamma^2\right) t$.
With the comparison theorem proved above, we can give an estimate of the process X_t *from above by a new process* Y_t*, satisfying the equation,*

$$Y_t = \frac{\rho}{\sigma}\left(\sigma(1-\alpha)Y + s^{1-\alpha}\right) \cdot A_t + M_t, \quad Y_0 = 0$$

which is essentially an Ornstein-Uhlenbeck process with left and right jumps. Applying the comparison theorem to X_t *and* Y_t *yields that,* $Y_t \geq X_t = F(S_t)$ *a.s. therefore*

$$S_t \leq F^{-1}(Y_t) \qquad a.s. \tag{4.2}$$

Now lets consider an increasing function f *with an option payoff* $f(S_T)$*. Assuming zero interest rates, the price of such option is given by* $\tilde{\mathbf{E}} f(S_T)$ *for an appropriate martingale measure* $\tilde{P}$ *(see [1] where existence of* $\tilde{P}$ *is discussed). Using inequality (4.2) we have that* $\tilde{\mathbf{E}} f(S_T) \leq \tilde{\mathbf{E}} f(F^{-1}(Y_T))$ *and thus we obtain an estimate for the option price for which* $\tilde{\mathbf{E}} f(F^{-1}(Y_T))$ *is easier to compute.*

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