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Research Article

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On Krylov's estimates for optional semimartingales

Abstract: The estimates of N. V. Krylov for distributions of stochastic integrals by means of L_d-norm of a measurable function are well-known and are widely used in the theory of stochastic differential equations and controlled diffusion processes. We generalize estimates of this type for optional semimartingales, then apply these estimates to prove the change of variables formula for a general class of functions from Sobolev space W_d^2. We also show how to use these estimates for investigation of L^2 covergence of solutions of optional SDE's.

Keywords: Krylov's estimates, laglad processes, optional semimartingales, Ito's formula

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1 Introduction

The estimates of N. V. Krylov have a great importance in the theory of controlled diffusion processes and stochastic differential equations (see [7], [9]). Anulova and Pragarauskas (see [2]) generalized this result to the Ito processes with Poisson random measures. Melnikov in [10] proved Krylov type estimates for continuous semimartingales on probability spaces under usual conditions. In this paper, we do not assume our probability space satisfy such technical conditions; and, our goal is to generalize Krylov's estimates to the class of optional semimartingales - *laglad* processes defined on a complete probability space such that the underlying filtration is not necessarily left nor right continuous nor complete.

Using these estimates Krylov obtained a generalization of Ito's formula for functions which have generalized derivatives up to and including the second order (see [7], [9]). In the same way, by using our obtained estimates we extend the change of variables formula for optional semimartingales (see [4], [6]) for class C^2 to the Sobolev class of functions W_d^2. Furthermore, we show how Krylov's estimates can be applied to the mean square convergence of optional solutions of SDE's under quite general assumptions on their coefficients.

Now we introduce some definitions and notation used throughout this paper.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given complete space and $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ be a corresponding filtration on it. The family $\mathbf{F}$ is not assumed right- or left-continuous, and it is not assumed to be complete. Denote $\mathbf{F}_+ = (\mathcal{F}_{t+})_{t\in[0,\infty)}$, where $\mathcal{F}_{t+} = \cap_{t<s}\mathcal{F}_s$.

On unusual stochastic basis two notions of stopping times exist: stopping times, $T \in \mathcal{T}$, are such that $(T \le t)$ is $\mathcal{F}_t$-measurable for all $t \ge 0$; wide (broad) sense stopping times, $U \in \mathcal{T}_+$, are such that $(U \le t)$ is $\mathcal{F}_{t+}$ measurable for all $t \ge 0$ (see [3], Def. 49.2). A process $X = (X_t)_{t\ge 0}$ belongs to the space $\mathcal{J}_{loc}$ if there is a localizing sequence of wide sense stopping times, (R_n), $n \in \mathbb{N}$, $R_n \in \mathcal{T}_+$, $R_n \uparrow \infty$ a.s. such that $X\mathbf{1}_{[0,R_n]} \in \mathcal{J}$ for all n, where $\mathcal{J}$ is a space of processes and $\mathcal{J}_{loc}$ is an extension of $\mathcal{J}$ by localization.

$\mathcal{A}_{loc}$ ($\mathcal{A}^c_{loc}$) is the set of all (continuous) processes $A = (A_t, \mathcal{F}_t)_{t\geq 0}$ having locally integrable variation, with $A_0 = 0$.

$\mathcal{M}^c_{loc}$ ($\mathcal{M}^{2,c}_{loc}$) is the set of all continuous optional local (square integrable) martingales $M = (M_t, \mathcal{F}_t)_{t\geq 0}$, $M_0 = 0$.

For functions $f : \mathbb{R}^d \times [0, \infty] \to \mathbb{R}$ we set

$$f_{x_i} = \frac{\partial f}{\partial x_i}, \quad f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad f_x = (f_{x_1}, ..., f_{x_d}).$$

For vectors $x = (x_i, ..., x_d), y = (y_1, ..., y_d) \in \mathbb{R}^d$: $|x| = \sum_{i=1}^d |x_i|$, $(x, y) = x_1 y_1 + ... + x_d y_d$. For a square matrix A: trA is a trace of A and $\det A$ is a determinant of A.

$L_d(U)$ is the space of measurable functions f, defined in the region $U \subset \mathbb{R}^d$, $d \geq 1$ such that

$$\|f\|_{d,U} = \left(\int_U |f(x)|^d \, dx \right)^{1/d} < \infty.$$

Let $E = \mathbb{R}^d \setminus \{0\}$ with σ-algebra of Borel subsets $\mathcal{E} = \mathcal{B}(E)$.

$B(\Gamma)$ denotes the set of bounded Borel functions on Γ with the norm

$$\|f\|_{B(\Gamma)} = \sup_{x\in\Gamma} |f(x)|$$

Let D be a bounded region in $\mathbb{R}^d$, and let $u(x)$ be a function in $\bar{D}$. We write $u \in W^2(D)$ ($u \in \bar{W}^2(D)$) if there exists a sequence of functions $u^n \in C^2(\bar{D})$ such that

$$\|u - u^n\|_{B(\bar{D})} \to 0, \quad \|u^n - u^m\|_{W^2(D)} \to 0 \quad \left(\|u^n - u^m\|_{\bar{W}^2(D)} \to 0 \right)$$

as $n, m \to \infty$, where

$$\|f\|_{W^2(D)} = \sum_{i,j=1}^d \|f_{x_i x_j}\|_{d,D} + \sum_{i=1}^d \|f_{x_i}\|_{d,D} + \|f\|_{B(\bar{D})} \quad \left(\|f\|_{\bar{W}^2(D)} = \|f\|_{W^2(D)} + \sum_{i=1}^d \|f_{x_i}\|_{2d,D} \right).$$

Definition 1.1. *Let $D \subset \mathbb{R}^d$, let v and h be Borel functions locally summable in D. The function h is said to be a generalized derivative (in the region D) of the function v of order n in the direction of coordinate vectors $r_1, ..., r_n$ and this function h is denoted by $v_{x_{r_1} \dots x_{r_n}}$ if for each $\phi \in C_0^\infty(D)$*

$$\int_D \phi(x) h(x) dx = (-1)^n \int_D v(x) \phi_{x_{r_1} \dots x_{r_n}} dx$$

The properties of generalized derivatives are well known (see [11]). We will apply some of them without proofs. Note first that a generalized derivative can be defined uniquely almost everywhere. The function $u \in W^2(D)$ has generalized derivatives up to and including the second order, and these derivatives belong to $L_d(D)$.

2 Krylov's estimates

We will consider the following form of a d-dimensional optional semimartingale X (see, for example, [5]), for $i = 1, ..., d$,

$$X^i_t = X^i_0 + a^i_t + m^i_t + \int_{0+}^t \int_{0<z\leq 1} z(\mu^r - \nu^r)(dt, dz) + \int_{0+}^t \int_{z>1} z\mu^r(dt, dz)$$
$$+ \int_0^{t-} \int_{0<z\leq 1} z(\mu^g - \nu^g)(dt, dz) + \int_0^{t-} \int_{z>1} z\mu^g(dt, dz), \quad (1)$$

where X_0^i is $\mathcal{F}_0$-measurable random variable, $a^i \in \mathcal{A}_{loc}^c$ and $m^i \in \mathcal{M}_{loc}^{2,c}$. The jump measures μ^r and μ^g are defined on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \times \mathcal{E})$ as follows

$$\mu^r(\Gamma) = \sum_{n \geqslant 1} \mathbf{1}_\Gamma(T_n, \Delta X_{T_n}), \qquad \mu^g(\Gamma) = \sum_{n \geqslant 1} \mathbf{1}_\Gamma(U_n, \Delta^+ X_{U_n}),$$

where $(T_n)_{n\geqslant 1}$, $(U_n)_{n\geqslant 1}$ are sequences of totally inaccessible stopping times and totally inaccessible wide sense stopping times, respectively; $\mathbf{1}_\Gamma(\cdot)$ is an indicator function of a set $\Gamma \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{E}$. The processes ν^j are respective compensators of μ^j, $j = r, g$.

The following facts from the theory of parabolic partial differential equations are necessary for our proof. We take an auxiliary non-negative smooth function $\varphi(x)$, $\varphi(x) = 0$ for $x \geq 1$, and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$.

For $\epsilon > 0$ we set $\varphi^\epsilon(x) = \epsilon^{-1}\varphi(x\epsilon^{-1})$.

Lemma 2.1. *(see [8]) For each $\lambda > 0, \epsilon > 0$ and for every continuous function $f : \mathbb{R}^d \to [0, \infty)$ with compact support, there exists a smooth function $u^\epsilon : \mathbb{R}^d \to [0, \infty)$ ($u^\epsilon(x) = \int_{\mathbb{R}^d} u(y)\varphi^\epsilon(x - y)dy$, see [8] for detailed explanation of $u(x)$) with the properties:*

(a) for each $l \in \mathbb{R}^d$

$$\sum_{i,j=1}^d u^\epsilon_{x_i x_j} l_i l_j \leq \lambda u^\epsilon |l|^2;$$

(b) $|u^\epsilon_x| \leq \sqrt{\lambda} u^\epsilon$

(c) for all symmetric nonegative definite $d \times d$ matrices A

$$\sum_{i,j=1}^d A_{ij} u^\epsilon_{x_i x_j} - \lambda(trA + 1)u^\epsilon \leq -(\det A)^{1/d} f^\epsilon,$$

$\left(f^\epsilon(x) = \int_{\mathbb{R}^d} f(y)\varphi^\epsilon(x - y)dy\right)$;

(d) for all $p \geq d, x \in \mathbb{R}^d$

$$|u^\epsilon(x)| \leq N(p, d, \lambda)\|f\|_{p,\mathbb{R}^d}.$$

We now present the main result of this paper.

Theorem 2.1. *Let $V \in \mathcal{A}_{loc}^c$ be an increasing process, and suppose the characteristics $a^i, \langle m^i \rangle, \nu^j, j = r, g$, of X satisfy the structural conditions:*

There exist densities ($dV \times dP$-a.s.)

$$\alpha^i = \frac{da^i}{dV}, \beta^{ik} = \frac{\langle m^i, m^k \rangle}{dV}, \beta = [\beta^{ik}],\ i, k = 1, 2, ..., d, \tag{2}$$

and measures $\bar{\nu}^j(\omega, t, \Gamma), \Gamma \in \mathcal{E}, j = r, g$ (see [5]), such that for all $t > 0$ ($dV \times dP$-a.s.)

$$\nu^r(\omega, (0, t], \Gamma) = \int_0^t \bar{\nu}^r(\omega, s, \Gamma)dV_s, \quad \nu^g(\omega, [0, t), \Gamma) = \int_0^{t-} \bar{\nu}^g(\omega, s, \Gamma)dV_s, \tag{3}$$

$$|\alpha_t| + \sum_{j=r,g} \int_{0<|z|\leq 1} z^2 \bar{\nu}^j(\omega, t, dz) + \sum_{j=r,g} \int_{|z|>1} \bar{\nu}^j(\omega, t, dz) \leq C(\omega, t), \tag{4}$$

where $C(\omega, t)$ is a predictable function such that $k_\infty = \mathbf{E}\int_0^\infty e^{-\phi_t} C(\omega, t)dV_t < \infty$.

Then for any measurable function $f \geq 0,\ \lambda > 0,\ p \geq d$

$$\mathbf{E}\int_0^\infty e^{-\lambda \int_0^t [\frac{1}{2} tr\beta_s + 1]dV_s} (\det \beta_t)^{1/d} f(X_{t-})dV_t \leq N(k_\infty, \lambda, d, p)\|f\|_{p,\mathbb{R}^d}. \tag{5}$$

Proof. We follow the same approach as in [10]. First, consider continuous nonnegative function $f = f(x)$ with compact support. Denote $\phi_t = \lambda \int_0^t [\frac{1}{2} tr\beta_s + 1] dV_s$. Applying the integration by parts formula (see [1], Lemma 3.4, see also [4]) to $u^\epsilon(X_t)e^{-\phi_t}$, we get

$$u^\epsilon(X_t)e^{-\phi_t} - u^\epsilon(X_0) = \int_{0+}^{t} e^{-\phi_s} du^{\epsilon,r}(X) + \int_{0}^{t-} e^{-\phi_s} du^{\epsilon,g}(X) + \int_{0+}^{t} u^\epsilon(X_{s-}) de^{-\phi_s},$$

where $u^{\epsilon,r}(X)$ and $u^{\epsilon,g}(X)$ are right- and left-continuous part of $u(X)$, respectively.

Next, using the change of variables formula to find $u^\epsilon(X_t)$ and $e^{-\phi_t}$, we find that

$$\begin{aligned}
&u^\epsilon(X_t)e^{-\phi_t} - u^\epsilon(X_0) \\
&= \int_{0+}^{t} e^{-\phi_s} \Bigg\{ \frac{1}{2} \sum_{i,j=1}^{d} u^\epsilon_{x_i,x_j}(X_{s-}) d\langle m^i, m^j \rangle_s + \sum_{i=1}^{d} u^\epsilon_{x_i}(X_{s-}) d(a^i_s + m^i_s) \\
&\quad + \int_{0<z\le 1} \left[u^\epsilon(X_{s-}+z) - u^\epsilon(X_{s-}) \right] (\mu^r - \nu^r)(ds, dz) \\
&\quad + \int_{z>1} \left[u^\epsilon(X_{s-}+z) - u^\epsilon(X_{s-}) \right] \mu^r(ds, dz) \\
&\quad + \int_{0<z\le 1} \left[u^\epsilon(X_{s-}+z) - u^\epsilon(X_{s-}) - \sum_{i=1}^{d} u^\epsilon_{x_i}(X_{s-}) z \right] \nu^r(ds, dz) \Bigg\} \\
&\quad + \int_{0}^{t-} e^{-\phi_s} \Bigg\{ \int_{0<z\le 1} \left[u^\epsilon(X_s+z) - u^\epsilon(X_s) \right] (\mu^g - \nu^g)(ds, dz) \\
&\quad + \int_{z>1} \left[u^\epsilon(X_s+z) - u^\epsilon(X_s) \right] \mu^g(ds, dz) \\
&\quad + \int_{0<z\le 1} \left[u^\epsilon(X_s+z) - u^\epsilon(X_s) - \sum_{i=1}^{d} u^\epsilon_{x_i}(X_s) z \right] \nu^g(ds, dz) \Bigg\} \\
&\quad - \int_{0+}^{t} e^{-\phi_s} \lambda \left(\frac{1}{2} tr\beta_s + 1 \right) u^\epsilon(X_{s-}) dV_s
\end{aligned} \tag{6}$$

Let $\{\sigma_n\}_{n\ge1}, \{\tau_n\}_{n\ge1}$ and $\{\xi_n\}_{n\ge1}$ be localizing sequences for $\int_{0+}^t e^{-\phi_s} \sum_{i=1}^d u^\epsilon_{x_i}(X_{s-}) dm_i$, $\int_{0+}^t e^{-\phi_s} \int_{0<z\le1} [u^\epsilon(X_{s-}+z) - u^\epsilon(X_{s-})] (\mu^r - \nu^r)(dz, ds)$ and $\int_0^{t-} e^{-\phi_s} \int_{0<z\le1} [u^\epsilon(X_s+z) - u^\epsilon(X_s)] (\mu^g - \nu^g)(dz, ds)$, respectively.

Define $\forall n \ge 1, R_n := t \wedge \sigma_n \wedge \tau_n \wedge \xi_n, R_n \in \mathcal{T}_+,\ R_n \uparrow \infty$ a.s. as $n \uparrow \infty$ and $t \uparrow \infty$. Taking expectation and applying structural conditions (2)-(3), we obtain from (6):

$$
\begin{aligned}
&\mathbf{E}u^\epsilon(X_{R_n})e^{-\phi_{R_n}} - \mathbf{E}u^\epsilon(X_0)\\
&\quad=\mathbf{E}\int_{0+}^{R_n} e^{-\phi_s}\left\{\frac{1}{2}\sum_{i,j=1}^{d} u^\epsilon_{x_i,x_j}(X_{s-})(\beta_s)_{ij} - \lambda\left(\frac{1}{2}tr\beta_s+1\right)u^\epsilon(X_{s-})\right.\\
&\qquad+\sum_{i=1}^{d}\alpha^i_s u^\epsilon_{x_i}(X_{s-}) + \int_{0<z\le 1}\left[u^\epsilon(X_{s-}+z)-u^\epsilon(X_{s-})-\sum_{i=1}^{d}u^\epsilon_{x_i}(X_{s-})z\right]\bar{\nu}^r(dz)\\
&\qquad+\int_{0<z\le 1}\left[u^\epsilon(X_s+z)-u^\epsilon(X_s)-\sum_{i=1}^{d}u^\epsilon_{x_i}(X_s)z\right]\bar{\nu}^g(dz)\\
&\qquad\left.+\int_{z>1}\left[u^\epsilon(X_{s-}+z)-u^\epsilon(X_{s-})\right]\bar{\nu}^r(dz)+\int_{z>1}\left[u^\epsilon(X_s+z)-u^\epsilon(X_s)\right]\bar{\nu}^g(dz)\right\}dV_s\\
&\quad:=\mathbf{E}\int_{0+}^{R_n} e^{-\phi_s}\left\{I_1+I_2+I_3+I_4+I_5+I_6+I_7\right\}dV_s
\end{aligned}
\tag{7}
$$

Using the properties (a)-(d) of the function u^ϵ in Lemma 2.1, we have

$$u^\epsilon(X_0)\le N_0\|f\|_{p,\mathbb{R}^d};$$

$$I_1+I_2=\frac{1}{2}\sum_{i,j=1}^{d}u^\epsilon_{x_i,x_j}(X_{s-})(\beta_s)_{ij}-\lambda\left(\frac{1}{2}tr\beta_s+1\right)u^\epsilon(X_{s-})\le-\left(det\frac{1}{2}\beta_s\right)^{1/d}f^\epsilon(X_{s-});$$

$$I_3=\sum_{i=1}^{d}\alpha^i_s u^\epsilon_{x_i}(X_{s-})\le|\alpha_s|\|f\|_{p,\mathbb{R}^d}.$$

Next, with the help of Taylor's decomposition for multivariate functions and Lemma 2.1-(d), we obtain

$$
\begin{aligned}
I_5 &= \int_{0<z\le 1}\left[u^\epsilon(X_s+z)-u^\epsilon(X_s)-\sum_{i=1}^{d}u^\epsilon_{x_i}(X_s)z\right]\bar{\nu}^g(dz)\\
&=\int_{0<z\le 1}\int_0^1(1-\theta)\sum_{i,j=1}^{d}u^\epsilon_{x_i,x_j}(X_s+\theta z)z^2d\theta\bar{\nu}^g(dz)\\
&\le\frac{\lambda}{2}\int_{0<z\le 1}z^2\bar{\nu}^g(dz)\sup_{x\in\mathbb{R}^d}u^\epsilon(x)\\
&\le\frac{\lambda N_2}{2}\|f\|_{p,\mathbb{R}^d}\int_{0<z\le 1}z^2\bar{\nu}^g(dz),
\end{aligned}
$$

where $\theta\in[0,1]$ is an auxiliary parameter.

We can also find similar inequality for the integral I_4:

$$I_4\le\frac{\lambda N_3}{2}\|f\|_{p,\mathbb{R}^d}\int_{0<z\le 1}z^2\bar{\nu}^r(dz).$$

Using property (d) of Lemma 2.1, we get

$$I_7=\int_{z>1}\left[u^\epsilon(X_s+z)-u^\epsilon(X_s)\right]\bar{\nu}^g(dz)\le\int_{z>1}\bar{\nu}^g(dz)\sup_{x\in\mathbb{R}^d}u^\epsilon(x)\le N_4\|f\|_{p,\mathbb{R}^d}\int_{z>1}\bar{\nu}^g(dz).$$

Similarly,

$$I_6 \leq N_5 \|f\|_{p,\mathbb{R}^d} \int_{z>1} \bar{\nu}^r(dz).$$

It follows from obtained inequalities and the relation (7) that

$$\mathbf{E}\int_0^{R_n} e^{-\phi_s}(\det\beta_s)^{1/d} f^\epsilon(X_{s-})dV_s \leq N(\lambda,d,p)\|f\|_{p,\mathbb{R}^d}$$
$$\times\,\mathbf{E}\int_0^{R_n} e^{-\phi_s}\left[|\alpha_s| + \sum_{j=r,g}\int_{0<|z|\leq 1} z^2\bar{\nu}^j(\omega,s,dz) + \sum_{j=r,g}\int_{|z|>1}\bar{\nu}^j(\omega,s,dz)\right]dV_s. \quad (8)$$

After applying condition (4) to (8), it becomes

$$\mathbf{E}\int_0^{R_n} e^{-\phi_s}(\det\beta_s)^{1/d} f^\epsilon(X_{s-})dV_s \leq N(\lambda,d,p)\|f\|_{p,\mathbb{R}^d}\mathbf{E}\int_0^{R_n} e^{-\phi_s}C(\omega,s)dV_s$$
$$\leq N(k_\infty,\lambda,d,p)\|f\|_{p,\mathbb{R}^d}.$$

Finally, we let $n\uparrow\infty$ then $t\uparrow\infty$ and $\epsilon\downarrow 0$ and reach estimate (5). Extension of the estimate to the Borel measurable function f is standard (see, for example, [2]). □

Corollary 2.1. *If, in addition to the structural conditions* (2)-(3) *of Theorem 2.1, there exist constants* $0<c_1\leq c_2<\infty$ *such that for all* $x\in\mathbb{R}^d$, $c_1|x|^2\leq(\beta_t x,x)\leq c_2|x|^2$ *(*$dV\times dP$*-a.s.) and*

$$|\alpha_t| \leq \frac{K}{2}tr\beta_t, \quad \int_{0<|z|\leq 1} z^2\bar{\nu}^j(\omega,t,dz) \leq \frac{K}{2}tr\beta_t, \quad \int_{|z|>1}\bar{\nu}^j(\omega,t,dz) \leq \frac{K}{2}tr\beta_t. \quad (9)$$

Then for any measurable function $f\geq 0$, $\lambda>0$, $p\geq d$

$$\mathbf{E}\int_0^\infty e^{-\lambda\int_0^t[\frac{1}{2}tr\beta_s+1]dV_s} f(X_{t-})dV_t \leq N(K,\lambda,d,p,c_1,c_2)\|f\|_{p,\mathbb{R}^d}.$$

3 Application: Change of Variables formula with Generalized Derivatives

Change of variables formula is an essential tool of Stochastic Calculus. In this section, we prove that in some cases the change of variables formula remains valid for functions whose generalized derivatives are ordinary functions.

Theorem 3.1. *Let* X_0 *be fixed,* $X_0\in\mathbb{R}^d$*. Let* τ_D *be the first exit time of the process* X_t *in* (1) *from a bounded region* $D\subset\mathbb{R}^d$*, and let* $\tau\in\mathcal{T},\tau<\tau_D$*. Suppose that* X *satisfies assumptions of Corollary 2.1.*

Then for any $v \in \bar{W}^2(D)$

$$
\begin{aligned}
&v(X_\tau)e^{-\phi_\tau} - v(X_0) \\
&\quad = \int_{0+}^{\tau} e^{-\phi_s} \Bigg\{ \frac{1}{2} \sum_{i,j=1}^{d} v_{x_i,x_j}(X_{s-}) d\langle m^i, m^j \rangle_s - \lambda \left(\frac{1}{2} tr\beta_s + 1 \right) v(X_{s-}) dV_s \\
&\qquad + \sum_{i=1}^{d} v_{x_i}(X_{s-}) da_s^i + \sum_{i=1}^{d} v_{x_i} dm_s^i + \int_{0<z\le 1} [v(X_{s-}+z) - v(X_{s-})](\mu^r - \nu^r)(dz, ds) \\
&\qquad + \int_{0<z\le 1} \left[v(X_{s-}+z) - v(X_{s-}) - \sum_{i=1}^{d} v_{x_i}(X_{s-}) z \right] \nu^r(dz, ds) \\
&\qquad + \int_{z>1} \left[v(X_{s-}+z) - v(X_{s-}) \right] \mu^r(dz, ds) \Bigg\} \\
&\qquad + \int_{0}^{\tau-} e^{-\phi_s} \Bigg\{ \int_{0<z\le 1} [v(X_s+z) - v(X_s)](\mu^g - \nu^g)(dz, ds) \\
&\qquad + \int_{0<z\le 1} \left[v(X_s+z) - v(X_s) - \sum_{i=1}^{d} v_{x_i}(X_s) z \right] \nu^g(dz, ds) \\
&\qquad + \int_{z>1} \left[v(X_s+z) - v(X_s) \right] \mu^g(dz, ds) \Bigg\} \quad (a.s.\ on\ \{0 \le \tau\}).
\end{aligned}
\tag{10}
$$

Proof. Let a sequence $v^n \in C^2(\bar{D})$ be such that

$$
\|v - v^n\|_{B(D)} \to 0, \ \|v - v^n\|_{W^2(D)} \to 0,
$$
$$
\| |(v_x - v_x^n)|^2 \|_{d,D} \to 0.
$$

For convenience rewrite (10) as following

$$
\begin{aligned}
&v(X_\tau)e^{-\phi_\tau} - v(X_0) \\
&\quad = \int_{0+}^{\tau} e^{-\phi_s} \{I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7\} \\
&\qquad + \int_{0}^{\tau-} e^{-\phi_s} \{I_8 + I_9 + I_{10}\}.
\end{aligned}
$$

We prove that the right side of (10) makes sense. For $s < \tau$

$$
\begin{aligned}
I_1 + I_2 + I_3 = \Bigg[&\frac{1}{2} \sum_{i,j=1}^{d} v_{x_i,x_j}(X_{s-}) \beta^{ij} - \lambda \left(\frac{1}{2} tr\beta_s + 1 \right) v(X_{s-}) \\
&+ \sum_{i=1}^{d} v_{x_i}(X_{s-}) \alpha^i \Bigg] dV_s
\end{aligned}
$$

From this, using Theorem 2.1 we obtain

$$\mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\left|\frac{1}{2}\sum_{i,j=1}^{d} v_{x_i,x_j}(X_{s-})\beta^{ij} - \lambda\left(\frac{1}{2}tr\beta_s + 1\right)v(X_{s-}) + \sum_{i=1}^{d} v_{x_i}(X_{s-})\alpha^i\right| dV_s$$
$$\leq N\mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\left[\sum_{i,j=1}^{d}|v_{x_i,x_j}(X_{s-})| + |v(X_{s-})| + \sum_{i=1}^{d}|v_{x_i}(X_{s-})|\right] dV_s \leq N\|v\|_{W^2(D)},$$

where N depends on $\lambda, p, d, K, c_1, c_2$.

Similarly,

$$\mathbf{E}\left|\int_{0+}^{\tau} e^{-\phi_s}\sum_{i}^{d} v_{x_i}(X_{s-})dm_s^i\right|^2 \leq N\mathbf{E}\int_{0+}^{\tau} e^{-2\phi_s}\,|v_x(X_{s-})|^2\,dV_s \leq N\||v_x|^2\|_{d,D}.$$

and

$$\mathbf{E}\left|\int_{0+}^{\tau} e^{-\phi_s}\int_{0<z\leq 1}[v(X_{s-}+z) - v(X_{s-})]\mu^r(dz,ds)\right|$$
$$\leq \mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\int_{0<z\leq 1}|v(X_{s-}+z) - v(X_{s-})|\bar{\nu}^r(dz)dV_s$$
$$\leq \mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\int_{0<z\leq 1}[|v(X_{s-}+z)| + |v(X_{s-})|]\,\bar{\nu}^r(dz)dV_s$$
$$\leq N\|v\|_{B(D)}.$$

Using the same technique, integrals I_7, I_8 and I_{10} are well-defined.

Since

$$\mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\int_{0<z\leq 1}\left|\sum_{i=1}^{d} v_{x_i}(X_{s-})z\right|\nu^r(dz,ds)$$
$$\leq N\mathbf{E}\int_{0+}^{\tau} e^{-\phi_s}\int_{0<z\leq 1}|v_x(X_{s-})|z\bar{\nu}^r(dz)dV_s$$
$$\leq N\|v_x\|_{d,D},$$

I_6 is well-defined. The same holds for I_9.

Further, we apply the change of variables formula for optional semimartingales (see [6]) to the expression $v^n(x_t)e^{-\phi_t}$. Then, we have almost surely

$$
\begin{aligned}
&v^n(X_\tau)e^{-\phi_\tau} - v^n(X_0) \\
&\quad= \int_{0+}^{\tau} e^{-\phi_s}\Bigg\{\frac{1}{2}\sum_{i,j=1}^{d} v^n_{x_i,x_j}(X_{s-})d\langle m^i, m^j\rangle_s - \lambda\left(\frac{1}{2}tr\beta_s + 1\right)v^n(X_{s-})dV_s \\
&\qquad + \sum_{i=1}^{d} v^n_{x_i}(X_{s-})da^i_s + \sum_{i=1}^{d} v^n_{x_i}dm^i_s \\
&\qquad + \int_{0<z\le 1} [v^n(X_{s-}+z) - v^n(X_{s-})](\mu^r - \nu^r)(dz,ds) \\
&\qquad + \int_{0<z\le 1}\left[v^n(X_{s-}+z) - v^n(X_{s-}) - \sum_{i=1}^{d} v^n_{x_i}(X_{s-})z\right]\nu^r(dz,ds) \\
&\qquad + \int_{z>1}[v^n(X_{s-}+z) - v^n(X_{s-})]\,\mu^r(dz,ds)\Bigg\} \\
&\qquad + \int_{0}^{\tau-} e^{-\phi_s}\Bigg\{\int_{0<z\le 1}[v^n(X_s+z) - v^n(X_s)](\mu^g - \nu^g)(dz,ds) \\
&\qquad + \int_{0<z\le 1}\left[v^n(X_s+z) - v^n(X_s) - \sum_{i=1}^{d} v^n_{x_i}(X_s)z\right]\nu^g(dz,ds) \\
&\qquad + \int_{z>1}[v^n(X_s+z) - v^n(X_s)]\,\mu^g(dz,ds)\Bigg\}
\end{aligned}
\tag{11}
$$

We pass to the limit in equality (11) as $n\to\infty$. By the Sobolev Theorem (see [13]) $v^n \to v$ uniformly in each finite region. From estimates similar to the estimates we found earlier it easily follows that the right side of (11) tends to the right side of (10). □

4 Application: Convergence of optional solutions of SDE

In this section we consider $(X^n_t)_{t\in[0,T]}$, $n = 0,1,2,...$ satisfying the following d-dimensional SDE's, respectively,

$$
\begin{aligned}
X^n_t =& X_0 + \int_0^t b^n(X^n_s)ds + \int_0^t \sigma^n(X^n_s)dW_s \\
&+ \int_{0+}^{t}\int_E c^n(X^n_{s-},z)\tilde{N}^r(ds,dz) + \int_0^{t-}\int_E h^n(X^n_s,z)\tilde{N}^g(ds,dz), \quad n = 0,1,2,...,
\end{aligned}
$$

where W_t is a d-dimensional Wiener process, $\tilde{N}^r(ds,dz)$ and $\tilde{N}^g(ds,dz)$ are, respectively, right-continuous and left-continuous modifications of 1-dimensional Poisson martingales with corresponding compensators $\lambda^r(ds,dz)$ and $\lambda^g(ds,dz)$, and b^n, $c^n, h^n \in \mathbb{R}^d$, and $\sigma^n \in \mathbb{R}^{d\times d}$. Hereafter, we write $X_t = X^0_t, b = b^0$ and so on.

Theorem 4.1. *Assume that*

(a) $|b^n(x)|^2 + |\sigma^n(x)|^2 + \int_E |c^n(x,z)|^2 \lambda^r(dz) + \int_E |h^n(x,z)|^2 \lambda^g(dz) \le k$ *and* $\mathbf{E}|X_0|^2 < k_0$, *where* k_0 *and* $k \ge 0$ *are constants;*
$(x-y)\cdot(b(x)-b(y)) \le F(s)\rho(|x-y|^2)$,

$$\sum_{i,j=1}^{d} |\sigma_{ij}(x) - \sigma_{ij}(y)|^2 + \int\limits_E |c(x,z)-c(y,z)|^2 \lambda^r(dz) + \int\limits_E |h(x,z)-h(y,z)|^2 \lambda^g(dz) \le F(s)\rho(|x-y|^2),$$

where $0 \le F(s)$ *satisfies that* $\forall t \ge 0$ $\int_0^t F(s)ds < \infty$, *and* $\rho(u)$ *is strictly increasing, continuous, and concave such that* $\rho(0)=0$, $\rho(u)>0$, *as* $u>0$; *and* $\int_{0+} du/\rho(u) = \infty$;
(b)

$$\begin{aligned} &\left\| |b^n(x)-b(x)|^2 \right\|_{p,\mathbb{R}^d} + \left\| |\sigma^n(x)-\sigma(x)|^2 \right\|_{p,\mathbb{R}^d} \\ &+ \left\| \int\limits_E |c^n(x,z)-c(x,z)|^2 \lambda^r(dz) \right\|_{p,\mathbb{R}^d} + \left\| \int\limits_E |h^n(x,z)-h(x,z)|^2 \lambda^g(dz) \right\|_{p,\mathbb{R}^d} \to 0, \end{aligned}$$

as $n \to \infty$, *where* $p \ge d+1$;
(c)there exists $k_1 > 0$ *and* $k_2 > 0$ *such that for all* $x \subset \mathbb{R}^d$, $n = 1,2,...,$ $k_1|x|^2 \le (\beta x, x) \le k_2|x|^2$, *where* $\beta = \sigma^n\sigma^{n*}$;
(d) $lim_{n\to\infty}\mathbf{E}|X_0^n - X_0|^2 = 0$.
Then we have $\forall t \ge 0$

$$\lim_{n\to\infty} \mathbf{E}|X_t^n - X_t|^2 = 0.$$

We will need the following two lemmas to prove the main theorem.

Lemma 4.1. *(see [12], Lemma 116) If for all* $t \ge 0$ *a real non-random funcion* y_t *satisfies*

$$0 \le y_t \le \int\limits_0^t \rho(y_s)ds < \infty,$$

where $\rho(u)$ *defined on* $u \ge 0$, *is non-negative, increasing such that* $\rho(0)=0$, $\rho(u)>0$, *as* $u>0$; *and* $\int_{0+} du/\rho(u) = \infty$, *then*

$$y_t = 0, \forall t \ge 0.$$

Lemma 4.2. *Suppose* $\mathbf{E}|X_0|^2 < k_0$ *and*

$$|b(x)|^2 + |\sigma(x)|^2 + \int\limits_E |c(x,z)|^2 \lambda^r(dz) + \int\limits_E |h(x,z)|^2 \lambda^g(dz) \le k,$$

where $k_0, k \ge 0$ *are constants. Then* $\mathbf{E}\sup_{t\in[0,T]}|X_t| \le k_T$ *for some constant* k_T.

Proof. First, note that

$$\begin{aligned} |X_t|^2 \le & 4\Bigg[|X_0|^2 + \left| \int\limits_0^t b(X_s)ds \right|^2 + \left| \int\limits_0^t \sigma(X_s)dW_s \right|^2 \\ &+ \left| \int\limits_{0+}^t c(X_{s-},z)\tilde{N}^r(ds,dz) \right|^2 + \left| \int\limits_0^{t-} c(X_s,z)\tilde{N}^r(ds,dz) \right|^2 \Bigg]. \end{aligned}$$

Next, using Doob's inequality and optional stochastic integral properties we obtain

$$\mathbf{E}\sup_{t\in[0,T]}\left|\int_0^t b(X_s)ds\right|^2 \le \mathbf{E}\sup_{t\in[0,T]}\int_0^t |b(X_s)|^2\, ds \le kT,$$

$$\mathbf{E}\sup_{t\in[0,T]}\left|\int_0^t \sigma(X_s)dW_s\right|^2 \le 4\mathbf{E}\int_0^t |\sigma(X_s)|^2\, ds \le 4kT,$$

$$\mathbf{E}\sup_{t\in[0,T]}\left|\int_{0+}^t\int_E c(X_{s-},z)d\tilde{N}^r(ds,dz)\right|^2 \le 8\mathbf{E}\int_{0+}^t\int_E |c(X_{s-},z)|^2\,\lambda^r(dz)ds \le 8kT,$$

$$\mathbf{E}\sup_{t\in[0,T]}\left|\int_0^{t-}\int_E h(X_s,z)d\tilde{N}^g(ds,dz)\right|^2 \le 8\mathbf{E}\int_0^{t-}\int_E |h(X_s,z)|^2\,\lambda^g(dz)ds \le 8kT$$

Thus, we conclude that

$$\mathbf{E}\sup_{t\in[0,T]}|X_t|^2 \le k_T,$$

where $k_T = 4(21kT + k_0)$.

□

Proof of Theorem.. By the change of variables formula

$$\begin{aligned}\mathbf{E}|X_t^n - X_t|^2 - \mathbf{E}|X_0^n - X_0|^2 =& 2\mathbf{E}\int_0^t (X_t^n - X_t)(b^n(X_s^n) - b(X_s))ds\\ &+\mathbf{E}\int_0^t\sum_{i,j=1}^d |\sigma_{ij}(X_s^n) - \sigma_{ij}(X_s)|^2 ds\\ &+\mathbf{E}\int_{0+}^t\int_E |c^n(X_{s-}^n,z) - c(X_{s-},z)|^2\lambda^r(dz)ds\\ &+\mathbf{E}\int_0^{t-}\int_E |h^n(X_s^n,z) - h(X_s,z)|^2\lambda^g(dz)ds\\ =&\sum_{i=1}^4 I^n(i).\end{aligned}$$

For the process X_t^n, we have $V_t = t, \alpha_t = b^n(X_t^n), \beta_t = \sigma^n\sigma^{n*}(X_t^n), \bar{\nu}^r(dz) = |c(X_t^n,z)|^2\lambda^r(dz)$, $\bar{\nu}^g(dz) = |h(X_t^n,z)|^2\lambda^g(dz)$. Furthermore,

$$k_\infty = \int_0^\infty e^{-\lambda\int_0^t[1/2tr\sigma^n\sigma^{n*}(X_s^n)+1]ds}kdt \le k\int_0^\infty e^{-\lambda t}dt = \frac{k}{\lambda} < \infty.$$

Therefore, condition (4) is satisfied. Thus, by the Krylov's estimate (Remark 2.3) and the assumption (b) we get

$$\begin{aligned}I^n(1) \le& 2\mathbf{E}\int_0^t (X_t^n - X_t)(b^n(X_s^n) - b(X_s^n))ds + 2\mathbf{E}\int_0^t (X_t^n - X_t)(b(X_s^n) - b(X_s))ds\\ \le&\mathbf{E}\int_0^t |X_t^n - X_t|^2 ds + N\left\||b^n(x) - b(x)|^2\right\|_{p,\mathbb{R}^d} + 2\int_0^t F(s)\rho\left(\mathbf{E}|X_s^n - X_s|^2\right)ds,\end{aligned}$$

where N depends on $\lambda, p, d, K, T, k_1, k_2$.

Similarly,

$$
\begin{aligned}
I^n(2) \leq & 2N \left\| |\sigma^n(x) - \sigma(x)|^2 \right\|_{p,\mathbb{R}^d} + 2 \int_0^t F(s)\rho\left(\mathbf{E}|X_s^n - X_s|^2\right) ds, \\
I^n(3) \leq & 2N \left\| \int_E |c^n(x,z) - c(x,z)|^2 \lambda^r(dz) \right\|_{p,\mathbb{R}^d} + 2 \int_{0+}^t F(s)\rho\left(\mathbf{E}|X_{s-}^n - X_{s-}|^2\right) ds, \\
I^n(4) \leq & 2N \left\| \int_E |h^n(x,z) - h(x,z)|^2 \lambda^g(dz) \right\|_{p,\mathbb{R}^d} + 2 \int_0^{t-} F(s)\rho\left(\mathbf{E}|X_s^n - X_s|^2\right) ds.
\end{aligned}
$$

Consequently, applying the assumptions (c) and (e) we have

$$
\mathbf{E}|X_t^n - X_t|^2 \leq \mathbf{E} \int_0^t |X_t^n - X_t|^2 ds + 8 \int_0^{t-} F(s)\rho\left(\mathbf{E}|X_s^n - X_s|^2\right) ds.
$$

Notice that by Lemma 4.1 for every $n = 0, 1, 2, ...$

$$
\mathbf{E} \sup_{t \in [0,T]} |X_t^n| \leq k_T < \infty
$$

Therefore, using Fatou's lemma, it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} \mathbf{E}|X_t^n - X_t|^2 \leq & \int_0^t \limsup_{n\to\infty} \mathbf{E}|X_t^n - X_t|^2 ds \\
& + 8 \int_0^t F(s)\rho\left(\limsup_{n\to\infty} \mathbf{E}|X_s^n - X_s|^2\right) ds.
\end{aligned}
$$

Thus, by Lemma 4.2

$$
\lim_{n\to\infty} \mathbf{E}|X_t^n - X_t|^2 = 0.
$$

□

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