Optional Processes on Unusual Probability Spaces and their Financial Applications

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March 2018

The Usual Basis and Strong Supermartingales

The Usual Stochastic Basis and RCLL Processes

- The stochastic basis $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$ is a probability space with a non-decreasing family of σ-algebras $\mathcal{F}_t \in \mathbf{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, for all $s \leq t$
- *Stochastic analysis* and *mathematical finance* have been comprehensively investigated under so-called "*usual conditions*":
  - Initial Completion: $\mathcal{F}_0$ is augmented with $\mathbf{P}$ all subset of null sets from $\mathcal{F}$
  - Progressive Completion: $\mathcal{F}_t$ is complete for all time t
  - Right-Continuity: $\mathcal{F}_t$ is right-continuous, $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$
- Under these conditions semimartingales are *RCLL*

Strong Supermartingales

Definition

A real optional process X is an optional strong supermartingale if (1) For every bounded stopping time T, X_T is integrable. (2) For every pair of bounded stopping times S, T such that $S \leq T$,

$$X_S \geq \mathbf{E}[X_T|\mathcal{F}_S].$$

Even under the usual conditions there exist many optional strong supermartingales which are not cadlag. For example,

1. The optional projection of a not necessarily right continuous decreasing process is always an optional strong supermartingale
2. The limit of a decreasing sequence of cadlag positive supermartingales is an optional strong supermartingale but is in general no longer cadlag

Mertens Decomposition of Strong Supermartingale

Theorem

X is an optional strong supermartingale if and only if it can be decomposed into

$$X = M - A,$$

where M is a cadlag local martingale and A an increasing predictable ladlag process. This decomposition then is unique.

Strong supermartingale decompsition found an application by Schachermayer (2014), "Admissible Trading Strategies under Transaction Costs", where the value process of a portfolio with transaction cost is an optional supermartingale

Example of *Un*usual Conditions

There are examples showing the existence of a stochastic basis without the "usual conditions"; Fleming & Harrington (2011) considered

$$X_t = \mathbf{1}_{t>t_0}\mathbf{1}_A,$$

where A is $\mathcal{F}$-measurable with $0 < \mathbf{P}(A) < 1$. Then, $\mathcal{F}_t = \sigma(X_s, s \leq t)$ is not right-continuous at t_0 because $A \notin \mathcal{F}_{t_0}$ but $A \in \mathcal{F}_{t_0+}$ and it is not possible to make it right continuous

The *Un*usual Basis

The *Un*usual Stochastic Basis and RLL Processes

- Dellacherie (1972) initiated the study of stochastic processes without the usual conditions and called it the "*unusual conditions*"
- Dellacherie began his study with the process

$$\text{"}\mathbf{E}\left[X|\mathcal{F}_t\right]\text{"}$$

  where X is a *bounded* random variable in $\mathcal{F}$ and $\mathcal{F}_t$ is not complete or right or left continuous
- The goal is to find out if there exist a reasonable adapted modification of the conditional expectation; And It turns out that there is one

Optional Projection Theorem by Dellacherie

Theorem

Let X be a bounded random variable then there is a version X_t of the martingale $\mathbf{E}\left[X|\mathcal{F}_t\right]$ possessing the following properties: X_t is an optional process and for every stopping time T, $X_T \mathbf{1}_{T<\infty} = \mathbf{E}\left[X \mathbf{1}_{T<\infty}|\mathcal{F}_T\right]$ a.s.

The optional processes in this case are not necessarily left or right continuous but have left and right limits

Few have contributed to the calculus of optional processes on *un*usual spaces such as Doob, Mertenz, Meyer, Dellacherie, Lenglart, Galchuk and Gasperyan

Outline

- Optional Calculus on Unusual Probability Spaces
  - Processes; Decomposition Results; Type of Jumps; Stochastic Integral and Calculus of Optional Processes; Stochastic Exponentials and Logarithms
- Markets of Optional Processes
  - Markets and Portfolios; Martingale Transforms; Example of a Jump-Diffusion Market
- Defaultable Markets
  - Default Time as Stopping-Time in the Broad Sense; Defaultable Claims and Cash-Flows
- Stochastic Equations
  - Gronwall Lemma; Nonhomogeneous Linear Stochastic Integral Equation; Existence and Uniqueness under Monotonicity Conditions; Comparison Theorem under Yamada conditions

Elements of Optional Calculus

The *Un*usual Stochastic Basis

Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in \mathbb{R}_+ = [0, \infty)$

- $(\Omega, \mathcal{F}, \mathbf{P})$ is complete but
- $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t \subseteq \mathcal{F}$ and $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s \leq t$
- $\mathbf{F}$ is not assumed to be complete right or left continuous
- $\mathbf{F}_+ = (\mathcal{F}_{t+})_{t\geq 0}$ right continuous
- $\mathbf{F}^{\mathbf{P}}$ is the completion of $\mathbf{F}$
- Define $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ be the optional and predictable σ-algebras and $\mathcal{O}(\mathbf{F}_+)$ and $\mathcal{P}(\mathbf{F}_+)$

Existence of Optional Modifications of Martingales

Galchuk (1977) proved an extension of Dellacherie (1972) optional projection theorem

Theorem

Let X be an integrable random variable then there exists a modification X_t of the martingale $(\mathbf{E}[X|\mathcal{F}_t])$ such that X_t is an optional process and for any Markov time T

$$X_T \mathbf{1}_{(T<\infty)} = \mathbf{E}[X\mathbf{1}_{(T<\infty)}|\mathcal{F}_T] \quad a.s., \quad (*)$$

and if another optional modification $(\tilde{X}_t)$ exists satisfying $(*)$ then X_t and $\tilde{X}_t$ are indistinguishable

Optional & Predictable Processes

- Optional processes, $X \in \mathcal{O}(\mathbf{F})$ have right and left limits but are not necessarily right or left continuous
- X is predictable if $X \in \mathcal{P}(\mathbf{F})$ and *strongly predictable*, that is X is in $\mathcal{P}_s(\mathbf{F})$, if $X \in \mathcal{P}(\mathbf{F})$ and $X_+ \in \mathcal{O}(\mathbf{F})$
- For processes on *un*usual spaces we can define right and left differentials
  - $\triangle X_t = X_t - X_{t-}$ and $\triangle^+ X_t = X_{t+} - X_t$
  - $X_{t-} = \liminf_{s \in \mathrm{D}, s \uparrow\uparrow t} X_s$ and $X_{t+} = \limsup_{s \in \mathrm{D}, s \downarrow\downarrow t} X_s$

Optional Martingales & Decomposition

Definition

M is an *optional martingale* (*supermartingale*, *submartingale*) if
(**a**) $M \in \mathcal{O}(\mathbf{F})$,
(**b**) The random variable $M_T \mathbf{1}_{T<\infty}$ is integrable for any stopping time $T \in \mathcal{T}(\mathbf{F})$
(**c**) There exists an integrable random variable $\mu \in \mathcal{F}_\infty$ such that $M_T = \mathbf{E}[\mu|\mathcal{F}_T]$ (respectively, $M_T \geq \mathbf{E}[\mu|\mathcal{F}_T]$, $M_T \leq \mathbf{E}[\mu|\mathcal{F}_T]$) a.s. for any stopping time $T \in \mathcal{F}_\infty$ such that $(T < \infty)$

Theorem (Decomposition of Optional Martingale)

If M is local optional martingale then it can be decomposed to

$$M = M^r + M^g, \quad M^r = M^c + M^d,$$

where M^c is continuous, M^d is right-continuous and M^g is left-continuous local optional martingales. M^d and M^g are orthogonal to each other and to any continuous (local) martingale

Increasing and Finite Variation Processes

- Let $\mathcal{V}^+$ the collection of increasing processes. An increasing process $A = (A_t)_{t\geq 0}$ is integrable if $\mathbf{E}A_\infty < \infty$ locally integrable if $\mathbf{E}A_{R_{n+}} < \infty$, $(R_n)_{n\geq 0} \subset \mathcal{T}(\mathbf{F}_+)$ and $R_n \uparrow \infty$. Let $\mathcal{A}^+$ ($\mathcal{A}^+_{loc}$ respectively) are collections of integrable (locally) integrable increasing processes
- $A = (A_t)_{t\geq 0}$ is finite variation if $\mathbf{Var}(A)_t < \infty$,

$$\mathbf{Var}(A)_t = \int_{0+}^{t} |dA^r_s| + \int_0^{t-} |\triangle^+ A_s|$$

  Let $\mathcal{V}$ the set finite variation processes
- $A = (A_t)_{t\geq 0}$ of finite variation belongs to the space $\mathcal{A}$ of integrable finite variation processes if $\mathbf{E}\left[\mathbf{Var}(A)_\infty\right] < \infty$ ($\mathcal{A}_{loc}$)
- A finite variation or an increasing process A can be *decomposed* to

$$A = A^r + A^g = A^c + A^d + A^g$$

  where A^c is continuous, A^r is a right-continuous, A^d is discrete right-continuous, A^g is discrete left-continuous

Decomposition of Supermartingales

Let M be an optional supermartingale then it is of *class* D if the family of random variables $M_T,\ T \in \mathcal{T}_+(\mathbf{F})$, is uniformly integrable and it is of class DL if the family of variables $M_T,\ T \in \mathcal{T}_+(\mathbf{F}),\ T \leq a$, is uniformly integrable for any $a,\ 0 \leq a < \infty$

Theorem

If M is an optional supermartingale of class D then

$$M = N - A$$

where N is optional martingale and $A,\ A_0 = 0$ is increasing strongly predictable integrable process. And if M is an optional supermartingale of class DL A is an increasing strongly predictable locally integrable process with $A_0 = 0$

Optional Semimartingales

- X is optional semimartingale ($X \in \mathcal{S}(\mathbf{F}, \mathbf{P})$) if it is representable in the form $X = M + A$, M is $O(\mathbf{F})$ optional local martingale & A an $\mathbf{F}$-adapted process of finite variation.
- X is special optional semimartingale, in $\mathcal{S}_p(\mathbf{F}, \mathbf{P})$, if A is strongly predictable

Jumps of Optional Semimartingales

For an optional semimartingale X there exist 3 sequences of jumps

- Predictable sequence of stopping times, $S_n \in \mathcal{T}(\mathbf{F}_-)$
- Totally inaccessible sequence of stopping times, $T_n \in \mathcal{T}(\mathbf{F})$
- Totally inaccessible stopping times in the broad sense, $U_n \in \mathcal{T}(\mathbf{F}_+)$

with mutually non-intersecting graphs within each sequence absorbing all jumps of X

Decomposition of Elementary Processes

- T is a *totally inaccessible s.t.* and ξ is $\mathcal{F}_T$*-measurable* integrable random variable (rv) then
$$\xi \mathbf{1}_{T \leq t} = A + Z$$
$Z \in \mathcal{M}$ and A is *unique, predictable, nondecreasing and continuous*
- T is a *predictable s.t.* and ξ is $\mathcal{F}_T$*-measurable* and integrable rv then
$$\xi \mathbf{1}_{T \leq t} = A + Z$$
$Z \in \mathcal{M}$ and A *unique right continuous predictable process*
- T is a *predictable* or *totally inaccessible s.t.* and ξ is $\mathcal{F}_{T+}$*-measurable* and integrable rv then
$$\xi \mathbf{1}_{T < t} = A + Z,$$
$Z \in \mathcal{M}$ and A is *unique, right continuous and strongly predictable process*
- T is a *totally inaccessible s.t.b.* and ξ is $\mathbf{F}_{T+}$*-measurable* and integrable rv then
$$\xi \mathbf{1}_{T < t} = A + Z,$$
$Z \in \mathcal{M}$ and A *continuous, unique, strongly predictable*

Stochastic Integral with respect to Optional Martingales

- Defined in terms of the decomposition $M = M^r + M^g$. The integral with respect to M^r is defined
- The Galchuck integral with respect to M^g is given by

$$H \odot M_t^g = \int_0^{t-} H_s dM_{s+}^g,$$

where $H \in L^2([M^g, M^g], \mathbf{P})$ approximated by $\sum_{j=0}^n H_j \mathbf{1}_{[t_i.t_{i+1}[}(t)$.
- Therefore, the stochastic integral with respect to optional martingale is given by

$$\begin{aligned} \int_0^t H_s dM_s &= \int_{0+}^t H_{s-} dM_s^r + \int_0^{t-} H_s dM_{s+}^g, \\ H \circ M &= H \cdot M^r + H \odot M^g. \end{aligned}$$

Integrals with respect to Optional Semimartingales

- Integral with respect to optional semimartingales is

$$H \circ X = H \cdot X^r + H \odot X^g = H \cdot M^r + H \odot M^g + H \cdot A^r + H \odot A^g,$$

  $H \cdot A^r$ and $H \odot A^g$ are interpreted in the Lebesgue sense
- The stochastic integral with respect to optional semimartingale X can be generalized to

$$Y_t = (f, g) \circ X_t = f \cdot X_t^r + g \odot X_t^g,$$

  where Y_t is again an optional semimartingale $f_- \in \mathcal{P}(\mathbf{F})$, and $g \in \mathcal{O}(\mathbf{F})$
- This integral is defined on a larger space, the product space $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$

Properties of the Optional Integral

- Isometry: $\left(f^2 \cdot [X^r, X^r]\right)^{1/2} \in \mathcal{A}_{loc}$ and $\left(g^2 \odot [X^g, X^g]\right)^{1/2} \in \mathcal{A}_{loc}$
- Linearity: $(f^1 + f^2, g^1 + g^2) \circ X_t = (f^1, g^1) \circ X_t + (f^2, g^2) \circ X_t$
- $\triangle^+ X^g$ is $\mathcal{O}(\mathbf{F}_+)$ and for its martingale part $\mathbf{E}\left[\triangle^+ M^g_T \mathbf{1}_{(T<\infty)} | \mathcal{F}_T\right] = 0$ a.s.
- Orthogonality: $X^r \perp X^g$
- Quadratic variation: $[X, X] = [X^r, X^r] + [X^g, X^g]$ where $[X^r, X^r]_t = \langle X^c, X^c\rangle_t + \int_{0+}^{t} (\triangle X_s)^2$ and $[X^g, X^g]_t = \int_0^{t-} (\triangle^+ X_s)^2$
- Differentials are independent: $\triangle Y = f \triangle X^r$ and $\Delta^+ Y_t = g \Delta^+ X^g_t$
- For any semimartingale Z the quadratic projection is $[Y, Z] = f \cdot [Y^r, Z^r] + g \odot [Y^g, Z^g]$

Functions of Semimartingales

Suppose $X = X_0 + A + M$ an optional semimartingale and $F(x)$ is a twice continuously differentiable function on $\mathbb{R}$. Then $F(x)$ is given by

$$\begin{aligned} F(X_t) &= F(X_0) + \int_{0+}^{t} \partial F(X_{s-}) d(A^r + M^r)_s + \frac{1}{2}\int_{0+}^{t} \partial^2 F(X_{s-}) d\langle M^c, M^c\rangle_s \\ &+ \sum_{0<s\leq t} [F(X_s) - F(X_{s-}) - \partial F(X_{s-})\Delta X_s] \\ &+ \int_{0}^{t-} \partial F(X_s) d(A^g + M^g)_{s+} \\ &+ \sum_{0\leq s<t} \left[F(X_{s+}) - F(X_s) - \partial F(X_s)\Delta^+ X_s\right], \end{aligned}$$

where ∂ is the differentiation operator

Stochastic Exponential

The stochastic exponential formula for optional semimartingale is given by

$$Z_t = Z_0 \exp\{X_t - \frac{1}{2}\langle X^c, X^c\rangle\} \times \prod_{0<s\leq t}(1+\Delta X_s)e^{-\Delta X_s}$$
$$\prod_{0\leq s<t}(1+\Delta^+ X_s)e^{-\Delta^+ X_s},$$

which is the solution of $Z_t = Z_0 + Z \cdot X_t^r + Z \odot X_t^g$

Stochastic Logarithm

Let Y be a real valued optional semimartingale such that the processes Y_- and Y do not vanish then the process

$$X_t = \frac{1}{Y} \circ Y_t = \int_{0+}^{t} \frac{1}{Y_{s-}} dY_s^r + \int_0^{t-} \frac{1}{Y_s} dY_{s+}^g, \quad X_0 = 0,$$

also denoted by $X = \mathcal{L}\text{og} Y$ is called the stochastic logarithm of Y, is the unique semimartingale X such that $Y = Y_0 \mathcal{E}(X)$. Moreover, if $\Delta X \neq -1$ and $\Delta^+ X \neq -1$ we also have

$$\begin{aligned} \mathcal{L}\text{og} Y_t \quad = \quad & \log \left| \frac{Y_t}{Y_0} \right| + \frac{1}{2Y^2} \circ \langle Y^c, Y^c \rangle_t - \sum_{0<s\leq t} \left(\log \left| 1 + \frac{\Delta Y_s}{Y_{s-}} \right| - \frac{\Delta Y_s}{Y_{s-}} \right) \\ & - \sum_{0\leq s<t} \left(\log \left| 1 + \frac{\Delta^+ Y_s}{Y_s} \right| - \frac{\Delta^+ Y_s}{Y_s} \right). \end{aligned}$$

It is important to note that the process Y need not be positive for $\mathcal{L}\text{og}(Y)$ to exist, in accordance with the fact that the stochastic exponential $\mathcal{E}(X)$ may take negative values

What about Financial Markets?

Are the usual conditions and RCLL semimartingales good enough to describe financial markets?

- Consider the notion of completion: completion requires that we know a-priori all the null sets of $\mathcal{F}$ and augment the initial σ-algebra $\mathcal{F}_0$ with these null sets!
- A σ-algebra $\mathcal{F}_t$ that is right continuous means that the immediate future is equivalent to the present which is different from the immediate past!
- Khun and Stroh (2009), "A note on Stchastic Integration with respect to Optional Semimartingale", redefined the optional stochastic integral on a subset of $\mathcal{P} \times \mathcal{O}$ with integrands having the form

$$H = H_0 + \sum H_i \mathbf{1}_{\{1\}\times]\tau_i,\tau_{i+1}]\cup\{2\}\times[\tau_i,\tau_{i+1}[}$$

Markets of Optional Semimartingales

Optional Semimartingale Market

Melnikov and A.N. Shiryaev (1996), "Criteria for the Absence of Arbitrage in the Financial Market", considered the case of the usual conditions

- Two securities x and X, $x_t > 0$ and $X_t \geq 0$ for all $t \geq 0$

$$
\begin{aligned}
x_t &= x_0 + x \cdot h_t^r + x \odot h_t^g, \quad X_t = X_0 + X \cdot H_t^r + X \odot H_t^g \\
h_t &= h_0 + a_t + m_t, \quad H_t = H_0 + A_t + M_t
\end{aligned}
$$

  a and A are locally bounded variation processes. If a and A are predictable then the semimartingales h and H are called special optional semimartingales. m and M are optional local martingales
- The solution for X is $X_t = X_0 \mathcal{E}_t(H)$, and x, is $x_t = x_0 \mathcal{E}_t(h)$
- We studied the properties the *ratio process* $R = X/x$

A Portfolio of Optional Semimartingales

- Let a portfolio $\pi = (\eta, \xi)$, be of *optional processes* η and ξ, describing the volume of reference asset x and traded securities X, respectively
- The value process associated with the *portfolio equation*, is given by $Y_t = \eta_t + \xi_t R_t$
- We restrict π to be *self-financing* that is $Y_t = Y_0 + \xi \circ R_t$ and $C_t = \eta_t + R \circ \xi_t + [\xi, R]_t = C_0$ where C is consumption with initial value C_0.
- The integral $\xi \circ R_t$ must be well defined then ξ must satisfy the following conditions: **I.** ξ evolves in the space $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$ with the predictable part determining the volume of R^r and the optional part determining the volume of R^g **II.** ξ must be R-integrable

$$\int_0^\infty \xi_s^2 d[R, R]_s \in \mathcal{A}_{loc}.$$

- Note that $(\eta, \xi) = (\eta^r, \eta^g, \xi^r, \xi^g)$ is not our usual predictable portfolio but contains predictable and optional parts

Transforming Optional Semimartingales to Optional local Martingales

First lets consider when the ratio R is a local optional martingale w.r.t. the initial measure $\mathbf{P}$? R is given as,

$$\begin{aligned} R_t &= \frac{X_t}{x_t} = R_0 \mathcal{E}(H)_t \mathcal{E}^{-1}(h)_t \\ &= R_0 \mathcal{E}(\Psi(h, H) = R_0 \mathcal{E}(H_t - h_t^* - [H, h^*]_t) \end{aligned}$$

where

$$h_t^* = h_t - \langle h^c, h^c \rangle_t - \sum_{0<s\leq t} \frac{(\triangle h_s)^2}{1 + \triangle h_s} - \sum_{0\leq s<t} \frac{(\triangle^+ h_s)^2}{1 + \triangle^+ h_s}$$

and

$$\begin{aligned} \Psi(h, H) &= H_t - h_t^* - [H, h^*]_t = H_t - h_t + \langle h^c, h^c - H^c \rangle_t \\ &+ \sum_{0<s\leq t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1 + \triangle h_s} + \sum_{0\leq s<t} \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1 + \triangle^+ h_s}. \end{aligned}$$

If $\Psi(h, H)$ is a local martingale then R is also a local martingale

Local Optional Martingale Deflator

Theorem

Given $R = R_0\mathcal{E}(\Psi(h,H))$ where $\Psi(h,H)$ as is as defined above and $Z = \mathcal{E}(N)$ then $ZR \in \mathcal{M}_{loc}(\mathbf{P},\mathbf{F})$ if and only if

$$(A-a) + \langle m^c - N^c, m^c - M^c\rangle + \tilde{K}^d + \tilde{K}^g = 0,$$

where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators' of the processes

$$K^d = \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)(\triangle h_s - \triangle H_s)}{1+\triangle h_s}, \;\; K^g = \sum_{0\leq s\leq t} \frac{(\triangle^+ h_s - \triangle^+ N_s)(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}.$$

Theorem

π is a self financing; ZR is a local optional martingale if and only if ZY^π_t is a local optional martingale

Black-Scholes with Left and Right Jumps

- Let us consider the augmented Black-Scholes model with left and right jumps, where the money market account is $x_t = x_0 \exp(rt)$ where $h_t = rt$ and

$$X_t = X_0 + \int_{0+}^{t} X_{s-} \left(\mu ds + \sigma dW_s + a dL^r_s\right) + \int_0^{t-} b X_s dL^g_{s+},$$

  where $L^r_t = L_t - \lambda t$, $L^g_t = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}_-$ are $\mathcal{O}(\mathbf{F})$ independent Poisson with constant intensity λ and γ respectively.
- We can write X as $X_t = X_0 \mathcal{E}(H)$ where $H_t = \mu t + \sigma W_t + a\left(L_t - \lambda t\right) + b\left(\gamma t - \bar{L}_{t-}\right)$ with $H_0 = 0$.

Black-Scholes with Left and Right Jumps

- For the ratio process $R_t = X_t / x_t$ and martingale transform $Z = \mathcal{E}(N)$ with $N_t = \varsigma W_t + c\left(L_t - \lambda t\right) + d\left(\gamma t - \bar{L}_{t-}\right)$ then

$$\begin{aligned}\Psi(h, H, N) = & \left(\varsigma + \sigma\right) W_t + \left(a + c + ac\right)\left(L_t - \lambda t\right) \\ & + \left(b + d - bd\right)\left(\gamma t - \bar{L}_{t-}\right) \\ & + \left(\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma\right) t\end{aligned}$$

  is local martingale if $\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma = 0$ $(*)$.

- We have to find (ς, c, d) such that (*) is true; i.e.

$$[\sigma, \quad 2a\lambda, \quad 2b\gamma]\,[\varsigma, \quad c, \quad d]^{\mathsf{T}} = r - \mu.$$

  One possible solution is $(\varsigma, c, d) = (\sigma, a, b) / |(\sigma, a, b)|^2$ another interesting solution is to let $d = 0$ which leads to RCLL martingale measure. Yet another solution is a one which will eliminate the effects of jumps on drift: let $d = -1/b\gamma$ and $c = 1/a\lambda$ in this case $\varsigma = \left(r - \mu\right)/\sigma$

Defaultable Markets

Introduction

- *Default risk* is the possibility that any *counterparty* in a financial agreement will not fulfill their obligations
- Credit risk modeling is concerned with the random time at which default risk event occurs, known as *default time*
- Default time is used to provide ways to price and to hedge financial contracts that are sensitive to credit risk events

Current Approaches

Consider the usual probability space $\left(\Omega, \mathcal{F}, \mathbf{F} = \left(\mathcal{F}_t\right)_{t\geq 0}, \mathbf{P}\right)$

- **Structural Models**
  - The value of the firm, X, determines default but X is not tradable
  - Default time τ is a predictable stopping time with respect to $\mathbf{F}$
  - Example default time is $\tau := \inf\{t > 0 : t \geq 0, X_t \leq B_t\}$ where B is a Barrier process
- **Reduced-Form Models**
  - Default time is a random time that arrives as a total surprise to all counterparties; occurs outside the market filtration $\mathbf{F}$
  - Is modeled as a *totally inaccessible stopping times* on an enlarged filtration $\mathbf{G}$ that encompasses the default-free market information $\mathbf{F}$ and information that is a result of default processes $\mathbf{H}$, $\mathcal{G}_t = \sigma\left(\mathcal{F}_t \vee \mathcal{H}_t\right)$
  - But *Enlargement* of the filtration $\mathbf{F}$ by $\mathbf{H}$ leads to changes the properties of martingales and semimartingale

Reduced-Form Models

- To establish rational pricing one has to invoke the invariance principles known as the **H** and **H'** hypothesis (immersion)
  - **H**: a local martingale in **F** is a local martingale in **G**
  - **H'**: a semimartingale in **F** is a semimartingale in **G**
- To price defaultable claims, compute the conditional expectation of payoffs given default such that immersion satisfied

Closer look at the Mechanics of Default

- Let X be the value of a defaultable asset and fix an instance of time t
- If default is predictable or inaccessible stopping time then $(\tau \leq t) \in \mathcal{F}_t$
- Otherwise $(\tau \leq t) \notin \mathcal{F}_t$ a random time the result of external factors
- However, after default takes place say at time at t all surprising information about it gets incorporated in future values of X
- So If X_t is RCLL and $\mathcal{F}_t = \mathcal{F}_{t+}$ then, obviously, $(\tau \leq t) \notin \mathcal{F}_{t+}$

Defaultable Markets on *Un*usual Spaces

To avoid constructing an enlarged filtrations and using the immersion properties, we propose a different approach:

- Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in \mathbb{R}_+$, be the *un*usual stochastic basis and market stays on this space
- A defaultable market will consist of at least of the following objects:
  - Y_t the asset subject to default is $\mathcal{F}_t$-measurable
  - And τ the time of default is such that

    $$\tau \textit{ is a totally inaccessible stopping time in the broad sense}, \ \tau \in \mathcal{T}(\mathbf{F}_+).$$

    and its associated default process H is $H_t = \mathbf{1}_{(\tau<t)}$
- H is optional, $\mathbf{F}$-measurable but left-continuous, however $\mathbf{1}_{(\tau\leq t)}$ is $\mathcal{F}_{t+}$-measurable

Defaultable Claims and Cash-Flows

Definition

The *dividend process* D of a defaultable claim $DCT = (A, \Lambda, \rho, R, H)$ equals

$$D_t = \tilde{X}\mathbf{1}_{(t \geq T)} + (1 - H) \circ A_t + R \circ H_t,$$

where $\tilde{X} = \Lambda (1 - H_T) + \rho H_T$. The process D is optional and **F**-measurable

- Λ the promised contingent claim redeemed at time expiration time T if default didn't occur
- A, $A_0 = 0$ is the promised dividends if there was no default prior to time T. A is **F** predictable
- ρ is the recovery claim; the payoff received at time T if default occurs prior to T
- R is the recovery process specifies the recovery payoff at time of default if it occurs prior to the maturity date T
- Finally the default process $H_t = \mathbf{1}_{(\tau < t)}$

Ex-dividend Price

Suppose there exist a martingale deflator measure $\mathbf{Q} \sim \mathbf{P}$. The realized value of a defaultable claim in this market is the discounted value of D

Definition

The *ex-dividend price process* $X(\cdot, T)$ of a defaultable claim $DCT = (A, \Lambda, \rho, R, H)$ which settles at time T is given by

$$
\begin{aligned}
X_t &= X(t, T) = B_t \mathbf{E}_{\mathbf{Q}} \left(B^{-1} \circ D_T - B^{-1} \circ D_t | \mathcal{F}_t \right) \\
&= B_t \mathbf{E}_{\mathbf{Q}} \left(\int_{t+}^{T} B_{u-}^{-1} dD_u + \int_{t}^{T-} B_u^{-1} dD_{u+} | \mathcal{F}_t \right), \quad \forall t \in [0, T].
\end{aligned}
$$

where B money market account

Valuation of a Defaultable Claim

- We need a convenient representation of the value of a defaultable claim in terms of the *probability of default*
- $D_t = (\Lambda(1-H_T) + \rho H_T)\mathbf{1}_{(t\geq T)} + (1-H)\circ A_t + R\circ H_t$ the value of *ex-dividend* defaultable claim is

$$\begin{aligned}
B_t^{-1}X_t &= \mathbf{E}\left[B_T^{-1}\left(\Lambda(1-H_T)+\rho H_T\right)|\mathcal{F}_t\right] \\
&+\mathbf{E}\left[\int_{t+}^{T} B_{u-}^{-1}(1-H_{u-})dA_u + \int_t^{T-} B_u^{-1}(1-H_u)dA_{u+}|\mathcal{F}_t\right] \\
&+\mathbf{E}\left[\int_t^{T-} B_u^{-1}R_u dH_{u+}|\mathcal{F}_t\right].
\end{aligned}$$

Valuation of a Defaultable Claim

Lemma

The value of $\tilde{\Lambda}_t = \mathbf{E}\left[B_T^{-1}\Lambda\left(1-H_T\right)|\mathcal{F}_t\right]$ *at time* t *is given by*

$$\tilde{\Lambda}_t = \mathbf{E}\left(B_T^{-1}\Lambda|\mathcal{F}_t\right)\left(1-H_t\right) + \mathbf{E}\left[\int_t^{T-}\left(\lambda_u + \triangle^+\lambda_u\right)dG_{u+}|\mathcal{F}_t\right]. \qquad (1)$$

where $\lambda_u = \mathbf{E}\left[B_T^{-1}\Lambda|\mathcal{F}_u\right]$ *and* $G_{u+} = G(u,u+) = \mathbf{E}\left(1-H_{u+}|\mathcal{F}_u\right)$

Lemma

The value $\tilde{\rho}_t = \mathbf{E}\left(\varrho_T H_T|\mathcal{F}_t\right)$ *is given by*

$$\tilde{\rho}_t = \varrho_t H_t - \mathbf{E}\left[\int_t^{T-}\left(\varrho_u + \triangle^+\varrho_u\right)dG_{u+}|\mathcal{F}_t\right]$$

where $\varrho_u = \mathbf{E}\left(B_T^{-1}\rho|\mathcal{F}_u\right)$

...

Zero-Coupon Defaultable Bond

- The price of a zero-coupon bond that may experience default is $B_t \mathbf{E}\left(B_T^{-1}\mathbf{1}_{(\tau \geq T)}|\mathcal{F}_t\right)$
- $B_t = e^{rt}$, hence $B(t,T) = e^{-r(T-t)} = B_t B_T^{-1}$, with a constant interest rate r
- The survival process admits a constant intensity γ such that

$$dG_{u+} = \mathbf{E}\left(H_u - H_{u+}|\mathcal{F}_u\right) = -\delta\left(u-\tau\right)\gamma e^{-\gamma u}du,$$

  where $\delta\left(u-\tau\right)$ is delta function at a *particular value* of default time τ.
- Then, the price is

$$X(t,T) = e^{-r(T-t)}\left[\mathbf{1}_{(\tau \geq t)} - \mathbf{1}_{(t \leq \tau < T)}e^{-\gamma t}\left(1 - e^{-\gamma(T-t)}\right)\right].$$

- At $t=0$, $X(0,T) = e^{-rT}\left[1 - \mathbf{1}_{(0 \leq \tau < T)}\left(1 - e^{-\gamma T}\right)\right]$ and at $t=T$, $X(T,T) = \mathbf{1}_{(\tau \geq T)}$.
- If $\tau < T$ then $X(0,T) = e^{-(r+\gamma)T}$
- default decreases the present value of the bond by a factor $e^{-\gamma T}$.

Stochastic Equations

Nonhomogeneous Linear Stochastic Equation

- $X = G + \int_{0+} X_{-} dH$, has a natural interpretation in finance: G is cash flow, H is the return process of a money market account and X is the time value of the cash flow G accumulated in a money market account
- In optional semimartingales setting $X_t = G_t + X \circ H_t$ which we showed that is has the solution

$$\begin{aligned} X_t &= \mathcal{E}_t(H)\left[G_0 + \int_0^t \mathcal{E}_s(H)^{-1} d\tilde{G}_s\right], \\ d\tilde{G}_t &= dG_t - d\left[G, \tilde{H}\right]_t, \\ \tilde{H}_t &= H_t^c + \sum_{0<s\le t} \frac{\triangle H_s}{1+\triangle H_s} + \sum_{0\le s<t} \frac{\triangle^+ H_s}{1+\triangle^+ H_s}. \end{aligned}$$

Gronwall Lemma for Optional Semimartingales

- Gronwall lemma allows us to put bounds on functions that satisfies an integral/differential inequality by a solution of the supposed equality
- Let X be an optional semimartingale, H be an optional increasing process and C an optional process such that

$$\begin{aligned} X_t &\leq C_t + \int_0^t X_s dH_s \\ &= C_t + \int_{0+}^t X_{s-} dH_s + \int_0^{t-} X_s dH_{s+} \end{aligned}$$

for all $t \in [0, \infty)$. *Then* $X_t \leq C_t \mathcal{E}_t(H)$.

Comparison Theorem

- Allows us to compare solutions of related stochastic equations
- We study comparison of solutions of stochastic equations driven by optional semimartingales in unusual probability spaces under Yamada conditions
- Consider the optional semimartingale Z that has the following *component representation*,

$$
\begin{aligned}
Z \quad = \quad & Z_0 + a + m \\
& + \int_{0+}^{t} \int_{\mathbb{E}} U(\mu^1 - \nu^1)(ds, du) + \int_{0+}^{t} \int_{\mathbb{E}} V\mu^1(ds, du) + \int_{0+}^{t} \int_{\mathbb{E}} up^1(ds, du) \\
& + \int_{0}^{t-} \int_{\mathbb{E}} U(\mu^2 - \nu^2)(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} V\mu^2(ds, du) + \int_{0}^{t-} \int_{\mathbb{E}} up^2(ds, du) \\
& + \int_{0}^{t-} \int_{\mathbb{E}} u\eta(ds, du).
\end{aligned}
$$

where $U = u\mathbf{1}_{|u|\leq 1}$, $V = u\mathbf{1}_{|u|>1}$ and $\eta = \eta^g$. $a \in \mathcal{A}_{1oc}$ with $a_0 = 0$ a continuous locally integrable process, $m \in \mathcal{M}^c_{1oc}$ with $m_0 = 0$ a continuous martingale and integer-valued measures μ^j, p^j for$j = 1, 2$ and η with predictable and optional projections ν^j, λ^j, and θ respectively

Comparison Equations

- Consider the equations

$$\begin{aligned} X_t^i &= X_0^i + f^i(X^i)\cdot a_t + g(X^i)\cdot m_t \\ &\quad + \sum_j Uh_j(X^i) * (\mu^j - \nu^j)_t + Vh_j^i(X^i) * \mu_t^j + \Big(k_j^i(X^i) + l_j^i(X^i)\Big) * p_t^j \\ &\quad + \Big(r^i(X^i) + w^i(X^i)\Big) * \eta_t, \end{aligned}$$

  where $U = \mathbf{1}_{|u|\leq 1}$ and $V = \mathbf{1}_{|u|>1}$ where all required integrability conditions are satisfied
- $X_0^2 \geq X_0^1$
- $f^2(s,x) > f^1(s,x)$ for any (s,x)
- $f^i(s,x)$ are continuous in (s,x)

Yamada Conditions

- There exists a *non-negative nondecreasing function* $\rho(x)$ on $\mathbb{R}_+$ and a $\mathcal{P}(\mathbf{F})$-measurable non-negative function G such that

$$|g(s,x) - g(s,y)| \leq \rho(|x-y|)G(s), \quad \int_0^{\epsilon} \rho^{-2}(x)dx = \infty$$

- There exists a non-negative $\widetilde{\mathcal{P}}(\mathbf{F})$-measurable functions H_1 and $\widetilde{\mathcal{O}}(\mathbf{F})$-measurable function H_2 such that

$$\begin{aligned} |h_1(s,u,x) - h_1(s,u,y)| &\leq \rho(|x-y|)H_1(s,u) \\ |h_2(s,u,x) - h_2(s,u,y)| &\leq \rho(|x-y|)H_2(s,u) \end{aligned}$$

- Weaker than Lipschitz

...

- For any (s,u,x,y) and $y \geq x$,

$$\begin{gathered}
h_1(s,u,y) \geq h_1(s,u,x), \\
h_2(s,u,y) \geq h_2(s,u,x), \\
y + h_1^2(s,u,y)\mathbf{1}_{|u|>1} \geq x + h_1^1(s,u,x)\mathbf{1}_{|u|>1}, \\
y - h_2^2(s,u,y)\mathbf{1}_{|u|>1} \geq x - h_2^1(s,u,x)\mathbf{1}_{|u|>1}, \\
y + h_1(s,u,y)\mathbf{1}_{|u|\leq 1} + (k_1^2 + l_1^2)(s,u,y) \geq x + h_1(s,u,x)\mathbf{1}_{|u|\leq 1} + (k_1^1 + l_1^1)(s,u,x), \\
y - h_2(s,u,y)\mathbf{1}_{|u|\leq 1} - (k_2^2 + l_2^2)(s,u,y) - \left(r^2 + w^2\right)(s,u,x) \\
\geq x - h_2(s,u,x)\mathbf{1}_{|u|\leq 1} - (k_2^1 + l_2^1)(s,u,x) - \left(r^1 + w^1\right)(s,u,x);
\end{gathered}$$

- The functions $\left(r^i + w^i\right)(s,u,x)$ and $(k_j^i + l_j^i)(s,u,x)$ are continuous in (s,u,x)

$$\begin{aligned}
(k_j^2 + l_j^2)(s,u,x) &> (k_j^1 + l_j^1)(s,u,x) \\
\left(r^2 + w^2\right)(s,u,x) &> \left(r^1 + w^1\right)(s,u,x)
\end{aligned}$$

Comparison Theorem

Theorem

Let there exist strong solutions X^i of the comparison equations and let all conditions listed above be satisfied then $X_t^2 \geq X_t^1$ for all t

Extension Lemma

Lemma

If the comparison theorem is valid for the following equations

$$\begin{aligned} Y_t^i &= X_0^i + f^i(Y^i) \cdot a_t + g(Y^i) \cdot m_t \\ &+ \sum_j U h_j(Y^i) * (\mu^j - \nu^j)_t + \left(k_j^i(Y^i) + l_j^i(Y^i)\right) * p_t^j \\ &+ \left(r^i(Y^i) + w^i(Y^i)\right) * \eta_t \end{aligned}$$

with functions X_0^i, f^i, g, h_j, k_j^i, l_j^i, r^i, and w^i satisfying all conditions stated above then it is also valid for the comparison equations

Outline of the Proof:

- Let (τ_n) be a nondecreasing sequence of stopping times absorbing the jumps of the processes $h_j^i(X^i) * \mu^j$. On $]\tau_k, \tau_{k+1}[$ the above equations and comparison equations coincide
- Prove comparison for the closure $[\tau_k, \tau_{k+1}]$ for every k by adjusting the solutions with left jump at τ_{k+1} and right jump at τ_k and validating "$\leq$"

Finally, prove Comparison for Y

- To do so let $\{\psi_n(x)\}_{n\in\mathbb{N}}$ be non-negative and continuous such that $\mathrm{supp}\psi_n \subseteq (a_n, a_{n-1})$,

$$\int_{a_n}^{a_{n-1}} \psi_n(x)dx = 1, \quad \psi_n(x) \leq \frac{2}{n}\rho^{-2}(|x|), \quad x \in \mathbb{R},$$

and the maximum of ψ_n is attained at $a_{n-1} - \epsilon_n$

...

- Set

$$\begin{aligned}\varphi_n(x) &= \int_0^{|x|} dy \int_0^y \psi_n(u)du, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \\ \varphi_n &\in C^2(R^1), \quad \varphi_n(x) \uparrow |x|, \quad n \to \infty, \quad |\varphi_n'| \leq 1\end{aligned}$$

- Consider $R_v = Y_v^2 - Y_v^1$ and use Galchuck-Ito on $\varphi_n(R_v)$ to arrive at

$$\begin{aligned}\mathbf{E}\varphi_n(R) &= \mathbf{E}\varphi_n(Y^2 - Y^1) \uparrow \mathbf{E}\left|Y^2 - Y^1\right| \\ 0 &\leq \mathbf{E}\left|Y^2 - Y^1\right| \leq \mathbf{E}\left(Y_v^2 - Y_v^1\right)\end{aligned}$$

Existence Uniqueness under Monotonicity Conditions

Theorem

Let H *be a separable Hilbert space and x an $\mathbb{R}^d$-valued adapted RLL process the equation*

$$x = \xi + a(x) \circ A + b(x) \circ M$$

where ξ be an $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^d$, $A \in \mathcal{V}^+(\mathbb{R})$, $M \in \mathcal{M}^2_{loc}(\mathrm{H})$ has a unique solution if certain conditions are satisfied

Structures and Conditions

- $V \in \mathcal{V}^+(\mathbb{R})$ such that $dV \geq dA$ and $dV \geq d\langle M \rangle$
- $\mathrm{L}_1(\mathrm{H},\mathrm{H})$ is the Banach space of the nuclear operators and $\mathrm{L}_2(\mathrm{H},\mathbb{R}^d)$ is the Hilbert space of the Hilbert-Schmidt operators
- Identify $\mathrm{L}_1(\mathrm{H},\mathrm{H})$ with $\mathrm{H} \otimes_1 \mathrm{H}$ and $\mathrm{L}_2(\mathrm{H},\mathbb{R}^d)$ with $\mathrm{H} \otimes_2 \mathbb{R}^d$ where $\mathrm{H} \otimes_1 \mathrm{H}$ is the projective-tensor product of H and $\mathrm{H} \otimes_2 \mathbb{R}^d$ the Hilbert-tensor product of H with $\mathbb{R}^d$

Furthermore,

- Let $Q := d\langle\langle M\rangle\rangle / dV$ where $\langle\langle M\rangle\rangle = \langle M \otimes_1 M\rangle$ and $\langle\langle M\rangle\rangle \in \mathcal{P}(\mathrm{H} \otimes_1 \mathrm{H})$
- If $Q \in \mathrm{L}_1(\mathrm{H},\mathrm{H}) \geq 0$ then $\mathrm{L}_Q(\mathrm{H},\mathbb{R}^d)$ is the set of all linear (not necessarily bounded) operators C mapping $Q^{1/2}(\mathrm{H})$ into $\mathbb{R}^d$ such that $CQ^{1/2} \in \mathrm{L}_2(\mathrm{H},\mathbb{R}^d)$
- a is $\mathbb{R}^d$-valued, $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable, continuous in x and locally integrable with respect to dA_t
- $\beta^1 := b_-\sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$ are $\mathrm{L}_2(\mathrm{H},\mathbb{R}^d)$-valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{O} \times \mathcal{B}(\mathbb{R}^d)$ measurable functions on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$

...

- For any $R \geq 0$ such that $|x| \leq R$, $|y| \leq R$ there exists a process $K_t(R)$ such that following inequalities hold:
  - **Monotonity Condition:**

$$2(x_- - y_-)(\alpha^1(t,x) - \alpha^1(t,y)) + \triangle V_t \left|\alpha^1(t,x) - \alpha^1(t,y)\right|^2 + |\beta^1(t,x) - \beta^1(t,y)|^2 \leq K_t(R)|x_- -$$

$$2(x - y)(\alpha^2(t,x) - \alpha^2(t,y)) + \triangle^+ V_t \left|\alpha^2(t,x) - \alpha^2(t,y)\right|^2 + |\beta^2(t,x) - \beta^2(t,y)|^2 \leq K_t(R)|x - y$$

  - **Restriction on Growth:**

$$2x_-\alpha^1(t,x) + \triangle V_t|\alpha^1(t,x)|^2 + |\beta^1(t,x)|^2 \leq K_t(R)(1 + |x_-|^2),$$
$$2x\alpha^2(t,x) + \triangle^+ V_t|\alpha^2(t,x)|^2 + |\beta^2(t,x)|^2 \leq K_t(R)(1 + |x|^2),$$

where $\alpha^1 := a_- \left(\frac{dA}{dV}\right)_-$, $\alpha^2 := a\left(\frac{dA}{dV}\right)$ and $\beta^1 := b_-\sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$

Uniqueness

- Let $\varphi_t = 1 + K\varphi \circ V$ and use the product rule on $\varphi^{-1}|x-y|^2$

$$\begin{aligned}\varphi^{-1}|x-y|^2 \quad = \quad & \varphi^{-1}\Big[2(x-y)\left(\alpha^1(x)-\alpha^1(y)\right)+\left(\alpha^1(x)-\alpha^1(y)\right)^2 \triangle V \\ & +\left(\beta^1(x)-\beta^1(y)\right)^2 - K|x_- - y_-|^2\Big]\cdot V \\ & +\varphi^{-1}\Big[2(x-y)\left(\alpha^2(x)-\alpha^2(y)\right)+\left(\alpha^2(x)-\alpha^2(y)\right)^2 \triangle^+ V \\ & +\left(\beta^2(x)-\beta^2(y)\right)^2 - K|x-y|^2\Big]\odot V + m''\end{aligned}$$

where

$$m'' = \varphi^{-1}2(x-y)(b(x)-b(y))\circ M + \varphi^{-1}(b(x)-b(y))^2 \circ ([M,M]-\langle M,M\rangle),$$

$\alpha^1 := a_-\Lambda_-$, $\alpha^2 := a\Lambda$ and $\beta^1 := b_-\sqrt{Q_-}$ and $\beta^2 := b\sqrt{Q}$

- The first two terms are negative by monotonicity conditions but $\varphi^{-1}|x-y|^2 \geq 0$ so we get

$$0 \leq \varphi^{-1}|x-y|^2 \leq m''$$

m'' is a non-negative local martingale so it is a non-negative supermartingale. Since $m''_0 = 0$ it follows that $m'' = 0$ consequently $\varphi^{-1}|x-y|^2 = 0$

Existence

- Let $\tau^n = \inf\left(t : |x_t^n| \geq n\right)$ and $\tau^{nm} = \tau^n \wedge \tau^m$ and consider the equation $z_t = \xi + a(z) \circ A_{t\wedge\tau^{nm}} + b(z) \circ M_{t\wedge\tau^{nm}}$
- The process $x_t^n = x_{t\wedge\tau^n}$ for every n follows the equation

$$x_{t\wedge\tau^n} = x_t^n = \xi + a(x^n) \circ A_{t\wedge\tau^n} + b(x^n) \circ M_{t\wedge\tau^n} \quad (*)$$

- $x_t^m = x_{t\wedge\tau^m}$ satisfy the same equation (*). By uniqueness $x_t^n = x_t^m$ on $[0, \tau^{nm}]$
- There exists a stopping time τ such that $\tau = \lim_{n\to\infty} \tau^n$ and the adapted ladlag process $x_t = \lim_{n\to\infty} x_t^n$ on $[0, \tau]$ a unique solution of

$$x_t = \xi + a(x) \circ A_t + b(x) \circ M_t$$

- For $\tau = \infty$ use $\psi_t = \varphi_t^{-1} \exp(-|\xi|)$ for an integrable random variable ξ; Then

$$\mathbf{E}\left[\left|x_{\tau^n}^n\right|^2 \psi_{\tau^n} \mathbf{1}_{(\tau^n<\infty)}\right] \leq \mathbf{E}\left[|\xi|^2 \exp\left(-|\xi|\right)\right] = const$$

hence $\mathbf{E}\psi_{\tau^n}\mathbf{1}_{(\tau^n<\infty)} \leq \frac{const}{n^2} \to 0$

The End