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Optional decomposition of optional supermartingales and applications to filtering and finance

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ABSTRACT
The classical Doob–Meyer decomposition and its uniform version the optional decomposition are stated on probability spaces with filtrations satisfying the usual conditions. However, the comprehensive needs of filtering theory and mathematical finance call for their generalizations to more abstract spaces without such technical restrictions. The main result of this paper states that there exists a uniform Doob–Meyer decomposition of optional supermartingales on *un*usual probability spaces. This paper also demonstrates how this decomposition works in the construction of optimal filters in the very general setting of the filtering problem for optional semimartingales. Finally, the application of these optimal filters of optional semimartingales to mathematical finance is presented.

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1. Introduction

Under the usual conditions, the Doob–Meyer decomposition and the uniform Doob–Meyer decomposition (UDMD), commonly known as the *optional decomposition*, are fundamental results in stochastic analysis with many applications. For example, in the area of mathematical finance, UDMD allows for the construction of superhedging strategies in incomplete markets.

UDMD asserts that if for a non-empty set, $\mathbb{Q}(X)$, of equivalent local martingale measures for the semimartingale, X, such that, the stochastic process Y is a local supermartingale with respect to all measures in $\mathbb{Q}(X)$ then there exists a unique predictable stochastic integrand φ such that the difference $Y - \varphi \cdot X$ is a decreasing optional process for all measures in $\mathbb{Q}(X)$. In contrast to the classical Doob–Meyer decomposition, the process $Y - \varphi \cdot X$ is optional and is not uniquely determined. Moreover, UDMD is *universal*, meaning that it holds simultaneously for all probability measures in $\mathbb{Q}(X)$.

In an incomplete market, the process Y is the market value of an $\mathcal{F}_T$ measurable contingent claim $H > 0$. Y can be defined as the essential supremum of the conditional expectation over the class of all equivalent martingale measures in $\mathbb{Q}(X)$, that is

$$Y_t = ess \sup_{\mathbf{Q} \in \mathbb{Q}(X)} \mathbf{E}_{\mathbf{Q}}[H|\mathcal{F}_t].$$

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It follows that Y is a cadlag supermartingale with respect to all $\mathbf{Q} \in \mathbb{Q}(X)$. UDMD is then used to identify Y as the value process of the superhedging strategy (φ, C) satisfying $Y - Y_0 = \varphi \cdot X - C$, where the integrand φ specifies the number of units of the underlying asset X and generates an increasing optional process, $-C$, of cumulative side payments with $C_0 = 0$. The strategy (φ, C) induces a perfect hedge in the class of investment strategies with terminal capital $Y_T = H$.

The existence of UDMD for supermartingales was first presented by El Karoui and Quenez [17] for diffusion processes. Kramkov [30] proved the existence of UDMD for locally bounded supermartingales. Kabanov and Foellmer [18] proved UDMD without the local boundedness assumption, thus permitting the inclusion of models with unbounded jumps. The authors also provided an interpretation of the integrand as a Lagrange multiplier for an optimization problem with constraints. Stricker and Yan [38] proved UDMD for supermartingales and submartingales in a general setting. Recently, Jacka [26] presented a simple proof of UDMD without integral representation of the local martingale in terms of the underlying assets and with the set of local martingale measures satisfying some closure property. Karatzas *et al.* [29] presented a specific treatment of the UDMD for continuous semimartingales and general filtrations where their method does not assume the existence of equivalent local martingale measure(s), only that of strictly positive local martingale deflator(s). All these results assumed that the stochastic basis satisfies the *usual conditions* and all the processes are RCLL. Therefore, the problem of finding an adequate form of the UDMD of *optional local supermartingales* on *un*usual probability spaces becomes an important question to be answered. In this paper, we extend the works of Kabanov and Foellmer and Stricker and Yan to derive UDMD theorems for *optional* super/submartingales on *un*usual probability spaces.

This paper is organized as follows. In Section 2, we present a summary of stochastic calculus of optional processes on *un*usual probability spaces. In Section 3, we prove the existence of UDMD on *un*usual probability spaces. Section 4 is devoted to showing how to apply this decomposition to the *filtering problem* for optional processes. Up to now, in filtering theory, only martingale representations were exploited for the construction of optimal filters. Here we show how UDMD can be used, instead of the traditionally used martingale representations to obtain optimal filters for optional processes. In Section 5, we apply UDMD and filtering of optional supermartingales to the problem of pricing and hedging of contingent claims in *incomplete* markets on *un*usual probability space where the flow of information is also *incomplete*. Other theoretical findings are supported by relevant examples as well.

2. Optional stochastic calculus: summary of results

In the middle of 1970, Dellacherie [14] started the study of stochastic processes without the usual conditions and called this case the *un*usual conditions. Further developments were carried by many mathematicians but mostly by [20–22]. In these publications, a stochastic calculus of processes on *un*usual probability spaces was constructed. For a comprehensive review of the stochastic calculus of processes on *un*usual probability spaces, current development and applications to finance, see [2, 3]. However, we provide a short summary here to facilitate understanding of the main findings of the paper.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, $t \in [0, \infty)$, where $\mathcal{F}_t \in \mathbf{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s \leq t$, be a *complete probability space* – $\mathbf{P}$ is a complete measure and $\mathcal{F}$ contains all $\mathbf{P}$ null sets. However, the family $\mathbf{F}$ is not assumed to be complete or right or left continuous. We introduce $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ as optional and predictable σ-algebras on $(\Omega, [0, \infty))$, respectively. $\mathcal{O}(\mathbf{F})$ is generated by all $\mathbf{F}$-adapted processes whose trajectories are right continuous and have limits from the left. $\mathcal{P}(\mathbf{F})$ is generated by all $\mathbf{F}$-adapted processes whose trajectories are left continuous and have limits on the right. A random process $X = (X_t)$, $t \in [0, \infty)$, is said to be *optional* if it is $\mathcal{O}(\mathbf{F})$-measurable. In general, an optional process have right and left limits but is not necessarily right or left continuous in $\mathbf{F}$. A random process (X_t), $t \in [0, \infty)$, is predictable if $X \in \mathcal{P}(\mathbf{F})$ and *strongly predictable*, that is X is in $\mathcal{P}_s(\mathbf{F})$, if $X \in \mathcal{P}(\mathbf{F})$ and $X_+ \in \mathcal{O}(\mathbf{F})$. In general, a predictable process has right and left limits but may not necessarily be right or left continuous in $\mathbf{F}$. For either optional or predictable processes, we can define the following processes: $X_- = (X_{t-})_{t\geq 0}$ and $X_+ = (X_{t+})_{t\geq 0}$, $\triangle X = (\triangle X_t)_{t\geq 0}$ such that $\triangle X_t = X_t - X_{t-}$ and $\triangle^+ X = (\triangle^+ X_t)_{t\geq 0}$ such that $\triangle^+ X_t = X_{t+} - X_t$.

On *un*usual stochastic basis, three canonical types of stopping times exist. Predictable stopping times, $S \in \mathcal{T}(\mathbf{F}_-)$, are such that $(S \leq t)$ is $\mathcal{F}_{t-}$ measurable for all t. Totally inaccessible stopping times, $T \in \mathcal{T}(\mathbf{F})$, are such that $(T \leq t)$ is $\mathcal{F}_t$ measurable for all t. However, we point out that $(T < t)$ is not necessarily $\mathcal{F}_t$ measurable since $\mathcal{F}_t$ is not right continuous. Finally, totally inaccessible stopping times in the broad sense, $U \in \mathcal{T}(\mathbf{F}_+)$, are such that $(U \leq t)$ is $\mathcal{F}_{t+}$ measurable for all t but since $\mathcal{F}_{t+}$ is right continuous $(U \leq t)$ is also $\mathcal{F}_{t+}$ measurable. A process $X = (X_t)_{t\geq 0}$ belongs to the space $\mathcal{J}_{loc}$ if there is a localizing sequence of stopping times in the broad sense, (R_n), $n \in \mathbb{N}$, $R_n \in \mathcal{T}(\mathbf{F}_+)$, $R_n \uparrow \infty$ a.s. such that $X\mathbf{1}_{[0,R_n]} \in \mathcal{J}$ for all n, where $\mathcal{J}$ is a space of processes and $\mathcal{J}_{loc}$ is an extension of $\mathcal{J}$ by localization.

A process $A = (A_t)_{t\geq 0}$ is increasing if it is non-negative, its trajectories do not decrease and for any t, the random variable A_t is $\mathcal{F}_t$-measurable. Let $\mathcal{V}^+(\mathbf{F}, \mathbf{P})$ ($\mathcal{V}^+$ for short) be the collection of increasing processes. An increasing process A is integrable if $\mathbf{E}A_\infty < \infty$ and locally integrable if there is a sequence $(R_n) \subset \mathcal{T}(\mathbf{F}_+)$, $n \in \mathbb{N}$, $R_n \uparrow \infty$ a.s. such that $\mathbf{E}A_{R_{n+}} < \infty$ for all $n \in \mathbb{N}$. The collection of such processes is denoted by $\mathcal{A}^+$ ($\mathcal{A}^+_{loc}$ respectively). A process $A = (A_t)$, $t \in \mathbb{R}_+$, is a finite variation process if it has finite variation on every segment $[0, t]$, $t \in \mathbb{R}_+$ a.s., that is $\mathbf{Var}(A)_t < \infty$, for all $t \in \mathbb{R}_+$ a.s., where

$$\mathbf{Var}(A)_t = \int_{0+}^{t} |\mathrm{d}A_s^r| + \sum_{0\leq s<t} |\triangle^+ A_s|.$$

We shall denote by $\mathcal{V}(\mathbf{F}, \mathbf{P})$ ($\mathcal{V}$ for short) the set of $\mathbf{F}$-adapted finite variation processes. A process $A = (A_t)_{t\geq 0}$ of finite variation belongs to the space $\mathcal{A}$ of integrable finite variation processes if $\mathbf{E}[\mathbf{Var}(A)_\infty] < \infty$. A process $A = (A_t)_{t\geq 0}$ belongs to $\mathcal{A}_{loc}$ if there is a sequence $(R_n) \subset \mathcal{T}(\mathbf{F}_+)$, $n \in \mathbb{N}$, $R_n \uparrow \infty$, such that $A\mathbf{1}_{[0,R_n]} \in \mathcal{A}$ for any $n \in \mathbb{N}$, i.e. for any n, $\mathbf{E}[\mathbf{Var}(A)_{R_n}] < \infty$. A finite variation or an increasing process A can be decomposed as $A = A^r + A^g = A^c + A^d + A^g$ where A^c is continuous, A^r is a right-continuous, A^d is discrete right-continuous, A^g is discrete left-continuous such that

$$A_t^d = \sum_{0<s\leq t} \triangle A_s \quad \text{and} \quad A_t^g = \sum_{0\leq s<t} \triangle^+ A_s,$$

where the series converges absolutely.

A process $M = (M_t)_{t\geq 0}$ is an optional martingale (supermartingale, submartingale) if $M \in \mathcal{O}(\mathbf{F})$, the random variable $M_T \mathbf{1}_{T<\infty}$ is integrable for any stopping time $T \in \mathcal{T}(\mathbf{F})$, and there exists an integrable random variable $\mu \in \mathcal{F}_\infty$ such that $M_T = \mathbf{E}[\mu|\mathcal{F}_T]$ (respectively, $M_T \geq \mathbf{E}[\mu|\mathcal{F}_T]$, $M_T \leq \mathbf{E}[\mu|\mathcal{F}_T]$) a.s. for any stopping time $T \in \mathcal{F}_\infty$ with $(T < \infty)$. Let $\mathcal{M}(\mathbf{F}, \mathbf{P})$ $\mathcal{M}$ for short denote the set of optional martingales and $\mathcal{M}_{loc}$ the set of optional local martingales. If M is a local optional martingale, then it can be decomposed as

$$M = M^r + M^g \quad \text{where } M^r = M^c + M^d,$$

M^c is continuous, M^d is right-continuous and M^g is left-continuous local optional martingales. M^d and M^g are orthogonal to each other and to any continuous local martingale. Moreover, M^d and M^g can be written as

$$M_t^d = \sum_{0<s\leq t} \triangle M_s \quad \text{and} \quad M_t^g = \sum_{0\leq s<t} \triangle^+ M_s.$$

An optional semimartingale $X = (X_t)_{t\geq 0}$ can be decomposed to an optional local martingale and an optional finite variation process,

$$X = X_0 + M + A,$$

where $M \in \mathcal{M}_{loc}$ and $A \in \mathcal{V}$. A semimartingale X is called special if the above decomposition exists with a strongly predictable process $A \in \mathcal{A}_{loc}$. Let $\mathcal{S}(\mathbf{F}, \mathbf{P})$ denote the set of optional semimartingales and $\mathcal{S}p(\mathbf{F}, \mathbf{P})$ the set of special optional semimartingales. If $X \in \mathcal{S}p(\mathbf{F}, \mathbf{P})$, then the semimartingale decomposition is unique. By optional martingale decomposition and decomposition of predictable processes, see [20, 22], we can decompose a semimartingale further to $X = X_0 + X^r + X^g$ with $X^r = A^r + M^r$, $X^g = A^g + M^g$ and $M^r = M^c + M^d$, where A^r and A^g are finite variation processes that are right and left continuous, respectively. $M^r \in \mathcal{M}_{loc}^r$ right continuous local martingales, $M^d \in \mathcal{M}_{loc}^d$ discrete right continuous local martingales and $M^g \in \mathcal{M}_{loc}^g$ a left continuous local martingales. This decomposition is useful for defining integration with respect to optional semimartingales.

A stochastic integral with respect to optional semimartingale was defined in [22] as

$$\begin{aligned}
\varphi \circ X_t &= \int_0^t \varphi_s \mathrm{d}X_s = \int_{0+}^t \varphi_{s-} \mathrm{d}X_s^r + \int_0^{t-} \varphi_s \mathrm{d}X_{s+}^g, \quad \text{where} \\
\int_{0+}^t \varphi_{s-} \mathrm{d}X_s^r &= \int_{0+}^t \varphi_{s-} \mathrm{d}A_s^r + \int_{0+}^t \varphi_{s-} \mathrm{d}M_s^r \quad \text{and} \\
\int_0^{t-} \varphi_s \mathrm{d}X_{s+}^g &= \int_0^{t-} \phi_s \mathrm{d}A_{s+}^g + \int_0^{t-} \phi_s \mathrm{d}M_{s+}^g.
\end{aligned}$$

The stochastic integral with respect to a finite variation process or a strongly predictable process, A^r over $(0, t]$ and A^g over $[0, t)$, is interpreted as usual, in the Lebesgue sense. The integral $\int_{0+}^t \varphi_{s-} dM_s^r$ over $(0, t]$ is our usual stochastic integral with respect to RCLL local martingale whereas $\int_0^{t-} \phi_s dM_{s+}^g$ over $[0, t)$ is Galchuk stochastic integral [21, 22] with respect to left continuous local martingale. In general, the stochastic integral with respect to optional semimartingale X can be defined as a bilinear form $(f,g) \circ X_t$ such that

$$Y_t = (f,g) \circ X_t = f \cdot X_t^r + g \odot X_t^g,$$
$$f \cdot X^r = \int_{0+}^{t} f_{s-} \mathrm{d}X_s^r, \quad g \odot X^g = \int_0^{t-} g_s \mathrm{d}X_{s+}^g,$$

where Y is again an optional semimartingale $f_- \in \mathcal{P}(\mathbf{F})$ and $g \in \mathcal{O}(\mathbf{F})$. Note that the stochastic integral over optional semimartingales is defined on a much larger space of integrands, the product space of predictable and optional processes, $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$. From now on, we are going to use the operator '$\circ$' to denote the stochastic optional integral, the operator '$\cdot$' to denote the regular stochastic integral with respect to RCLL semimartingales, and the operator '$\odot$' for the Galchuk stochastic integral $g \odot X^g$ with respect to left continuous semimartingales. The properties of optional stochastic integral are: First, isometry is satisfied with

$$\left(f^2 \cdot \left[X^r, X^r\right]\right)^{1/2} \in \mathcal{A}_{loc} \quad \text{and} \quad \left(g^2 \odot \left[X^g, X^g\right]\right)^{1/2} \in \mathcal{A}_{loc}.$$

The quadratic variations are defined as

$$[X, X] = \left[X^r, X^r\right] + \left[X^g, X^g\right] \quad \text{where}$$
$$\left[X^r, X^r\right]_t = \left\langle X^c, X^c\right\rangle_t + \sum_{0<s\le t} (\triangle X_s)^2 \quad \text{and}$$
$$\left[X^g, X^g\right]_t = \sum_{0\le s<t} \left(\triangle^+ X_s\right)^2.$$

Linearity is also satisfied with $(f^1 + f^2, g^1 + g^2) \circ X_t = (f^1, g^1) \circ X_t + (f^2, g^2) \circ X_t$ for any (f^1, g^1) and (f^2, g^2) in the space $\mathcal{P}(\mathbf{F}) \times \mathcal{O}(\mathbf{F})$; $\triangle^+ X^g$ is $\mathcal{O}(\mathbf{F}_+)$ with its martingale part satisfying $\mathbf{E}[\triangle^+ M_T^g \mathbf{1}_{(T<\infty)} | \mathcal{F}_T] = 0$ a.s. for any stopping time T in the broad sense and $\triangle X^r$ is $\mathcal{O}(\mathbf{F})$ with its martingale part satisfying $\mathbf{E}[\triangle M_T^r \mathbf{1}_{(T<\infty)} | \mathcal{F}_T] = 0$ a.s. for any stopping time T. Moreover, orthogonality is as such that $X^r \perp X^g$ are orthogonal, in the sense that their product is a local optional martingale. Also, differentials are independent: $\triangle Y = f \triangle X^r$ and $\triangle^+ Y = g \triangle^+ X^g$. Lastly, for any semimartingale Z the quadratic projection is $[Y, Z] = f \cdot [X^r, Z^r] + g \odot [X^g, Z^g]$.

Next, we present a proof of UDMD on *un* usual stochastic basis.

3. Uniform Doob–Meyer decomposition

Theorem 3.1: *Let X be an $\mathbb{R}^d$-valued RLL optional semimartingale on the unusual stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), \mathbf{P})$. Let $\mathbb{Q}(X)$ be the set of all probability measures $\mathbf{Q} \sim \mathbf{P}$ such that X is a $\mathbf{Q}$-local optional martingale. Suppose that $\mathbb{Q}(X) \neq \emptyset$ and Y is a non-negative process. Then Y is a $\mathbf{Q}$ local optional supermartingale for each $\mathbf{Q} \in \mathbb{Q}(X)$ if and only if there exist a unique optional process φ integrable with respect to X and an adapted optional increasing process C with $C_0 = 0$ such that $Y = Y_0 + \varphi \circ X - C$ for each $\mathbf{Q} \in \mathbb{Q}(X)$.*

Proof: In the proof, we will be using auxiliary facts formulated as lemmas right after the formal proof. Let $\mathbb{Z}(X)$ be the set of all strictly positive local optional martingales Z with $Z_0 = 1$ such that ZX is a local optional martingale. Corresponding to $\mathbb{Z}(X)$ is $\mathbb{Q}(X)$ the set of all probability measures $\mathbf{Q}$ such that $\mathbf{Q} \sim \mathbf{P}$ and X is a local optional martingale with respect to $\mathbf{Q}$.

Suppose $\mathbb{Z}(X) \neq \emptyset$ and consider the right continuous version $\mathcal{F}_{t+}$ of the filtration $\mathcal{F}_t$ where X_{t+}, Y_{t+}, $Z_{t+}Y_{t+}$ and $Z_{t+}X_{t+}$ are RCLL $\mathcal{F}_{t+}$-measurable processes. ZX is a local optional martingale and ZY is a local optional supermartingale. As well, $Z_{t+}X_{t+}$ is a local martingale and $Z_{t+}Y_{t+}$ a local supermartingale (see Lemma 3.3).

Let $\mathbb{Z}^+(X)$ be the collection of all Z_{t+} RCLL version of the local martingale deflator Z of X. With the collection $\mathbb{Z}^+(X)$, X_{t+} is the RCLL $\mathcal{F}_{t+}$-measurable local martingale and Y_+ is RCLL local supermartingale (see Lemma 3.4). Let us re-label processes and filtrations that are subscripted with '$t+$', with the superscript '$+$' to avoid any possible confusion as to the meaning of the plus sign. So, let $X_t^+ := X_{t+}$, $Y_t^+ := Y_{t+}$, $Z_t^+ := Z_{t+}$ and $\mathcal{F}_t^+ := \mathcal{F}_{t+}$.

Since $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space and $\mathbf{F}_+ = (\mathcal{F}_{t+})$ is right continuous filtration, then by way of Remark 3.5 and Lemma 3.6 the probability space $(\Omega, \mathcal{F}, \mathbf{F}_+, \mathbf{P})$ satisfies the *usual* conditions. Therefore, by UDMD in the *usual* case, we find (φ, C^+), where, in this case, φ is $\mathcal{F}_t$-measurable and C^+ is $\mathcal{F}_{t+}$-measurable increasing process such that

$$Y_t^+ = Y_0 + \varphi \cdot X_t^+ - C_t^+, \tag{1}$$

with $C_0^+ = 0$ for any Z^+ in $\mathbb{Z}^+(X)$. By taking optional projection of Equation (1) on $\mathcal{F}_t$ with the expectation operator '$\mathbf{E}$' we arrive at the result we desire. In what follows, we show the steps in detail. Z^+Y^+

is a local supermartingale for any $Z^+ \in \mathbb{Z}^+(X)$ whereas Z^+X^+ is a local martingale under the right continuous filtration $\mathcal{F}_t^+$ and complete space $(\Omega, \mathcal{F}, \mathbf{P})$. By UDMD in the usual case, we get

$$\begin{aligned} Z^+Y^+ &= Y_0 + \varphi \cdot Z^+X^+ - C^+ \Rightarrow \\ Z^+Y^+ + C^+ &= Y_0 + \varphi \cdot Z^+X^+. \end{aligned} \tag{2}$$

The right-hand side of Equation (2) is a local martingale, therefore the left-hand side is also a local martingale. Taking projections on the *un*usual σ-algebra $\mathcal{F}_t$, we get

$$\mathbf{E}\left[Z_t^+Y_t^+ + C_t^+|\mathcal{F}_t\right] = Y_0 + \mathbf{E}\left[\varphi \cdot \left(Z^+X^+\right)_t|\mathcal{F}_t\right] \tag{3}$$

which, by the argument we will present next, implies

$$Z_tY_t + C_t = Y_0 + \mathbf{E}\left[\varphi \cdot \left(Z^+X^+\right)_t|\mathcal{F}_t\right] \Rightarrow \tag{4}$$

$$Z_tY_t = Y_0 + \varphi \circ (ZX)_t - C_t, \tag{5}$$

is true for any Z, where $C_t = \mathbf{E}[C_t^+|\mathcal{F}_t]$.

Equation (4) leads to Equation (5) in which the usual stochastic integral $\varphi \cdot Z^+X^+$ under projection on $\mathcal{F}_t$ has been transformed to the optional stochastic integral with respect to an RLL process, that is $\mathbf{E}[\varphi \cdot Z^+X^+|\mathcal{F}_t] = \varphi \circ ZX$. To show how we arrived to this result, let $M^+ := Z^+X^+$ and consider the statement

$$\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+|\mathcal{F}_s\right] = \mathbf{E}\left[\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+|\mathcal{F}_s^+\right]|\mathcal{F}_s\right] = 0,$$

since $\mathcal{F}_s^+ \supseteq \mathcal{F}_s$ for all s. Then,

$$\begin{aligned}
\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+|\mathcal{F}_s\right] &= \mathbf{E}\left[\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+|\mathcal{F}_s^+\right]|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{(0,t]} \varphi_{u-}\mathrm{d}M_u^+ - \int_{(0,s]} \varphi_{u-}\mathrm{d}M_u^+|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{(0,s]} \varphi_{u-}\mathrm{d}M_u^+ + \int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^+ - \int_{(0,s]} \varphi_{u-}\mathrm{d}M_u^+|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^+|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\left(\mathrm{d}M_u^{r,+} + \triangle M_u^{g,+}\right)|\mathcal{F}_s\right], \\
&\text{since } M_{u+} = M_{u+}^r + M_{u+}^g, \\
&= \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^{r,+}|\mathcal{F}_s\right] + \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\triangle M_u^{g,+}|\mathcal{F}_s\right].
\end{aligned} \tag{6}$$

But since $M_u^{g,+}$ (i.e. M_{u+}^g) is evolving in the interval $(s,t]$, it follows that M_u^g is evolving in the interval $[s,t)$ and (6) can be written as

$$\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+|\mathcal{F}_s\right] = \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^{r,+}|\mathcal{F}_s\right] + \mathbf{E}\left[\int_{[s,t)} \varphi_v\triangle M_v^{g,+}|\mathcal{F}_s\right].$$

But since $\triangle M_v^{g,+} = M_{v+}^g - M_v^g = \triangle^+ M_v^g$ and $M_u^{r,+} = M_u^r$, then

$$\begin{aligned}
&= \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^r + \int_{[s,t)} \varphi_v\triangle^+ M_v^g|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{(s,t]} \varphi_{u-}\mathrm{d}M_u^r + \int_{[s,t)} \varphi_v\mathrm{d}M_{v+}^g|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{[s,t]} \varphi_u\mathrm{d}M_u|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{[0,t]} \varphi_u\mathrm{d}M_u|\mathcal{F}_s\right] + \mathbf{E}\left[\int_{[0,s]} \varphi_u\mathrm{d}M_u|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\varphi \cdot M_t^+|\mathcal{F}_s\right] - \mathbf{E}\left[\varphi \circ M_s|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \circ M_s|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[\varphi \cdot M_t^+|\mathcal{F}_s\right] - \varphi \circ M_s = 0 \Rightarrow \mathbf{E}\left[\varphi \cdot Z^+X_t^+|\mathcal{F}_s\right] = \varphi \circ ZX_s.
\end{aligned} \tag{7}$$

Alternatively, the same result (7) can be proved by using the definition of the integral by limits of sums. We will demonstrate this result as follows. In the limit of the partitions Π_n

of $[s,t]$ for any s and t and for any $u \in [s,t]$ the integrand φ is the limit of simple functions,

$$\varphi_t \longleftarrow \varphi_s + \sum \varphi_{t_{i-1}} \mathbf{1}_{]t_{i-1},t_{i+1}]}(u).$$

Note that we have chosen the intervals $]t_{i-1}, t_{i+1}]$ where $\varphi_{t_{i-1}} = \varphi_{t_i}$. Since the stochastic integral is well defined under any refinement, we are justified in using this interval.

Consider a partition Π_n of $[s,t]$. The projection of the integral $\int_s^t \phi_{u-} \mathrm{d}M_u^+$ on $\mathcal{F}_s$ is

$$\begin{aligned}
\mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \cdot M_s^+ | \mathcal{F}_s\right] &= \mathbf{E}\left[\int_s^t \varphi_{u-} dM_u^+ | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\lim \sum \varphi_{t_{i-1}} \left(M_{t_{i+1}}^+ - M_{t_{i-1}}^+\right) | \mathcal{F}_s\right], \\
&\text{by localization and integrability,} \\
&= \lim \mathbf{E}\left[\sum \varphi_{t_{i-1}} \left(M_{t_{i+1}}^+ - M_{t_{i-1}}^+\right) | \mathcal{F}_s\right] \\
&= \lim \mathbf{E}\left[\sum \varphi_{t_{i-1}} \left(M_{t_{i+1}}^+ - M_{t_i}^+\right) + \varphi_{t_{i-1}} \left(M_{t_i}^+ - M_{t_{i-1}}^+\right) | \mathcal{F}_s\right], \\
&\text{by } \varphi_{t_{i-1}} = \varphi_{t_i}, \\
&= \lim \mathbf{E}\left[\sum \varphi_{t_i} \left(M_{t_{i+1}}^+ - M_{t_i}^+\right) + \varphi_{t_{i-1}} \left(M_{t_i}^+ - M_{t_{i-1}}^+\right) | \mathcal{F}_s\right].
\end{aligned} \tag{8}$$

Using the properties of conditional expectations and using the fact that $\mathcal{F}_{t_i} \supseteq \mathcal{F}_s$ for any t_i and that φ_t is $\mathcal{F}_t$ measurable, the chain of equalities (8) continues as

$$\begin{aligned}
&= \mathbf{E}\left[\lim \sum \begin{array}{c} \mathbf{E}\left[\varphi_{t_i} M_{t_{i+1}}^+ | \mathcal{F}_{t_{i+1}}\right] - \mathbf{E}\left[\varphi_{t_i} M_{t_i}^+ | \mathcal{F}_{t_i}\right] \\ +\mathbf{E}\left[\varphi_{t_{i-1}} M_{t_i}^+ | \mathcal{F}_{t_i}\right] - \mathbf{E}\left[\varphi_{t_{i-1}} M_{t_{i-1}}^+ | \mathcal{F}_{t_{i-1}}\right] \end{array} | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\lim \sum \begin{array}{c} \varphi_{t_i} \mathbf{E}\left[M_{t_{i+1}}^+ | \mathcal{F}_{t_{i+1}}\right] - \varphi_{t_i} \mathbf{E}\left[M_{t_i}^+ | \mathcal{F}_{t_i}\right] \\ +\varphi_{t_{i-1}} \mathbf{E}\left[M_{t_i}^+ | \mathcal{F}_{t_i}\right] - \varphi_{t_{i-1}} \mathbf{E}\left[M_{t_{i-1}}^+ | \mathcal{F}_{t_{i-1}}\right] \end{array} | \mathcal{F}_s\right], \\
&\text{by } M_u^+ = M_{u+} = M_u + \triangle^+ M_u \\
&= \mathbf{E}\left[\lim \sum \begin{array}{l} \varphi_{t_{i-1}} \left(M_{t_i} - M_{t_{i-1}}\right) + \varphi_{t_i} \left(M_{t_{i+1}} - M_{t_i}\right) \\ +\varphi_{t_{i-1}} \left(\triangle^+ M_{t_i} - \triangle^+ M_{t_{i-1}}\right) + \varphi_{t_i} \left(\triangle^+ M_{t_{i+1}} - \triangle^+ M_{t_i}\right) \end{array} | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\lim \sum \begin{array}{l} \varphi_{t_{i-1}} \left(M_{t_i} - M_{t_{i-1}}\right) + \varphi_{t_i} \left(M_{t_{i+1}} - M_{t_i}\right) \\ +\varphi_{t_{i-1}} \left(\triangle^+ M_{t_i} - \triangle^+ M_{t_{i-1}}\right) + \varphi_{t_i} \left(\triangle^+ M_{t_{i+1}} - \triangle^+ M_{t_i}\right) \end{array} | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{s+}^t \varphi_{u-} \mathrm{d}M_u^r + \int_s^{t-} \varphi_u \mathrm{d}M_{u+}^g | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_{0+}^t \varphi_{u-} \mathrm{d}M_u^r + \int_0^{t-} \varphi_u \mathrm{d}M_{u+}^g - \int_{0+}^s \varphi_{u-} \mathrm{d}M_u^r - \int_0^{s-} \varphi_u \mathrm{d}M_{u+}^g | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\int_0^t \varphi_{u-} \mathrm{d}M_u^+ - \int_{0+}^s \varphi_{u-} \mathrm{d}M_u^r - \int_0^{s-} \varphi_u \mathrm{d}M_{u+}^g | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\varphi \cdot M_t^+ - \varphi_- \cdot M_s^r - \varphi \odot M_{s+}^g | \mathcal{F}_s\right] = \mathbf{E}\left[\varphi \cdot M_t^+ - \varphi \circ M_s | \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\varphi \cdot M_t^+ | \mathcal{F}_s\right] - \varphi \circ M_s = 0.
\end{aligned}$$

Therefore, $\mathbf{E}[\varphi \cdot M_t^+ | \mathcal{F}_s] = \varphi \circ M_s = \varphi \cdot M_s^r + \varphi \odot M_{s+}^g$ for any $s \leq t$. Note that in the limit $\triangle^+ M_{t_i} - \triangle^+ M_{t_{i-1}} = \triangle^+ M_{t_{i+1}} - \triangle^+ M_{t_i} = 0$.

Having carried optional projection on the right-hand side of Equation (2), we consider next the left-hand side, $Z^+Y^+ + C^+$. Let $N_t^+ := Z_{t+}Y_{t+} + C_t^+ = Z_t^+ Y_t^+ + C_t^+$ which is an $\mathbf{F}_+$ RCLL local martingale. Using optional projection of N^+ on $\mathbf{F}$, we find the local optional martingale $N_u := \mathbf{E}[N_t^+ | \mathcal{F}_u]$ for all $u \leq t$ by Lemma 3.2 or simply by equality (3).

The process Z^+Y^+ is a local supermartingale for which the projection on $\mathbf{F}$ gives $\mathbf{E}[Z_{t+}Y_{t+} | \mathcal{F}_t] \leq Z_t Y_t$. Since $N_t = \mathbf{E}[N_t^+ | \mathcal{F}_t]$, we have

$$N_{t+} = \mathbf{E}\left[N_t^+ | \mathcal{F}_{t+}\right] = \mathbf{E}\left[Z_{t+}Y_{t+} + C_t^+ | \mathcal{F}_{t+}\right] = Z_{t+}Y_{t+} + C_t^+$$

and $N_{t+} = N_t^+$ is RCLL. Consequently, $N_t = Z_t Y_t + C_{t-}^+$ where C_{t-}^+ is optional $\mathcal{F}_t$ adapted increasing process.

Alternatively, one can consider the difference $N_{t+} - Z_{t+}Y_{t+}$, which is an increasing process. Hence, $N_t - Z_t Y_t$ is a local optional submartingale, i.e. $N_t - Z_t Y_t = \mathbf{E}[N_{t+} - Z_{t+}Y_{t+} | \mathcal{F}_t]$ having the decomposition, $N_t - Z_t Y_t = \tilde{N}_t + A_t$, where A is an increasing optional process and $\tilde{N}_t$ is a local optional martingale. Therefore, $N_t - \tilde{N}_t = Z_t Y_t + A_t$ and $N_{t+} - \tilde{N}_{t+} = Z_{t+}Y_{t+} + A_{t+}$ which imply that $C_t^+ = \tilde{N}_{t+} + A_{t+}$. This leads us to conclude that since C_t^+ is increasing, then it must be that $\tilde{N}_{t+} = 0$ and $\tilde{N}_t = \mathbf{E}[\tilde{N}_{t+} | \mathcal{F}_t] = 0$ for all t. ■

3.1. Supporting lemmas

Lemma 3.2: *Suppose that N_{t+} is an $\mathbf{F}_+$ local martingale, then N_t is an $\mathbf{F}$ local optional martingale.*

Proof: Let $N_t^+ := N_{t+}$. For any $\mathbf{F}_+$ stopping time $\tau \geq t$, $\mathbf{E}[N_\tau^+ | \mathcal{F}_t^+] = N_t^+$, since N_{t+} is an $\mathbf{F}_+$ local martingale. Then, N_t is an $\mathbf{F}$ local optional martingale, for that

$$N_t := \mathbf{E}\left[N_\tau^+ | \mathcal{F}_t\right] \Rightarrow$$
$$\mathbf{E}\left[N_t | \mathcal{F}_s\right] = \mathbf{E}\left[\mathbf{E}\left[N_\tau^+ | \mathcal{F}_t\right] | \mathcal{F}_s\right] = \mathbf{E}\left[N_\tau^+ | \mathcal{F}_s\right] = N_s$$

for any $s \leq t$. ■

Lemma 3.3: *Given that $Z_t X_t$ is a local optional martingale and $Z_t Y_t$ is a local optional supermartingale on $\mathbf{F}$, then $Z_{t+}X_{t+}$ is a local martingale and $Z_{t+}Y_{t+}$ a local supermartingale on $\mathbf{F}_+$.*

Proof: First we show that, if ZY is a local optional supermartingale under $\mathbf{F}$, then Z^+Y^+ is a local supermartingale under $\mathbf{F}_+$, that is $\mathbf{E}[Z_t^+ Y_t^+ | \mathcal{F}_s^+] \leq Z_s^+ Y_s^+$ for any $s \leq t$.

Let $S_t = Z_t Y_t$, then $\mathbf{E}[S_t | \mathcal{F}_s] \leq S_s$ and $\mathbf{E}[S_{t+\epsilon} | \mathcal{F}_{s+\delta}] \leq S_{s+\delta}$ for any $s + \delta \leq t + \epsilon$ and $\delta > 0$ and $\epsilon > 0$. Then,

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \mathbf{E}\left[S_{t+\epsilon} | \mathcal{F}_{s+\delta}\right] \leq \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} S_{s+\delta} \Rightarrow$$
$$\lim_{\delta \downarrow 0} \mathbf{E}\left[S_{t+} | \mathcal{F}_{s+\delta}\right] \leq S_{s+} \Rightarrow \mathbf{E}\left[S_{t+} | \mathcal{F}_{s+}\right] \leq S_{s+} \Rightarrow$$

$$\mathbf{E}\left[S_t^+|\mathcal{F}_s^+\right] \leq S_s^+ \Rightarrow \mathbf{E}\left[Z_t^+ Y_t^+|\mathcal{F}_s^+\right] \leq Z_s^+ Y_s^+.$$

Similarly, ZX is a local optional martingale under $\mathbf{F}$, then Z^+X^+ is a local martingale under $\mathbf{F}_+$. ■

Corollary 3.4: *Let the collection $\mathbb{Z}^+(X)$ be of all RCLL Z_{t+} versions of the local optional martingale deflators Z of X. With the collection $\mathbb{Z}^+(X)$, X_{t+} is RCLL local martingale and Y_+ is RCLL local supermartingale.*

Proof: By way of the above lemma. ■

Remark 3.5: Under the *un*usual conditions, the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is complete – $\mathcal{F}$ is a complete σ-algebra and $\mathbf{P}$ is a complete probability measure. Hence, the expectation operator $\mathbf{E}$ is also complete as a measure. However, under the *un*usual conditions, the filtrations $\mathcal{F}_t$ and $\mathcal{F}_{t+}$ are not complete. In the proof of UDMD theorem, we have used the optional projection of stochastic process on $\mathcal{F}_t$ and $\mathcal{F}_{t+}$ where in certain cases the conditioning of an optional process on an *un*usual filtration leads to subsets of null sets that are not measurable in neither $\mathcal{F}_t$ nor $\mathcal{F}_{t+}$. But since $\mathbf{P}$ has been completed these sets pose no problem as their probability values are zero, hence their expectation.

Lemma 3.6: *A complete measure completes an incomplete sub-σ-algebra.*

Proof: Given a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$ that is not complete, then, for any zero measure set $A \in \mathcal{F}$, i.e. $\mathbf{E}[\mathbf{1}_A] = 0$, the expectation $\mathbf{E}[\mathbf{1}_A|\mathcal{G}] = \mathbf{E}[\mathbf{E}[\mathbf{1}_A|\mathcal{F}]|\mathcal{G}] = 0$ a.s. $\mathbf{P}$. Moreover, for any subset B of C such that $\mathbf{E}[\mathbf{1}_C|\mathcal{G}] = 0$, the expectation $\mathbf{E}[\mathbf{1}_B|\mathcal{G}] = \mathbf{E}[\mathbf{E}[\mathbf{1}_B|\mathcal{F}]|\mathcal{G}] = 0$. Therefore, the measure $\mathbf{E}_{\mathcal{G}}[\cdot] := \mathbf{E}[\cdot|\mathcal{G}]$ maps subsets of zero measure sets in $\mathcal{G}$ to zero. Let

$$\mathcal{N}_{\mathcal{G}} := \{B : \mathbf{E}_{\mathcal{G}}[B] = 0\}.$$

If we augment $\mathcal{G}$ with $\mathcal{N}_{\mathcal{G}}$, we complete $\mathcal{G}$ and obtain a complete measurable space $(\Omega, \mathcal{G} \vee \mathcal{N}_{\mathcal{G}}, \mathbf{E}_{\mathcal{G}})$. ■

3.2. Submartingale decomposition

Stricker and Yan [38] proved UDMD of optional submartingale under the usual conditions. An extension to the *un*usual case is straightforward by employing the same methods we have developed in the proof Theorem 3.1. Below a version of UDMD for local optional submartingales under the *un*usual conditions is presented.

Theorem 3.7: *Let X be $\mathbb{R}^d$-valued RLL optional semimartingale on the unusual stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Let $\mathbb{Q}(X)$ be the set of all probability measures $\mathbf{Q} \sim \mathbf{P}$ such that X is a $\mathbf{Q}$ local optional martingale. Suppose $\mathbb{Q}(X) \neq \emptyset$ and let U be a nonnegative process. Then U admits the decomposition $U = U_0 + \psi \circ X + A$, where ψ is X-integrable optional process and A is an adapted increasing process with $A_0 = 0$, if and only if U is a local optional submartingale for each $\mathbf{Q} \in \mathbb{Q}(X)$.*

Proof: By way of the procedure, we have developed in Theorem 3.1 and submartingale UDMD in the usual conditions proved in [38], the result of the theorem is established. ■

As an application of the UDMD, we will look at the filtering of optional semimartingales. Filtering theory has a vast literature investigating different aspects of the filtering problem and its applications, see [6, 10, 25, 27, 28, 31, 32, 39, 42] .

4. Filtering of optional supermartingales by optional decomposition

Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a partially observable stochastic process where Y is the unobservable component and X is the observed process. The filtering problem is understood as constructing for each $t \geq 0$ an optimal estimate of Y_t by observations of X_t.

On the *usual* stochastic basis, Khadjiev [25] was the first to study the filtering of semimartingales, Y , by observations of a point process, X, by means of dual projection and martingale representation theorems. Using the same approach, Vetrov [39] (see also [32]) investigated the filtering problem when both X and Y are semimartingales under some conditions. The most important among the conditions is that the probability measure $\mathbf{P}^X$ restricted to the filtration generated by X (i.e. $\mathbf{F}^X$) is unique. It turns out that the uniqueness of $\mathbf{P}^X$ is equivalent to the filtration $\mathbf{F}^X$ being right continuous and complete (see Chapter 4, Section 10 of [32]).

Although right continuity of filtration is a desirable property, in stochastic analysis there are many examples of stochastic processes whose associated filtrations are not right continuous, see [19, p. 24]. Here we are going to extend filtering theory to the spaces of optional processes on *un*usual probability spaces by the use of UDMD. In doing so, we are able to relax the assumptions of right continuity of the filtration $\mathbf{F}^X$ and the requirement that $\mathbf{P}^X$ is unique.

We are going to do this in two ways. In the first way, we are going to start anew, with an *un*usual probability space and develop a theory for the filtering of optional local supermartingales using UDMD. In the second approach, we will begin with the usual filtering theory of special semimartingale but allow for the filtration $\mathbf{F}^X$ not to be right continuous and employ the optional projection on *un*usual stochastic basis in conjunction with UDMD to develop a filtering theory for optional local supermartingale. We begin with the first method then develop the second one.

4.1. The first approach: filtering on unusual stochastic basis

Let X be an $\mathbb{R}^d$ valued RLL optional semimartingale on the *un*usual stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Let Y be an $\mathbb{R}^d$ valued RLL optional semimartingale on the same space. The processes X and Y are related; $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a partially observable stochastic process where Y is the unobservable component and X is the observed process. Let $\mathbf{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ be the filtration generated by X which is not necessarily right continuous nor complete, obviously $\mathbf{F}^X \subseteq \mathbf{F}$. However, $\mathcal{F}^X := \mathcal{F}_\infty^X \vee \mathcal{N}$ is complete and $\mathbf{P}^X$ is the restriction of the measure $\mathbf{P}$ on $\mathcal{F}^X$. Assume that X remains an optional semimartingale under $(\Omega, \mathcal{F}^X, \mathbf{F}^X, \mathbf{P}^X)$.

Lemma 4.1: *For the partially observable system of optional semimartingales $(X_t, Y_t)_{t\geq 0}$, let $ZX \in \mathcal{M}_{1oc}(\mathbf{F}^X, \mathbf{P}^X)$ be a local optional martingale for any strictly positive local optional* *martingale Z with $Z_0 = 1$, $Z \in \mathbb{Z}(X)$. There are, corresponding to any Z the optional local martingale measures $\mathbf{Q} \in \mathbb{Q}(X)$ such that $\mathbf{Q}_t \sim \mathbf{P}^X_t \ll \mathbf{P}^X_{t+}$ for all $t \geq 0$ where $\mathbf{P}^X_t$ is the restriction of $\mathbf{P}^X$ on $\mathcal{F}^X_t$ and $\mathbf{P}^X_{t+}$ is the restriction of $\mathbf{P}^X$ on $\mathcal{F}^X_{t+}$. If ZY is a local optional supermartingale for every $Z \in \mathbb{Z}(X)$, then the optimal estimate of Y given $\mathcal{F}^X_t$ is the optional projection $\pi_t(Y) := \mathbf{E}[Z_t Y_t | \mathcal{F}^X_t]$ satisfying the following stochastic equation:*

$$\pi_t(Y) = \pi_0(Y) + \phi \circ X - C, \quad \forall\, \mathbf{Q}\ a.s.,$$

where ϕ is $\mathcal{P}(\mathbf{F}^X) \times \mathcal{O}(\mathbf{F}^X)$ is unique X integrable process and C is an optional process. ϕ is given by

$$\begin{aligned}\phi_t = {}& \mathbf{E}_\mathbf{Q}\left[\frac{\mathrm{d}\,\langle Y^c, X^c\rangle_t}{\mathrm{d}\,\langle X^c, X^c\rangle_t} | \mathcal{F}^X_{t-}\right] + \mathbf{E}_\mathbf{Q}\left[\frac{\triangle Y^d_t - \triangle C^d_t}{\triangle X^d_t} | \mathcal{F}^X_{t-}\right] \\ &+ \mathbf{E}_\mathbf{Q}\left[\frac{\triangle^+ Y^g_t - \triangle^+ C^g_t}{\triangle^+ X^g_t} | \mathcal{F}^X_t\right]\end{aligned}$$

and $-C_t = \pi_t(Y) - \pi_0(Y) - \phi \circ X$. Moreover, given that $Y = Y_0 + M - A$ then $C_t = \mathbf{E}_\mathbf{Q}[A_t | \mathcal{F}^X_t]$ and $\mathbf{E}_\mathbf{Q}[M_t | \mathcal{F}^X_t] = \phi \circ X_t$.

Proof: Let $\mathbb{Z}(X)$ be the set of all strictly positive local optional martingales Z with $Z_0 = 1$ such that ZX is a local optional martingale, i.e. ZX belongs to $\mathcal{M}_{1oc}(\mathbf{F}^X, \mathbf{P}^X)$. Denote by $\mathbb{Q}(X)$ the set of all probability measures $\mathbf{Q}$ such that $\mathbf{Q} \ll \mathbf{P}^X_+$ and $\mathbf{Q} \sim \mathbf{P}^X$ where X is a local optional martingale with respect to $\mathbf{Q}$. Assume $\mathbb{Q}(X) \neq \emptyset$ or that $\mathbb{Z}(X)$ is not empty. We assume that the unobserved process Y is an RLL local supermartingale with respect to any $\mathbf{Q} \in \mathbb{Q}(X)$. Consequently, the optional project of Y on $\mathcal{F}^X$ under $\mathbf{Q}$ is $\pi_t(Y) := \mathbf{E}_\mathbf{Q}[Y_t | \mathcal{F}^X_t]$ is a local optional supermartingale that is $\mathcal{F}^X_t$ measurable. By UDMD of Theorem 3.1, we arrive at

$$\pi_t(Y) = \pi_0(Y) + \phi \circ X - C.$$

Let us identify $\hat{Y} := \pi_t(Y)$. Using the definition of the quadratic variation of optional processes, we get

$$\begin{aligned}\left[\hat{Y}, X\right] &= \phi_t \circ [X, X]_t - [C, X]_t\,, \\ &\Rightarrow \phi_t \circ [X, X]_t = \left[\hat{Y} - C, X\right].\end{aligned}$$

Consider each part of the quadratic variation $[\hat{Y}, X]$, continuous, discrete right continuous and discrete left continuous independently:

$$\begin{aligned}\left\langle \hat{Y}^c, X^c \right\rangle_t &= \phi^c \cdot \langle X^c, X^c\rangle_t\,, \quad \text{since } \langle C^c, X^c\rangle = 0, \\ \left[\hat{Y}^d, X^d\right] &= \phi^d \cdot \left[X^d, X^d\right]_t - \left[C^d, X^d\right]_t\,, \\ \left[\hat{Y}^g, X^g\right] &= \phi^g \circ \left[X^g, X^g\right]_t - \left[C^g, X^g\right]_t\,.\end{aligned}$$

Therefore,

$$\phi_t^c = \frac{\mathrm{d}\left\langle \hat{Y}^c, X^c \right\rangle_t}{\mathrm{d}\left\langle X^c, X^c \right\rangle_t}, \quad \phi_t^d = \frac{\mathrm{d}\left[\hat{Y}^d - C^d, X^d\right]_t}{\mathrm{d}\left[X^d, X^d\right]_t},$$

$$\phi_t^g = \frac{\mathrm{d}\left[\hat{Y}^g - C^g, X^g\right]_t}{\mathrm{d}\left[X^g, X^g\right]_t} \Rightarrow$$

$$\phi_t^c = \mathbf{E_Q}\left[\frac{\mathrm{d}\left\langle Y^c, X^c \right\rangle_t}{\mathrm{d}\left\langle X^c, X^c \right\rangle_t} | \mathcal{F}_{t-}^X\right], \quad \phi_t^d = \mathbf{E_Q}\left[\frac{\triangle Y_t^d - \triangle C_t^d}{\triangle X_t^d} | \mathcal{F}_{t-}^X\right],$$

$$\phi_t^g = \mathbf{E_Q}\left[\frac{\triangle^+ Y_t^g - \triangle^+ C_t^g}{\triangle^+ X_t^g} | \mathcal{F}_t^X\right]. \tag{9}$$

Since $Y = Y_0 + M - A$ is $\mathbf{Q}$ local optional supermartingale, then it must be that $\mathbf{E_Q}[M_t|\mathcal{F}_t^X] = \phi \circ X_t$ and $\mathbf{E_Q}[A_t|\mathcal{F}_t^X] = C_t$. ■

Remark 4.2: The filtering equation developed in the above theorem assumes that the dominant measure $\mathbf{P}^X$ is known or that its restriction $\mathbf{P}_{t+}^X$ to $\mathcal{F}_{t+}^X$ can be constructed from observations of X_t. However, the measure $\mathbf{P}$ is not necessarily known even though we have assumed it is defined and complete. A consequence of this is that the equations we have developed here are different from the ones developed by Vectrov [39], Lipster and Shiryaev [32] where they have assumed that $\mathbf{P}$ is known.

Remark 4.3: In Theorem 4.1, we have made the stronger assumption that ZY is local optional supermartingale for every $Z \in \mathbb{Z}(X)$. However, we only needed $\pi_t(Y) := \mathbf{E}[Z_t Y_t|\mathcal{F}_t^X]$ to be a local optional supermartingale.

4.2. The second approach: filtering on mixed stochastic basis

Consider the usual probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Let X and Y be $\mathbb{R}^d$ valued RCLL semimartingales but Y is special. The processes X and Y are related; $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a partially observable stochastic process where Y is the unobservable component and X is the observed process. X is a semimartingle, i.e. $X \in \mathcal{S}(\mathbf{F}, \mathbf{P})$. Then $X \in \mathcal{S}(\mathbf{F}_+^X, \mathbf{P}^X)$ where $\mathbf{P}^X$ is the restriction of the measure $\mathbf{P}$ to the σ-algebra $\mathcal{F}_\infty^X = \sigma(\cup_{t\geq 0}\mathcal{F}_t^X)$ and $\mathbf{F}_+^X = (\mathcal{F}_{t+}^X)_{t\geq 0}$ the family of σ-algebras $\mathcal{F}_{t+}^X = \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}^X$ where $\mathcal{F}_t^X = \sigma\{X_s, s \leq t\} \vee \mathcal{N}$. If $\tilde{\mathbf{P}}^X$ is any probability measure on $(\Omega, \mathcal{F}_\infty^X)$ such that $\tilde{\mathbf{P}}^X \ll \mathbf{P}^X$, then $X \in \mathcal{S}(\mathbf{F}_+^X, \tilde{\mathbf{P}}^X)$. If there is more than one probability measure $\{\tilde{\mathbf{P}}^X\}$, then $\mathbf{F}^X$ is not right continuous. Moreover, X is a different semimartingale under the different measure $\tilde{\mathbf{P}}^X$.

At this point, we are going to construct the probability space $(\Omega, \mathcal{F}^X, \mathbf{F}^X, \mathbf{P}^X)$ which is *un*usual. That is $\mathbf{F}^X$ may not be right continuous or complete. Let $\tilde{\mathbb{P}}^X = \{\tilde{\mathbf{P}}^X : \tilde{\mathbf{P}}^X \ll \mathbf{P}^X\}$, then $X \in \mathcal{S}(\mathbf{F}_+^X, \tilde{\mathbf{P}}^X)$ is an RCLL semimartingale and the projection of X on $\mathbf{F}^X$ is an optional semimartingale in $\mathcal{S}(\mathbf{F}^X, \tilde{\mathbf{P}}^X)$. Also, for every $\tilde{\mathbf{P}}^X$ there exist a Doob–Meyer decomposition of the optional semimartingale X.

Consider that a subset of $\tilde{\mathbb{P}}^X$ exist, such that $\tilde{\mathbf{P}}^X > 0$ and under which X is a local optional martingale. It is the set of equivalent optional local martingale measures,

$$\mathbb{Q}^X := \left\{ \mathbf{Q}^X : \mathbf{Q}^X \sim \tilde{\mathbf{P}}^X \right\} \subseteq \tilde{\mathbb{P}}^X$$

or

$$\mathbb{Q}^X := \left\{ \begin{array}{c} \mathbf{Q}^X : \mathbf{Q}^X > 0,\ \mathbf{Q}^X \ll \mathbf{P}^X, \\ \text{and } X \text{ is local optional martingale under } \mathbf{Q}^X \end{array} \right\}.$$

Furthermore, assume that the projection of Y on $\mathbf{F}^X$, $\pi_t(Y) := \mathbf{E}_\mathbf{Q}[Y_t|\mathcal{F}_t^X]$, is a local optional supermartingale that is $\mathcal{F}_t^X$-measurable. Hence, by optional UDMD we get

$$\pi_t(Y) = \pi_0(Y) + \phi \circ X - C.$$

This brings us to our next corollary.

Corollary 4.4: *Given the above exposition, the filtering equation of $\pi_t(Y)$ is*

$$\pi_t(Y) = \pi_0(Y) + \phi \circ X - C, \quad \forall \mathbf{Q}\ a.s.,$$

where ϕ is $\mathcal{P}(\mathbf{F}^X) \times \mathcal{O}(\mathbf{F}^X)$ is unique, X integrable process and C an optional process. ϕ is given by

$$\begin{aligned} \phi_t = {} & \mathbf{E}_\mathbf{Q}\left[\frac{\mathrm{d}\left\langle Y^c, X^c\right\rangle_t}{\mathrm{d}\left\langle X^c, X^c\right\rangle_t}|\mathcal{F}_{t-}^X\right] + \mathbf{E}_\mathbf{Q}\left[\frac{\triangle Y_t^d - \triangle C_t^d}{\triangle X_t^d}|\mathcal{F}_{t-}^X\right] \\ & + \mathbf{E}_\mathbf{Q}\left[\frac{\triangle^+ Y_t^g - \triangle^+ C_t^g}{\triangle^+ X_t^g}|\mathcal{F}_t^X\right] \end{aligned} \tag{10}$$

and $-C_t = \pi_t(Y) - \pi_0(Y) - \phi \circ X$. Since $Y = Y_0 + M - A$ is a $\mathbf{P}$ semimartingale, then $C_t = \mathbf{E}_\mathbf{Q}[A_t|\mathcal{F}_t^X]$ and $\mathbf{E}_\mathbf{Q}[M_t|\mathcal{F}_t^X] = \phi \circ X_t$.

Proof: Similar to Lemma 4.1. ∎

Remark 4.5: In Lemma 4.1 and Corollary 4.4, we have assumed that in determining the integrand ϕ of the observation process X, that C is known. However, Equations (9) and (10) can be thought of as equations relating the processes ϕ with C, X and Y . It is important to note that in Foellmer and Kabanov [18] the integrator in optional decomposition was determined by an optimization procedure where it is identified as the optimal Lagrange multiplier. As a consequence of determining the optimal Lagrange multiplier, the optional decreasing process was determined. In Lemma 4.1 and Corollary 4.4, C can be determined as a consequence of the optimization procedure.

5. Applications to mathematical finance

Filtering theory has a long history of applications to finance, economics and engineering [6–9, 12, 37, 41]. However, some of the most fundamental connections between filtering and finance are in studying the problems of insider trading [11, 33, 40], hedging and pricing under incomplete information and asymmetric information existing between market agents [5, 13, 15, 24, 34].

Here we study a particular model of financial markets based on the calculus of optional semimartingales on a *mixed probability spaces*, i.e. usual and *un*usual, where some elements of this market may not be observed. In other words, we are going to study an optional semimartingale market under incomplete information. Examples of these markets are: the pricing and hedging of derivatives of non-tradable assets that are however correlated with tradable ones, and derivatives of assets that are illiquid such as over the counter traded assets or private funds and dark pools [1, 4, 16, 23, 36]. In any one of these cases, the *market-value* of the underlying assets is not observed directly or can't be hedged with.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, $t \in [0, \infty)$, be the *usual* stochastic basis and that the financial market stays on this space. Let $Y_t = Y_0\mathcal{E}_t(y)$ be the unobserved asset where the stochastic exponential $\mathcal{E}_t(y)$ is given by

$$\mathcal{E}_t(y) = \exp\left\{y_t - y_0 - \frac{1}{2}\left\langle y^c, y^c\right\rangle\right\} \prod_{0<s\leq t} \left[\left(1 + \triangle y_s\right) \mathrm{e}^{-\triangle y_s}\right]$$

and $X_t = X_0\mathcal{E}_t(x)$ be the observed asset that is *related to* Y. Both X and Y are positive for all $t \geq 0$. The initial values X_0 and Y_0 are $\mathcal{F}_0$-measurable random variables. $x = (x_t)_{t\geq 0}$ and $y = (y_t)_{t\geq 0}$ are RCLL semimartingales admitting the representation, $x_t = x_0 + a_t + m_t$ and $y_t = y_0 + b_t + n_t$ with respect to $\mathbf{P}$. $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$ are locally finite variation processes. $m = (m_t)_{t\geq 0}$ and $n = (n_t)_{t\geq 0}$ are local martingales.

Remark 5.1: Under the usual conditions, the stochastic exponential technique in the context of semimartingale markets was developed in Melnikov et al. [35]. This technique was extended to *un*usual spaces by Abdelghani and Melnikov [2].

A financial derivative based on Y is a function of Y. Let h be positive and continuous function and consider the financial derivative $H_t = h(Y_t)$. Suppose Y is unknown, however, the process X, which is related to Y, is observed. We are going to consider the pricing rule given by the projection of $h(Y_t)$ on our knowledge, $\mathcal{F}_t^X$, under the risk neutral measure $\mathbf{Q}$,

$$\pi_t(H) := \mathbf{E}_\mathbf{Q}\left[H_t|\mathcal{F}_t^X\right].$$

Note that the filtration $\mathbf{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ may not be right continuous. Furthermore, the measure $\mathbf{P}$ may not be accessible. In this case, we have to construct an *un*usual probability space $(\Omega, \mathcal{F}^X, \mathbf{F}^X, \mathbf{P}^X)$. In accordance with the second approach of Section 4.2, we suppose the set of risk neutral measures

$$\mathbb{Q}^X := \left\{\begin{matrix} \mathbf{Q} : \mathbf{Q} > 0,\ \mathbf{Q} \ll \mathbf{P}^X, \\ \text{and } X \text{ is local optional martingale under } \mathbf{Q}\end{matrix}\right\}$$

is not empty. It was shown in [2] that one can find a non-empty set of positive local (optional) martingale measures, respectively, for the stochastic exponential process $X_t = X_0\mathcal{E}_t(x)$ under some conditions on x. Therefore, by Corollary 4.4 we can price and hedge

H by

$$\pi_t(H) = \pi_0(H) + \phi \circ X - C, \quad \forall\, \mathbf{Q}\; a.s.,$$

where ϕ is given by

$$\phi_t = \mathbf{E_Q}\left[\frac{\mathrm{d}\,\langle H^c, X^c\rangle_t}{\mathrm{d}\,\langle X^c, X^c\rangle_t}|\mathcal{F}^X_{t-}\right] + \mathbf{E_Q}\left[\frac{\triangle H^d_t - \triangle C^d_t}{\triangle X^d_t}|\mathcal{F}^X_{t-}\right]$$
$$+ \mathbf{E_Q}\left[\frac{\triangle^+ H^g_t - \triangle^+ C^g_t}{\triangle^+ X^g_t}|\mathcal{F}^X_t\right].$$

Remark 5.2: In a market of derivatives of non-tradable assets, such as temperature derivatives, it is possible to measure temperature but its market value is unknown. Temperature market value can be estimated by observations through proxies – other tradable assets that are correlated with temperature such as the price of gas. Even though temperature is an observed variable its market value is not. Therefore, temperature derivatives can only be evaluated by optional projection on observed related assets.

Remark 5.3: In OTC markets, the traded value of the asset is the observed market price which may or may not contain information about the value of the asset traded over the counter. Therefore, a contingent claim written on an illiquid asset is subject to asymmetric information where the writer could have access to information in the OTC market while the buyer does not, resulting in illiquidity premium.

Example 5.4: Suppose the unobserved asset is $Y_t = Y_0\mathcal{E}(\mu t + \sigma \tilde{W}_t)$, where μ and σ are constants, evolving on the **usual** probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbf{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}, \mathbf{P})$. On the other hand, the observed asset follows the stochastic equation

$$X_t = X_0\mathcal{E}\left(\alpha t + \eta W_t + \gamma L_t\right),$$

where α, η and γ are constants. $L_t = N_{t-} - \lambda t$ is compensated left continuous Poisson process $(N_t)_{t\geq 0}$ which is $\mathcal{F}_{t+}$-measurable with constant intensity λ. Also, $\triangle^+ L > -\frac{1}{\gamma}$ since $X > 0$. X is evolving on the **un**usual but **observable** probability space $(\Omega, \mathcal{F}^X, \mathbf{F}^X = (\mathcal{F}^X_t)_{t\geq 0}, \mathbf{P})$, where $\mathcal{F}^X_t := \sigma(X_s, s \leq t)$ is not right continuous as a result of X dependence on L. X and Y are correlated by the Wiener processes W and $\tilde{W}$ with $\langle W, \tilde{W}\rangle = qt$, where q is the covariation constant, and by the time trends αt and μt. The overall probability space $(\Omega, \mathcal{G}, \mathbf{G} = (\mathcal{G}_t)_{t\geq 0}, \mathbf{P})$ generated by $(\tilde{W}, W, N)$ may be constructed in the **usual** way. Even though the overall probability space maybe a usual probability space the observed space is unusual.To compute a local martingale deflator for X it makes sense to start with the guess $Z_t = \mathcal{E}(aW_t + bL_t)$ an optional positive local martingale such that ZX is an optional local martingale:

$$\begin{aligned} ZX &= X_0\mathcal{E}\left(aW_t + bL_t\right)\mathcal{E}\left(\alpha t + \eta W_t + \gamma L_t\right) \\ &= X_0\mathcal{E}\left(aW_t + bL_t + \alpha t + \eta W_t + \gamma L_t + a\eta t + b\gamma L - \lambda b\gamma t\right) \\ &= X_0\mathcal{E}\left(\left(\alpha + a\eta - \lambda b\gamma\right)t + \left(a + \eta\right)W_t + \left(b + \gamma + b\gamma\right)L_t\right). \end{aligned}$$

Hence, for ZX to be a local optional martingale we must have $\alpha + a\eta - \lambda\gamma b = 0$ or that $(\eta, -\lambda\gamma)(a, b)^\top = -\alpha$ which results in many solutions. Let $H = Y_T$ be a contingent claim (i.e. a future), at time T that we like to price and hedge knowing $\mathcal{F}_t^X$. To do so we must first find a Z such that ZY is a local optional supermartingale:

$$\begin{aligned} ZY &= Y_0\mathcal{E}\left(\mu t + \sigma \tilde{W}_t\right)\mathcal{E}\left(aW_t + bL_t\right) \\ &= Y_0\mathcal{E}\left(\left(\mu + \sigma qa\right)t + \sigma \tilde{W}_t + aW_t + bL_t\right) \\ &= Y_0\mathcal{E}\left(\left(\mu + \sigma qa\right)t\right)\mathcal{E}\left(\sigma \tilde{W}_t + aW_t + bL_t\right) \\ &= Y_0 \exp\left(\left(\mu + \sigma qa\right)t\right)\mathcal{E}\left(\sigma \tilde{W}_t + aW_t + bL_t\right), \end{aligned}$$

where $\mathcal{E}(\sigma \tilde{W}_t + aW_t + bL_t)$ is a local optional martingale and also a supermartingale by Fatou lemma, i.e. $\mathbf{E}[Z_{t\wedge\tau} Y_{t\wedge\tau}] \leq \mathbf{E}[Y_0 \exp((\mu + \sigma qa)(t \wedge \tau))]$ for any stopping time τ. Because $Z > 0$ and $ZY > 0$, then it must be that $\mu + \sigma qa \geq 0$ and $\triangle^+ L > -\frac{1}{b}$. But since $\triangle^+ L > -\frac{1}{\gamma}$ then we must at least choose $b \leq \gamma$. Putting all these conditions together we arrive at the set of equations

$$(\eta, -\lambda\gamma)\,(a, b)^\top = -\alpha, \quad a \geq -\frac{\mu}{\sigma q},\ b \leq \gamma.$$

The pricing formula is

$$\begin{aligned} &\mathbf{E}\left[Z_T Y_T|\mathcal{F}_t^X\right] \\ &\quad = Y_0 \exp\left(\left(\mu + \sigma qa\right)T\right)\mathbf{E}\left[\mathcal{E}\left(\sigma \tilde{W}_T + aW_T + bL_T\right)|\mathcal{F}_t^X\right]. \end{aligned}$$

Considering the sup over all Z,

$$\begin{aligned} &\sup_{Z\in\mathbb{Z}(X)} \mathbf{E}\left[Z_T Y_T|\mathcal{F}_t^X\right] \\ &\quad = Y_0 \sup_{(a,b)}\left\{\exp\left(\left(\mu + \sigma qa\right)T\right)\mathbf{E}\left[\mathcal{E}\left(\sigma \tilde{W}_T + aW_T + bL_T\right)|\mathcal{F}_t^X\right]\right\}, \end{aligned}$$

$$\text{such that } (\eta, -\lambda\gamma)\,(a, b)^\top = -\alpha, \quad a \geq -\frac{\mu}{\sigma}, \quad b \leq \gamma.$$

Using a geometric view of the equations above, one arrives at the solution $b = \gamma$ and $a = \eta^{-1}(\lambda\gamma^2 - \alpha)$ that gives us the maximum supermartingale

$$\begin{aligned} &\sup_{Z\in\mathbb{Z}(X)} \mathbf{E}\left[Z_T Y_T|\mathcal{F}_t^X\right] = Y_0 \exp\left(\left(\mu + \eta^{-1}\sigma q\left(\lambda\gamma^2 - \alpha\right)\right)T\right) \times \\ &\mathbf{E}\left[\mathcal{E}\left(\sigma \tilde{W}_T + \eta^{-1}\left(\lambda\gamma^2 - \alpha\right)W_T + \gamma L_T\right)|\mathcal{F}_t^X\right] \\ &\quad = Y_0 \exp\left(\left(\mu + \eta^{-1}\sigma q\left(\lambda\gamma^2 - \alpha\right)\right)T\right) \times \\ &\mathcal{E}\left(\left(\sigma\sqrt{q} + \eta^{-1}\left(\lambda\gamma^2 - \alpha\right)\right)W_t + \gamma L_t\right). \end{aligned}$$

Consequently, the maximizing local optional martingale is $\tilde{Z}_t = \mathcal{E}(\eta^{-1}(\lambda\gamma^2 - \alpha)W_t + \gamma L_t)$ for which we write $\tilde{H}_t = \mathbf{E}[\tilde{Z}_T H|\mathcal{F}_t^X]$ as $\hat{H}_t = \hat{H}_0 + \phi \circ X_t - C_t$ by UDMD under

$\tilde{\mathbf{Q}} = \tilde{Z} \circ \mathbf{P}$. In our particular market framework, we have that the supremum value of the contingent claim is

$$\sup_{Z \in \mathbb{Z}(X)} \mathbf{E}\left[Z_T Y_T | \mathcal{F}_t^X\right] = \phi \circ \tilde{Z}_t X_t - C_t \quad \text{or that}$$

$$\begin{aligned} \hat{H}_t &= \mathbf{E}\left[\tilde{Z}_T Y_T | \mathcal{F}_t^X\right] = \mathbf{E}\left[\tilde{Z}_T \mathbf{E}\left[Y_T | \mathcal{F}_T^X\right] | \mathcal{F}_t^X\right] \\ &= \mathbf{E}\left[\tilde{Z}_T \pi_T(Y) | \mathcal{F}_t^X\right] = \mathbf{E}\left[\tilde{Z}_T \hat{Y}_T | \mathcal{F}_t^X\right] \\ &= \hat{H}_0 + \phi \circ \tilde{Z}_t X_t - C_t, \end{aligned}$$

where ϕ is chosen as the minimum value such that $\hat{H}_t - \phi \circ \tilde{Z}_t X_t \leq \hat{H}_0$ for all t, thereby the process C is the maximum difference.

6. Concluding remarks

The UDMD for optional supermartingales on *un*usual probability spaces is a natural and important generalization of the classical Doob–Meyer decomposition and optional decomposition. We expect that this decomposition will play the same role played by optional decomposition in cadlag markets, but for financial markets evolving on *un*usual spaces (for example, see [2]). Clearly, we have demonstrated how such a decomposition can be used to build optimal filters in filtering of optional supermartingale, further generalizing the theory of optimal filtering of semimartingales.

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References

[1] M.N. Abdelghani and A.V. Melnikov, *On macrohedging problem in semimartingale markets*, Front. Appl. Math. Stat. 1 (2015), pp. 3. ISSN 2297-4687. doi:10.3389/fams.2015.00003. URL http://journal.frontiersin.org/article/10.3389/fams.2015.00003.

[2] M.N. Abdelghani and A.V. Melnikov, *Financial markets in the context of the general theory of optional processes. Mathematical and Computational Approaches in Advancing Modern Science and Engineering*, 2016, pp. 519–528. Springer.

[3] M.N. Abdelghani and A.V. Melnikov, *On linear stochastic equations of optional semimartingales and their applications*, Stat. Probab. Lett. 125 (2017), pp. 207–214.

[4] A. Ang, A.A. Shtauber and P.C. Tetlock, *Asset pricing in the dark: The cross-section of otc stocks*, Rev. Financ. Stud. 26(12) (2013), pp. 2985–3028.

[5] T. Berrada, *Incomplete information, heterogeneity, and asset pricing*, J. Financ. Econ. 4(1) (2006), pp. 136–160.

[6] R. Bhar, Stochastic Filtering with Applications in Finance, World Scientific, Singapore, 2010.

[7] A.V. Borisov, B.M. Miller and K.V. Semenikhin, *Filtering of the Markov jump process given the observations of multivariate point process*, Autom. Remote Control 76(2) (2015 Feb), pp. 219–240. ISSN 1608-3032. doi:10.1134/S0005117915020034. URL https://doi.org/10.1134/S0005117915020034.
[8] D. Brigo and B. Hanzon, *On some filtering problems arising in mathematical finance*, Insur. Math Econ. 22(1) (1998), pp. 53–64.
[9] P. Christoffersen, C. Dorion, K. Jacobs and L. Karoui, *Nonlinear Kalman filtering in affine term structure models*, Manage. Sci. 60(9) (2014), pp. 2248–2268.
[10] S. Cohen and R.J. Elliott, Stochastic Calculus and Applications, Birkhäuser, New York, NY, 2015.
[11] A. Danilova, *Stock market insider trading in continuous time with imperfect dynamic information*, Stochast. Int. J. Probab. Stochast. Proc. 82(1) (2010), pp. 111–131.
[12] P. Date and K. Ponomareva, *Linear and non-linear filtering in mathematical finance: A review*, IMA J. Manage. Math. 22(3) (2011), pp. 195–211.
[13] De Meyer B., *Price dynamics on a stock market with asymmetric information*, Games Econ. Behav. 69(1) (2010), pp. 42–71.
[14] C. Dellacherie, *Sem. Probabilites XI Lecture Notes in Math.*, Vol. 581, Chapter Deux remarques sur la separabilite optionelle, pp. 47–50. Springer-Verlag, 1975.
[15] J.B. Detemple, *Asset pricing in a production economy with incomplete information*, J. Finance 41(2) (1986), pp. 383–391.
[16] D. Duffie, Dark Markets: Asset Pricing and Information Transmission in Over-the-Counter Markets, Princeton University Press, Princeton, NJ, 2012.
[17] N. ElKaroui and M.C. Quenez, *Dynamic programming and pricing of contingent claims in an incomplete market*, SIAM J. Control Optim. 33(1) (1995), pp. 27–66.
[18] H. Foellmer and Y.M. Kabanov, *Optional decomposition and lagrange multipliers*, Financ Stochast 2(1) (1997), pp. 69–81.
[19] T.R. Fleming and D.P. Harrington, Counting Processes and Survival Analysis, Vol. 169. John Wiley & Sons, Hoboken, NJ, 2011.
[20] L.I. Galchuk, *On the existence of optional modifications for martingales*, Theory Prob. Appl. 22 (1977), pp. 572–573.
[21] L.I. Galchuk, *Decomposition of optional supermartingales*, Math. USSR Sbornik 43(2) (1982), pp. 145–158.
[22] L.I. Galchuk, *Stochastic integrals with respect to optional semimartingales and random measures*, Theory Probab. Appl. XXIX 1 (1985), pp. 93–108.
[23] H. Geman and M. Leonardi, *Alternative approaches to weather derivatives pricing*, Manage. Finance 31(6) (2005), pp. 46–72.
[24] X. Guo, R.A. Jarrow and Y. Zeng, *Credit risk models with incomplete information*, Math. Oper. Res. 34(2) (2009), pp. 320–332.
[25] D.I. Hadjiev, *On the filtering of semimartingales in case of observations of point processes*, Theory Probab. Appl. 23(1) (1978), pp. 169–178.
[26] S. Jacka, *A simple proof of Kramkov's result on uniform supermartingale decompositions*, Stochast. Int. J. Probab. Stoch. Proc. 84(5–6) (2012), pp. 599–602.
[27] A.H. Jazwinski, Stochastic Processes and Filtering Theory, Courier Corporation, Mineola, NY, 2007.
[28] G. Kallianpur, Stochastic Filtering Theory, Vol. 13. Springer Science & Business Media, New York City, NY, 2013.
[29] I. Karatzas and C. Kardaras, *Optional decomposition for continuous semimartingales under arbitrary filtrations*, Electron. Commun. Probab. 20 (2015), pp. 59–69.
[30] D.O. Kramkov, *Optional decomposition of supermartingales and hedging in incomplete security markets*, Probab. Theory Related Fields 105 (1996), pp. 459–479.
[31] H. Kushner, *Nonlinear filtering: The exact dynamical equations satisfied by the conditional mode*, IEEE Trans. Automat. Control 12(3) (1967), pp. 262–267.
[32] R.S. Lipster and A.N. Shiryaev, Theory of Martingales, Kluwer, Dordrecht, 1986.
[33] S. Luo and Q. Zhang, *Dynamic insider trading*, AMS IP Stud. Adv. Math. 26 (2002), pp. 93–104.

[34] R.C. Merton, *A simple model of capital market equilibrium with incomplete information*, J. Finance 42(3) (1987), pp. 483–510.
[35] A.V. Melnikov, S.N. Volkov and M.L. Nechaev, Mathematics of Financial Obligations, Vol. 194. American Mathematical Society, Providence, RI, 2002.
[36] H. Mi and S. Zhang, *Dynamic valuation of options on non-traded assets and trading strategies*, J. Sys. Sci. Complex. 26(6) (2013), pp. 991–1001.
[37] W.J. Runggaldier, *Estimation via stochastic filtering in financial market models*, Contemp. Math. 351 (2004), pp. 309–318.
[38] C. Stricker and J.A. Yan, *Some remarks on the optional decomposition theorem*. In Seminaire de Probabilites XXXII, 1998, pp. 56–66.
[39] L.G. Vetrov, *On linearization of stochastic differential equations of optimal nonlinear filtering*, Theory Probab. Appl. 25(2) (1981), pp. 393–401.
[40] P. Vitale, *Insider trading in sequential auction markets with risk-aversion and time-discounting*, Eur. J. Finance 23(2016), pp. 1–13.
[41] C. Wells, The Kalman Filter in Finance, Vol. 32. Springer Science & Business Media, Berlin, 2013.
[42] M. Zakai, *On the optimal filtering of diffusion processes*, Zeitschr Wahrscheinl Verwandt Gebiet 11(3) (1969), pp. 230–243.