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On comparison theorem for optional SDEs via local times and applications*

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ABSTRACT
In this paper, we study SDEs with respect to optional semimartingales or optional SDEs. Our leading idea is to explore the concept and technique of local time of optional processes to extend several results on comparison and pathwise uniqueness of solutions of such stochastic equations. We also obtain a comparison result for optional stochastic equations with different jump-diffusions. Moreover, we apply our comparison theorem to calculate option price bounds in mathematical finance. Our findings are supported by numerical examples.

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Comparison theorem; local time; laglad processes; non-Lipschitz condition

1. Introduction and motivation

Optional semimartingales are *laglad* processes defined on *un*usual probability space – a complete probability space such that the underlying filtration is not necessarily left nor right continuous nor complete, Dellacherie [9]. The study of optional semimartingales was advanced by many mathematicians Lepingle [24], Horowitz [19], Lenglart [23], and Gal'chuk [10,11,13].

A growing amount of literature in Mathematical Finance shows a necessity for the development of the theory of optional semimartingales. A comprehensive exposition of optional processes and their different applications can be found in a recent book by Abdelghani and Melnikov [4].

This paper is devoted to the study of *comparison* of solutions of stochastic equations of optional semimartingales on *un*usual probability space, and the study of *pathwise uniqueness* of these solutions using local times.

A comparison theorem for stochastic equations with respect to continuous semimartingales was proved by Melnikov [25] who developed the Yamada method [28]. Later a similar result was given by Yan [29] using the local time technique. The case of SDEs with integer-valued random measures where the coefficients are not Lipschitz but satisfy weaker conditions similar to those of Yamada were considered by Gal'chuk [12]. Interesting applications of path-wise comparison theorem to mathematical finance were given in [20] and

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further developed in [21]. Recently, a comparison theorem for optional semimartingales on *un*usual probability space was given in [2] when coefficients satisfy the Yamada conditions. Therefore, our goal here is to study comparison of optional SDEs on *un*usual stochastic basis under a more general condition, *local time condition*, placed on the diffusion coefficient. Besides, we extend a version of the comparison theorem for solutions of SDEs with different diffusion coefficients (see [14,26]) to the laglad jump-diffusion case.

Even though the stochastic calculus of optional semimartingales is well developed, little is done in showing pathwise uniqueness of solutions of stochastic equations driven by optional semimartingales on *un*usual probability spaces, except, for the works of Gasparyan [16] and Abdelghani and Melnikov [3] on the existence and uniqueness of strong solutions under Lipschitz conditions and monotonicity conditions, respectively. On the other hand, Perkins [27] proved the pathwise uniqueness of solutions of stochastic equations of continuous semimartingales using the local time technique. As a result, we consider the questions of pathwise uniqueness under one-sided Lipschitz continuity on a drift function and local time condition on the diffusion coefficient for *laglad* optional semimartingale using the method of local time, which was not done before.

Besides a purely theoretical interest, the topic is motivated by the needs of the energy market. In many electricity markets, retailers buy electricity at an unregulated price and sell it to consumers at a regulated price. Therefore, the occurrence of price spikes due to sudden changes in electricity demand or supply in these markets represents a major source of risk to retailers. Hence, accurate modelling of price spikes is important. As a result, we have modelled spikes in spot price in a way so that each upward jump is accompanied by an immediate downward jump. The flexibility, modelling capacity, and accuracy of laglad processes cannot be achieved by using cadlag processes, because they are right-continuous and, consequently, cannot have immediate downward jumps. Moreover, even if we use a sequence of right jumps, it is hard to control times at which downward jumps happen after an upward jump, and, thus, even if we tried to model 'almost' immediate downward jumps after upward jumps for cadlag processes, we would not succeed.

We calibrated a widely used mean-reverting jump-diffusion model for electricity spot prices [7] and our proposed optional model on data obtained from Alberta Electric System Operator during the period 1 January 2000–30 November 2019. After that, we used a standard model check – model fit (MSE) and comparison of model-based statistical moments of simulated log returns to the empirical ones.

Although the standard deviation of the distribution simulated by the jump-diffusion model is slightly closer to the empirical one, its skewness and kurtosis are significantly overshot (see Table 1). The advantage of the optional model is most prominent when we look at kurtosis. In fact, the jump-diffusion model-based kurtosis is more than two times greater than the optional model-based one, indicating an excessive number of jumps. Overall, the optional model has a lower MSE and matched the first four moments much better than the jump-diffusion model, thus, supporting our theoretical arguments.

Table 1. Statistics of log returns.

Moment	Mean	Standard deviation	Skewness	Kurtosis	MSE
Empirical	0.0005	0.5103	−0.0294	7.6285	
Jump-diffusion model	0.0000	0.5294	−1.9880	19.3170	1.9572
Optional model	0.0000	0.5590	−0.4515	8.9404	1.8482

The rest of the paper consists of the following sections. Section 2 provides a brief introduction to the theory of optional processes and sets up the main SDE and associated assumptions required for subsequent discussion. In Section 3, we prove the comparison theorem. Further, in Section 4, we demonstrate a comparison result for stochastic equations with different jump-diffusion coefficient functions. In Section 5, we give an illustrative example of an application of comparison theorems to finance by computing bounds for the price of an option where the underlying asset is an optional semimartingale. Finally, in Section 6, a result on pathwise uniqueness is proved.

2. Preliminaries

Below we provide a short list of preliminaries from the optional stochastic analysis.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ which is not complete, right- or left-continuous. We introduce $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$ as the optional and predictable σ-algebras on $\Omega \times [0, \infty)$, respectively. $\mathcal{O}(\mathbf{F})$ (resp., $\mathcal{O}(\mathbf{F}_+)$, where $\mathbf{F}_+ = (\mathcal{F}_{t+})_{t\geq 0}$ and $\mathcal{F}_{t+} = \bigcap_{t<s} \mathcal{F}_s$) is generated by all $\mathbf{F}$ (resp., $\mathbf{F}_+$)-adapted processes whose trajectories are right-continuous and have left limits. $\mathcal{P} = \mathcal{P}(\mathbf{F}) = \mathcal{P}(\mathbf{F}_+)$ is generated by all $\mathbf{F}$-adapted processes whose trajectories are left-continuous and have right limits. A process X is called optional (resp., predictable) if $(\omega, t) \mapsto X_t(\omega)$ is $\mathcal{O}$- (resp., $\mathcal{P}$-)measurable. For either optional or predictable processes we can define the following processes: $X_- = (X_{t-})_{t\geq 0}$ and $X_+ = (X_{t+})_{t\geq 0}$, $\Delta X = (\Delta X_t)_{t\geq 0}$ such that $\Delta X_t = X_t - X_{t-}$, and $\Delta^+ X = (\Delta^+ X_t)_{t\geq 0}$ such that $\Delta^+ X_t = X_{t+} - X_t$.

Consider the Lusin space $(E, \mathcal{E})$ where $E = \mathbb{R}^d \setminus \{0\} \cup \{\delta\}$; δ is some supplementary point; $\mathcal{E} = \mathcal{B}(E)$ is σ-algebra of Borel subsets. Define $\tilde{\mathcal{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathcal{E}$, $\tilde{\mathcal{O}}(\mathbf{F}) = \mathcal{O}(\mathbf{F}) \times \mathcal{E}$, $\tilde{\mathcal{O}}(\mathbf{F}_+) = \mathcal{O}(\mathbf{F}_+) \times \mathcal{E}$, $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{E}$.

$\mathcal{A}_{\text{loc}}$ is the set of all processes of locally integrable variation, $\mathcal{V}$ is the set of all processes of finite variation, $\mathcal{M}_{\text{loc}}$ ($\mathcal{M}^2_{\text{loc}}$) is the set of all optional (square integrable) local martingales. Furthermore, a process Y is called an optional semimartingale if $Y = A + M$, where $A \in \mathcal{V}$, $M \in \mathcal{M}_{\text{loc}}$, $M_0 = 0$.

On unusual stochastic basis three canonical types of stopping times exist: predictable stopping times, S, are such that $\{S \leq t\}$ is $\mathcal{F}_{t-}$-measurable for all t; totally inaccessible stopping times, T, are such that $\{T \leq t\}$ is $\mathcal{F}_t$-measurable for all t, however, we note that $\{T < t\}$ is not necessarily $\mathcal{F}_t$-measurable since $\mathcal{F}_t$ is not right-continuous; finally, totally inaccessible wide sense stopping times, U, are such that $\{U \leq t\}$ is $\mathcal{F}_{t+}$-measurable for all t, but since $\mathcal{F}_{t+}$ is right continuous, $\{U < t\}$ is also $\mathcal{F}_{t+}$-measurable.

Let Y be a one-dimensional optional semimartingale and $(S_n)_{n\geqslant 1}$, $(T_n)_{n\geqslant 1}$, $(U_n)_{n\geqslant 1}$ be sequences of predictable, totally inaccessible stopping times and totally inaccessible wide sense stopping times, respectively, exhausting all jumps of the process Y, i.e. the set $\{\Delta Y \neq 0\} \cup \{\Delta^+ Y \neq 0\}$, such that the graphs of these stopping times do not intersect within each sequence. Define integer random measures on $(\mathbb{R}_+ \times E, \tilde{\mathcal{E}})$

$$\mu^d(\Gamma) = \sum_{n\geqslant 1} \mathbf{1}_\Gamma(T_n, \beta^d_{T_n}), \quad \mu^g(\Gamma) = \sum_{n\geqslant 1} \mathbf{1}_\Gamma(U_n, \beta^g_{U_n}),$$

$$p^d(\Gamma) = \sum_{n\geqslant 1} \mathbf{1}_\Gamma(S_n, \beta^d_{S_n}), \quad p^g(\Gamma) = \sum_{n\geqslant 1} \mathbf{1}_\Gamma(S_n, \beta^g_{S_n}),$$

$$\eta(\Gamma) = \sum_{n \geqslant 1} \mathbf{1}_\Gamma(T_n, \beta^g_{T_n}),$$

where $\mathbf{1}_\Gamma(\cdot)$ is an indicator function of a set $\Gamma \in \tilde{\mathcal{E}}$, $\beta^d_t = \Delta Y_t$ if $\Delta Y_t \neq 0$ and $\beta^d_t = \delta$ if $\Delta Y_t = 0$, $\beta^g_t = \Delta^+ Y_t$ if $\Delta^+ Y_t \neq 0, \beta^g_t = \delta$ if $\Delta^+ Y_t = 0$, $t > 0$.

Under the *un*usual conditions on probability space Gasparyan [15, Theorem 1] showed that Y can be decomposed as follows:

$$\begin{aligned} Y_t = Y_0 + a_t + m_t &+ \int_{]0,t]\times E} u\mathbf{1}_{(|u|\leq 1)}\, \mathrm{d}(\mu^d - \nu^d) + \int_{[0,t[\times E} u\mathbf{1}_{(|u|\leq 1)}\, \mathrm{d}(\mu^g - \nu^g) \\ &+ \int_{]0,t]\times E} u\mathbf{1}_{(|u|>1)}\, \mathrm{d}\mu^d + \int_{[0,t[\times E} u\mathbf{1}_{(|u|>1)}\, \mathrm{d}\mu^g \\ &+ \int_{]0,t]\times E} u\, \mathrm{d}p^d + \int_{[0,t[\times E} u\, \mathrm{d}p^g + \int_{[0,t[\times E} u\, \mathrm{d}\eta \end{aligned}$$

or in short notation

$$Y = Y_0 + a + m + \sum_{j=d,g} \left[u\mathbf{1}_{(|u|\leq 1)} * (\mu^j - \nu^j) + u\mathbf{1}_{(|u|>1)} * \mu^j + u * p^j \right] + u * \eta, \quad (1)$$

where Y_0 is $\mathcal{F}_0$-measurable random variable, $a \in \mathcal{A}_{\mathrm{loc}}, a_0 = 0$, and $m \in \mathcal{M}^2_{\mathrm{loc}}, m_0 = 0$, are both continuous; ν^j are the compensators of μ^j. Whenever we mention index j in this paper, $j = d, g$. Hereafter, we refer a reader to the book [4, Section 2.4, Chapter 5, Chapter 7], for additional explanation of notation and definitions from the theory of optional semimartingales.

Consider the following SDE:

$$\begin{aligned} X = X_0 + f(X)\cdot a + g(X)\cdot m &+ \sum_{j=d,g} [Uh_j(X) * (\mu^j - \nu^j) + (k_j + l_j)(X) * p^j] \\ &+ (r + w)(X) * \eta, \end{aligned} \quad (2)$$

where X_0 is $\mathcal{F}_0$-measurable random variable; $u \mapsto U(u) = \mathbf{1}_{(|u|\leq 1)}$; $a \in \mathcal{A}_{\mathrm{loc}}$ is continuous increasing, $m, \mu^j, \nu^j, p^j, \eta$ as in (1). Further, for convenience we use the notation $f(X) = f(\omega, t, X_{t-}), g(X) = g(\omega, t, X_{t-}), h_d(X) = h_d(\omega, t, u, X_{t-}), h_g(X) = h_g(\omega, t, u, X_t)$ and similarly for k_j, l_j, w and r whenever this does not lead to confusion.

To guarantee the well-posedness of the integrals in (2), we make the following assumptions.

Assumption 2.1:
- For $j = d, g$, $|f(X)|\cdot a \in \mathcal{A}_{\mathrm{loc}}$, $(g(X))^2 \cdot \langle m\rangle \in \mathcal{A}_{\mathrm{loc}}$, $|h(X)|^2 * \nu^j \in \mathcal{A}_{\mathrm{loc}}$, $|l_j(X)| * p^j \in \mathcal{A}_{\mathrm{loc}}$, $[|k_j(X)|^2 * p^j]^{1/2} \in \mathcal{A}_{\mathrm{loc}}$, $|r(X)| * \eta \in \mathcal{A}_{\mathrm{loc}}$, $[|w(X)|^2 * \eta]^{1/2} \in \mathcal{A}_{\mathrm{loc}}$, and $E[k_r(S, \beta^d_S, X_0)|\mathcal{F}_{S-}] = 0$ a.s. for any predictable stopping time S on $\{S < \infty\}$ and $E[k_g(T, \beta^g_T, X_0)|\mathcal{F}_T] = 0$, $E[w(T, \beta^g_T, X_0)|\mathcal{F}_T] = 0$ a.s. for any stopping time T on $\{T < \infty\}$.
- $f(\omega, s, x)$ and $g(\omega, s, x)$ are defined on $(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ and are $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable;

- $h_d(\omega,s,u,x), k_d(\omega,s,u,x),\ l_d(\omega,s,u,x)$ are defined on $(\Omega\times\mathbb{R}_+\times E\times\mathbb{R})$ and $\tilde{\mathcal{P}}\times\mathcal{B}(\mathbb{R})$-measurable;
- $h_g(\omega,s,u,x), k_g(\omega,s,u,x),\ l_g(\omega,s,u,x),\ r(\omega,s,u,x),\ w(\omega,s,u,x)$ are defined on $(\Omega\times\mathbb{R}_+\times E\times\mathbb{R})$ and $\tilde{\mathcal{O}}\times\mathcal{B}(\mathbb{R})$-measurable.

For convenience, we state here sufficient conditions for the existence and the uniqueness of the strong solution of (2).

Definition 2.1: Let Assumptions 2.1 hold. We say that the functions f,g,h_j,k_j,l_j,r,w in (2) satisfy the $\mathbf{L(Y,X_0)}$ **conditions** if:

(L1) there exists non-negative functions $F,G,H^j,L^j,K^j,R,W,\ j=d,g$, such that
  (a) (a)$F(\omega,s),\ \ G(\omega,s)$ are $\mathcal{P}$-measurable; $H^d(\omega,s,u)$ is $\mathcal{P}\times\mathcal{B}(E\cap(|u|\le 1))$-measurable; $H^g(\omega,s,u)$ is $\mathcal{O}\times\mathcal{B}(E\cap(|u|\le 1))$-measurable; $L^d(\omega,s,u),K^d(\omega,s,u)$ are $\tilde{\mathcal{P}}$-measurable; $L^g(\omega,s,u),K^g(\omega,s,u),R(\omega,s,u),W(\omega,s,u)$ are $\tilde{\mathcal{O}}$-measurable.
  (b) (b)$F\cdot a_t+G\cdot\langle m\rangle_t+\sum_{j=d,g}[H^jU*\nu^j+(K^j+L^j)*\lambda^j]+(R+W)*\zeta<\infty$ a.s. for any $t>0$, where λ^j and ζ are compensators of p^j and η, respectively.
  (c) (c)For any $x,y\in\mathbb{R},\ s\in\mathbb{R}_+$ and $j=d,g$,

$$
\begin{aligned}
&|f(x)-f(y)|\cdot a_t\le(F|x-y|)\cdot a_t\\
&(g(x)-g(y))^2\cdot\langle m\rangle_t\le(G|x-y|^2)\cdot\langle m\rangle_t\\
&|h_j(x)-h_j(y)|*\nu^j_t\le(H^j|x-y|)*\nu^j_t,\\
&|l_j(x)-l_j(y)|*\lambda^j_t\le(L^j|x-y|)*\lambda^j_t,\\
&|k_j(x)-k_j(y)|^2*\lambda^j_t\le(K^j|x-y|^2)*\lambda^j_t,\\
&|r(x)-r(y)|*\zeta_t\le(R|x-y|)*\zeta_t,\\
&|w(x)-w(y)|^2*\zeta_t\le(W|x-y|^2)*\zeta_t,
\end{aligned}
$$

(L2)

$$
\begin{aligned}
&(g(X_0))^2\cdot\langle m\rangle_t+f(X_0)\cdot a_t+[r(X_0)+(w(X_0))^2]*\zeta\\
&\quad+\sum_{j=d,g}[(h_j(X_0))^2U*\nu^j+[(k_j(X_0))^2+l_j(X_0)]*\lambda^j]<\infty
\end{aligned}
$$

a.s. for any $t>0$.

Theorem 2.2 (see [16, Theorem 1], [17, Theorem 3.3.1]): *Let Y be an optional semimartingale and suppose that f,g,h_j,k_j,l_j,r,w satisfy the* $\mathbf{L(Y,X_0)}$ *conditions. Then the strong solution of* (2) *exists and is unique.*

Remark 2.1: Results in this paper can be easily generalized to Equation (2) with an additional term $\mathbf{1}_{(|u|>1)}h'_j*\mu^j$ due to its simple structure (see [2, Lemma 3.2]).

Next, we discuss the notion of a local time for an optional semimartingale which was first introduced in [23, VI.3.4]. This concept is crucial for our proof of comparison of solutions and pathwise uniqueness. A local time at a of an optional semimartingale X is denoted by $L_t^a(X)$ and given by

$$L_t^a(X) = |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sign}(X_{s-} - a)X_s$$
$$- \sum_{0<s\le t} [|X_s - a| - |X_{s-} - a| - \operatorname{sign}(X_{s-} - a)\Delta X_s]$$
$$- \sum_{0\le s<t} [|X_{s+} - a| - |X_s - a| - \operatorname{sign}(X_s - a)\Delta^+ X_s],$$

where $\operatorname{sign}(x) = 1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x \le 0$.

Definition 2.3 (cf. [6, Definition 1], [5, Section 2]): We say that a coefficient function g of equation (2) satisfies the **LT** condition, if for any two solutions X^1 and X^2 of equation (2), the local time at level 0 satisfies,

$$\forall t \ge 0 \quad L_t^0(X^1 - X^2) = 0. \tag{3}$$

Finally, we state formula of occupation density. Let X be an optional semimartingale with a local time $(L^a)_{a\in\mathbb{R}}$ and let g be a bounded Borel measurable function. Then a.s.

$$\int_{-\infty}^{\infty} L_t^a g(a)\,\mathrm{d}a = \int_0^t g(X_{s-})\,\mathrm{d}\langle X^c\rangle_s.$$

3. Comparison theorem in terms of local times

Let us investigate comparison of solutions of stochastic differential equations driven by optional semimartingales. In this section, we consider two processes given by SDE's of the same type as Equation (2):

$$X^i = X_0^i + f^i(X^i)\cdot a + g(X^i)\cdot m + \sum_{j=d,g} [Uh_j(X^i) * (\mu^j - \nu^j)$$
$$+ (k_j^i + l_j^i)(X^i) * p^j] + (r^i + w^i)(X^i) * \eta, \quad i = 1, 2. \tag{4}$$

We are going to present a general version of a Comparison Theorem with **LT** condition on g and the following conditions on functions $f^i, h_j, k_j^i, l_j^i, r^i, w^i, \imath = 1, 2$:

D Conditions. Suppose that

(D1) $X_0^2 \ge X_0^1$;
(D2) $f^2(s,x) \ge f^1(s,x)$ for any $s \in \mathbb{R}_+,\ x \in \mathbb{R}$;
(D3) For any $s \in \mathbb{R}_+,\ u \in E,\ x, y \in \mathbb{R}, y \ge x$

$$y + h_j(s,u,y) \ge x + h_j(s,u,x),$$
$$y + k_j^2(s,u,y) + l_j^2(s,u,y) \ge x + k_j^2(s,u,x) + l_j^2(s,u,x),$$
$$y + r^2(s,u,y) + w^2(s,u,y) \ge x + r^2(s,u,x) + w^2(s,u,x);$$

(D4) For any $s \in \mathbb{R}_+,\ u \in E,\ x \in \mathbb{R}$

$$k_j^2(s,u,x) + l_j^2(s,u,x) \geq k_j^1(s,u,x) + l_j^1(s,u,x),$$
$$r^2(s,u,x) + w^2(s,u,x) \geq r^1(s,u,x) + w^1(s,u,x);$$

Theorem 3.1: *Suppose that $f^i, g, h_j, k_j^i, l_j^i, r^i, w^i$ in (4) satisfy $\mathbf{L}(\mathbf{Y}, \mathbf{X_0^i})$, $i = 1, 2$, and conditions **D** and **LT** hold. Then there exist unique strong solutions X^1 and X^2, and $X_t^1 \leq X_t^2$ for all $t \in \mathbb{R}_+$ a.s. ($X^1 \leq X^2$).*

Proof: Let $Y := X^1 - X^2$ and

$$I_1 := \mathbf{1}_{(Y_->0)}(f^1(X^1) - f^2(X^2)) \cdot a_t,$$
$$I_2 := \mathbf{1}_{(Y_->0)}(g(X^1) - g(X^2)) \cdot m_t + [(Y_- + h_d(X^1) - h_d(X^2))^+ - Y_-^+]U * (\mu^d - \nu^d)_t$$
$$+ [(Y + h_g(X^1) - h_g(X^2))^+ - Y^+]U * (\mu^g - \nu^g)_t,$$
$$I_3 := [(Y_- + h_d(X^1) - h_d(X^2))^+ - Y_-^+ - (h_d(X^1) - h_d(X^2))\mathbf{1}_{(Y_->0)}]U * \nu_t^d$$
$$+ [(Y + h_g(X^1) - h_g(X^2))^+ - Y^+ - (h_g(X^1) - h_g(X^2))\mathbf{1}_{(Y>0)}]U * \nu_t^g,$$
$$I_4 := [(Y_- + k_d^1(X^1) + l_d^1(X^1) - k_d^2(X^2) - l_d^2(X^2))^+ - Y_-^+] * p_t^d$$
$$+ [(Y + k_g^1(X^1) + l_g^1(X^1) - k_g^2(X^2) - l_g^2(X^2))^+ - Y^+] * p_t^g,$$
$$I_5 := [(Y + r^1(X^1) + w^1(X^1) - r^2(X^2) - w^2(X^2))^+ - Y^+] * \eta_t.$$

By Theorem A.1, Y^+ is expressed in the following form:

$$Y_t^+ = Y_0^+ + \frac{1}{2}L_t^0(Y) + I_1 + I_2 + I_3 + I_4 + I_5. \tag{5}$$

After using (D1) and **LT** conditions, equation (5) becomes

$$Y_t^+ = I_1 + I_2 + I_3 + I_4 + I_5.$$

Next, we examine each of $I_1 - I_5$ separately. By (D2) and (L1) conditions,

$$I_1 = \mathbf{1}_{(Y_->0)}[(f^1(X^1) - f^2(X^1)) + (f^2(X^1) - f^2(X^2))] \cdot a_t$$
$$\leq FY_-^+ \cdot a_t.$$

Now, consider I_3

$$I_3 = [(Y_- + h_d(X^1) - h_d(X^2))^+ - (Y_- + h_d(X^1) - h_d(X^2))\mathbf{1}_{(Y_->0)}]U * \nu_t^d$$
$$+ [(Y + h_g(X^1) - h_g(X^2))^+ - (Y + h_g(X^1) - h_g(X^2))\mathbf{1}_{(Y>0)}]U * \nu_t^g.$$

Further, applying the identity $I = I^+ - I^-$ to the terms $Y_- + h_d(X^1) - h_d(X^2)$ and $Y + h_g(X^1) - h_g(X^2)$, we get

$$I_3 = [(Y_- + h_d(X^1) - h_d(X^2))^-\mathbf{1}_{(X_-^1>X_-^2)} + (Y_- + h_d(X^1) - h_d(X^2))^+\mathbf{1}_{(X_-^1\leq X_-^2)}]U * \nu_t^d$$
$$+ [(Y + h_g(X^1) - h_g(X^2))^-\mathbf{1}_{(X^1>X^2)} + (Y + h_g(X^1) - h_g(X^2))^+\mathbf{1}_{(X^1\leq X^2)}]U * \nu_t^g.$$

It follows from (D3), that $I_3 = 0$.

Using (D4), we get

$$\begin{aligned}
I_5 &= \mathbf{1}_{(Y>0)}[r^2(X^1)+w^2(X^1)-r^2(X^2)-w^2(X^2))]*\eta_t \\
&\quad + [(Y+r^1(X^1)+w^1(X^1)-r^2(X^1)-w^2(X^1)+r^2(X^1)+w^2(X^1)-r^2(X^2) \\
&\quad - w^2(X^2))^+ - Y^+ - \mathbf{1}_{(Y>0)}(r^2(X^1)+w^2(X^1)-r^2(X^2)-w^2(X^2)))]*\eta_t \\
&\le \mathbf{1}_{(Y>0)}[r^2(X^1)+w^2(X^1)-r^2(X^2)-w^2(X^2)]*\eta_t \\
&\quad + [(Y+r^2(X^1)+w^2(X^1)-r^2(X^2)-w^2(X^2))^+ \\
&\quad - \mathbf{1}_{(Y>0)}(Y+r^2(X^1)+w^2(X^1)-r^2(X^2)-w^2(X^2)))]*\eta_t.
\end{aligned}$$

Due to (D3) and (L1) conditions

$$I_5 \le RY^+ * \eta_t + \mathbf{1}_{(Y>0)}[w^2(X^1)-w^2(X^2)]*\eta_t.$$

Repeating the same calculations for I_4, we obtain that

$$\begin{aligned}
I_4 &\le L^d Y_-^+ * p^d + L^g Y^+ * p^g + \mathbf{1}_{(Y_->0)}[k_d^2(X^1)-k_d^2(X^2)]*p_t^d \\
&\quad + \mathbf{1}_{(Y>0)}[k_g^2(X^1)-k_g^2(X^2)]*p_t^g.
\end{aligned}$$

By combining all the estimates for $I_1,\ I_3 - I_5$ with (5), we have

$$Y_t^+ \le M_t + Y^+ \circ C_t,$$

where $\circ$ means an optional stochastic integral (see [4, Section 7.1]), $C := F\cdot a_t + W * \eta_t + L^d * p^d + L^g * p^g$ is a non-negative increasing process, and

$$\begin{aligned}
M_t :&= I_2 + \mathbf{1}_{(Y>0)}[w^2(X^1)-w^2(X^2)]*\eta_t \\
&\quad + \mathbf{1}_{(Y_->0)}[k_d^2(X^1)-k_d^2(X^2)]*p_t^d + \mathbf{1}_{(Y>0)}[k_g^2(X^1)-k_g^2(X^2)]*p_t^g, \quad M_0 = 0.
\end{aligned}$$

Using Assumptions 2.1, we have

$$\begin{aligned}
&\mathbf{1}_{(Y_->0)}(g(X^1)-g(X^2))^2\cdot\langle m\rangle_t \le 2(g(X^1))^2\cdot\langle m\rangle_t + 2(g(X^2))^2\cdot\langle m\rangle_t \in \mathcal{A}_{\mathrm{loc}}, \\
&[(Y_- + h_d(X^1)-h_d(X^2))^+ - Y_-^+]^2 U * \nu_t^d \le 2[h_d(X^1)]^2 * \nu_t^d + 2[h_d(X^2)]]^2 * \nu_t^d \in \mathcal{A}_{\mathrm{loc}} \\
&[(Y + h_g(X^1)-h_g(X^2))^+ - Y^+]^2 U * \nu_t^g \le 2[h_g(X^1)]^2 * \nu_t^g + 2[h_g(X^2)]]^2 * \nu_t^g \in \mathcal{A}_{\mathrm{loc}} \\
&[\mathbf{1}_{(Y>0)}[w^2(X^1)-w^2(X^2)]^2 * \eta_t]^{1/2} \\
&\quad \le [2[w^2(X^1)]^2 * \eta_t]^{1/2} + [2[w^2(X^2)]^2 * \eta_t]^{1/2} \in \mathcal{A}_{\mathrm{loc}}, \\
&[\mathbf{1}_{(Y_->0)}[k_d^2(X^1)-k_d^2(X^2)]^2 * p_t^d]^{1/2} \\
&\quad \le [2[k_d^2(X^1)]^2 * p_t^d]^{1/2} + [2[k_d^2(X^2)]^2 * p_t^d]^{1/2} \in \mathcal{A}_{\mathrm{loc}}, \\
&[\mathbf{1}_{(Y>0)}[k_g^2(X^1)-k_g^2(X^2)]^2 * p_t^g]^{1/2} \\
&\quad \le [2[k_g^2(X^1)]^2 * p_t^g]^{1/2} + [2[k_g^2(X^2)]^2 * p_t^g]^{1/2} \in \mathcal{A}_{\mathrm{loc}}.
\end{aligned}$$

Thus, M is an optional local martingale (see [4, Section 7.4.2, p. 234]).

Now, by Grownwall lemma (see [1, Lemma 3.2])

$$Y_t^+ \leq \mathcal{E}_t(C)M_t.$$

Since C_t is an increasing process, $\mathcal{E}_t(C) \geq 0$. Thus, $M_t \geq 0$ since $Y_t^+ \geq 0$. Therefore, M_t is a non-negative optional local martingale and, consequently, by Lemma A.2 it is a non-negative supermartingale starting from 0. It follows that $M_t = 0$ for all $t \in \mathbb{R}_+$ a.s. Hence, $Y^+ \leq 0$, and $X^1 \leq X^2$. ■

Next, we show generality of **LT** condition imposed on function g.

Definition 3.2 (Yamada condition (see [28, Theorem 1.1], [2, Theorem 3.2]): There exists a non-negative non-decreasing function $\rho(u)$ on $\mathbb{R}_+$ and a $\mathcal{P}$-measurable non-negative function G such that

$$|g(x) - g(y)| \leq \rho(|x-y|)G(s),$$

$$G^2 \cdot \langle m\rangle_s < \infty \text{ a.s. } \int_0^\epsilon \rho^{-2}(u)\,\mathrm{d}u = \infty \quad \text{for any } s \in \mathbb{R}_+, \epsilon > 0, x, y \in \mathbb{R}.$$

Lemma 3.3: *If g satisfies Yamada condition then **LT** condition holds.*

Proof: Using the formula of occupation density, we have

$$\begin{aligned}
&\int_0^\infty L_t^a(X^1 - X^2)\rho^{-2}(a)\,\mathrm{d}a \\
&\quad = \int_0^t \mathbf{1}_{(X^1 - X^2 > 0)}\rho^{-2}(X_s^1 - X_s^2)\,\mathrm{d}\langle X^c\rangle_s \\
&\quad = \int_0^t \mathbf{1}_{(X^1 - X^2 > 0)}\rho^{-2}(X_s^1 - X_s^2)[g(X^1) - g(X^2)]^2\,\mathrm{d}\langle m\rangle_s < \infty.
\end{aligned}$$

Thus, since $a \mapsto L_t^a(X^1 - X^2)$ is right-continuous and $\int_0^\epsilon \rho^{-2}(u)\,\mathrm{d}u = \infty, \forall \epsilon > 0$, it follows that g satisfies **LT** condition. ■

Example 3.4: We provide an example of the function $g(x)$ which satisfies **LT** condition but does not satisfy Yamada condition. Let $g(x) = 1 + [\log(|x|^{-1} \vee 2)]^{-p}$ $(p > 0)$, it can be shown (see [27, Example 3]) that Yamada condition does not hold. On the other hand, by the mean value theorem for some $c > 0$, and for all $0 < y < \frac{1}{4}$,

$$\begin{aligned}
(g(x+y) - g(x))^2 \leq cy|\log y|(1 + [\log(|x|^{-1} \vee 2)]^{-p})^2|x|^{-1} \\
\times (\log(|x|^{-1}))^{-(2p+1)}\mathbf{1}_{[-3/4,1/2]}(x).
\end{aligned}$$

Let $\mathrm{d}X_t^1 = g(X_t^1)\,\mathrm{d}W_t$ and $\mathrm{d}X_t^2 = g(X^2)\,\mathrm{d}W_t$ be two strong solutions (W is a Wiener process), then by using the formula of occupation density and the above inequality we prove that

$$\int_0^{1/4} \frac{1}{a\log a} L_t^a(X^1 - X^2)\,\mathrm{d}a = \int_0^t \mathbf{1}_{(\frac{1}{4} > X^1 - X^2 > 0)} \frac{[g(X^1) - g(X^2)]^2}{(X_s^1 - X_s^2)\log(X_s^1 - X_s^2)}\,\mathrm{d}s$$

$$\leq c \int_0^t \frac{(1 + [\log(|X^2|^{-1} \vee 2)]^{-p})^2}{|X^2|(\log(|X^2|^{-1}))^{(2p+1)}} \mathbf{1}_{[-3/4,1/2]}(X^2)\, \mathrm{d}s < \infty$$

because the expression under the integral sign is Lebesgue integrable over compacts. Thus, since $a \mapsto L_t^a(X^1 - X^2)$ is right-continuous and $\int_0^{1/4} \frac{1}{a \log a}\, \mathrm{d}a = \infty$ it follows that g satisfies **LT** condition.

Remark 3.1: The local time technique allows us to prove the comparison theorem in a short and concise way. As shown in Example 3.4, **LT** condition in Theorem 3.1 is generally weaker than Yamada condition in [2]. Notice further that conditions on functions f^i, l_j^i and r^i are weakened in the sense that inequalities in (D2) and (D4) are not strict as the ones given in [2, Theorem 3.1]. In addition, we have not used conditions (A4) and (A8) from [2, Theorem 3.1] in our proof.

Remark 3.2: Note that the condition on the function g in (L1)-(c) guarantees fulfilment of the (**LT**) condition by Lemma 3.3, but no other way around.

4. Comparison of solutions of SDEs with different jump-diffusions

In this section, we expand the comparison theorem proved for SDEs with different diffusions in [14] to the optional jump-diffusion case. For the sake of brevity, here we want to compare two processes following a simplified version of SDE (4) in the form of

$$X_t^i = X_0^i + f^i(X^i) \cdot a_t + g_i(X^i) \cdot m_t + \sum_{j=d,g} [U h_j^i(X^i) * (\mu^j - \nu^j)_t], \quad i = 1, 2, \tag{6}$$

with initial condition $X_0^i = x_0^i$. Since now g_i and h_j^i can be all different, we should put stronger conditions on f^i, g_i, h_j^i and x_0^i, $i = 1, 2$.

Denote

$$F_i(z) := \int_{x_0^i}^{z} \frac{\mathrm{d}x}{g_i(x)},$$

$$\tilde{f}^i(z) := \frac{f^i(z)}{g_i(z)} - \frac{1}{2} g_i'(z)\alpha - \sum_{j=d,g} \int_E \left[\frac{h_j^i(z,u)}{g_i(z)} \right] U \tilde{\nu}^j(\mathrm{d}u),$$

$$\tilde{h}_j^i(z) := \int_z^{z+h_j^i(z)} \frac{\mathrm{d}x}{g_i(x)},$$

where α and $\tilde{\nu}^j$ are given below (see B1).

Let us introduce the following

B Conditions.

(B1) (structural conditions): There exist densities

$$\alpha = \frac{\mathrm{d}\langle m \rangle_t}{\mathrm{d}a_t},$$

$$\nu^d(\omega,(0,t],\Gamma) = \int_{0+}^{t} \tilde{\nu}^d(\omega,s,\Gamma)\,\mathrm{d}a_s,$$
$$\nu^g(\omega,[0,t),\Gamma) = \int_{0}^{t-} \tilde{\nu}^g(\omega,s,\Gamma)\,\mathrm{d}a_{s+}.$$

(B2) g_i $i=1,2$, is positive and continuously differentiable in z such that for any $s \in \mathbb{R}_+$, $z \in \mathbb{R}$

$$F_1(s,z) \geq F_2(s,z). \tag{7}$$

(B3) For any $s \in \mathbb{R}_+,\ u \in E,\ z,y \in \mathbb{R}, y \geq z$

$$\begin{aligned}\tilde{f}^1(s,z) &\leq \tilde{f}^2(s,y),\\ \tilde{h}^1_j(s,u,z) &\leq \tilde{h}^2_j(s,u,y),\\ z+\tilde{h}^2_j(s,u,z) &\leq y+\tilde{h}^2_j(s,u,y).\end{aligned}$$

(B4) $\tilde{f}^2(z)$ and $\tilde{h}^2_j(z)$ are Lipschitz continuous.

Theorem 4.1: *Suppose that f^i, g_i, h^i_j in (6) satisfy $\mathbf{L}(\mathbf{Y},\mathbf{X^i_0})$, $i=1,2$, and conditions **B** hold. Then there exist unique strong solutions X^1 and X^2, and $X^1 \leq X^2$.*

Proof: We transform the processes X^i with the help of the change of variables formula (see [4, Lemma 7.4.5]) and structural conditions, $i = 1, 2$,

$$\begin{aligned}\tilde{X}^i_t :&= \int_{x^i_0}^{X^i_t} \frac{\mathrm{d}x}{g_i(x)}\\ &= \frac{f^i(X^i_{t-})}{g_i(X^i_{t-})}\cdot a_t + m_t - \frac{1}{2}g_i'(X^i_{t-})\cdot\langle m\rangle_t\\ &\quad+\int_{X^i_{t-}}^{X^i_{t-}+h^i_d(X^i_{t-})}\frac{\mathrm{d}x}{g_i(x)}U*\mu^d_t + \int_{X^i_t}^{X^i_t+h^i_g(X^i_t)}\frac{\mathrm{d}x}{g_i(x)}U*\mu^g_t\\ &\quad-\frac{h^i_d(X^i_{t-})}{g_i(X^i_{t-})}U*\nu^d_t - \frac{h^i_g(X^i_t)}{g_i(X^i_t)}U*\nu^g_t\\ &=\tilde{f}^i(F_i^{-1}(\tilde{X}^i_{t-}))\cdot a_t + m_t + \tilde{h}^i_d(F_i^{-1}(\tilde{X}^i_{t-}))U*\mu^d_t + \tilde{h}^i_g(F_i^{-1}(\tilde{X}^i_t))U*\mu^g_t.\end{aligned}$$

From the condition (B2), it follows that $F_1^{-1}(z) \leq F_2^{-1}(z)$ and, consequently, for any z

$$\tilde{f}^1(F_1^{-1}(z)) \leq \tilde{f}^2(F_2^{-1}(z)), \tag{8}$$
$$\tilde{h}^1_j(F_1^{-1}(z)) \leq \tilde{h}^2_j(F_2^{-1}(z)) \tag{9}$$
$$z+\tilde{h}^2_j(s,u,z) \leq y+\tilde{h}^2_j(s,u,y) \tag{10}$$

by applying (B3).

Functions $\tilde{f}^2(F_2^{-1}(x))$ and $\tilde{h}_j^2(F_2^{-1}(x))$ are obviously Lipschitz continuous since F_2^{-1} is continuously differentiable transformation and (B4).

Now, we cannot directly use Theorem 3.1 because $\tilde{h}_j^1(F_1^{-1}(z)) \neq \tilde{h}_j^2(F_2^{-1}(z))$ in general. Instead, notice that

$$\begin{aligned} Y^+ &:= (\tilde{X}_t^1 - \tilde{X}_t^2)^+ = \mathbf{1}_{Y_->0}[\tilde{f}^1(F_1^{-1}(\tilde{X}_{t-}^1)) - \tilde{f}^2(F_2^{-1}(\tilde{X}_{t-}^1)) + \tilde{f}^2(F_2^{-1}(\tilde{X}_{t-}^1)) \\ &\quad - \tilde{f}^2(F_2^{-1}(\tilde{X}_{t-}^2))] \cdot a_t + [(Y + \tilde{h}_d^1(F_1^{-1}(\tilde{X}_{t-}^1)) - \tilde{h}_d^2(F_2^{-1}(\tilde{X}_{t-}^2)))^+ - Y^+]U * \mu_t^d \\ &\quad + [(Y + \tilde{h}_g^1(F_1^{-1}(\tilde{X}_t^1)) - \tilde{h}_g^2(F_2^{-1}(\tilde{X}_t^2)))^+ - Y^+]U * \mu_t^g \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By (8) and Lipschitz continuity, $I_1 \leq$ const. $Y^+ \cdot a_t$. Next, applying the same approach as in finding inequality for I_5 in the proof of Theorem 3.1 and using (9), (10) and Lipschitz continuity, we get $I_2 \leq$ const. $Y^+U * \mu_t^d$ and $I_3 \leq$ const. $Y^+U * \mu_t^g$. Consequently, by Gronwall Lemma we prove that $\tilde{X}^1 \leq \tilde{X}^2$.

This, together with (B2), implies that

$$\int_{x_0^2}^{X_t^1} \frac{\mathrm{d}x}{g_2(x)} \leq \int_{x_0^1}^{X_t^1} \frac{\mathrm{d}x}{g_1(x)} \leq \int_{x_0^2}^{X_t^2} \frac{\mathrm{d}x}{g_2(x)}.$$

Since $g_2(x) > 0$ we conclude that $X^1 \leq X^2$. ■

Remark 4.1: Proceeding with the same technique for jump measures as in the above proof, Theorem 4.1 can be directly extended to solutions of (2).

Let us give specific examples.

Example 4.2: Let X^1 and X^2 follows the equation (6) with $m_t = W_t, a_t = t$. For $A \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{E}$, the Poisson random measures μ^d and μ^g are defined as

$$\begin{aligned} \mu^d(A \times \Gamma) &:= \#\{(t, \Delta L_t^d) \in A \times \Gamma \,|\, t > 0 \text{ such that } \Delta L_t^d \neq 0\}, \\ \mu^g(A \times \Gamma) &:= \#\{(t, \Delta^+ L_t^g) \in A \times \Gamma \,|\, t > 0 \text{ such that } \Delta^+ L_t^g \neq 0\}, \end{aligned}$$

where L_t^1 and L_t^2 are a Poisson process and a left-continuous modification of a Poisson process with constant intensities $\gamma^d = 1$ and $\gamma^g = 2$, respectively, and compensators $\nu^d = \gamma^d t$ and $\nu^g = \gamma^g t$. Furthermore, we assume that L^1 and L^2 are independent. We have

$$\begin{aligned} &f^1(z) = 0, \quad g_1(z) = 1, \quad x_0^1 = 0; \\ &f^2(z) = \frac{0.15\cos(z)}{(1 - 0.3\sin(z))^3}, \quad g_2(z) = (1 - 0.3\sin(z))^{-1}, \quad x_0^2 = 0. \end{aligned}$$

Firstly, we consider the case with no jumps, i.e. $h_j^i = 0,\ i = 1, 2$. It is easy to check that all assumptions of Theorem 4.1 hold, and, thus, $X^1 \leq X^2$ (see Figure 1).

Next, let us assume that $h_j^i = 1,\ i = 1, 2$. Intuitively, by adding the compensated jumps with the same magnitude, we anticipate the same result as in the continuous case. However,

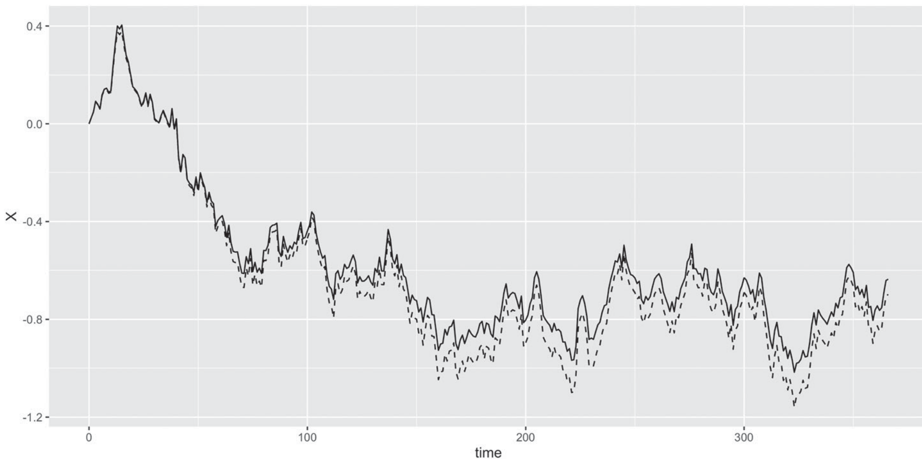

Figure 1. Simulated paths of X^1 (dashed) and X^2 (solid). $X^1 \leq X^2$.

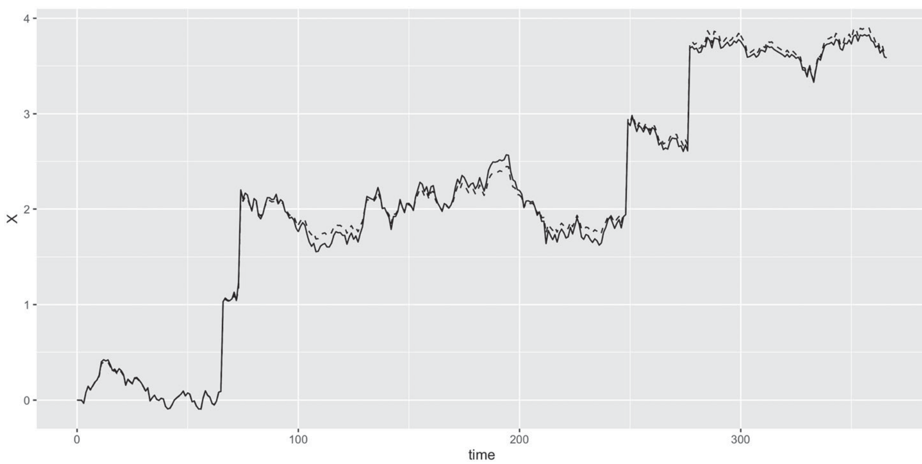

Figure 2. Simulated paths of X^1 (dashed) and X^2 (solid). $X^1 \nleq X^2$.

condition (B3) of Theorem 4.1 is clearly not satisfied. Therefore, $X^1 \leq X^2$ is not necessarily true (see Figure 2).

Finally, assume that $h_j^1 = 0.7, h_j^2 = 1$ and $f^1(z) = -1.8$, other functions stay the same. Since

$$\tilde{f}^1(z) = -3.9,$$

$$\tilde{f}^2(y) = -3(1 - 0.3\sin(y)),$$

$$\tilde{h}_j^1(z) = \int_z^{z+0.7} \frac{\mathrm{d}x}{1} = 0.7,$$

$$\tilde{h}_j^2(y) = \int_y^{y+1} \frac{\mathrm{d}x}{(1 - 0.3\sin(x))^{-1}} = 1 + 0.3\cos(y+1) - 0.3\cos(y)$$

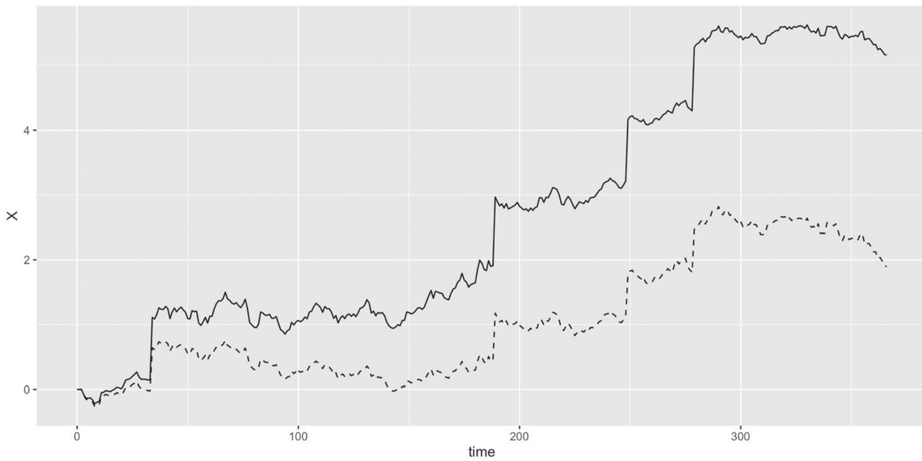

Figure 3. Simulated paths of X^1 (dashed) and X^2 (solid). $X^1 \leq X^2$.

then (B3) holds. In addition, $\tilde{f}^2$ and $\tilde{h}_j^2$ are obviously Lipschitz continuous. Thus, by Theorem 4.1, $X^1 \leq X^2$ (see Figure 3).

Example 4.3: Let X^1 and X^2 follows Equation (6) with the same $a_t, m_t, \mu_t^j, \lambda_t^j$, as in Example 3.2 and

$$f^1(z) = -0.5e^{-2z}, \quad g_1(z) = e^{-z}, \quad h_j^1 = 0, \quad x_0^1 = 0;$$
$$f^2(z) = 0.3 + z + z^3, \quad g_2(z) = 1 + z^2, \quad h_j^2 = 1, \quad x_0^2 = 0.$$

It is not hard to show that all assumptions of Theorem 4.1 hold, and, thus, $X^1 \leq X^2$.

5. Approximation of option price bounds using comparison property

The comparison theorems considered in the previous sections can be applied to find boundaries of option prices in the case of so-called Constant Elasticity of Variance (CEV) model. This idea was introduced in [20] and developed further in [21]. Here, we extend it to jump-diffusion CEV model and solve the problem formulated in [21].

CEV model was proposed by Cox and Ross [8]. It is often used in mathematical finance to capture leverage effects and stochasticity of volatility. It is also widely used by practitioners in the financial industry for modelling equities and commodities. Consider a more general version of the jump-diffusion CEV model [30] where the stock price is said to satisfy the following integral equation:

$$S_t = \rho \int_0^t S_{s-}\, ds + \sigma S^\alpha \cdot W_t + S_{t-} U * (\mu^1 - \nu^1)_t + S_t U * (\mu^2 - \nu^2)_t, \quad S_0 = s, \quad (11)$$

where ρ and σ are constants. W_t is a Wiener process, $\mu^1 - \nu^1$ is a compensated measure of left jumps and $\mu^2 - \nu^2$ is a compensated measure of right jumps. For $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{E}$ the jump measures are defined as follows:

$$\mu^1(B \times \Gamma) := \#\{(t, \Delta L_t^1) \in B \times \Gamma | t > 0 \text{ such that } \Delta L_t^1 \neq 0\},$$

$$\mu^2(B \times \Gamma) := \#\{(t, \Delta^+ L_t^2) \in B \times \Gamma | t > 0 \text{ such that } \Delta^+ L_t^2 \neq 0\},$$

where L_t^1 and L_t^2 are a Poisson process and a left-continuous modification of a Poisson process with constant intensities γ^1 and γ^2, respectively, and L^1 and L^2 are independent. Hence, $\nu^1 = \gamma^1 t$ and $\nu^2 = \gamma^2 t$.

Consider the function

$$F(x) = \frac{1}{\sigma}\int_s^x u^{-\alpha}\,\mathrm{d}u = \frac{x^{1-\alpha} - s^{1-\alpha}}{\sigma(1-\alpha)}, \quad \text{and find}$$

$$F'(x) = \frac{x^{-\alpha}}{\sigma}, \quad F''(x) = \frac{-\alpha x^{-\alpha-1}}{\sigma},$$

where $0 < \alpha < 1$.

Denote $X_t = F(S_t)$ and apply the change of variables formula (see [4, Lemma 7.4.5]). We have

$$\begin{aligned}
X_t &= F'(S)\left[\rho\int_0^t S_{s-}\,\mathrm{d}s + \sigma S^\alpha \cdot W_t\right] + \int_0^t \frac{\sigma^2}{2}F''(S_{s-})S_{s-}^{2\alpha}\,\mathrm{d}s \\
&\quad + [F(2S_{t-}) - F(S_{t-})]U * (\mu^1 - \nu^1)_t + [F(2S_t) - F(S_t)]U * (\mu^2 - \nu^2)_t \\
&\quad + [F(2S_{t-}) - F(S_{t-}) - S_{t-}F'(S_{t-})]U * \nu_t^1 + [F(2S_t) - F(S_t) - S_tF'(S_t)]U * \nu_t^2 \\
&\quad = \int_0^t \left[kS_{s-}^{1-\alpha} - \frac{\sigma\alpha}{2}S_{s-}^{\alpha-1}\right]\mathrm{d}s + W_t \\
&\quad + cS_{t-}^{1-\alpha}U * (\mu^1 - \nu^1)_t + cS_t^{1-\alpha}U * (\mu^2 - \nu^2)_t \\
&= \int_0^t \left[k(X_{s-}\sigma(1-\alpha) + s^{1-\alpha}) - \frac{\sigma\alpha}{2}(X_{s-}\sigma(1-\alpha) + s^{1-\alpha})^{-1}\right]\mathrm{d}s + W_t \\
&\quad + c(X_{t-}\sigma(1-\alpha) + s^{1-\alpha})U * (\mu^1 - \nu^1)_t \\
&\quad + c(X_t\sigma(1-\alpha) + s^{1-\alpha})U * (\mu^2 - \nu^2)_t,
\end{aligned}$$

where $k := \frac{\rho}{\sigma} + (2^{1-\alpha} - 1 - \frac{1}{\sigma})\lambda_1 U + (2^{1-\alpha} - 1 - \frac{1}{\sigma})\lambda_2 U$ and $c := \frac{2^{1-\alpha}-1}{\sigma(1-\alpha)}$.

With the Comparison Theorem 3.1, we can give an estimate of the process X_t from above by a new process Y_t, satisfying the equation

$$\begin{aligned}
Y_t &= \int_0^t [k(Y_{s-}\sigma(1-\alpha) + s^{1-\alpha})]\,\mathrm{d}s + W_t \\
&\quad + c(Y_{t-}\sigma(1-\alpha) + s^{1-\alpha})U * (\mu^1 - \nu^1)_t \\
&\quad + c(Y_t\sigma(1-\alpha) + s^{1-\alpha})U * (\mu^2 - \nu^2)_t, \\
Y_0 &= 0.
\end{aligned}$$

The process Y is an Ornstein–Uhlenbeck process with left and right jumps. The explicit solution for the above non-homogeneous linear stochastic integral equation is given by the following formula (see [1, Theorem 3.1]):

$$Y_t = \mathcal{E}_t(H)[\mathcal{E}(H)^{-1}\tilde{G}_t],$$

where $\mathcal{E}$ is an optional stochastic exponent and

$$H_t = k\sigma(1-\alpha)t + (2^{1-\alpha}-1)U * (\mu^1 - \nu^1)_t + (2^{1-\alpha}-1)U * (\mu^2 - \nu^2)_t,$$
$$\tilde{G}_t = s^{1-\alpha}(k - c(1-2^{\alpha-1})\gamma^1 U - c(1-2^{\alpha-1})\gamma^2 U)t$$
$$+ c(s2)^{1-\alpha} U * [(\mu^1-\nu^1)_t + (\mu^2-\nu^2)_t] + W_t.$$

Applying the comparison theorem to X_t and Y_t yields that $Y_t \geq X_t = F(S_t)$ a.s. Since $F(x)$ is monotonically increasing function, we have

$$S_t \leq F^{-1}(Y_t) \quad \text{a.s.} \tag{12}$$

Now, let us consider a function f with an option payoff $f(S_T)$, where f is increasing. Assuming zero interest rates, the price of such option is given by $\tilde{\mathbf{E}}f(S_T)$ for an appropriate martingale measure $\tilde{\mathbf{P}}$ (see [1, Section 4], where the existence of $\tilde{P}$ is discussed). Using inequality (12), we have that $\tilde{\mathbf{E}}f(S_T) \leq \tilde{\mathbf{E}}f(F^{-1}(Y_T))$ and thus we obtain an estimate for the option price for which $\tilde{\mathbf{E}}f(F^{-1}(Y_T))$ is easier to compute.

6. Pathwise uniqueness in terms of local times

In the last section, as a supplementary result being of independent interest, we demonstrate how the local time technique used in the proof of the Theorem 3.1 can similarly be utilized to prove the pathwise uniqueness of (2). We say that solution of (2) is pathwise unique if whenever X and Z are any two solutions of (2) defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the same $m \in \mathcal{M}_{\text{loc}}$ and the same measures μ^j, p^j, η such that $X_0 = Z_0$ a.s. then $X_t = Z_t$ for all t a.s.

To prove the pathwise uniqueness of solutions of (2), we require several assumptions on its coefficient functions. To begin with, we present a slightly more general condition than the one-sided Lipschitz condition (cf. [22, Chapter 1.1]).

Definition 6.1 (One Sided Lipschitz Condition): We say that a coefficient function f of Equation (2) satisfies one-sided Lipschitz condition with respect to x if there exists predictable function $G(\omega, s)$ such that for any $x, y \in \mathbb{R}, s \in \mathbb{R}_+, \omega \in \Omega$

$$(x-y)(f(\omega,s,x) - f(\omega,s,y)) \leq G(\omega,s)(x-y)^2.$$

Let us introduce the following conditions:

C Conditions.

We say that the functions $f, g, h_j, k_j, l_j, r, w$ satisfy the **C** conditions if:

(C1) f is one-sided Lipschitz continuous,

(C2) g satisfies **LT** condition,

(C3) there exists non-negative functions $H^d(\omega, s, u) \in \mathcal{P} \times \mathcal{B}(E \cap (|u| \leq 1))$, $H^g(\omega, s, u) \in \mathcal{O} \times \mathcal{B}(E \cap (|u| \leq 1))$, $L^d(\omega, s, u), K^d(\omega, s, u) \in \tilde{\mathcal{P}}$; $L^g(\omega, s, u), K^g(\omega, s, u)$, $R(\omega, s, u), W(\omega, s, u) \in \tilde{\mathcal{O}}$ such that for any $x, y \in \mathbb{R}$, $s \in \mathbb{R}_+, u \in E \cap (|u| \leq 1)$, $\omega \in \Omega$:

$$|h_j(\omega,s,u,x) - h_j(\omega,s,u,y)| \leq H^j(\omega,s,u)|x-y|, \quad \text{and any } u \in E$$

$$
\begin{aligned}
&|l_j(\omega,s,u,x)-l_j(\omega,s,u,y)| \le L^j(\omega,s,u)|x-y|,\\
&|k_j(\omega,s,u,x)-k_j(\omega,s,u,y)| \le K^j(\omega,s,u)|x-y|,\\
&|r(\omega,s,u,x)-r(\omega,s,u,y)| \le R(\omega,s,u)|x-y|,\\
&|w(\omega,s,u,x)-w(\omega,s,u,y)| \le W(\omega,s,u)|x-y|,\\
&G\cdot a_t[L^d+K^d]*p_t^d+[L^g+K^g]*p_t^g+2H^dU*\nu_t^d+2H^gU*\nu_t^g\\
&\quad +[R+W]*\zeta_t<\infty,
\end{aligned}
$$

is increasing.

Theorem 6.2: *Suppose that functions f, g, h_j, l_j, k_j, w and r satisfy* **C** *conditions, then if the solution of Equation* (2) *exists then it is pathwise unique.*

Proof: Assume that there are two solutions X and Z of equation (2), and let $Y := X-Z$. Applying the formula (A1) to Y and using identity $|Y| = 2Y^+ - Y$, we get

$$
\begin{aligned}
|Y_t| = {}& \operatorname{sign}(Y_-)[(f(X)-f(Z))\cdot a_t+(g(X)-g(Z))\cdot m_t]+L_t^0(Y)\\
&+[|Y_-+h_d(X)-h_d(Z)|-|Y_-|]U*(\mu^d-\nu^d)_t\\
&+[|Y+h_g(X)-h_g(Z)|-|Y|]U*(\mu^g-\nu^g)_t\\
&+[|Y_-+h_d(X)-h_d(Z)|-|Y_-|-(h_d(X)-h_d(Z))\operatorname{sign}(Y_-)]U*\nu_t^d\\
&+[|Y+h_g(X)-h_g(Z)|-|Y|-(h_g(X)-h_g(Z))\operatorname{sign}(Y)]U*\nu_t^g\\
&+[|Y_-+l_d(X)-l_d(Z)+k_d(X)-k_d(Z)|-|Y_-|]*p_t^d\\
&+[|Y+l_g(X)-l_g(Z)+k_g(X)-k_g(Z)|-|Y|]*p_t^g\\
&+[|Y+r(X)-r(Z)+w(X)-w(Z)|-|Y|]*\eta_t.
\end{aligned}
$$

Let

$$
\begin{aligned}
M_t := {}& \operatorname{sign}(Y_-)(g(X)-g(Z))\cdot m_t+[|Y_-+h_d(X)-h_d(Z)|-|Y_-|]U*(\mu^d-\nu^d)_t\\
&+[|Y+h_g(X)-h_g(Z)|-|Y|]U*(\mu^g-\nu^g)_t\in\mathcal{M}_{\mathrm{loc}}.
\end{aligned}
$$

By using one-sided Lipschitz condition on the drift coefficient function *f*, **LT** condition and simple algebraic inequalities, we have

$$
\begin{aligned}
|Y| \le {}& |Y_-|G\cdot a_t+M_t\\
&+\sum_{j=d,g}[|l_j(X)-l_j(Z)|+|k_j(X)-k_j(Z)|]*p_t^j+2|h_j(X)-h_j(Z)|U*\nu_t^j\\
&+[|r(X)-r(Z)|+|w(X)-w(Z)|]*\eta_t
\end{aligned}
\tag{13}
$$

Further, we apply (C3) to (13) and get

$$
\begin{aligned}
|Y_t| \le {}& M_t+|Y_-|G\cdot a_t+|Y_-|[L^d+K^d]*p_t^d+|Y|[L^g+K^g]*p_t^g\\
&+2|Y_-|H^dU*\nu_t^d+2|Y|H^gU*\nu_t^g+|Y|[R+W]*\eta_t.
\end{aligned}
\tag{14}
$$

If we now define a process $C_t := G \cdot a_t + [L^d + K^d] * p_t^d + [L^g + K^g] * p_t^g + 2H^d U * \nu_t^d + 2H^g U * \nu_t^g + [R + W] * \zeta_t$, then Equation (14) can be rewritten as

$$|Y_t| \leq M_t + |Y| \circ C_t.$$

Finally, by Gronwall lemma (see [1, Lemma 3.2]) we get that $|Y_t| \leq \mathcal{E}_t(C)M_t$. Since C_t is increasing process, $\mathcal{E}_t(C) \geq 0$. Thus, $M_t \geq 0$ because $|Y| \geq 0$. Therefore, M_t is a non-negative optional local martingale and, consequently, by Lemma A.2 it is a non-negative supermartingale starting from 0. It follows that $M = 0$. Hence, $Y_t = 0$ for all t a.s. and the pathwise uniqueness follows. ■

Remark 6.1: Existence and uniqueness theorems for differential equations under one-sided Lipschitz condition on the drift coefficient function were explored by several authors (see, for example, [3,18]). One-sided Lipschitz continuity is weaker than Lipschitz continuity, and an example illustrating this relation is a function $f(x) = e^{-x}$.

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Appendix

Theorem A.1: *Let X be an optional semimartingale given in (2) and Assumptions 2.1 hold. Then X^+ is an optional semimartingale and has the following representation:*

$$
\begin{aligned}
X_t^+ &= X_0^+ + \mathbf{1}_{(X_->0)} f(X)\cdot a_t + \mathbf{1}_{(X_->0)} g(X)\cdot m_t + \frac{1}{2}L_t^0 \\
&\quad + [(X_- + h_d(X))^+ - X_-^+]U * (\mu^d - \nu^d)_t \\
&\quad + [(X + h_g(X))^+ - X^+]U * (\mu^g - \nu^g)_t \\
&\quad + [(X_- + h_d(X))^+ - X_-^+ - \mathbf{1}_{(X_->0)} h_d(X)]U * \nu_t^d \\
&\quad + [(X + h_g(X))^+ - X^+ - \mathbf{1}_{(X>0)} h_g(X)]U * \nu_t^g \\
&\quad + [(X_- + (k_d + l_d)(X))^+ - X_-^+] * p_t^d + [(X + (k_g + l_g)(X))^+ - X^+] * p_t^g \\
&\quad + [(X + (r + w)(X))^+ - X^+] * \eta_t.
\end{aligned} \tag{A1}
$$

Proof: Apply the change of variables formula (see [23, VI.3.4]) to the optional semimartingale (1)

$$
\begin{aligned}
X_t^+ &= X_0^+ + \mathbf{1}_{(X_->0)} f(X)\cdot a_t + \mathbf{1}_{(X_->0)} g(X)\cdot m_t + \frac{1}{2}L_t^0 \\
&\quad + \mathbf{1}_{(X_{s-}>0)}[h_d(X)U * (\mu^d - \nu^d) + [k_d(X) + l_d(X)] * p^d + [r(X) + w(X)] * \eta] \\
&\quad + \sum_{0<s\le t} [X_s^+ - X_{s-}^+ - \mathbf{1}_{(X_{s-}>0)}\Delta X_s] \\
&\quad + \mathbf{1}_{(X_s>0)}[h_g(X)U * (\mu^g - \nu^g) + [k_g(X) + l_g(X)] * p^g] \\
&\quad + \sum_{0\le s<t} [X_{s+}^+ - X_s^+ - \mathbf{1}_{(X_s>0)}\Delta^+ X_s].
\end{aligned} \tag{A2}
$$

Define $C_t = \sum_{0<s\le t}[X_s^+ - X_{s-}^+ - \mathbf{1}_{(X_{s-}>0)}\Delta X_s]$, $B_t = \sum_{0\le s<t}[X_{s+}^+ - X_s^+ - \mathbf{1}_{(X_s>0)}\Delta^+ X_s]$ and represent them in the form $C_t = \sum_{i=1}^2 C_t^i$, $B_t = \sum_{j=1}^3 B_t^j$, where

$$
\begin{aligned}
C_t^1 &= \sum_{T_n\le t} [X_{T_n}^+ - X_{T_n-}^+ - \mathbf{1}_{(X_{T_n-}>0)}\Delta X_{T_n}]\mathbf{1}_{|\Delta X_{T_n}|\le 1}, \\
C_t^2 &= \sum_{S_n\le t} [X_{S_n}^+ - X_{S_n-}^+ - \mathbf{1}_{(X_{S_n-}>0)}\Delta X_{S_n}], \\
B_t^1 &= \sum_{U_n<t} [X_{U_n+}^+ - X_{U_n}^+ - \mathbf{1}_{(X_{U_n}>0)}\Delta^+ X_{U_n}]\mathbf{1}_{|\Delta^+ X_{U_n}|\le 1}, \\
B_t^2 &= \sum_{S_n<t} [X_{S_n+}^+ - X_{S_n}^+ - \mathbf{1}_{(X_{S_n}>0)}\Delta^+ X_{S_n}], \\
B_t^3 &= \sum_{T_n<t} [X_{T_n+}^+ - X_{T_n}^+ - \mathbf{1}_{(X_{T_n}>0)}\Delta^+ X_{T_n}].
\end{aligned}
$$

In [23, VI.3.5], it is shown that C, B belong to $\mathcal{V}$. Applying Assumptions 2.1, we have

$$
\begin{aligned}
&|F_{h_d}^{(d)}|^2 U * \nu^d \le |h_d(X_-)|^2 * \nu^d \in \mathcal{A}_{\text{loc}}, \quad |F_{h_j}^{(d)}|^2 U * \nu^g \le |h_g(X)|^2 * \nu^g \in \mathcal{A}_{\text{loc}}, \\
&|\mathbf{1}_{(X_->0)} h_d(X_-)|^2 U * \nu^d \le |h_d(X_-)|^2 * \nu^d \in \mathcal{A}_{\text{loc}}, \\
&|\mathbf{1}_{(X>0)} h_d(X)|^2 U * \nu^g \le |h_g(X)|^2 * \nu^g \in \mathcal{A}_{\text{loc}},
\end{aligned}
$$

where $F_{h_d}^{(d)} = (X_- + h_d(X_-))^+ - X_-^+$, $F_{h_g}^{(g)}(X) = (X + h_g(X))^+ - X^+$. Thus, we can decompose C^1 and B^1 in the following form:

$$C^1 = F_{h_d}^{(d)} U * (\mu^d - \nu^d) - [h_d(X)\mathbf{1}_{(X_{S_n-}>0)}]U * (\mu^d - \nu^d) + [F_{h_d}^{(d)} - h_d(X)\mathbf{1}_{(X_{S_n-}>0)}]U * \nu^d,$$

$$B^1 = F_{h_g}^{(g)} U * (\mu^g - \nu^g) - [h_g(X)\mathbf{1}_{(X_{S_n}>0)}]U * (\mu^g - \nu^g) + [F_{h_g}^{(g)} - h_g(X)\mathbf{1}_{(X_{S_n}>0)}]U * \nu^g,$$

where the first two terms in each formula are in $\mathcal{M}_{\mathrm{loc}}^2$ and last terms are in $\mathcal{A}_{\mathrm{loc}}$.

Since the processes $\sum_{S_n \le t} \mathbf{1}_{(X_{S_n-}>0)} \Delta X_{S_n}$, $\sum_{S_n < t} \mathbf{1}_{(X_{S_n}>0)} \Delta^+ X_{S_n}$, $\sum_{T_n \le t} \mathbf{1}_{(X_{T_n}>0)} \Delta^+ X_{T_n}$ are semimartingales, then we represent the processes C^2, B^2 and B^3 as

$$C^2 = F_{k_d+l_d}^{(d)} * p^d - [k_d(X) + l_d(X)]\mathbf{1}_{(X_{S_n-}>0)} * p^d,$$

$$B^2 = F_{k_g+l_g}^{(g)} * p^g - [k_g(X) + l_g(X)]\mathbf{1}_{(X_{S_n}>0)} * p^g,$$

$$B^3 = F_{r+w}^{(g)} * \eta - [r(X) + w(X)]\mathbf{1}_{(X_{S_n}>0)} * \eta,$$

where all terms on the right side are semimartingales. By plugging $\sum_{i=1}^2 C_t^i$, $\sum_{j=1}^3 B_t^j$ back into (A2) we get (A1). ∎

Lemma A.2: *A non-negative optional local martingale X is a supermartingale.*

Proof: Let $X \in \mathcal{M}_{\mathrm{loc}}$, $X \ge 0$. Then by definition of optional local martingale there exist $X^n \in \mathcal{M}$, $X^n \ge 0$, and a sequence of wide sense stopping times R_n, $R_n \uparrow \infty$ a.s. such that for any $n \ge 1: \ X = X^n \mathbf{1}_{[0,R_n]}$. Next, for any $t \ge s$ and $A \in \mathcal{F}_s$, we have

$$\begin{aligned}\mathbf{E}X_t\mathbf{1}_A &= \lim_{n\to\infty} \mathbf{E}X_t\mathbf{1}_A\mathbf{1}_{(t\le R_n)} = \lim_{n\to\infty} \mathbf{E}X_t^n\mathbf{1}_A\mathbf{1}_{(t\le R_n)} \\ &\le \lim_{n\to\infty} \mathbf{E}X_t^n\mathbf{1}_A\mathbf{1}_{(s\le R_n)} = \lim_{n\to\infty} \mathbf{E}X_s^n\mathbf{1}_A\mathbf{1}_{(s\le R_n)} = \mathbf{E}X_s\mathbf{1}_A.\end{aligned}$$

Hence, the process X_t is a non-negative supermartingale (in the usual sense). ∎