Kinetic theory and the Einstein-Vlasov system

Mohamed Abdelghani
Department of Physics
University of Toronto, Toronto
Ontario, Canada
mohamed@physics.utoronto.ca
http://www.physics.utoronto.ca/

April 10, 2003

Abstract

The Vlasov-Einstein equations are a closed self-consistent collision-less system of kinetic equations. They were introduced to model the dynamics of self gravitating multi-particles system. The particles could be a neutral gas, plasma, interstellar dust, stars, galaxies or even clusters of galaxies. In 1990 Rendall and Rein initiated the mathematical study of the Einstein-Vlasov system. Since then many properties of its solution have been established. In General-Relativity hydrodynamic models are commonly used. These models are naive and may give rise to unphysical phenomenas such as shell-crossing singularities. Coupling the Vlasov to the Einestin field equations results in a better description which rules out these singularities. In this paper, I will introduce kinetic theory and then discuss the Vlasov-Einstein system, It's Lagrangian formalism and properties of its solutions.

Contents

1 Introduction to Kinetic theory

Kinetic theory is an important field of research and has been for several decades. Kinetic theory models the time evolution of a collection of particles. The main focus of kinetic theory has been on non-relativistic and special relativistic physics. From the mathematical point of view there are fundamental advantages in using the kinetic models in general relativity over the commonly used hydrodynamic models.

Kroning, Calusius, Maxwell, and Boltzmann initiated the study of kinetic theory, based on the assumption that a gas consists of particles each of which behaves according to the laws of mechanics. Mechanical analysis leads to a large number of coupled nonlinear equations that describe the motion of each particle in the collection. This is a many-body problem which is hard to solve. In the last century Poincare solved the equations governing the motion of planets using the method of coarse-graining. In which, the solutions are a set of coarse trajectories with a correction factors. Poincare method is compatible with the method devised by Maxwell and Boltzmann to establish kinetic theory. For further discussion see [1].

Coarse-Graining or what is now known as the Contracted-Description of the many-body problem resulted in a stochastic interpretation of Kinetic theory. A particular physical system can be described at different levels of detail. Each level of description is characterized by a set of relevant variables with typical spacetime scales of variation. The selection of one level or another in order to describe the same system depends essentially on the typical spatial or time scales that one wishes to explore, usually dictated by experimental requirements. In the Mesoscopic; the relevant variables are the positions and velocities of the particles. And in the macroscopic level; The relevant variable are particles distribution around a point in spacetime.

The first level of description in kinetic theory is the Liouville equation, which we describe next.

1.1 The Liouville equation

Suppose we have a collection of N point particles. Let Γ be the set of all system points $(t, q(t), p(t))$. The space Γ has $6N + 1$ dimensions. q and p are each of dimension $3N$. The system state at time t is given by (q, p). We define the density function $\rho(t, q, p) : \Gamma \to \mathbb{R}$. $\rho(t, q, p)dqdp$ is defined as the number of points (t, q, p) in the volume $dqdp$ at time t. Assuming that system points are conserved then the time rate of change of $\int_V \rho(t, q, p)dV$ must be equal to the net flux through the surface ∂V given by $\int_{\partial V} v^a \rho(t, q, p)dS_a$. $a = 1...6N$ and v^a is a component in the state vector $\partial_t(..., q^b..., ..., p^c, ...)$. ∇_a is a component in $(..., \nabla_{q^b}, ..., \nabla_{p^c}, ...)$ and dS_a is the differential surface element. There is an implicit sum over a. It follows then

$$\partial_t \int_V \rho(t, q, p)dV = -\int_{\partial V} v^a \rho(t, q, p)dS_a = -\int_V \nabla_a(v^a \rho(t, q, p))dV. \quad (1)$$

In the limit $V \to 0$, and using the mean value theorem

$$\partial_t \rho(t, q, p) + \nabla_a(v^a \rho(t, q, p)) = 0. \quad (2)$$

The equation above is the continuity equation. Lets rewrite ∇_a in terms of its momentum and position basis as ∇_{q^α} and ∇_{p^α}, $\alpha = 1...3N$. Then equation 2 becomes

$$\partial_t \rho(t, q, p) + \nabla_{q^\alpha}(\dot{q^\alpha} \rho(t, q, p)) + \nabla_{p^\alpha}(\dot{p^\alpha} \rho(t, q, p)) = 0. \quad (3)$$

Let $H = H(t, q, p)$ be the Hamiltonian of the system then we substitute for $\dot{p^\alpha} = -\nabla_{q^\alpha} H$ and $\dot{q^\alpha} = \nabla_{p^\alpha} H$ into 3 to obtain the following result

$$\begin{aligned} \partial_t \rho(t, q, p) &= \nabla_{p^\alpha} \rho(t, q, p) \nabla_{q^\alpha} H - \nabla_{p^\alpha} H \nabla_{q^\alpha} \rho(t, q, p) && (4) \\ &= -v^a \nabla_a \rho(t, q, p) = [H, \rho(t, q, p)] && (5) \end{aligned}$$

$[,]$ are the Poisson brackets. The equation above is know as the Liouville equation.

1.2 BBGKY Hierarchy

In kinetic theory, we are most interested in measuring the macroscopic state of a system. The Liouville equation is not a statistical description and is not useful in determining macroscopic properties of a system. BBGKY Hierarchy, which was proposed independently by Born, Bogoliubov, kirkwood, Green and Yvon, is the first attempt at applying probabilistic analysis to the Liouville equation.

Let $f^N = f^N(t, p, q)$ be the N-particle distribution function, which represents the probability that the particle is in a volume $dp^1 dp1...dp^N dq^N$

around the point $(t, p^1 ..., p^N, q^1, ..., q^N)$. f^N is defined in terms of the density function as

$$f^N(t, q, p) = \frac{\rho(t, q, p)}{\int \rho(t, q, p) dq dp} \int dq^1 dp^1 ... dq^N dp^N. \tag{6}$$

We also define the reduced or the s-particle distribution function as $f^s(t, q, p) = (\int f dq^{s+1} dp^{s+1} ... dq^N dp^N)/(\int dq^{s+1} dp^{s+1} ... dq^N dp^N)$, where s is the number of particles under consideration. We also assume that the distribution functions are symmetric with respect to the interchange of any two particles coordinates. The normalization takes care of this symmetry. Now, we are ready to derive the BBGKY equations.

Let H be the total hamiltonian of the system of N particle. It is defined as

$$H(t, q, p) = \sum_{i=1}^{N} [K_i + \sum_{j>i}^{N} U_{ij}] \tag{7}$$

$K_i = |p^i|^2/2m$ is the kinetic energy of the i^{th} particle. $U_{ij} = U(t, q^i, q^j, p^i, p^j)$ is the interaction energy between any two particles in the system. the sum can be re-arranged to yield

$$H(t, q, p) = \sum_{i=1}^{s} [K_i + \sum_{j>i}^{s} U_{ij}] + \sum_{i=s+1}^{N} [K_i + \sum_{j>i}^{N} U_{ij}] + \sum_{i=1}^{s} \sum_{j>s}^{N} U_{ij}. \tag{8}$$

In equation 8 we defined the hamiltonian of the system of N particles in terms of a hamiltonian of s particles. This is needed for the derivation of the BBGKY equations. Then we can rewrite the total system hamiltonian in the following way

$$H_s = \sum_{i=1}^{s} [K_i + \sum_{j>i}^{s} U_{ij}]. \tag{9}$$

$$H_r = \sum_{i=s+1}^{N} [K_i + \sum_{j>i}^{N} U_{ij}]. \tag{10}$$

$$U_I = \sum_{i=1}^{s} \sum_{j>s}^{N} U_{ij}. \tag{11}$$

$$H(t, q, p) = H_s + H_r + U_I. \tag{12}$$

H_s is the hamiltonian of s particles, while H_r is the hamiltonian of the remainder of particles. U_I is the interaction potential between the two subsystems.

By applying the normalization procedure to the density function in Liouville equation we obtain the equation in terms of the N-particle distribution function

$$\partial_t f^N(t,q,p) = [H, f^N(t,q,p)] = [H_s + H_r + U_I, f^N(t,q,p)]. \tag{13}$$

Integrating over the coordinates $dq^{s+1}dp^{s+1}...dq^N dp^N$ will yield

$$\begin{gathered}\partial_t f^s(t,q,p) - [H_s, f^s(t,q,p)] = \\ V^s \int [H_r + U_I, f^N(t,q,p)] dq^{s+1}dp^{s+1}...dq^N dp^N.\end{gathered} \tag{14}$$

$V^s = \int dq^{s+1}dp^{s+1}...dq^N dp^N$ The volume of the subsystem. The integral over the $[H_r, f^N]$ vanished see [3] for proof. Therefore, the above equation becomes

$$\begin{gathered}\partial_t f^s(t,q,p) - [H_s, f^s(t,q,p)] = \\ V^s \int [U_I, f^N(t,q,p)] dq^{s+1}dp^{s+1}...dq^N dp^N.\end{gathered} \tag{15}$$

¿From the equation above, considering U_I to be independent of the momentum of particles, the right hand side is the average over all the forces exerted by the rest of the particles of the system on the subsystem of s particles. This fact is the essential quality of the BBGKY Hierarchy. Later we shall see that for $s = 1$ leads to equation of a 1-particle distribution function. The Boltzmann and Vlasov equations are equations of the 1-particle distribution function. They are deduced after considering assumptions on the interaction function U_I between particles. The method above is equivalent to a perturbation expansion of the Liouville equation from which The Boltzmann equation could also be obtain by expanding in terms $nr_0^3 << 1$, where $n = N/V$ is the density of particles and r_0 the average range of the interaction potentials [4]. While the Vlasov equation is obtained for $1/nr_0^3 << 1$ and consideration of collision less particles system. The Boltzmann equation describe a system of low particle density while the Vlasov equations a system of high particle hight density. For further discussion see [4].

1.3 The Boltzman equation

A characteristic feature of kinetic theory is its statistical description of many-particle systems by distribution function $f = f(t, q, p)$. Let $f(t, q, p)$ be the 1-particle distribution function. We can think of f as describing the dynamics of a single particle influenced by a smeared out or average particle density or a force field exerted by all other particles in the system. Knowing the distribution function we can calculate many macroscopic quantities. The Boltzmann equation is 1-particle distribution function equation that takes into account the collisions between particles. It assumes that the dominant collisions are elastic binary collisions. In this section we describe the relativistic boltzman equation.

In Minkowski spacetime with a metric of signature $(-, +, +, +)$, consider a collection of neutral particles with rest mass $m_0 = 1$. Normalize the speed of light, $c = 1$. The four-momentum of a particle is denoted by $p^a = (-p^0, p)$. Let p stand for the 3-momentum (p^1, p^2, p^3) then $p^0 = \sqrt{(1 + |p|)^2}$ is the energy of a particle. The relativistic velocity v is given by

$$v = \frac{p}{\sqrt{1 + |p|^2}}. \tag{16}$$

Note that $|v| < 1 = c$.

The relativistic Boltzmann equation models the space-time behavior of the 1-particle distribution function $f = f(t, q, p)$. It can be derived from 15 by setting $s = 1$ and $U_I = 0$.

$$(\partial_t + \frac{p}{p^0} \cdot \nabla_q) f = Q(f, f), \tag{17}$$

$Q(f, f) = -q.\nabla_p f$ is the relativistic collision operator see [7] for more information. Dudyński and Ekiel-Jezewska proved the existence of solutions to the relativistic Boltzmann equation [5]. At present there is no result on the global existence of the relativistic Boltzmann equation and is an interesting open problem [2]. For more information on the relativistic Boltzmann equation see [6], [7] and [8].

2 Vlasov-Einestien System

In this section we will consider a self-gravitating collision less gas and we will write down the Einstein-Vlasov system and describe its general mathematical features. To a large extent, I follow the Rendall [10] and Andréasson

[2].

Let M be a four-dimensional manifold and let g_{ab} be a metric with Lorentz signature $(-,+,+,+)$ so that (M, g_{ab}) is a spacetime. The metric is assumed to be time-orientable so that we can distinguish between future and past directed vectors on spacetime. The worldline of a particle with non-zero rest mass m_0 is a timelike curve and the unit future directed tangent vector v^a to this curve is the four-velocity of the particle. The four-momentum p^a is given by $m_0 v^a$. We assume that all particles have equal rest mass which we set $m_0 = 1$. The possible values of the four-momentum are all future directed unit timelike vectors and they constitute a hypersurface $SM \in TM$, which is called the mass shell.

$f(t, q, p)$ is the 1-particle distribution function. The Vlasov equation is an equation for f. To get an expression for this equation We choose local coordinates on M such that the hypersurfaces $t = q^0$ =constant are spacelike such that t is a time coordinate and $q^j,\ j = 1, 2, 3$ are spatial coordinates. A timelike vector is future directed if and only if its zero component is positive. Local coordinates on SM can then be taken as q^a together with the components of the four-momentum p^a in these coordinates. The Vlasov equation then reads

$$\partial_t f + \frac{p^j}{p^0}\partial_{x^j} f - \frac{1}{p^0}\Gamma^j_{ab} p^a p^b \partial_{p^j} f = 0. \tag{18}$$

Here $a, b = 0, 1, 2, 3$ and $j = 1, 2, 3$ and Γ^j_{ab} are the Christoffel Connection. It is understood that p^0 is expressed in terms of p^j and the metric g_{ab} using the relation $g_{ab}p^a p^b = -1$

2.1 The Lagrangian Rational

An attempt at obtaining the Einestien-Vlasov equations from the Lagrangian formalism. In this section I will follow a different approach than what I have presented in the previous sections concerning the derivation of kinetic equations. There is not in any relevant scientific material regarding the Lagrangian derivation of the Vlasov-Einstein equations in the literature . Therefore, In this section of the paper I will present an analysis in the Lagrangian framework of these equations.

Consider a collection of free particles where collisions are rare and can be ignored. Let (Q, g_{ab}) be a 4-dimensional spacetime manifold and (P, η_{ab})

be the 4-dimension energymomentum manifold. We form the space $S = (Q \times P, g_{ab}, \eta_{ab})$. Now we equip S with a probability measure $d\mu(t,q,p) = f(t,q,p)dtdQdP$ which is bounded and compactly supported with the additional property

$$\int_S d\mu = 1. \tag{19}$$

The equation above is essential for deriving the continuity equation (i.e. Vlasov equation) on the manifold, which we show below

$$\begin{aligned} \int_S d\mu = \int_S f(t,q,p)dtdQdP = 1. \\ d\int_S f(t,q,p)dtdQdP = 0. \\ df(t,q,p) = 0 \Rightarrow \frac{df}{dt}(t,q,p) = 0. \\ \partial_t f + \dot{q}^j \partial_{q^j} f + \dot{p}^j \partial_{p^j} f = 0. \\ \dot{p}^j = -\Gamma^j_{ik} p^i p^k. \\ \partial_t f + \dot{q}^j \partial_{q^j} f - \Gamma^j_{ik} p^i p^k \partial_{p^j} f = 0. \end{aligned} \tag{20}$$

Notice that the form of equation 20 is different from 18. In the derivation I presented here, I have assume the momentum vector $p = (p^1, p^2, p^3)$ to be normalized by the energy p^0, so that the 4-momentum will read $(-1, p^1, p^2, p^3)$. The factor p^0 will be absorbed in the metric η_{ab}.

We have derived the Vlasov equation. Now we proceed to the formulation of the Vlasov-Einstein Lagrangian. $f(t,q,p)$ can be thought of as a probability density, scalar field or a rank-zero tensor. The particles have equal non-zero rest mass m_0 which we set to one. Let N be the total number of particles and V is the volume in the phase space (q,p). The Lagrangian of a single particle is just the kinetic energy of the particle $L_\alpha = g_{ab} p^a_\alpha p^b_\alpha$, p_α is just the momentum the α-particle (p^1, p^2, p^3). The the total Lagrangian for the collection of all free particles is given by

$$L_M = \sum_\alpha^N L_\alpha = \sum_\alpha^N g_{ab} p^a_\alpha p^b_\alpha \tag{21}$$

In the continuum limit? I don't know what does this mean but we physicists love to use it and can't really justify it. So, When $V/N \to 0$ or the density of particles tends to infinity! the sum over the variable α can be converted to an integral on the Manifold S as

$$\sum_{\alpha}^{N} g_{ab} p_{\alpha}^{a} p_{\alpha}^{b} \rightarrow L_M(t,p,q) =$$
$$\int_{Q'} \int_{P'} f(t, q+q', p+p')(g_{ab}(p^a + p'^a)(p^b + p'^b)) dQ' dP' \tag{22}$$

Where $dQ'dP'$ is the element of volume on phase space. $dP' = \sqrt{-\eta} dp'^1 dp'^2 dp'^3$ and $dQ' = \sqrt{-g} dq'^1 dq'^2 dq'^3$

Alternatively, We can define the notion of a Lagrangian at a point (t,q,p) as a function $\lambda(t,q,p) = g_{ab} p^a p^b$. The total Lagrangian of the manifold evaluated at that point is given by the integral in equation

$$L(t,p,q) = \int_{Q'} \int_{P'} f(t, q+q', p+p') \lambda(t, q+q', p+p') dQ' dP' \tag{23}$$

The basic idea behind this definition is motivated by the notion that we could think of a collection of particles as a single particle with a cloud or matter field around it.

We are ready to obtain the Einstein-Vlasov equations. The action for the Einstein-Vlasov Lagrangian is give by

$$\mathcal{I} = \int_{Q} L_G + L_M dQ \tag{24}$$

The variation of the action leads to $\delta(L_G + L_M) = 0$. Thus we get the following equations

$$\delta L_G = - \int_{Q'} \int_{P'} (\delta f \lambda + f \delta \lambda) dQ' dP' \tag{25}$$

Since $\delta f = 0$, by the conservation of the measure μ on the manifold. Since $\delta \lambda = \frac{\delta \lambda}{\delta g_{ab}} \delta g_{ab} = p^a p^b$ and $\frac{\delta L_G}{\delta g_{ab}} \delta g_{ab}$ leads to the Einstein field equation.

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} =$$
$$- \int_{Q'} \int_{P'} f(t, q+q', p+p')(p_a + p'_a)(q^b + q'_b) dQ' dP'. \tag{26}$$

Substituting for $dQ'dP'$, the momentum energy tensor is then

$$T^{ab}(t,p,q) = -\int_{Q'}\int_{P'} f(t,q+q',p+p')(p^a+p'^a)(q^b+q'^b) \sqrt{(-g)(-\eta)}dq'^1dq'^2dq'^3dp'^1dp'^2dp'^3 \quad (27)$$

Considering a cloud of particles at the origin of the coordinate system defined on the manifold S we get the energy momentum tensor to be

$$\begin{aligned} T^{ab}(t,0,0) &= -\int_Q\int_P f(t,q,p)p^ap^b\sqrt{g\eta}dq^1dq^2dq^3dp^1dp^2dp^3 \\ &= -\int_Q f(t,p)p^ap^b\sqrt{g\eta}dp^1dp^2dp^3 \end{aligned} \quad (28)$$

Which is the same as the one reported in [2]. Finally, I am done.

2.2 Analysis of Einstein-Vlasov equation

The following discussion is primely due to Andréasson [2]. Consider a fixed spacetime where the matter exits at the origin of the coordinate system. The energy-momentum tensor is

$$T_{ab} = -\int_P f(t,p)p_ap_b|g\eta|^{1/2}dp^1dp^2dp^3,$$

The Vlasov equation is a linear hyperbolic equation for f and we can solve it by solving the characteristic system

$$\begin{aligned} \partial_t f + \dot{q}^j\partial_{q^j} f = 0 \Rightarrow & \\ dX^i/ds &= -P^i, \quad (29) \\ \partial_t f - \Gamma^j_{ik}p^ip^k\partial_{p^j} f = 0 \Rightarrow & \\ dP^i/ds &= \Gamma^i_{jk}P^aP^b. \quad (30) \end{aligned}$$

In terms of initial data f_0 the solution to the Vlasov equation can be written

$$f(q^j,p^i) = f_0(Q^i(0,q^j,p^i),P^i(0,q^j,p^i)), \quad (31)$$

where $Q^i(s,q^j,p^i)$ and $P^i(s,q^j,p^i)$ solve (41) and (42) and where $Q^i(t,q^j,p^i) = q^i$ and $P^i(t,q^j,p^i) = p^i$.

The Einstein-Vlasov system where as usual $p_a = g_{ab}p^b$ and $|g|$ denotes the absolute value of the determinant of g.

$$G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab},$$

Here G_{ab} is the Einstein tensor, R_{ab} the Ricci tensor and R is the scalar curvature. We also define the particle current density

$$N^a = -\int_{\mathbb{R}^3} f\, p^a |g\eta|^{1/2} dp^1 dp^2 dp^3.$$

Using normal coordinates at a given point and assuming that f is compactly supported it is not hard to see that T_{ab} is divergence-free which is a compatibility condition with G_{ab} being divergence-free by the Bianchi identities [9]. N^a is also, divergence-free which expresses the fact that the number of particles is conserved [2]. The definitions of T_{ab} and N^a immediately give us a number of inequalities. If V^a is a future directed timelike or null vector then we have $N_a V^a \leq 0$ with equality if and only if $f = 0$ at the given point. Hence N^a is always future directed timelike if there are particles at that point. Moreover, if V^a and W^a are future directed timelike vectors then $T_{ab}V^aW^b \geq 0$, which is the dominant energy condition. If X^a is a spacelike vector then $T_{ab}X^aX^b \geq 0$. This is the non-negative pressure condition. These last two conditions together with the Einstein equations imply that $R_{ab}V^aV^b \geq 0$ for any timelike vector V^a, which is the strong energy condition. That the energy conditions hold for Vlasov matter is one reason that the Vlasov equation defines a well-behaved matter model in general relativity [2].

2.3 Spherically Symmetric spacetime

In general relativity If an isolated body is studied the data are called asymptotically flat. Far away from the body one expects spacetime to be topologically $\mathbb{R}^3$. An asymptotically flat spacetime means that if there exist global coordinates q^i such that as $|q|$ tends to infinity $g_{ij} \to \delta_{ij}$. Important asymptotically flat spacetime is the spherically symmetric or asymptotically flat except in one direction, namely cylindrical spacetimes. Global properties of solutions to spherically symmetric Einstein-Vlasov system was studied by Rein and Rendall in 1990. They choose Schwarzschild metric of the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $t \in \mathbb{R}$, $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. Asymptotic flatness is expressed by the boundary conditions

$$\lim_{r\to\infty} \lambda(t,r) = \lim_{r\to\infty} \mu(t,r) = 0, \ \forall t \geq 0.$$

A regular center is also required and is guaranteed by the boundary condition

$$\lambda(t,0) = 0.$$

With

$$x = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

as spatial coordinate and

$$v^j = p^j + (e^\lambda - 1)\frac{x\cdot p}{r}\frac{x^j}{r}$$

as momentum coordinates the Einstein-Vlasov system reads

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\sqrt{1+v^2}}\cdot\partial_x f - (\lambda_t\frac{x\cdot v}{r} + e^{\mu-\lambda}\mu_r\sqrt{1+v^2})\frac{x}{r}\cdot\partial_v f = 0, \quad (32)$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2\rho, \quad (33)$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p. \quad (34)$$

The matter quantities are defined by

$$\rho(t,x) = \int_{\mathbb{R}^3}\sqrt{1+|v|^2}f(t,x,v)\,dv, \quad (35)$$
$$p(t,x) = \int_{\mathbb{R}^3}\Big(\frac{x\cdot v}{r}\Big)^2 f(t,x,v)\,\frac{dv}{\sqrt{1+|v|^2}}. \quad (36)$$

Let the square of the angular momentum be denoted by L, i.e.

$$L := |x|^2|v|^2 - (x\cdot v)^2.$$

A consequence of spherical symmetry is that angular momentum is conserved along the characteristics of (32). Introducing the variable

$$w = \frac{x\cdot v}{r},$$

the Vlasov equation for $f = f(t, r, w, L)$ becomes

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - (\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{F}{r^3 E})\partial_w f = 0, \tag{37}$$

where

$$E = E(r, w, L) = \sqrt{1 + w^2 + L/r^2}.$$

The matter terms take the form

$$\rho(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} Ef(t, r, w, L)\, dwdL, \tag{38}$$

$$p(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{w^2}{E} f(t, r, w, L)\, dwdL. \tag{39}$$

Let us write down a couple of known facts about the system (33),(34),(37),(38) and (39). A solution to the Vlasov equation can be written

$$f(t, r, w, L) = f_0(R(0, t, r, w, L), W(0, t, r, w, L), L), \tag{40}$$

where R and W are solutions to the characteristic system

$$\frac{dR}{ds} = e^{(\mu-\lambda)(s,R)}\frac{W}{E(R, W, L)}, \tag{41}$$

$$\frac{dW}{ds} = -\lambda_t(s, R)W - e^{(\mu-\lambda)(s,R)}\mu_r(s, R)E(R, W, L) + e^{(\mu-\lambda)(s,R)}\frac{L}{R^3 E(R, W, L)}, \tag{42}$$

such that the trajectory $(R(s, t, r, w, L), W(s, t, r, w, L), L)$ goes through the point (r, w, L) when $s = t$. An Important conservation laws for the Einstein-Vlasov system is the conservation of the number of particles

$$4\pi^2\int_0^{\infty} e^{\lambda(t,r)}\left(\int_{-\infty}^{\infty}\int_0^{\infty} f(t, r, w, L) dLdw\right) dr,$$

3 Acknowledgment

I acknowledge that this review paper, to a large extent is primely based on [2].

References

[1] Toyoki KogaIntroduction to Kinetic theory stochastic processes in Gaseous Systems, *Pergamon Press,* (1970)

[2] H. Andréasson, The Einstein-Vlasov system/Kinetic theory, *"http://www.livingreviews.org/Articles/Volume5/2002-7andreasson/index.html"*, (18 October 2002).

[3] Liboff Introduction to the theory of kinetic equations, *John Wiley and Sons, Inc.,* (1968)

[4] Duderstadt Transport Theory, *John Wiley and Sons, Inc.,* (1979)

[5] M. Dudyński and M. Ekiel Jezewska, Global existence proof for relativistic Boltzmann equation, *J. Stat. Phys.,* 66, 991-1001, (1992).

[6] R. Glassey, *The Cauchy problem in kinetic theory,* SIAM, Philadelphia, 1996.

[7] S.R. de Groot, W.A. van Leeuwen, C.G. van Weert, *Relativistic Kinetic Theory.* North-Holland, Amsterdam, 1980.

[8] J.M. Stewart, *Non-equilibrium relativistic kinetic theory.* Lecture notes in physics 10, Springer-Verlag, Berlin 1971.

[9] R. Wald, *General relativity,* The University of Chicago Press, Chicago, 1984.

[10] Alan D. Rendall, *An Introduction to the Einstein-Vlasov System,* arxiv:gr-qc/9604001, v1, 1 Apr 1996.