Financial Markets in the Context of the General Theory of Optional Processes

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Abstract A probability space is considered ”unusual” if the information flow is not right or left continuous or is not complete. On these probability spaces lives certain type of stochastic processes known as optional processes including optional semimartingales. Optional processes have right and left limits but are not necessarily right or left continuous. Here, we present a short summary of the calculus of optional processes and define stochastic logarithms and present some of its properties. Moreover, we develop a financial market model based on optional semimartingales and a methods for finding local martingale deflators for this market. Finally, we present some financial examples.

1 Introduction

The assumption that the stochastic basis $\left(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t \geq 0}, \mathbf{P}\right)$ satisfy the usual conditions, where $\mathbf{F}$ is complete and right-continuous, is a foundation concept in the theory of stochastic processes. Stochastic processes that are adapted to this basis form a large class of processes known as semimartingales whose paths are right-continuous with left limits (RCLL). This theory has been instrumental in generating many important results in mathematical finance and in the theory of stochastic processes. It is difficult to conceive of a theory of stochastic processes without the usual conditions and RCLL processes.

However, it turns out that it is not difficult to give some examples of stochastic basis without the usual conditions. In 1975 Dellacherie [5] started to study the theory of stochastic processes without the assumption of the usual conditions, termed ”unusual conditions”.

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Further developments of this theory were carried on by Lepingle (1977) [13], Horowitz (1978) [8], Lenglart (1980) [12], and mostly by Galtchouk [6, 7].

We believe that the theory optional processes will offer a natural foundation and a versatile set of tools for modeling financial markets. We can mention here few research problems that were not treated with the methods of the calculus of optional processes but possibly should be. The first one, is a recent development in mathematical finance specially in pricing of derivative contracts and hedging under transaction costs (see [2] for details) that hints to the needed application of the calculus of optional processes to price derivatives and hedge under transaction costs. Furthermore, in models [1] with stochastic dividends paid at random times, there is an opportunity to treat these problems in the context of optional semimartingale theory in a natural way. Duffie [3] presented a new approach to modeling term structures of bonds and contingent claims that are subject to default risk. Perhaps Duffie's method could be studied with the methods of optional calculus. The aim of this paper is to present the theory of optional processes on unusual stochastic basis, develop new results and bring its methods to mathematical finance. The paper is organized as follows. Section 2 presents foundation material on optional processes. Section 3 introduces stochastic exponentials and logarithms. Section 4 describes optional semimartingale model of a financial market and two methods for finding local martingale deflators for these markets. Section 5 presents examples of optional semimartingale markets. Finally, we give some concluding remarks.

2 Foundation

Suppose we are given $\left(\Omega,\mathscr{F},\mathbf{F}=(\mathscr{F}_t)_{t\geq 0},\mathbf{P}\right)$, $t\in[0,\infty)$, where $\mathscr{F}_t\in\mathbf{F}$, $\mathscr{F}_s\subseteq\mathscr{F}_t$, $s\leq t$, a complete probability space. It is complete because $\mathscr{F}$ contains all $\mathbf{P}$ null sets. But this space is unusual because the family $\mathbf{F}$ is not assumed to be complete, right or left continuous. On this space, we introduce $\mathscr{O}(\mathbf{F})$ and $\mathscr{P}(\mathbf{F})$ the σ-algebras of optional and predictable processes, respectively (see [7]). A random process $X=(X_t)$, $t\in[0,\infty)$, is said to be *optional* if it is $\mathscr{O}(\mathbf{F})$-measurable. In general, optional processes have right and left limits but are not necessarily continuous. For an optional process we can define the following: (a) $X_-=(X_{t-})_{t\geq 0}$, a left continuous version of the process X and $X_+=(X_{t+})_{t\geq 0}$, the right continuous version of X; (b) The jump processes $\triangle X=(\triangle X_t)_{t\geq 0}$ and $\triangle X_t=X_t-X_{t-}$ and (c) $\triangle^+X=(\triangle^+X_t)_{t\geq 0}$, $\triangle^+X_t=X_{t+}-X_t$. A random process (X_t), $t\in[0,\infty)$, is predictable if $X\in\mathscr{P}(\mathbf{F})$ and *strongly predictable* if $X\in\mathscr{P}(\mathbf{F})$ and $X_+\in\mathscr{O}(\mathbf{F})$. We denote by $\mathscr{P}_s(\mathbf{F})$ the set of strongly predictable processes. An optional semimartingale $X=(X_t)_{t\geq 0}$ is an optional process that can be decomposed to an optional local martingale $M\in\mathscr{M}_{loc}$ and an optional finite variation processes $A\in\mathscr{V}$, i.e. $X=X_0+M+A$, [7]. A semimartingale X is called special if the above decomposition exists but with A being a strongly predictable process ($A\in\mathscr{A}_{loc}$ the set of locally integrable finite variation processes [7]). Let $\mathscr{S}$ denote the set of optional semimartingales and $\mathscr{S}p$ the set of special optional semimartingales. If $X\in\mathscr{S}p$ then the semimartingale decomposition is unique. Using the decomposition optional martingales and of finite variation processes we can decompose a semimartingale further to $X = X_0 + X^r + X^g = X_0 + (A^r + M^r) + (A^g + M^g)$ where A^r and A^g are right and left continuous finite variation processes, respectively. $M^r \in \mathscr{M}^r_{loc}$ is a right-continuous local martingale and $M^g \in \mathscr{M}^g_{loc}$ is a left-continuous local martingale.

The stochastic integral with respect to optional semimartingale X is defined as a bilinear form $(\varphi, \phi) \circ X_t$ where

$$Y_t = (\varphi, \phi) \circ X_t = Y_0 + \varphi \cdot X^r_t + \phi \odot X^g_t = \int_{0+}^{t} \varphi_{s-} dX^r_s + \int_0^{t-} \phi_s dX^g_{s+}.$$

where Y_t is again an optional semimartingale $\varphi_- \in \mathscr{P}(\mathbf{F})$, and $\phi \in \mathscr{O}(\mathbf{F})$, such that $\left(\varphi^2 \cdot [X^r, X^r]\right)^{1/2} \in \mathscr{A}_{loc}$ and $\left(\phi^2 \odot [X^g, X^g]\right)^{1/2} \in \mathscr{A}_{loc}$. The integral $\int_{0+}^t \varphi_{s-} dX^r_s$ is our usual stochastic integral with respect to RCLL semimartingale however the integral $\int_0^{t-} \phi_s dX^g_{s+}$ is Galchuk stochastic integral [7] with respect to left-continuous semimartingale.

3 Stochastic Exponentials and Logarithms

Stochastic exponentials and logarithms are indispensable tools of financial mathematics, for example, (see, Melnikov et al. 2002 [14]). They describe relative returns, link hedging with the calculation of minimal entropy and therefore utility indifference. Moreover, they determine the structure of the Girsanov transformation.

For optional semimartingales the stochastic exponential was defined by Galchuk [7]. If $X \in \mathscr{S}$ then there exists a unique semimartingale $Z \in \mathscr{S}$ such that

$$Z_t = Z_0 + Z \circ X_t = Z_0 + \int_{0+}^t Z_{s-} dX^r_s + \int_0^{t-} Z_s dX^g_{s+} = Z_0 \mathscr{E}(X)_t, \tag{1}$$

Two useful properties of stochastic exponentials are the inverse and product formulas. The inverse of stochastic exponential is $\mathscr{E}^{-1}(h) = \mathscr{E}(-h^*)$, such that,

$$h^*_t = h_t - \langle h^c, h^c \rangle_t - \sum_{0<s\le t} \frac{(\triangle h_s)^2}{1 + \triangle h_s} - \sum_{0 \le s < t} \frac{(\triangle^+ h_s)^2}{1 + \triangle^+ h_s}.$$

And, the product of stochastic exponentials is $\mathscr{E}(U)\mathscr{E}(V) = \mathscr{E}(U + V + [U,V])$ where $[U,V]_t = \langle U^c, V^c \rangle_t + \sum_{0<s\le t} \triangle U_s \triangle V_s + \sum_{0\le s<t} \triangle^+ U_s \triangle^+ V_s$.

The stochastic logarithm is defined by the following theorem.

Theorem 1. *Let Y be a real valued optional semimartingale such that the processes Y_- and Y do not vanish then the process*

$$X_t = \frac{1}{Y} \circ Y_t = \int_{0+}^t \frac{1}{Y_{s-}} dY^r_s + \int_0^{t-} \frac{1}{Y_s} dY^g_{s+}, \quad X_0 = 0, \tag{2}$$

also denoted by $X = \mathscr{L}\text{og}Y$ is called the stochastic logarithm of Y. The process X is a unique semimartingale such that $Y = Y_0\mathscr{E}(X)$. Moreover, if $\Delta X \neq -1$ and $\Delta^+ X \neq -1$ we also have

$$\mathscr{L}\text{og}Y_t = \log\left|\frac{Y_t}{Y_0}\right| + \frac{1}{2Y^2}\circ\langle Y^c, Y^c\rangle_t - \sum_{0<s\leq t}\left(\log\left|1+\frac{\Delta Y_s}{Y_{s-}}\right| - \frac{\Delta Y_s}{Y_{s-}}\right) \quad (3)$$
$$- \sum_{0\leq s<t}\left(\log\left|1+\frac{\Delta^+ Y_s}{Y_s}\right| - \frac{\Delta^+ Y_s}{Y_s}\right).$$

It is important to note that the process Y need not be positive for $\mathscr{L}\text{og}(Y)$ to exist, in accordance with the fact that the stochastic exponential $\mathscr{E}(X)$ may take negative values.

Proof. The assumptions that Y_- and Y don't vanish implies that: $S_n = \inf\left(t : |Y_{t-}| \leq \frac{1}{n}\right) \uparrow \infty$, hence $1/Y_-$ is locally bounded; likewise, $T_n = \inf\left(t : |Y_t| \leq \frac{1}{n}\right) \uparrow \infty$, hence $1/Y$ is also locally bounded. Therefore, the stochastic integral in (2) makes sense. Let $\tilde{Y} = Y/Y_0$ then $\tilde{Y}_0 = 1$. By equation (2) we have that $X = (1/\tilde{Y})\circ\tilde{Y}$. Therefore, $1+\tilde{Y}\circ X = 1+\tilde{Y}\circ\left(\frac{1}{\tilde{Y}}\circ\tilde{Y}\right) = 1+\left(\tilde{Y}\frac{1}{\tilde{Y}}\right)\circ\tilde{Y} = \tilde{Y}$, i.e. $\tilde{Y} = \mathscr{E}(X)$. Furthermore, $\Delta X = \Delta Y/Y_- \neq -1$ and $\Delta^+ X = \Delta^+ Y/Y \neq -1$. To obtain uniqueness let $\tilde{X}$ be any other semimartingale satisfying $Y = \mathscr{E}(\tilde{X})$. Since $\tilde{Y} = Y/Y_0$ then $\tilde{Y} = \mathscr{E}(\tilde{X})$. Therefore $\tilde{Y} = 1+\tilde{Y}\circ\tilde{X}$ and $Y = Y_0 + \tilde{Y}\circ\tilde{X}$. But since $X_0 = 0$ we have $\tilde{X} = \frac{\tilde{Y}}{\tilde{Y}}\circ\tilde{X} = \frac{1}{\tilde{Y}}\circ(Y - Y_0) = \frac{1}{\tilde{Y}}\circ Y = X$ and we obtain uniqueness.

To deduce equation (3) we apply Gal'chuk-Ito's lemma (see [7]) for the optional semimartingale $\log|Y|$. But the log function explodes at 0. To circumvent this problem consider for each n the C^2 functions $f_n(x) = \log|x|$ on $\mathbb{R}$ such that $|x| \geq 1/n$. Consequently for all n, $t < T_n$ and $t < S_n$ we get $\log|Y_t| = \log|Y_0| + \frac{1}{Y}\circ Y_t - \frac{1}{2Y^2}\circ\langle Y^c, Y^c\rangle_t + \sum_{0<s\leq t}\left(\Delta\log|Y_s| - \frac{\Delta Y_s}{Y_{s-}}\right) + \sum_{0\leq s<t}\left(\Delta^+\log|Y_s| - \frac{\Delta^+ Y_s}{Y_s}\right)$. This result together with equation (2) yields equation (3) for $t < T_n$ and $t < S_n$. Since, $T_n \uparrow \infty$ and $S_n \uparrow \infty$ we obtain equation (3) everywhere.

Now, we present some of the properties of stochastic logarithms.

Lemma 1. *(a) If X is a semimartingale satisfying $\Delta X \neq -1$ and $\Delta^+ X \neq -1$ then $\mathscr{L}\text{og}(\mathscr{E}(X)) = X - X_0$. (b) If Y is a semimartingale such that Y and Y_- do not vanish, then $\mathscr{E}(\mathscr{L}\text{og}(Y)) = Y/Y_0$. (c) For any two optional semimartingales X and Z we get the following identities: $\mathscr{L}\text{og}(XZ) = \mathscr{L}\text{og}X + \mathscr{L}\text{og}Z + [\mathscr{L}\text{og}X, \mathscr{L}\text{og}Z]$; $\mathscr{L}\text{og}\left(\frac{1}{X}\right) = 1 - \mathscr{L}\text{og}(X) - \left[X, \frac{1}{X}\right]$.*

Proof. (a) $\mathscr{L}\text{og}(\mathscr{E}(X)) = \mathscr{E}(X)^{-1}\circ\mathscr{E}(X) = \frac{\mathscr{E}(X)}{\mathscr{E}(X)}\circ X = X - X_0$. (b) Let $Z = \mathscr{E}(\mathscr{L}\text{og}(Y))$ then by (a) we find that $\mathscr{L}\text{og}(Z) = \mathscr{L}\text{og}(\mathscr{E}(\mathscr{L}\text{og}(Y))) = \mathscr{L}\text{og}(Y) - \mathscr{L}\text{og}(Y_0) = \mathscr{L}\text{og}(Y/Y_0)$. Therefore, $Z = Y/Y_0$. (c) By the integral definition of the stochastic logarithm we find $\mathscr{L}\text{og}(XZ) = \frac{1}{XZ}\circ(X\circ Z + Z\circ X + [X,Z]) = \mathscr{L}\text{og}X + \mathscr{L}\text{og}Z + [\mathscr{L}\text{og}X, \mathscr{L}\text{og}Z]$. Using integration by parts and the integral definition the stochastic logarithm, $\mathscr{L}\text{og}\left(\frac{1}{X}\right) = X\circ\left(\frac{1}{X}\right) = 1 - \frac{1}{X}\circ X - \left[X, \frac{1}{X}\right] = 1 - \mathscr{L}\text{og}(X) - \left[X, \frac{1}{X}\right]$.

4 Markets of Optional Processes

Let $\left(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbf{P}\right)$, $t \in [0,\infty)$, be the unusual stochastic basis and that the financial market stays on this space. The market consists of two types of securities x and X. A portfolio $\pi = (\eta, \xi)$, is composed of the optional processes η and ξ. η is the volume of the reference asset x while ξ is the volume of the security X. Suppose $x_t > 0$ and $X_t \geq 0$ for all $t \geq 0$ and write the ratio process $R_t = X_t/x_t$. Then, the value of the portfolio is

$$Y_t = \eta_t + \xi_t R_t. \tag{4}$$

We restrict the portfolio π to be self-financing that is we must have,

$$Y_t = Y_0 + \xi \circ R_t. \tag{5}$$

Reconciling equations (4) and (5) we obtain $C_t = \eta_t + R \circ \xi_t + [\xi, R]_t = Y_0 = C_0$ where C_t is the consumption process with its initial value C_0. Since the ratio process R is optional semimartingale then ξ, evolves in the space $\mathscr{P}(\mathbf{F}) \times \mathscr{O}(\mathbf{F})$ with the predictable part determining the volume of R^r and the optional part determining the volume of R^g. Also, η belongs to the space $\mathscr{O}(\mathbf{F})$. Furthermore, for the integral in equation (5) to be well defined ξ must be R-integrable, $\int_0^\infty \xi_s^2 d[R,R]_s \in \mathscr{A}_{loc}$.

One can interpret the trading strategy ξ in several different ways. First, eventhough ξ gives us the option to trade different parts of the ratio-process, R^r predictably and R^g optionally, we can still trade both parts in the same way, predictably. Alternatively, in a portfolio of a sum of left-continuous and right-continuous assets (see example 5) we can trade each asset independently. The left-continuous asset will be traded with an optional strategy while the right-continuous asset will be traded by a predictable one. This certainly a possible portfolio in the current structure of financial markets. Moreover, with a theory of optimal portfolio with consumption and endowment streams we can interpret the predictable trades of R^r as the optimal portfolio while the optional trades of R^g as an optimal consumption or endowment streams. Yet another way to think about optional portfolios in financial markets is in terms of market-orders. Market-orders can either be predictable or optional. Optional in the sense that trades can be executed based on a condition, for example the stock price passing some boundary. Conditional trades are optional processes that act on the left-continuous part of the ratio-process R^g.

Finally, note that the ratio process is an optional semimartingale and must be transformed to a local optional martingale for any pricing and hedging theory to be viable. In the next section we show how to find local martingale transforms that change the ratio process to some optional local martingales. A special subset of these martingale deflators can transform RLL optional semimartingales to RCLL local martingale. This special subset of transforms can be useful in cases where markets only allow for predictable trading strategies. We will illustrate this procedure in example 5.

As in Melnikov et al. 2002 [14] for RCLL We suppose that the dynamics of securities in our market follows the stochastic exponential, $X_t = X_0 \mathscr{E}_t(H)$ and

$x_t = x_0\mathscr{E}_t(h)$ where x_0 and X_0 are $\mathscr{F}_0$-measurable random variables. $h = (h_t)_{t\geq 0}$ and $H = (H_t)_{t\leq 0}$ are optional semimartingales admitting the representations, $h_t = h_0 + a_t + m_t$ and $H_t = H_0 + A_t + M_t$ with respect to (w.r.t) $\mathbf{P}$. $a = (a_t)_{t\geq 0}$ and $A = (A_t)_{t\geq 0}$ are locally bounded variation processes and predictable. $m = (m_t)_{t\geq 0}$ and $M = (M_t)_{t\geq 0}$ are optional local martingales. A local martingale deflator is a strictly positive supermartingale multiplier used in mathematical finance to transform the value process of a portfolio to a supermartingale (i.e. a local martingale). Here we will develop methods for finding local martingale deflators. We can write R as,

$$\begin{aligned}
R_t &= \frac{X_t}{x_t} = R_0\mathscr{E}(H)_t\mathscr{E}^{-1}(h)_t = R_0\mathscr{E}(H_t)\mathscr{E}(-h_t^*) = R_0\mathscr{E}(\Psi(h_t,H_t)), \qquad (6)\\
\Psi_t &= \Psi(h_t,H_t) = H_t - h_t^* - [H,h^*]_t = H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J^d + J^g,\\
J^d &= \sum_{0<s\leq t} \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s}, \quad J^g = \sum_{0\leq s<t} \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}.
\end{aligned}$$

if Ψ is a local optional martingale then R is a local optional martingale and we are done. Otherwise, we have to find a strictly positive transformation $Z \in \mathscr{M}_{loc}$ that will render $ZR \in \mathscr{M}_{loc}$. Z is known as the local martingale deflator. For a strictly positive Z, we can define $N \in \mathscr{M}_{loc}$ with $N = \mathscr{L}\text{og}(Z) = Z^{-1}\circ Z$ or $Z = \mathscr{E}(N)$. To find N we have the following theorem;

Theorem 2. *Given $R = R_0\mathscr{E}(\Psi(h,H))$ where $\Psi(h,H)$ as in equation (6) and $Z = \mathscr{E}(N)$ where $Z, N \in \mathscr{M}_{loc}(\mathbf{P},\mathbf{F})$ and $Z > 0$ then $ZR \in \mathscr{M}_{loc}$ is a local optional martingale if and only if $(A-a) + \langle m^c - N^c, m^c - M^c\rangle + \tilde{K}^d + \tilde{K}^g = 0$, where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of the processes*

$$K^d = \sum_{0<s\leq t} \frac{(\triangle h_s - \triangle N_s)\,(\triangle h_s - \triangle H_s)}{1+\triangle h_s}, K^g = \sum_{0\leq s<t} \frac{(\triangle^+ h_s - \triangle^+ N_s)\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}.$$

Proof. Suppose $Z_t = \mathscr{E}(N)_t \in \mathscr{M}_{loc}$, $Z_t > 0$ for all t such that $ZR \in \mathscr{M}_{loc}$ then$ZR = R_0\mathscr{E}(N)\mathscr{E}(\Psi(h,H)) = R_0\mathscr{E}(\Psi(h,H,N))$, where $\Psi(h,H,N) = N_t + H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J_t^d + J_t^g + [N,H] - [N,h] + [N,J^d] + [N,J^g]$, hence,

$$\begin{aligned}
\Psi(h,H,N) = N_t &+ H_t - h_t + \langle h^c, h^c - H^c\rangle_t + J_t^d + J_t^g\\
&+ \langle N^c, H^c\rangle_t + \sum_{0<s\leq t} \triangle N_s \triangle H_s + \sum_{0\leq s<t} \triangle^+ N_s \triangle^+ H_s\\
&- \langle N^c, h^c\rangle_t - \sum_{0<s\leq t} \triangle N_s \triangle h_s - \sum_{0\leq s<t} \triangle^+ N_s \triangle^+ h_s\\
&+ \sum_{0<s\leq t} \triangle N_s \frac{\triangle h_s(\triangle h_s - \triangle H_s)}{1+\triangle h_s}\\
&+ \sum_{0\leq s<t} \triangle^+ N_s \frac{\triangle^+ h_s\,(\triangle^+ h_s - \triangle^+ H_s)}{1+\triangle^+ h_s}.
\end{aligned}$$

therefore, $\Psi(h,H,N) = N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t + K^d + K^g$. So, if $\Psi(h,H,N) \in \mathscr{M}_{loc}$ then $ZR \in \mathscr{M}_{loc}$. And if $\triangle^+\Psi(h,H,N) \neq -1$ and $\triangle\Psi(h,H,N) \neq -1$ then $\Psi(h,H,N) \in \mathscr{M}_{loc} \Leftrightarrow ZR \in \mathscr{M}_{loc}$. Now, let us take into consideration the decomposition of H and h and write $\Psi(h,H,N) = (A-a) + (M-m+N) + \langle (m-N)^c, (m-M)^c\rangle + K^d + K^g$. So, $\Psi(h,H,N)$ is a local optional martingale under $\mathbf{P}$ if $(A-a) + \langle m^c - N^c, m^c - M^c\rangle + \tilde{K}^d + \tilde{K}^g = 0$ where $\tilde{K}^d$ and $\tilde{K}^g$ are the compensators of K^d and K^g, respectively.

By finding all $N \in \mathscr{M}_{loc}$ such that the above equation (4) is valid and $\mathscr{E}(N) > 0$ we find the set of all appropriate local optional martingale transforms Z such that ZR is a local optional martingale. Note that if Z is a local martingales transform such that ZR is a local martingale then it is true for all self financing strategies π.

Theorem 3. *If Z is a local martingale transform of R, that is ZR is a local optional martingale, and π is a self financing portfolio which is R-integrable then ZY_t^π is a local optional martingale.*

Proof. Z is a local martingale transform of R therefore $Z > 0$. $\pi = (\eta, \xi)$ is self financing and R-integrable, then $Y_t^\pi = Y_0 + \xi \circ R_t$ and $Z_t Y_t^\pi$ can be written as, $d(Z_t Y_t^\pi) = \xi_t [Z_t dR_t + dZ_t R_t + d[Z_t, R_t]] + dZ_t \eta_t = \xi_t d(Z_t R_t) + dZ_t \eta_t$. This leads us to the following result $Z_t Y_t^\pi = \xi \circ Z_t R_t + \eta \circ Z_t$. $\eta \circ Z_t$ and $\xi \circ Z_t R_t$ are local optional martingales therefore their sum $Z_t Y_t^\pi$ is a well defined local optional martingale. Note, that we have implicitly used the fact that η is bounded, i.e. comes from the fact that π is a self financing and also that, $\int_0^\infty \xi_t^2 d[ZR]_t \in \mathscr{A}_{loc}$.

On the other hand, if we know that there exist a Z such that ZY^π is a local optional martingale then what can we say about the portfolio π and the product ZR? It is reasonable to suppose that $Z = \mathscr{E}(N) > 0$, π-self-financing, ξ is R-integrable and η is bounded. In this case, $\xi \circ Z_t R_t = Z_t Y_t^\pi - \eta \circ Z_t$ is a sum of two local optional martingales and therefore a local optional martingale it self, for any optional process ξ, in particular for $\xi = 1$; therefore ZR is a local optional martingale.

An alternative approach can be developed using stochastic logarithms. What is interesting about this approach is that we don't have to define the process R as a stochastic exponential of an underlying process Ψ. All that is required is that the ratio process R and its predictable version R_- don't vanish, except on sets of measure zero. This approach is technically based on the following lemmas and results.

Lemma 2. *Suppose $X = X_0 + A + M$, $x = x_0 + a + m$, $R = X/x$ and $R_- \neq 0$ and $R \neq 0$ a.s. $\mathbf{P}$, then $\mathscr{L}\mathrm{og}(R)$ is a local optional martingale if and only if $\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2}[m,m] - \frac{1}{xX} \circ [M,m] = -1$.*

Lemma 3. *Suppose that R_- and R don't vanish then $\mathscr{L}\mathrm{og}(R) \in \mathscr{M}_{loc} \Leftrightarrow R \in \mathscr{M}_{loc}$.*

If R is not a local martingale then there exists a local martingale $Z > 0$ such that ZR is a local martingale. The following lemma helps us with finding Z.

Lemma 4. *Let $X = X_0 + A + M$, $x = x_0 + a + m$, $R = X/x$ and suppose that R_- and R don't vanish then $\mathscr{L}\mathrm{og}(ZR)$ is a local optional martingale if and only if $1 + \frac{1}{X} \circ$*

$A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m,m] - \frac{1}{xX} \circ [M,m] + \frac{1}{ZX} \circ [Z,M] - \frac{1}{xZ} \circ [Z,m] = -1$ *furthermore if* $Z = \mathscr{E}(N) > 0$ *then* $\frac{1}{X} \circ A - \frac{1}{x} \circ a + \frac{1}{x^2} \circ [m,m] - \frac{1}{xX} \circ [M,m] + \frac{1}{X} \circ [N,M] - \frac{1}{x} \circ [N,m] = -1$.

Corollary 1. *Suppose that* R_- *and* R *don't vanish then* $\mathscr{L}\mathrm{og}(ZR) \in \mathscr{M}_{loc} \Leftrightarrow ZR \in \mathscr{M}_{loc}$.

5 Illustrative Examples

Consider a market composed of a bond x and an asset X evolving according to $x_t = x_0 \mathscr{E}(h)_t$ and $X_t = X_0 \mathscr{E}(H)_t$ where $h_t = rt + bL_t^g$, $h_0 = 0$, $H_t = \mu t + \sigma W_t + aL_t^d$, $H_0 = 0$. $L_t^d = L_t - \lambda t$, $L_t^g = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are Poisson with constant intensity λ and γ respectively. Let $\mathscr{F}_t$ be the natural filtration that is neither right or left continuous. Here the bond is modeled by a left continuous process for which we have assumed that its jumps don't necessarily avoid the jumps of the asset (see also [3] for bonds that can experience defaults). We believe our model gives a better description of a portfolio of stocks and bonds than models that assume RCLL processes on usual probability space.

Given x and X the ratio process is $R_t = \frac{X_0}{x_0} \mathscr{E}(H_t - h_t^* - [H, h^*]_t)$. We want to find $Z = \mathscr{E}(N)$ such that ZR is a local martingale. In section 4 we showed that associated with the product $ZR = \frac{X_0}{x_0} \mathscr{E}(\Psi(h,H,N))$ is the process $\Psi(h,H,N)$. To compute a reasonable form for $\Psi(h,H,N)$ we suppose that $N_t = \varsigma W_t + cL_t^d + \theta L_t^g$ an optional local martingale for which Z an optional local martingale deflator, and hence

$$\begin{aligned}
\Psi(h,H,N) &= \left[\varsigma W_t + cL_t^d + \theta L_t^g\right] + \left[\mu t + \sigma W_t + aL_t^d\right] - \left[rt + bL_t^g\right] \\
&+ \left\langle \left[rt + bL_t^g\right]^c - \left[\varsigma W_t + cL_t^d + \theta L_t^g\right]^c, \left[rt + bL_t^g\right]^c - \left[\mu t + \sigma W_t + aL_t^d\right]^c \right\rangle \\
&+ \sum_{0<s\le t} ac\left(\triangle L_s^d\right)^2 + \sum_{0\le s<t} \frac{b(b-\theta)\left(\triangle^+ L_s^g\right)^2}{1 + b\triangle^+ L_s^g} \\
&= (\mu - r + \varsigma\sigma)t + (\varsigma + \sigma)W_t + (c+a)L_t^d + (\theta - b)L_t^g + acL_t \\
&+ (b-\theta)L_t^g + b(\theta - b)\left[L_t^g, L_t^g\right] \\
&= (\mu - r + \varsigma\sigma)t + (\varsigma + \sigma)W_t + (c+a)L_t^d + acL_t + b(\theta - b)\bar{L}_{t-}.
\end{aligned}$$

because W_t and L_t^d are martingales, $\Psi(h,H,N)$ is a martingale if and only if $\mu - r + \varsigma\sigma + ca\lambda + \theta b - b^2\gamma = 0$. The solution of this equation leads to infinitely many solutions which means the market is incomplete. Here are some solutions. Let $\theta = 0$ which leads to right continuous local martingale deflator. Another solution is a one which will eliminate the effects of jumps on drift that is by letting $\theta = -1/b$ and $c = 1/a\lambda$, and hence $\varsigma = \left(r - \mu + b^2\gamma\right)/\sigma$.

This is a simple but important example that we have alluded to in Section 4. Suppose that the market participants can't trade the left-continuous part of the market ratio process R with an optional trading strategy. A way around this problem is to transform RLL semimartingles to RCLL local optional martingales. Consider a market ratio process $R = \mathscr{E}(M) > 0$, $M_t = rt + aW_t + bD_t + cG_t$ where W is a diffusion process, D a right-continuous compensated Poisson with intensity λ and G a left continuous compensated Poisson with intensity γ. Let $N = \alpha W + \beta D + \kappa G$. We like to find $Z = \mathscr{E}(N) > 0$ such that ZR is RCLL. $ZR = \mathscr{E}(N)\mathscr{E}(M) = \mathscr{E}(N + M + [N,M])$. Therefore, $ZR = \mathscr{E}((a+\alpha)W + (b+\beta)D + (c+\kappa)G + (r + a\alpha + b\lambda\beta + c\gamma\kappa)t)$. If we choose $\kappa = -c$ we get rid of the left-continuous part of ZR. And, if we choose α and β such that $r + a\alpha + b\lambda\beta - \gamma c^2 = 0$ we obtain the RCLL local optional martingale ZR we are searching for. However, we still need to make sure that our choice $Z = \mathscr{E}(\alpha W + \beta D - cG) > 0$. This is true, if and only if $(1 + \beta \triangle D_t)(1 - c \triangle^+ G_t) > 0$ or $(1+\beta)(1-c) > 0$ for all time t.

6 Conclusion

Optional processes including left continuous ones are a natural occurrence in financial markets. For example, a defaultable bond is modeled by a left continuous process in [3], stochastic dividends in [17] and transaction costs in [2, 4]. Also optional processes appears for optimal consumption from investment [16], for consumption from investment with random endowment [10, 18, 15]. Furthermore, optional processes naturally arise in the context of super-replication in incomplete markets as a result of optional decomposition theorem [11]. Therefore, to study financial markets with optional processes, left-continuous processes and a mixture of right and left continuous processes the calculus of optional processes is going to be an indispensable tool in the future study of mathematical finance. We have introduced the notions of the unusual basis and optional semimartingales, presented a summary of the calculus of optional processes and introduced new results of stochastic logarithms. We have also described the optional semimartingale model of financial market and described a procedure of finding local martingale deflators for this market. Finally we presented illustrative examples. The first example is of a portfolio of a left continuous bond and right continuous stock where we showed how to find local martingale deflators for this market. The second examples demostrates a method for finding a subset of local martingale deflators that transforms a ladlag semimartingale to cadlag local optional martingale.

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