

Relations

Chapter 9

Chapter Summary

- Relations and Their Properties
- Representing Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

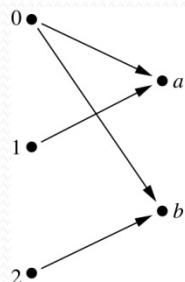
- Relations and Functions
- Properties of Relations
 - Reflexive Relations
 - Symmetric and Antisymmetric Relations
 - Transitive Relations
- Combining Relations

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A .

Binary Relation on a Set

Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$.

The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are

$(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A ?

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A ?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A .

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid |a| = |b|\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

For each of these pairs, determine to which relations it belongs

(1,1), (1, 2), (2, 1), (1, -1), and (2, 2)

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

For each of these pairs, determine to which relations it belongs

(1,1), (1, 2), (2, 1), (1, -1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1 , R_3 , R_4 , and R_6 : (1,2) is in R_1 and R_6 : (2,1) is in R_2 , R_5 , and R_6 : (1, -1) is in R_2 , R_3 , and R_6 : (2,2) is in R_1 , R_3 , and R_4 .

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in A \rightarrow (x,x) \in R]$$

Example: The following relations on the set of integers are reflexive:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_3 = \{(a,b) \mid |a| = |b|\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

If $A = \emptyset$ then the empty relation is reflexive vacuously: The empty relation on an empty set is reflexive!

The following relations are not reflexive:

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 3 > 3\text{)},$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 3 \neq 3 + 1\text{)},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that } 4 + 4 \not\leq 3\text{)}.$$

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y [(x,y) \in R \rightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$$R_3 = \{(a,b) \mid |a| = |b|\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

The following are not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \text{ (note that } 3 \leq 4, \text{ but } 4 \not\leq 3\text{)},$$

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 4 > 3, \text{ but } 3 \not> 4\text{)},$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1\text{)}.$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

- **Example:** The following relations on the integers are antisymmetric:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_4 = \{(a, b) \mid a = b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\}.$$

For any integer, if $a \leq b$ and $a \leq b$, then $a = b$.

The following relations are not antisymmetric:

$$R_3 = \{(a, b) \mid |a| = |b|\}$$

(note that both $(1, -1)$ and $(-1, 1)$ belong to R_3),

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$
 (note that both $(1, 2)$ and $(2, 1)$ belong to R_6).

Transitive Relations

Definition: A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$$

- **Example:** The following relations on the set of integers are transitive:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid |a| = |b|\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

For every integer, $a \leq b$
and $b \leq c$, then $b \leq c$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that both $(4,3)$ and $(3,2)$ belong to R_5 , but not $(4,2)$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(2,1)$ and $(1,2)$ belong to R_6 , but not $(2,2)$).

Combining Relations

- Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- Example:** Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \quad R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Composition

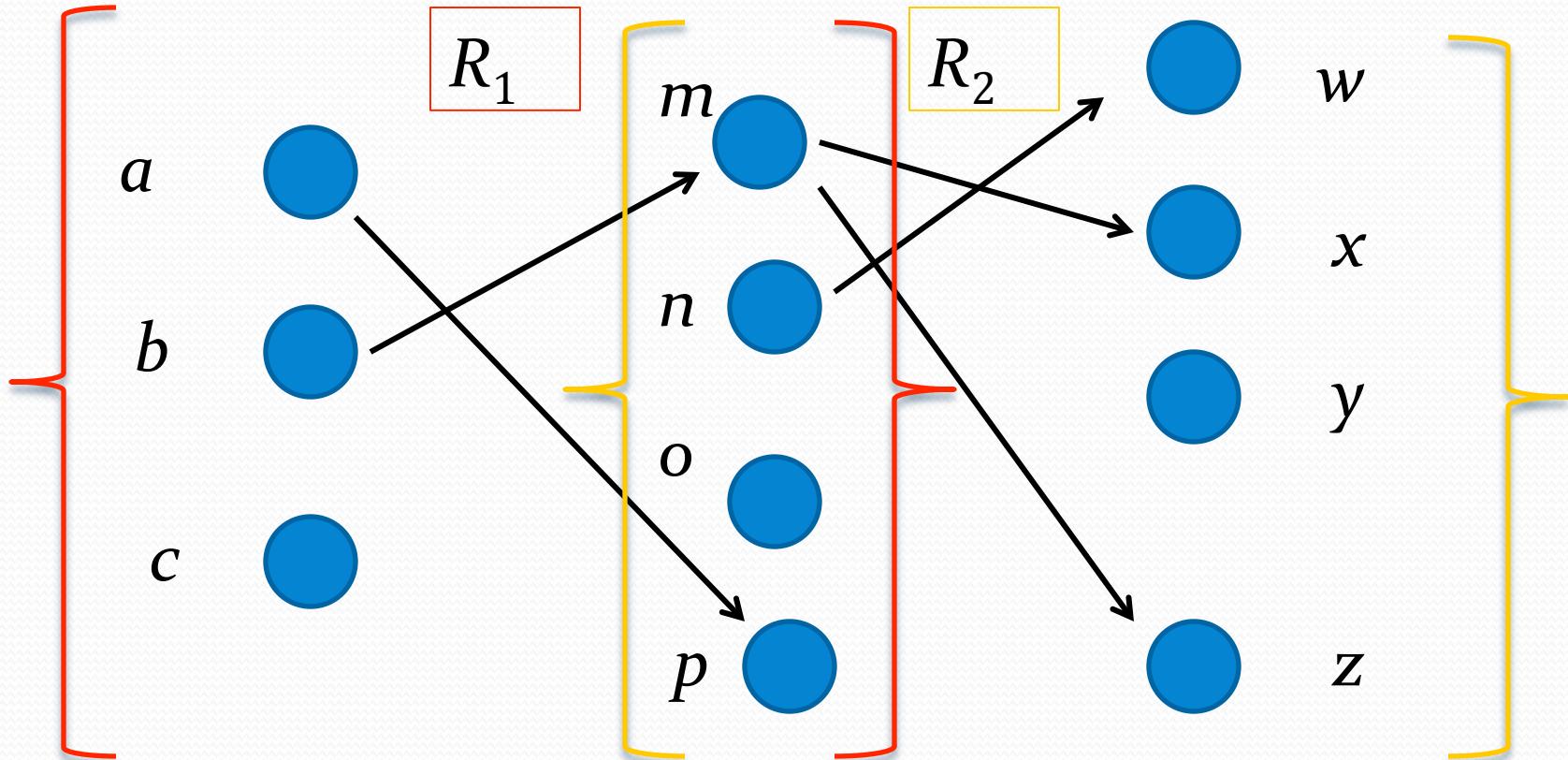
Definition: Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from B to a set C .

Then the *composition* (or *composite*) of R_2 with R_1 , is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2 , then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



Composition of a relation with itself

Definition: Suppose R is a relation on a set A .

Then the *composition* (or *composite*) of R with R , denoted by $R \circ R$, is a relation on A where

- if (x,y) is a member of R and (y,z) is a member of R , then (x,z) is a member of $R \circ R$.

Example: Let R be the relation on the set of all people such that (a, b) is in R if person a is parent of person b . Then (a, c) is in $(R \circ R)$ iff there is a person b such that (a, b) is in R and (b, c) is in R . In other words, (a, c) is in R if and only if a is a grandparent of c .

Powers of a Relation

Definition: Let R be a binary relation on A . Then the powers R^n of the relation R can be defined recursively by:

- Basis Step: $R^1 = R$
- Recursive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for all positive integers n .

(see the text for a proof via mathematical induction)

N-ary relations

- **Definition:** Let A_1, A_2, \dots, A_n be sets. An n -ary relation on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$. The sets A_1, A_2, \dots, A_n are called the domains of the relation and n is called the degree.
- **Example:** Let R be the ternary relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_+$ consisting of triples (a, b, m) where $a \equiv b \pmod{m}$
Then $(-1, 9, 5)$, $(8, 2, 3)$ and $(14, 0, 7)$ belong to R , but $(7, 2, 3)$ does not.
- **Example:** Let R' be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination and T is the departure time. The degree of this relation is 5.

Databases and Relations

- Relations are often used to represent **databases**
- Relations used to represent databases are also called *tables*, because they are often displayed as tables
- The database query language **SQL (Structured Query Language)** is used to carry out operations on tables
- Each column of the table corresponds to an **attribute**
- The **natural join** binary operator combines two tables into one, when these tables share some identical fields.
- The result of the **natural join** between R and S is the set of all combinations of tuples in R and S that are equal on their common attribute names.

Natural Join Example

- A table contains fields for airline, flight number, and gate
- A second table contains fields for flight number, gate, and departure time.
- One can use *natural join* to combine these two relations into a third one containing fields for airline, flight number, gate and departure time.

TABLE 5 Teaching_assignments.

<i>Professor</i>	<i>Department</i>	<i>Course_number</i>
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

<i>Department</i>	<i>Course_number</i>	<i>Room</i>	<i>Time</i>
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

(Table 5) natural join (Table 6) = (Table 7)

TABLE 7 Teaching_schedule.

<i>Professor</i>	<i>Department</i>	<i>Course_number</i>	<i>Room</i>	<i>Time</i>
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a **zero-one matrix**.
- Suppose R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.
 - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the $m \times n$ matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j .

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Examples of Representing Relations Using Matrices (cont.)

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

- If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

- R is a **symmetric** relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an **antisymmetric** relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

$$\begin{bmatrix} & & 1 & \\ 1 & & & 0 \\ & 0 & & \end{bmatrix}$$

(a) Symmetric

$$\begin{bmatrix} & & 1 & \\ 0 & & & 0 \\ & 0 & & \end{bmatrix}$$

(b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Matrices for combinations of relations

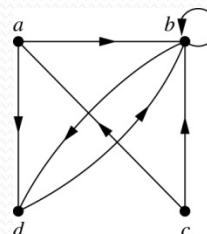
- The matrix of the **union** of two relations is the **join** (Boolean OR) between the matrices of the component relations
- The matrix of the **intersection** of two relations if the **meet** (Boolean AND) between the matrices of the component relations
- The matrix of the **composite relation** ($R \circ R$) is the **Boolean Product** between the matrix representing R and itself.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b) , and the vertex b is called the *terminal vertex* of this edge.

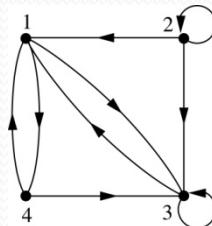
- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a , b , c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?

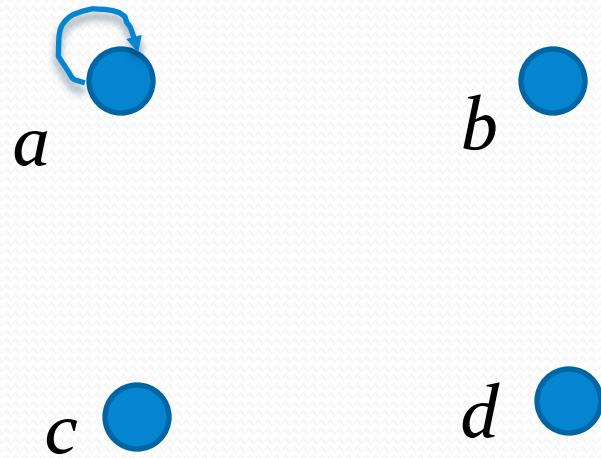


Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

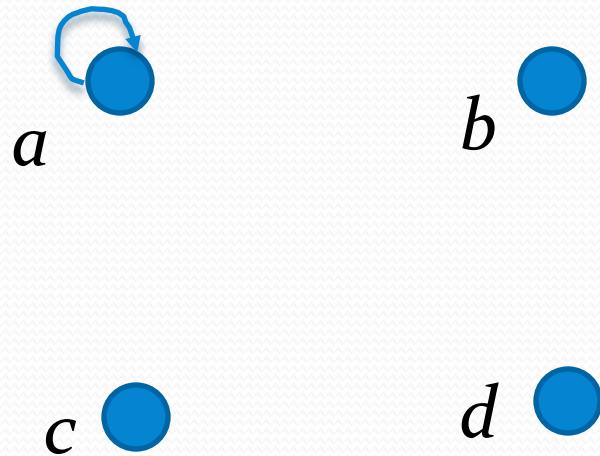
Determining Properties of a Relation from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x) .
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z) .

Determining Properties of a Relation from its Digraph – Example 1

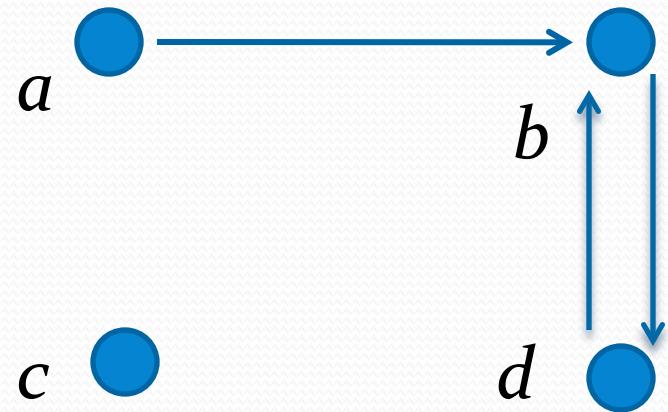


Determining Properties of a Relation from its Digraph – Example 1

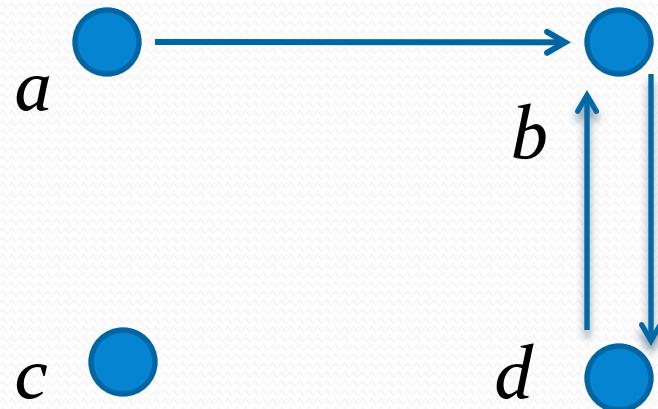


- *Reflexive?* No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from any one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from any one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from any one vertex to another

Determining Properties of a Relation from its Digraph – Example 2

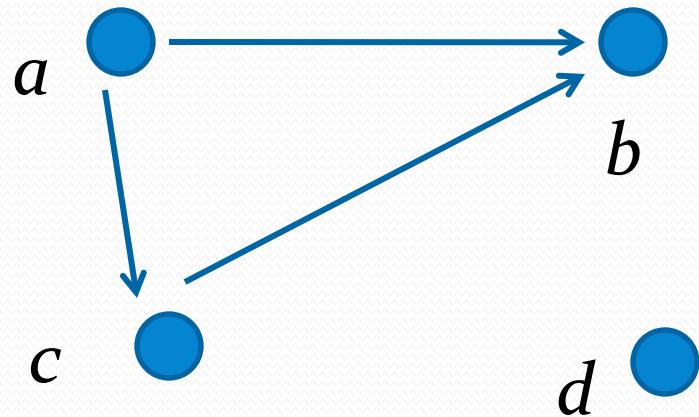


Determining Properties of a Relation from its Digraph – Example 2

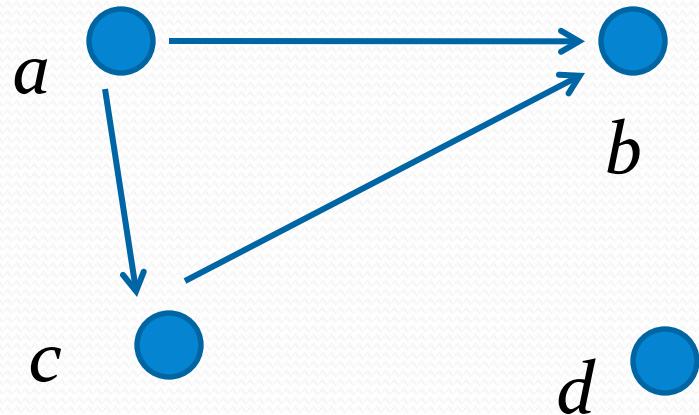


- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from a to b , but not from b to a
- *Antisymmetric?* No, there is an edge from d to b and b to d
- *Transitive?* No, there are edges from a to c and from c to b ,
but there is no edge from a to d

Determining Properties of a Relation from its Digraph – Example 3



Determining Properties of a Relation from its Digraph – Example 3



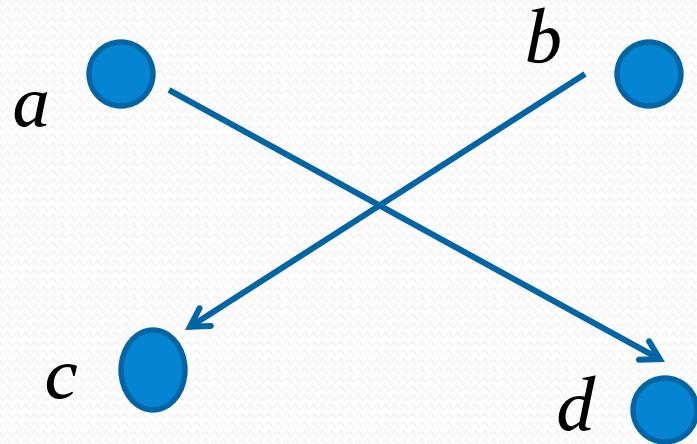
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from c to a

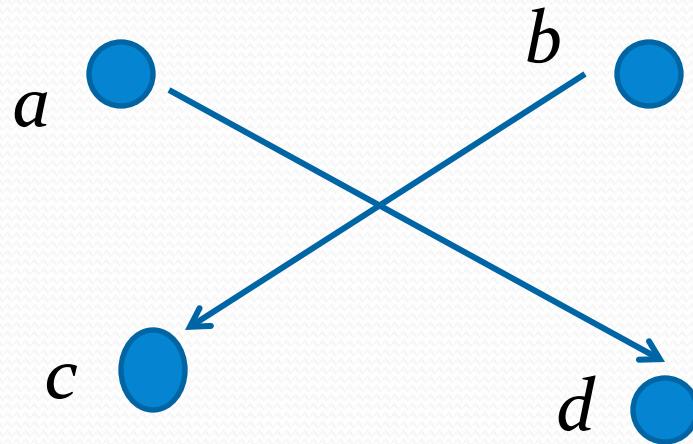
Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

Transitive? Yes

Determining Properties of a Relation from its Digraph – Example 4

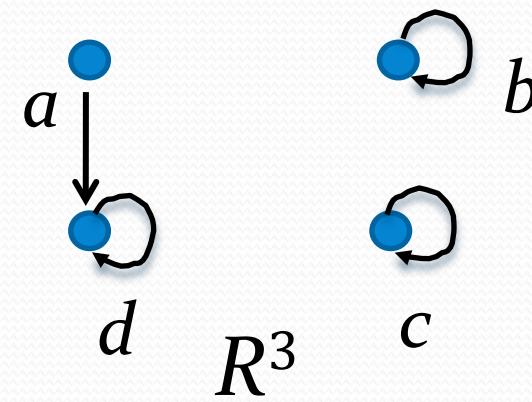
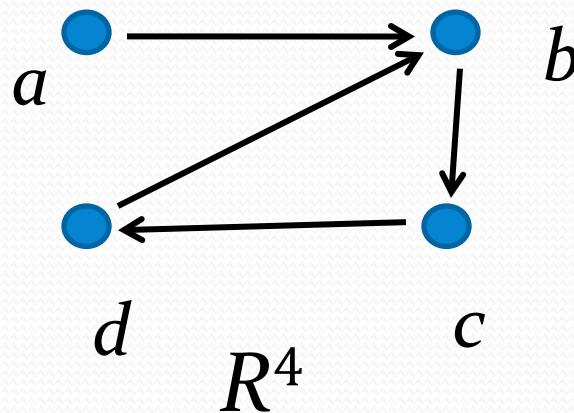
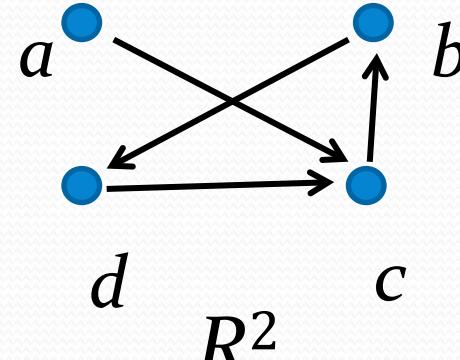
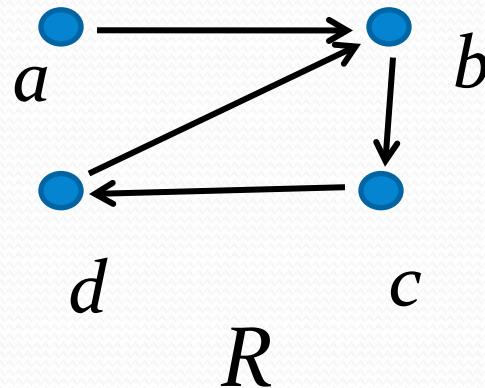


Determining Properties of a Relation from its Digraph – Example 4



- *Reflexive?* No, there are no loops
- *Symmetric?* No, for example, there is no edge from d to a
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is **reflexive**, **symmetric**, and **transitive**.

Definition 2: Two elements a , and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity:* Because $l(a) = l(a)$, it follows that aRa for all strings a .
- *Symmetry:* Suppose that aRb . Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa .
- *Transitivity:* Suppose that aRb and bRc . Since $l(a) = l(b)$, and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc .

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity:* $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry:* Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m , and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.
- *Transitivity:* Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the “divides” relation on the set of positive integers is not an equivalence relation.

Divides

Example: Show that the “divides” relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, “divides” is not an equivalence relation.

- *Reflexivity:* $a \mid a$ for all a .
- *Not Symmetric:* For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity:* Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a . The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R , for this equivalence class.

Note that $[a]_R = \{s | (a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m* . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{..., a-2m, a-m, a+m, a+2m, ... \}$. For example,

$$[0]_4 = \{..., -8, -4, 0, 4, 8, ...\}$$

$$[1]_4 = \{..., -7, -3, 1, 5, 9, ...\}$$

$$[2]_4 = \{..., -6, -2, 2, 6, 10, ...\}$$

$$[3]_4 = \{..., -5, -1, 3, 7, 11, ...\}$$

Equivalence Classes and Partitions

Theorem 1: Let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

- (i) aRb
- (ii) $[a] = [b]$
- (iii) $[a] \cap [b] \neq \emptyset$

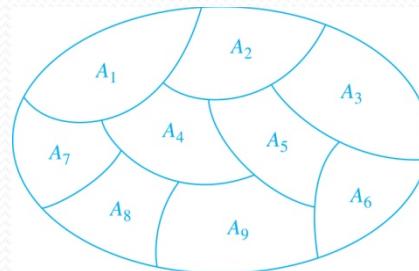
Proof: We show that (i) implies (ii). Assume that aRb . Now suppose that $c \in [a]$. Then aRc . Because aRb and R is symmetric, bRa . Because R is transitive and bRa and aRc , it follows that bRc . Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(see text for proof that (ii) implies (iii) and (iii) implies (i))

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A . The union of all the equivalence classes of R is all of A , since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A , because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S . Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity:* For every $a \in S$, $(a, a) \in R$, because a is in the same subset as itself.
- *Symmetry:* If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.
- *Transitivity:* If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c . Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.

Example

- Let n be a positive integer and S be a set of strings.
- Suppose that R_n is the relation on S such that $s R_n t$ if and only if $s = t$ or both s and t have at least n characters and the first n characters of s and t are the same.
- Show that for every set S and every positive integer n , R_n is an equivalence relation on S .

Identifiers in C programming

- In the C programming language, an **identifier** is the name of a variable, a function or another type of entity
- Each **identifier** is a nonempty string of characters: each character is lowercase/uppercase English letter, a digit, or underscore and the first character is a lowercase/uppercase English letter.
- Identifiers can be of any length
- However, compilers for some versions of C, have a limit on the number of characters checked when two names are compared to see if they refer to the same thing

Identifiers in C programming

- Standard C compilers consider two identifiers the same when they agree on their first 31 characters
- Two identifiers are the same if they are in relation R_{31} (see previous slide) which is an equivalence relation
- What are the equivalence classes of each of the identifiers
 - Number_of_tropical_storms
 - Number_of_named_tropical_storms
 - Number_of_named_tropical_storms_in_the_Atlantic_in_2005

Partial Orderings

Section 9.6

Section Summary

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is **reflexive**, **antisymmetric**, and **transitive**.

A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.

- *Reflexivity:* $a \geq a$ for every integer a .
- *Antisymmetry:* If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity:* If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (See Appendix 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation ($|$) is a partial ordering on the set of positive integers.

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (\mid) is a partial ordering on the set of positive integers.

- *Reflexivity:* $a \mid a$ for all integers a . (see Example 9 in Section 9.1)
- *Antisymmetry:* If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (see Example 12 in Section 9.1)
- *Transitivity:* Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.
- (\mathbb{Z}^+, \mid) is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation (\subseteq) is a partial ordering on the power set of a set S .

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation (\subseteq) is a partial ordering on the power set of a set S .

- *Reflexivity:* $A \subseteq A$ whenever A is a subset of S .
- *Antisymmetry:* If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity:* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset (S, \leq) are *comparable* if either $a \leq b$ or $b \leq a$. When a and b are elements of S so that neither $a \leq b$ nor $b \leq a$, then a and b are called *incomparable*.

The symbol \leq is used to denote the relation in any poset.

Definition 3: If (S, \leq) is a partially ordered set and *every two elements of S are comparable*, S is called a *totally ordered* or *linearly ordered set*, and \leq is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Definition 4: (S, \leq) is a *well-ordered* set if it is a poset such that \leq is a total ordering and *every nonempty subset of S has a least element*.

Example: (\mathbb{Z}, \leq) is a totally ordered set.

$(\mathbb{Z}, |)$ is a partially ordered but not totally ordered set.

(\mathbb{N}, \leq) is a well-ordered set.

Lexicographic Order

Definition: Given two partially ordered sets (A_1, \leq_1) and (A_2, \leq_2) , the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2) , that is,

$$(a_1, a_2) < (b_1, b_2),$$

either if $a_1 <_1 b_1$ or if $a_1 = b_1$ and $a_2 <_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings .

Example: Consider strings of lowercase English letters. A *lexicographic ordering* can be defined using the ordering of the letters in the alphabet.

This is the same ordering as that used in dictionaries.

- *discreet* < *discrete*, because these strings differ in the seventh position and *e* < *t*.
- *discreet* < *discreteness*, because the first eight letters agree, but the second string is longer.